MOVEMENT IN QUANTUM COMBINATORIAL GAMES

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Abstract. For any combinatorial game, there may exist multiple, natural definitions for movement on a game position that will yield isomorphic gameplay under classical conditions; which we shall call 'quantum interpretations' of a classical game. These quantum interpretations of combinatorial games are highly sensitive to how moves are defined, meaning that multiple games may result from 'natural' quantum lifts performed on any classical combinatorial game. In this paper, we discuss this phenomenon with examples from Nim and Hackenbush.

1. Introduction to Combinatorial Game Theory

In this paper, we study a variety of games called combinatorial games. These games are generally confined to the conditions that there are no element of chance, no information hidden from any player, and no possibilities for a tie as a game outcome [1, 2]. Combinatorial game analysis is frequently restricted to the case of two-player games - which is the case in this paper.

During a game, players will take turns performing legal moves on a game position, until the game reaches a terminating point. This terminating point is often defined to be a point in the game such that no legal moves exist for the current player. Under the conventions of “normal play” (which will be the implicit assumption for all games within this paper, unless stated otherwise), the player who is unable to make any legal moves will lose [2].

This condition also gives rise to an important consideration in combinatorial game theory: whether the game is “partisan” or “impartial”. This paper will focus primarily on impartial games, though discussion of partisan games is not entirely excluded. An impartial game is a game in which all players are allowed to make the same moves, for any position in the game. These games have deep history in combinatorial game theory; some of the pioneering theorems for the discipline revolved around such games [3, 4].

Partisan games, in contrast, are games where players may not necessarily share an equivalent set of moves for any given position. Describing a game in a succinct manner was one of the many techniques developed during the study of partisan games. First introduced by John Conway [5], the set-wise technique of describing a game is now used almost universally. This notation describes a game, $\mathcal{G}$, as a set of the possible moves following from the position. This is denoted

$$\mathcal{G} = \{\mathcal{G}'_L \mid \mathcal{G}'_R\}$$

where $\mathcal{G}'_L$ and $\mathcal{G}'_R$ distinguish the positions available to the “left” and “right” player, respectively. In an impartial game (where $\mathcal{G}'_L = \mathcal{G}'_R$ by definition), the separating bar is removed. Importantly, this notation implies that one can fully describe a position in a game by enumerating the available options in said position. Because of this convention, combinatorial game theorists will often use tree structures to describe games, since trees can be directly constructed from the recursive set structure illustrated above, and are amenable to succinctly encoding many important qualities that a game may possess. A node of any game tree is a legal positon within the game, and, during a turn, a player will change the game position to one of the children of said node.

The final introductory concept needed for this paper is the idea of outcome classes and values. Any game may be classified by one of four outcome classes: $\mathcal{P}, \mathcal{N}, \mathcal{L},$ and $\mathcal{R}$. A game is $\mathcal{P}$ if the first player to move will lose, and is $\mathcal{N}$ if the first player to move will win (assuming perfect play). $\mathcal{L}$ and $\mathcal{R}$ denote a left or right player win occurring regardless of who goes first. The latter two outcomes only occur in partisan games.

During his work on partisan games, Conway also developed a method of assigning numerical values to games, called the surreal number system [5]. Since our focus will be largely dedicated to impartial games, we will not leverage the full extent of the surreal number system – instead, we shall use the Grundy number system [3, 4], which is used to quantify impartial games.

In the Grundy number system, numbers are constructed recursively, and are commonly denoted with a "$*$" to distinguish them from "ordinary" numbers. We construct the numbers as follows:

$$* \, 0 = \{\}$$

which is the game with no available options (and thus $\mathcal{P}$ under normal play). From this, the value of some impartial game $\mathcal{G}$ may be defined with the following:

$$\mathcal{G} = \{\mathcal{G}'_1, \mathcal{G}'_2, ...\} = \{* \, n_1, * \, n_2, ...\} = * \, (\mathbf{mex}(n_1, n_2, ...))$$

where **mex** is the "minimum excluded value" of a set (operating in $\mathbb{N}$). Thus, we can say that

$$* \, 1 = \{* \, 0\} = \{* \, 0, * \, 2\}$$

since $1 = \mathbf{mex}(0, 2) = \mathbf{mex}(0)$. From this definition, we can assert that $* \, 0$ is a $\mathcal{P}$ game, and $* \, n$ is a $\mathcal{N}$ game if $n > 0$ (since an $\mathcal{N}$ game is a game with a $\mathcal{P}$ position as an available option).

2. Quantum Game Definition

With an understanding of the prerequisite concepts for this paper, we may now venture into the realm of quantum combinatorial games. The idea of applying quantum concepts to a game setting extends far beyond the scope of combinatorial game theory specifically, so we shall be restricting our basis for quantum combinational games to the framework proposed by Dorbec and Mhalla in 2018 [6]. Broadly, this framework revolves around the idea of allowing "superpositions" within positions and moves.

To formalize a game for use in a quantum setting, we shall borrow the notation used in [6]. Consider a set of positions in a game, $\mathbb{G}$, and an alphabet of moves Σ. Define a game as the map

$$\Gamma : \mathbb{G} \times \Sigma \to \mathbb{G} \cup \{\text{NULL}\}$$

where NULL indicates an incompatible position-move pair. To implement a superposition, we shall borrow the quantum mechanical notation (used in [7]) to denote a quantum position $\boldsymbol{\mathcal{G}}^Q$ as

$$\boldsymbol{\mathcal{G}}^Q = \langle \boldsymbol{\mathcal{G}}_1, \boldsymbol{\mathcal{G}}_2, ...\rangle, \boldsymbol{\mathcal{G}} \in \mathbb{G}$$

and a superimposed move $\boldsymbol{\mathcal{M}}^Q$ as

$$\boldsymbol{\mathcal{M}}^Q = \langle \boldsymbol{m}_1, \boldsymbol{m}_2, ...\rangle, \boldsymbol{m} \in \Sigma$$

Applying $\boldsymbol{\mathcal{M}}^Q$ to $\boldsymbol{\mathcal{G}}^Q$ is then performed through the following:

$$\boldsymbol{\mathcal{G}}^Q \times \boldsymbol{\mathcal{M}}^Q = \left\{\Gamma(g, m) : g \in \boldsymbol{\mathcal{G}}^Q, m \in \boldsymbol{\mathcal{M}}^Q, \Gamma(g, m) \neq \text{NULL}\right\}$$

, which is analogous to a "filtered" (NULL-exclusionary) cross product of classical positions and moves. Following classical conventions, $\boldsymbol{\mathcal{M}}^Q$ is only legal if $\boldsymbol{\mathcal{G}}^Q \times \boldsymbol{\mathcal{M}}^Q$ is not empty; and if no such superposition of moves exists, the game is over (and the current player loses, as per normal play). Let us denote this game $\Gamma^Q : \mathbb{G}^Q \times \Sigma^Q \to \mathbb{G}^Q$, or the "quantum lift" of Γ.

It is worth noting that a considerable amount of work [6–8] has been dedicated to studying different quantum "rulesets" that change the legality of superimposed moves. These rulesets are originally labeled A, B, C, C', and D in Dorbec and Mhalla's paper [6]. This paper is operating under ruleset D, which can be thought of as the most permissive ruleset for a quantum game. The other rulesets A through C' make restrictions on what type of superimposed movements are allowed – for example, ruleset A never allows a superimposed move composed with less than two unique moves (i.e, no "classical" moves). We will study ruleset D in this paper largely because it encompasses all other variants, and provides the most flexibility during gameplay.

2.1. **A Closer Look at Movement.**

In classical combinatorial game theory, the recursively defined structure of any game effectively allows for the conceptual dismissal of what a "move" actually is. Movement is simply the selection of an option, with no necessary regard to how that transition between game states may correlate to other, potentially comparable movements along the game tree.

The recursive definition is undeniably efficient, and enables the analysis of combinatorial games from a very abstract perspective, agnostic of ruleset. The ruleset endows a game tree with structure, and from there, the specifics of a game can be almost entirely ignored. However, this technique cannot be as widely used in the context of quantum combinatorial games.

Formally, we know that

$$\boldsymbol{\mathcal{G}} = \{\boldsymbol{\mathcal{G}}'_1, \boldsymbol{\mathcal{G}}'_2, ..., \boldsymbol{\mathcal{G}}'_k\} \Rightarrow \forall n \in [1, ..., k], \exists \boldsymbol{m} \in \Sigma, \Gamma(\boldsymbol{\mathcal{G}}, \boldsymbol{m}) = \boldsymbol{\mathcal{G}}'_n$$

. In the classical setting, we need not worry about how Σ is structured, or how some $\boldsymbol{m} \in \Sigma$ may apply to other positions $\boldsymbol{\mathcal{G}} \in \mathbb{G}$. We simply care about the reachable positions from a particular position, and any other information contained in Σ is irrelevant. However, in the quantum setting, some move $\langle \boldsymbol{m} \rangle \in \Sigma^Q$ may apply to multiple positions in $\mathbb{G}$. For example, we could see that

$$\Gamma^Q(\langle \boldsymbol{\mathcal{G}}_1, \boldsymbol{\mathcal{G}}_2 \rangle, \langle \boldsymbol{m} \rangle) = \langle \boldsymbol{\mathcal{G}}_1', \boldsymbol{\mathcal{G}}_2' \rangle$$

or that

$$\Gamma^Q(\langle \boldsymbol{\mathcal{G}}_1, \boldsymbol{\mathcal{G}}_2 \rangle, \langle \boldsymbol{m} \rangle) = \langle \boldsymbol{\mathcal{G}}_1' \rangle$$

, depending on if $\boldsymbol{m}$ is legal for $\boldsymbol{\mathcal{G}}_2$ or not. This means that the way that we define and structure Σ is pivotal to the resulting behavior of the game in a quantum setting.

3. Lessons from Nim and Hackenbush

The game of Nim [9] is an excellent example of how subtle changes to the definition of movement can impact gameplay. In the classical setting, Nim is played with piles of counters. On every turn, a player may select any pile, and remove a nonzero number of counters from that pile. The player to remove the last counter from the game wins, per normal play. Within the papers by Burke [7, 8] and Dorbec [6], movement in Nim is defined as a "selection" of a pile, alongside a number signifying how many counters the player would like to take. This is a perfectly acceptable interpretation, as it encompasses all legal moves outlined by Nim's rules. This particular interpretation of Nim can be shown to diverge from classical Nim when it is lifted into a quantum context. In Burke's paper [7], $\texttt{Nim}^Q(2,2)$ is shown to be $\mathcal{N}$, rather than the $\mathcal{P}$ which would arise under classical gameplay.

However, let us embark on a brief motivational interlude. Consider the game Hackenbush [2, 5, 10], a game played on "tree" structures where players take turns pruning edges from the tree, with the additional rule that any tree segment disconnected from the ground will vanish. For our purposes, we will only consider the impartial (commonly called "green") version of Hackenbush. It is a commonly known fact that any game of Nim can be "embedded" into green Hackenbush - that is, for any Nim position $\boldsymbol{N}$, there exists some Hackenbush position whose game tree is isomorphic to $\boldsymbol{N}$. This Hackenbush position is easily constructed by forming "towers" with heights corresponding to the size of the piles in the equivalent Nim game.

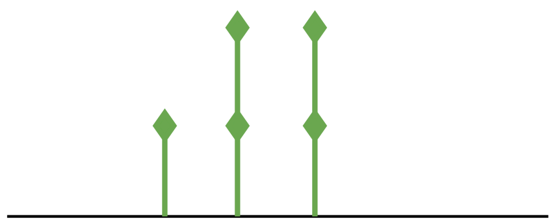

FIGURE 1. A Hackenbush position corresponding to the game $\texttt{Nim}(1, 2, 2)$

The equivalence between paired Nim and Hackenbush games can be proven by examining the options in each game. Specifically, the options of a tower of height $\boldsymbol{n}$ are equivalent to the options of a pile of size $\boldsymbol{n}$, meaning that the two games are structurally identical.

With this ability to embed Nim into Hackenbush, a curious mathematician may wonder if a similar trick can be applied within the quantum variants of each game. However, we quickly run into a conceptual roadblock: given the definition of movement as an act of subtraction (which was suggested before), there is no equivalent action of a Hackenbush position that represents "subtraction". To illustrate this issue with an example, consider the superposition $\boldsymbol{\mathcal{G}}^Q = \langle \texttt{Nim}(1, 2), \texttt{Nim}(3, 4)\rangle$, and the move $\boldsymbol{\mathcal{M}}^Q = \langle \texttt{Subtract 1 from the first pile}\rangle$. Intuitively, we see that

$$\boldsymbol{\mathcal{G}}^Q \times \boldsymbol{\mathcal{M}}^Q = \langle \texttt{Nim}(0, 2), \texttt{Nim}(2, 4)\rangle$$

.

Let us perform a similar construction for Hackenbush. Let $\texttt{Tower}(\texttt{a}, \texttt{b}, \texttt{c}, ...)$ be a Hackenbush position consisting of parallel "towers" of respective heights a, b, c, etc. If we construct an analogous superposition $\boldsymbol{\mathcal{H}}^Q = \langle \texttt{Tower}(1, 2), \texttt{Tower}(3, 4)\rangle$. But if we try and come up with a move that will yield $\langle \texttt{Tower}(0, 2), \texttt{Tower}(2, 4)\rangle$ as an outcome, we encounter a problem. We don't have a well-defined notion of "subtraction" in Hackenbush! Consider the following Hackenbush diagram:

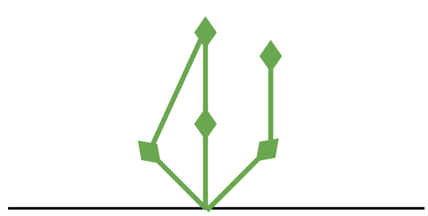

FIGURE 2. A complex Hackenbush position

What does it mean to "subtract 1" from this position? Intuitively, the operation simply does not make much sense. However, it makes sense to talk about cutting specific, identifiable branches as moves. But if we define the move $\boldsymbol{\mathcal{M}}^Q = \langle\texttt{Cut the bottom branch of the first tower}\rangle$, our resulting position

$$\boldsymbol{\mathcal{H}}^Q \times \boldsymbol{\mathcal{M}}^Q = \langle \texttt{Tower}(0,2),\, \texttt{Tower}(0,4)\rangle$$

is still not the same as the outcome with our Nim move. Instead of a subtraction operation, the equivalent interpretation of this move in a Nim context would be to "set" a pile to a particular, legal value.

3.1. **Setter vs Taker Nim.**

Let us further explore the interpretation of moves on Nim being "set" operations instead of "take" operations. We may first make the following assertion:

Theorem 3.1.1. *Quantum Setter Nim may be embedded into Quantum Green Hackenbush*

Proof. Given a superimposed Nim position $\boldsymbol{\mathcal{N}}^Q$, construct an analogous superposition of Hackenbush `Towers`, $\boldsymbol{\mathcal{H}}^Q$. For any legal superimposed move on $\boldsymbol{\mathcal{N}}^Q$, there exists an isomorphic move on $\boldsymbol{\mathcal{H}}^Q$, since a mapping exists between the legal moves on each position (the conversion of `Cut the n-th edge on the K-th tower` to `Set the K-th pile to n-1`). □

As is evident from our previous example, the move alphabet for the original implementation of Nim, Σ_{take}, is incompatible with the alphabet of our new interpretation, Σ_{set}. The former is composed of moves of the form `Take N from pile M`, whereas the latter moves have the form `Set pile M to N`. In the classical setting, these two move alphabets are still compliant with Nim gameplay – for any legal subtractor move, there is an equivalent setter move, and vice versa. It is only under the quantum lift of each respective game, $\texttt{Nim}^Q_{\text{set}}$ vs $\texttt{Nim}^Q_{\text{take}}$, that differences in gameplay may emerge. This last remark is also a reflection of the broader fact:

Theorem 3.1.2. *For all games* Γ*,* $\Gamma \subseteq \Gamma^Q$

Proof. Consider all superimposed moves and positions on Γ^Q with a size of 1 (also called "1-wide" positions and moves [7]). Applying only moves on a 1-wide starting position will yield the classical possibilities of Γ, thus, it is embedded in Γ^Q □

To further illustrate the differences between "setter" and "taker" interpretations, we present a table of computed Grundy values for various 2-pile configurations of quantum Nim in Figure 3. Unfortunately, computation of these positions quickly becomes intractable, so the sizes of these tables is limited. To illustrate the differences for more Nim configurations, we also present the computed Grundy values for games restricted to 2-wide moves ($\boldsymbol{m} \in \boldsymbol{\mathcal{M}}^Q, |m| \leq 2$).

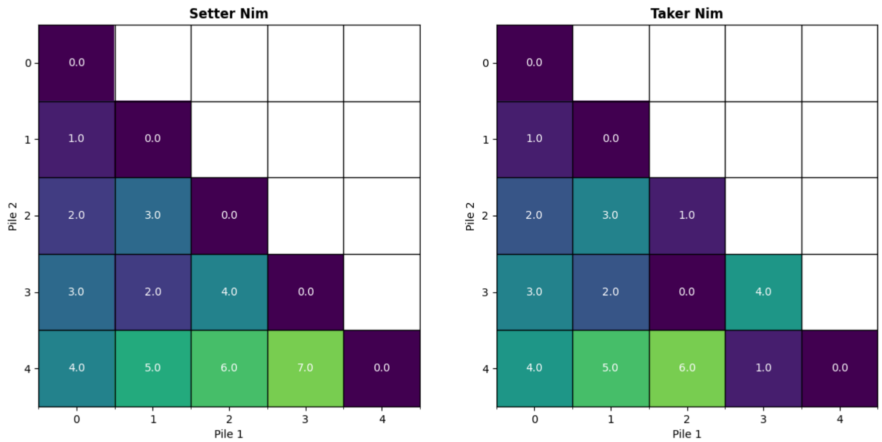

FIGURE 3. Tables of values for quantum Nim interpretations

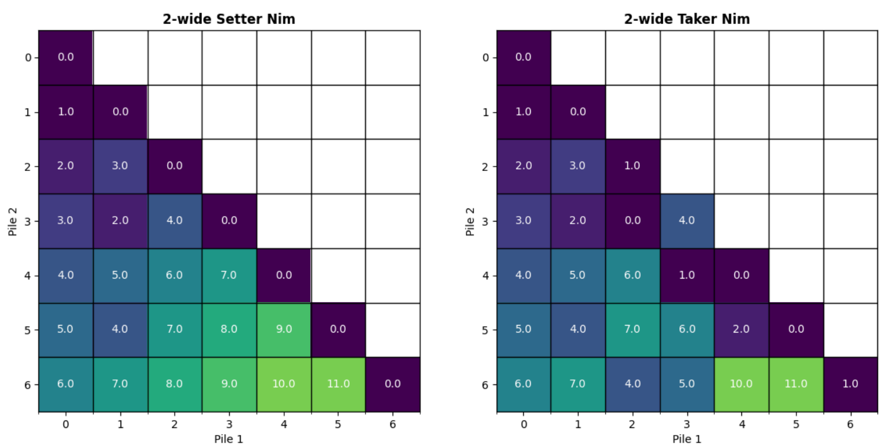

FIGURE 4. Tables of values for quantum Nim, with only 2-wide moves

There are a number of fascinating patterns within these two results alone; some of which may certainly warrant more than the few sentences that will be provided as commentary here. Curiously, the setter Nim outcomes are remarkably close to the outcomes of classical Nim; the only game of (unrestricted) setter Nim that differed in outcome value was the game $\texttt{Nim}_{\texttt{Setter}}(3, 2)$. Taker Nim, in contrast, demonstrates greater deviation from classical Nim, which may suggest that this particular interpretation of Nim is actually less faithful to the classical game in some measures. The significance (or lack thereof) of these computational results is left open for further inquiry.

4. Interpretations of Quantum Games

An attentive reader may notice the repeated use of the word "interpretation" when discussing the differences between setter Nim and taker Nim. We believe this to be an apt term for any particular family of games that have equivalent gameplay under a classical restriction, but differ in their quantum gameplay. The term is primarily motivated by how physicists and philosophers have devised many interpretations of quantum theory in an attempt to form connections between quantum behaviors and broader, well-understood classical phenomena.

To formalize the notion of an interpretation, we can construct the set of all valid move alphabets for a particular game. Let us call this set $\boldsymbol{\xi}$. As an example, in the case of Nim, $\Sigma_{\text{Take}}, \Sigma_{\text{Set}} \in \boldsymbol{\xi}_{\texttt{Nim}}$.

For a particular alphabet to be in this set, the following must hold:

$$\begin{gathered}\Sigma \in \boldsymbol{\xi} \Leftrightarrow \\ \forall \boldsymbol{\mathcal{G}} = \{\boldsymbol{\mathcal{G}}'_1, \boldsymbol{\mathcal{G}}'_2, ..., \boldsymbol{\mathcal{G}}'_k\} \in \mathbb{G}, \forall n \in [1, ..., k], \exists \boldsymbol{m} \in \Sigma, \boldsymbol{\mathcal{G}} \times \boldsymbol{m} = \boldsymbol{\mathcal{G}}_n \\ \wedge\, \boldsymbol{\mathcal{G}}' \notin \boldsymbol{\mathcal{G}} \Rightarrow \nexists \boldsymbol{m}' \in \Sigma, \boldsymbol{\mathcal{G}} \times \boldsymbol{m}' = \boldsymbol{\mathcal{G}}' \\ \wedge\, \boldsymbol{m}_1, \boldsymbol{m}_2 \in \Sigma, \boldsymbol{\mathcal{G}} \times \boldsymbol{m}_1 = \boldsymbol{\mathcal{G}} \times \boldsymbol{m}_2 \Rightarrow \boldsymbol{m}_1 = \boldsymbol{m}_1\end{gathered}$$

That is, the necessary and sufficient condition for $\Sigma \in \boldsymbol{\xi}$ is that, for a position $\boldsymbol{\mathcal{G}}$ with options $\{\boldsymbol{\mathcal{G}}_1, ..., \boldsymbol{\mathcal{G}}_k\}$, the only legal moves that exist in Σ are those that will result in one of the listed options, and, furthermore, that no two unique moves will map $\boldsymbol{\mathcal{G}}$ to the same option. With this definition of $\boldsymbol{\xi}$, we can formalize our previously touched-upon assertion:

Theorem 4.1. *For any move alphabet Σ in the interpretation set $\boldsymbol{\xi}$ of a game, all interpretations are equivalent under classical gameplay*

Proof. This follows directly from our definition of $\boldsymbol{\xi}$. Say we have two arbitrary alphabets $\Sigma_1, \Sigma_2 \in \boldsymbol{\xi}$ If we consider the game $\Gamma_1 : \mathbb{G} \times \Sigma_1 \to \mathbb{G} \cup \{\text{NULL}\}$, the options of a position $\boldsymbol{\mathcal{G}}$ is the set $\{\Gamma_1(\boldsymbol{\mathcal{G}}, \boldsymbol{m}) \neq \text{NULL} : \boldsymbol{m} \in \Sigma_1\}$, which must be equivalent to the corresponding option set generated by Γ_2 using Σ_2 by definition of $\boldsymbol{\xi}$. Thus, the options for any position are equivalent under any two arbitrary movement alphabets Σ_1, Σ_2, and thus classical gameplay is equivalent. $\square$

With our new definition of an interpretation $\boldsymbol{\xi}$ for any game, we can explore various other ways of constructing alphabets that may reside in $\boldsymbol{\xi}$.

4.1. **Exploring Interpretations.**

We may begin our exploration with a particular observation about setter Nim and taker Nim: moves in taker Nim may be repeated, but moves in setter Nim may never be repeated. In an abstract sense, one might think of taker Nim as possessing greater “flexibility” than setter Nim as a consequence of this fact. This intuition is difficult to formalize, but it can point us in a direction to discover a new possible interpretation – one that seeks “maximum rigidity”.

4.1.1. *The Disjoint Interpretation.*

Consider an alphabet Σ where, for all valid moves $\boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{G}}'$, there is a single, unique move corresponding to this transition of positions that cannot apply to any other position. Formally, enforcing that

$$\forall \boldsymbol{\mathcal{G}}_1, \boldsymbol{\mathcal{G}}_2 \in \mathbb{G}, \boldsymbol{m} \in \Sigma, \boldsymbol{\mathcal{G}}_1 \neq \boldsymbol{\mathcal{G}}_2, \Gamma(\boldsymbol{\mathcal{G}}_1, \boldsymbol{m}) \neq \text{NULL} \Rightarrow \Gamma(\boldsymbol{\mathcal{G}}_2, \boldsymbol{m}) = \text{NULL}$$

will yield an alphabet that we can call $\Sigma_{\texttt{disjoint}}$.

This alphabet has a number of important properties. For instance, in the quantum lift of any disjoint interpretation of a game $\Gamma^Q_{\texttt{disjoint}}$, the number of reachable positions in the game tree is maximal among all $\Sigma \in \boldsymbol{\xi}$.

Theorem 4.1.1.2. *The positions reachable in* $\Gamma^Q_{\texttt{disjoint}}$ *is maximal among all interpretations.*

Proof. Consider an arbitrary alphabet Σ in $\boldsymbol{\xi}$. Let some quantum move $\boldsymbol{m}^Q \in \Sigma^Q$ operate on a superposition $\boldsymbol{\mathcal{G}}^Q \in \mathbb{G}^Q$. The resulting superposition will be

$$\boldsymbol{\mathcal{G}}^Q \times \boldsymbol{m}^Q = \langle g \times m : g \in \boldsymbol{\mathcal{G}}^Q,\ \ m \in \boldsymbol{m}^Q,\, g \times m \neq \text{NULL} \rangle$$

. We may construct an equivalent move

$$\boldsymbol{M}^Q_{\texttt{disjoint}} \in \Sigma^Q_{\texttt{disjoint}} = \langle g \to g \times m : g \in \boldsymbol{\mathcal{G}}^Q,\ \ m \in \boldsymbol{m}^Q,\, g \times m \neq \text{NULL} \rangle$$

, that, by construction, will yield the same outcome superposition. Thus, the number of reachable positions in $\Gamma^Q_{\texttt{disjoint}}$ cannot be lower than those of Γ^Q_{Σ}. □

With $\Sigma_{\texttt{disjoint}}$, we also acquire a new perspective on how all other alphabets in $\boldsymbol{\xi}$ may be constructed. Consider a segment of our disjoint map $\Gamma_{\texttt{disjoint}}$:

	$\boldsymbol{m_1}$	$\boldsymbol{m_2}$	...	$\boldsymbol{m_a}$	$\boldsymbol{m_b}$	...
$\boldsymbol{\mathcal{G}}_1$	$\boldsymbol{\mathcal{G}}'_{1,1}$	$\boldsymbol{\mathcal{G}}'_{1,2}$				
$\vdots$			$\ddots$			
$\boldsymbol{\mathcal{G}}_n$				$\boldsymbol{\mathcal{G}}'_{\text{n,a}}$	$\boldsymbol{\mathcal{G}}'_{\text{n,b}}$	
$\vdots$						$\ddots$

Each move applies to a single position, as is the condition for our disjoint interpretation. Now consider some other non-disjoint alphabet $\Sigma_{\texttt{other}}$

	$\boldsymbol{m_x}$	$\boldsymbol{m_y}$	...
$\mathcal{G}_1$	$\mathcal{G}'_{\text{1,x}}$	$\mathcal{G}'_{1,2}$	
$\vdots$			$\ddots$
$\mathcal{G}_n$	$\mathcal{G}'_{\text{n,x}}$	$\mathcal{G}'_{\text{n,y}}$	
$\vdots$			$\ddots$

This particular interpretation is constructed by a surjective mapping $\varphi : \Sigma_{\texttt{disjoint}} \to \Sigma_{\texttt{other}}$, where $\boldsymbol{m_1}, \boldsymbol{m_a} \to \boldsymbol{m_x}$ and $\boldsymbol{m_2}, \boldsymbol{m_b} \to \boldsymbol{m_y}$. This example is arbitrary, but we can very naturally extend this into our examples with Nim. Applying the surjective map that sends all moves setting some arbitrary pile K to a value N to a single unique move will yield the setter Nim interpretation. Likewise, the map from all moves that remove some N from a pile K to a single move will yield taker Nim. Following from this observation, we can assert the following:

Theorem 4.1.1.3. *Within some finite game (or a finite subset of an infinite game),* $|\Sigma_{\texttt{disjoint}}| \geq |\Sigma| \; \forall \Sigma \in \boldsymbol{\xi}$

Proof. A surjective map can exist from $\Sigma_{\texttt{disjoint}}$ to any other $\Sigma \in \boldsymbol{\xi}$ via the technique described above. Such a mapping cannot increase cardinality, and thus, the proposition must hold. $\square$

This result also explains why disjoint game interpretations are computationally intractable in a quantum context. The large amount of moves available in $\Sigma_{\texttt{disjoint}}$ results in rapid growth for the number of possible combinations in a superimposed move.

4.1.2. *The Anti-Disjoint Interpretation.*

With our disjoint interpretation of any particular game, we may now cast our gaze to the opposite end of the "flexibility spectrum". What would an interpretation that is "maximally flexible" look like? One reasonable condition might be that, for any move $\boldsymbol{m} \in \Sigma$, $\boldsymbol{m}$ must be applicable to all $\mathcal{G} \in \mathbb{G}$ (assuming that $\mathcal{G}$ is not a terminal position). We may construct a lower bound on the size of an anti-disjoint interpretation $\Sigma_{\texttt{anti}}$ with the maximum number of options for any $\mathcal{G} \in \mathbb{G}$. If $\Sigma_{\texttt{anti}}$ were to possess fewer than this number of moves, it would not fully represent the original game, and would thus be excluded from $\boldsymbol{\xi}$.

A natural follow-up question is whether this $\Sigma_{\texttt{anti}}$ can even exist. Such an interpretation can exist, but is not necessarily guaranteed to. If there are positions where the number of options is nonzero but is less than the maximal number of options, no such $\Sigma_{\texttt{anti}}$ can exist. Via the pigeonhole principle, for any position with less-than-maximal option count, there would need to exist two $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \Sigma$ that would map to the same option, which would imply that $\boldsymbol{m}_1 = \boldsymbol{m}_2$ by the condition for membership in $\boldsymbol{\xi}$. This would then violate the lower bound for the number of elements needed in a $\Sigma_{\texttt{anti}}$ – and thus, such an interpretation cannot exist. One such game that would satisfy this requirement is the game "love-petal" [2], as there is only a single option from any position N: the position $N - 1$.

Alternatively, we may loosen are requirements posed for an interpretation. The requirement that a single, unique move must correspond to a single option is the condition allowing us to draw a surjective mapping from $\Sigma_{\texttt{disjoint}}$ to any other $\Sigma \in \boldsymbol{\xi}$. However, if we discard this property, we may now consider interpretations for $\Sigma_{\texttt{anti}}$ that are classically consistent, but which may have the un-intuitive property that multiple moves may map to a single option from any arbitrary position.

Consider the Nim game $\texttt{Nim}(\texttt{a}, \texttt{b}, \texttt{c}, ...)$. There are $N = ((a-1)+(b-1)+(c-1)+...)$ options available from this position, meaning that any $\Sigma_{\texttt{anti}}$ interpretation must have at least N many moves. For Nim, both setter and taker Nim meet this lower bound. However, neither setter nor taker meet the requirement that each possible move is applicable to all game states. The solution is simple: for any move $\boldsymbol{m} \in \Sigma$ that would have been illegal on a position $\boldsymbol{\mathcal{G}}$, $(\Gamma(\boldsymbol{\mathcal{G}}, \boldsymbol{m}) = \text{NULL})$, map $\Gamma(\boldsymbol{\mathcal{G}}, \boldsymbol{m}) = \texttt{Arbitrary option } \boldsymbol{\mathcal{G}}' \in \boldsymbol{\mathcal{G}}$. Our conditions for anti-disjointness are now satisfied (all moves apply to all positions), and the classical gameplay requirement is also satisfied (all $\boldsymbol{\mathcal{G}}$ only have moves that map to a legal option).

This conclusion is less satisfying that that of the disjoint interpretation, but still holds the possibility of further investigation. Even if there are a large number of feasible anti-disjoint interpretations with the loosened restriction, their gameplay under quantum lift may still hold interesting properties.

4.1.3. *Beyond Classical Fesibility.*

To a reckless enthusiast of combinatorial games, the loosening of requirements demonstrated within the previous segment may have spurred further curiosity. Such a mind may come to the conclusion that we have been rather reserved with our construction of surjective mappings from $\Sigma_{\texttt{disjoint}} \to \Sigma \in \boldsymbol{\xi}$. After all, these should remind the reader of equivalence classes being constructed upon a set of moves. Alongside the partition of a set into entirely disjoint equivalence classes, the other, trivial equivalence relation is that in which every element is equal; so we could just send every move in $\Sigma_{\texttt{disjoint}}$ to a single move!

This is ill-advised. If you wish for this interpretation to be classically playable, this is not the way to do it. In this interpretation, every position $\boldsymbol{\mathcal{G}}$ is sent to every single one of its options, all at the same time. This makes absolutely no sense in a classical setting, so this is clearly not a classically feasible interpretation of the game. However, in the quantum lift of a game, this interpretation isn't entirely useless.

Let us denote this interpretation as $\Sigma_{\texttt{single}}$, consisting of a single move $\boldsymbol{M}$. On any position $\boldsymbol{\mathcal{G}} = \{\boldsymbol{\mathcal{G}}'_1, \boldsymbol{\mathcal{G}}'_2, ..., \boldsymbol{\mathcal{G}}'_k\}$, the move $\boldsymbol{\mathcal{G}} \times \boldsymbol{M} = \langle \boldsymbol{\mathcal{G}}'_1,\ \ \boldsymbol{\mathcal{G}}'_2,\ ..., \boldsymbol{\mathcal{G}}'_k \rangle$. With this definition, we can prove the following:

Theorem 4.1.3.4. *Given a classical game Γ on some position $\boldsymbol{\mathcal{G}} \in \mathbb{G}$, the interpretation $\Gamma^{Q}_{\texttt{single}}$ will have an outcome value equivalent to that of Love-petal on the maximum tree depth of $\boldsymbol{\mathcal{G}}$ in Γ*

Proof. Performing a movement of $\boldsymbol{M}$ on $\boldsymbol{\mathcal{G}}$ will traverse every single branch rooted at $\boldsymbol{\mathcal{G}}$. Repeated application of $\boldsymbol{M}$ will traverse the entire game tree. This process can only terminate when no sequence of moves in Γ could lead to a non-terminal position – which occurs at maximal tree depth; since this particular sequence of moves in Γ must be encompassed in the sequence of repeated movement by $\boldsymbol{M}$

(because $\boldsymbol{M}$, by definition, will traverse every available option), the number, n, of consecutive movements, $\boldsymbol{M}^n$, must be equal to the depth of Γ at $\boldsymbol{\mathcal{G}}$. A guaranteed sequence of n moves occurring is then outcome-equivalent to `Love-petal`(n) □

On a related and similarly intuitive note, it is also known that the game-tree depth of any quantum game Γ^Q at a 1-wide position $\boldsymbol{\mathcal{G}} \in \mathbb{G}^Q$ cannot exceed the classical depth of the tree when playing Γ on $\boldsymbol{\mathcal{G}}$ [7].

The existence of $\Sigma_{\texttt{single}}$ also suggests a spectrum of other classically-infeasible but quantum-viable interpretations for any game. Given the relaxed restrictions suggested in sections 4.1.3 and 4.1.2, let us call these variants of games "informal interpretations", denoted $\boldsymbol{\xi^i}$. For an informal interpretation,

$$\begin{gathered}\Sigma \in \boldsymbol{\xi^i} \Leftrightarrow \\ \forall \boldsymbol{\mathcal{G}} = \{\boldsymbol{\mathcal{G}}'_1, \boldsymbol{\mathcal{G}}'_2, ..., \boldsymbol{\mathcal{G}}'_k\} \in \mathbb{G}, \forall n \in [1, ..., k], \exists \boldsymbol{m} \in \Sigma, \boldsymbol{\mathcal{G}} \times \boldsymbol{m} \ni \boldsymbol{\mathcal{G}}_n \\ \wedge\, \boldsymbol{\mathcal{G}}' \notin \boldsymbol{\mathcal{G}} \Rightarrow \nexists \boldsymbol{m}' \in \Sigma, \boldsymbol{\mathcal{G}} \times \boldsymbol{m}' \ni \boldsymbol{\mathcal{G}}'\end{gathered}$$

,

with the notable distinctions from $\boldsymbol{\xi}$ being that moves need not yield single options when applied to a position $\boldsymbol{\mathcal{G}}$, and that multiple moves are permitted to map to a single option. However, the restriction that moves may not map to positions outside of the classical options of $\boldsymbol{\mathcal{G}}$ is still inherited. With this definition, we also know that $\boldsymbol{\xi} \subseteq \boldsymbol{\xi^i}$.

5. Conclusions and Open Questions

The world of quantum combinatorial games is fascinating and vast. From the conclusions drawn in this paper, the existence of countless unique interpretations existing for any particular classical game hints at the near-infinite number of opportunities for further inquiry within the field. Considerable effort has been put into studying the computational complexity of various quantum games [7, 8], and we believe that there may be a non-negligible connection between the "complexity" of a particular interpretation, and the resulting computational complexity of the quantum game.

Construction of non-intuitive interpretations may be performed computationally through construction from the disjoint interpretation, as we discussed in section 4.1.1. Other interpretations may also be discovered or motivated by the problem of embedding classically related games in a quantum setting, as was the case with Hackenbush and Nim.

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