L7 Lie Algebra

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Canonical Poisson Bracket $\dot{q}^i = \frac{\partial H}{\partial p_i} = \{p_i, H\}$ $\dot{p}_i = -\frac{\partial H}{\partial q_i} = \{p_i, H\}$

$$\{F, H\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q^i}$$

刚体: $\dot{\Pi}_1 = \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3$, ...

$\dot{F} = \{F, H\}$. $\{F, H\}(\Pi) = -\Pi(\nabla F \times \nabla H)$.

李代数: - vector Space $\mathfrak{g}$ - 双线性的 反对称括号.

对于(ξ, η), $[\xi, \eta] = \xi\eta - \eta\xi$

还满足. 给定 ξ, η, μ.

$$[[\xi, \eta], \mu] + [[\eta, \mu], \xi] + [[\mu, \xi], \eta] = 0.$$

例: ① 给定 $(\mathbb{R}^3, \times)$. $[\vec{a}, \vec{b}] = \vec{a} \times \vec{b}$ 符合条件.

② 给定 $(\mathbb{R}^{n\times n}, \cdot)$ $[A, B] = AB - BA$

③ Skew Symmetric - SO(n). 反对称/斜对称. $(A^T = -A)$

Lie Algebra Rotation Group SO(n)

$[A, B]$ both skew. $(AB - BA)^T = B^T A^T - A^T B^T = BA - AB$

所以 $[B, A] = -[A, B]$.

而对称矩阵并非李代数.

vector space. - basis: $e_1, e_2, \dots e_n$

定义 $[e_a, e_b] = \sum_{d=1}^{n} C_{ab}^{d} e_d$

例: 对SO(3) Skew: $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ 一个基 $\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{e_1}$ $\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{e_2}$ $\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}}_{e_3}$

我们有 $[e_1, e_2] = -e_3$　因此 $c_{12}^3 = -1$. 其它全为 0.

Rigid Body Equation $\dot{\Pi} = \Pi \times \omega$

Generalize to Lie Algebra: $\dot{\Pi} = [\Pi, \omega]$, $\Pi, \omega \in \mathfrak{g}$.

Lie - Poisson Bracket.

之前 $\{F, G\}(\pi) = -\pi \cdot (\nabla F \times \nabla G)$ 广泛化至矩阵.

$\{F, G\}(\pi) = -\mathrm{Tr}\ \underset{\sim}{\Pi}\,(\nabla F \times \nabla G)$.

把之前的点乘换成了迹

Kenemetic of Rigid Body.

Rotation: Euler Angles　Dynamics locally interms of (θ, φ, ψ)

$$\underset{(\Omega_1, \Omega_2, \Omega_3)}{\underset{\sim}{\Omega}} = \begin{bmatrix} \dot\theta \cos\psi + \dot\varphi \sin\psi \sin\theta \\ -\dot\theta \sin\psi + \dot\varphi \cos\psi \sin\theta \\ \dot\varphi \cos\theta + \dot\psi \end{bmatrix}$$

对于 $L = T = \frac{I_1}{2}\Omega_1^2 + \frac{I_2}{2}\Omega_2^2 + \frac{I_3}{2}\Omega_3^2$

Lagrange for ψ. $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\psi}\right) - \frac{\partial T}{\partial \psi} = 0$. 可以有 $\frac{\partial \dot\Omega_3}{\partial \dot\psi} = 1$. $\frac{\partial \Omega_1}{\partial \psi} = -\Omega_2$　$\frac{\partial \Omega_2}{\partial \psi} = \Omega_1$

$\Rightarrow \frac{\partial T}{\partial \dot\psi} = I_3 \Omega_3$.　$\frac{\partial T}{\partial \psi} = I_1 \Omega_1 \Omega_2 - I_2 \Omega_2 \Omega_1$

$\Rightarrow$ 可以用 $L = T = \sum \frac{1}{2} I_i \Omega_i^2$ 推出　$I_3 \dot\Omega_3 = (I_1 - I_2)\Omega_1 \Omega_2$.

刚体: $I\dot\Omega = I\Omega \times \Omega$.

Claim: 我们写成 $\hat{I}\dot{\hat{\Omega}} = [\hat{I}\hat{\Omega}, \hat{\Omega}]$ 其中 $\hat{\Omega} \in so(3)$　Skew - Sym

$$((\Omega_1, \Omega_2, \Omega_3), \times) \longrightarrow \left(\begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}, [\ ] \right)$$

$$\left[\begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} \right] = \widehat{(\Omega \times \theta)}$$

(待验证)

3 dim $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \Omega}\right) = \frac{\partial L}{\partial \Omega} \times \Omega$

SO(3) $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \hat{\Omega}}\right) = \left[\frac{\partial L}{\partial \hat{\Omega}}, \hat{\Omega} \right]$