Real analysis

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Preface

This is a brief note of *Real Analysis* by Elias M. Stein & Rami Shakarchi.

Chapter 1

Measure Theory

1.1 Preliminaries

The **open ball** in $\mathbb{R}^d$ centered at x and of radius r is defined by

$$B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r\}.$$

A subset $E \subset \mathbb{R}^d$ is **open** if for every $x \in E$ there exists $r > 0$ with $B_r(x) \subset E$. By definition, a set is **closed** if its complement is open.

We note that any (not necessarily countable) union of open sets is open, while in general the intersection of only finitely many open sets is open. A similar statement holds for the class of closed sets, if one interchanges the roles of unions and intersections.

A set E is **bounded** if it is contained in some ball of finite radius. A bounded set is **compact** if it is also closed. Compact sets enjoy the **Heine-Borel covering property**:

- Assume E is compact, $E \subset \bigcup_\alpha O_\alpha$, and each O_α is open. Then there are finitely many of the open sets, $O_{\alpha_1}, O_{\alpha_2}, \dots, O_{\alpha_N}$, such that $E \subset \bigcup_{j=1}^{n} O_{\alpha_j}$.

In words, any covering of a compact set by a collection of open sets contains a finite subcovering.

A point $x \in \mathbb{R}^d$ is a **limit point** of the set E if for every $r > 0$,the ball $B_r(x)$ contains points of E. This means that there are points in E which are arbitrarily close to x. An **isolated point** of E is a point $x \in E$ such that there exists an $r > 0$ where $B_r(x) \cap E$ is equal to $\{x\}$.

A point $x \in E$ is an **interior point** of E if there exists $r > 0$ such that $B_r(x) \subset E$. The set of all interior points of E is called the **interior** of E. Also, the **closure** $\overline{E}$ of the E consists of the union of E and all its *limit points.* The **boundary** of a set E, denoted by ∂E, is the set of points which are in the closure of E but not in the interior of E.

Note that the closure of a set is a closed set; every point in E is a limit point of E; and a set is closed if and only if it contains all its limit points. Finally, a closed set E is **perfect** if E does not have any isolated points.

Lemma 1.1. *If a rectangle is the almost disjoint union of finitely many other rectangles, say* $R = \bigcup_{k=1}^{N} R_k$*, then*

$$|R| = \sum_{k=1}^{N} |R_k|$$

Lemma 1.2. *If* $R, R_1, \ldots, R_N$ *are rectangles, and* $R \subset \bigcup_{k=1}^{N} R_k$*, then*

$$|R| \leq \sum_{k=1}^{N} |R_k|$$

Proof.

Every open subset O of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.

Every open subset O of $\mathbb{R}^d$, $d \geq 1$, can be written uniquely as a countable union of almost disjoint closed cubes.

The **Cantor set** C is by definition the intersection of all C_k's:

$$\mathcal{C} = \bigcap_{k=0}^{\infty} C_k.$$

The set C is not empty, since all end-points of the intervals in C_k (all k) belong to C. Despite its simple construction, the Cantor set enjoys many interesting topological and analytical properties. For instance, C is closed and bounded, hence compact. Also, C is totally disconnected: given any $x, y \in C$ there exists $z \notin C$ that lies between x and y. Finally, C is perfect: it has no isolated points.

In terms of cardinality the Cantor set is rather large: it is not countable. Since it can be mapped to the interval $[0, 1]$, the Cantor set has the cardinality of the continuum. However, from the point view of "length" the size f C is small. Roughly speaking, it has length zero.

1.2 The exterior measure

Definition 1.1. If E is any subset of $\mathbb{R}^d$, the **exterior measure** of E is

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|,$$

where the infimum is taken over all countable coverings $E \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes. In general we have $0 \leq m_*(E) \leq \infty$.

- It would not suffice to allow finite sums in the definition of $m_*(E)$. The quantity that would be obtained if one considered only coverings of E by finite unions of cubes is in general larger than $m_*(E)$.
- One can, however, replace the coverings by cubes, with coverings by rectangles; or with coverings by balls.

Example 1.1. The exterior measure of a closed cube is equal to its volume.

Proof.

We consider an arbitrary covering $Q \subset \bigcup_{j=1}^{\infty} Q_j$ by cubes, where Q_j is closed for $j = 1, 2, \ldots$. Then it suffices to prove that

$$|Q| \leq \sum_{j=1}^{\infty} |Q_j|.$$

Example 1.2. The exterior measure of a open cube is equal to its volume.

Example 1.3. The exterior measure of a rectangle is equal to its volume.

Proof.

Arguing as in the first example, we see that $|R| \leq m_*(R)$. To obtain the reverse inequality, consider a grid in $\mathbb{R}^d$ formed by cubes of side length $1/k$, then let k tend to infinity yields $m_*(R) \leq |R|$.

Example 1.4. The Cantor set C has exterior measure 0.

1.2.1 Properties of the exterior measure

From the definition we know that: - For every $\epsilon > 0$, there exists a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$ with

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon.$$

Proposition 1.1.

1. *(Monotonicity) If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$.*
2. *(Countatble sub-additivity) If $E = \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.*
3. *If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf m_*(O)$, where the infimum is taken over all open sets O containing E.*
   Proof.
   It suffices to prove that $m_*(E) \geq \inf m_*(O)$. Notice that for any closed cube Q_j, there exists an open set S_j which contains Q_j and such that $|S_j| \leq (1+\epsilon)|Q_j|$ for a fixed $\epsilon > 0$.

4. *If* $E = E_1 \cup E_2$,*and* $d(E_1, E_2) > 0$, *then*

$$m_*(E) = m_*(E_1) + m_*(E_2).$$

5. *If a set* E *is the countable union of almost disjoint cubes* $E = \bigcup_{j=1}^{\infty} Q_j$, *then*

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j|.$$

Proof.
Consider constructing a family of cubes that are at a finite distance from one another, so we can use proposition 4.

1.3 Measurable sets and the Lebesgue measure

The notion of measurability isolates a collection of subsets in $\mathbb{R}^d$ for which the exterior measure satisfies all our desired properties, including additivity (and in fact countable additivity) for disjoint unions of sets.

There are a number of different ways of defining measurability, but these all turn out to be equivalent. Probably the simplest and most intuitive is the following: A subset E of $\mathbb{R}^d$ is **Lebesgue measurable**, or simply **measurable**, if for any $\epsilon > 0$ there exists an open set O with $E \subset O$ and

$$m_*(\mathcal{O} - E) \leq \epsilon.$$

This should be compared to Proposition 1.1 3, which holds for all sets E. If E is measurable, we define its **Lebesgue measure** (or **measure**) $m(E)$ by

$$m(E) = m_*(E).$$

Immediately from the definition, we find:

Proposition 1.2.

1. *Every open set in* $\mathbb{R}^d$ *is measurable.*
2. *If* $m_*(E) = 0$, *then* E *is measurable. In particular, if* F *is a subset of a set of exterior measure* 0, *then* F *is measurable.*
3. *A countable union of measurable sets is measurable.*
4. *Closed sets are measurable.*
   Proof.
   Since any closed set F *can be written as the union of compact sets, say* $F = \bigcup_{k=1}^{\infty} F \cap B_k$, *where* B_k *denotes the closed ball of radius* k *centered at the origin, it suffices to prove that compact sets are measurable.*

5. *The complement of a measurable set is measurable.*
   Proof.
   For every positive integer n we choose an open set O_n with $E \subset O_n$ and $m_*(\mathcal{O}_n - E) \leq 1/n$. Notice that

$$(E^c - \bigcup_{n=1}^{\infty} \mathcal{O}_n^c) \subset (\mathcal{O}_n - E)$$

   then E^c is measurable since $E^c = (E^c - \bigcup_{n=1}^{\infty} \mathcal{O}_n^c) \cup \bigcup_{n=1}^{\infty} \mathcal{O}_n^c$.
6. *A countable intersection of measurable sets is measurable.*

To prove the fourth proposition, we need the following lemma.

Lemma 1.3. *If F is closed, K is compact, and these sets are disjoint, then $d(F, K) > 0$.*

Theorem 1.1. *If $E_1, E_2, ...,$ are disjoint measurable sets, and $E = \bigcup_{j=1}^{\infty} E_j$, then*

$$m(E) = \sum_{j=1}^{\infty} m(E_j).$$

Proof.

If $F_1, F_2, \ldots, F_N$, are compact and disjoint, then obviously for any $j, k, j \neq k$, $d(F_j, F_k) > 0$, so $m\left(\bigcup_{j=1}^{N} F_j\right) = \sum_{j=1}^{N} m(F_j)$. If each E_j is bounded, we can choose a closed subset F_j for E_j with $m_*(E_j - F_j) \leq \epsilon/2^j$ for each j. Then

$$m(E) \geq \sum_{j=1}^{N} m(F_j) \geq \sum_{j=1}^{N} m(E_j) - \epsilon$$

Letting N tend to infinity, since ϵ is arbitrary we find that

$$m(E) \geq \sum_{j=1}^{\infty} m(E_j).$$

In the general case, we select any sequence of cubes $\{Q_k\}_{k=1}^{\infty}$ that increases to $\mathbb{R}^d$, in the sense that $Q_k \subset Q_{k+1}$ for all $k \geq 1$ and $\bigcup_{k=1}^{\infty} Q_k = \mathbb{R}^d$. We then let $S_1 = Q_1$ and $S_k = Q_k - Q_{k-1}$ for $k \geq 2$. If we define measurable sets by $E_{j,k} = E_j \cap S_k$, then

$$m(E) = \sum_{j,k} m(E_{j,k}) = \sum_{j} \sum_{k} m(E_{j,k}) = \sum_{j} m(E_j).$$

Corollary 1.1. *Suppose $E_1, E_2, ...$ are measurable subsets of $\mathbb{R}^d$.*

- *If $E_k \nearrow E$, then $m(E) = \lim_{N\to\infty} m(E_N)$.*

- *If $E_k \searrow E$ and $m(E_k) < \infty$ for some k, then $m(E) = \lim_{N\to\infty} m(E_N)$.*

Theorem 1.2. *Suppose E is measurable subset of $\mathbb{R}^d$. Then for every $\epsilon > 0$:*

1. *There exists an open set O with $E \subset O$ and $m(\mathcal{O} - E) \leq \epsilon$.*
2. *There exists a closed set F with $F \subset E$ and $m(E - F) \leq \epsilon$.*
3. *If $m(E)$ is finite, there exists a compact set K with $K \subset E$ and $m(E-K) \leq \epsilon$.*
4. *If $m(E)$ is finite, there exists a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes such that*
$$m(E \triangle F) \leq \epsilon.$$

Proof.
Choose a family of closed cubes $\{Q_j\}_{j=1}^{\infty}$ so that

$$E \subset \bigcup_{j=1}^{\infty} Q_j \text{ and } \sum_{j=1}^{\infty} |Q_j| \leq m(E) + \epsilon/2.$$

Since $m(E) < \infty$, then series converges and there exists $N > 0$ such that $\sum_{j=n+1}^{\infty} |Q_j| \leq \epsilon/2$. If $F = \bigcup_{j=1}^{N} Q_j$, then

$$\begin{aligned} m(E \triangle F) &= m(E-F) + m(F-E) \\ &\leq m\left(\bigcup_{j=1}^{\infty} Q_j - F\right) + m\left(\bigcup_{j=1}^{\infty} Q_j - E\right) \\ &\leq m\left(\bigcup_{j=n+1}^{\infty} Q_j\right) + m\left(\bigcup_{j=n+1}^{\infty} Q_j\right) - m(E) \\ &\leq \sum_{j=n+1}^{\infty} |Q_j| + \sum_{j=1}^{\infty} |Q_j| - m(E) \\ &\leq \epsilon. \end{aligned}$$

1.3.1 Invariance properties of Lebesgue measure

1.3.2 σ-algebra and Borel sets

Borel σ-algebra in $\mathbb{R}^d$, denoted by $B_{\mathbb{R}^d}$, is the smallest σ-algebra in $\mathbb{R}^d$ that contains all open sets. Elements of this σ-algebra are called **Borel sets**. Since we observe that any intersection (not necessarily countable) of σ-algebra is again a σ-algebra, we may define $B_{\mathbb{R}^d}$ as the intersection of all σ-algebras that contain the open sets. This shows the existence and uniqueness of the Borel σ-algebra.

Remark. There exists Lebesgue measurable sets that are not Borel sets. (See Exercise ???ref(exr: #35)

1.3.3 Construction of a non-measurable set

1.3.4 Axiom of choice

1.4 Measurable functions

1.4.1 Definition and basic properties

The starting point is the notion of a **characteristic function** of a set E, which is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

For the Riemann integral it is in effect theclass of **step functions** that build the blocks of integration theory , with each give as a finite sum

$$f = \sum_{k=1}^{N} a_k \chi_{R_k}$$

where each R_k is a rectangle, and the a_k are constants.

For the lebesgue integral we need a more general notion. A **simple function** is a finite sum

$$f = \sum_{k=1}^{N} a_k \chi_{E_k}$$

where each E_k is a measurable set of finite measure, and the a_k are constants.

A function f defined on a measurable subset E of $\mathbb{R}^d$ is **measurable**, if for all $a \in \mathbb{R}$, the set

$$f^{-1}([-\infty, a)) = \{x \in E : f(x) < a\}$$

is measurable. Note that this definition applies to extended-valued functions, so we use $f^{-1}([-\infty, a))$ instead of $f^{-1}((-\infty, a))$.

Proposition 1.3.

1. *The finite-valued function f is measurable iff $f^{-1}(O)$ is measurable for every open set O, and iff $f^{-1}(F)$ is measurable for every closed set F. (Note that this property also applies to extended-valued functions, if we make the additional hypothesis that both $f^{-1}(\infty)$ and $f^{-1}(-\infty)$ are measurable sets since $[-\infty, a) = \{-\infty\} \cup \left(\bigcup_{n=1}^{\infty}(-n, a)\right)$.)*
2. *If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued, and Φ is continuous, then $\Phi \circ f$ is measurable. (Note that it is not true that $f \circ \Phi$ is measurable. See exercise 35.)*
   Proof.
   Φ is continuous, so $\Phi^{-1}((-\infty, a))$ is open set O.

3. *Suppose* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of measurable functions. Then*

$$\sup_n f_n(x), \quad \inf_n f_n(x), \quad \limsup_{n\to\infty} f_n(x) \quad \text{and} \quad \liminf_{n\to\infty} f_n(x)$$

are measurable.

4. *If* $\{f_n\}_{n=1}^{\infty}$ *is a collection of measurable functions, and*

$$\lim_{n\to\infty} f_n(x) = f(x),$$

then f *is measurable.*

5. *If* f *and* g *are measurable, then*

(i) *The integer powers* f^k, $k \geq 1$ *are measurable.*
(ii) $f + g$ *and* fg *are measurable if both* f *and* g *are finite-valued.*

6. *Suppose* f *is measurable, and* $f(x) = g(x)$ *for a.e.* x*. Then* g *is measurable.*

1.4.2 Approximation by simple functions or step functions

Theorem 1.3.

Suppose f *is a non-negative measurable function on* $\mathbb{R}^d$*. Then there exists an increasing sequence of non-negative simple functions* $\{\varphi_n\}_{n=1}^{\infty}$ *that converges pointwise to* f*, namely,*

$$\varphi_k(x) \leq \varphi_{k+1}(x) \text{ and } \lim_{k\to\infty} \varphi_k(x) = f(x), \text{ for all } x.$$

Proof.

Φ *is continuous, so* $\Phi^{-1}((-\infty, a))$ *is open set* O.

Theorem 1.4.

Suppose f *is a measurable function on* $\mathbb{R}^d$*. Then there exists a sequence of simple functions* $\{\varphi_n\}_{n=1}^{\infty}$ *that satisfies*

$$|\varphi_k(x)| \leq |\varphi_{k+1}(x)| \text{ and } \lim_{k\to\infty} \varphi_k(x) = f(x), \text{ for all } x.$$

Proof.

We use the decomposition of f*:* $f(x) = f^+(x) - f^-(x)$.

Theorem 1.5.

Suppose f *is measurable on* $\mathbb{R}^d$*. Then there exists an sequence of step functions* $\{\psi_n\}_{n=1}^{\infty}$ *that converges pointwise to* $f(x)$ *for almost every* x*,*

$$\varphi_k(x) \leq \varphi_{k+1}(x) \text{ and } \lim_{k\to\infty} \varphi_k(x) = f(x), \text{ for all } x.$$

Proof.

By Theorem 1.4, it suffices to show that if E is a measurable set with finite measure, then $f = \chi_E$ can be approximated by step functions. This can be proven by split E into cubes and then rectangles with Theorem 1.2.

1.4.3 Littlewood's three principles

Theorem 1.6 (Egorov)**.** *Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, and assume that $f_k \to f$ a.e. on E and $f_1, f_2, \dots, f_k, f$ are finite valued a.e. on E. Given $\epsilon > 0$, we can find a closed set $A_\epsilon \subset E$ such that $m(E - A_\epsilon) \le \epsilon$ and $f_k \to f$ uniformly on A_ϵ.*

Remark. Note that $f_1, f_2, \dots, f_k, f$ are finite valued a.e. on E. Indeed, if it is not satisfied, then we cannot construct $\{E_k^n\}_{k=1}^{\infty}$ such that $E_k^n \nearrow E$. A counterexample is that

$$f_k(x) = \begin{cases} k & |x| \le k, \\ \infty & |x| > k \end{cases}$$

and $f(x) = \infty$ on $\mathbb{R}$.

Remark. Note that $m(E) < \infty$ and it is easy to construct counterexamples when $m(E) = \infty$. Indeed, if $m(E) = \infty$, then we cannot find k_n such that $m(E - E_{k_n}^n) < 1/2^n$ since $m(E - E_{k_n}^n) = m(E) - m(E_{k_n}^n)$.

Theorem 1.7 (Lusin)**.** *Suppose f is measurable and finite valued a.e. on E with E of finite measure. Then for every $\epsilon > 0$ there exists a closed set F_ϵ, with*

$$F_\epsilon \subset E, \text{ and } m(E - F_\epsilon) \le \epsilon$$

and such that $f|_{F_\epsilon}$ is continuous.

1.5 The Brunn-Minkowski inequality

Remark.

Let $a, b \ge 0$, then

$$(a+b)^\gamma \ge a^\gamma + b^\gamma \text{ if } \gamma \ge 1,$$
$$(a+b)^\gamma \le a^\gamma + b^\gamma \text{ if } 0 < \gamma < 1$$

Proof.

Let $f(\gamma) = (1+x)^\gamma - (1+x^\gamma)$, where $x > 0$. Then

$$\begin{aligned} f'(\gamma) &= (1+x)^\gamma \ln(1+x) - x^\gamma \ln x \\ &= [(1+x)^\gamma - x^\gamma] \ln(1+x) + x^\gamma \ln(1+1/x) > 0, \end{aligned}$$

notice that $f(1) = 0$, so when $\gamma \ge 1$, $(1+x)^\gamma \ge (1+x^\gamma)$ and when $0 < \gamma < 1$, $(1+x)^\gamma < (1+x^\gamma)$. With this result, the original inequality is obvious.

???ref(exr:19)

Theorem 1.8. *Suppose A and B are measurable sets in $\mathbb{R}^d$ and their sum $A+B$ is also measurable. Then*

$$m(A+B)^{1/d} \geq m(A)^{1/d} + m(B)^{1/d}.$$

1.6 Exercise

1.7 Problem

Chapter 2

Integration Theory

2.1 The lebesgue integral: basic properties and convergence theorems

2.1.1 Stage one: simple functions

Proposition 2.1. *The integral of simpe functions defined bve satifies the following properties:*

1. *Independence of the representation. If $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ is any representation of φ, then*
$$\int \varphi = \sum_{k=1}^{N} a_k m(E_k).$$
2. *Linearity.*
3. *Additivity.*
4. *Monotonicity.*
5. *Triangle inequality. If φ is a simple function, then so is $|\varphi|$, and*
$$\left| \int \varphi \right| \leq \int |\varphi|.$$

2.1.2 Stage two: bounded functions supported on a set of finite measure

Lemma 2.1. *Let f be a bounded function supported on a set E of finite measure. If $\{\varphi_n\}_{n=1}^{\infty}$ is any sequence of simple functions bounded by M, supported on E, and with $\varphi_n(x) \to f(x)$ for a.e. x, then:*

1. *The limit* $\lim_{n\to\infty}\int\varphi_n$ *exists.*
2. *If* $f=0$ *a.e., then the limit* $\lim_{n\to\infty}\int\varphi_n$ *equals 0.*

Proof.

Setting $I_n=\int\varphi_n$ and applying Egorov's theorem which is proven in Chapter 1 we have that for and large n and m

$$\begin{aligned}|I_n-I_m| &\leq \int_E|\varphi_n-\varphi_m| \\ &= \int_{A_\epsilon}|\varphi_n-\varphi_m|+\int_{E-A_\epsilon}|\varphi_n-\varphi_m| \\ &\leq \int_{A_\epsilon}\epsilon\ dx+\int_{E-A_\epsilon}2M\ dx \\ &\leq m(E)\epsilon+2M\epsilon.\end{aligned}$$

given any $\epsilon>0$. This proves that $\{I_n\}$ os a Cauchy sequence nd hence converges. If $f=0$, letting m tend to infinity we have $|I_n-f|=|I_n|\leq m(E)\epsilon+2M\epsilon$, which yields $\lim_{n\to\infty}I_n=0$.

For a bounded function f that is supported on sets of finite measure, we define its **Lebesgue integral** by

$$\int f=\lim_{n\to\infty}\int\varphi_n.$$

where $\{\varphi_n\}$ is any sequence of simple functions satisfying: $|\varphi_n|\leq M$, each φ_n is supported on the support of f, and $\varphi_n(x)\to f(x)$ for a.e. x as n tends to infinity.

Next, we must show that $\int f$ is independent of the limiting sequence $\{\varphi_n\}$ used, in order for the integral to be well-defined. Suppose that $\{\psi_n\}$ is another sequence of simple functions that satisfies the properties above. Then, if $\eta_n=\varphi_n-\psi_n$, the sequence $\{\eta_n\}$ consists of simple functions bounded by $2M$, supported on a set of finite measure, and such that $\eta_n\to 0$ a.e. as n tends to infinity. Applying the lemma we find

$$\lim_{n\to\infty}\int\varphi_n=\lim_{n\to\infty}\int\psi_n+\lim_{n\to\infty}\int\eta_n=\lim_{n\to\infty}\int\psi_n$$

as desired.

Proposition 2.2. *Suppose f and g are bounded functions supported on sets of finite measure. Then the following properties hold.*

1. *Linearity.*
2. *Additivity.*
3. *Monotonicity.*

4. Triangle inequality.

Theorem 2.1 (Bounded convergence theorem)**.**

Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by M, are supported on a set E of finite measure, and $f_n(x) \to f(x)$ a.e. x as $n \to \infty$. Then f is measurable, bounded ,supported on E for a.e. x, and

$$\int |f_n - f| \to 0 \text{ as } n \to \infty.$$

Consequently,

$$\int f_n \to \int f \text{ as } n \to \infty.$$

Proof.

The proof is a reprise of the argument in Lemma 2.1. Given $\epsilon > 0$, we may find, by Egorov' theorem,

$$\begin{aligned}\int |f_n - f| &\leq \int_{A_\epsilon} |f_n - f| + \int_{E-A_\epsilon} |f_n - f| \\ &\leq m(E)\epsilon + 2M\epsilon.\end{aligned}$$

for all large n.

2.1.3 Return to Riemann integrable functions

Theorem 2.2.

Suppose f os Riemann integrable on the closed interval $[a, b]$. Then f is measurable, and

$$\int_{[a,b]}^{R} f(x)\ dx = \int_{[a,b]}^{L} f(x)\ dx.$$

Proof.

By definition of Riemann integrability, f is bounded, say $|f(x)| \leq M$, and we may construct two consequences of step functions $\{\varphi_k\}$ and $\{\psi_k\}$ that satisfy the following properties: $|\varphi_k(x) \leq M|$ and $|\psi_k(x) \leq M|$ for all $x \in [a, b]$ and $k \geq 1$,

$$\varphi_1(x) \leq \varphi_2(x) \leq \cdots \leq f \leq \cdots \leq \psi_2(x) \leq \psi_1,$$

and

$$\lim_{k\to\infty} \int_{[a,b]}^{R} \varphi_k(x)\ dx = \lim_{k\to\infty} \int_{[a,b]}^{R} \psi_k(x)\ dx = \int_{[a,b]}^{R} f(x)\ dx.$$

Notice that

$$\int_{[a,b]}^{R} \varphi_k(x)\ dx = \int_{[a,b]}^{L} \varphi_k(x)\ dx,$$

and

$$\int_{[a,b]}^{R} \psi_k(x)\ dx = \int_{[a,b]}^{L} \psi_k(x)\ dx.$$

Consider Theorem 2.1 (Bounded convergence theorem) and you will complete the proof.

2.1.4 Stage three: non-negative functions

In the case of such a function f we define its **Lebesgue integral** by

$$\int f = \sup_g \int g.$$

Proposition 2.3. *The integral of non-negative measurable functions enjoys the following properties:*

1. *Linearity.*
2. *Additivity.*
3. *Monotonicity.*
4. *If g is integrable and $0 \le f \le g$, then f is integrable.*
5. *If f is integrable, then $f(x) < \infty$ for a.e. x.*
6. *If $\int f = 0$, then $f(x) = 0$ for a.e. x.*

   Proof.

   We just prove the first assertion. Let φ, ψ and η be non-negative functions bounded and supported on sets of finite measure, where $\varphi \le f$, $\psi \le g$ and $\eta \le f+g$, then $\varphi + \psi \le f + g$. Consequently,

$$\int f + \int g \le \int (f+g).$$

   On the other hand, if we define $\eta_1 = \min(f(x), \eta(x))$ and $\eta_2 = \eta - \eta_1$, then

$$\int \eta = \int (\eta_1 + \eta_2) = \int \eta_1 + \int \eta_2 \le \int f + \int g,$$

   which means that

$$\int (f+g) \le \int f + \int g.$$

Lemma 2.2 (Fatou)**.**

Suppose $\{f_n\}$ is a sequence of measurable functions with $f_n \ge 0$. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x, then

$$\int f \le \liminf_{n\to\infty} \int f_n.$$

Proof.

Suppose $0 \le g \le f$*, where* g *is bounded and supported on a set* E *of finite measure. If we set* $g_n(x) = \min(g(x), f_n(x))$*, then* $g_n \to g$ *a.e. as* $n \to \infty$*. By Theorem 2.1 (Bounded convergence theorem) we have*

$$\int g_n \to \int g.$$

Since $g_n \le f_n$*, we have* $\int f_n \le \int g_n$*, so that*

$$\int g \le \liminf_{n\to\infty} \int f_n.$$

Taking the supremum over all g *yields the desired inequality.*

Corollary 2.1. *Suppose* f *is a non-negative measurable function, and* $\{f_n\}$ *a sequence of non-negative measurable functions with* $f_n(x) \le f(x)$ *and* $f_n(x) \to f(x)$ *for a.e.* x*. Then*

$$\lim_{n\to\infty} \int f_n = \int f.$$

Corollary 2.2 (Monotone convergence theorem)**.** *Suppose* $\{f_n\}$ *a sequence of non-negative measurable functions with* $f_n(x) \nearrow f(x)$*. Then*

$$\lim_{n\to\infty} \int f_n = \int f.$$

Corollary 2.3. *Consider a series* $\sum_{k=1}^{\infty} a_k(x)$*, where* $a_k(x) \ge 0$ *is measurable for every* $k \ge 1$*. Then*

$$\int \sum_{k=1}^{\infty} a_k(x)\ dx = \sum_{k=1}^{\infty} \int a_k(x)\ dx.$$

2.1.5 General form

In this case, we define the **Lebesgue integral** of f by

$$\int f = \int f^+ - \int f^-.$$

Proposition 2.4. *The integral of Lebesgue integrable functions is linear, additive, monotonic, and satisfies the triangle inequality.*

Proposition 2.5. *Suppose* f *is integral on* $\mathbb{R}^d$*. Then for every* $\epsilon > 0$*:*

1. *There exists a set of finite measure* B *(a ball, for example) such that*

$$\int_{B^c} |f| < \epsilon.$$

2. *There is a* $\delta > 0$ *such that*

$$\int_E |f| < \epsilon \quad \text{whenever } m(E) < \delta.$$

Proof.
Assume that $f \geq 0$*: 1.* $B_N = (-N, N)$*; 2.* $E_N = \{x : f(x) \leq N\}$.

Theorem 2.3 (Dominated convergence theorem)**.**

Suppose $\{f_n\}$ *is a sequence of measurable functions such that* $f_n \to f$ *a.e.* x *as* n *tends to infinity. If* $f_n(x) \leq g(x)$*, where* g *is integrable, then*

$$\int |f_n - f| \to 0 \quad \text{as } n \to \infty,$$

and consequently

$$\int f_n \to \int f.$$

Proof.

For each $N \geq 0$ *let* $E_N = \{x : |x| \leq N, g(x) \leq N\}$.

$$\begin{aligned} \int |f_n - f| &= \int_{E_N} |f_n - f| + \int_{E_N^c} |f_n - f| \\ &\leq \int_{E_N} |f_n - f| + 2\int_{E_N^c} g \\ &\leq \epsilon + 2\epsilon \end{aligned}$$

for all large n*. This prove the theorem.*

2.1.6 Complex-valued functions

The collection of all complex-valued integrable functions on a measurable subset $E \subset \mathbb{R}^d$ forms a vector space over $\mathbb{C}$.

2.2 The space L^1 of integrable functions

For any integrable function f on $\mathbb{R}^d$ we define the L^1-norm of f,

$$\|f\| = \|f\|_{L^1} = \|f\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f|.$$

Proposition 2.6. *Suppose* f *and* g *are two functions in* $L^1(\mathbb{R}^d)$.

1. *$\|af\| = |a|\|f\|$ for all $a \in \mathbb{C}$.*
2. *$\|f + g\| \leq \|f\| + \|g\|$.*
3. *$\|f\| = 0$ iff $f = 0$ a.e.*
4. *$d(f, g) = \|f - g\|$ defines a metric on $L^1(\mathbb{R}^d)$.*

Theorem 2.4 (Riesz-Fischer)**.** *The vector space L^1 is complete in its metric.*

Corollary 2.4. *If $\{f_n\}_{n=1}^{\infty}$ converges to f in L^1, the nthere exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that*

$$f_{n_k}(x) \to f(x) \quad a.e.x.$$

We say that a family G of integrable functions is **dense** in L^1 if for ant $f \in L^1$ and $\epsilon > 0$, there exists $g \in G$ so that $\|f - g\| < \epsilon$.

Theorem 2.5. *The following families of functions are dense in $L^1(\mathbb{R}^d)$:*

1. *The simple functions.*
2. *The step functions.*
3. *The continuous functions of compact support.*

2.2.1 Invariance properties

2.2.2 Translations and continuity

Proposition 2.7.

Suppose $f \in L^1(\mathbb{R}^d)$. Then

$$\|f_h - f\| \to 0 \qquad as\ h \to 0.$$

Proof.

By Theorem 2.5, find a continuous function with compact support to approximate f.

2.3 Fubini's theorem

2.4 A Fourier inversion formula

2.5 Exercise

2.6 Problem

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Chapter 3

Differentiation and Integration

3.1 Differentiation of the integral

We may ask whether