Math 658 HW1 Mo Yang

Exercises

◇ **1.2-1.** Consider the Lagrangian

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz.$$

Compute the equations of motion in both Lagrangian and Hamiltonian form. Verify that the Hamiltonian (energy) is conserved along the flow. Are there other conserved quantities?

◇ **1.2-2.** Consider a Lagrangian of the form $L = \frac{1}{2}\sum_{k,l=1}^{n} g_{kl}(q)\dot{q}^k\dot{q}^l$, where g_{kl} is a symmetric matrix. Show that the Lagrange equation of motion are

$$\sum_s g_{rs}\ddot{q}^s + \sum_{l,m} \Gamma_{rlm}\dot{q}^l\dot{q}^m = 0$$

for suitable symbols Γ. Verify conservation of energy directly for this system.

Exercise 1. (1.2-1)

Lagrangian: $L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$

As $\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$ $\Rightarrow \begin{cases} x: & 0 - m\ddot{x} = 0 \\ y: & 0 - m\ddot{y} = 0 \\ z: & -mg - m\ddot{z} = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x} = 0 \\ \ddot{y} = 0 \\ \ddot{z} = -g \end{cases}$ So $\vec{v}(x,y,z) = c_1 t\,\hat{x} + c_2 t\,\hat{y} + (c_3 t - \frac{1}{2}gt^2)\hat{z} + \vec{v_0}$

Hamitonian: $p_i = \frac{\partial L}{\partial \dot{q}_i} \Rightarrow p_x = m\dot{x},\ p_y = m\dot{y},\ p_z = m\dot{z}$

$H = \sum p_i \dot{q}^i - L = m\dot{x}^2 + m\dot{y}^2 + m\dot{z}^2 - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$

We can see $H = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$ is the total Energy which include kenetic energy and potential energy which will conserve.

equations of motion: $\frac{\partial H}{\partial p_i} = \dot{q}^i$ & $-\frac{\partial H}{\partial q^i} = \dot{p}_i$

$\begin{cases} m\ddot{x} = 0. \\ m\ddot{y} = 0 \\ m\ddot{z} = -gm \end{cases} \Rightarrow \begin{cases} \ddot{x} = 0 \\ \ddot{y} = 0 \\ \ddot{z} = -g \end{cases}$ $\Rightarrow \vec{v}(x,y,z) = c_1 t\hat{x} + c_2 t\hat{y} + (c_3 t - \frac{1}{2}gt^2)\hat{z} + \vec{v_0}$

Besides Hamitonion, there are other conserved quantity as L is irrelevant with x and y.

for q^i which is irrelevant to L $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q} = 0$. So $\dot{x} = 0$. $\dot{y} = 0$ Are conservative.

Exercise 2. (1.2-2)

$L = \frac{1}{2}\sum_{k,l=1}^{n} g_{kl}(q)\cdot\dot{q}^k\cdot\dot{q}^l$ Lagrangian motion equation $-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} = 0$. So

$\frac{\partial L}{\partial \dot{q}^i} = \frac{1}{2}\sum_{k=1}^{n} g_{ki}(q)\dot{q}^k + \frac{1}{2}\sum_{l=1}^{n} g_{il}(q)\dot{q}^l = \sum_{k=1}^{n} g_{ki}(q)\dot{q}^k$ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \sum_{k=1}^{n} g_{ki}(q)\ddot{q}^k + \underbrace{\sum_{k=1}^{n} g_{ki}'(q)^T \dot{q}\,\dot{q}^k}$

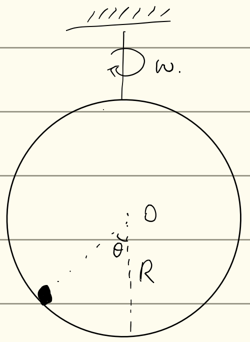
$\frac{\partial L}{\partial q} = 0$. So $\sum_{k=1}^{n} g_{ki}(q)\ddot{q}^k + \sum_{k,j=1}^{n}\frac{\partial g_{ki}}{\partial q^j}\dot{q}^j\dot{q}^k = 0$. $= \sum_{k=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial g_{ki}}{\partial q^j}\cdot\dot{q}^j\right)\dot{q}^k = \sum_{k,j=1}^{n}\frac{\partial g_{ki}}{\partial q^j}\dot{q}^j\dot{q}^k$

We can see for any r. $\sum_s g_{sr}(q)\ddot{q}^s + \sum_{l,m}\Gamma_{rlm}\dot{q}^l\dot{q}^m = 0$ where $\Gamma_{rlm} = \frac{\partial g_{rl}}{\partial q^m}$

Using the basic principles of mechanics given in Chapter 1, we start with the Lagrangian function for this problem (the kinetic energy in an inertial frame minus the gravitational potential energy). Then the associated Euler–Lagrange equations *with forces* are given by

$$mR^2\ddot{\theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta - \nu R\dot{\theta}, \qquad (2.1.1)$$

Exercise 3 (2.1-1).



$L = \frac{1}{2} m(\omega^2 R^2 \sin^2\theta + \dot{\theta}^2 R^2) - (-mg\cos\theta)$ (Setting the centre of circle's potential to be zero).

$\frac{\partial L}{\partial \theta} = m\omega^2 R^2 \sin\theta\cos\theta - mg\sin\theta.$

$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}$ $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mR^2\ddot{\theta}$ $\quad$ As $\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = F_i$ $\quad$ for friction, $F_i = f_i = \nu R\dot{\theta}$

So $mR^2\ddot{\theta} = m\omega^2 R^2 \sin\theta\cos\theta - mg\sin\theta - \nu R\dot{\theta}$

◇ **1.3-2** (Rosenberg [1977]). Consider the Lagrangian

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

with the constraints

$$\dot{z} - y\dot{x} = 0\,.$$

(a) Write down the dynamic nonholonomic equations.
(b) Write down the variational nonholonomic equations.
(c) Are these two sets of equations the same?

Exercise 4.

(a). Dynamics nonholonomic equations $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \sum_{j=1}^{m} \lambda_j a_i^j$

$\frac{\partial L}{\partial \dot{x}} = \dot{x}$ $\quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \ddot{x}$ $\quad \frac{\partial L}{\partial x} = 0.$ $\quad$ And for constraint $\dot{z} - y\dot{x} = 0.$ $\quad \underbrace{a_1(q^i)}_{-y}\dot{x} + \underbrace{a_2(q^i)}_{0.}\dot{y} + \underbrace{a_3(q^i)}_{1}\dot{z} = 0.$

So $\begin{cases} \ddot{x} = -\lambda y. \\ \ddot{y} = 0 \\ \ddot{z} = \lambda. \end{cases}$

(b) Variational Nonholonomic Equations $\quad L_A = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \mu(\dot{z} - y\dot{x})$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$

So $\begin{cases} \ddot{x} - \dot{\mu}y - \mu\dot{y} = 0. \\ \ddot{y} - \mu\dot{x} = 0. \\ \ddot{z} + \dot{\mu} = 0. \end{cases}$

(c) The two equations are different. $\dot{z} - y\dot{x}$ is a nonholonomic control.

5. Show the energy and momentum are preserved for the rigid body equations discussed in class.

Exercise 5.

$$\begin{cases} I_1 \dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3 \\ I_2 \dot{\Omega}_2 = (I_3 - I_1)\Omega_3\Omega_1 \\ I_3 \dot{\Omega}_3 = (I_1 - I_2)\Omega_1\Omega_2 \end{cases}$$

let $\Pi_i = I_i \Omega_i$. Then we have $\dot{\Pi}_1 = I_1 \dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3 = \Pi_2\Omega_3 - \Pi_3\Omega_2 = \frac{(I_2 - I_3)\Pi_2\Pi_3}{I_2 I_3}$ which means $\dot{\Pi} = \Pi \times \Omega$.

So for $M = \sum \Pi_i^2$ $\quad \dot{M} = (\Pi^T \Pi)' = \Pi \cdot \dot{\Pi} = \Pi \cdot (\Pi \times \Omega) = 0$

$H = \frac{1}{2}\sum I_i \Omega_i^2 = \frac{1}{2}\Pi \cdot \Omega \quad \dot{H} = \frac{1}{2}\dot{\Pi} \cdot \Omega = \frac{1}{2}(\Pi \times \Omega) \cdot \Omega = 0.$

So as $\dot{M} = \dot{H} = 0$. Momentum and Energy conserves.

6. Consider the central force problem in two dimensions: we have Lagrangian

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

Compute the Hamiltonian equations of motion.

Exercise 6.

Hamotonian: (r, θ) are coordinates. $\quad q_1 = r. \quad q_2 = \theta.$

$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad p_2 = \frac{\partial L}{\partial \dot{q}_2} = mr^2\dot{\theta} \qquad L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$

$H = \sum p_i \dot{q}_i - L = m\dot{r}\dot{r} + mr^2\dot{\theta}\dot{\theta} - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$

So $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$ is constant

7. Show that Hamilton's equations may be written using the canonical Poisson bracket.

8. Show that the canonical Poisson bracket satisfies the Jacobi identity.

Exercise 7

For Hamotonian. $\dot{q}^i = \frac{\partial H}{\partial p_i}$ $\quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$

$$\{ p_i , H \} = \sum_j \left(\frac{\partial p_i}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q_j} \frac{\partial p_i}{\partial p_j} \right) = \sum_j \left(-\dot{p}_j \cdot \delta_{ij} \right) = \dot{p}_i$$

As $\frac{\partial p_i}{\partial q_j} = 0$. $\quad \frac{\partial p_i}{\partial p_j} = 0$ when $i \neq j$ $\quad \frac{\partial p_i}{\partial p_i} = 1$.

Similarly
$$\{ q_i , H \} = \sum_j \left(\frac{\partial q_i}{\partial q^j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q^j} \frac{\partial q^i}{\partial p_j} \right) = \sum_j \delta_{ij} \dot{q}_j = \dot{q}_i$$

As $\frac{\partial q_i}{\partial q_j} = \delta_{ij}$, $\quad \frac{\partial q^i}{\partial p_j} = 0$.

So we have hamotonian in the form of Poisson Bracket: $\dot{p}_i = \{ p_i , H \}$, $\quad \dot{q}^i = \{ q^i , H \}$

Exercise 8.

$$\{F, \{G,H\}\} = \left\{ F, \sum_{i=1}^{n} \frac{\partial G}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i} \frac{\partial H}{\partial q^i} \right\} = \frac{\partial F}{\partial q^i} \frac{\partial}{\partial p_i} \left(\frac{\partial G}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i} \frac{\partial H}{\partial q^i} \right) - \frac{\partial F}{\partial p_i} \frac{\partial}{\partial q^i} \left(\frac{\partial G}{\partial q^i} \frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i} \frac{\partial H}{\partial q^i} \right)$$

$$= \frac{\partial F}{\partial q^i} \left(\underbrace{\frac{\partial^2 G}{\partial p_i \partial q^i} \frac{\partial H}{\partial p_i}}_{1} + \underbrace{\frac{\partial^2 H}{\partial p_i^2} \frac{\partial G}{\partial q^i}}_{2} - \underbrace{\frac{\partial^2 G}{\partial p_i^2} \frac{\partial H}{\partial q^i}}_{3} - \underbrace{\frac{\partial G}{\partial p_i} \frac{\partial^2 H}{\partial p_i \partial q^i}}_{4} \right) - \frac{\partial F}{\partial p_i} \left(\underbrace{\frac{\partial^2 G}{\partial {q^i}^2} \frac{\partial H}{\partial p_i}}_{5} + \underbrace{\frac{\partial G}{\partial q^i} \frac{\partial^2 H}{\partial q^i \partial p_i}}_{6} - \underbrace{\frac{\partial^2 G}{\partial q^i \partial p_i} \frac{\partial H}{\partial q^i}}_{7} - \underbrace{\frac{\partial G}{\partial p_i} \frac{\partial^2 H}{\partial {q^i}^2}}_{8} \right)$$

Similarly
$$\{G, \{H,F\}\} = \frac{\partial G}{\partial q^i} \left(\underbrace{\frac{\partial^2 H}{\partial p_i \partial q^i} \frac{\partial F}{\partial p_i}}_{6} + \underbrace{\frac{\partial^2 F}{\partial p_i^2} \frac{\partial H}{\partial q^i}}_{9} - \underbrace{\frac{\partial H}{\partial p_i^2} \frac{\partial F}{\partial q^i}}_{2} - \underbrace{\frac{\partial H}{\partial p_i} \frac{\partial^2 F}{\partial p_i \partial q^i}}_{10} \right) - \frac{\partial G}{\partial p_i} \left(\underbrace{\frac{\partial^2 H}{\partial {q^i}^2} \frac{\partial F}{\partial p_i}}_{8} + \underbrace{\frac{\partial H}{\partial q^i} \frac{\partial^2 F}{\partial q^i \partial p_i}}_{11} - \underbrace{\frac{\partial^2 H}{\partial q^i \partial p_i} \frac{\partial F}{\partial q^i}}_{4} - \underbrace{\frac{\partial H}{\partial p_i} \frac{\partial^2 F}{\partial {q^i}^2}}_{12} \right)$$

$$\{H, \{F,G\}\} = \frac{\partial H}{\partial q^i} \left(\underbrace{\frac{\partial^2 F}{\partial p_i \partial q^i} \frac{\partial G}{\partial p_i}}_{11} + \underbrace{\frac{\partial^2 G}{\partial p_i^2} \frac{\partial F}{\partial q^i}}_{3} - \underbrace{\frac{\partial F}{\partial p_i^2} \frac{\partial G}{\partial q^i}}_{9} - \underbrace{\frac{\partial F}{\partial p_i} \frac{\partial^2 G}{\partial p_i \partial q^i}}_{7} \right) - \frac{\partial H}{\partial p_i} \left(\underbrace{\frac{\partial F}{\partial {q^i}^2} \frac{\partial G}{\partial p_i}}_{12} + \underbrace{\frac{\partial F}{\partial q^i} \frac{\partial^2 G}{\partial q^i \partial p_i}}_{1} - \underbrace{\frac{\partial^2 F}{\partial q^i \partial p_i} \frac{\partial G}{\partial q^i}}_{10} - \underbrace{\frac{\partial F}{\partial p_i} \frac{\partial^2 G}{\partial {q^i}^2}}_{5} \right)$$

$\therefore \{F, \{G,H\}\} + \{G, \{H,F\}\} + \{H, \{F,G\}\} = 0.$ $\qquad$ Proved 8.