Recitation Class for Vector Calculus

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1 Surface

1.1 Parameterization

A smooth parametrized m-surface in $\mathbb{R}^n$ is a subset $\mathcal{S} \subset \mathbb{R}^n$ together with a locally bijective, continuously differentiable map (parametrization)

$$\varphi : \Omega \to \mathcal{S}, \qquad \Omega \subset \mathbb{R}^m \tag{1}$$

such that

$$\operatorname{rank} D\varphi|_x = m \tag{2}$$

for almost every $x \in \Omega$. If $m = n - 1$, $\mathcal{S}$ is said to be a *parametrized hyper-surface*.

1.2 Tangent Spaces of Surface

Let $\mathcal{S} \subset \mathbb{R}^n$ be a parametrized m-surface with parametrization $\varphi : \Omega \to \mathcal{S}$. Then

$$t_k(p) = \frac{\partial}{\partial x_k} \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix} \Bigg|_{x=\varphi^{-1}(p)}, \qquad k = 1, \dots, m. \tag{3}$$

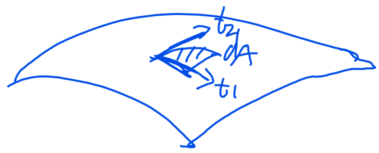
is called the kth tangent vector of $\mathcal{S}$ at $p \in \mathcal{S}$ and

$$T_p\mathcal{S} := \operatorname{ran} D\varphi|_x = \operatorname{span}\{t_1(p), \dots, t_m(p)\} \tag{4}$$

is called the tangent space to $\mathcal{S}$ at p. The vector field

$$t_k : \mathcal{S} \to \mathbb{R}^n, \qquad p \mapsto t_k(p) \tag{5}$$

is called the kth tangent vector field on $\mathcal{S}$.



1.3 Flux

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field defined in a neighborhood of a hypersurface $\mathcal{S}$ with parametrization $\varphi : \Omega \to \mathbb{R}^n, \Omega \subset \mathbb{R}^{n-1}$. Then we define the*flux of F through* $\mathcal{S}$ by

$$\begin{aligned} \int_{\mathcal{S}} F \, d\vec{A} &:= \int_{\mathcal{S}} < F, N > dA \qquad (= \int_{\mathcal{S}} < F, d\vec{A} > \\ &= \int_{\Omega} < F \circ \varphi(x), N \circ \varphi(x) > \sqrt{g(x)} \, dx_1 \dots dx_{n-1} \end{aligned}$$

where

$$g(x) = \det \begin{pmatrix} < t_1, t_1 >, \dots < t_1, t_m > \\ \dots, \dots, \dots \\ < t_m, t_1 >, \dots, < t_m, t_m > \end{pmatrix} \tag{6}$$

Especially, in $\mathbb{R}^3$

$$A = \int \int_{\Omega} \parallel t1 \times t2 \parallel \circ \varphi(x) dx_1 dx_2 \tag{7}$$

Where t1, t2 are the tangent vector at point (x1, x2).

Note that you don' t need to normalize the tangent vector!

Another point of view see the problem somehow similar to substitution. (Note this is only another point which is not include in slides!)

$$\begin{aligned} \vec{n} &= \frac{\vec{r'_u} \times \vec{r'_v}}{|\vec{r'_u} \times \vec{r'_v}|} \\ dS &= |\vec{r'_u} \times \vec{r'_v}| dudv \\ d\vec{S} = \vec{n} dS = \det \begin{pmatrix} \vec{i} \ \vec{j} \ \vec{k} \\ x'_u \ y'_u \ z'_u \\ x'_v \ y'_v \ z'_v \end{pmatrix} &= (\frac{\partial(y,z)}{\partial(u,v)} dudv)\vec{i} + (\frac{\partial(x,z)}{\partial(u,v)} dudv)\vec{j} + (\frac{\partial(x,y)}{\partial(u,v)} dudv)\vec{k} \end{aligned} \tag{8}$$

So we can define

$$\begin{aligned} dy \wedge dz &= \frac{\partial(y,z)}{\partial(u,v)} dudv \\ dx \wedge dz &= \frac{\partial(x,z)}{\partial(u,v)} dudv \\ dx \wedge dy &= \frac{\partial(x,y)}{\partial(u,v)} dudv \end{aligned} \tag{9}$$

So for $\vec{v} = (P, Q, R)$

$$\int\int \vec{v} \cdot d\vec{S} = \int\int_S P dy \wedge dz + Q dx \wedge dz + R dx \wedge dy \tag{10}$$

1.4 Example

1. For

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \tag{11}$$

Find

$$\int\int_S x^3 dydz + y^3 dzdx \tag{12}$$

Let $x = a\sin\theta\cos\varphi$ $\quad y = b\sin\theta\sin\varphi.$ $\quad c = z\cos\theta.$

$0 \leq \varphi \leq 2\pi.$ $\quad 0 \leq \theta \leq \frac{\pi}{2}$

$$\begin{cases} \dfrac{\partial(y,z)}{\partial(\theta,\varphi)} = bc\sin^2\theta\cos\varphi \\ \\ \dfrac{\partial(z,x)}{\partial(\theta,\varphi)} = ac\sin^2\theta\sin\varphi \end{cases}$$

$$\begin{aligned} \iint_S x^3 dydz &= \iint a^3\sin^3\theta\cos^3\varphi \cdot bc\sin^2\theta\cos\varphi \, d\theta d\varphi \\ &= a^3bc\int_0^{\frac{\pi}{2}}\int_0^{2\pi} \sin^5\theta\cos^4\varphi \, d\theta d\varphi \\ &= \frac{2}{5}\pi a^3 b. \end{aligned}$$

$$\iint_S y^3 dxdy = \frac{2}{5}\pi ab^3 c.$$

So Answer is $\frac{2}{5}\pi(a^3b + ab^3c).$

2 Gradient, Divergence and Rotation

2.1 Gradient

Let's consider a function **f(x, y)** that depends on variables **x** and **y**. The gradient of **f** at a point (x_0, y_0) is defined as:

$$\nabla f(x_0, y_0) = (\partial f/\partial x, \partial f/\partial y) \tag{13}$$

Here, ∂f/∂x represents the partial derivative of **f** with respect to **x** evaluated at (x_0, y_0), and ∂f/∂y represents the partial derivative of **f** with respect to **y** evaluated at (x_0, y_0).

2.2 Divergence

$$\operatorname{div} F := \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n} \tag{14}$$

is called the divergenceof F.

Remark: The total flux density at a point x is given by the divergence of the field at x.

In triangle calculus, the divergence of a vector field F can be expressed as

$$div\, F =< \nabla, F >= \left\langle \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \begin{pmatrix} F_1 \\ \vdots \\ F_n \end{pmatrix} \right\rangle = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}. \tag{15}$$

In some other notations, one also uses $\nabla \cdot F$ to indicates the divergence of F. This type of notation is more common in a physical textbooks.

2.3 Rotation

In triangle calculus, the rotation of a vector field F can be expressed as

$$rotF = \nabla \times F = (\partial F_z/\partial y - \partial F_y/\partial z)\vec{i} + (\partial F_x/\partial z - \partial F_z/\partial x)\vec{j} + (\partial F_y/\partial x - \partial F_x/\partial y)\vec{k} \tag{16}$$

Let $\Omega \subset \mathbb{R}^n$ be an open set, $F : \Omega \to \mathbb{R}^n$ a continuously dif{}ferentiable vector field and $\mathcal{C}^*$ an oriented closed curve in $\mathbb{R}^n$. Then

$$\int_{\mathcal{C}^*} \langle F, T \rangle ds \tag{17}$$

is called the (total) circulation of F along $\mathcal{C}$.

If a field is of no rotation, then there exists a potential.

3 Stokes' theorem

3.1 Newton-Leibniz formula

$$\int_b^a f(x)dx = F(b) - F(a) \tag{18}$$

3.2 Green's theorem

$\Omega = [a, b]$ $\partial\Omega = \{a, b\}$.

Let $R \subset \mathbb{R}^2$ be a bounded, simple region and $\Omega \supset R$ an open set containing R. Let $F : \Omega \to \mathbb{R}^2$ be a continuously differentiable vector field. Then

$$\int_{\partial R^*} F\, d\vec{s} = \int_R \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \right) dx \tag{19}$$

where ∂R^* denotes the boundary curve of R with positive (counter-clockwise) orientation.

3.3 Stokes theorem

Let $\Omega \subset \mathbb{R}^3$ be an open set, $\mathcal{S} \subset \Omega$ a parametrized, admissible surface in $\mathbb{R}^3$ with boundary $\partial\mathcal{S}$ and let $F : \Omega \to \mathbb{R}^3$ be a continuously differentiable vector field. Then

$$\int_{\partial\mathcal{S}^*} F\, d\vec{s} = \int_{\mathcal{S}^*} \operatorname{rot} F\, d\vec{A} \tag{20}$$

where the orientations of the boundary curve $\partial\mathcal{S}^*$ and the surface $\mathcal{S}^*$ are chosen according to right hand law.

3.4 Guass's theorem

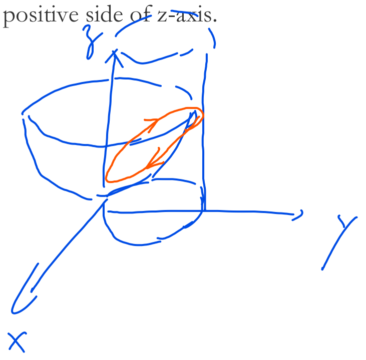
Let $R \subset \mathbb{R}^n$ be an admissible region and $F : \overline{R} \to \mathbb{R}^n$ a continuously dif ferentiable vector field. Then

$$\int_R \operatorname{div} F\, dx = \int_{\partial R^*} F\, d\vec{A}. \tag{21}$$

3.5 Example

2. L is the intersection line of $z = 3x^2 + 4y^2$ and $4x^2 + y^2 = 4y$, and is clockwise seeing from the positive side of z-axis.

   Please find the circulation of $\vec{v} = y(z+1)\cdot\vec{i} + xz\cdot\vec{j} + (xy - z)\cdot\vec{k}$ on L



$$\oint_L \vec{v}\cdot\vec{t}\,dl = \iint_S (\nabla\times\vec{v})\,d\vec{S}$$

$$= \iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y(z+1) & xz & xy-z \end{vmatrix}$$

$$= -\iint_S dxdy$$ (which is the projection of area on x-y surface)

$$= 2\pi$$

3. Suppose $\nabla \cdot E = \frac{\rho}{\epsilon_0}$, try to illustrate the Guass's law in physics.

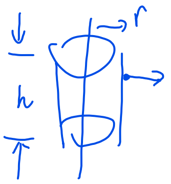
Guass' Law: $\oiint_S \vec{E}\cdot d\vec{s} = \frac{1}{\varepsilon_0}\sum Q_i$

$$\oiint_S \vec{E}\cdot d\vec{s} = \iiint_V (\nabla\cdot\vec{E})\,dV = \iiint_V \cdot\left(\frac{\rho}{\varepsilon_0}\right)dV = \frac{1}{\varepsilon_0}\sum Q_i$$

Next. we'll try to find the $\vec{E}$ of an infinity uniformly charged wire.

ρ density of electrons $= \rho$. find E.

As it is symmetric. We find a Guass Surface.



we know there is no $\vec{E}$ point upward or downward.

So $\vec{E}$ can only pointing from the wire.

As $\oint_S \vec{E}\cdot ds = \frac{1}{\varepsilon_0}\sum Q_i \Rightarrow E\cdot 2\pi r\cdot h = h\cdot \rho \cdot \frac{1}{\varepsilon_0} \Rightarrow E = \frac{\rho}{2\pi r \varepsilon_0}$

3.6 General Case

For $v = P\vec{i} + Q\vec{j} + R\vec{k}$, we denote

$$\int_L \vec{v}\cdot d\vec{r} = \int_L Pdx + Qdy + Rdz = \int_L \omega_{\vec{v}}^1 \tag{22}$$

$$\int\int_s \vec{v}\cdot d\vec{S} = \int\int_S Pdy\wedge dz + Qdx\wedge dz + Rdx\wedge dy = \int \omega_{\vec{v}}^2 \tag{23}$$

So we can know

$$\begin{gathered}\oint_{\partial S} \vec{v}\cdot d\vec{S} = \int\int_S \nabla\times\vec{v}\cdot dS \\ \oint_{\partial S}\omega_{\vec{v}}^1 = \int_S \omega_{\vec{v}}^2\end{gathered} \tag{24}$$

Notice that

$$d\omega_{\vec{v}}^1 = \omega_{\nabla\times\vec{v}}^2 \tag{25}$$

We know

$$\int_{\partial S}\omega_{\vec{v}}^1 = \int_S d\omega_{\vec{v}}^1 \tag{26}$$

Similarly, we can deduce from Guass's theorem that

$$\int_{\partial S}\omega_{\vec{v}}^2 = \int_S d\omega_{\vec{v}}^2 \tag{27}$$

Generally, to n-dimention, we have

$$\int_{\partial\Omega}\omega = \int_\Omega d\omega \tag{28}$$

where Ω is a r dimention surface in $\mathbb{R}^n$

4 Green Identity

Let $R \subset \mathbb{R}^n$ be an admissible region and $u, v : \overline{R} \to \mathbb{R}$ be twice continuously dif{}ferentiable potential functions. Then

$$\int_R \langle\nabla u, \nabla v\rangle\, dx = -\int_R u\cdot\Delta v\, dx + \int_{\partial R^*} u\frac{\partial v}{\partial n}\, dA \tag{29}$$

and

$$\int_R (u\cdot\Delta v - v\cdot\Delta u)\, dx = \int_{\partial R^*}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dA. \tag{30}$$

(29) is commonly called *Green's first identity* and (30) *Green's second identity.*

Illustration:

① let $F = \psi \nabla \phi$ (which is used in electromagnetics and electrodynamics)

Divergence: $\int_U \nabla \cdot F dV = \oint_{\partial U} F \cdot n dS$

$$\Rightarrow \int_U (\psi \nabla^2 \phi + \nabla \phi \cdot \nabla \psi) = \oint_{\partial U} \psi \frac{\partial \phi}{\partial n} dS.$$

$$\Rightarrow \int_U \nabla \phi \cdot \nabla \psi = - \int_U \psi \Delta \phi + \int_{\partial U^*} \psi \frac{\partial \psi}{\partial n} dS.$$

② As $\int_V \nabla \cdot (\psi \nabla \varphi) d\tau = \oint \psi \nabla \varphi \, d\vec{s}$ (i)

$\int_V \nabla (\varphi \nabla \psi) d\tau = \oint \varphi \nabla \psi \, d\vec{s}$ (ii)

(i)-(ii): $\int [\nabla \cdot (\psi \nabla \varphi) - \nabla \cdot (\varphi \nabla \psi)] d\tau = \oint (\psi \nabla \varphi - \varphi \nabla \psi) d\vec{s}$

$$\Rightarrow \int [\psi \nabla^2 \varphi - \varphi \nabla^2 \psi] d\tau = \oint (\psi \nabla \varphi - \varphi \nabla \psi) d\vec{s}$$

$$= \oint_{\partial U} \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right) dS$$

5 References

- VV285, slide. Horst Hohberger
- RC, slides. TA-Pingbang Hu
- RC, slides. TA-Chen Yuxiang

For further questions, you can contact me through wechat.

Fragments

芬兰

扫一扫上面的二维码图案，加我为朋友。

The End !