L9 L-P Bracket

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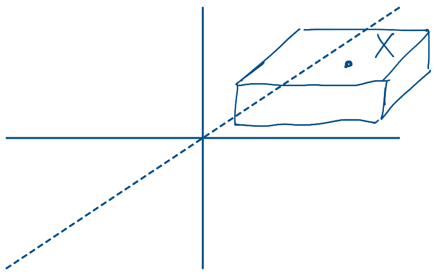
Rigid Body Dynamics $\quad \dot{\Pi} = \Pi \times \omega$

在n维情况下 $\quad M = [M, \Omega]$

Kinematics. 旋转矩阵.

$R(t)$.

$\dot{x} = \dot{R}(t) X$



$R^{-1}\dot{R}$ 和 $\dot{R}R^{-1}$ 均为 Skew 的

旋转矩阵: $R^T R = R R^T = I$.

$\dot{x} = \dot{R}X = \dot{R}R^{-1}x$ $\quad$ Skew $\quad so(3)$

$= \omega \times x = \hat{\omega} x$. $\quad \hat{\omega} = \dot{R}R^{-1} \Rightarrow \dot{R} = \omega R$.

Body Angular Velocity. $\quad$ why no derivative to R^{-1}

$X = R^{-1}x \qquad (R^{-1}x = X)$.

$\dot{X} = R^{-1}\dot{R}R^{-1}x = R^{-1}\dot{R}X = \hat{\Omega}X$.

$\hat{\Omega} = R^{-1}\dot{R} \qquad \dot{R} = R\hat{\Omega} \qquad \hat{\omega} = R\hat{\Omega}R^{-1}$

RB Dynamics & Bracket.

是一个在李代数空间中的一个特殊情况

用μ表示 $\mathcal{G}^*$中的元素. $\mathcal{G}^*$为$\mathcal{G}$的对偶空间. 对于函数 F.

$$\frac{\delta F}{\delta \mu} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[F(\mu + \delta\mu) - F(\mu) \right] = \langle \delta_\mu, \frac{\delta F}{\delta \mu} \rangle$$

而括号 $\langle , \rangle$ 将 $\mathcal{G}$ 和 $\mathcal{G}^*$联系了起来.

Determine Lie-Poisson 括号

$$\{F, G\}(\mu) = \pm \langle \mu, [\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}] \rangle.$$

对应到 RB Dynamics. $SO(3)$ $\stackrel{+}{=}$ $\Pi \cdot \langle \nabla F \times \nabla G \rangle$.

Locally in co-ordinates

$$[\xi_i, \xi_j] = C_{ij}^k \xi_k$$

$$\{F, G\} = \pm C_{ij}^k M_k \frac{\partial F}{\partial \mu_i} \frac{\partial G}{\partial \mu_j}$$

Note: L-P Bracket 和 Canonical bracket

1) $\{F, G\}$ 是 real, bilinear 的.

2). $\{F, G\} = -\{G, F\}$.

3) $\{F, G\}$ 有 Jacobian.

4). $\{FG, H\} = F\{G, H\} + \{F, H\} G$. (- Leibniz Rule)

对于 Manifolds P together with a bracket on $C^\infty(P)$: Poisson Manifold.

e.g. ① $(\mathbb{R}^{2n}, \{\}_{can})$

② $\mathbb{R}^3, \times$

③ $SO(n)$, Lie-Poisson Bracket

Special case: Symplectic Manifolds

Bracket是 non degegenerate 的. $\Rightarrow$ ie. no Casimirs

例: $\mathbb{R}^{2n}$ $\{\}_{can}$ - Symplectic.

例: not $SO(3)$ with L-P. $\Pi\cdot(\nabla F\times\nabla G)$.

给定一个 Poisson Manifold $(P, \{\})$

Vector Field X_H.

$\dot{F} = \{F, H\} = \langle dF, X_H\rangle$

在 RB中 $X_H(\Pi) = -\Pi \times \nabla H$.

Canonical $X_H(q, p) = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i}\right)$

可以从一个括号映射到另一个括号

e.g. Euler's Rules

$(\theta, \varphi, \psi, p_\theta, p_\varphi, p_\psi) \to (\Pi_1, \Pi_2, \Pi_3)$

$\Pi_1 = \frac{1}{\sin\theta}\left[(p_\varphi - p_\psi\cos\theta)\sin\psi + p_\theta\sin\theta\cos\psi\right]$

$\Pi_2 = \frac{1}{\sin\theta}\left[(p_\varphi - p_\psi\cos\theta)\cos\psi - p_\theta\sin\theta\sin\psi\right]$

$\Pi_3 = p_\psi$.

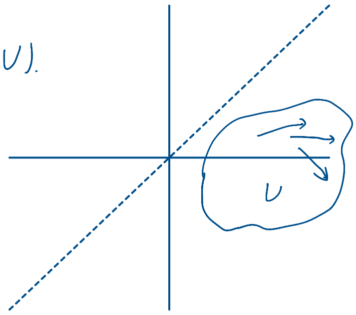
可以把 Canonical Bracket 映射到 Π的 L-P 括号

定义 令$r \geq 0$. r-integer C^r vector field 在一个开集 $U \subset \mathbb{R}^n$上.

$X: U \to \mathbb{R}^n$ of class C^r

C^r 向量场，在U上．用 $\mathcal{X}^r(U)$.

C^∞ $\mathcal{X}(U)$ 和 $\mathcal{X}(U)$



定义：A integral curve of $X \in \mathcal{X}(U)$ 是个可微曲线 $c(t)$.

用于定义 m. 一些 interval $I \subset \mathbb{R}$ containing 0 使得 $c(0)=X_0$.

$c'(t) = X(c(t)) \quad \forall\ t \in I.$

Existence & Uniqueness

(Lipschitz enough) C^1 more than enough.

可能会在有限时间内 blow up. 例 ($\dot{x} = x \Rightarrow x = x_0 e^t$)

关于 V. fields 的记号

$\dot{x} = f(X)$. Given $F(x)$ - Differencial v. field.

$\frac{d}{dt} F(x) \quad \frac{\partial F}{\partial x_1}\frac{dx_1}{dt} \cdots \frac{\partial F}{\partial x_n}\frac{dx_n}{dt} \quad = \sum_i \frac{dx}{dt}\frac{\partial}{\partial x_i} F$

$X_f(F)$

$$\boxed{X_f = \sum_i \dot{x}_i \frac{\partial}{\partial x_i}}$$

General v. Fields on $\mathbb{R}^n$ $\quad \sum_i g_i(x)\frac{\partial}{\partial x_i} \equiv \dot{x}_i = g(x) = X_g$

由g给出的向量场

eg. Heisenberg Ctrl Problem.

$$\begin{cases} \dot{x} = u_1, \\ \dot{y} = u_2 \\ \dot{z} = y u_1 - x u_2 \end{cases}$$

$\rightarrow$ 2个 Control 的向量场

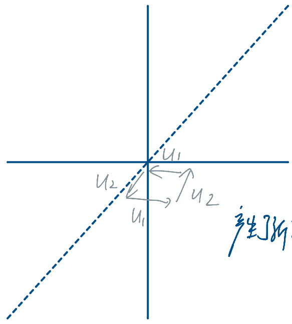
$$\begin{cases} X_{u_1} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \\ X_{u_2} = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}. \end{cases}$$

Lie-Bracket of X_{u_1}, X_{u_2}.

$$[X_{u_1}, X_{u_2}] = X_{u_1}X_{u_2} - X_{u_2}X_{u_1} = -\frac{\partial}{\partial z} - \frac{\partial}{\partial z}$$

$[X_{u_1}, X_{u_2}]$ new v. field. $X_3 = -2\frac{\partial}{\partial z}$

$\equiv$ Controllability - 沿z轴移动.



产生了新的方向.

定义 Lie-Bracket of v. fields f, g.

$$[f,g]^i = (f\cdot\nabla)g - g\cdot\nabla f.$$

$$= \sum_{j=1}^{n} \left(f^j \frac{\partial g^i}{\partial x^j} - g^j \frac{\partial f^i}{\partial x^j}\right).$$

Optimal Control & Hamstonian System.

$$\begin{cases} \dot{x} = u_1 \\ \dot{y} = u_2 \\ \dot{z} = y u_1 - x u_2. \end{cases} \qquad \min \int_0^T (u_1^2 + u_2^2)\, dt.$$

$$L = \dot{x}_1^2 + \dot{x}_2^2 + \lambda(\dot{z} - y\dot{x} + x\dot{y}).$$