

Exit distribution and exit time

2023年9月21日 星期四 上午8:05

- 撞到一个集合的频率.

- Profit ratio of assets (资产组合) 撞到目标或者破产.

- 物理学 (到达临界能量).

- 生物学 (物种存活的可能性)

定义: 对于一个集合 $A \subset S$ 定义 $V_A = \inf \{n \geq 0, X_n \in A\}$

如果 $A = \{a\}$ 是 singleton (单例) 我们使用 V_a 代替 A .

性质: 考虑 马尔科夫链, 其状态空间为 S . 对于 $A, B \subset S$, $C = S \setminus \{A \cup B\}$ 是有限的

假设存在 $h: S \rightarrow [0, 1]$ 使得 $h(a) = 1$ 对 $\forall a \in A$ $h(b) = 0$ 对 $\forall b \in B$ 且对 $\forall c \in C$

$$h(c) = \sum_{y \in S} p(c, y) h(y). \quad \text{如果 } P_C(V_A \cap V_B < \infty) > 0 \text{ 则有 } h(x) = P(V_A < V_B)$$

证明: 让 $T = V_A \wedge V_B$. 因为 C 有限 $P_C(T < \infty) = 1$ 对 $\forall c \in C$

因为 Lemma 2.11. $P_T(T \leq k) \geq a \quad \forall y \Rightarrow P_T(T > m_k) \leq (1-a)^m$

$$\text{有 } h(c) = \sum_{y \in A \cup B} p(c, y) h(y) + \sum_{y \in C} p(c, y) h(y)$$

$$= \sum_{y \in A \cup B} p(c, y) h(y) + \sum_{y \in C} \sum_{z \in S} p(c, y) p(y, z) h(z).$$

(一个很取小值)

$$= \mathbb{E}_c h(X_{T \wedge 2}) \mathbf{1}_{\{T=1\}} + \mathbb{E}_c h(X_{T \wedge 2}) \mathbf{1}_{\{T>2\}} = \mathbb{E}_c h(X_{T \wedge 2})$$

通过迭代这个式子, 有 $h(c) = \mathbb{E}_c h(X_{T \wedge 1})$

$$\because h(X_{T \wedge n}) \rightarrow h(X_T) \quad \text{由收敛定理} \quad \mathbb{E}_c h(X_{T \wedge n}) \rightarrow \mathbb{E}_c h(X_T)$$

$$= \mathbb{E}_c \mathbf{1}_{\{X_T \in A\}} = P_C(V_A < V_B)$$

例子: Gambler's Ruin

$$S = \{0, 1, \dots, N\}, \quad p(x, x+1) = p, \quad \text{设 } h(x) = P_X(V_N < V_0)$$

$S = \{0, 1, \dots, N\}$, $p(x, x+1) = p$. 让 $h(x) = P_x(V_N < V_0)$

递推公式为: $h(x) = p h(x+1) + (1-p) h(x-1)$

$$h(x+1) - h(x) = \frac{1-p}{p} (h(x) - h(x-1)) \quad \approx \left(\frac{1-p}{p}\right)^x h(0).$$

$$h(N) - h(0) = h(0) \sum_{n=0}^{N-1} \left(\frac{1-p}{p}\right)^x = h(0) \begin{cases} \frac{1 - \left(\frac{1-p}{p}\right)^N}{1 - \left(\frac{1-p}{p}\right)} & (p \neq \frac{1}{2}) \\ N & (p = \frac{1}{2}) \end{cases}$$

$$h(x) = \begin{cases} \frac{\left(\frac{1-p}{p}\right)^x}{1 - \left(\frac{1-p}{p}\right)^N} & (p \neq \frac{1}{2}) \\ \frac{x}{N} & (p = \frac{1}{2}) \end{cases}$$

Example 1.40 (Two Year College). At a local two year college, 60 % of freshmen become sophomores, 25 % remain freshmen, and 15 % drop out. 70 % of sophomores graduate and transfer to a four year college, 20 % remain sophomores and 10 % drop out. What fraction of new students eventually graduate?

We use a Markov chain with state space 1 = freshman, 2 = sophomore, G = graduate, D = dropout. The transition probability is

	1	2	G	D
1	0.25	0.6	0	0.15
2	0	0.2	0.7	0.1
G	0	0	1	0
D	0	0	0	1

Let $h(x)$ be the probability that a student currently in state x eventually graduates. By considering what happens on one step

$$h(1) = 0.25h(1) + 0.6h(2)$$

$$h(2) = 0.2h(2) + 0.7$$

To solve we note that the second equation implies $h(2) = 7/8$ and then the first that

$$h(1) = \frac{0.6}{0.75} \cdot \frac{7}{8} = 0.7$$

定理: 有状态空间 S 的马尔科夫链, 让 $A \subset S$ 使得 $C = S \setminus A$ 是有限的 $P_C(V_A < \infty) > 0$

对 $\forall C \subset C$, 如果存在一个有界的 $g: S \rightarrow \mathbb{R}$ 使得 $g(a) = 0 \forall a \in A$.

且 $g(c) = 1 + \sum_{y \in S} p(c, y) g(y)$ 对 $\forall c \in C$. 则有 $g(x) = \mathbb{E}_x(V_A)$

证明：与上相同 $P_c(V_A < \infty) = 1$. $g(c) = 1 + \sum_{y \in C} p(c, y) g(y)$

$$g(y) = 1 + \sum_{y \in C} p(c, y) + \sum_{y \in C} \sum_{z \in C} p(c, y) p(y, z) g(z)$$

$$= \dots = \sum_{k=1}^n P_c(V_A \geq k) + E_c[g(X_n) \mathbf{1}_{\{V_A > n\}}]$$

用归纳法原理，得证。 \square