Recitation Class for Multi-variable Integration

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1 Cuboids

Let $a_k, b_k, k = 1, \dots, n$ be pairs of numbers with $a_k < b_k$. Then the set $Q \subset \mathbb{R}^n$ given by

$$Q = [a_1, b_1] \times \cdots \times [a_n, b_n] = \{x \in \mathbb{R}^n : x_k \in [a_k, b_k], k = 1, \dots, n\} \tag{1}$$

is called an n-*cuboid*. We define the volume of Q to be

$$|Q| := \prod_{k=1}^{n} (b_k - a_k) \tag{2}$$

We will denote the set of all n-cuboids by $\mathcal{Q}_n$.

Remark: Clearly, an n-cuboid is a compact set in $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded non-empty set. We define the outerand inner volume of Ω by

$$\begin{gathered} \overline{V}(\Omega) := \inf\left\{\sum_{k=0}^{r} |Q_k| : r \in \mathbb{N},\, Q_0, \dots, Q_r \in \mathcal{Q}_n, \Omega \subset \bigcup_{k=1}^{r} Q_k\right\}, \\ \underline{V}(\Omega) := \sup\left\{\sum_{k=0}^{r} |Q_k| : r \in \mathbb{N},\, Q_0, \dots, Q_r \in \mathcal{Q}_n, \Omega \supset \bigcup_{k=1}^{r} Q_k, \bigcap_{k=1}^{r} Q_k = \emptyset\right\}. \end{gathered} \tag{3}$$

It is easy to see that $0 \le \underline{V}(\Omega) \le \overline{V}(\Omega)$.

We can now define Jordan Measurable based on outer and inner volume of a set.

Let $\Omega \subset \mathbb{R}^n$ be a bounded set. Then Ω is said to be Jordan measurable if either

- $\overline{V}(\Omega) = 0$ or
- $\overline{V}(\Omega) = \underline{V}(\Omega)$.

In the first case, we say that Ω has Jordan measure zero, in the second case we say that

$$|\Omega| := \overline{V}(\Omega) = \underline{V}(\Omega) \tag{4}$$

is the Jordan measure of Ω.

(Refering to Hu Pingbang's VV285 RC)

2 Some Properties about multi-variable integration

2.1 Fubini's Theorem

Let Q_1 be an n_1-cuboid and Q_2 an n_2-cuboid so that $Q := Q_1 \times Q_2 \subset \mathbb{R}^{n_1+n_2}$ is an (n_1+n_2)-cuboid. Assume that $f : Q \to \mathbb{R}$ is integrable on Q and that for every $x \in Q_1$ the integral

$$g(x) = \int_{Q_2} f(x, \cdot) \tag{5}$$

exists. Then

$$\int_Q f = \int_{Q_1 \times Q_2} f = \int_{Q_1} g = \int_{Q_1} \left(\int_{Q_2} f \right) \tag{6}$$

Remark: This is a very powerful tool. So that we can divide and conquer a integral in $\mathbb{R}^n$.

2.2 Linearity

$$\int\int_R \lambda f(x,y) + \mu g(x,y) dA = \lambda \int\int_R f(x,y) dA + \mu \int\int_R g(x,y) dA. \tag{7}$$

2.3 Area additivity

If region R can be split into two disjoint regions D_1, D_2 with $D_1 \cap D_2 = \phi$, then

$$\int\int_R f(x,y) dA = \int\int_{D_1} f(x,y) dA + \int\int_{D_2} f(x,y) dA \tag{8}$$

2.4 Integral mean value theorem

If f is continuous on D, and D is a connected bounded closed interval, then there exists x_0 that

$$\int_D f = f(x_0) \cdot |D| \tag{9}$$

2.5 Comparability

If $f(x) > g(x)$ is always true on D, then

$$\int\int_D f(x) dA > \int\int_D g(x) dA \tag{10}$$

2.6 Examples

1. Find the intersecting volume of $x^2 + y^2 \le a^2$ and $x^2 + z^2 \le a^2$

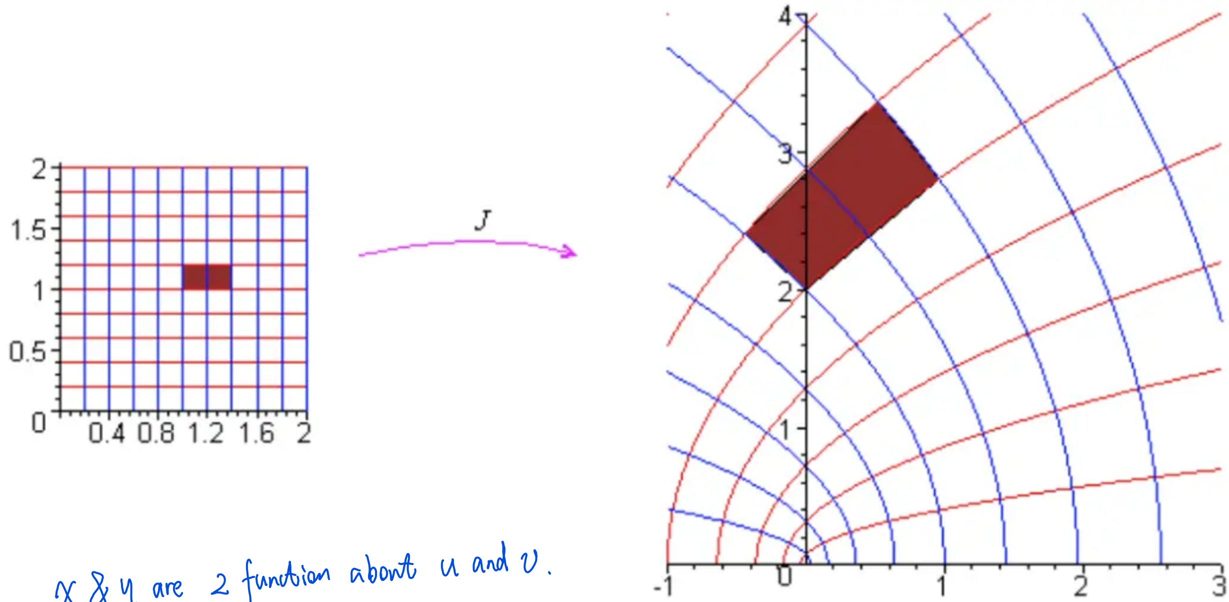
We only consider the first quadrat first

$\frac{1}{8}V = \iint_D \sqrt{a^2-x^2}\, dx\, dy \qquad D: x^2+y^2 \le a^2$

$= \int_0^a dx \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\, dy = \int_0^a (\sqrt{a^2-x^2})^2 = \frac{2}{3}a^3$

$\Rightarrow V = \frac{16}{3}a^3$

3 Integration by substitution



x & y are 2 function about u and v.

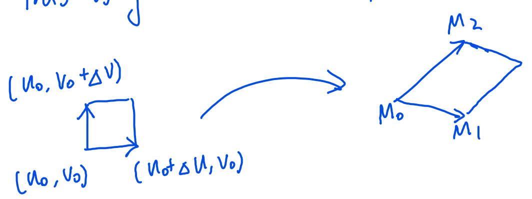
Now we consider a 2-d problem. Suppose $x = \cancel{f}\,x(u, v)$ and $y = \cancel{g}\,y(u, v)$

$$\int\int_D f(x,y)dxdy = \int\int_{D'} f(x,y)|\frac{\partial(x,y)}{\partial(u,v)}|dudv \tag{11}$$

where

$$|\frac{\partial(x,y)}{\partial(u,v)}| = |\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}| \tag{12}$$

This is just a intuitive explation in 2-D case:



$M_0: (x(u_0,v_0), y(u_0,v_0))$

$M_1: (x(u_0+\Delta u, v_0), y(u_0+\Delta u, v_0))$

$M_2: (x(u_0, v_0+\Delta v), y(u_0, v_0+\Delta v))$

Mean Value Therom

$$\underbrace{\Delta x \Delta y}_{\text{Area in }(x,y)} = |\overrightarrow{M_0M_1} \times \overrightarrow{M_0M_2}| = |(x_1-x_0)(y_2-y_0) - (x_2-x_0)(y_1-y_0)| = |(\frac{\partial x}{\partial v}\Delta u + o(\Delta u))(\frac{\partial y}{\partial u}\Delta v + o(\Delta v)) - \cdots|$$

$$= |\frac{\partial(x,y)}{\partial(u,v)}| \underbrace{\Delta u \Delta v}_{\text{Area in }(u,v)}$$

3.1 Examples

2. Find the volume of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \tag{13}$$

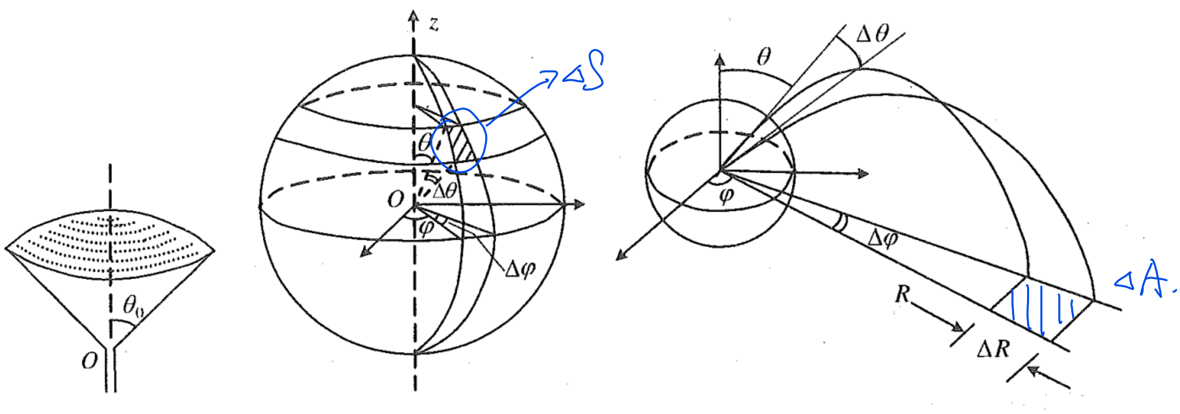
let $x' = \frac{x}{a}$, $y' = \frac{y}{b}$, $z' = \frac{z}{c}$ $\qquad x'^2 + y'^2 + z'^2 \leq 1$

(You can practice integration yourselves)

$$|J| = \begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{vmatrix} = \frac{1}{abc} \quad \Rightarrow V' = \frac{1}{abc} V$$

$$V' = \frac{4}{3}\pi \qquad \Rightarrow V = \frac{4}{3}\pi abc.$$

3. The sprinkler head used for sprinkling irrigation is shown in Figure 2-Example 3(a). Small holes with the same diameter are distributed on the spherical surface to spray water. The radius of the spherical surface is r, and the small holes are relative to the axis of symmetry. The distribution range of the polar angle 0 is: $0 \leqslant \theta \leqslant \theta_0 = \pi/4$. In order to make the water injection energy sprayed to the ground evenly distributed (uniform irrigation), the distribution of the number of small holes per unit area on the spherical surface of the sprinkler is calculated, that is, the expression of the small hole number density n.



(a) (b) (c)

Maybe someone of you have met this physics problem before.

From physics. You know $R = \frac{1}{g} \cdot 2v_0^2 \sin(\frac{\pi}{2} - \theta)\cos(\frac{\pi}{2} - \theta) = \frac{v_0^2}{g}\sin 2\theta$

So. For ΔR. $\qquad \Delta R = \frac{v_0^2}{g} \Delta \sin 2\theta$

$$\Rightarrow \Delta A = \Delta R \cdot (\Delta\varphi \cdot R) = \left(\frac{v_0^2}{g}\right)^2 \sin 2\theta \cdot \Delta \sin 2\theta \cdot \Delta\varphi.$$

On the other hand.

$$\Delta S = r^2 \sin\theta \, \Delta\theta \, \Delta\varphi.$$

We want the water be spread in a even way. $\Rightarrow$ That is $\frac{\Delta N}{\Delta A} = \text{Const}$

$$\Rightarrow \frac{n(\theta) \cdot r^2 \sin\theta \, \Delta\theta \, \Delta\varphi}{\left(\frac{v_0^2}{g}\right)^2 \sin 2\theta \, \Delta \sin 2\theta \, \Delta\varphi} = \text{Const} \qquad \Rightarrow n(\theta) = C_2 \cos\theta \frac{\Delta \sin 2\theta}{\Delta\theta} = C_3 \cos\theta \cos 2\theta.$$

4. Find

$$\int\int_{x^2+y^2\le R^2} e^{x^2+y^2}\,dxdy \tag{14}$$

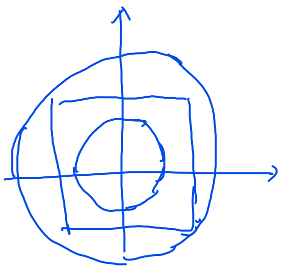
$$\iint_{x^2+y^2\le R^2} e^{x^2+y^2}\,dx\,dy \underset{r^2=x^2+y^2}{=\!=} \int_0^R e^{-r^2}\,dr\,d\theta = \int_0^R dr\, e^{-r^2}\, r \int_0^{2\pi} d\theta = 2\pi \int e^{-r^2}\, r\,dr$$

Change into Cardinal Cordinate

$$= \pi \left(e^{-r^2}\right)\Big|_0^R = \pi\left(1-e^{-R^2}\right)$$

Interestingly. Do you remember $\int_0^\infty e^{-x^2}dx$?

You cannot find $\int e^{-x^2}dx$ but $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$



Consider $\pi\left(1-e^{-R^2}\right) \le \left(\int_{-R}^{R} e^{-x^2}dx\right)^2 \le \pi\left(1-e^{-2R^2}\right)$

Small Circle Square Big Circle

$\Rightarrow \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

3.2 Cylindrical coordinates

Cylindrical coordinates in $\mathbb{R}^3$ are given through a map

$$\Phi : (0,\infty)\times[0,2\pi)\times\mathbb{R}\to\mathbb{R}^3\setminus\{0\}, \qquad (r,\phi,\zeta)\mapsto(x,y,z) \tag{15}$$

defined by

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z=\zeta \tag{16}$$

In this case

$$|\det J_\Phi(r,\phi,\zeta)| = \left|\det\begin{pmatrix}\cos\phi & -r\sin\phi & 0\\ \sin\phi & r\cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\right| = r \tag{17}$$

3.3 Spherical coordinates

Spherical coordinates in $\mathbb{R}^3$ are often defined through a map

$$\begin{gathered}\Phi : (0,\infty)\times[0,2\pi)\times(0,\pi)\to\mathbb{R}^3\setminus\{0\}, \quad (r,\phi,\theta)\mapsto(x,y,z),\\ x = r\cos\phi\sin\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\theta.\end{gathered} \tag{18}$$

Of course, there is a certain freedom in defining θ and ϕ, so there are alternative formulations. The modulus of the determinant of the

Jacobian is given by

$$|\det J_\Phi(r,\phi,\theta)| = \left|\det\begin{pmatrix}\cos\phi\sin\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta\\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta\\ \cos\theta & 0 & -r\sin\theta\end{pmatrix}\right| = r^2\sin\theta \tag{19}$$

4 Integration on higher dimensions

4.1 n-dimentional Ball

$$B_n(a) = \{(x_1, x_2, \ldots, x_n) | x_1^2 + x_2^2 + \ldots + x_n^2 \le a^2\} \tag{20}$$

$$\int_{B_n(a)} \cdots \int dx_1 dx_2 \cdots dx_n = a^n \underbrace{\int_{x_1^2+\cdots+x_n^2 \le 1} \cdots \int dx_1 \cdots dx_n}_{B_n(1)}$$

$$\underset{\text{The volume}}{\mu}(B_n(1)) = \int_{x_1^2+\cdots+x_{n-2}^2 \le 1-x_{n-1}^2-x_n^2} \cdots \int dx_1 \cdots dx_n = \int_{x_{n-1}^2+x_n^2\le 1} \cdots \int dx_{n-1}\, dx_n \cdot \mu\left(B_{n-2}\left(\sqrt{1-x_n^2-x_{n-1}^2}\right)\right)$$

$$= \iint_{x_{n-1}^2+x_n^2 \le 1} dx_{n-1}\, dx_n \cdot \left(\sqrt{1-x_n^2-x_{n-1}^2}\right)^{n-2} \cdot \mu(B_{n-2}(1))$$

$$= \mu(B_{n-2}(1)) \iint_{x_{n-1}^2+x_n^2 \le 1} dx_{n-1}\, dx_n \left(1-x_n^2-x_{n-1}^2\right)^{\frac{n-2}{2}}$$

(Change to cardinal coordinate) $= \mu(B_{n-2}(1)) \cdot \int_0^1 dr \cdot \int_0^{2\pi} d\theta\, (1-r^2)^{\frac{n-2}{2}}\, r = \frac{2\pi}{n}\mu(B_{n-2}(1))$

$$\Rightarrow \begin{cases} \mu(B_{2n}(a)) = a^{2n}\dfrac{\pi^n}{n!} \\ \mu(B_{2n-1}(a)) = a^{2n-1}\cdot 2^n \dfrac{\pi^{n-1}}{(2n-1)!} \end{cases}$$

For further questions, you can contact me on wechat.

Fragments

芬兰

扫一扫上面的二维码图案，加我为朋友。