L18 Hamitonian Vector Field

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$(M, \omega) \to \{F, G\} = \omega(X_H, X_G)$.   Poisson Bracket：函数的李代数.

$\{FG, H\} = \{F, H\} G + F\{G, H\}$

Symplectic $\{F, G\} = \omega(X_F, X_G) = dF \cdot X_G$

$\{FG, H\} = X_H[FG] = F\{G, H\} + G\{F, H\}$

性质：令 P是一个 Poisson $H \in \mathcal{F}(P)$. 则 $\exists!$ 向量场 X_H. $X_H[G] = \{G, H\}$ $\forall G \in \mathcal{F}(P)$.

证明：这是 $\mathcal{F}(P)$上面 Derivation property 的后果

所有$\mathcal{F}(P)$上面的 deviation 可以被一个 vector field 代替

Fix $H \to G \to \{G, H\}$ 是一个 deviation $\Rightarrow$ Can define X_H uniques by $X_H[G] = \{G, H\}$

Poisson Tensor： $\{F, G\} = \sum B^{IJ}(z) \frac{\partial F}{\partial z^I} \frac{\partial G}{\partial z^J}$ $F(z)$. $G(z)$ on P.

B^{IJ} - Poisson Tensor 有 $\{z^I, z^J\} = B^{IJ}(z)$.

Jacobian implied by special cases $\{z^I, z^J, z^K\} + \cdots = 0$

$$B^{LI} \frac{\partial B^{JK}}{\partial z^L} + B^{LJ} \frac{\partial B^{KI}}{\partial z^L} + B^{LK} \frac{\partial B^{IJ}}{\partial z^L} = 0 \quad \forall k.$$

$X_H[F] = \{F, H\}$.

$$X_H^I \frac{\partial F}{\partial z^I} = B^{JK} \frac{\partial F}{\partial z^J} \frac{\partial H}{\partial z^K}$$

$$\Rightarrow X_H^I = B^{IJ} \frac{\partial H}{\partial z^J} \qquad \text{Canonical Case } B^{IJ} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

$$\Omega(X_H, \xi) = dH \cdot \xi$$

$$\Omega_{IJ} X_H^I \xi^J = \frac{\partial H}{\partial z^J} \xi^J \Rightarrow \Omega_{IJ} X_H^I = \frac{\partial H}{\partial z^J}$$

如果 Ω^{IJ} 的逆是 Ω^{JI}. $X_H^I = \Omega^{JI} \frac{\partial H}{\partial z^J}$ $\Rightarrow \boxed{B^{IJ} = -\Omega^{IJ}}$ 因为 Skew

$$\{F, H\} = B(dF, dH) = \Omega(X_F, X_H)$$

Poisson Tensor for Rigid Body.

$$\dot{\Pi} = \Pi \times w = \underset{\sim}{\hat{\Pi}} \cdot w$$

Skew Matrix

所以 $B = \hat{\Pi} = \begin{bmatrix} 0 & -\Pi_3 & \Pi_2 \\ \Pi_3 & 0 & -\Pi_1 \\ -\Pi_2 & \Pi_1 & 0 \end{bmatrix}$

如果 B 可逆: Sympletic

如果 B 不可逆: 存在 Casimir. ∃ 函数 C, $\{C, H\} = 0$ 对于所有的 H

例如对于 Rigid Body $C = \frac{1}{2}\Pi^T\Pi$.

Sympletic form: Casimir 是一个平凡的守恒量. non-trivial 守恒量是那些只对 H 有 $\{F, H\} = 0$ 的.

e.g. non-trivial conservation law

Toda Lattice. $\begin{bmatrix} b & a_1 & 0 & & \\ a_1 & b & a_2 & & \\ 0 & a_2 & b & & \\ \vdots & & & \ddots & \\ & & & & b \end{bmatrix}$ $B = \begin{bmatrix} 0 & a_1 & & \\ -a_1 & 0 & \ddots & \\ & \ddots & & a_{i-1} \\ & & a_{i-1} & 0 \end{bmatrix}$ $\dot{L} = [B, L]$

Sympletic form. $\sum \frac{da_i \wedge db_i}{a_i} = \sum d(\ln a_i) \wedge db_i$ $a_i = \exp \frac{1}{2}(x_i - x_{i+1})$

$$H = \sum_{i=1}^{n-1} \frac{a_i^2 + b_i^2}{2} = \frac{1}{2}\operatorname{Tr} L^2$$

Also $H_k = \operatorname{Tr} L^k$ 对 $\forall k$ 也是对于 flow 上守恒的 $\{X_H, X_{H_k}\} = 0 \Leftrightarrow dH_k \cdot X_k = 0$.

性质: $H \to X_H$ 是一个 Lie-algebra anti-homomorphism. i.e. $[X_H, X_K] = -X_{\{H,K\}}$

用 Jacobian: $[X_H, X_K][F] = X_H[X_K[F]] - X_K[X_H[F]] = \{\{F,K\},H\} - \{\{F,H\},K\}$

$$= \{\{F,K\},H\} - \{\{F,H\},K\} = -\{F,\{H,K\}\} = -X_{\{H,K\}}[F]$$

Control Systems

e.g. $\dot{x} = \sum_{i=1}^{m} u_i(t) f_i(x(t))$ $x(t) \in \mathbb{R}^n$ — Kinematic Control Sys.

$v_i(t)$: Smooth over time. f_i: smooth

Question: 当我们找到一个平滑的 p 维矢即阵 subset M. 在 $\mathbb{R}^n$ 上. f_i spans tangent space of M

i.e. 可以在 M 上任何位置而非 M 之外移动.

假设我们让 $\dot{x} = f_1(x)$. 然后 $f_2(x)$. $-f_1(x)$. $-f_2(x)$.

Formally $\underbrace{(\exp -tf_2)}_{\text{Locally 解是 } e^{-tf_2}}(\exp -tf_1)(\exp -tf_2)(\exp tf_1)(x_0)$.

Locally 解是 e^{-tf_2} 假设 f_1. f_2 是一个标量.

Locally 这是 $x_0 + \frac{1}{2} x_0^2 [f_1, f_2] + O(x_0^3)$.

Recall $\mathcal{L}_X[f] = X[f] = df \cdot X = X^i \frac{\partial f}{\partial x_i}$

Lie-bracket $[X, Y]^j = X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial y^i}$