Exit distribution and exit time

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- 撞到一个集合的概率.
- Portfolio of assets (资产组合) 撞到目标或者破产.
- 物理学 (到达临界能量).
- 生物学 (物种存活的可能性)

定义: 对于一个集合 $A \subset S$ 定义 $V_A = \inf\{n \geq 0, X_n \in A\}$

如果 $A = \{a\}$ 是 singleton (单例) 我们使用 V_a 代表 A.

定理: 考虑 马尔科夫链, 其状态空间为 S. 对于 $A, B \subset S$. $C = S \setminus \{A \cup B\}$ 是有限的

假设存在 h $S \to [0,1]$ 使得 $h(a) = 1$ 对 $\forall a \in A$ $h(b) = 0$ 对 $\forall b \in B$ 且 对 $\forall c \in C$

$h(c) = \sum_{y \in S} p(c,y) h(y)$. 如果 $P_c(V_A \wedge V_B < \infty) > 0$ 则有 $h(x) = P(V_A < V_B)$.

证明: 让 $T = V_A \wedge V_B$. 因为 C 有限 $P_c(T < \infty) = 1$ 对 $\forall c \in C$

因为 Lemma 2.11. $P_y(T \leq k) \geq a \ \forall y \Rightarrow P_y(T > mk) \leq (1-a)^m$

有 $h(c) = \sum_{y \in A \cup B} p(c,y) h(y) + \sum_{y \in C} p(c,y) h(y)$

$= \sum_{y \in A \cup B} p(c,y) h(y) + \sum_{y \in C} \sum_{z \in S} p(c,y) p(y,z) h(z)$.

($\wedge$ 代表最小值)

$= E_c h(X_{T \wedge 2}) \mathbb{1}\{T = 1\} + E_c h(X_{T \wedge 2}) \mathbb{1}\{T \geq 2\} = E_c h(X_{T \wedge 2})$

通过迭代 n 个元素. 有 $h(c) = E_c h(X_{T \wedge n})$

$\because h(X_{T \wedge n}) \to h(X_T)$ 由 收敛定理 $E_c h(X_{T \wedge n}) \to E_c h(X_T)$

$= E_c \mathbb{1}\{X_T \in A\} = P_c(V_A < V_B)$

例子: Gambler's Ruin

$S = \{0, 1, \cdots N\}$. $p(x, x+1) = p$. 记 $h(x) = P_x(V_N < V_0)$

$S=\{0,1,\cdots N\}$. $p(x,x+1)=p$. 记 $h(x)=\mathbb{P}_x(V_N<V_0)$

到达X状态: $h(x)=p\,h(x+1)+(1-p)\,h(x-1)$

$$h(x+1)-h(x)=\frac{1-p}{p}\left(h(x)-h(x-1)\right)=\left(\frac{1-p}{p}\right)^x h(1)$$

$$1=h(N)-h(0)=h(1)\sum_{n=0}^{N-1}\left(\frac{1-p}{p}\right)^x=h(1)\begin{cases}\dfrac{1-\left(\frac{1-p}{p}\right)^x}{1-\left(\frac{1-p}{p}\right)^N} & (p\neq\frac{1}{2})\\ N & (p=\frac{1}{2})\end{cases}$$

$$h(x)=\begin{cases}\dfrac{1-\left(\frac{1-p}{p}\right)^x}{1-\left(\frac{1-p}{p}\right)^N} & (p\neq\frac{1}{2})\\ \dfrac{x}{N} & (p=\frac{1}{2})\end{cases}$$

Example 1.40 (Two Year College). At a local two year college, 60 % of freshmen become sophomores, 25 % remain freshmen, and 15 % drop out. 70 % of sophomores graduate and transfer to a four year college, 20 % remain sophomores and 10 % drop out. What fraction of new students eventually graduate?

We use a Markov chain with state space 1 = freshman, 2 = sophomore, G = graduate, D = dropout. The transition probability is

	1	**2**	**G**	**D**
1	0.25	0.6	0	0.15
2	0	0.2	0.7	0.1
G	0	0	1	0
D	0	0	0	1

Let $h(x)$ be the probability that a student currently in state x eventually graduates. By considering what happens on one step

$$h(1)=0.25h(1)+0.6h(2)$$
$$h(2)=0.2h(2)+0.7$$

To solve we note that the second equation implies $h(2)=7/8$ and then the first that

$$h(1)=\frac{0.6}{0.75}\cdot\frac{7}{8}=0.7$$

定理: 有状态空间 S 的马尔科夫链, 记 $A\subset S$ 使得 $C=S\setminus A$ 是有限的 $P_c(V_A<\infty)>0$ 对 $\forall c\in C$. 如果存在一个有界的 $g: S\to\mathbb{R}$ 使得 $g(a)=0$ 对 $\forall a\in A$. 并且 $g(c)=1+\sum_{y\in S}p(c,y)g(y)$ 对 $\forall c\in C$. 则有 $g(x)=\mathbb{E}_x(V_A)$

证明：与上相同 $P_c(V_A<\infty)=1$. $g(c)=1+\sum_{y\in C} p(c,y)g(y)$

$$g(y)=1+\sum_{y\in C} p(c,y)+\sum_{y\in C}\sum_{z\in C} p(c,y)p(y,z)g(z)$$

$$=\cdots=\sum_{k\geq 1}^{n} P_c(V_A\geq k)+E_c[g(X_n)\mathbb{1}_{\{V_A>n\}}]$$

因为收敛定理，得证． $\boxtimes$