L21 Lyapnov Function

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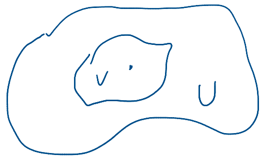
Lyapunov stability: 在 V 出发，最后也一直在范围 U 里面



Lyapnov function $V>0$. $\dot{V}\le 0$.

Non-linear stability : for Hamotonian System.

① neutral / spectral stability.

系统 $\dot{x}=X(u)$ 的平衡点被称为 neutural / spectual stable 如果该线性系统的 Spectrum 是纯虚的.

~ time evalution oscllation.

② Linearized Stability.

平衡被称为 Linearly Stable. 相对于一个 norm $\|\delta u\|$ 对 $\forall\varepsilon>0$. $\exists\delta>0$. 若 $t=0$ 时, $\|\delta u\|<\delta$, 则 $\|\delta u\|<\varepsilon$ 对于 $\forall t$

例: $\ddot{x}=u$. 是一个 spectrually stable 的例子. 但可以一直加速 而不 Linearized stability.

对于一个有限维度下 $DX(u_e)$ 有不同的 eigen value 在虚轴上 ⇒ Linear stability.

③ Formal Stability.

说对系统 $\dot{x}=X(u)$ 的平衡点是 formal stable 如果一个守恒量可以存在.

守恒量 的 deveation 在平衡点 vanish, 并且其在这个点的二次型是正定 / 负定的

④ non-linear stability.

对于 $\forall$ neighbourhood V of u_e $\exists\, a$. 从 a出发 所有的轨迹都在 V 里面

In terms of a norm $\|u\|$ non-lin stability ⇒ $\forall\varepsilon>0$. $\exists\delta>0$. s.t. $\|u(0)-u_e\|\le\delta$ = $\|u(t)-u_e\|<\varepsilon$

<u>Remark</u> : 对于 有限维空间中的

e.g. Ham Sys with 3 自由度 — linear stability for non-lin unstable.

— Arnold diffusion.

而在 finite dimension 里面. formal stability $\Rightarrow$ Nonlinear stability. 而无限维不成立

例: Polar coordinates (r,θ).

$\dot r = r^3(1-r^2)$ $\quad \dot\theta = 1$

对于 $r>0$ 且 r 很小

Eigenvalue : $\pm\sqrt{-1}$.

但并非 conservative 的

(x,y) coordinates $\Rightarrow \begin{cases} \dot x = x(x^2+y^2)(1-x^2-y^2) - y. \\ \dot y = y(x^2+y^2)(1-x^2-y^2) + x. \end{cases}$

Cherry's example.

spectural stable. linearily stable. 但不满足 non-linear stable.

$$H = \frac{1}{2}(q_1^2+p_1^2) - (q_2^2+p_2^2) + \frac{1}{2}p_2(p_1^2-q_1^2) - q_1 q_2 p.$$

Linearized Sys : 因为是两个谐振器所以是 stable 的. exact solution $\begin{cases} q_1 = -\sqrt{2}\,\dfrac{\cos(t-\tau)}{t-\tau} \\ q_2 = \dfrac{\cos(t-\tau)}{t-\tau} \\ p_1 = \dfrac{\sqrt{2}\sin(t-\tau)}{t-\tau} \\ p_2 = \dfrac{\sin 2(t-\tau)}{t-\tau} \end{cases}$

在 $t=0$ 时 distance $\sqrt{3}/\tau$. 而当 τ large 时 可以无限接近.

然而在有限时间下 solution blows up.

Stability Algorithm

Energy - Casimir Method / Energy-Momentum method

System on a space P (Banach Space).

A $\dot u = X(u)$ ① 找一个 conserved function H for ①. i.e. $\frac{d}{dt}H(u)=0$ along flow.

Remark 如果 P. Poisson 则有 $\dot F = \{F,H\}$

B 找到一个族的 constants of the motion C on P 使得 $\frac{d}{dt}C(u)=0$. along flow of (1).

often: Casimirs - $\{C,G\}=0$ $\forall G$ on P.

C First variation relate equib u_e of (1). of motion C Requiring $H_C = H+C$. 有个 critical pt at u_e.

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D 有限维. 选择一个 C 使得 $D^2 H_C(u_e)$ 在点 u_e 是正定/负定的.

More generally Find quadratic form Q_1, Q_2 on P 使得

(3). $Q(\Delta u) \leq H(u_e + \Delta u) - H(u_e) - DH(u_e)\cdot \Delta u$.

(4). $Q_2(\Delta u) \leq C(u_e + \Delta u) - C(u_e) - DC(u_e)\Delta u$ $\quad \forall \Delta u$ on P.

(5). $Q_1(\Delta u) + Q_2(\Delta u) > 0$ 对 all Δu on P 且 $\Delta u \neq 0$.

R-B Dynamics

$\dot{\Pi} = \Pi \times w \qquad \Pi_i = I_i w_i$

$I = (I_1, I_2, I_3) \quad I_i > 0. \quad H(\Pi) = \frac{1}{2}\Pi\cdot w = \sum_i \frac{1}{2}\frac{\Pi_i^2}{I_i}$

Lie-Poisson Bracket. $\{F, G\}(\Pi) = -\Pi(\nabla F(\Pi) \times \nabla G(\Pi))$ Casimir $\phi\left(\frac{\|\Pi\|^2}{2}\right)$.

- Constructing Lyapnov function by good choice of ϕ.

假设 $I_1 > I_2 > I_3$

考虑 $H + \phi\left(\frac{\|\Phi\|^2}{2}\right)$.

① Critical pt at $(1,0,0)$ $\quad \delta\left(H + \phi\left(\frac{\|\Pi\|^2}{2}\right)\right) = \sum_i \frac{\Pi_i}{I_i}\delta\Pi_i + \phi'\left(\frac{\|\Pi\|^2}{2}\right)$

如果 $\frac{\Pi_i}{I_i} + \phi'\left(\frac{\|\Pi\|^2}{2}\right) = 0$. 在点 $(1,0,0)$ $\phi'\left(\frac{1}{2}\right) = -\frac{1}{I_1}$

$$\delta^2\left(H + \phi\left(\frac{\|\Pi\|^2}{2}\right)\right) = \sum_i \frac{(\delta\Pi_i)^2}{I_i} + \phi'\left(\frac{\|\Pi\|^2}{2}\right)\left(\sum_i(\delta\Pi_i)^2\right) + \phi''\left(\frac{\|\Pi\|^2}{2}\right)\left(\sum_i \Pi_i\delta\Pi_i\right)^2$$

在 $(1,0,0)$ 点 $\quad = \sum_i \left(\frac{\delta\Pi_i}{I_i}\right)^2 - \frac{1}{I_1}\left(\sum_i(\delta\Pi_i)^2\right) + \phi'\left(\frac{1}{2}\right)(\delta\Pi_1)^2$

$$= (\delta\Pi_2)^2\underbrace{\left(\frac{1}{I_2} - \frac{1}{I_1}\right)}_{>0} + (\delta\Pi_3)^2\underbrace{\left(\frac{1}{I_3} - \frac{1}{I_2}\right)}_{>0} + \underbrace{\phi''\left(\frac{1}{2}\right)}(\delta\Pi_1)^2$$

因此需要的是 $\phi''\left(\frac{1}{2}\right) > 0$

故可以选择 $\phi(x) = -\frac{1}{I_1}x + \left(x - \frac{1}{2}\right)^2$ 因此 $I_1 > I_2 > I_3$ 是稳定的.