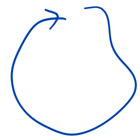
RC 2: Static Electric Field

By Mo Yang

1 Before RC

- My office hour: 10:00 a.m - 12:00 a.m. Thursday, Shanghai time.
- Please sent me a message if you want to attend the office hour.

closed curl



2 Review of Vector Calculus

- Stokes' Theorem:

3D vector field

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} \tag{1}$$

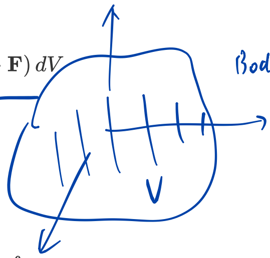
line. (curl).

$\nabla \times F$. surface.

$\int_D (\nabla \times F) \cdot dS = \int_{\partial D} F dr$.

- Gauss' Theorem (Divergence Theorem):

$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_V (\nabla \cdot \mathbf{F})\, dV \tag{2}$$

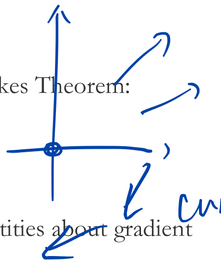


Body.

$\oint_S F dS = \int_V (\nabla \cdot F) dV$.

$S = \partial V$.

- Generalized Stokes Theorem:

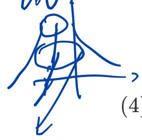


divergent

curl

$$\int_M d\omega = \int_{\partial M} \omega \tag{3}$$

$d\omega$ dr. dV. V.



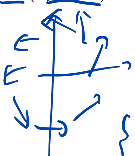
$=V$

- Two useful identities about gradient

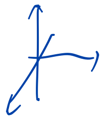
$$\nabla \times (\nabla V) \equiv 0$$
$$\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0 \tag{4}$$

∇V

$\nabla \times A$. → curl



$\dot{x} = -y$, $\dot{y} = x$



3 Coulomb' s Law

3.1 Point Charge

$F_{21} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}$ $F_{12} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}$

Force on a Point Charge in the Field of Another Point Charge

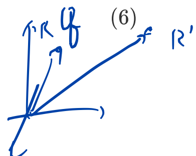
q_1 · r · q_2

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\mathbf{F}_{12} = q_2 \mathbf{E}_{12} = \hat{\mathbf{a}}_R \frac{q_1 q_2}{4\pi\epsilon_0 R^2} \tag{5}$$

Actually, we define "field" from forces. Getting rid of "force" makes it possible to consider potential without interaction. For a single point charge \$ \$q\$ \$ located at the origin, the electric field \$ \$\mathbf{E}\$ \$ is given by:

$$\mathbf{E} = \hat{\mathbf{a}}_R \frac{q}{4\pi\epsilon_0 R^2} \quad (\mathrm{V/m}) \tag{6}$$



Single Point Charge (Charge Not at the Origin):

Charge at R.

$$\mathbf{E}_p = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^3} \quad (\mathrm{V/m}) \tag{7}$$

Several Point Charges:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k (\mathbf{R} - \mathbf{R}'_k)}{|\mathbf{R} - \mathbf{R}'_k|^3} \tag{8}$$

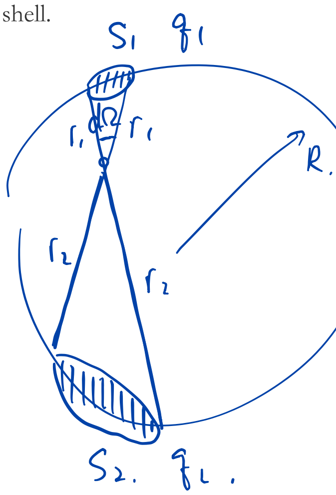
q_1, q_2 . q_k

Note that all the things are inversely proportional to the square of distance.

q'

3.1.1 Exercise 1:

A total charge Q is uniformly put on a thin spherical shell of radius R. Determine the electric field intensity at an arbitrary point inside the shell.

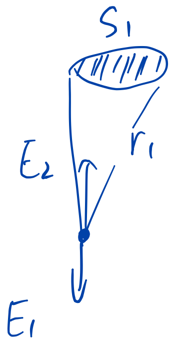


shell Cone.

$\sigma = \frac{Q}{4\pi R^2}$

$q_1 = \sigma S_1 = \sigma \cdot r_1^2 \cdot d\Omega$

$q_2 = \sigma S_2 = \sigma\, r_2^2 \cdot d\Omega$



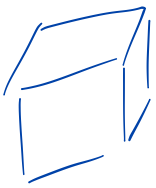
$\begin{cases} S_1 = d\Omega \cdot r_1^2 \\ S_2 = d\Omega \cdot r_2^2 \end{cases}$

$E_1 = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_1^2} = \frac{1}{4\pi\varepsilon_0} \frac{\sigma r_1^2 d\Omega}{r_1^2} = \frac{\sigma d\Omega}{4\pi\varepsilon_0}$

$E_2 = \frac{\sigma d\Omega}{4\pi\varepsilon_0}$. $E_1 + E_2 = 0$. $E = 0$

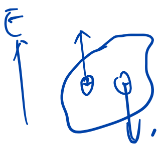
3.2 Dipole

Why we need dipole? A model of physics when external electric field appears.

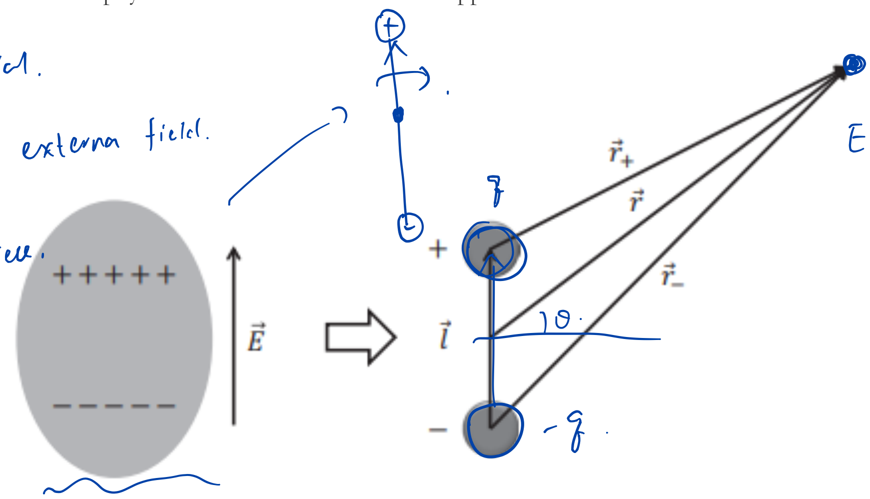


material.

↑ E external field.



small piece.



The electric field $\mathbf{E}$ due to an electric dipole is given by:

$$\boxed{\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left(\frac{\mathbf{R} - \frac{\mathbf{d}}{2}}{|\mathbf{R} - \frac{\mathbf{d}}{2}|^3} - \frac{\mathbf{R} + \frac{\mathbf{d}}{2}}{|\mathbf{R} + \frac{\mathbf{d}}{2}|^3}\right)} \tag{9}$$

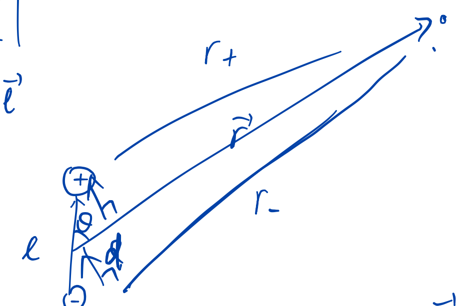
Approximation for $d \ll R$:

$$\boxed{\mathbf{E} \approx \frac{q}{4\pi\epsilon_0 R^3}\left(3\frac{\mathbf{R} \cdot \mathbf{d}}{R^2}\mathbf{R} - \mathbf{d}\right)} \tag{10}$$

$\vec{p}$: dipole. $\vec{p} = q\vec{l}$

3.3 Exercise 2:

Derive the electric field E for the dipole.



Method 1: (derivate the potential field)

$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$

$\vec{d} + \vec{r} = r_-$ $\vec{d} + r_+ = \vec{r}$.

$$= \frac{q}{4\pi\epsilon_0}\frac{r_- - r_+}{r^2} = \frac{q}{4\pi\epsilon_0}\frac{l\cos\theta}{r^2} = \frac{q}{4\pi\epsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

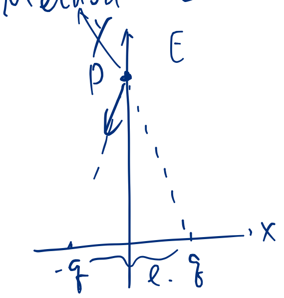
$$\vec{E}(r) = -\nabla\varphi = -\frac{1}{4\pi\epsilon_0}\left[\frac{\nabla\cdot(\vec{p}\cdot\vec{r})}{r^3} + \vec{p}\cdot\vec{r}\cdot\nabla\frac{1}{r^3}\right] = -\frac{1}{4\pi\epsilon_0}\frac{\vec{p}-3(\vec{p}\cdot\vec{r})\hat{r}}{r^3}$$

Add up electric fied: vector calculus

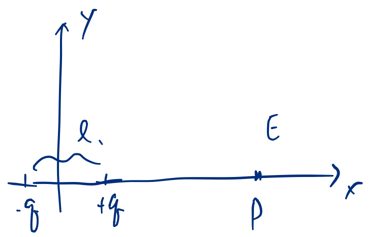
Add up potential: Simply adding up.

Potential.

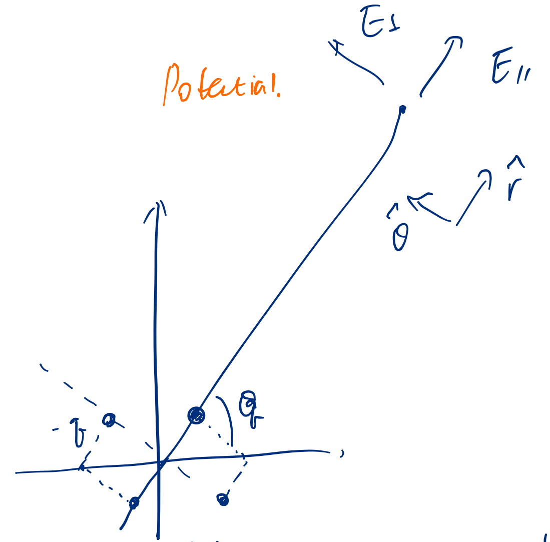
Method 2 electrical field.



(a)



(b)



(c).

$$E_a = -\frac{1}{4\pi\epsilon_0}\frac{ql}{r^3} \qquad E_b = \frac{1}{4\pi\epsilon_0}\frac{2ql}{r^3}$$

$$E_\parallel = \frac{1}{4\pi\epsilon_0}(2p_\parallel)\cdot\frac{1}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{2ql\cos\theta}{r^3}$$

$$E_\perp = \frac{1}{4\pi\epsilon_0}(p_\perp)\frac{1}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{ql\sin\theta}{r^3}$$

$$E = E_\parallel\cdot\hat{r} + E_\perp\,\hat{\theta}$$

Electric Dipole Moment Definition:

$$p = qd \tag{11}$$

where q is the charge, vector d goes from $-q$ to $+q$.

$$p = a_z p = p(a_R\cos\theta - a_\theta\sin\theta) \tag{12}$$

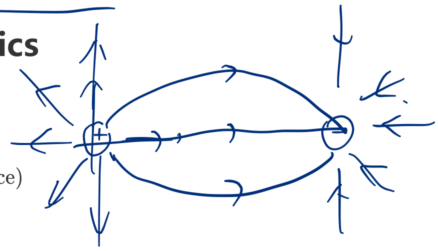
Electric Field: (spherical coordinate)

$$E = \frac{p}{4\pi\epsilon_0 R^3}(2a_R\cos\theta + a_\theta\sin\theta) \tag{13}$$

3.4 Fundamental Conclusions of Electrostatics

Differential form (of Maxwell Equations):

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{(divergence)}$$
$$\nabla\times\mathbf{E} = 0 \quad \text{(curl)} \tag{14}$$



Integral form:

$$\oint_S \mathbf{E}\cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$
$$\oint_C \mathbf{E}\cdot d\ell = 0 \tag{15}$$

(from mathematical form of Stokes).

4 Gauss's Theorem

The total outward flux of the electric over any closed surface in free space is equal to the total charge enclosed in the surface divided by ϵ_0. (Note that we can choose an arbitrary surface S for our convenience.)

$$\oint_S \mathbf{E}\cdot d\mathbf{s} = \frac{Q}{\epsilon_0} \tag{16}$$

$$\oint_{\partial V}\vec{E}\,ds = \frac{1}{\epsilon_0}\sum Q_i$$

$\iiint_V \rho \, dV,$

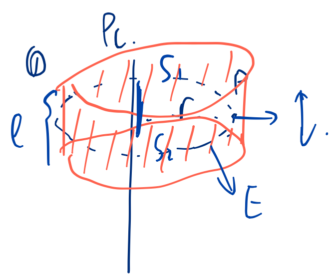
4.1 Some useful models

(Put them on your CTPP!)

Model (Uniformly Charged)	Electric Field E (Magnitude)
① Infinitely Long, Line Charge	$E = \frac{\rho_l}{2\pi r\epsilon_0}$
② Infinite Planar Charge	$E = \frac{\rho_s}{2\epsilon_0}$
③ Uniform Spherical Surface Charge with Radius R	$\begin{cases} E = 0 & \text{if } r < R \\ E = \frac{Q}{4\pi r^2\epsilon_0} & \text{if } r > R \end{cases}$
④ Uniform Sphere Charge with Radius R	$\begin{cases} E = \frac{Qr}{4\pi R^3\epsilon_0} & \text{if } r < R \\ E = \frac{Q}{4\pi r^2\epsilon_0} & \text{if } r > R \end{cases}$
⑤ Infinitely Long, Cylindrical Charge with Radius R	$\begin{cases} E = \frac{\rho_v r}{2\epsilon_0} & \text{if } r < R \\ E = \frac{\rho_v R^2}{2r\epsilon_0} & \text{if } r > R \end{cases}$

4.2 Exercise 3:

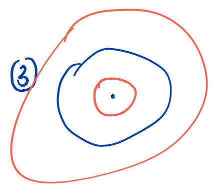
Derive the conclusions above with Gauss's theorem.



$\int S_1 = 0.$

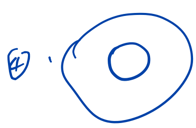
$E \cdot 2\pi r \cdot l = l \cdot \rho_l \cdot \frac{1}{\varepsilon_0}$

$\Rightarrow E = \frac{\rho_l}{2\pi\varepsilon_0}.$

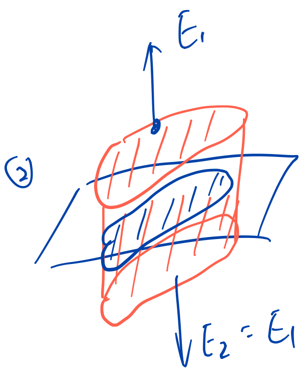


inside $\to 0$

outside $\to \frac{Q}{4\pi r^2\varepsilon_0}.$

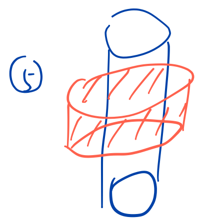


$\begin{cases} \frac{Qr}{4\pi R^3\varepsilon_0} \\ \frac{Q}{4\pi r^2\varepsilon_0}. \end{cases}$



$E_1 \cdot S + E_2 \cdot S = \sigma \cdot S \cdot \frac{1}{\varepsilon_0}$

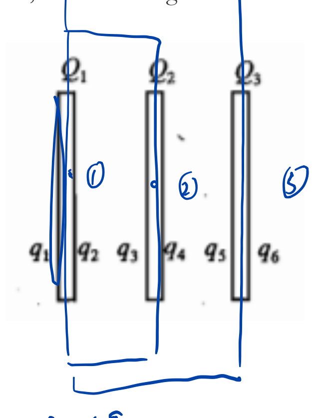
$\Rightarrow E = \frac{\sigma}{2\varepsilon_0} = \frac{\rho_s}{2\varepsilon_0}.$



4.3 Exercise 4:

Three metal plates with uniform thickness d, length L, and width W are arranged in the thickness direction with equal spacing. The length and width of the metal plates are much larger than the distance between the plates.The known charges on the metal plates are Q_1, Q_2, and Q_3. Without considering edge effects, find the charges on both sides of the plates: $q_1, q_2, q_3, q_4, q_5, q_6$.

Inside metal $\to$ no E



$q_2 + q_3 = 0.$

Similarly $q_4 + q_5 = 0.$

① $\frac{q_1}{2\varepsilon_0 S} = \frac{q_2+q_3+\cdots+q_6}{2\varepsilon_0 S}$ C

C

② $\frac{q_1+q_2+q_3}{2\varepsilon_0 S} = \frac{q_4+q_5+q_6}{2\varepsilon_0 S}$

③ $\frac{q_1+\cdots+q_5}{2\varepsilon_0} = \frac{q_6}{2\varepsilon_0 S}$

$q_1 = \frac{1}{2}(Q_1+Q_2+Q_3) = q_6.$

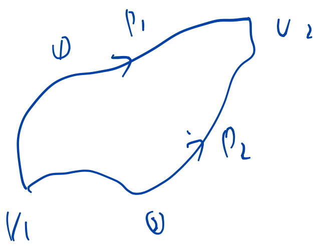
$q_2 = \frac{1}{2}(Q_2+Q_3-Q_1) = -q_3$

$q_4 = -q_5 = \frac{1}{2}(Q_1+Q_2-Q_3)$

5 Electrical Potential

E: $\oint_C E\,dl = 0.$ static field.

The potential of an electric field can be defined as follows:



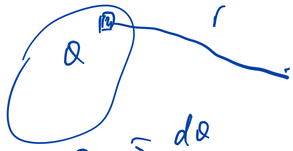
$$\mathbf{E} = -\nabla V, \tag{17}$$

$$V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell}. \tag{18}$$

$\int_{V_1}^{V_2} E\,dl - \int_{V_1}^{V_2}$ $E\,dl = 0.$

① ②

Charge Distribution	Electric Potential V
Point Charge	$V = \frac{q}{4\pi\epsilon_0 R}$
Line Charge	$V = \frac{1}{4\pi\epsilon_0}\int_{L'} \frac{\rho_\ell}{R} d\ell'$
Surface Charge	$V = \frac{1}{4\pi\epsilon_0}\int_{S'} \frac{\rho_s}{R} ds'$
Volume Charge	$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho}{R} dv'$



$\varphi = \sum \frac{dQ}{r}$

If there are n point charges $q_1, q_2, \ldots, q_n$ located at positions $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$, respectively, the total electric potential V at a point $\mathbf{r}$ is given by:

$$V(\mathbf{r}) = \sum_{i=1}^{n} V_i(\mathbf{r}) = \sum_{i=1}^{n} \frac{kq_i}{|\mathbf{r} - \mathbf{r}_i|} \tag{19}$$

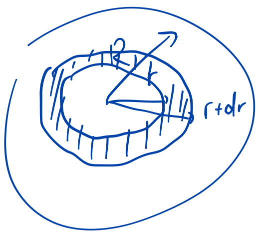
Q_1 p Q_2

5.1 Exercise 5:

body.

For a uniformly charged sphere with radius R and total charge Q, find the electric potential U_0 at the center of the sphere.

Sphere.



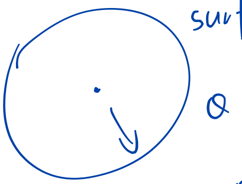
$\rho = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi R^3}$

$Q_s = 4\pi r^2 \cdot dr \cdot \rho$

$dU = \frac{Q_s}{4\pi\varepsilon_0 \cdot r} dr = \frac{1}{4\pi\varepsilon_0} 4\pi \rho r \cdot dr$

$= 2\pi\rho \cdot \frac{1}{4\pi\varepsilon_0} \cdot 2r.$

surface.



$\varphi = \frac{Q}{4\pi\varepsilon_0 \cdot r}$

$U = \sum dU = \frac{1}{4\pi\varepsilon_0} \frac{3q}{2R}.$

Thanks!