L8 Rigid Body Mechenics

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Rigid Body Lagrangian $L(\Omega) = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)$

$$\frac{d}{dt}\underbrace{\frac{\partial L}{\partial \Omega}}_{M} = \frac{\partial L}{\partial \Omega} \times \Omega$$

$\delta\int_a^b L\,dt = 0$ $\delta\Omega = \dot{\Sigma} + \Omega\times\Sigma$, Σ任意, 在 endpoint vanish

Variational case 和实际情况是一样的

$\underbrace{I\dot{\Omega}}_{\Pi} = I\Omega\times\Omega$

$\dot{\Pi} = \Pi\times\Omega$

Map: $(\Omega_1, \Omega_2, \Omega_3) \longrightarrow \begin{bmatrix} 0 & \Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$

$\underset{\sim}{\Omega} \longrightarrow \hat{\Omega}$

$[\underset{\sim}{V_1}, \underset{\sim}{V_2}] = [\hat{V_1}, \hat{V_2}]$

因为 $\underset{\sim}{V_1}\cdot\underset{\sim}{V_2}$ 有点积运算. 令 $V_1\cdot V_2 = -\frac{1}{2}\operatorname{Tr}(\hat{V_1}\hat{V_2})$

因此刚体有 Lagrangian $\frac{1}{2}\Pi\Omega = -\frac{1}{4}\operatorname{Tr}\hat{M}\hat{\Omega}$

同样的有 $\underset{\sim}{\hat{\Omega}}\underset{\sim}{V} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -\Omega_3 V_2 + \Omega_2 V_3 \\ \vdots \\ \vdots \end{bmatrix}$

相当于叉乘.

Rigid Body. $\dot{\underset{\sim}{\Pi}} = \underset{\sim}{\Pi}\times\Omega$

$\dot{\hat{\Pi}} = [\hat{\Pi}, \hat{\Omega}]$

$\underset{\sim}{\Pi} = (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3)$.

我们可以把 Rigid Body 写成 $\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}[\hat{\Omega}] + [\hat{\Omega}]\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$

n维刚体. $\dot{M} = [M, \Omega]$ $\Omega \in SO(n)$. - skew symmetric matrix

$M = J(\Omega) = I\Omega + \Omega I$

$L = E = T = -\frac{1}{4}\operatorname{Tr} M\Omega$

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$\mathrm{Tr} M$, $\mathrm{Tr} M^2$, $\cdots$ 都是守恒量.

eigenvalue 在 M的 flow 上保持不变.

Lie-Algebra Picture

$[\xi, \eta]$ - Lie Bracket

e.g. $SO(n)$.

Basis of e_a. $[e_a, e_b] = \sum_{d=1}^{r} c_{ab}^{d} e_d$.

$$\frac{\partial \hat{L}}{\partial \Omega} = \left[\frac{\partial \hat{L}}{\partial \Omega}, \hat{\Omega} \right]. \qquad \Omega = \sum_{i=1}^{3} \xi^i e_i.$$

$$\frac{d}{dt} \frac{\partial L}{\partial \xi^d} = C_{ad}^{b} \frac{\partial L}{\partial \xi^b} \xi^a$$ 约定：重复的下标都要求和

例: Toda Lattice (线上的粒子).

$$H = \sum_{k=1}^{n} \frac{1}{2} p_k^2 + \sum_{k=1}^{n-1} e^{(x_k - x_{k+1})}$$

最近邻居势.

Particular Case: $e^{x_0 - x_1} = 0$. $e^{x_n - x_{n+1}} = 0$. $x_0 = -\infty$. $x_{n+1} = +\infty$.

$$\dot{x}_k = \frac{\partial H}{\partial p_k} = p_k. \qquad \dot{p}_k = -\frac{\partial H}{\partial x_k} = e^{x_{k-1} - x_k} - e^{x_k - x_{k+1}}$$

$$b_k = -\frac{1}{2} p_k$$
$$a_k = \frac{1}{2} e^{(x_k - x_{k+1})/2}$$

$$\begin{cases} \dot{a}_k = a_k (b_{k+1} - b_k) & k = 1, 2, \cdots n-1 \\ \dot{b}_k = 2 (a_k^2 - a_{k-1}^2) & k = 1, 2, \cdots n \end{cases}$$

写为 $\dot{L} = [B, L]$.

$$L = \begin{bmatrix} b_1 & a_1 & & 0 \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 & a_1 & & \\ a_1 & 0 & \ddots & \\ & \ddots & \ddots & a_{n-1} \end{bmatrix}$$

$$L = \begin{bmatrix} b_1 & a_1 & & 0 \\ a_1 & b_2 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ 0 & & a_{n-1} & b_n \end{bmatrix} \qquad B = \begin{bmatrix} 0 & a_1 & & \\ a_1 & 0 & \ddots & \\ & \ddots & \ddots & a_{n-1} \\ & & a_{n-1} & 0 \end{bmatrix}$$

Symmetric　　　　Skew.

Also check:

$Tr\,L$, $Tr\,L^2$, $\cdots$ $Tr\,L^{n-1}$ 均为守恒量.

$$Ham = \frac{1}{2} Tr\,L^2$$

Rewrite as $\dot{L} = [B, L]$.

$$\frac{d}{dt} \frac{Tr\,L^k}{k} = Tr\,L^{k-1}\dot{L} \quad \text{(因为 Tr 可以有交换律)}.$$

$$= Tr\,L^{k-1}[B, L] = Tr\,[B, L]\,L^{k-1} = 0.$$

因为对 A, B, C. $Tr\,AB = Tr\,BA$.

$$Tr\,A[B,C] = Tr\,A[BC - CB] = 0.$$

Central Force Problem

$$L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r).$$

$H = T + V$ 是守恒的.

而 L 和 θ 无关. 因此, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 = mr^2\dot{\theta}$ - 角动量

令 $\ell = mr^2\dot{\theta}$　　$f(r) = -\frac{\partial V}{\partial r}$.

$$E = \frac{1}{2} m(\dot{r})^2 + \frac{1}{2}\frac{\ell^2}{mr^2} + V = \text{const.}$$

$$\Rightarrow \dot{r} = \sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}$$

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}} \Rightarrow t = \int \frac{dr}{\sqrt{\quad}}$$

Rigid Body Kinematics

刚体的转动 用一个旋转矩阵 $R=R(t)\in SO(3)$

3×3 正交阵.且行列式为1.

$x=RX$. x是X经过R旋转所得.

$\dot{x}=\dot{R}X$

因为X和时间无关. $\dot{x}=\dot{R}X=\dot{R}R^{-1}x$. $\longrightarrow$ Skew.

为什么 $\dot{R}R^{-1}$ Skew?

$\frac{d}{dt}(RR^{-1})=0.$ $\Rightarrow \dot{R}R^{-1}+R(\dot{R^{-1}})=(\dot{R}R^{-1}+R(-R^{-1}\dot{R}R^{-1}))$

$\| $

$\dot{R}R^{-1}+(\dot{R}R^{-1})^{T}=0.$ 所以是 Skew 的.

次回预告: $\dot{R}=R\hat{\Omega}$