Recitation Class Outline: Curves

Provided by Yang Mo

1 What is Curve?

1.1 Definition

Let $(V; \|\cdot\|)$ be a normed vector space and $I \subset R$ an interval.

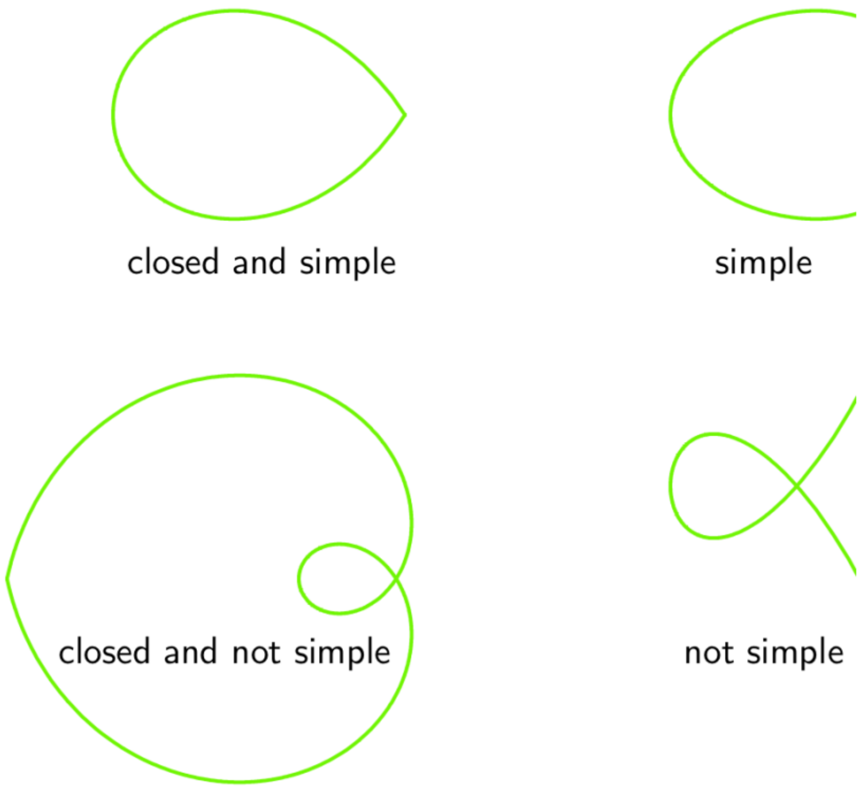
A set $C \subset V$ for which there exists a continuous, surjective and locally injective map γ: $I \to C$ is called a curve.

The map γ, is called a parametrization of C.

A curve C together with a parametrization γ,, i.e., the pair $(C; \gamma)$, is called a parametrized curve.

(From Horst's Slides, Page 352)

1.2 Simple, Open and Closed Curves



(From Horst's Slides, Page 358)

2 Some Characters to describe Curve

2.1 Tangent Vector

Tangent line at $t_0 : T_pC = x \in V : x = \gamma(t_0) + \gamma\prime(t_0)t, t \in R$

It is an approximation of the curve!

Unit tangent vector: $T \circ \gamma(t) := \frac{\gamma\prime(t)}{\|\gamma\prime(t)\|}$

2.2 Curve's Length

Suppose C is a curve in V and $\gamma : [b, s] \to C$ the parameterization.

$$L = \int_b^s ||\gamma'(t)||dt \tag{1}$$

This is actually induced with the help of Lagrange's Mean Value Theory:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \tag{2}$$

2.3 Curvature

$$\kappa \circ \gamma(t) = \frac{\| (T \circ \gamma)\prime(t) \|}{\| \gamma\prime(t) \|} \tag{3}$$

2.3.1 Example 1

Calculate the curvature of the curve with parameterization $f : [0, 1] \to R, f(x) = (cos(2\pi x), sin(2\pi x))$

(From Chen YuXiang and Zhang Leyang)

$f(x) = \begin{pmatrix} \cos(2\pi x) \\ \sin(2\pi x) \end{pmatrix}$ $\quad f'(x) = \begin{pmatrix} -2\pi \sin(2\pi x) \\ 2\pi \cos(2\pi x) \end{pmatrix}$ $\quad \|f'(x)\| = 2\pi$.

$$(T\circ f)(x) = \frac{f'(x)}{\|f'(x)\|} = \begin{pmatrix} -\sin(2\pi x) \\ \cos(2\pi x) \end{pmatrix}$$

$$(T\circ f)'(x) = \begin{pmatrix} -2\pi \cos(2\pi x) \\ -2\pi \sin(2\pi x) \end{pmatrix}$$

$$\kappa \circ f(x) = \frac{\|(T\circ f)'(x)\|}{\|f'(x)\|} = \frac{2\pi}{2\pi} = 1$$

For a circle, $\kappa = \frac{1}{r}$

2.3.2 Example 2

Calculate the curvature of the curve $y = f(x)$ (Consider the curve consist of all (x,y))

$g(x) = \begin{pmatrix} x \\ f(x) \end{pmatrix}$, $\quad g'(x) = \begin{pmatrix} 1 \\ f'(x) \end{pmatrix}$ $\quad \|g'(x)\| = \sqrt{1+f'^2(x)}$

$$(T\circ g)(x) = \frac{f(x)}{\|f(x)\|} = \begin{pmatrix} \frac{1}{\sqrt{1+f'^2(x)}} \\ \frac{f'(x)}{\sqrt{1+f'^2(x)}} \end{pmatrix} \qquad (T\circ g)'(x) = \begin{pmatrix} -\frac{f'(x) f''(x)}{(1+f'^2(x))^{3/2}} \\ \frac{f''(x)}{(1+f'(x))^{1/2}} - \frac{f'^2(x) f''(x)}{(1+f'(x))^{3/2}} \end{pmatrix}$$

$$(T\circ g)'(x) = \begin{pmatrix} -\dfrac{f'(x)f''(x)}{(1+f'(x)^2)^{3/2}} \\ \dfrac{f''(x)}{(1+f'(x)^2)^{3/2}} \end{pmatrix}.$$

$$\Rightarrow \kappa\circ f(x) = \frac{\|(T\circ f)'(x)\|}{\|f'(x)\|} = \boxed{\frac{|f''(x)|}{(1+f'(x))^{3/2}}}$$

3 Line Integral and Potential Fields

3.1 Line Integral

line integral of a potential function: the line integral of f along C^* by

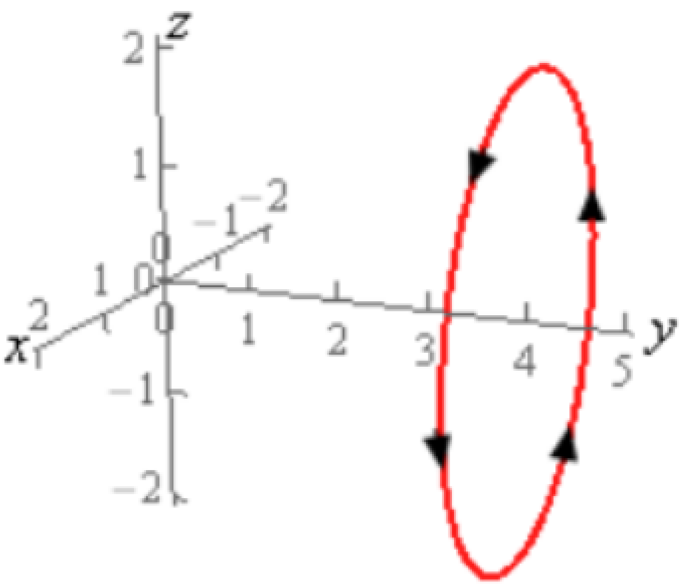
$$\int_{C^*} f\,dl := \int_I (f\circ\gamma)(t)\cdot \parallel \gamma''(t) \parallel dt \tag{4}$$

If we calculate the line integral using a concrete parametrization $\gamma : I \to C$, we obtain

$$\int_{C^*} F d\vec{l} = \int_{C^*} \langle F, T\rangle dl = \int_I \langle F\circ\gamma(t), T\circ\gamma(t)\rangle \parallel \gamma'(t) \parallel dt \tag{5}$$

3.1.1 Example 3

Evaluate $\int_{C^*} x^2y^2 dl$ where C is the circle centered at (0, 4, 0) with radius 2. The orientation is shown in following figure.



$\gamma : [0, 2\pi]$. Set a parameter θ.

$$\gamma(\theta) = \begin{pmatrix} 2\cos\theta \\ 4 \\ -2\sin\theta \end{pmatrix}. \qquad \|r'(\theta)\| = \left\| \begin{pmatrix} -2\sin\theta \\ 0 \\ -2\cos\theta \end{pmatrix} \right\| = 2.$$

$$\int_{C^*} x^2y^2\, d\vec{l} = \int_0^{2\pi} 64\cos^2(t)\cdot 2\cdot dt = 128\pi.$$

Don't forget multiplying $\|r'(\theta)\|$!

3.2 Potential Fields

Condition: There exist U(x) such that $F(x) = \nabla U(x)$

In physics there is a negative sign

Conservative Field:

$$\oint_{\mathcal{C}} F \, d\vec{\ell} = 0.$$

Potential Fields are Conservative!

3.2.1 Example 4

Prove the gravity field is conservative.

Gravity: $\vec{F} = -\frac{GMm}{r^2}\hat{r}$

There exist a $U(\vec{r}) = -\frac{GMm}{r}$

$\vec{F} = -\nabla U(\vec{r})$ So it is conservative!

The end.