L17 Lie Direvatives

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Sympletic manifolds 辛流形

(M, ω) - ω : non-degenerating closed 2-form

$d\omega = 0$

X_H. $\quad i_{X_H} \omega = dH$. $\quad \begin{cases} H \text{ conserve} : \omega \text{ skew} \\ \omega \text{ conserves} : d\omega = 0. \end{cases}$

在 $\mathbb{R}^3$ 中 $\underset{\sim}{v} \in \mathbb{R}^3 = v^1 \underset{\sim}{e_1} + v^2 \underset{\sim}{e_2} + v^3 \underset{\sim}{e_3}$ - v 对应的 1-form

令 $v^\flat = v^1 dx + v^2 dy + v^3 dz$ 类似于 gradient (flatten)

$(v^\flat)^\# = \underset{\sim}{v}$ i.e. $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$, $\rightarrow \alpha^\# = \alpha_1 \underset{\sim}{e_1} + \alpha_2 \underset{\sim}{e_2} + \alpha_3 \underset{\sim}{e_3}$

Musical Notation $\#$ & $\flat$.

$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots = (\nabla f)^\flat$

$*$ Operator : map k-form to $n-k$ form.

$\mathbb{R}^3$

$*1 = dx \wedge dy \wedge dz$	$*(dy \wedge dz) = dx$
$*dx = dy \wedge dz$	$*(dx \wedge dz) = -dy$
$*dy = -dx \wedge dz$	$*(dx \wedge dy) = dz$
$*dz = dx \wedge dy$	$*(dx \wedge dy \wedge dz) = 1$

由 $*$ 的定义. 有以下 ∇ 的性质 $\quad F = F_1 \underset{\sim}{e_1} + F_2 \underset{\sim}{e_2} + F_3 \underset{\sim}{e_3}$

$\nabla \times F = [* \, dF^\flat]^\# \qquad \nabla \cdot F = * \, d(* F^\flat) \qquad v \times w = [*(v^\flat \wedge w^\flat)]^\#$

Dynamics Lie-Derivative

让 α 是一个 k-form $\quad X$: vector field.

令 X 有一个 flow φ_t. 随着 X 的李级分是. $\mathcal{L}_X \alpha = \lim_{t \to 0} \frac{1}{t} [\varphi_t^* \alpha - \alpha] = \frac{d}{dt} \varphi_t^* \alpha \Big|_{t=0}$

可以 check : $\frac{d}{dt} \varphi_t^* \alpha = \varphi_t^* \mathcal{L}_X \alpha$

在流形M上定义一个函数 f $\mathcal{L}_X f = X[f] = df \cdot X$

如果 Y 是一个在流形 N 上的向量场. $\varphi: M \to N$.

(diffeo)

Pull back $\varphi^* Y$. vector field M defined as $\varphi^* Y(x) = T_x \varphi^{-1} \circ Y \circ \varphi$

Locally $\bar{x} = \varphi(x)$. $(\varphi_* X)^j(\bar{x}) = \frac{\partial \varphi^j}{\partial x_i}(x) \; x^i(x)$ (Jacobian 逆)

M. C^∞. $f \to X[Y(f)] - Y[X(f)]$

where $X[f] = df \cdot X$ $Y[f] = df \cdot Y$ 定义 $\mathcal{L}_X = [X, Y]$

Lie derivative of X along Y.

Lie bracket of X and Y.

Can show $\frac{d}{dt} \varphi_t^* Y = \varphi_t \mathcal{L}_X Y$ φ_t 是 X 的 flow. $\frac{d}{dt}\Big|_{t=0} \varphi_t^* Y = \mathcal{L}_X Y$.

$$(\mathcal{L}_X Y)^j = X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial y^i} = (X \cdot \nabla) Y^j - (Y \cdot \nabla) X^j$$

Algebraic Approach to $\mathcal{L}$

扩展 Lie derivative 的定义. 从向量场上的函数到 形式

$$\mathcal{L}_X \langle \alpha, Y \rangle = \langle \mathcal{L}_X \alpha, Y \rangle + \langle \alpha, \mathcal{L}_X Y \rangle$$

where X, Y. 向量场 在 $\langle \alpha, Y \rangle = \alpha(Y)$.

$\mathcal{L}_X \; \alpha(Y_1, \cdots Y_k)$: α k-form.

$$= (\mathcal{L}_X \alpha)(Y_1, \cdots Y_k) + \sum_{i=1}^{k} \alpha(Y_1, \cdots \mathcal{L}_X Y_i, \cdots Y_k)$$

Prop: Dynamics along Lie-derivative

Cartan's (Magic) Formula : $\boxed{\mathcal{L}_X \alpha = d i_X \alpha + i_X d\alpha.}$

Sketch of Proof.

令 α 是一个 0-form. $\mathcal{L}_X f = i_X df + d i_X f = df \cdot X$

假设 $n = k$ 时成立 要证明 $n = k+1$ 时也成立 $\sum df_i \wedge w_i$

Then $\mathcal{L}_X (df \wedge w) = \mathcal{L}_X df \wedge w + df \wedge \mathcal{L}_X w$ $- k$-form.

Application.

n-manifold 被称为 orientible 如果 存在 a. nowhere vanishes n-form w on it

ω – volume form.

– it is a basis for $\Omega^n(M)$ – Space of n-form on M.

$\mathbb{R}^n$, $dx^1 \wedge dx^2 \cdots \wedge dx^n$

Since $L_X \omega \in \Omega^n(M)$, $\exists$ function 称为 divergene of X s.t. $\mathcal{L}_X \omega = \mathrm{div}_\omega(X)\,\omega$

From Dynamics $\mathrm{div}_\omega(X) = 0$ iff $F_t^* \omega = 0$ where F_t flow of X

$\Rightarrow F_t$: volume preserving. (之前说明当 div $= 0$ 时有体积守恒)

另一个有用的 identity :

θ : 1-form. X, Y 为向量场

$$\underbrace{d\theta}_{\text{2-form}}(X,Y) = X[\theta(Y)] - Y[\theta(X)] - \theta[X,Y]$$

Stokes' Theorem

ω n-form on an oriented n-manifold. $\Omega = f(x^1, \cdots x^n)\, dx^1 \wedge \cdots \wedge dx^n$

令 M 是一个 compact, oriented k 维流形. 边界为 ∂M. let α be smooth $(k-1)$ form.

$$\int_M d\alpha = \int_{\partial M} \alpha.$$

e.g. M, ω, X_H – Ham System and F_t – flow of X_H

Claim $F_t^* \omega = \omega$

$$\frac{d}{dt} F_t^* \omega = F_t^* \mathcal{L}_{X_H} \omega = F_t^* (\underbrace{i_{X_H} d\omega}_{d\omega = 0} + d \underbrace{i_{X_H} \omega}_{= d\, dH = 0})$$

Poisson Manifolds.

定义: 一个流形上面的 Poisson Bracket on 流形 P

是一个双线性算子 $\{ \}$ 在 $F(p) = C^\infty(p)$ 上. s.t.

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(i) $(\mathcal{F}(P), \{\})$ 是一个李代数.

(ii) $\{,\}$ 是一个 derivation 对于每个 factor. i.e. $\{FG, H\} = \{F,H\}G + F\{G,H\}$

e.g. Symplectic Manifold with Poisson Bracket.

$$\{F, G\} = \omega(X_F, X_G)$$

因此有性质 (ii) $\{FG, H\} = X_H[F, G] = F X_H[G] + G X_H[F] = F\{G,H\} + G\{F,H\}$.