Math 658 Homework 2 Mo Yang.

1. Show that the rigid body equations are variational with respect to the reduced variational principle discussed in class i.e. with respect to the variations

$$\delta\Omega = \dot{\Sigma} + \Omega \times \Sigma$$

$L = \frac{1}{2} \Omega^T I \Omega \qquad \delta L = I\Omega\delta\Omega$

$\delta \int_a^b L \, dt = \int_a^b \delta L \, dt = \int_a^b I\Omega\delta\Omega \, dt$ Plug in $\delta\Omega = \Omega \times \Sigma + \dot{\Sigma}$

$= \int_a^b I\Omega(\dot{\Sigma} + \Omega\times\Sigma) \, dt = \int_a^b I\Omega\dot{\Sigma} + I\Omega(\Omega\times\Sigma) \, dt$

$= \int_a^b I\Omega \, d\Sigma + \int_a^b I\Omega(\Omega\times\Sigma) \, dt = I\Omega\Sigma \Big|_a^b - \int_a^b I(d\Omega)\Sigma + \int_a^b I\Omega(\Omega\times\Sigma) dt$

As Σ vanishes at a and b. $I\Omega\Sigma\Big|_a^b = 0$.

$\delta \int_a^b L \, dt = \int_a^b \left(I\Omega(\Omega\times\Sigma) \, dt - I(d\Omega)\Sigma \right)$

As $I\Omega(\Omega\times\Sigma) = \Pi(\Omega\times\Sigma) = (\Pi\times\Omega)\Sigma = \dot{\Pi}\Sigma = (I\dot{\Omega})\Sigma = I\dot{\Omega}\Sigma$

So $I\Omega(\Omega\times\Sigma) \, dt = I\dot{\Omega}\Sigma \, dt = I(\dot{\Omega} \, dt)\Sigma = I(d\Omega)\Sigma$

So $\delta \int_a^b L \, dt = 0.$

2. Linearize the rigid body equations about their equilibria and compute the eigenvalues of the linearized flow in each case. What can you say about stability?

Rigid Body Equations :
$$\begin{cases} \dot{\Omega}_1 = \frac{I_2 - I_3}{I_1} \Omega_2 \Omega_3 \\ \dot{\Omega}_2 = \frac{I_3 - I_1}{I_2} \Omega_3 \Omega_1 \\ \dot{\Omega}_3 = \frac{I_1 - I_2}{I_3} \Omega_1 \Omega_2 \end{cases}$$

Linearize: We want to have a form where
$$\dot{\Omega} = A \Omega$$

$$\dot{\Omega} = f(\Omega) \qquad A = \begin{bmatrix} \frac{\partial f_1}{\partial \Omega_1} & \frac{\partial f_1}{\partial \Omega_2} & \frac{\partial f_1}{\partial \Omega_3} \\ \frac{\partial f_2}{\partial \Omega_1} & \frac{\partial f_2}{\partial \Omega_2} & \frac{\partial f_2}{\partial \Omega_3} \\ \frac{\partial f_3}{\partial \Omega_1} & \frac{\partial f_3}{\partial \Omega_2} & \frac{\partial f_3}{\partial \Omega_3} \end{bmatrix}$$

So we have
$$\frac{\partial f_1}{\partial \Omega_1} = 0. \quad \frac{\partial f_1}{\partial \Omega_2} = \frac{I_2 - I_3}{I_1} \Omega_3 \quad \frac{\partial f_1}{\partial \Omega_3} = \frac{I_2 - I_3}{I_1} \Omega_2$$

So
$$A = \begin{bmatrix} 0 & \frac{I_2 - I_3}{I_1} \Omega_3 & \frac{I_2 - I_3}{I_1} \Omega_2 \\ \frac{I_3 - I_1}{I_2} \Omega_3 & 0 & \frac{I_3 - I_1}{I_2} \Omega_1 \\ \frac{I_1 - I_2}{I_3} \Omega_2 & \frac{I_1 - I_2}{I_3} \Omega_1 & 0 \end{bmatrix}$$

We can know the equilibria is when the rigid body spins round one of its main axis.

Suppose the rigid body rotates round axis 1. So $\Omega_2 = \Omega_3 = 0$. $\Omega_1 =$ constant

$$A^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{I_3 - I_1}{I_2} \Omega_1 \\ 0 & \frac{I_1 - I_2}{I_3} \Omega_1 & 0 \end{bmatrix} \quad \text{Similarly} \quad A^2 = \begin{bmatrix} 0 & 0 & \frac{I_2 - I_3}{I_1} \Omega_2 \\ 0 & 0 & 0 \\ \frac{I_1 - I_2}{I_3} \Omega_2 & 0 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0 & \frac{I_2 - I_3}{I_1} \Omega_3 & 0 \\ \frac{I_3 - I_1}{I_2} \Omega_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For stability, we take A^1 as example.

$$\det(A^1 - \lambda I) = -\lambda^3 + \lambda \frac{I_3 - I_1}{I_2} \frac{I_1 - I_2}{I_3} \Omega_1^2 \quad \Rightarrow \lambda_1 = 0. \quad \lambda_2 = \sqrt{\frac{I_3 - I_1}{I_2} \frac{I_1 - I_2}{I_3}} \Omega_1 \quad \lambda_3 = -\sqrt{\left|\frac{I_3 - I_1}{I_2} \frac{I_1 - I_2}{I_3}\right|} \Omega_1$$

If
$$\begin{cases} ① (I_3 - I_1)(I_1 - I_2) > 0. & \text{Not Stable as } \lambda_2 > 0. \\ ② (I_3 - I_1)(I_1 - I_2) < 0 & \text{Not sure whether stable.} \end{cases}$$

Situation is similar for I_2 and I_3

3. Solve the symmetric rigid body equations, i.e. when I1 = I2.

$$\begin{cases} I_1 \dot{\Omega}_1 = (I_2 - I_3)\Omega_2 \Omega_3 \\ I_2 \dot{\Omega}_2 = (I_3 - I_1)\Omega_3 \Omega_1 \\ I_3 \dot{\Omega}_3 = (I_1 - I_2)\Omega_1 \Omega_2 . \end{cases}$$

For $I_1 = I_2$. we know $\dot{\Omega}_3 = 0$ So Ω_3 is a constant

$$\begin{cases} I_1 \dot{\Omega}_1 = (I_1 - I_3)\Omega_2 \Omega_3 \\ I_1 \dot{\Omega}_2 = (I_3 - I_1)\Omega_1 \Omega_3 \end{cases}$$

As $I_2 \ddot{\Omega}_2 = (I_3 - I_1)\Omega_3 \dot{\Omega}_1 = (I_3 - I_1)\Omega_3 \cdot \frac{1}{I_1}(I_1 - I_3)\Omega_2 \Omega_3$

$\Rightarrow \ddot{\Omega}_2 + \frac{(I_1 - I_3)^2 \Omega_3^2}{I_1 I_2} \Omega_2 = 0.$ (which is the simple harmonic viberation)

So
$$\begin{cases} \Omega_1 = A \cos\left(\frac{|I_1 - I_3|\Omega_3}{I_1} t + \phi\right). \\ \Omega_2 = A \cos\left(\frac{|I_1 - I_3|\Omega_3}{I_1} t + \phi\right). \\ \Omega_3 = \text{const} \end{cases}$$

which is the solution.

4. Prove the hat map in a Lie algebra homomorphism.

Hat map $f: \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ So $\vec{v} = (v_1, v_2, v_3) \to \hat{v} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$

① Linearity: for $\hat{v}$ and $\hat{w}$

$$a\hat{v} + b\hat{w} = a\begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -av_3-bw_3 & av_2+bw_2 \\ av_3+bw_3 & 0 & -av_1-bw_1 \\ -av_2-bw_2 & av_1+bw_1 & 0 \end{pmatrix} = \widehat{av+bw}$$

② Lie-Bracket: define $[\hat{v}, \hat{w}] = \hat{v}\hat{w} - \hat{w}\hat{v}$ want to show $f[v,w] = [f(v), f(w)] = [\hat{v}, \hat{w}]$

for vectors $[\vec{v}, \vec{w}] = \vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2w_3 - v_3w_2,\ v_3w_1 - v_1w_3,\ v_1w_2 - v_2w_1)$

$$f([\vec{v}, \vec{w}]) = \begin{pmatrix} 0 & -v_1w_2+v_2w_1 & v_3w_1-v_1w_3 \\ v_1w_2-v_2w_1 & 0 & -v_2w_3+v_3w_2 \\ -v_3w_1+v_1w_3 & v_2w_3-v_3w_2 & 0 \end{pmatrix}$$

Then for $[\hat{v}, \hat{w}] = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} -v_3w_3 - v_2w_2 & v_2w_1 & v_3w_1 \\ v_1w_2 & -v_3w_3-v_1w_1 & v_3w_2 \\ v_1w_3 & v_2w_3 & -w_2v_2-w_1v_1 \end{pmatrix} - \begin{pmatrix} -w_3v_3 - w_2v_2 & v_1w_2 & v_1w_3 \\ v_2w_1 & -v_3w_3-v_1w_1 & v_2w_3 \\ v_3w_1 & v_3w_2 & -w_2v_2-v_1w_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -v_1w_2+v_2w_1 & v_3w_1-v_1w_3 \\ v_1w_2-v_2w_1 & 0 & -v_2w_3+v_3w_2 \\ -v_3w_1+v_1w_3 & v_2w_3-v_3w_2 & 0 \end{pmatrix}$$

So $[f(\vec{v}), f(\vec{w})] = f([\vec{v}, \vec{w}])$. The hat map in Lie algebra is homomorphism.

5. Write down the structure constants for the Lie algebra of vectors on R^3 with the cross product and the Lie algebra of skew symmetric 3 by 3 matrices with the matrix bracket.

Structual constants: $[e_i, e_j] = \sum_{k=1}^{n} C_{ij}^{k} e_k.$

① For vectors on $\mathbb{R}^3$ e_1, e_2 and e_3

$[e_1, e_2] = e_1 \times e_2 = e_3 \qquad [e_2, e_3] = e_1 \qquad [e_3, e_1] = e_2$

So $C^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad C^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad C^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

② For Skew Symmetric matrix on $\mathbb{R}^3$

$e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad e_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$[e_3, e_2] = e_3 e_2 - e_2 e_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = -e_3$

$\Rightarrow [e_3, e_2] = -e_1. \quad [e_2, e_3] = e_1 \quad$ Similarly $\quad [e_1, e_2] = e_3 \quad [e_3, e_1] = e_2.$

So $C^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad C^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad C^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6. Compute the Lie bracket for the control vector fields of the Heisenberg system discussed in class. Compute the distance travelled along the z-axis when a you traverse a unit square in the positive quadrant of $x-y$-plane with corner at the origin (in the anti-clockwise direction).

Heisenburg System: $\begin{cases} \dot{x} = u \\ \dot{y} = v \\ \dot{z} = yu - xv \end{cases}$ So the dynamics $\begin{cases} X_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \\ X_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z} \end{cases}$

So the Lie-bracket for the system: $[X_1, X_2] = X_1X_2 - X_2X_1 = \left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right) - \left(\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right) = -2\frac{\partial}{\partial z}$

① From $(0,0)$ to $(1,0)$. $x' = u = 1$. $y' = v = 0$. $z' = yu - xv = 0$

② From $(1,0)$ to $(1,1)$ $x' = u = 0$ $y' = v = 1$ $z' = yu - xv = -x$ $\Rightarrow \Delta z = \int_0^1 z' = -\frac{1}{2}$

③ From $(1,1)$ to $(0,1)$ $x' = u = 1$. $y' = v = 0$ $z' = yu - xv = y$ $\Rightarrow \Delta z = \int_1^0 z' = -\frac{1}{2}$

④ From $(0,1)$ to $(0,0)$. $x' = u = 0$ $y' = v = 1$ $z' = 0$.

Distance travelled: -1

7.

2.2-1. Using the submersion criterion, show that the level set $x_1^2 + \cdots + x_n^2 - 1 = 0$ is a differentiable manifold of dimension $n-1$.

As $(x_1, x_2, \cdots x_n) \in \mathbb{R}^n$.

For the level set. Set $f(x_1 \cdots x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 - 1 = 0$.

So $\text{grad } f = (2x_1, 2x_2, \cdots 2x_n)$.

As $\sum x_i^2 = 1$. So $\text{grad } f \neq 0$. will not vanish.

As there are only 1 independent constraint. So Manifold M will be $(n-1)$-dimension

8.

◇ **2.2-2.** Show that the set $\{(x,y) \mid x^2(x+1) - y^2 = 0\}$ in $\mathbb{R}^2$ is *not* a differentiable manifold.

for $x^2(x+1) - y^2 = 0$.

$\{(x,y) \mid x^2(x+1) - y^2 = 0\} = \{(x,y) \mid y = x\sqrt{x+1}\} \cup \{(x,y) \mid y = -x\sqrt{x+1}\}$.

For $f(x) = x\sqrt{x+1}$ the derivative at $x=0$ is $f'(0) = \left(\sqrt{x+1} + \frac{x}{2(x+1)^{\frac{1}{2}}}\right)\Big|_{x=0} = 1$

$g(x) = -x\sqrt{x+1}$. $g'(0) = -1$.

For set $\{(x,y) \mid y = x\sqrt{x+1}\}$, the tangent space at $(0,0)$ is $x - y = 0$.

For set $\{(x,y) \mid y = -x\sqrt{x+1}\}$. $x + y = 0$.

It is impossible for $\{(x,y) \mid x^2(x+1) - y^2 = 0\}$ to have 2 different tangent space at point $(0,0)$. if it is a manifold.

◇ **2.2-3.** Let $\mathcal{S} = \{X \in \mathrm{GL}(n, \mathbb{R}) \mid X^T A X = A\}$, as in the text. Note that the $n \times n$ identity matrix I is in $\mathcal{S}$, and show that for any pair of matrices $V_1, V_2 \in T_I\mathcal{S}$ we have $V_1 V_2 - V_2 V_1 \in T_I\mathcal{S}$.

For a vector $V \in T_I S$. $I + tV \in S$ for a small t.

$$\text{So } (I+tV)^T A (I+tV) = A \Rightarrow A + tV^T A + tAV + t^2 V^T A V = A.$$

$$\Rightarrow V^T A + AV = 0.$$

$$\text{So } T_I S = \{ V \mid V^T A + AV = 0 \}$$

for $V_1, V_2 \in T_I S$. $V_1^T A + AV_1 = 0$ $V_2^T A + AV_2 = 0$. $\Rightarrow V_1^T A = -AV_1$. $V_2^T A = -AV_2$

$$(V_1V_2 - V_2V_1)^T A + A(V_1V_2 - V_2V_1) = V_2^T V_1^T A - V_1^T V_2^T A + AV_1V_2 - AV_2V_1$$

$$= -V_2^T AV_1 + V_1^T AV_2 + AV_1V_2 - AV_2V_1 = AV_2V_1 - AV_1V_2 + AV_1V_2 - AV_2V_1 = 0.$$

$$\text{So } (V_1V_2 - V_2V_1) \in \{V \mid V^T A + AV = 0\} = T_I S.$$