MATH 2850 Mid2 RC PART I

Mo Yang

2023.7

Content

- Norm, Open Sets and Close Sets
- Continuous Functions
- First Derivatives
- Curves

Norm, Open Sets, and Close Sets

Norms

The most common norm that we're familiar with is on $\mathbb{R}^n$

$$||(x_i)||_2 = \sqrt{\sum_i x_i^2}$$

$$||(x_i)||_\infty = max\{|x_i| : 1 \leq i \leq n\}$$

Suppose $||\cdot||$ is a norm on V, then for any element x, y belongs to vector space V, it follows:

- $||x|| \geq 0$
- $k \cdot ||x|| = ||k \cdot x||$
- $||x + y|| \leq ||x|| + ||y||$

8.5. Definition. Let V be a vector space on which we may define two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then the two norms are called ***equivalent*** if there exists two constants $C_1, C_2 > 0$ such that

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1 \qquad \text{for all } x \in V. \tag{8.3}$$

Therefore, the following theorem is of fundamental importance:

8.11. Theorem. In a finite-dimensional vector space, all norms are equivalent.

Open and Close Sets

2.1.2. Definition. Let $(V, \|\cdot\|)$ be a normed vector space. A set $U \subset V$ is called *open* if for every $a \in U$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(a) \subset U$.

2.1.18. Definition. Let $(V, \|\cdot\|)$ be a normed vector space and $M \subset V$. Then M is said to be *closed* if its complement $V \setminus M$ is open.

- Open set (e.g. open ball, empty set, entire set, M=int M)
- Close set: complement of open set (e.g. single point, $\partial M \subset M$)

Suppose X is a complete normed space and $S_1, S_2, ..., S_n$ is a finite collection of subsets of X. Judge T or F:

- If S_i 's are open, then $\cup_i S_i$ is open.
- If S_i 's are open, then $\cap_i S_i$ is open.
- If S_i 's are closed, then $\cup_i S_i$ is closed.
- If S_i 's are closed, then $\cap_i S_i$ is closed.

The set $\Omega = \{A \in \mathrm{Mat}(2 \times 2) \colon \det A = 1\}$ is

- ☐ bounded.
- ☐ open.
- ☒ closed.
- ☐ compact.

For example, $A = \begin{pmatrix} n & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \in \Omega$. Take the max norm for determinant $\|\cdot\| = \max\limits_{i,j} |a_{ij}|$ and let $n \to \infty$, we can see that Ω is not bounded.

Since Ω is not bounded, then Ω is not compact.

We argue that Ω is not open. Take $A = \begin{pmatrix} n & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$ $(n > 1) \in \Omega$ and the max norm, we find $B_\varepsilon(A)$ for $\varepsilon > 0$, $B = \begin{pmatrix} n + \frac{\varepsilon}{2} & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \in B_\varepsilon(A)$, but $B \notin \Omega$.

Continuous Functions

Continuous Functions

Theorem

Let $(U, ||\cdot||_1)$ and $(V, ||\cdot||_2)$ be normed vector spaces and $f : U \to V$ a function. Then f is continuous at $a \in U$ if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in U}{\forall}\quad \|x-a\|_1 < \delta \quad \Rightarrow \quad \|f(x)-f(a)\|_2 < \varepsilon.$$

Question.

Continuous or not: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} (1 - \cos \frac{x^2}{y})\sqrt{x^2 + y^2} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

Comment.

- $y = 0$, $f(x, y)$ is continuous.
- $y \neq 0$, we want to show $\lim\limits_{\sqrt{x^2+y^2}\to 0} f(x, y) = 0$. Let $\sqrt{x^2 + y^2} \to 0$, then

$$|f(x, y)| \leq |1 - \cos \frac{x^2}{y}|\sqrt{x^2 + y^2} \leq 2\sqrt{x^2 + y^2} \to 0$$

First Derivatives

First Derivatives

Definition

There is a linear map $L_x \in \mathcal{L}(X, V)$ (called derivative) such that

$$f(x+h) = f(x) + L_x h + o(h) \qquad \text{as } h \to 0.$$

Comment.

We may thus regard the derivative as a linear map

$$D\colon C^1(\Omega, V) \to C(\Omega, \mathcal{L}(X, V)), \qquad f \mapsto Df.$$

Exercise 2. Calculate the first derivative of the function

$$f\colon \operatorname{Mat}(n \times n; \mathbb{R}) \to \operatorname{Mat}(n \times n; \mathbb{R}), \qquad f(A) = A^3$$

Find the second derivative, i.e., the derivative of $Df|_x$ as a function of x. **(6 Marks)**

We have

$$\begin{aligned} f(A+H) &= (A+H)^3 = (A+H)(A+H)(A+H) = (A+H)(A^2+HA+AH+H^2) \\ &= A^3 + AHA + A^2H + HA^2 + o(H) \end{aligned}$$

(1 Mark) so

$$Df|_A H = AHA + A^2H + HA^2$$

(1 Mark) Then

$$\begin{aligned} Df|_{A+J}H &= (A+J)H(A+J) + (A+J)^2H + H(A+J)^2 \\ &= AHA + A^2H + HA^2 + JHA + AHJ + AJH + JAH + AHJ + AJH + o(J) \end{aligned}$$

(1 Mark) so

$$D^2f|_A(H,J) = JHA + AHJ + AJH + JAH + AHJ + AJH$$

(1 Mark) and the second derivative is interpreted as a bilinear map.

Curves

Curves

Definition

Let $(V; \|\cdot\|)$ be a normed vector space and $I \subset R$ an interval.

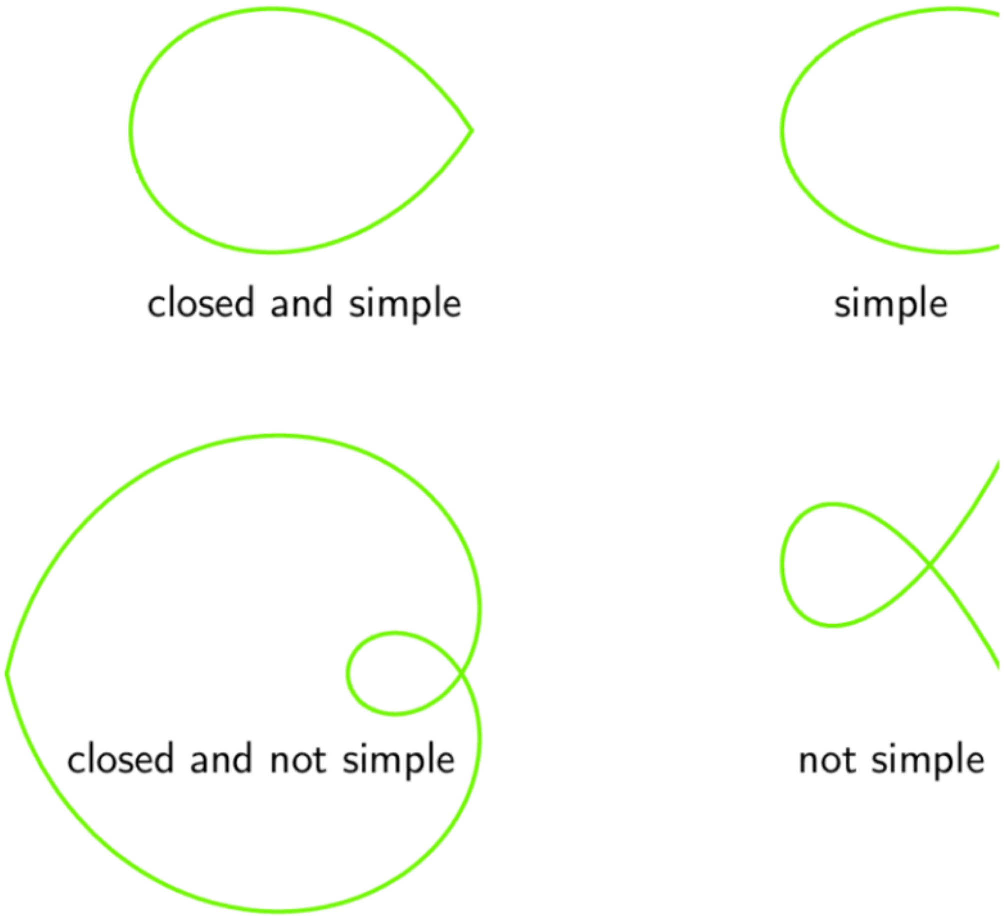
A set $C \subset V$ for which there exists a continuous, surjective and locally injective map $\gamma: I \to C$ is called a curve.

The map γ, is called a parametrization of C.

A curve C together with a parametrization γ,, i.e., the pair $(C; \gamma)$, is called a parametrized curve.

(From Horst's Slides, Page 352)

Simple, Open and Closed Curves



(From Horst's Slides, Page 358)

2.3.9. Definition. Let $\mathcal{C} \subset V$ be a curve with parametrization $\gamma \colon I \to \mathcal{C}$.

(i) Let $J \subset \mathbb{R}$ be an interval. A continuous, bijective map $r \colon J \to I$ is called a *reparametrization*.of the parametrized curve $(\mathcal{C}, \gamma)$.

Comment.

Given any two parametrizations $\gamma, \widetilde{\gamma}$ of an open curve, one can always find a reparametrization by setting $r = \gamma^{-1} \circ \widetilde{\gamma}$ (the continuity and local injectivity is enough for this definition to make sense).

Tangent Vector: $T \circ \gamma(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$

Normal Vector: $N \circ \gamma(t) = \frac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|}$

Binormal vector: $B \circ \gamma(t) = T \circ \gamma(t) \times N \circ \gamma(t)$(Only in $\mathbb{R}^3$)

Curve length: $l(\mathcal{C}) = \int_a^b \|\gamma'(t)\| \, dt$

Curve function: $l \circ \gamma(t) = \int_a^t \|\gamma'(t)\| \, dt$

Curvature: $\kappa \circ l^{-1}(s) = \left\| \frac{d}{ds} T \circ l^{-1}(s) \right\| = \frac{\left\|(T \circ \gamma)'(t)\right\|}{\|\gamma'(t)\|}$

Curvature in $\mathbb{R}^3$: $\kappa \circ l^{-1}(s) = \frac{\left\|\gamma'(t) \times \gamma''(t)\right\|}{\|\gamma'(t)\|^3}$

Torsion: $\frac{d(B \circ l^{-1}(s))}{ds} = -\tau(s)(N \circ l^{-1}(s))$, then

$\tau(s) = -\frac{d(B \circ l^{-1}(s))}{ds} \cdot (N \circ l^{-1}(s))$

Question.

Find the tangent vector, normal vector, curve length function, curvature of cycloid $\gamma(t),\ t \in (0, 2\pi)$

$$\gamma(t) = \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}$$

$$\gamma'(t) = \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix}$$

$$T \circ \gamma(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{1}{2\sin(t/2)} \begin{pmatrix} 1 - \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} \sin(t/2) \\ \cos(t/2) \end{pmatrix}$$

$$(T \circ \gamma)'(t) = \frac{1}{2} \begin{pmatrix} \cos(t/2) \\ -\sin(t/2) \end{pmatrix}$$

$$N \circ \gamma(t) = \frac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|} = \begin{pmatrix} \cos(t/2) \\ -\sin(t/2) \end{pmatrix}$$

$$\|\gamma'(t)\| = 2\sin(t/2)$$

$$l \circ \gamma(t) = \int_0^t \|\gamma'(t)\|\, dt = 4(1 - \cos t/2), \quad t \in (0, 2\pi)$$

$$l \circ \gamma(2\pi) = 8$$

$$\kappa \circ l^{-1}(s) = \frac{\|(T \circ \gamma)'(t)\|}{\|\gamma'(t)\|} = \frac{1}{4\sin(t/2)}$$

Curve Length

Suppose $\mathcal{C}$ is a curve in V and $\gamma\colon [b, s] \to \mathcal{C}$ the parameterization. The curve length formula:

$$l = \int_b^s \|\gamma'(t)\| \, dt$$

Calculate the length of the curve length of the cardioid, i.e,

r = 1 – sin(θ), using a suitable parameterization

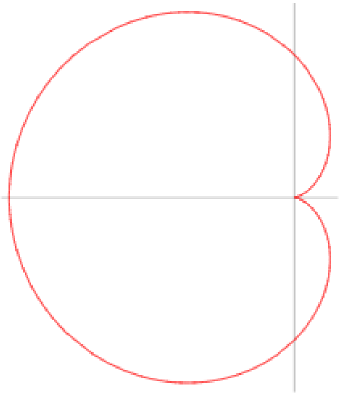
We let $x = r\cos(\theta)$ and $y = r\sin(\theta)$

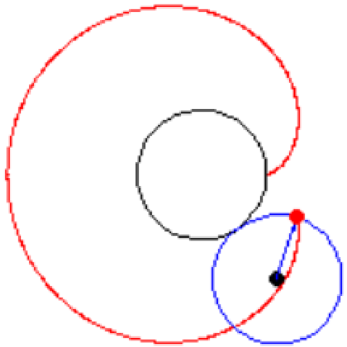
So for $\gamma(\theta) = ((1-\sin(\theta))\cos(\theta)), (1-\sin(\theta))\sin(\theta))$

$\gamma'(\theta) = (-\sin(\theta) + \sin^2(\theta) - \cos^2(\theta), \cos(\theta) - 2\sin(\theta)\cos(\theta))$

The length should be

$$\int_0^{2\pi} \sqrt{(-\sin(\theta) + \sin^2(\theta) - \cos^2(\theta))^2 + (\cos(\theta) - 2\sin(\theta)\cos(\theta))^2 d\theta} = 8$$





Thank You!

Reference

- VV285, slide. Horst Hohberger
- Sample Exam. Horst Hohberger
- RC, slides. TA-Pingbang Hu
- RC, slides. TA-Chen Yuxiang
- RC, slides. Yahoo