STABILITY IN DISCRETE SYSTEMS

In the design and analysis of control systems the primary objective is to ensure that the required specifications are satisfied and so it is useful to identify desirable regions in the z-plane where the poles can or cannot lie. Knowledge of the stability boundary is important in this respect, as is the relationship between the s-plane and the z-plane. We start with a discussion on establishing the stability boundary in the z-plane.

Stability Boundary

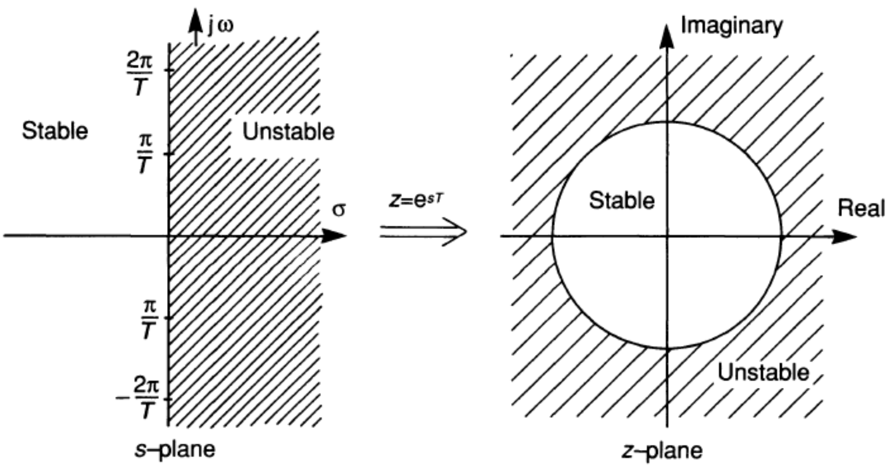
The stable region in the z-plane can be ascertained using the properties of the s-plane together with the definition of the z-transformation. Since $z = e^{sT}$ and s is a complex number $(= \sigma + j\omega)$, z also defines a complex plane with $z = e^{(\sigma + j\omega)T}$. Now

$$|z| = e^{\sigma T}$$

And

$$< z = \omega T$$

For $\sigma = 0$ we have continuous oscillations, or the *undamped* case in the s-plane. This corresponds to $|z| = 1$ and $< z$ changing by 2π radians (anti-clockwise) whenever ω changes by $\frac{2\pi}{T}$, or some multiple. Therefore the imaginary axis in the s-plane, which defines the stability boundary, is transformed into the circumference of the unit circle centred at the origin in the z-plane. For all $\sigma > 0$ (unstable in the s-plane) we have $|z| > 1$, and all corresponding points lie outside this unit circle in the z-plane. For all $\sigma < 0$ (stable in the s-plane) we have $|z| < 1$ and the points lie inside the unit circle in the z-plane. Therefore we can deduce that a digital system is stable if all its poles lie inside the unit circle centred at the origin in the z-plane, the circumference of this circle thus defining the stability boundary. A pictorial representation of this relationship between the continuous and discrete cases is shown in Figure



Relationship between the s- and z-planes

$$-1 < z < 1$$

Having established the stable and unstable regions in the z-plane it is worth analysing the numerical integration approximations discussed previously, to determine how well the methods perform and why. It was seen that starting with a stable continuous system, the forward rectangular method yielded an unstable digital realisation. Also, the other two methods considered, namely the backward rectangular rule and Tustin's rule, gave differing degrees of accuracies in the approximations for the same system.

Forward Rectangular Rule

$$s = \frac{z-1}{T}$$

Backward Rectangular Rule

$$s = \frac{z-1}{Tz}$$

Trapezium Rule

$$s = \left(\frac{2}{T}\right)\left(\frac{z-1}{z+1}\right)$$

Forward Rectangular Rule

$$z = 1 + Ts$$

Backward Rectangular Rule

$$z = \frac{1}{1-Ts}$$

Trapezium Rule

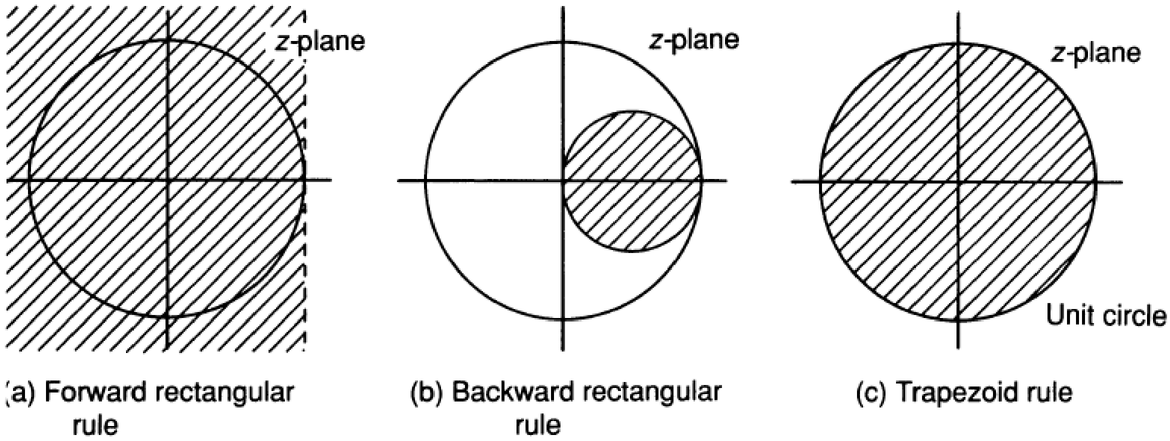
$$z = \frac{1 + {}^{Ts}/_{2}}{1 - {}^{Ts}/_{2}}$$

Therefore, for: $s = -0.2 \pm j0.8$

i. Forward Rectangular Rule: $z = 0.8 \pm j0.8$
ii. Backward Rectangular Rule: $z = 0.58 \pm j0.38$
iii. Trapezium Rule: $z = 0.61 \pm j0.58$
iv. For exact transformation: $\mathrm{z} = 0.57 \pm j0.59$

To understand the reasons for these results each of the three transformations may be viewed graphically to determine how the stability region in the s-plane maps into the z-plane.

The shaded regions show the transformations of the left half s-plane to the z-plane when different approximations are used.



Stability of numerical integration approximations

Since the unit circle gives the stability boundary in the z-plane, it is obvious from the Figure that the forward rectangular rule can give an unstable transformation for a stable continuous transfer function (as illustrated by the example considered). The other two transformations will always give stable digital realisations for stable continuous systems. It is also important to note that, since Tustin's rule maps the stable region of the s-plane exactly into the stable region of the z-plane, it gives rise to better approximations as seen in the example. However this does not imply perfect results because the entire $j\omega$-axis of the s-plane is squeezed into the 2π length of the unit circle in the z-plane. Clearly a distortion takes place in the mapping in spite of the agreement of the stability regions.

Since the stability boundary in the z-plane is the unit circle centred at the origin, it appears straightforward to determine whether a particular system is stable or not, simply by determining the location of its poles. In practice the situation can be rather more complicated. The main difficulty is that for large order polynomials the roots are not readily available, and use of computer packages is necessary for the factorisation. Such difficulties have sled to the development of methods that do not require explicit evaluation of the roots of the characteristic polynomials, and will be discussed next.

ROUTH'S METHOD

It is well known that for the continuous case the method developed by Routh allows us to assess system stability by the construction of a simple array, known as the Routh array. Here, by calculating various determinants, it is possible to determine how many system poles are in the unstable right-half s-plane.

The Routh method can be extended to discrete systems if the different stability boundary in the z-domain can be modified, that is, the unit circle at the origin in the z-domain needs to be converted to the imaginary axis.

To illustrate this technique, we consider the simple example of a discrete system whose characteristic equation is

$$F(z) = 3z^4 + z^3 - z^2 - 2z + 1 = 0$$

To apply the Routh method we firstly apply the transformation:

$$z = \left(\frac{\omega + 1}{\omega - 1}\right)$$

to give $F(\omega)$.

$$F(\omega) = 3\left(\frac{\omega + 1}{\omega - 1}\right)^4 + \left(\frac{\omega + 1}{\omega - 1}\right)^3 - \left(\frac{\omega + 1}{\omega - 1}\right)^2 - 2\left(\frac{\omega + 1}{\omega - 1}\right) + 1 = 0$$

$$F(\omega) = \frac{3(\omega + 1)^4}{(\omega - 1)^4} + \frac{(\omega + 1)^3}{(\omega - 1)^3} - \frac{(\omega + 1)^2}{(\omega - 1)^2} - \frac{2(\omega + 1)}{(\omega - 1)} + \frac{1}{1} = 0$$

$$\frac{3(\omega + 1)^4 + (\omega + 1)^3(\omega - 1) - (\omega + 1)^2(\omega - 1)^2 - 2(\omega + 1)(\omega - 1)^3 + (\omega - 1)^4}{(\omega - 1)^4} = 0$$

$$\begin{aligned}
3(\omega + 1)^4 + (\omega + 1)^3(\omega - 1) - (\omega + 1)^2(\omega - 1)^2 - 2(\omega + 1)(\omega - 1)^3 + (\omega - 1)^4 \\
= 3\omega^4 + 12\omega^3 + 18\omega^2 + 12\omega + 3 + \omega^4 + 2\omega^3 - 2\omega - 1 - \omega^4 + 2\omega^3 - 1 \\
- 2\omega^4 + 4\omega^3 - 4\omega + 2 + \omega^4 - 4\omega^3 + 6\omega^2 - 4\omega + 1 \\
= 2\omega^4 + 14\omega^3 + 26\omega^2 + 2\omega + 4
\end{aligned}$$

$$F(\omega) = 2\omega^4 + 14\omega^3 + 26\omega^2 + 2\omega + 4 = 0$$

ω^4	$+2$	26	4
ω^3	$+14$	2	0
ω^2	$\frac{(14 \times 26) - (2 \times 2)}{14} = +25.7$	$\frac{(14 \times 4) - (2 \times 0)}{14} = 3.9$	0
ω^1	$\frac{(25.7 \times 2) - (3.9 \times 14)}{25.7} = -0.12$	0	0
ω^0	$\frac{(-0.12 \times 3.9) - (25.7 \times 0)}{-0.12} = +3.9$	0	0

Forming the Routh array we obtain the Table, to which the Routh stability result can be applied. This states that the system has no unstable roots if all the numbers in the first column have the same sign, and also that none of the numbers vanishes. In our example there are two sign changes, and so there are two unstable poles.

In fact, $F(z)$ and $F(\omega)$ have the following roots:

$F(z)$: $z = -0.74 \pm j0.69$ (unstable poles), or $0.57 \pm j0.08$

$F(\omega)$: $\omega = 0.003 \pm j0.39$ (unstable poles), or $-3.5 \pm j0.79$

JURY'S METHOD

An alternative, array based, stability assessment method that addresses the discrete characteristic polynomial directly was developed by Jury and Blanchard. In this case, though there is no need to transform to another domain, the array construction and assessment conditions are more involved. Consider the characteristic polynomial:

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + a_1 z + a_0 = 0$$

where $a_0, a_1, \dots, a_n$ are real coefficients and $a_n > 0$. To construct Jury's table, the a_i coefficients are inserted on the first two rows of the array and are then used to calculate the complete array shown in the Table

Row	$\mathbf{z^0}$	$\mathbf{z^1}$	$\mathbf{z^2}$	**…..**	$\mathbf{z^{n-2}}$	$\mathbf{z^{n-1}}$	$\mathbf{z^n}$
1 2	a_0 a_n	a_1 a_{n-1}	a_2 a_{n-2}	…. ….	a_{n-2} a_2	a_{n-1} a_1	a_n a_0
3 4	b_0 b_{n-1}	b_1 b_{n-2}	b_2 b_{n-3}	…. ….	b_{n-2} b_1	b_{n-1} b_0	
5 6	c_0 c_{n-2}	c_1 c_{n-3}	c_2 c_{n-4}	…. ….	c_{n-2} c_0		
. . .	. . .	. . .	. . .	. . .			

In forming the array we have:

$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$ for $k = 0,1,\dots,n-1$,

$c_i = \begin{vmatrix} b_0 & b_{n-1-i} \\ b_{n-1} & b_i \end{vmatrix}$ for $i = 0,1,\dots,n-2$,

Having constructed the above table we can apply Jury's stability test which states the necessary and sufficient conditions for $F(z)$ to have no roots on, or outside, the unit circle. The conditions are as follows:

(i)

$$[F(z)]_{z=1} > 0$$

(ii)

$$[F(z)]_{z=-1} > 0 \quad for\ n\ even$$

$$[F(z)]_{z=-1} < 0 \quad for\ n\ odd$$

(iii)

$$|a_0| < |a_n|$$

$$|b_0| > |b_{n-1}|$$

$$|c_0| \qquad > \qquad |c_{n-2}|$$

As an example, we again consider the system defined by the characteristic polynomial:

$$F(z) = 3z^4 + z^3 - z^2 - 2z + 1 = 0$$

but now assess its stability by Jury's method.

Since conditions (i) and (ii) do not depend on the table, it is advisable to check them before constructing the array.

- $F(z = 1) = 3(1)^4 + (1)^3 - (1)^2 - 2(1) + 1 = 2 > 0$ and so condition (i) is satisfied.
- $F(z = -1) = 3(-1)^4 + (-1)^3 - (-1)^2 - 2(-1) + 1 = 4 > 0$, $n = 4$, is even, and so condition (ii) is satisfied.

$$b_0 = \begin{vmatrix} a_0 & a_4 \\ a_4 & a_0 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -8$$

$$b_{n-1} = b_3 = \begin{vmatrix} a_0 & a_1 \\ a_4 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} = 7$$

$$c_0 = \begin{vmatrix} b_0 & b_3 \\ b_3 & b_0 \end{vmatrix} = \begin{vmatrix} -8 & 7 \\ 7 & 8 \end{vmatrix} = 15$$

$$c_2 = \begin{vmatrix} b_0 & b_1 \\ b_3 & b_2 \end{vmatrix} = \begin{vmatrix} -8 & b_1 \\ 7 & b_2 \end{vmatrix} = \begin{vmatrix} -8 & -5 \\ 7 & 2 \end{vmatrix} = 19$$

$$b_1 = \begin{vmatrix} a_0 & a_3 \\ a_4 & a_1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} = -5$$

$$b_2 = \begin{vmatrix} a_0 & a_2 \\ a_4 & a_2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} = 2$$

$$c_1 = \begin{vmatrix} b_0 & b_2 \\ b_3 & b_1 \end{vmatrix} = \begin{vmatrix} -8 & 2 \\ 7 & -5 \end{vmatrix} = 26$$

We can therefore construct Jury's array, from which we have

$|a_0| = 1,$ $\qquad |a_4| = 3$ $\qquad |a_0| < |a_n|$ satisfied

$|b_0| = -8,$ $\qquad |b_3| = 7$ $\qquad |b_0| > |b_{n-1}|$ is *not* satisfied

$|c_0| = 15,$ $\qquad |c_{n-2}| = 19$ $\qquad |c_0| > |c_{n-2}|$ is *not* satisfied

and so we can deduce that the system is unstable as discovered earlier.

Row		$\boldsymbol{z^0}$	$\boldsymbol{z^1}$	$\boldsymbol{z^2}$	$\boldsymbol{z^3}$	$\boldsymbol{z^4}$
1	a_0	1	-2	-1	1	3
2	a_4	3	1	-1	-2	1
3	b_0	-8	-5	2	7	
4	b_3	7	2	-5	-8	

5	c_0	15	26	19
6	c_2	19	26	15

RAIBLE'S METHOD

A simplification for the construction of Jury's array by hand was suggested by Raible. Considering:

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + a_1 z + a_0 = 0$$

with the a_i values real and $a_n > 0$ as before, the Raible table is constructed as shown in the Table, where

$$k_a = \frac{a_0}{a_n}, \qquad k_b = \frac{b_{n-1}}{b_0}, \qquad k_c = \frac{c_{n-2}}{c_0}, \qquad k_q = \frac{q_1}{q_0},$$

$$b_i = a_{n-i} - k_a a_i \qquad for \quad i = 0,1,2, \dots \dots, n-1$$

$$c_j = b_j - k_b b_{n-j-1} \qquad for \quad j = 0,1,2, \dots \dots, n-2$$

z^n	z^{n-1}	z^{n-2}	...	z^2	z^1	z^0	k
a_n	a_{n-1}	a_{n-2}	...	a_2	a_1	a_0	k_a
b_0	b_1	b_2	...	b_{n-2}	b_{n-1}		k_b
c_0	c_1	c_2	...	c_{n-2}			k_c
. . .	. . .	. . .					. . .
p_0	p_1	p_2					k_p
q_0	q_1						k_q
r_0							

We then have the following result:

Raible's Criterion

The number of roots of $F(z) = 0$ inside the unit circle is equal to the number of positive calculated elements in the first column

$$[b_0, c_0, d_0, \dots\ q_0, r_0]^T$$

or *equivalently*, the number of negative calculated elements in the first column gives the number of roots outside the unit circle.

Again, to illustrate the method, we consider the system which has the characteristic polynomial

$$F(z) = 3z^4 + z^3 - z^2 - 2z + 1 = 0$$

$$k_a = \frac{a_0}{a_n} = \frac{a_0}{a_4} = \frac{1}{3} = 0.33$$

$$b_0 = a_4 - k_a a_0 = 3 - 0.33 = 2.67$$

$$b_1 = a_3 - k_a a_1 = 1 + 0.67 = 1.67$$

$$b_2 = a_2 - k_a a_2 = -1 + 0.33 = -0.67$$

$$b_3 = a_1 - k_a a_3 = -2 - 0.33 = -2.33$$

$$k_b = \frac{b_3}{b_0} = \frac{-2.33}{2.67} = -0.87$$

$$c_0 = b_0 - k_b b_3 = 2.67 - 2.03 = 0.64$$

$$c_1 = b_1 - k_b b_2 = 1.67 - 0.38 = 1.09$$

$$c_2 = b_2 - k_b b_1 = -0.67 + 1.46 = 0.79$$

$$k_c = \frac{c_2}{c_0} = \frac{0.79}{0.64} = 1.23$$

$$d_0 = c_0 - k_c c_2 = 0.64 - 0.97 = -0.33$$

$$d_1 = c_1 - k_c c_1 = 1.09 - 1.34 = -0.25$$

$$k_d = \frac{d_1}{d_0} = \frac{-0.25}{-0.33} = 0.76$$

$$e_0 = d_0 - k_d d_1 = -0.33 + 0.19 = -0.14$$

	$\boldsymbol{z^4}$	$\boldsymbol{z^3}$	$\boldsymbol{z^2}$	$\boldsymbol{z^1}$	$\boldsymbol{z^0}$	$\boldsymbol{k}$
$a's$ $-k_a a_i$	$\mathbf{3}$ -0.33	1 0.67	-1 0.33	-2 -0.33	1	0.33
$b's$ $-k_b b_{3-j}$	$\mathbf{2.67}$ -2.03	1.67 -0.58	-0.67 1.46	-2.33		-0.87
$c's$ $-k_c c_{2-j}$	$\mathbf{0.64}$ -0.97	1.09 -1.34	0.79			1.23
$d's$ $-k_d d_{1-j}$	$\mathbf{-0.33}$ 0.19	-0.25				0.76
e	$\mathbf{-0.14}$					

Note that in the calculation of the array we have inserted intermediate rows (for example, $-k_a a_i$) to assist the hand calculations. The calculated first column is $[2.67, 0.64, -0.33, -0.14]^T$ from which we can deduce that $F(z)$ has two roots inside the unit circle, and two outside (hence the system is unstable).