L23 Dissipation

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Dissipation $\quad M\ddot{q} + S\dot{q} + Vq = 0.$

$M, V.$ symmetric $\quad S:$ Skew. 旋转的力/磁场. 一般 $M. V.$ 都为正定的.

$$\{F,K\} = \sum_i \frac{\partial F}{\partial q_i}\frac{\partial K}{\partial p_i} - \frac{\partial K}{\partial q_i}\frac{\partial F}{\partial p_i} - S^{ij}\frac{\partial K}{\partial p_i}\frac{\partial K}{\partial p_j}$$

如果 V 有负特征值的话 $\to$ Unstable. 否则 S 可能 spectrally stable $\leftarrow$ (gyroscope stablization)

如果加上一个 dissipation $\longrightarrow$ Linearly unstable always (加上一项 $R\dot{q}$. R 是正定的).

对于系统

$$H = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + \frac{1}{2}(\alpha x^2+\beta y^2). \quad \Omega = dx\wedge d\dot{x} + dy\wedge d\dot{y} - g\, dx\wedge dy$$

增加 dissapation.
$$\begin{cases} \ddot{x} - g\dot{y} + \alpha x + \gamma\dot{x} = 0. \\ \ddot{y} + g\dot{x} + \beta y + \delta\dot{y} = 0. \end{cases}$$

$$p(\lambda) = \lambda^4 + (\gamma+\delta)\lambda^3 + (g^2+\alpha+\beta+\gamma\delta)\lambda^2 + (\gamma\beta+\delta\alpha)\lambda + \alpha\beta$$

Undamped case

$p_0(\lambda) = \lambda^4 + (g^2+\alpha+\beta)\lambda^2 + \alpha\beta$ 特征多项式

Roots: (1) $\alpha>0, \beta>0$ 特征值在虚轴上.

(2). $\alpha\beta<0.$ 一组特征值在虚轴上. 一组在实轴上. $\Rightarrow$ "Unstable"

(3). $\alpha,\beta<0 \quad \lambda^2 = \frac{1}{2}\left[-(g^2+\alpha+\beta) \pm \sqrt{D}\,\right] \quad D = (g^2+\alpha+\beta)^2 - 4\alpha\beta$

(a). $D<0.$ 2 roots 在右半平面, 2个在左边

(b). $D=0.$ $g^2+\alpha+\beta>0.$ roots 在虚轴. $g^2+\alpha+\beta<0.$ 根在实轴.

(c). $D>0.$ $g^2+\alpha+\beta>0.$ roots on 虚轴. $g^2+\alpha+\beta<0$ roots on 实轴

如果 $D>0.$ g small $\to$ unstable $\quad g$ suff large: Stable

$\longrightarrow$ Gyroscopic Stablization.

如果 Add damping.

$$p(\lambda) = \lambda^4 + \rho_1\lambda^3 + \rho_2\lambda^2 + \rho_3\lambda + \rho_4.$$

Routh - Hurwitz Criterian

在右半平面上面 p的零点数量等于 下面的序列 符号变化的数量

$$\left\{ 1,\ p_1,\ \frac{p_1 p_2 - p_3}{p_1},\ \frac{p_3 p_1 p_2 - p_3^2 - p_4 p_1^2}{p_1 p_2 - p_3},\ p_4 \right\}$$

Apply to the case.

$\alpha < 0.\ \beta < 0.\ g^2 + \alpha + \beta > 0.\ \gamma, \delta > 0 \Rightarrow$ System spectually unstable!

Fiber translation @ Magnetic terms.

Momentum Shifting.

令 A 是一个 Q 上面的 1-形式. 令 $t_A : T^*Q \to T^*Q$,

defined by $\alpha_q \to \alpha_q + A(q)$ where $\alpha_q \in T \in T_q^* Q$.

令 θ $(p\,dq)$ 是 T^*Q 上面的 Canonical 1-form $\qquad t^*_A \theta = \theta + \pi_Q^* A$

$\pi_Q^* : T^*Q \to Q$ projection

$t_A^* \Omega = \Omega - \pi_Q^* dA.$ $\Omega = -d\theta$ 是 Canonical Symplectic form.

在 Local chart 里面 Shifted 2-form

$$\Big((u, \beta), (v, \gamma) \Big) = \langle \gamma, u \rangle - \langle \beta, v \rangle - \beta(w)(u, v) \quad \text{at point } (w, \alpha).$$

Particle in a magnetic field.

令 B 是一个 closed 2-form $\quad \underset{\sim}{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \quad$ B closed iff B div. free.

$i_{\underset{\sim}{B}} (dx \wedge dy \wedge dz) = B.$ $\quad B = B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z + dx \wedge dy.$

Equations of momentum for a particle in a magnetic field. charge e mass m

Equations of momentum for a particle in a magnetic field.

$$m\frac{dv}{dt} = \frac{e}{c}\, \underset{\sim}{v} \times \underset{\sim}{B} \quad \text{— Lorentz Equation}$$

$\underset{\sim}{v} = (\dot{x}, \dot{y}, \dot{z})$. $\quad \mathbb{R}^3 \times \mathbb{R}^3$ 上的 Hamiltonian $\quad (\underset{\sim}{x}, \underset{\sim}{v})$.

Symplectic form $\quad \Omega_B = m\,(dx \wedge d\dot{x} + dy \wedge d\dot{y} + dz \wedge d\dot{z}) - \frac{e}{c} B$

Claim $\quad H = \frac{1}{2} m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ 为哈密顿量.

Set $\quad dH = i_{X_H} \omega_B = m(\dot{x}\, d\dot{x} + \dot{y}\, d\dot{y} + \dot{z}\, d\dot{z}) = m\,(u\, d\dot{x} - \dot{u}\, dx + v\, d\dot{y} - \dot{v}\, dy + w\, d\dot{z} - \dot{w}\, dz)$

$$- \frac{e}{c} [\, B_x v\, dz - B_x w\, dy - B_y u\, dz + B_y w\, dx + B_z u\, dy - B_z v\, dx \,]$$

$$(u = \dot{x} \quad v = \dot{y} \quad w = \dot{z})$$

$$\begin{cases} m\dot{u} = \frac{e}{c}\,(B_z v - B_y w) \\ m\dot{v} = \frac{e}{c}\,(B_x w - B_z u) \\ m\dot{w} = \frac{e}{c}\,(B_y u - B_x v) \end{cases} \qquad \text{Lorentz 方.}$$