L6 Hamotonian II

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$(q^1, \cdots q^n, \dot{q}^1, \cdots \dot{q}^n)$

$(q^1, \cdots q^n, p_1, \cdots p_n)$

$p_i = \frac{\partial L}{\partial \dot{q}^i}$ Legendre Transformation

对于 $f(x)$ $df = \frac{\partial f}{\partial x} dx$. (下面是为什么勒让德变换)

令 $df = p\,dx$. $d(xp - f) = dx\, p + x dp - \frac{\partial f}{\partial x} dx = x\,dp$.

定义 $f^* = xp - f$. $\therefore \frac{df^*}{dp} = x$.

Variational Non-holonomic & Non-holonomic

$L_A = L + \lambda_1 (\quad) + \lambda_2 (\quad) + \cdots$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} + \sum_{j=1}^{n-m} \dot{\lambda}_j a_j$

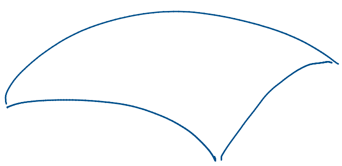
如果约束是 Holonomic 的. $a_{ji} = \frac{\partial \phi^j}{\partial q^i}$ 那么两者一样.

例: 圆上的粒子.

$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$ 限制在 $x^2 + y^2 = 1$ 上.

$L_A = \frac{1}{2} m (\dot{x} + \dot{y}) + \lambda (\dot{x} x + \dot{y} y) \Rightarrow \begin{cases} m\ddot{x} = \lambda x \\ m\ddot{y} = \lambda y. \end{cases}$

$T = \frac{1}{2} \sum_{ji} g_{ij}(q)\, \dot{q}_i \dot{q}_j$ $\mathbb{R}^n$: Riemanian Mechanism.



- dynamics due to toke.
- geometric flow

"Least" Action

- Modified Hamilton Principal

$\delta \int_{t_1}^{t_2} L\, dt$ Hamotonian: $\delta \int_{t_1}^{t_2} \left(\Sigma p_i \dot{q}_i - H(q,p) \right) dt$

$$\int_{t_1}^{t_2} \left(\sum_i \delta p_i \left(\dot{q} - \frac{\partial H}{\partial p_i} \right) - \delta q^i \left(\dot{p} + \frac{\partial H}{\partial q_i} \right) \right) dt = 0.$$

所以 $\dot{q} = \frac{\partial H}{\partial p_i}$ $\dot{p} = -\frac{\partial H}{\partial q_i}$ (这里 p 和 q 是独立的).

历史上的 Least Action.

$A = \int_{t_1}^{t_2} \sum_i p_i \dot{q}_i \, dt$ L_A (critical) $= \Delta \int_{t_1}^{t_2} \sum_i p_i q_i \, dt$

- 允许改变在终点的时间, 但约束能量

如果 $L = T - V$ 而 $V = 0$. $T = \frac{1}{2} \sum_{ij} g_{ij} \dot{q}^i \dot{q}^j$ $\Sigma p_i \dot{q}_i = 2T$

Least Action $\Delta \int T\, dt$ $\Delta \int dT \Rightarrow$ 用最短的时间   (从费马原理出发的)

Hamilton: $\dot{q}^i = \frac{\partial H}{\partial p_i}$ $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ $H(q^i, p^i)$

定义: 对于两个任意的光滑的函数 $F(q^i, p_i)$, $G(q^i, p_i)$

我们定义泊松括号 $\sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p^i} - \frac{\partial F}{\partial p^i} \frac{\partial G}{\partial q^i} \right) = \{F, G\}$

Note: ① 泊松括号是双线性的 $\{F, G+H\} = \{F, G\} + \{F, H\}$

② (Skew) $\{F, G\} = -\{G, F\}$.

③ (Jacobi identity) $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$.

我们发现 $\begin{cases} \dot{q}^i = \{q^i, H\} = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q^i} \end{cases}$

对于任何的守恒量. 我们可以用 $\{G, H\}$ 来检测 G 是否守恒

Rigid Body

Euler Equations：转动惯量 I_1, I_2, I_3. 角动量 $\Omega_1, \Omega_2, \Omega_3$.　令 $I_1 > I_2 > I_3$

$$\begin{cases} I_1 \dot{\Omega}_1 = (I_2 - I_3)\,\Omega_2 \Omega_3 \\ I_2 \dot{\Omega}_2 = (I_3 - I_1)\,\Omega_3 \Omega_1 \\ I_3 \dot{\Omega}_3 = (I_1 - I_2)\,\Omega_1 \Omega_2 . \end{cases}$$

Quadratic.

Energy 和 Momentum 都守恒.

$H = \frac{1}{2} \sum I_i \Omega_i^2$　　$M = \sum I_i^2 \Omega_i^2$　　$\dot{H} = \dot{M} = 0$ along flow

$\pi_i = I_i \Omega_i$, $i = 1, 2, 3$.　Rewrite $\dot{\pi}_1 = \dfrac{(I_2 - I_3)\,\pi_2 \pi_3}{I_2 I_3}$　$\Rightarrow$　$\dot{\underset{\sim}{\pi}} = \underset{\sim}{\pi} \times \underset{\sim}{\Omega}$

Hamitonian with respect to bracket

$$\{F, G\}(\underset{\sim}{\pi}) = -\underset{\sim}{\pi} \cdot (\nabla F \times \nabla G).$$

Euler equations　$\dot{F} = \{F, H\}_{\underset{\sim}{L\text{-}P}}$

Lie-Poisson

对于 Momentum $\sum_i \pi_i^2 = \underset{\sim}{C}(\pi)$

Casimir　$\Rightarrow$ 满足括号的性质，

$\dot{C} = \{C, H\} = 0$　因为 $C = F$.　$\nabla C = \nabla F = \underset{\sim}{\pi}$　所以垂直. 所以为零.