RC4: Capacitors and Electrostatic Solutions

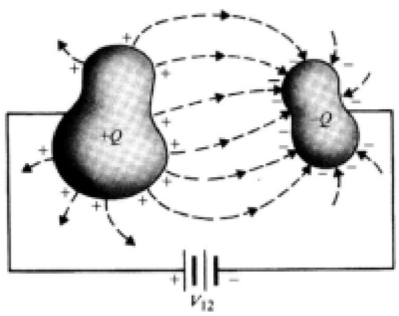
By Mo Yang

Capacitors

Definition: The capacitance of isolated conducting body is the electric charge that must be added to the body per unit increase in its electric potential.

$$C = \frac{Q}{V} \tag{1}$$

Components: Two conductors with arbitrary shapes are separated by free space or dielectric medium.



Specifically, for the connections of different capacitors, we can have

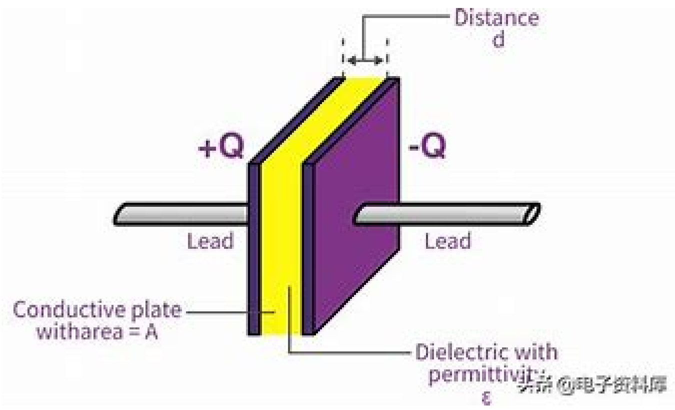
Series:

$$\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n} \tag{2}$$

Parallel:

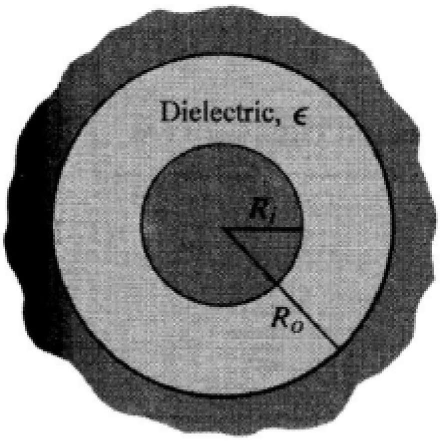
$$C_{pr} = C_1 + C_2 + \ldots + C_n \tag{3}$$

Use of media in capacitor



Exercise 1

A spherical capacitor consists of an inner conducting sphere of radius R_i and an outer conductor with a sphere inner wall of radius R_o. The space in between is filled with a dielectric of permitivity ϵ. Determine the capacitance.



Energy in Capacitors and Electric Fields

The potential energy of N discrete charges at rest:

$$W_e = \frac{1}{2} \sum_{k=1}^{N} Q_k V_k \tag{4}$$

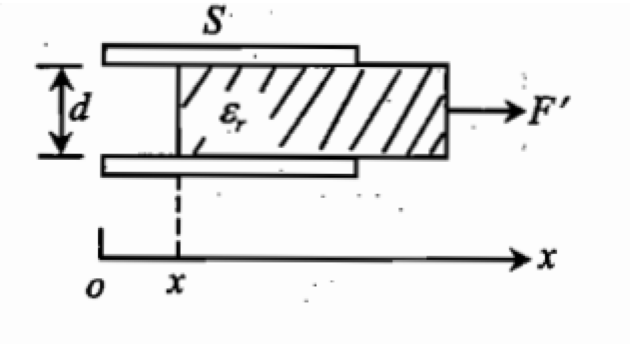
We can also deduce the case for continuous distribution.

$$W_e = \frac{1}{2}\int_{R^3} \epsilon E^2 dV \tag{5}$$

Exercise 2

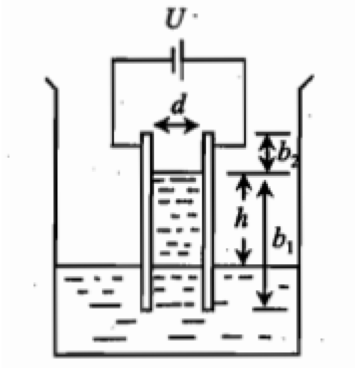
A parallel plate capacitor has a plate area of S and a plate separation of d. The space between the plates is filled with a dielectric material with a relative permittivity ϵ_r. Under the following two conditions, determine how much external force F is needed to completely remove the dielectric from the capacitor:

1. The voltage U across the capacitor remains constant.
2. The charge Q on the capacitor remains constant.



Exercise 3

A parallel plate air capacitor is vertically inserted into a liquid dielectric with relative permittivity ϵ_r and density ρ. The capacitor plates have an area S (where $S = ab$), and a separation distance d. The voltage U between the two plates is kept constant. Find the height h that the liquid level rises in the capacitor.



Uniqueness Theorem:

A solution of Poisson's equation $\nabla^2 V = -\frac{\rho_f}{\epsilon}$ that satisfies the given boundary conditions is a unique solution.

Poisson's equation:

$$\nabla^2 V = -\frac{\rho_f}{\epsilon} = -\frac{\rho}{\epsilon_0} \tag{6}$$

where ρ_f is the free charge density, ϵ is the absolute permittivity, and ρ is the total charge density (free charge density + induced charge density).

Laplace's equation:

$$\nabla^2 V = 0 \tag{7}$$

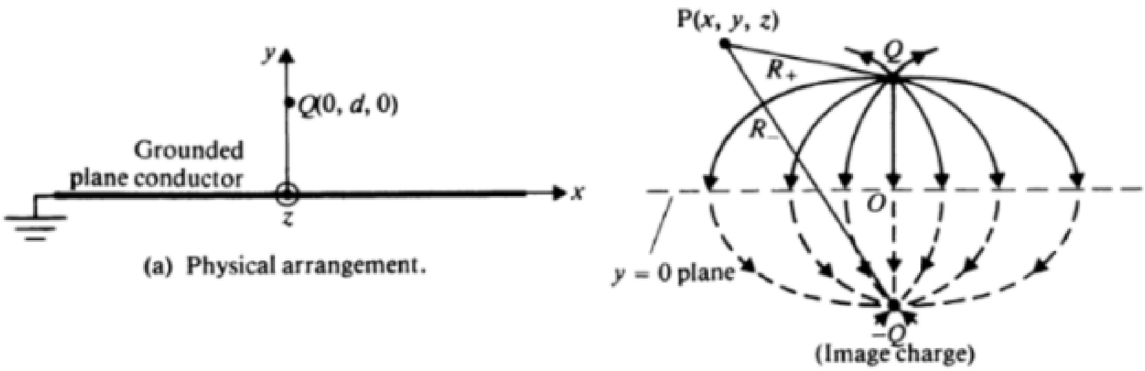
which is a special case of Poisson's equation ($\rho = 0$ everywhere).

We can try to show the proof of the uniqueness theorem.

Method of Images

Methods of images is a smart way to solve electrostatics to satisfy certain boundary conditions, utilizing equivalent image charge.(e.g. The voltage potential of a plate is 0 everywhere) The use of image charge is actually based on the uniqueness theorem of electrostatic solution.

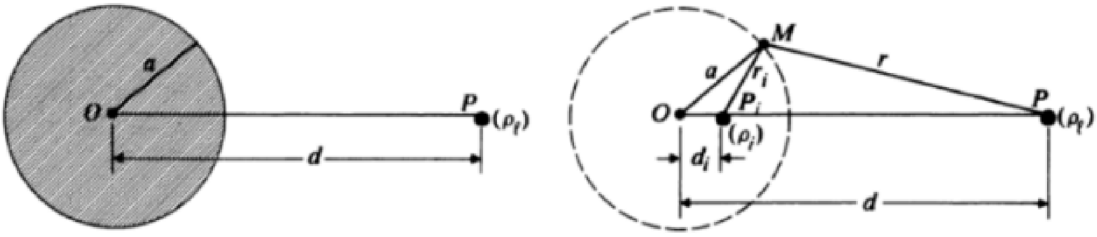
Case 1: Point Charge and Grounded Plane Conductor



(a) Physical arrangement.

(b) Image charge and field lines.

Case 2: Line Charge and Parallel Conducting Cylinder



(a) Line charge and parallel conducting cylinder. (b) Line charge and its image.

Reference

Nana Liu, VE 230 slides.

Fan Hu, VE 230 RC slides.

Jiafu Cheng, Electro-megnetics.

Thanks!