L16 Tensor

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$\varphi: M \to N$. α: 在 N上的一个 k-form

Pull back: $\varphi^*\alpha$ — 一个在M上面的 k-form

$(\varphi^*\alpha)(x)(v_1, \cdots v_k) = \alpha(\varphi(x))(T_x\varphi_{v_1}, \cdots T_x\varphi_k)$

Note: $\varphi^*(\alpha\wedge\beta) = \varphi^*\alpha \wedge \varphi^*\beta$

Push forward: $\varphi_* = (\varphi^{-1})^*$

内积 Interior Produt.

α是一个向量场 X 上面的 K form

$i_X\alpha = i(X)\alpha$ 被定义为 $i_X\alpha(x)(v_2, \cdots v_k) = \alpha(x)(X(x), v_2, \cdots v_k)$

例如: Hamitonian 向量场 $w(X_H, v) = dH \cdot v \iff i_{X_H} \underset{\text{2-form}}{\underset{\sim}{w}} = \underset{\text{1-form}}{dH}$

Exterior derivative

$\Rightarrow$ Generalize df. $\rightarrow$ product rule

Note: α: k-form. $d\alpha$: $(k+1)$-form

性质: $\exists!$ mapping d from k-form on M $\rightarrow$ $k+1$ forms on M. s,t.

(i) $f \in C^\infty(m)$ df

(ii) $d\alpha$ 是 α 上面线性的 $d(a\alpha + b\beta) = a\,d\alpha + b\,d\beta$

(iii) d 满足 product rule $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^k\alpha\wedge d\beta$

(iv) $d(d\alpha) = 0$.

(v) Dislocal: $d(\alpha|_V) = (d\alpha)\big|_V$

In coordinates

$\alpha = \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_k} \quad dx^{i_1}\wedge dx^{i_2} \cdots \wedge dx^{i_k}$

$$d\alpha = \frac{\partial \alpha_{i_1} \cdots}{\partial x^j}$$

例: 1, 2 form, α, β — 1-form

$$\alpha \wedge \beta = 2!\ A(\alpha \otimes \beta)$$

$$\alpha \wedge \beta (v_1, v_2) = 2! \ \frac{1}{2!} \left(1 \cdot \alpha(v_1)\, \beta(v_2) + (-1)\, \alpha(v_2)\, \beta(v_1) \right)$$

$$= \alpha(v_1)\, \beta(v_2) - \alpha(v_2)\, \beta(v_1)$$

在 $\mathbb{R}^2$ 上. e.g. $\alpha = dx$. $\beta = dy$ $\alpha \wedge \beta = dx \wedge dy$ $\Rightarrow$ $\mathbb{R}^2$ 上的 Sympletic Form

Exercise: e_i 是 $\mathbb{R}^3$ 上的基. e^i 是 dual space 的基

令 $\alpha = \alpha_1 e^1 + \alpha_2 e^2 + \alpha_3 e^3$, $\beta = \beta_1 e^1 + \beta_2 e^2 + \beta_3 e^3$

Check $\alpha \wedge \beta = (\alpha_2 \beta_3 - \alpha_3 \beta_2)\, e^2 \wedge e^3 + (\alpha_3 \beta_1 - \alpha_1 \beta_3)\, e^3 \wedge e^1 + (\alpha_1 \beta_2 - \alpha_2 \beta_1)\, e^1 \wedge e^2$

$\Rightarrow$ 叉积.

General Tensor

E, V Space 定义一个多线性的映射.

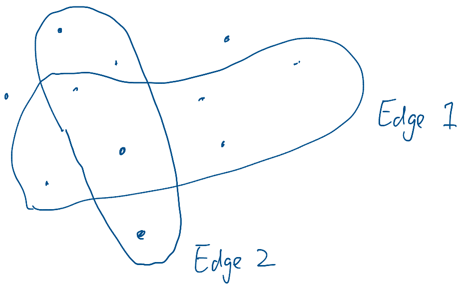
$$T^r_s(E) = L^{(r+s)}(\underbrace{E^*, \cdots E^*}_{r\text{个 covector}}, \underbrace{E, \cdots E}_{s\text{个 vector}})$$

r个 form

Contravariant order s

Covariant order r

e.g. Hypergraph: 每个 edge 可以超过两个点



Defn: 无向Hypergraph G

pair $G(V,E)$ $V=\{1,2,\cdots n\}$

$E\{e_1,\cdots e_p\}$: hyperedge $e_\ell \subseteq V$.

如果每个hyperedge 有着一样的node, i.e. $|e_p|=k$, $k\leq n$. G称为 k-uniform Hypergraph

Defn. 令 $G=\{E,V\}$ be a k-uniform hypergraph. 定义 Adjacent Tensor $A\in\mathbb{R}^{n\times\cdots\times n}$

$$A_{j_1\cdots j_k} = \begin{cases} \frac{1}{(k-1)!} & (j_1,j_2,\cdots j_k)\in E \\ 0 & \text{otherwise.} \end{cases}$$

$\frac{1}{(k-1)!}$: 这样一个点其它的加起来是1.

e.g. Ham Mechanics on Sympletic manifolds

Defn 设 M^{2n}是一个偶数维的流形

流形M上的辛结构是一个 closed, non-degenerate M上的 2-form

Here closed: $d\omega=0$ and $\forall\xi\neq0$. $\exists\eta$ $\omega(\xi,\eta)\neq0$. $\xi,\eta\in T_x M$

例如: $(\mathbb{R}^{2n}\ dq^1\wedge dp_1 \cdots dq^n\wedge dp_n)$

$d\omega=0$

$\omega=[dq^1\cdots dq^n, dp_1,\cdots dp_n]$

$$\omega=[dq^1\cdots dq^n, dp_1 \cdots dp_n]\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} dq^1 \\ \vdots \\ dp_n \end{bmatrix}$$

Remark: Defn $\{f,g\}$ on M. by $\omega(X_f, X_g)=\{f,g\}$

f是 f的 Hamitonian 向量场 $i_{X_f}\omega = dH$

给定 (M,ω) 定义 X_H Ham向量场 by $\omega(X_H, \xi) = dH\cdot\xi$. $\quad i_{X_H}\omega = dH$

Ham System

Triple (M, ω, H). e.g. $\mathbb{R}^n$: $\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}$

e.g. T^*M = locally same

e.g. Darboux = locally same.

Prop: $\mathbb{R}^{2n}\ (q^1, \cdots q^n, p_1, \cdots p_n) \quad \omega = \sum dq^i \wedge dp_i \quad X_H\left(\frac{\partial H}{\partial p_i}\ \ -\frac{\partial H}{\partial q_i}\right) = J\, dH$

$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$

Proof: X_H 如前定义. Check $i_{X_H}\omega = dH$.

$i_{X_H} dq^i = \frac{\partial H}{\partial p_i} \qquad i_{X_H} dp_i = -\frac{\partial H}{\partial q^i}$

$i_{X_H}\omega = \sum i_{X_H}(dq^i \wedge dp^i) = \sum (i_{X_H} dq^i)\wedge dp^i - \sum dq_i \wedge i_{X_H} dp_i$

$= \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i = dH.$

Prop: Let (M, ω, X_H) 是 Hamotonian Sys 令 $c(t)$ 是 X_H 上的一个积分曲线 Then $H(c(t))$ 是 constant

Pf: $\frac{d}{dt} H(c(t)) = dH(c(t))\cdot c'(t) = dH(c(t))\cdot X_H(c(t))$

↙ 定义.

$= \omega\big(X_H(c(t)),\ X_H(c(t))\big) = 0.$ 而 ω 是 Skew 的

Prop: 令 (M, ω, X_H) be a Hamotonian. 对每个 t. $F_t^*\omega = \omega$ (volume conservation)

Pf $\frac{d}{dt} F_t^*\omega = F_t^* \cdot \underbrace{(L_{X_H}\omega)}_{\text{Lie-directive.}} = F_t^*(i_{X_H} \underbrace{d\omega}_{=0} + d i_{X_H}\omega) = F_t^*\, d(i_{X_H}\omega)$

$\underline{\text{Pf}}$ $\quad \frac{d}{dt} F_t^* \omega = F_t^* (\underbrace{L_{X_H} \omega}_{\text{Lie-direvative.}}) = F_t^* (i_{X_H} \underset{\overset{\shortparallel}{0}}{d\omega} + d\, i_{X_H} \omega) = F_t^* d (i_{X_H} \omega)$

$$= F_t^* d (dH) = F_t^* d^2 H = 0.$$