MATH 2860 Midterm RC

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Contents

First-Order Linear Equation

For the initial value problem:

$$y' = f(x)g(y), \quad y(\xi) = \eta$$

The necessary conditions for solving the initial value problem is:
1. Continuity: The function f and g are continuous in some $I_x \subset \mathbb{R}, I_y \subset \mathbb{R}$
2. $\xi \in I_x$ and $\eta \in I_y$

For different initial values at $g(\eta)$, we have different conditions for the answers:
1. $g(\eta) \neq 0$ Therefore, the solution to the initial value problem is obtained from

$$\int_\eta^y \frac{ds}{g(s)} = \int_\xi^x f(t)dt$$

2. $g(\eta) = 0$ Intuitive Answer: $y(x) = \eta$. Other Answers: Depends on whether the integration on the left hand side exists. You should always include the intuitive answer.

Ex1

$(1 + 2x)y' = 3 + y$ with $y(0) = -2$

Ex1

(1 + 2x)y ′ = 3 + y with y(0) = −2

$$\int_{y}^{g} \frac{ds}{g(s)} = \int_{\xi}^{x} f(t)\,dt$$

$$(1+2x)\frac{dy}{dx} = 3 + y$$

$$\frac{dy}{3+y} = \frac{dx}{1+2x}$$

$$\int_{-2}^{y} \frac{ds}{3+s} = \int_{0}^{x} \frac{dt}{1+2t}$$

$$\ln(3+s)\Big|_{-2}^{y} = \frac{1}{2}\ln(2x+1)\Big|_{0}^{x}$$

$$\ln(3+y) = \frac{1}{2}\ln(2x+1)$$

$$\Longrightarrow\ y = \sqrt{2x+1} - 3$$

General Solution to Linear FODE

The solution to

$$a_1(x)y' + a_0(x)y = f(x), \quad y(\xi) = \eta$$

is given by

$$y(x) = \eta e^{-G(x)} + e^{-G(x)} \int_{\xi}^{x} \frac{f(s)}{a_1(s)} e^{G(s)} ds$$

for x in a neighborhood of ξ, where

$$G(x) = \int_{\xi}^{x} \frac{a_0(t)}{a_1(t)} dt$$

Ex2

$y' - y\sin(x) = 2\sin(x)$ with $y(\frac{\pi}{2}) = -3$

Ex2

y ′ − ysin(x) = 2sin(x) with y(π 2) = −3

$$y - y\sin(x) = 2\sin(x) \text{ with } y\left(\frac{\pi}{2}\right) = -3$$

$$a_0(x) = -\sin(x)$$

$$f(x) = 2\sin(x)$$

$$a_1 = 1$$

$$y(x) = -3e^{-\int_{\pi/2}^{x} -\sin s\, ds} + e^{-\cos x}\int_{\pi/2}^{x} 2\sin(s)\cdot e^{\cos s}\, ds$$

$$= -3e^{-\cos x} + e^{-\cos x}\left(-2e^{\cos s}\Big|_{\pi/2}^{x}\right)$$

$$= -3e^{-\cos x} - 2e^{-\cos x}(e^{\cos x} - e^{0})$$

$$y(x) = -e^{-\cos x} - 2$$

Linear System of Equations

A system of first order ODE is the form

$$\begin{aligned}
\dot{x}_1 &= F_1(x_1, x_2, \ldots, x_n, t)\\
\dot{x}_2 &= F_2(x_1, x_2, \ldots, x_n, t)\\
&\vdots\\
\dot{x}_n &= F_n(x_1, x_2, \ldots, x_n, t)
\end{aligned}$$

Given an explicit ODE of order n, $x^{(n)}(t) = f(t, x, \dot{x}, \ldots, x^{n-1})$, we can introduce new variables $x_i = x^{(i-1)}$ $(i = 1, 2, \ldots, n)$, and we will get a system of linear equation:

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= x_3\\
&\vdots\\
\dot{x}_n &= f(x_1, x_2, \ldots, x_n, t)
\end{aligned}$$

A linear system of ODE is in the form of:

$$\frac{dx}{dt} = A(t)x + b(t), t \in I$$

where $A : I \to R^{n\times n}$ and $b : I \to R^n$.

- If $b = 0$, we call it homogeneous linear system, otherwise inhomogeneous.
- If A and b are constant functions, we call the system constant coefficients, otherwise variable coefficients.
  We basically focus on constant coefficients.

- superposition Principle: Assume x_1 and x_2 are two solutions of homogeneous linear equation $\frac{dx}{dt} = A(t)x$, then $\lambda x_1 + \mu x_2$ is also a solution.
- solution space: all solution of $\frac{dx}{dt} = A(t)x$ forms a linear space. It's basis is called fundamental system of solutions.
- For a basis $\mathscr{B} = \{b_1, b_2, \ldots, b_n\}$ of R^n, denote x_i to be the unique solution of equation $\frac{dx}{dt} = A(t)x, x(0) = b_i$, then $\mathscr{F} = \{x_1, x_2, \ldots, x_n\}$ is a fundamental system.

For the linear system

$$\frac{dx}{dt} = Ax, x(0) = x_0, x : R \to R^n$$

The solution is $x(t) = e^{At}x_0$. Here A is the $n \times n$ matrix, so we need to define the matrix exponential function.

The matrix exponential is defined as

$$e^{At} := I + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!}$$

If A is diagonalizable, then $A = UDU^{-1}$, $D = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$ and we have

$$e^{At} = Ue^{Dt}U^{-1}$$

and $e^{Dt} = diag(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$

Ex3

Exercise 9.

Consider the initial value problem

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Calculate $e^{\left(\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right)t}$ and use this to solve the above initial value problem.
(4 Marks)

Solution 9.

We first calculate the eigenvalues from the characteristic polynomial:

$$p(\lambda) = \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 4-\lambda \end{pmatrix} = (2-\lambda)(4-\lambda) - 1 = \lambda^2 - 6\lambda + 7,$$

so $p(\lambda) = 0$ if

$$\lambda = 3 \pm \sqrt{9-7} = 3 \pm \sqrt{2}.$$

Thus the eigenvalues are $\lambda_1 = 3 + \sqrt{2}$ and $\lambda_2 = 3 - \sqrt{2}$. We now the corresponding eigenvectors:

λ_1 : We solve

$$\begin{pmatrix} -1-\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

which gives

$$v_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$$

λ_2 : We solve

$$\begin{pmatrix} -1+\sqrt{2} & 1 \\ 1 & 1+\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

which gives

$$v_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha \begin{pmatrix} 1+\sqrt{2} \\ -1 \end{pmatrix}$$

Choosing in both cases $\alpha = 1/\sqrt{(1+\sqrt{2})^2+1}$ we see that v_1 and v_2 are orthonormal. Set

$$U = \alpha \begin{pmatrix} 1 & 1+\sqrt{2} \\ 1+\sqrt{2} & -1 \end{pmatrix}.$$

Then $U = U^* = U^{-1}$.

$$
\begin{aligned}
e^{\left(\begin{smallmatrix}2&1\\1&4\end{smallmatrix}\right)t} &= U \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} U \\
&= \alpha^2 \begin{pmatrix} 1 & 1+\sqrt{2} \\ 1+\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} 1 & 1+\sqrt{2} \\ 1+\sqrt{2} & -1 \end{pmatrix} \\
&= \alpha^2 \begin{pmatrix} 1 & 1+\sqrt{2} \\ 1+\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & (1+\sqrt{2})e^{\lambda_1 t} \\ (1+\sqrt{2})e^{\lambda_2 t} & -e^{\lambda_2 t} \end{pmatrix} \\
&= \alpha^2 \begin{pmatrix} e^{\lambda_1 t} + (1+\sqrt{2})^2 e^{\lambda_2 t} & (1+\sqrt{2})(e^{\lambda_1 t} - e^{\lambda_2 t}) \\ (1+\sqrt{2})(e^{\lambda_1 t} - e^{\lambda_2 t}) & e^{\lambda_2 t} + (1+\sqrt{2})^2 e^{\lambda_1 t} \end{pmatrix} \\
&= \frac{e^{3t}}{(1+\sqrt{2})^2+1} \begin{pmatrix} e^{\sqrt{2}t} + (1+\sqrt{2})^2 e^{-\sqrt{2}t} & (1+\sqrt{2})(e^{\sqrt{2}t} - e^{-\sqrt{2}t}) \\ (1+\sqrt{2})(e^{\sqrt{2}t} - e^{-\sqrt{2}t}) & e^{-\sqrt{2}t} + (1+\sqrt{2})^2 e^{\sqrt{2}t} \end{pmatrix}
\end{aligned}
$$

The solution to the IVP is given by

$$
\begin{aligned}
e^{\left(\begin{smallmatrix}2&1\\1&4\end{smallmatrix}\right)t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} &= \frac{e^{3t}}{(1+\sqrt{2})^2+1} \begin{pmatrix} e^{\sqrt{2}t} + (1+\sqrt{2})^2 e^{-\sqrt{2}t} & (1+\sqrt{2})(e^{\sqrt{2}t} - e^{-\sqrt{2}t}) \\ (1+\sqrt{2})(e^{\sqrt{2}t} - e^{-\sqrt{2}t}) & e^{-\sqrt{2}t} + (1+\sqrt{2})^2 e^{\sqrt{2}t} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\
&= \frac{e^{3t}}{(1+\sqrt{2})^2+1} \begin{pmatrix} 3e^{\sqrt{2}t} + 3(1+\sqrt{2})^2 e^{-\sqrt{2}t} + 2(1+\sqrt{2})(e^{\sqrt{2}t} - e^{-\sqrt{2}t}) \\ 3(1+\sqrt{2})(e^{\sqrt{2}t} - e^{-\sqrt{2}t}) + 2e^{-\sqrt{2}t} + 2(1+\sqrt{2})^2 e^{\sqrt{2}t} \end{pmatrix}
\end{aligned}
$$

On the other hand, if A is not diagonalizable, then assume $A = UJU^{-1}$, J is the corresponding Jordan normal form and we have

$$e^{At} = Ue^{Jt}U^{-1}$$

Since $J = D + N$, N is a nilpotent matrix, this can be done easily.

Refer to exercise 10 in sample to explore more.

We need to find a particular solution for the equation $\frac{dx}{dt} = Ax + b(t)$. The particular solution is given by $x_{part}(t) = e^{At} \int_{t_0}^{t} e^{-As} b(s) ds$. Then the solution to the initial value problem $(x(t_0) = x_0)$ is

$$x(t) = e^{A(t-t_0)} x_0 + e^{At} \int_{t_0}^{t} e^{-As} b(s) ds$$

The general solution is the general solution for homogeneous equation plus the x_part, i.e.,

$$\sum_{k=1}^{n} c_k x_k + x_{part}(t)$$

$\{x_1, x_2, \ldots, x_n\}$ is a fundamental system for the homogeneous part.

- The Wronskian of n solutions of a system. $x^{(1)}, \ldots, x^{(n)}$ are n arbitrary solutions of the homogeneous system

$$\frac{dx}{dt} = A(t)x.$$

Then the *Wronskian* is given by

$$W_{x_1,\ldots,x_n}(t) := \det(x^{(1)}(t), \ldots, x^{(n)}(t)).$$

- 1.10.8. Lemma and Abel's equation.

$$\frac{dW}{dt} = a(t)W, \quad a(t) = \operatorname{tr} A(t), \quad W(t) = W(t_0)e^{-\int_{t_0}^{t} a(s)ds}.$$

W (t) is non-zero if and only if the functions are linearly independent.

Second Order Linear Equations

$$y'' + p(t)y' + q(t)y = g(t), \qquad t \in I.$$

- Constant coefficients: $ay'' + by' + cy = g(t)$.
  - Homogeneous: $ay'' + by' + cy = 0$.
    1. $b^2 \neq 4ac, \lambda_1, \lambda_2 \in \mathbb{R}$.
    $$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \qquad c_1, c_2 \in \mathbb{R}.$$
    2. $b^2 \neq 4ac, \lambda_1, \lambda_2 \in \mathbb{C}$.
    $$y(t) = c_1 e^{\mathrm{Re}\lambda_i t} \sin(\mathrm{Im}\lambda_i t) + c_2 e^{\mathrm{Re}\lambda_i t} \cos(\mathrm{Im}\lambda_i t), \; c_1, c_2 \in \mathbb{R}.$$
    3. $b^2 = 4ac, \lambda_1 = \lambda_2 \in \mathbb{R}$.
    $$y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}, \qquad c_1, c_2 \in \mathbb{R}.$$

$$y'' + p(t)y' + q(t)y = g(t), \qquad t \in I.$$

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- Constant coefficients: $ay'' + by' + cy = g(t)$.
  - Inhomogeneous: $ay'' + by' + cy = g(t)$.
    1. Find two independent solutions y_1, y_2 of the homogeneous equation.
    2. Find particular solution $ay'' + by' + cy = 0$.

$$y_{\text{part}}(t) = -y^{(1)}(t) \int \frac{g(t)y^{(2)}(t)}{W(y^{(1)}(t), y^{(2)}(t))} dt + y^{(2)}(t) \int \frac{g(t)y^{(1)}(t)}{W(y^{(1)}(t), y^{(2)}(t))} dt.$$

    3. $y_{\text{inhom}}(t; c_1, c_2) = y_{\text{hom}}(t; c_1, c_2) + y_{\text{part}}(t)$.

For variable coefficient homogeneous second-order linear equation $y'' + p(t)y' + q(t)y = 0$, it's hard to solve in general. However, if we know one of its solution somehow, we can get the other one, and form a fundamental system.

If we know a solution $y_1(t)$. Set $y_2 = v(t)y_1(t)$, and plug into the equation, we get

$$y_1 v'' + (2y_1' + py_1)v' = 0$$

Which is a first-order linear equation with respect to v'

Ex4

Exercise 11.
Find the general solution of

$$y'' - 4y' + 4y = te^{2t}$$

Solution 11.

We transform into a system by setting $x_1 = y$, $x_2 = y'$. Then

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}}_{=:A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ te^{2t} \end{pmatrix}$$

The eigenvalues of A are determined from

$$-\lambda(4-\lambda) + 4 = \lambda^2 - 4\lambda + 4 = 0$$

giving $\lambda = 2$. Computing eigenvectors,

$$\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

gives $-2u + v = 0$, so

$$u_1 = \alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

for $\alpha \in \mathbb{R}$. We choose an independent eigenvector from

$$\ker(A-\lambda)^2 = \ker \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}^2 = \mathbb{R}^2,$$

taking

$$u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}, \qquad u_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\end{pmatrix}.$$

The vectors are orthonormal, so

$$U = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}$$

is orthogonal, $U^{-1} = U^T$. Then

$$J = U^T A U = \frac{1}{5}\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\-4 & 4\end{pmatrix}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix} = \begin{pmatrix}2 & 5\\0 & 2\end{pmatrix}$$

We can obtain a fundamental solution of the associated homogeneous equation from

$$e^{At}U = Ue^{Jt} = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}e^{2t} & 5te^{2t}\\0 & e^{2t}\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}e^{2t} & (5t-2)e^{2t}\\2e^{2t} & (10t+1)e^{2t}\end{pmatrix}$$

The Wronskian of this solution is

$$W(t) = \frac{1}{5}e^{4t}\det\begin{pmatrix}1 & (5t-2)\\2 & (1+10t)\end{pmatrix} = e^{4t}$$

We obtain a particular solution of the inhomogeneous equation by replacing columns in the Wronskian with the inhomogeneity, i.e., we calculate

$$c_1'(t)\frac{1}{\sqrt{5}W(t)}\det\begin{pmatrix}0 & (5t-2)e^{2t}\\te^{2t} & (1+10t)e^{2t}\end{pmatrix} = \frac{1}{\sqrt{5}}\det\begin{pmatrix}0 & 5t-2\\t & 1+10t\end{pmatrix} = (-5t^2+2t)/\sqrt{5}$$

so

$$c_1(t) = \frac{1}{\sqrt{5}}(t^2 - 5t^3/3)$$

and

$$\frac{1}{\sqrt{5}W(t)}\det\begin{pmatrix} e^{2t} & 0 \\ 2e^{2t} & te^{2t} \end{pmatrix} = t/\sqrt{5}.$$

so

$$c_2(t) = \frac{1}{\sqrt{5}}(t^2/2)$$

Hence a particular solution of the inhomogeneous system of ODEs is given by

$$y_{\text{part}}(t) = \frac{t^2 - 5t^3/3}{5}\begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + \frac{t^2/2}{5}\begin{pmatrix} (5t-2)e^{2t} \\ (1+10t)e^{2t} \end{pmatrix} = \frac{1}{6}\begin{pmatrix} t^3e^{2t} \\ (2t^3+3t^2)e^{2t} \end{pmatrix} = \begin{pmatrix} x_{\text{part}}(t) \\ \dot{x}_{\text{part}}(t) \end{pmatrix}.$$

Hence a particular solution of the 2nd order ODE is

$$x_{\text{part}}(t) = \frac{1}{6}t^3e^{2t}.$$

We obtain two homogneous solutions of the second order ODE from the system, namely

$$x_{\text{hom}}(t) = \frac{1}{\sqrt{5}}e^{2t} \qquad \text{and} \qquad x_{\text{hom}}(t) = \frac{1}{\sqrt{5}}(5t-2)e^{2t}$$

Thus the general solution of the 2nd order ODE is

$$x_{\text{inhom}}(t) = \frac{1}{6}t^3e^{2t} + c_1e^{2t} + c_2te^{2t}$$

The meaning of the world, is outside the world

By Wittgenstein

The meaning of math, is outside of mathematics

By Mo Yang

Good Luck!