RC4: Capacitors and Electrostatic Solutions

By Mo Yang

Capacitor $+Q$ $-Q$ U $\frac{Q}{U}$ Ball Q U Stores electricity

Capacitors

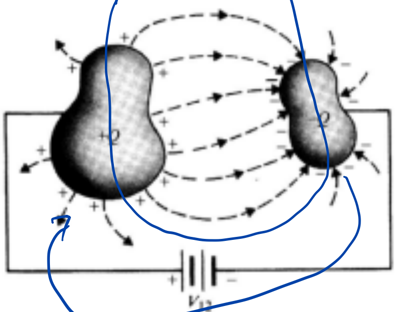
Definition: The capacitance of isolated conducting body is the electric charge that must be added to the body per unit increase in its electric potential.

$C = \frac{Q}{U}$

$$C = \frac{Q}{V} \tag{1}$$

Components: Two conductors with arbitrary shapes are separated by free space or dielectric medium.

$C = \frac{\varepsilon S}{d}$



C_1 C_2 C_3 C_n

Specifically, for the connections of different capacitors, we can have

Series:

$$\boxed{\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}} \tag{2}$$

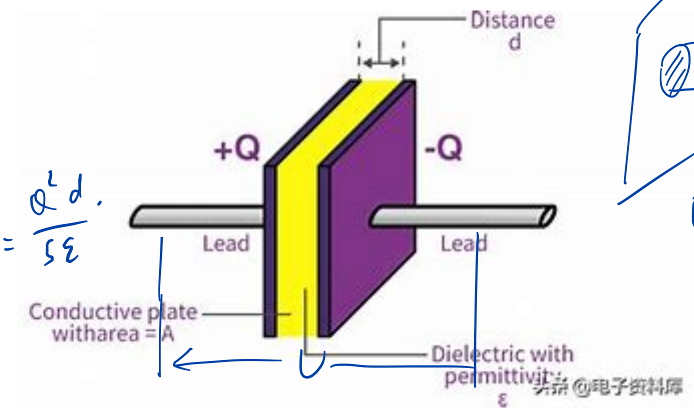
Parallel:

$$\boxed{C_{pr} = C_1 + C_2 + \ldots + C_n} \tag{3}$$

Use of media in capacitor

$p = \varepsilon E^2$.

$\text{Energy} = \varepsilon E^2 \cdot S \cdot d = \varepsilon \cdot \frac{Q^2}{\varepsilon^2 S^2} S d = \frac{Q^2 d}{\varepsilon S}$.

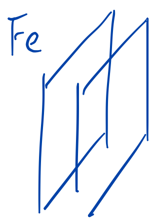


$E \Delta S = \frac{\sigma \Delta S}{\varepsilon} \Rightarrow E = \frac{\sigma}{\varepsilon}$.

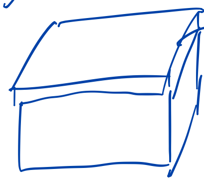
Q fixed. $E = \frac{Q}{\varepsilon S}$ $U = E \cdot d = \frac{Qd}{\varepsilon S} \Rightarrow C = \frac{Q}{U} = \frac{\varepsilon S}{d}$.

fill into dielectric $\rightarrow$ $C = \frac{\varepsilon_0 S}{d}$.

$C' = \frac{\varepsilon_0 \varepsilon_r S}{d}$ $\varepsilon_r > 1$. $C' > C$.



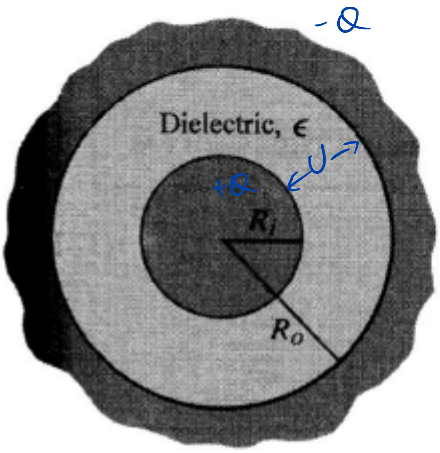
electricity $\rightsquigarrow$ oxygen



$\rightarrow$ a layer of FeO / Fe_2O_3

Exercise 1

A spherical capacitor consists of an inner conducting sphere of radius R_i and an outer conductor with a sphere inner wall of radius R_o. The space in between is filled with a dielectric of permitivity ϵ. Determine the capacitance.



$D \cdot 4\pi R^2 = Q$. (Gauss) $(D = \varepsilon E)$

$D = \frac{Q}{4\pi R^2}$

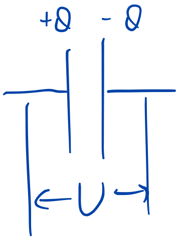
$E = \frac{D}{\varepsilon} = \frac{Q}{4\pi R^2 \varepsilon}$

$U = \int_{R_1}^{R_2} E\, dl$

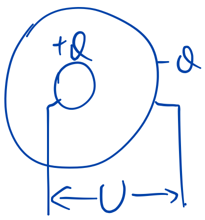
$= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$

$C = \frac{Q}{U} = \frac{4\pi\varepsilon}{\frac{1}{R_1} - \frac{1}{R_2}}$

①



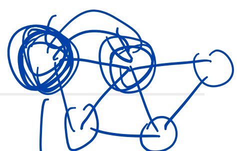
②



③ Cyllinder & Ball

Capacity $\rightarrow$ shape

$\rightarrow$ calculated all the things 2 times



Energy in Capacitors and Electric Fields

The potential energy of N discrete charges at rest:

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \tag{4}$$

$\sum$ $\sum_{k=1}^{N} Q_k \cdot V_k$

contain itself

q $\oplus$ $\longrightarrow$

U_1 U_2

$\Delta W = (U_2 - U_1)\cdot q$.

We can also deduce the case for continuous distribution.

Defn $W_e = \frac{1}{2}\int \rho \cdot V dv$

$$W_e = \frac{1}{2}\int_{R^3} \epsilon E^2 dV \tag{5}$$

integration

$W_e = \frac{1}{2}\int (\nabla\cdot D) V dv$ $\quad \nabla\cdot(VD) = V\cdot\nabla D + D\cdot\nabla V$

$= \frac{1}{2}\int_{V'} \nabla\cdot(VD) - \frac{1}{2}\int_{V'} D\cdot\nabla V dV$

$\rho = \varepsilon E^2$

energy density

$= \frac{1}{2}\oint_{S'} V D\cdot a_n ds + \frac{1}{2}\int_{V'} D\cdot E dv.$

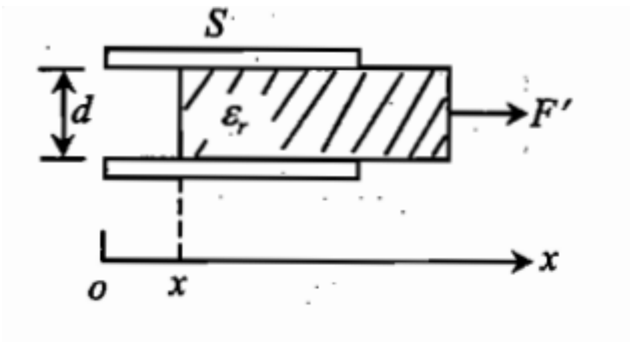
$W_e = \frac{1}{2}\int_{R^3} D\cdot E dv = \frac{1}{2}\int_{R^3} \varepsilon E^2 dv.$

Exercise 2

A parallel plate capacitor has a plate area of S and a plate separation of d. The space between the plates is filled with a dielectric material with a relative permittivity ϵ_r. Under the following two conditions, determine how much external force F is needed to completely remove the dielectric from the capacitor:

1. The voltage U across the capacitor remains constant.
2. The charge Q on the capacitor remains constant.

For capacitor, Energy $= \frac{1}{2}CU^2 = \frac{1}{2}\frac{Q^2}{C}$,



① F'. A_1' → work done by F.

$A_1' = \underbrace{\Delta W_1}_{\text{energy change}} - A_{power} = \frac{1}{2}U^2(C_2 - C_1) - U\cdot\Delta Q$

$= \frac{1}{2}U^2(C_2 - C_1) - U^2(C_2 - C_1) = \frac{(\varepsilon_r - 1)\varepsilon_0 S}{2d}U^2.$

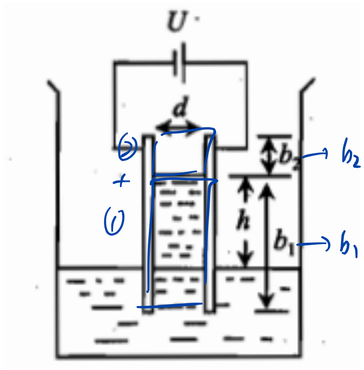
② $A_2' = \underbrace{\Delta W_2}_{\text{Energy change}} = \frac{1}{2}Q^2\left(\frac{1}{C_2} - \frac{1}{C_1}\right) = \frac{(\varepsilon_r - 1)dQ^2}{2\varepsilon_r\varepsilon_0 S}$

No other thing work

Notice $A_1' \neq A_2'$

Exercise 3

A parallel plate air capacitor is vertically inserted into a liquid dielectric with relative permittivity ϵ_r and density ρ. The capacitor plates have an area S (where $S = ab$), and a separation distance d. The voltage U between the two plates is kept constant. Find the height h that the liquid level rises in the capacitor.



Capacity = $C = \frac{\varepsilon_r \varepsilon_0 a b_1}{d} + \frac{\varepsilon_0 a b_2}{d} = \frac{\varepsilon_0 [b + b_1(\varepsilon_r - 1)] a}{d}$

Static $\underbrace{F}_{\text{force electric give to liquid}} = mg. = ahd\rho g.$

Suppose our liquid height goes down a bit Δx.

$$\underbrace{mg\Delta x}_{\text{Gravity}} = \underbrace{\frac{1}{2}U^2(C_2 - C_1)}_{\text{energy change}} - \underbrace{U\cdot U(C_2 - C_1)}_{\text{power source.}}$$

$$\Rightarrow mg\Delta x = \frac{\varepsilon_0(\varepsilon_r - 1) a\Delta x}{2d}U^2 \quad \Rightarrow h = \frac{\varepsilon_0(\varepsilon_r - 1)}{2d^2\rho g}U^2.$$

Uniqueness Theorem:

A solution of Poisson's equation $\nabla^2 V = -\frac{\rho_f}{\epsilon}$ that satisfies the given boundary conditions is a unique solution.

Poisson's equation:

$$\nabla^2 V = -\frac{\rho_f}{\epsilon} = -\frac{\rho}{\epsilon_0} \tag{6}$$

where ρ_f is the free charge density, ϵ is the absolute permittivity, and ρ is the total charge density (free charge density + induced charge density).

Laplace's equation:

$$\nabla^2 V = 0 \quad (\rho = 0) \tag{7}$$

which is a special case of Poisson's equation ($\rho = 0$ everywhere).

We can try to show the proof of the uniqueness theorem.

Proof Uniqueness : Suppose we can have 2 different solutions. φ' φ''

$$\begin{cases} \vec{E}' = -\nabla\varphi' \\ \vec{E}'' = -\nabla\varphi'' \end{cases} \quad D' \ D'' \ldots$$

According to defn $\varphi' = \varphi''$ or $\vec{D}' \cdot \vec{e}_n = \vec{D}'' e_n$ boundary
same boundary conditions

$\vec{Z}(r) = (\varphi' - \varphi'')(\vec{D}' - \vec{D}'')$ $Z = 0$ on boundary case.

$0 = \oint_S Z(r) = \int_V \nabla \cdot \vec{Z}(\vec{r})\, d\tau.$ $\nabla \cdot Z(r) = (\varphi' - \varphi'')(\underbrace{\nabla \cdot \vec{D}' - \nabla \cdot \vec{D}''}_{=0}) + (\nabla\varphi' - \nabla\varphi'')(\vec{D}' - \vec{D}'')$

Plug in

we're talking about same sys $\nabla \cdot \vec{D}' = \rho = \nabla \cdot \vec{D}''$ $\Rightarrow \nabla \cdot \vec{Z}(r) = -(\vec{E}' - \vec{E}'')(\vec{D}' - \vec{D}'')$

$\int_V (\vec{E}' - \vec{E}'')(\vec{D}' - \vec{D}'')\, dv = 0.$

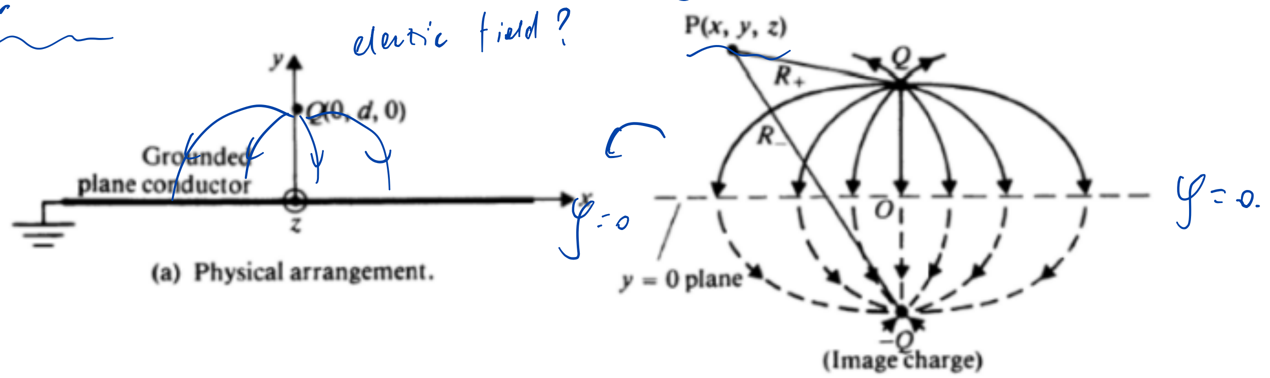
$\int_V \varepsilon (\vec{E}' - \vec{E}'')^2 dv = 0.$ $\varepsilon \neq 0.$ $(\vec{E}' - \vec{E}'') \geq 0$ $\rightarrow E' \equiv E''$ Unique !

Method of Images

Methods of images is a smart way to solve electrostatics to satisfy certain boundary conditions, utilizing equivalent image charge.(e.g. The voltage potential of a plate is 0 everywhere) The use of image charge is actually based on the uniqueness theorem of electrostatic solution.

Case 1: Point Charge and Grounded Plane Conductor

$P(x, y, z)$.

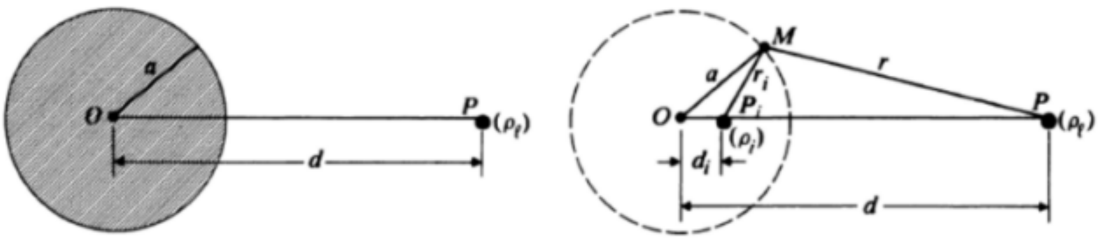


(a) Physical arrangement.

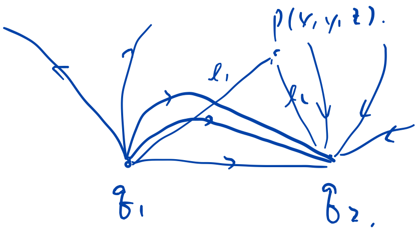
(b) Image charge and field lines.

Shares boundary condition

Case 2: Line Charge and Parallel Conducting Cylinder



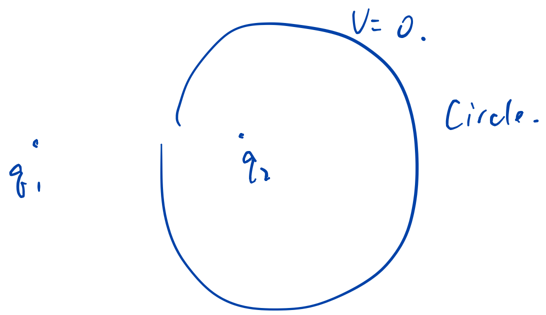
(a) Line charge and parallel conducting cylinder. (b) Line charge and its image.



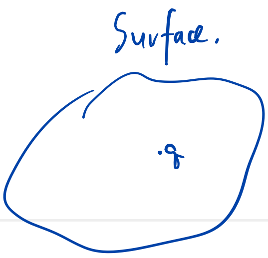
$V_p = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{l_1} + \frac{q_2}{l_2}\right)$

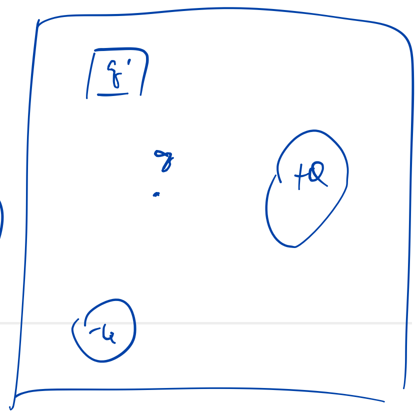
$V_p = 0$? $\frac{q_1}{l_1} + \frac{q_2}{l_2} = 0$.

$\frac{l_1}{q_1} = -\frac{l_2}{q_2}$. $\frac{l_1}{l_2} = -\frac{q_1}{q_2}$



treat the circle as a mirror





Calculate distribution

find V=0 surface,

claim find find solution..

Reference

Nana Liu, VE 230 slides.

Fan Hu, VE 230 RC slides.

Jiafu Cheng, Electro-megnetics.

Thanks!