L5 Hamitonion I

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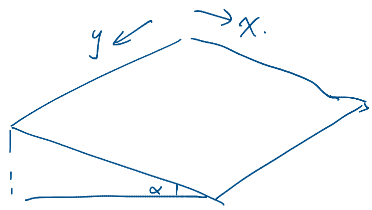
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right) - \frac{\partial L}{\partial q_i} = F_i$ F_i 可以为保守力. 保守力可以进势能项.

约束 $\sum_{k=1}^{n} a_k^i(q)\,\dot q^k = 0$.

对于 Knife Edge 问题.



$L = \frac{1}{2}m(\dot x^2+\dot y^2) + \frac{1}{2}J\dot\varphi^2 + mgx\sin\alpha.$

$$\begin{cases} m\ddot x = \lambda\sin\varphi + mg\sin\alpha. \\ m\ddot y = -\lambda\cos\varphi. \\ J\ddot\varphi = 0. \quad \Rightarrow \dot\varphi = \omega \text{ 为常数.} \end{cases}$$

Trick $E(0) = \frac{1}{2}J\omega^2 = \frac{1}{2}m(\dot x^2+\dot y^2) + \underbrace{\frac{1}{2}J\dot\varphi^2}_{=\frac{1}{2}J\omega^2} - mgx\sin\alpha.$ 而 $\dot x\sin\varphi = \dot y\cos\varphi$

$\Rightarrow \frac{\dot x^2}{2\cos^2\varphi} - mgx\sin\alpha = 0$ 答案为 $x = 2g\sin\alpha\sin^2\omega t.$

关于拉格朗日-达朗贝尔原理其它写法.

$\delta\int_a^b L(q,\dot q)\,dt = 0$ 其中 δq 满足 $\sum_{k=1}^{n} a_k^j\,\delta q^k = 0.$

这里我们先做了微分, 然后再解 δq^i

而不是我们把 δq 先扔到积分号当中 (只有在没有约束时可以这样做).

Conservation Laws

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right) - \frac{\partial L}{\partial q^i} = 0.$ 如果 L 和 q^k 无关 (但和 $\dot q^k$ 有关). 则有

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^k}\right) = 0.$ $\frac{\partial L}{\partial \dot q^k}$ is conserved along flow 我们称 q^k cyclic

例: $L = \sum_{i=1}^{n} m_i\dot q_i^2 - U(q)$ $m\dot q^k$ = constant. $\Rightarrow$ 诺特定理.

Hamiltonians

我们引入 conjunction momenta. 令 $p_i = \frac{\partial L}{\partial \dot q^i}$ ($\underset{\sim}{q}$ vector & $\underset{\sim}{p}$ covector)

Change $(q^i, \dot q^i) \to (q^i, p_i)$

定义 $\boxed{H(\underset{\sim}{q}, \underset{\sim}{p}) = \sum_{j=1}^{n} p_j\dot q^j - L(q^i, \dot q^i, t)}$

change ...

定义 $\boxed{H(\underset{\sim}{q}, \underset{\sim}{p}) = \sum_{j=1}^{n} p_j \dot{q}^j - L(q^i, \dot{q}^i, t)}$

这样 $\dot{q}^i = \frac{\partial H}{\partial p_i}$ $(i=1,2,\cdots n)$

$\dot{p}_i = -\frac{\partial H}{\partial q_i}$

$$\frac{\partial H}{\partial p_i} = \underbrace{\dot{q}^i + \sum_{j=1}^{n}}_{\text{Product Rule}} \left(p_j \frac{\partial \dot{q}^j}{\partial p^i} - \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}^j}{\partial p_i} \right)$$

而 $p_j = -\frac{\partial L}{\partial \dot{q}^j}$ 所以 $\frac{\partial H}{\partial p_i} = \dot{q}^i$

$$\frac{\partial H}{\partial q^i} = \sum_{j=1}^{n} p_i \frac{\partial \dot{q}^j}{\partial q^i} - \frac{\partial L}{\partial q^i} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial q^i}$$

$$\Rightarrow \frac{\partial H}{\partial q^i} = -\frac{\partial L}{\partial q^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = -\frac{d}{dt} p_i$$

定理：$\frac{dH}{dt} = 0$. (在没有外力的情况下).

$$\frac{dH}{dt} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i} \dot{q}^i + \frac{\partial H}{\partial p_i} \dot{p}_i \right) = 0$$

定理：令 $L = T - V$ $T = \frac{1}{2} \sum_{i,j=1}^{n} g_{ij}(q) \dot{q}^i \dot{q}^j$ 即 $T = \frac{1}{2} \dot{q}^T G(q) \dot{q}$

$V(q)$. 其中 G 为正定的

$\underset{\sim}{p} = \frac{\partial L}{\partial \dot{q}} = G\dot{q}$ $\Rightarrow \dot{q} = G^{-1} p$.

$H = p^T \dot{q} - \frac{1}{2} \dot{q}^T G \dot{q} + V(q). = \frac{1}{2} \dot{q}^T G \dot{q} + V(q) = T + V.$

$H = T + V$ (Total Energy)

这也是为什么 H 是守恒的.

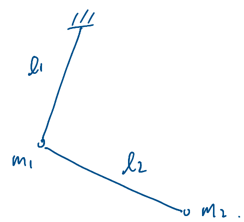
我们需要假设 $\frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j} \neq 0$. hyper-regular Lagrangian.

当我们有一个 Singular L $\Rightarrow$ Dirac - Mechanics. 例如 $L = \frac{1}{2} m \dot{x}^2 + \frac{k}{2}(x^2 + y^2)$.

Hamilton: $\dot{q}^i = \frac{\partial H}{\partial p_i}$ $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

诺特定理：如果 H 和 q^i 无关，那么 p_i 守恒

例：复摆



确认空间：$Q = S^2_{l_1} \times S^2_{l_2}$ (三维空间).

$$L = L(q_1, q_2, \dot{q}_1, \dot{q}_2) = \frac{1}{2} m_1 \|\dot{q}_1\|^2 + \frac{1}{2} m_2 \|\dot{q}_1 + \dot{q}_2\|^2$$

$$- m_1 g \underset{\sim}{q_1} k - m_2 g (\underset{\sim}{q_1} + \underset{\sim}{q_2}) k$$