RC5: Electronic Solutions and Steady Electric Currents

by Mo Yang

Boundary Value Problem in Cartesian Coordinates

We have Laplace’s equation:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0 \tag{1}$$

If we assume $V(x,y,z)=X(x)Y(y)Z(z)$, then we have

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=0 \tag{1}$$

Let $f(x)=\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}$, $f(y)=\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}$, $f(z)=\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}$, since $f(x)$, $f(y)$, and $f(z)$ are independent of each other, to make Eq.(1) always exist, $f(x)$, $f(y)$, and $f(z)$ must all be constants. Therefore,

$$\frac{df(x)}{dx}=0,\quad \frac{df(y)}{dy}=0,\quad \frac{df(z)}{dz}=0 \tag{2}$$

Then after simplifying, we know that

$$\frac{d^2X(x)}{dx^2}+k_x^2X(x)=0,\quad \frac{d^2Y(y)}{dy^2}+k_y^2Y(y)=0,\quad \frac{d^2Z(z)}{dz^2}+k_z^2Z(z)=0 \tag{3}$$

where k_x^2, k_y^2, and k_z^2 are constants and

$$k_x^2+k_y^2+k_z^2=0 \tag{4}$$

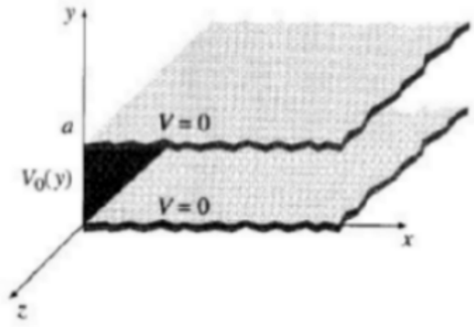
Possible solutions of $X''(x)+k_x^2X(x)=0$:

k_x^2	k_x	$X(x)$
0	0	A_0x+B_0
+ k	$A_1\sin kx+B_1\cos kx$	$C_1e^{jkx}+D_1e^{-jkx}$
- jk	$A_2\sinh kx+B_2\cosh kx$	$C_2e^{kx}+D_2e^{-kx}$

Where k is real. And constant A and B should be determined by boundary conditions.

Exercise 1

Two infinite grounded metal plates lie parallel to the xz plane, one at y = 0, the other at y = a. The left end, at x = 0, is closed off with an infinite strip insulated form the two plates and maintained at a specific potential V0(y). Find the potential inside this ”slot”.



Boundary Value Problem in Spherical Coordinates

We have Laplace’s equation:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \tag{5}$$

We assume that the solution is independent of ϕ, then we have

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \tag{6}$$

Assume $V(r,\theta) = R(r)\Theta(\theta)$. Putting this into the above equation, and dividing by V,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0 \tag{7}$$

Since the first term depends only on r, and the second only on θ, it follows that each must be a constant:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1), \qquad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1) \tag{8}$$

Here $l(l+1)$ is just a fancy way of writing the separation constant—you'll see in a minute why this is convenient.

As always, separation of variables has converted a partial differential equation into ordinary differential equations. The radial equation,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R \tag{9}$$

has the general solution

$$R(r) = Ar^l + \frac{B}{r^{l+1}} \tag{10}$$

A and B are the two arbitrary constants to be determined by boundary conditions. The angular equation is:

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\Theta \tag{11}$$

Its solutions are Legendre polynomials in the variable $\cos\theta$:

$$\Theta(\theta) = P_l(\cos\theta) \tag{12}$$

$P_l(x)$ is most conveniently defined by the Rodrigues formula:

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l \tag{13}$$

The first few Legendre polynomials are listed below:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{3x^2-1}{2} \\ P_3(x) &= \frac{5x^3-3x}{2} \\ P_4(x) &= \frac{35x^4-30x^2+3}{8} \\ P_5(x) &= \frac{63x^5-70x^3+15x}{8} \end{aligned} \tag{14}$$

In the case of azimuthal symmetry, then, the most general separable solution to Laplace's equation, consistent with minimal physical requirements, is

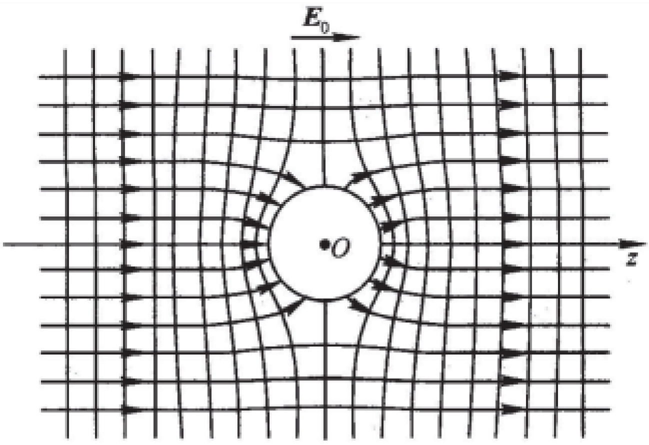
$$V(r,\theta) = \left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{15}$$

As before, separation of variables yields an infinite set of solutions, one for each l. The general solution is the linear combination of separable solutions:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{16}$$

Exercise 2

A grounded conductor ball of radius R is placed in a uniform outer field E_0, find the score of the space field Cloth.



Boundary Value Problem in Cylindrical Coordinates

Laplace's Equation in Cylindrical Coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0 \tag{17}$$

Assuming V has no z dependence,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}=0 \tag{18}$$

Assume

$$V(r,\phi)=R(r)\Phi(\phi) \tag{19}$$

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)+\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=0 \tag{20}$$

Therefore,

$$\frac{r}{R(r)}\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)=k^2 \tag{21}$$

$$\frac{d^2\Phi(\phi)}{d\phi^2}+k^2\Phi(\phi)=0 \tag{22}$$

has solution

$$R(r)=A_r r^n+B_r r^{-n} \tag{23}$$

$$\Phi(\phi)=A_\phi\sin n\phi+B_\phi\cos n\phi \tag{24}$$

Therefore,

$$V_n(r,\phi)=r^n(A_n\sin n\phi+B_n\cos n\phi)+r^{-n}(A_n'\sin n\phi+B_n'\cos n\phi),\quad n\neq 0 \tag{25}$$

In the special case where $k=0$,

$$\frac{d^2\Phi(\phi)}{d\phi^2}=0 \tag{26}$$

$$\Phi(\phi)=A_0\phi+B_0,\quad k=0 \tag{27}$$

and $A_0=0$ if there is no circumferential variations. Meanwhile,

$$\frac{d}{dr}\left(r\frac{dR(r)}{dr}\right)=0 \tag{28}$$

$$R(r)=C_0\ln r+D_0,\quad k=0 \tag{29}$$

Therefore,

$$V(r)=C_1\ln r+C_2 \tag{30}$$

Thus, the general solution is

$$V(r,\phi)=a_0+b_0\ln r+\sum_{k=1}^{\infty}\left(r^k(a_k\cos k\phi+b_k\sin k\phi)+r^{-k}(c_k\cos k\phi+d_k\sin k\phi)\right) \tag{31}$$

Steady Electric Currents

Current Density and Ohm's Law

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \quad (\mathrm{A}) \tag{32}$$

where $\mathbf{J}$ is the volume current density or current density, defined by

$$\mathbf{J} = Nqu \quad (\mathrm{A/m^2}) \tag{33}$$

where N is the number of charge carriers per unit volume, each of charge q moves with a velocity u.

Since Nq is the free charge per unit volume, by $\rho = Nq$, we have:

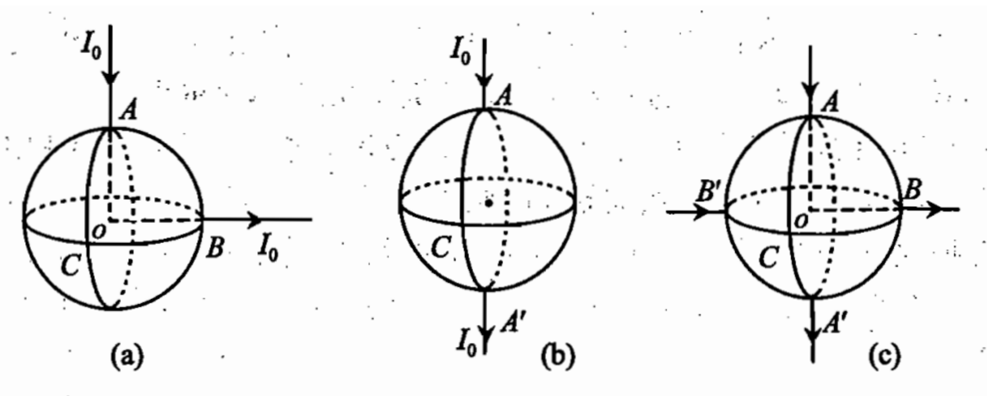
$$\mathbf{J} = \rho u \quad (\mathrm{A/m^2}) \tag{34}$$

For conduction currents,

$$\mathbf{J} = \sigma \mathbf{E} \quad (\mathrm{A/m^2}) \tag{35}$$

Exercise 3

A thin spherical conductor with radius R is centered at point O. On the surface of the sphere, there are three points A, B, and C with semi-diameters OA, OB, and OC being mutually perpendicular. There are thin conductive wires connected at points A and B on the sphere's surface, and these two thin conductive wires are connected to a power source. It is known that an electric current I_0 flows into the sphere at point A and flows out of the sphere at point B, as shown in Figure 2-5(a). Determine the current density at point C (i.e., the current per unit length flowing through the direction of the electric current at point C).



Referece

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- Prof. Nana Liu, Slides, VE230.
- Jiafu Cheng, Electromegnetics.
- Lei Zhou, Electrodynamics.

Thanks for Attending!