

Assignment 01: Bayes Rule and Computer Generation of Random Variables

1. A posteriori probability:

(10 points)

There are $K = 11$ urns labeled by $u \in \{0, 1, 2, \dots, 10\}$, each containing $L = 10$ balls. Urn u contains u black balls and $10 - u$ white balls. Fred selects an urn u at random and draws N times with replacement from that urn, obtaining N_B blacks and $N - N_B$ whites. Fred's friend, Bill, looks on. If after $N = 10$ draws $n_B = 3$ blacks have been drawn, what is the probability that the urn Fred is using is urn u , from Bill's point of view? (Bill doesn't know the value of u .) Calculate the numerical values of the probability for $u = 0, 1, \dots, 10$ and tabulate the probability values. Indicate the value of u for which the probability is maximum.

2. Computer Generation of Random Variables

(30 points)

Suppose that $p(u)$ denotes a valid probability density function (PDF) for a (continuous) real-valued random variable and let $F(u) = \int_{-\infty}^u p(v)dv$ denote the corresponding cumulative distribution function. Let X be a random variable that is uniformly distributed over the interval between 0 and 1, i.e., its probability density function (PDF) is given by

$$p_X(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Now let $G(\cdot)$ denote a function for which the function $F(\cdot)$ is an inverse¹, i.e. for each $t \in [0, 1]$,

$$F(G(t)) = t \quad (2)$$

and define the random variable Y as a function of X by the relation:

$$Y = G(X) \quad (3)$$

- (a) Show that the PDF $p_Y(y)$ for the random variable Y is given by $p_Y(y) = p(y)$, i.e., the PDF of the random variable matches the probability density function $p(\cdot)$. In order to simplify your proof, you may assume that $G(t)$ is differentiable with a non-zero derivative for all $t \in [0, 1]$ (though this condition is not required).

Hint: You can solve this problem either by computing the cumulative distribution function for Y and computing the PDF from that or by using the formula for the PDF of a function of a random variable along with the chain rule for derivatives.

- (b) **Matlab:** For computer simulation, the above observation suggests a method for generating a random variable with a desired PDF $p(u)$, using the mapping $Y = G(X)$ on a uniformly distributed random variable X . Consider the PDF

$$p(u) = \begin{cases} 2u & 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

Generate $N = 10^4$ realizations of the random variable with the above PDF using the method suggested above. Plot and compare the *suitably normalized* histogram of these realizations you generate against the above PDF in a single plot. Explain your normalization procedure. Repeat this exercise for $N = 10^6$

¹Note that for a reasonable (i.e. measurable) function $p(u)$ that can be the PDF of a continuous random variable, it can be shown that such a $G(\cdot)$ exists.