L12 Linear System

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嵌入式流形 (Embeded Manifolds) $f_1=0, \cdots f_{n-k}=0$ 并且 independent

TM_x - Orthogonal Complement of grad f_i $TM = \bigcup T_xM$ $\pi: T_xM$ $x\in M$.

Lagrangian Mechanics / Gradient Flow.

-需要一个 Reimanian 度量.

$\langle , \rangle$ 是一个 T_xM 对每个 x.

$g(,)$ $g(v,v) = \sum_{i,j=1}^{n} g_{ij}(q)\dot{q}_i\dot{q}_j$ 在 $M\subset Q$ 上.

给定一个函数 f 和一个度量 $\langle , \rangle = g(,)$. 定义梯度流 $df(v) = dfv = \langle \text{grad } f, v\rangle$

$\because df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots$ $\therefore df(v) = \left(v_1\frac{\partial}{\partial x_1}, v_2\frac{\partial}{\partial v_2}, \cdots v_n\frac{\partial}{\partial x_n}\right)$

例: $M=\mathbb{R}$. Standard Metric $\langle v_1, v_2\rangle = \sum_i v_1^i v_2^i$

Set $\langle v_1, v_2\rangle_a = \sum_i v_1^i a_i v_2^i = v_1^T \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} v_2$.

而 grad f 是通过度量定义的 $\frac{\partial f}{\partial x}\cdot v = (\text{grad} f)^T A v$

因此 grad $f = \left(\frac{1}{a_1}\frac{\partial f}{\partial x_1}, \frac{1}{a_2}\frac{\partial f}{\partial x_2}, \cdots \frac{1}{a_n}\frac{\partial f}{\partial x_n}\right)$

对于任何可逆矩阵 G. 有 grad $f = G^{-1}\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots \frac{\partial f}{\partial x_n}\right)$

例: $T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}(q)\dot{q}^i\dot{q}^j$ 被用来定义动能, G 可以等于 $\begin{bmatrix} m_1 & & & 0 \\ & m_2 & & \\ & & \ddots & \\ 0 & & & m_n \end{bmatrix}$

$\Rightarrow$ Lagrange $\sim$ geodesic flow

Locally $\underbrace{\sum_{i=1}^{n} g_{rj}}_{\text{类似于质量矩阵}}\ddot{q}_j + \sum_{\ell m}\begin{bmatrix} \ell m \\ r \end{bmatrix}\dot{q}^\ell\dot{q}^m$ (Lagrange)

Hamotonian System

Linear Hamotonian System

$H = \underbrace{\frac{1}{2} x^T S x}_{二次形}$ $\quad S$: fixed symmetric matrix.

$\underset{\sim}{x}$: 2n维. Flow: $\dot{x} = J \nabla H = J S x$ $\quad$ 令 $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

$\underset{\sim}{x} = (q, p)$. $\sim$ $\dot{q} = \frac{\partial H}{\partial p}$. $\quad \dot{p} = -\frac{\partial H}{\partial p}$.

线性系统: $\dot{x} = Ax$. 其中 $J = JS$

其有 JS 满足 $A^T J + JA = 0$.

Check: $(JS)^T J + J(JS) = S J^T J + J^2 S$

$\because J$ 具有性质: $J^T = -J$. $J^2 = -I$. $\quad = S(-J)J + (-I)S = S - S = 0$

定理: 若 A 是一个 Halmotonian Matrix. 若 λ 是一个特征值. 则 $-\lambda$, $\bar{\lambda}$, $-\bar{\lambda}$ 亦为特征值.

pf: 看一看 $\det(A - \lambda I) = 0$. 它是多项式

定义: A是 Hamoltonian 的. said be elements of the symbotic Lie-Algebra $Sp(2n)$

$[A_1, A_2]$ $\quad A_1, A_2$ 均为 Hamoltonian. $\quad A_i J + J A_i = 0$

Symplectic Matrices

Hamotonian 矩阵是 inf version of Sympletic matrix.

$Sp(2n, \mathbb{R}) = \{ \underbrace{A \in GL(2, \mathbb{R})}_{\text{可逆 } 2n\times 2n \text{ 矩阵}} \}$，使得 $A^T J A = J$，并且 $\det A = 1 \}$

定理：若 λ 是一个特征值，$A \in Sp(2n, \mathbb{R})$，则有 $\frac{1}{\lambda}$，$\bar{\lambda}$，$\frac{1}{\bar{\lambda}}$ 均为特征值

引理：$A \in Sp(2n)$ iff $e^A \in Sp(2n)$

$A^T J A = J$　如果 $A = e^{Bt}$ 则 $(e^{Bt})^T J (e^{Bt}) = J$.

$\frac{d}{dt}\Big|_{t=0} \left[(e^{Bt})^T J (e^{Bt}) \right] = 0 \Rightarrow B^T J + J B = 0.$

$\dot{x} = Ax$. 若 A 是 Hamiltonian. 解 e^{At} 为 Sympletic $\Rightarrow$ Sympletic Flow.