MATH 2860 RC 2

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Matrix diagonalization

Steps for Matrix Diagonalization

1. **Find the Eigenvalues and Eigenvectors**: For a given $n \times n$ matrix (A), we first find its eigenvalues and eigenvectors.
2. **Construct the Diagonal Matrix and Transformation Matrix**: Use the found eigenvalues to construct a diagonal matrix (D) and the eigenvectors to create a transformation matrix (P).
3. **Verify Diagonalization**: We can verify that we obtained the correct (D) by computing $P^{-1}AP$.

Example

Consider the following matrix:

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

Step 1: Find the eigenvalues and eigenvectors

To find the eigenvalues, we solve the equation:

$$\det(A - \lambda I) = 0$$

where

$$A - \lambda I = \begin{bmatrix} 6-\lambda & 2 \\ 2 & 3-\lambda \end{bmatrix}$$

Solving this gives the eigenvalues $\lambda_1 = 7$ and $\lambda_2 = 2$.

Now we find the eigenvectors. For $\lambda = 7$, we have:

$(A - 7I)v = 0$

which gives an eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda = 2$, we find another eigenvector $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Step 2: Construct the diagonal matrix and transformation matrix

Using the eigenvalues and eigenvectors we found, we can construct the diagonal matrix (D) and transformation matrix (P):

$$D = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Step 3: Verify diagonalization

Now we verify $D = P^{-1}AP$:

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} = D$$

Thus, we have correctly diagonalized the matrix (A).

Conditions to Diagonalize

The condition for a transformation A in the n dimensional space V can be diagonalized is can be:

1. A has n independent eigenvectors
2. $dim\ V = dim\ V_{\lambda 1} + dim\ V_{\lambda 2} + \ldots + dim\ V_{\lambda n}$, where λ_i stands for the i^{th} eigenvalue, and V_{λ_i} stands for the space expanded through the eigenvectors corresponding to the i^{th} eigenvalue.
3. V can be written as the sum of direct sum of $V_{\lambda 1}$, $V_{\lambda 2}$,..., $V_{\lambda n}$

However, in some cases you cannot find enough eigen-vectors to reach the dimension of V. In this cases, you need to find some vectors that can be led to eigen-vectors through the matrix.

Finding Generalized Eigenvectors "Top-down"

Clearly, $v^{(m-1)}$ satisfies

$$(A - \lambda \mathbb{1})^{m-1} v^{(m-1)} = 0 \qquad \text{and} \qquad (A - \lambda \mathbb{1})^{m-2} v^{(m-1)} \neq 0.$$

We iteratively set

$$\begin{aligned} v^{(m-1)} &:= (A - \lambda \mathbb{1}) v^{(m)}, \\ v^{(m-2)} &:= (A - \lambda \mathbb{1}) v^{(m-1)}, \\ &\vdots \\ v^{(1)} &:= (A - \lambda \mathbb{1}) v^{(2)}. \end{aligned}$$

The set of generalized eigenvectors $\{v^{(m)}, v^{(m-1)}, \dots, v^{(1)}\}$ is called a ***chain of length m*** of generalized eigenvectors. Note that $v^{(1)}$ is always an eigenvector.

Example for the "Top-down" Method

7.5. Example. Let us return to the matrix A of Example 7.3,

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & -1 \\ 4 & -1 & 3 \end{pmatrix}.$$

We already know that A has a single eigenvalue $\lambda = 2$ of algebraic multiplicity 3 and geometric multiplicity 1. Therefore, we need at most a generalized eigenvalue of rank 3 to extend V_λ to E_3. The top-down method requires us to first solve

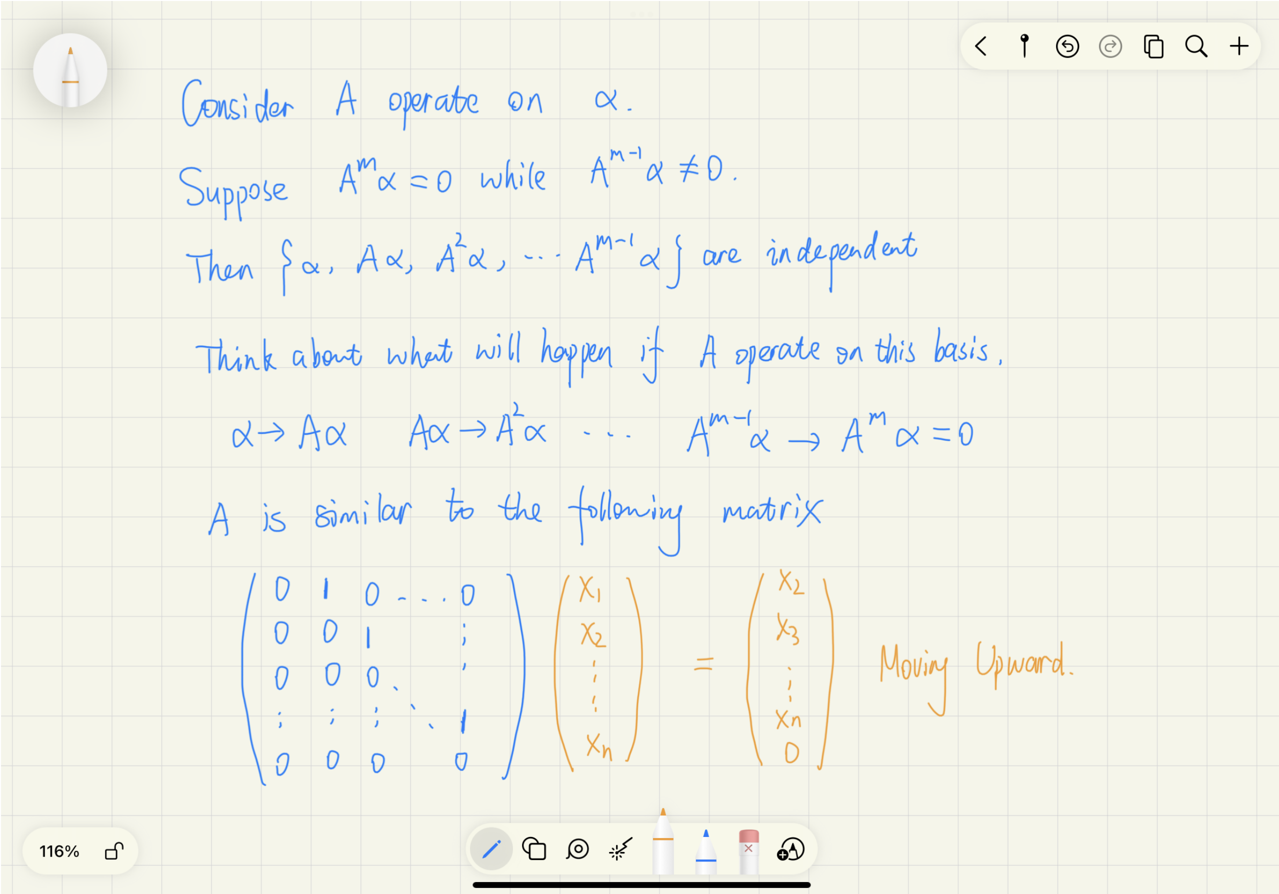
$$(A - 2\mathbb{1})^3 v = 0 \qquad \text{with} \qquad (A - 2\mathbb{1})^2 v \neq 0.$$

We will again use Mathematica for all calculations.

Nilpotent Operators

Suppose $\exists l \in N^*$ and linear transformation A satisfies that $A^l = 0$, we take the smallest l as the least nilpotent exponent. This suggest that for A, $A^{l-1} \neq 0$ but $A^l = 0$. We say A is a nilpotent transformation.

Lemma: Suppose A is a linear transformation on F, if $A^m\alpha \neq 0$ while $A^{m-1} = 0$, it can be concluded that $\alpha, A\alpha, \ldots, A^{m-1}\alpha$ is independent with each other.



Jordan Normal Form

Steps to Find the Jordan Normal Form of a Matrix

1. **Find the eigenvalues**: As with finding eigenvectors, we first find the matrix's eigenvalues.
2. **Find the eigenvectors and generalized eigenvectors**: In addition to the regular eigenvectors, we need to find so-called “generalized eigenvectors” which will help us form Jordan chains.
3. **Form the Jordan chains**: Using the eigenvectors and generalized eigenvectors, we form Jordan chains.
4. **Construct the Jordan blocks**: Each Jordan chain corresponds to a Jordan block. A Jordan block is a square matrix with the eigenvalue on the diagonal and ones just above the diagonal.
5. **Construct the Jordan Normal Form**: We place all the Jordan blocks into a larger diagonal matrix to form the Jordan Normal Form.

For a 3×3 matrix A:

1. Write out the characteristic polynomial and the eigenvalues

2. If there is only one eigenvalue λ, calculate $(A - \lambda I)^3$ and $(A - \lambda I)^2$
3. Solve $(A - \lambda I)^3 v = 0$ and choose a $v^{(3)}$ so that $(A - \lambda I)^2 v \neq 0$
4. Obtain $v^{(2)}$ by $v^{(2)} = (A - \lambda I)v^{(3)}$
5. Obtain $v^{(1)}$ by $v^{(1)} = (A - \lambda I)v^{(2)}$
6. Set $U = (v^{(1)}, v^{(2)}, v^{(3)})$ and calculate $U^{-1}AU$.

Check whether the matrix A below is diagonalizable and find its Jordan normal form.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

```
(* Step 1: Define matrix A *)
A = {{1, 0, 3}, {0, 1, 4}, {3, 4, 1}};

(* Step 2: Find the eigenvalues and eigenvectors *)
{eigenvalues, eigenvectors} = Eigensystem[A];

(* Step 3: Construct the transformation matrix P *)
P = Transpose[eigenvectors];

(* Step 4: Construct the diagonal matrix D *)
D = DiagonalMatrix[eigenvalues];

(* Step 5: Verify the diagonalization *)
PInverse = Inverse[P];
DiagonalizedA = PInverse . A . P;

(* Output the results *)
Print["Eigenvalues: ", eigenvalues];
Print["Eigenvectors: ", eigenvectors];
Print["Transformation matrix P: ", P];
Print["Diagonal matrix D: ", D];
Print["Verification of diagonalization (should equal D): ",
DiagonalizedA];
```

```
Eigenvalues: {6, -4, 1}
Eigenvectors: {{3, 4, 5}, {-3, -4, 5}, {-4, 3, 0}}
Transformation matrix P: {{3, -3, -4}, {4, -4, 3}, {5, 5, 0}}
Diagonal matrix D: {{6, 0, 0}, {0, -4, 0}, {0, 0, 1}}
```

Verification of diagonalization (should equal D): $\left\{\left\{\frac{3}{10}, \frac{7}{10}, -\frac{12}{5}\right\}, \left\{\frac{7}{10}, \frac{3}{10}, \frac{12}{5}\right\}, \left\{-\frac{24}{5}, \frac{24}{5}, \frac{12}{5}\right\}\right\}$

Note: For all self-adjoint matrix, it is diagnolizable.

###
Consider the following 2×2 matrix:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

Step 1: Find the eigenvalues

We find the eigenvalues by solving the equation $\det(A - \lambda I) = 0$, where$I$ is the identity matrix, and λ is the eigenvalue.

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)^2 = 0$$

This gives a repeated eigenvalue of $\lambda = 3$.

Steps 2 and 3: Find the eigenvectors and form the Jordan chains

We find the eigenvectors by solving $(A - \lambda I)v = 0$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We get an eigenvector as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Now we need to find a vector that is not an eigenvector but satisfies $(A - \lambda I)v =$ eigenvector. We solve the equation:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We find another vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ that forms a Jordan chain with the previously found eigenvector.

Steps 4 and 5: Construct the Jordan blocks and Jordan Normal Form

We end up with the following Jordan block:

$$J = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

This also happens to be our Jordan Normal Form.

Matrix for Jordon Normal Form

Multiple eigenvalues:
A 5×5 matrix A has the eigenvalues 1, 1, 4, 4, 4 (counted with algebraic multiplicities). The eigenvalue 1 has geometric multiplicity 1 and the eigenvalue 4 has geometric multiplicity 1. Give two different matrices that constitute a Jordan normal form for A.

ex.

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

★ it makes no sense to seperate a Jordan block

Let λ be an eigenvalue of a matrix A and denote by a_λ the algebraic multiplicity of λ. Suppose that the algebraic multiplicity is greater than its geometric multiplicity, i.e.

$$dim V_\lambda < a_\lambda$$

For $\lambda \in \mathbb{C}$ we define the **Jordan block** of size $k \in \mathbb{N}\backslash\{0\}$ by

$$J_1(\lambda) := \lambda \qquad J_k(\lambda) := \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} \in Mat(k \times k, \mathbb{C}).$$

References

- VV286, slide. Horst Hohberger
- RC, slides. TA-Huang YuCheng

For further questions, you can contact me through WeChat.

Fragments

芬兰

扫一扫上面的二维码图案，加我为朋友。