L24 Integrable System

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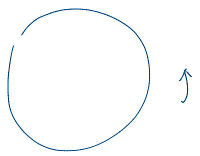
Energy + Casimir Energy - Momentum Method.

+ Dissapation ?



gyroscopic stablization

旋转后可以增加稳定性 $M\ddot{q} + S\dot{q} + kq = 0$

Particle in a magnetic field.

$\underset{\sim}{B} = (B_x, B_y, B_z)$. $i_{\underset{\sim}{B}}(dx \wedge dy \wedge dz) = B_x\, dy \wedge dz + B_y\, dz \wedge dx + B_z\, dx \wedge dy$

particle : $m\frac{dv}{dt} = \frac{e}{c} v \times B$

$$\boxed{\begin{cases} H = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ \Omega_B = m(dx \wedge d\dot{x} + dy \wedge d\dot{y} + dz \wedge d\dot{z}) - \frac{e}{c} B \end{cases} \qquad dH = i_{X_H} \Omega_B \Rightarrow \text{Lorentz Equation}}$$

如果 $B = dA \equiv B = \nabla \times A$. $\Rightarrow \underset{\sim}{A}^\flat = A$ A - 1-form.

如果 $\underset{\sim}{A} = (A_x, A_y, A_z)$. $\underset{\sim}{A}^\flat = A_x dx + A_y dy + A_z dz$ 从 vector → form.

Map $t_A(x, v) = (\underset{\sim}{x}, \underset{\sim}{p})$ where $\underset{\sim}{p} = m\underset{\sim}{v} + \frac{e}{c}\underset{\sim}{A}$

momentum 从 canonical 变成加了 Lorentz term

Lorentz Equas 相对于 (x, p) 空间的 Canonical form 也是一个 Hamotonian

with $\boxed{H = \frac{1}{2m} \| p - \frac{e}{c} \underset{\sim}{A} \|^2}$

<u>Remark</u> 不是所有的场都可以被写成 $B = \nabla \times A$. 磁单极子 : $B(r) = \frac{\underset{\sim}{r}}{\|\underset{\sim}{r}\|^3} g$.

<u>Kaluza - Klein.</u>

电荷是守恒的. 介绍一个新的 cyclic variable. conjugate momentum 是 charge.

$\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$ $H(\underset{\sim}{q}, \underset{\sim}{p}) = \frac{1}{2} \|\underset{\sim}{p} - \frac{e}{c}\underset{\sim}{A}\|^2$

Set $L(q, \dot{q}) = \frac{1}{2} m \|\dot{q}\|^2 + \frac{e}{c}\underset{\sim}{A} \cdot \underset{\sim}{\dot{q}}$ $\Rightarrow$ $\underset{\sim}{p} = \frac{\partial L'}{\partial \dot{q}} = m\dot{q} + \frac{e}{c}\underset{\sim}{A}$

$H(\underset{\sim}{q}, \underset{\sim}{p}) = \underset{\sim}{p} \cdot \underset{\sim}{\dot{q}} - L(q, \dot{q}) = (m\underset{\sim}{\dot{q}} + \frac{e}{c}\underset{\sim}{A}) \cdot \dot{q} - \frac{1}{2}\|\dot{q}\|^2 - \frac{e}{c}\underset{\sim}{A} \cdot \dot{q} = \frac{1}{2} m \|\dot{q}\|^2 = \frac{1}{2m}\|\underset{\sim}{p} - \frac{e}{c}\underset{\sim}{A}\|^2$

Let configuration space be extended to $Q_K = \mathbb{R}^3 \times S^1$ with variable $(\underset{\sim}{q}, \theta)$

define $A = \underset{\sim}{A}^{\flat}$ 1-form on $\mathbb{R}^3$ Set $\omega = A + d\theta$ on Q_K.

(Connection 1-form)

Kaluza-Kelvin Lagrangian:

$$L_K(q, \dot{q}, \theta, \dot{\theta}) = \frac{1}{2} m \|\dot{q}\|^2 - \frac{1}{2}\| \langle \omega, (q, \dot{q}, \theta, \dot{\theta}) \rangle \|^2$$

$$= \frac{1}{2} m \|\dot{q}\|^2 + \frac{1}{2}(A\dot{q} + \dot{\theta})$$

Conjugate momentum

$$\begin{cases} \underset{\sim}{p} = m\dot{q} + (A\dot{q} + \dot{\theta})\underset{\sim}{A} \\ p = A\dot{q} + \dot{\theta} \qquad \text{scalar } p. \end{cases}$$

E-L equations. — geodesic on $\mathbb{R}^3 \times S^1$

因为E是关于θ cyclic的 p是随时间不变的.

定义 charge $p = \frac{e}{c}$. $\Rightarrow \underset{\sim}{p} = m\underset{\sim}{\dot{q}} + \underbrace{(\underset{\sim}{A} \cdot \underset{\sim}{\dot{q}} + \dot{\theta})}_{\text{Conserved.}} \underset{\sim}{A} = m\underset{\sim}{\dot{q}} + \frac{e}{c}\underset{\sim}{A}$.

Hamotonian on $T^* Q_K$.

$$H_K = \frac{1}{2} \| \underset{\sim}{p} - \underset{\overset{\|}{\frac{e}{c}}}{p} A \|^2 + \frac{1}{2} p^2$$

除了常数 $\frac{1}{2}p^2$ 以外 与之前的 Hamotonian 是一样的.

example for <u>Reduction</u> 从四维降回到三维.

$\mathbb{R}^3 \times S^1 \to \mathbb{R}^3$. charge can conserve.   Yang-Mill's Theory.

<u>Integrable System</u>

Theorem: (Liouville-Arnold). 给定一个 $2n$维的辛流形, 给定n个函数 $F_1, \cdots F_n$.

这些函数在 M上 involution $\{F_i, F_j\} = \omega(F_i, F_j) = 0$ $i, j = 1, 2, \ldots n$.

Let $M_f = \{x, F_i(x) = f_i, i = 1, 2, \cdots n\}$

— level set of integrals

假设 F_i 是独立于 M_f 的 i.e. dF_i (1-form) 是 LI at each pt on M_f.

Then

(i) M_f 是一个光滑流形, 可逆 under phase flow of $H = F$.

(ii) 如果 M_f 是 compact 和 connected, 它是 diffeomorphic to the n-tours

$$T^n = \{(\varphi_1, \cdots \varphi_n) \bmod 2\pi\} \quad \text{i.e. } S^1 \times S^2 \times \cdots \times S^1$$

(iii) Phase flow with Hamotonian $H = F_i$ conditionally perodic motion on M_f.

i.e. in co-ords $\varphi = (\varphi_1, \varphi_2, \cdots \varphi_n)$ motion on T^n given by

$$\frac{d\underset{\sim}{\varphi}}{dt} = \underset{\sim}{\omega} = (\omega_1, \cdots \omega_n) \qquad \underset{\sim}{\omega} = \underset{\sim}{\omega}(f)$$

perodic if ω_i 是 rationally related. e.g. M^4 $5\omega_1 = 3\omega_2$.

(iv) Eqns of motion with hamotonian H are solvable by quadratures

by inverting functions & along relations

例: ① Kepler Problem 2-body (r, θ)

② Rigid Body - Euler equations on $T^* SO(3)$

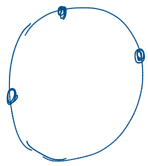
③ Toda Lattice

Remark 如果 ω_i 不是成比例相关,

preturbed Hamotonian of 可积系统 behave similarly!

(KAM 定理)

Toda on a ring



$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \exp(-(\phi_1 - \phi_2)) + \exp(-(\phi_2 - \phi_3)) + \exp(-(\phi_3 - \phi_1)) - 3$$

Integrals $H_1 = H$ $\quad H_2 = P_3 = p_1 + p_2 + p_3$

$I = H_3$? Transform to new variables. $P_3 = 0$.

$$\bar{H} = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\exp(2y + 2\sqrt{3}x) + \exp(2y - 2\sqrt{3}x) + \exp(-4y) - \frac{1}{8}$$

$$H_3 = 8p_x(p_x^2 - 3p_y^2) + (p_x + \sqrt{3}p_y)\exp(2y - 2\sqrt{3}x) - 2p_x\exp(-4y) + (p_x - \sqrt{3}p_y)\exp(2y + 2\sqrt{3}x).$$

FPU Fermi Pasta Ulam – Los Alamos.