VE230 Final RC

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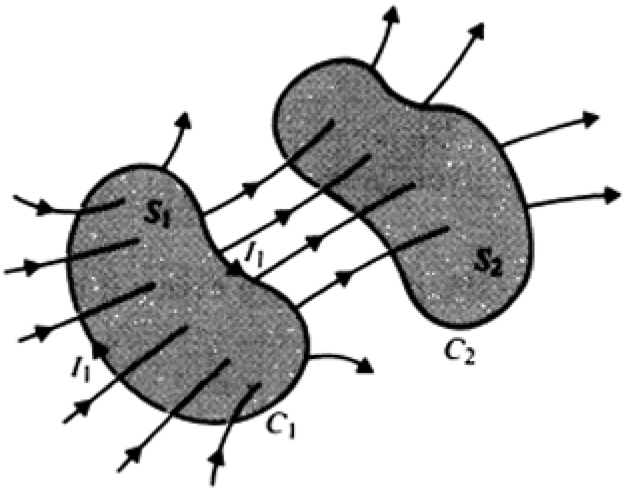
Inductances and Magnetic Energy

Inductances and Inductors

For two loops C_1, C_2:



For linear media where permeability does not change with I_1,

$$L_{12} = \frac{\Lambda_{12}}{I_1} \tag{1}$$

where Λ_{12} is the flux linkage, L_{12} is the mutual inductance between loops C_1 and C_2. A more general definition for L_{12} is then:

$$L_{12} = \frac{d\Lambda_{12}}{dI_1} \tag{2}$$

The total flux linkage with C_1 caused by I_1 is calculated as:

$$\Lambda_{11} = N_1\Phi_{11} > N_1\Phi_{12} \tag{3}$$

The self-inductance of loop C_1 could be calculated as:

$$L_{11} = \frac{\Lambda_{11}}{I_1} \tag{4}$$

For linear medium, in general:

$$L_{11} = \frac{d\Lambda_{11}}{dI_1} \tag{5}$$

A conductor arranged in an appropriate shape (e.g., a conducting wire wound as a coil) to supply a certain amount of self-inductance is called an inductor. It can store magnetic energy.

Method to determine the self-inductance of an inductor:

1. Choose an appropriate coordinate system for the given geometry.
2. Assume a current I in the conducting wire.
3. Find B from I by Ampere's circuital law,

$$\oint_C B \cdot dl = \mu_0 I \tag{6}$$

if symmetry exists; if not, Biot-Savart law

$$B = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{dl' \times \mathbf{a}_R}{R^2} \tag{7}$$

must be used.

4. Find the flux linking with each turn, Φ, from B by integration:

$$\Phi = \int_S B \cdot ds \tag{8}$$

where S is the area over which B exists and links with the assumed current.

5. Find the flux linkage Λ by multiplying Φ by the number of turns.
6. Find L by taking the ratio $L = \frac{\Lambda}{I}$.

To determine the mutual inductance L_{12} between two circuits, after choosing an appropriate coordinate system, assume $I_1 \rightarrow$ Find $B_1 \rightarrow$ Find Φ_{12} by integrating B_1 over surface $S_2 \rightarrow$ Find flux linkage $\Lambda_{12} = N_2\Phi_{12} \rightarrow$ Find $L_{12} = \frac{\Lambda_{12}}{I_1}$.

In high-frequency applications, current tends to concentrate in the "skin" of the inner conductor as a surface current, internal self-inductance tends to zero. Neumann formula for mutual inductance:

$$L_{12} = L_{21} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{dl_1 \cdot dl_2}{R} \tag{9}$$

Magnetic Energy

For a system of N loops carrying currents,

$$W_m = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} L_{jk} I_j I_k \tag{10}$$

For a single inductor,

$$W_m = \frac{1}{2} L I^2 \tag{11}$$

$$W_m = \frac{1}{2} \sum_{k=1}^{N} I_k \Phi_k \tag{12}$$

Magnetic Energy In Terms of Field Quantities

Generally,

$$W_m = \frac{1}{2} \int_{V'} \mathbf{A} \cdot \mathbf{J} \, dv' = \frac{1}{2} \int_{V'} \mathbf{H} \cdot \mathbf{B} \, dv' = \frac{1}{2} \int_{V'} \frac{B^2}{\mu} \, dv' = \frac{1}{2} \int_{V'} \mu H^2 \, dv' \tag{13}$$

If we define a magnetic energy density w_m such that

$$W_m = \int_{V'} w_m \, dv' \tag{14}$$

then

$$w_m = \frac{1}{2} \mathbf{H} \cdot \mathbf{B} = \frac{B^2}{2\mu} = \frac{1}{2} \mu H^2 \tag{15}$$

And we have

$$L = \frac{2W_m}{I^2} \tag{16}$$

Faraday's Law of Electromagnetic Induction

In static case, E and B can exist together (in a conducting medium), but they won't influence each other. The relationship between induced emf and the negative rate of change of flux linkage:

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{17}$$

$$\oint_C E \cdot d\ell = -\int_S \frac{\partial B}{\partial t} \cdot ds \tag{18}$$

If we define emf

$$V = \oint_C E \cdot d\ell \tag{19}$$

and magnetic flux

$$\Phi = \int_S B \cdot ds \tag{20}$$

then Faraday's law can be rewritten as

$$V = -\frac{d\Phi}{dt} \quad \text{(Faraday's law of electromagnetic induction)} \tag{21}$$

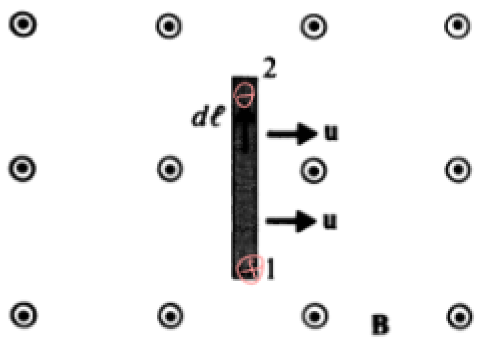
If there are N turns of wire, then the total magnetic flux is $N\Phi$, and we have

$$V = -N\frac{d\Phi}{dt} \tag{22}$$

Lenz's Law: The induced emf will cause a current to flow in the closed loop in such a direction as to oppose the change in the linking magnetic flux.

- Motional EMF:

$$V_0 = \oint_C (u \times B) \cdot d\ell \tag{23}$$



For moving circuit in a time-varying magnetic field

- Lorentz's Force Equation:

$$F = q(E + u \times B) \tag{24}$$

- Effective Electric Field: If an observer has the same movement with q, Lorentz's force on q can be seen as effective electric field

$$E' = E + u \times B \tag{25}$$

- General Form of Faraday's Law:

$$\oint_C E' \cdot d\ell = -\int_S \frac{\partial B}{\partial t} \cdot ds + \oint_C (u \times B) \cdot d\ell \tag{26}$$

where the left side talks about emf induced in a moving frame of reference, and on the right side, transformer emf equals to

$$V_t = -\int_S \frac{\partial B}{\partial t} \cdot ds \tag{27}$$

and motional emf equals to

$$V_m = \oint_C (u \times B) \cdot d\ell \tag{28}$$

Potential Function

- Electric field in time-varying field

$$E = -\nabla V - \frac{\partial A}{\partial t} \tag{29}$$

where $-\nabla$ comes from charge distribution, and $-\frac{\partial A}{\partial t}$ comes from time-varying current.

- Quasi-static fields
  If ρ and J vary slowly with time and the range of R is small in comparison with the wavelength (low frequency, long wavelength), we can use the below 2 equations to find quasi-static fields.

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R} dv' \tag{30}$$

$$A = \frac{\mu_0}{4\pi} \int_{V'} \frac{J}{R} dv' \tag{31}$$

Note:

1. These above equations are solutions of Poisson's equation in static case.
2. If there are high-frequency sources, we need to consider time-retardation effects, which means as source changes in time, it takes time to change the potential at a certain distance from the source.

- Non-homogeneous wave equation for vector potential
  If we choose the divergence of A such that

$$\nabla \cdot A + \mu\epsilon \frac{\partial V}{\partial t} = 0 \tag{32}$$

which is called the Lorentz condition/gauge for potentials, we can find the non-homogeneous wave equation for vector potential A as

$$\nabla^2 A - \mu\epsilon \frac{\partial^2 A}{\partial t^2} = -\mu J \tag{33}$$

- Non-homogeneous wave equation for scalar potential

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{34}$$

Exercise

Substitute equations $B = \nabla \times A$ and $E = -\nabla V - \frac{\partial A}{\partial t}$ in Maxwell's equations to obtain wave equations for scalar potential V and vector potential A for a linear, isotropic but inhomogeneous medium. Show that these wave equations reduce to equations $\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$ and $\nabla^2 A - \mu\epsilon \frac{\partial^2 A}{\partial t^2} = -\mu J$ for simple media. (Hint: Use the following gauge condition for potentials in an inhomogeneous medium:

$$\nabla \cdot (\epsilon A) + \mu\epsilon^2 \frac{\partial V}{\partial t} = 0 \tag{35}$$

Exercise

Prove by direct substitution that any twice differentiable function of $(t - R\sqrt{\mu\epsilon})$ or of $(t + R\sqrt{\mu\epsilon})$ is a solution of the homogeneous wave equation.

$$\frac{\partial^2 U}{\partial R^2} - \mu\epsilon\frac{\partial^2 U}{\partial t^2} = 0. \tag{36}$$

Wave Equations and Solutions

For given charge and current distribution ρ and J, in order to get E and B, we first need to find solutions for V and A in non-homogeneous wave equation.

- Solution for scalar potential:

$$V(R,t) = \frac{1}{4\pi\epsilon}\int_{V'}\frac{\rho(t-R/u)}{R}dv' \tag{37}$$

- which is called the retarded scalar potential, indicating that it takes time R/u for the effect of ρ to be felt at distance R.

$$\tag{38}$$

- Solution for vector potential:

$$A(R,t) = \frac{\mu}{4\pi}\int_{V'}\frac{J(t-R/u)}{R}dv' \tag{39}$$

For source-free wave equations in simple non-conducting media

- Source-free: $\rho = 0, J = 0$.
- Simple non-conducting media: ϵ and μ are constant, and $\sigma = 0$.
- Rewrite the Maxwell's equations:

$$\nabla \times E = -\mu\frac{\partial H}{\partial t} \tag{40}$$

$$\nabla \times H = \epsilon\frac{\partial E}{\partial t} \tag{41}$$

$$\tag{42}$$

$$\nabla \cdot E = 0 \tag{43}$$

$$\tag{44}$$

- Obtain wave equations for E and H:

$$\nabla^2 E - \frac{1}{u^2}\frac{\partial^2 E}{\partial t^2} = 0 \tag{45}$$

$$\nabla^2 H - \frac{1}{u^2}\frac{\partial^2 H}{\partial t^2} = 0 \tag{46}$$

where $u = \frac{1}{\sqrt{\mu\epsilon}}$.

Thanks for Coming and Good Luck for Exam!