L4 D'Alembert Principle

2024年1月24日 星期三 10:04

$$\begin{cases} \dot{x} = u_1 \\ \dot{y} = u_2 \\ \dot{z} = y u_1 - x u_2 \end{cases}$$

Heisenberg Problem.

<u>Optimal Control</u> $\min \frac{1}{2}\int_0^T (u_1^2 + u_2^2)\, dt$. s.t. $q(0) = (0,0,0)$. $q(T) = (0,0,a)$.

$\min \frac{1}{2}\int_0^T (\dot{x}^2 + \dot{y}^2)$ s.t. $\dot{z} = y\dot{x} - x\dot{y}$

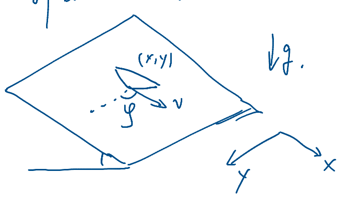
$L_a = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \lambda(\dot{z} - y\dot{x} + x\dot{y})$

而对 Lagrangian 有 $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q^i} = 0$.

x: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \begin{cases} \ddot{x} - 2\lambda\dot{y} = 0. \\ \ddot{y} + 2\lambda\dot{x} = 0 \\ \dot{\lambda} = 0. \end{cases}$ λ是t的函数. 但代入 $\dot{\lambda} = 0$.

Try $x = a\cos t$ $y = b\sin t$ $\Rightarrow a = b$. $\lambda = -\frac{1}{2}$ ⇒螺旋线

斜坡上的刀面.



$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\varphi}^2 + mgx\sin\alpha$ 约束: $\dot{x}\sin\varphi = \dot{y}\cos\varphi$

因为刀片只能直着走.

Nonholonomic motion.

$$\begin{cases} m\ddot{x} = \lambda\sin\varphi + mg\sin\alpha \\ m\ddot{y} = -\lambda\cos\varphi \\ J\ddot{\varphi} = 0. \end{cases}$$

初条件:

$x(0) = \dot{x}(0) = y(0) = \dot{y}(0)$

$\varphi(0) = 0$ $\dot{\varphi}(0) = \omega$ (Const)

$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\varphi}^2 - mgx\sin\alpha$

能量守恒: $E(0) = E(t) + \frac{1}{2}J\omega^2 \Rightarrow E = \frac{1}{2}J\omega^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\varphi}^2 - mgx\sin\alpha$.

$\Rightarrow \frac{1}{2}\frac{\dot{x}^2}{\cos^2\varphi} - mgx\sin\alpha = 0$.

$\begin{cases} x = \frac{g}{2\omega^2}\sin\alpha\sin^2\omega t \end{cases}$

$$\begin{cases} x = \frac{g}{2\omega^2}\sin\alpha \sin^2\omega t \\ y = \frac{g}{2\omega^2}\sin\alpha\,(\omega t - \frac{1}{2}\sin 2\omega t) \end{cases}$$

但如果我们用 variational way (最小值那个东西) ⇒ 会一路让 knife 下降.

带力的机制：

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right) - \frac{\partial L}{\partial q^i} = F$$

Lagrange D' Alembet.

约束 - holonomic & non-holonomic.

m个约束： $\sum_{k=1}^{n} a_k^i(q)\cdot \dot q^k = 0. \quad i=1,2,\cdots m.$

如果以用位置状态写出来，存在 $b^j(q)=0$,

使上式能写成 $\sum_{k=1}^{n} \frac{\partial b^j}{\partial q^k}\dot q^k = 0,$ (例如 $x^2+y^2=0, \Rightarrow x\dot x + y\dot y = 0$).

具有 non-holonomic 约束的动力系统

Principal：约束力不做功，

$F_1\delta q^1 + F_2\delta q^2 + \cdots + F_n\delta q^n = 0,$ δq^i 虚位移

non-holonomic 约束： $\sum_{k=1}^{n} a_k^i(q)\,\underset{\delta q^k}{\dot q^k} = 0$ 有 m个 vector $(a_1^1, a_2^1, \cdots a_n^1)$ $\vdots$ $(a_1^m \cdots a_n^m)$

因为约束力不做功 $\underset{\sim}{F}\cdot V = 0$, 任何位移.

而 $\underbrace{A}_{约束}\cdot\underbrace{V}_{位移} = 0$. 所以 V在 A 的零空间 ⇒ $F \perp \ker A$. $F \in \text{Range}(A^T)$

$$\boxed{\therefore\ F_i = \sum_{j=1}^{n}\lambda_j a_i^j}$$

F是 a的一个线性组合，

例： Disk

$$L = \frac{1}{2}m(\dot x^2+\dot y^2) + \frac{1}{2}I\dot\theta^2 + \frac{1}{2}J\dot\varphi^2$$

$$\begin{cases} \dot x = R\cos\varphi\,\dot\theta \Rightarrow \dot x - R\cos\varphi\,\dot\theta = 0. \equiv a^1\,(\dot x, \dot y, \dot\varphi, \dot\theta)^T = 0, \\ \dot y = R\sin\varphi\,\dot\theta \Rightarrow \dot y - R\sin\varphi\,\dot\theta = 0 \equiv a^2\,(\dot x\ \dot y\ \dot\varphi\ \dot\theta)^T = 0. \end{cases}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \lambda_1 a^1 + \lambda_2 a^2 + u_\varphi f^\varphi + u_\theta f^\theta$$

$$f^\varphi = (0,0,1,0)^T \quad f^\theta = (0,0,0,1)^T$$

$\Rightarrow$ 这个是我们的 Control

Variational case.

$$L_c = L + \mu_1(\dot{x} - R\cos\varphi\,\dot{\theta}) + \mu_2(\dot{y} - R\sin\varphi\,\dot{\theta}).$$