L3 Least Action Principal

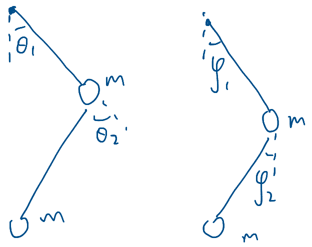
2024年1月22日 星期一　10:00

Least Action　广义坐标.

用 q 代表 Configuration Space.　$(q^1, \cdots q^n)$

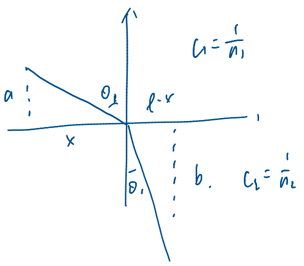
我们希望不要多余的坐标.. ⇒ 我们希望自由度 = 最小坐标数.

Simple Kinematic Chain. 如果我们有一个平面刚体由无质量杆子连接



可以选择不一样的 generalized coordinates

Snell: Least Action Principal.



$$\min T = \frac{\sqrt{a^2+x^2}}{c_1} + \frac{\sqrt{b^2+(l-x)^2}}{c_2}$$

$$\frac{dT}{dx} = \frac{1}{c_1}\frac{x}{\sqrt{a^2+x^2}} - \frac{1}{c_2}\frac{l-x}{\sqrt{b^2+(l-x)^2}}$$

$$\Rightarrow n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

$$\delta \int_a^b L(q,\dot q)\, dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q \right) dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\right) \delta q\, dt + \underbrace{\frac{\partial L}{\partial \dot q}\delta q \Big|_{t_1}^{t_2}}_{=0}$$

分部积分

因为 δq 在 t_1 . t_2 为零.

$$= \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right)\right) \delta q^i = 0.$$

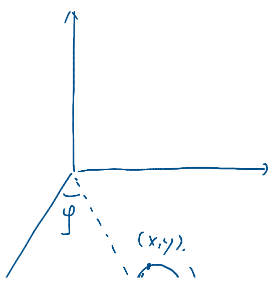
因为这对于任何的 δq^i 成立. 则有 $\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right) = 0.$

例：对于 Free Particle.　$L = \frac{1}{2}m\dot x^2 + \frac{1}{2}m\dot y^2$　$\begin{cases} m\ddot x = 0 \\ m\ddot y = 0. \end{cases}$

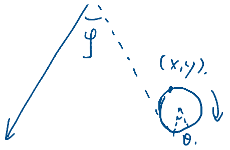
滚动硬币.

$$L = \frac{1}{2}m(\dot x^2 + \dot y^2) + \frac{1}{2}I\dot\theta^2 + \frac{1}{2}J\dot\varphi^2$$

约束条件 $\dot x = R\cos\varphi\,\dot\theta$　$\dot y = R\sin\varphi\,\dot\theta$



$\delta x = R\cos\varphi\,\delta\theta.$



$$\begin{cases} \delta x = R\cos\varphi\,\delta\theta \\ \delta y = R\sin\varphi\,\delta\theta \end{cases}$$ 换进去

$$\delta\int L\,dt = m\ddot{x}\,\delta x + m\ddot{y}\,\delta y + I\ddot{\theta}\,\delta\theta + J\ddot{\varphi}\,\delta\varphi = 0.$$

$$= m\ddot{x}\,R\cos\varphi\,\delta\theta + m\ddot{y}\,R\sin\varphi\,\delta\theta + I\ddot{\theta}\,\delta\theta + J\ddot{\varphi}\,\delta\varphi.$$

其中 $\delta\varphi$ 和 $\delta\theta$ 是独立的. $\begin{cases} m\ddot{x}R\cos\varphi + m\ddot{y}R\sin\varphi + I\ddot{\theta} = 0. \\ J\ddot{\varphi} = 0. \end{cases}$

换进去.

而 $\begin{cases} \dot{x} = R\cos\varphi\,\dot{\theta} \\ \dot{y} = R\sin\varphi\,\dot{\theta} \end{cases} \Rightarrow \begin{cases} \ddot{x} = R\cos\varphi\,\ddot{\theta} - R\sin\varphi\,\dot{\varphi}\,\dot{\theta} \\ \ddot{y} = R\sin\varphi\,\ddot{\theta} + R\cos\varphi\,\dot{\varphi}\,\dot{\theta} \end{cases}$

$$\boxed{(I+mR^2)\ddot{\theta} = 0. \quad J\ddot{\varphi} = 0.}$$

这一项只是在关心速度

对于力: $\left.\begin{aligned} (I+mR^2)\ddot{\theta} &= F_1 = u_\theta. \\ J\ddot{\varphi} &= F_2 = u_\varphi. \end{aligned}\right\}$ Base Space. 二阶导数

$$\begin{cases} \dot{x} = R\cos\varphi\,\dot{\theta} \\ \dot{y} = R\sin\varphi\,\dot{\theta} \end{cases}$$ 1^{st} order $\Rightarrow$ Fiber.

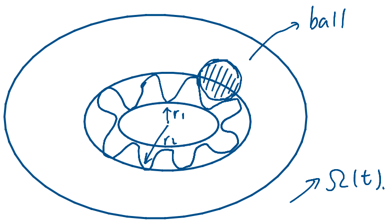
在 base 里面是可控的 Space (θ,φ). 而关心在 $m(x,y)$ - fiber 中的行为.

Fiber-Bundle.

Variational Constrained Dynamics.

$$L = \frac{1}{2}(m\dot{x}^2 + m\dot{y}^2) + \frac{1}{2}J\dot{\varphi}^2 + \frac{1}{2}I\dot{\theta}^2 + \mu_1(t)(\dot{x} - R\cos\varphi\,\dot{\theta}) + \mu_2(t)(\dot{y} - R\cos\varphi\,\dot{\theta})$$

但这样的方程是错误的.



— 一个旋转的桌子. 球会在上面在 r_1 和 r_2 之间滚动.

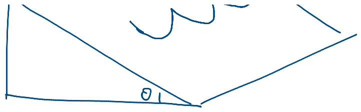
用上面的方程：球会飞出去.

但实际上不会. 要用 达朗贝尔原理处理.

对于一个刀片. Knife Edge.

初条件如果是 $\dot{x} = \dot{y} = 0$. $\dot{\theta} = const.$

出现的情况并非刀片会下降. 而是如图中轨迹



出现的情况并非刀片会下降. 而是如图中轨迹

例: Nonholonomic 积分器, 海森堡问题.

$$\begin{cases} \dot{x} = u_1 \\ \dot{y} = u_2 \\ \dot{z} = y u_1 - x u_2 \end{cases}$$

一阶问题.

显然. 可以在 x、y 轴随意移动,

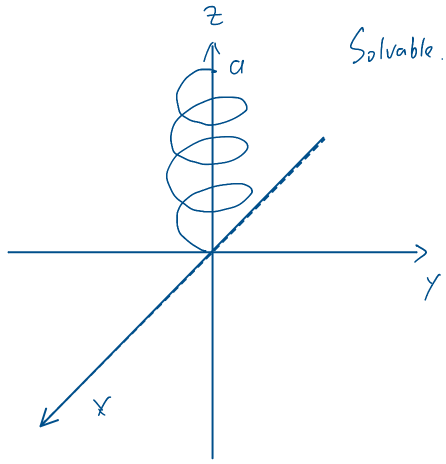
问题: 是否能在 z 轴随意移动?

<u>Optimal Control</u>

要 $\min \int_0^T (u_1^2 + u_2^2)\, dt$. s.t. $q(0) = (x, y, z)\ (0,0,0) = (0,0,0)$

$q(t) = (0, 0, a)$.

即要求 $\min \int_0^T (\dot{x}^2 + \dot{y}^2)\, dt$, s.t. $\dot{z} = y\dot{x} - x\dot{y}$



Solvable.

Sub-Rienmanian Optimal Control Problem.

$$\min \int_0^T \dot{x}^2 + \dot{y}^2 + \lambda(z - y\dot{x} + x\dot{y})\, dt.$$