L20 Stability

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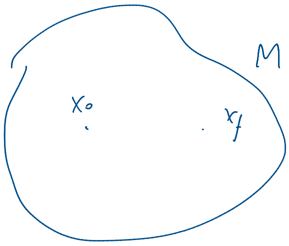
Dynamical System. $\dot{x} = f(x)$.

Control System $\dot{x} = f(x, u(t))$



$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x).$$

Lie-Algebra rank condition $\{f, g_1, \cdots g_m\}_{L.A.}$

rank n 对于可控性来说是必要的. 而对于 accessibility 而言是充分的

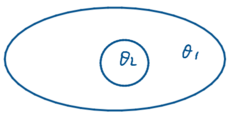
accessibility : reachable set from x_0

线性系统 : $\dot{x} = Ax + Bu$.

Kalman rank condition 退化成 $[B, AB, A^2B, \cdots A^{n-1}B]$

Control of Elroy's beanie.

- Sys on $S^1 \times S^1$



$I\dot{\theta}_1 + I\dot{\theta}_2 = \mu$. 角动量守恒

Denote by S the shape space.

$\psi = \theta_2 - \theta_1$

变换坐标来包含形状, $\theta = \theta_1$, $\psi = \theta_2 - \theta_1$

$$I_1\dot{\theta} + I_2(\dot{\theta} + \dot{\psi}) = \mu.$$

改写成 form: $d\theta + \frac{I_2}{I_1+I_2} d\psi = \frac{\mu}{I_1+I_2} dt$

而对于$\mu=0$的情况 $\Delta\theta = -\frac{I_2}{I_1+I_2}\int_0^{2\pi} d\psi$. $\quad I_1 \gg I_2$.

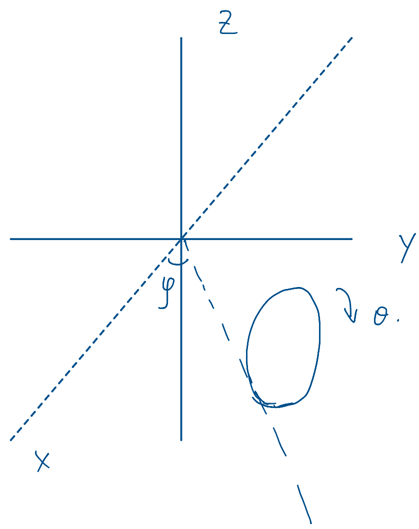
Geometrical interpretation $\quad$ A mech = mechanical connection

$$A_{mech} = d\theta + \frac{I_2}{I_1+I_2} d\psi.$$

"Flat" connection for trivial S^1 bundle. $\pi_1 : S^1 \times S^1 - S^1$

$\Delta\theta$: holonomy of connection.

Rolling Penny



$z_1 = \theta$. $z_2 = \varphi$. $z_3 = x$ $z_4 = y$ $z_5 = \dot{\theta}$ $z_6 = \dot{\varphi}$

Dynamics

$\dot{z}_1 = z_5$ $\dot{z}_2 = z_6$. $\dot{z}_3 = z_5 \cos z_2$.

$\dot{z}_4 = z_5 \sin z_2$ $\dot{z}_5 = \frac{1}{2} u_1$ $\dot{z}_6 = u_2$

Rewrite as $\dot{x}_1 = x_3$. $\dot{x}_2 = -A(x_1)\, x_3$. $\dot{x}_3 = v \to$ controls

$\left(x_1 = (z_1, z_2). \quad x_2 = (z_3, z_4) \quad x_3 = (z_5, z_6) \right)$

Holonomy $x_2^T = \oint_\gamma A(x_1)\, dx_1$

Controllability : 如果两个控制向量场 $g_1 . g_2$ 和一个 drift f.

L. A. $g_1, g_2, [g_1, f]$. $[g_2, [f, [g_1, f]]]$ $[g_2 [f [g_1 [f [g_2, f]]]]]$ 可控

Control : Step 1: 在原点附近

i.e. transfer $(x_1^0, x_2^0, x_3^0) \to (0, x_2^T, 0)$

Step 2 : Trace closed path in base space to get to $(0, x_2^T, 0) \to (0, 0, 0)$

Stability (Asymptotic).

Heisenburg : $\begin{cases} \dot{x} = u \\ \dot{y} = v \\ \dot{z} = xv - yu. \end{cases}$

Problem：选择 u, v 使得 $(x, y, z) \to (0, 0, 0)$

令 $\begin{cases} u = -\alpha x + \beta y \ \mathrm{sign}(z) \\ v = -\alpha y - \beta x \ \mathrm{sign}(z). \end{cases}$

$$V = \frac{1}{2}(x^2 + y^2)$$

$$\dot{V} = x\dot{x} + y\dot{y} = x(-\alpha x + \beta y \ \mathrm{sign}(z)) + y(-\alpha y - \beta x \ \mathrm{sign}(z))$$

当 $x \to 0$, $y \to 0$ 时有 $\alpha(x^2 + y^2) \to 0$.

$$\dot{z} = xv - yu = -\beta(x^2 + y^2)\,\mathrm{sign}(z) = -2\beta V \,\mathrm{sign}(z)$$

$|z|$ decreases, reaches 0 in finite time.

if $2\beta \int_0^\infty V(z)\, dz > |z(0)|$

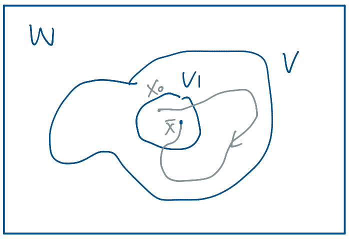
if $z(\,)$ is such that $2\beta \int_0^\alpha V(t)\, dz = |z(0)|$

Stability

对于一个 $\dot{x} = f(x)$ 是一个 $\mathbb{R}^n$ 上面光滑的动力系统. 令 $\bar{x}$ 是一个平衡态 $f(\bar{x}) = 0$.

$\bar{x}$ 是稳定的. 当对于任何的邻域 U, f 被定义在 W 上.

s.t. 所有 $x(x_0, t)$ $(x_0 \in V_1)$ 全部都在 V 里面. 对 $\forall t > 0$.



Stab that is not asymptotic stable is called neutral (不converge到平点).

Asymptotic stable

a fixed pt which is not stable is unstable.

if all eigvalues of $Df(\bar{x})$ have negative part. $\bar{x}$ is a sink.

定理: 假设 $Df(\bar{x})$ 有特征值 实部 $< -c$. 其中 $c > 0$.

则存在 邻域 $U \subset W$ (对于 $\bar{x}$) 使得

a). $\phi_t(x)$ is defined and in U $\forall x \in U$. $t > 0$

b). $\exists$ 一个 norm 在 $\mathbb{R}^n$ 中. s.t. $|\phi_t(x) - \bar{x}| < e^{-tc} |x - \bar{x}|$. $x \in U$ t

c). 对任何的 norm. $\exists \beta > 0$. $|\phi(t) - x| \leq \beta e^{-tc} |x - \bar{x}|$. $\forall t > 0$. $x \in U$

Lyapunov functions

令 $V: U \to \mathbb{R}$ 是一个可微的函数. 定义在 邻域 $U \subset W$ 之中. ($\bar{x}$ 的邻域). 定义 $\dot{V}(x) = \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial x} f(x)$.

定理: let $V: U \to \mathbb{R}$ 在 $U \subset W$ 上连续, 在 $U - \bar{x}$ 上面可微

s.t. (a) $V(\bar{x}) = 0$ $V(x) > 0$. $x = \bar{x}$

(b). $\dot{V} \leq 0$ in $U - \bar{x}$ 则有 $\bar{x}$ 是稳定的

如果还有 (c) $V < 0$ on $U - \bar{x}$. $\bar{x}$ 是 assymptotic stable.

例: $\begin{cases} \dot{x} = 2y(z-1) \\ \dot{y} = -x(z-1) \\ \dot{z} = -z^3 \end{cases}$ 对于点, $\bar{x} = 0$

$$\begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

2个 eigenvalue 和 0.

$$\begin{cases} \dot{y} = -x(z-1) \\ \dot{z} = -z^3 \end{cases} \qquad \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

令 $V(x,y,z) = a x^2 + b y^2 + c z^2$.

$\dot{V} = 2(a x \dot{x} + b y \dot{y} + c z \dot{z}) \Rightarrow \frac{1}{2}\dot{V} = 2axy(z-1) - bxy(z-1) - c z^4$.

令 $2a=b$, 而 $c=1$. (为了使得 $\dot{V} \le 0$). 所以 $\frac{1}{2}\dot{V} = -c z^4$, $c>0$.

Lyapov stable — stable. 但不一定是 assymptotic stable

- Hamitonian System. 是 Lyapnov 但不一定是 assymptotic stable (damping).

e.g. Ham + Momentum / Casimir

Pf. 令 $\delta > 0$ 很小 使得 $B_\delta(\bar{x}) \subset U$ 是 $\bar{x}$ 的一个邻域. 令 α 是 V 在边界上的最小值. $B_\delta(\bar{x}) = S_\delta(\bar{x})$.

令 $V_1 = \{x \in B_\delta(\bar{x}) \mid V(x) < \alpha\}$

则 no solution can meet $S_\delta(\bar{x})$.

因为 V 是 non increasing along solution curves. 因此 不会离开 $B_\delta(\bar{x})$.

assymptotic $\dot{V} < 0$ 会有耗散项