How to Solve Integration Problems

Learn what integration problems are. Discover how to find integration sums and how to solve integral calculus problems using calculus example problems.

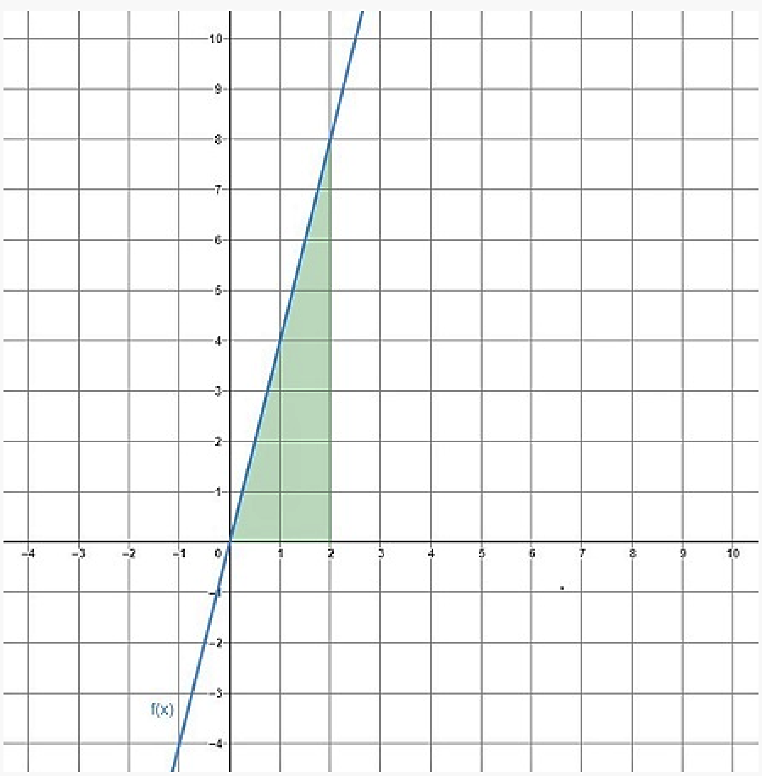
Integral Calculus Problems

What is integral calculus? While **differential calculus** focuses on rates of change, **integral calculus** deals with totals of lengths, areas, and volumes. An integral is a numerical value equal to the area under a graph, but it could also be the equation of an original function that was derived. Recall that a **function** is an equation where every input has exactly one unique output. For example, $f(x) = 3x^2 + 4$ is a parabolic function, and the ordered pair $(2, 16)$ is one of many solutions to $f(x)$, but there are no other solutions that will also have 16 as an output. Integral calculus problems are used to undo derivatives, or work backward, starting from the differential equation towards finding the original function that was derived.

The notation used to represent an integral involves an elongated S, $\int$, called an integral sign followed by some type of function and then a variable of integration. Such as $\int 4x\,dx$. Here $4x$ represents the function and dx represents that the function is being integrated with respect to the variable x. There are two types of integrals, definite and indefinite.

Definite Integrals

Definite integrals can be solved analytically or graphically. A definite integral will result in a numerical value and involves limits of integration within the notation. For example, $\int_0^2 4x\,dx$. To break this down, the **limits of integration** are represented by the 0 and the 2. These endpoints are used to evaluate (or solve) the integral. What this means graphically is that the numerical value is a result of the area under the graph of $f(x) = 4x$, but strictly the region between $x = 0$ and $x = 2$.



Graph of f(x)=4x with the area under the graph between x=0 and x=2 shaded in.

The numerical value of $\int_0^2 4x\,dx$ is 8, because that is the calculated area found under the graph of $f(x)$ between $x=0$ and $x=2$. Looking at the graph of $\int_0^2 4x\,dx$, the green region representing the area below the graph and above the x-axis is in the shape of a triangle. The area of a triangle can be found by using the formula $A=\frac{1}{2}\cdot b\cdot h$. Since the triangle has a base length of 2 and a height of 8 units, the $area$ is $\frac{1}{2}\cdot 2\cdot 8=\frac{1}{2}\cdot 16=8$. Definite integrals can also be evaluated analytically (or algebraically). To evaluate integrals analytically, a specific set of rules are followed.

Indefinite Integrals

Finding **indefinite** integrals will result in an equation of a general function. This type of integral is less exact, or not definite, because a part of the function cannot be deduced so a constant of integration is used. The **constant of integration**, or $+C$, is used for the portion of the function that is unknown. This term is a constant value. Since the constant could represent any number (even zero), $+C$ is used as a placeholder. For the most part, indefinite integrals are evaluated analytically.

For example: $\int 4x\,dx$ becomes $2x^2+C$

Integration Problems

Integration problems can involve different types of functions and different rules of integration. To evaluate integration problems, first identify the type of function. That will be useful to determine what rule to use. It is also important to have background knowledge on solving derivative problems since integration involves derivatives.

How to Solve Integration Problems

Recall that integration problems can be solved using the graphs of the functions and the area under the graph. However, the focus here will be how to solve integration problems algebraically. The most used rule will be the **power rule**, though there's also the **constant rule**, rules for reciprocal and exponential functions, the substitution rule for composite functions, and specific rules for trigonometric functions.

Monomials

The most basic function is made up of a single term, for example $f(x)=4x$ only has one term, $4x$. This single term is called a **monomial**. When integrating monomials, the **degree** of the function is used. The degree of the function is the highest power in the function. For $f(x)=4x$, the x has a power of one thus the function has a degree of one. To integrate this function the power rule is used. The power rule is as follows:

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+C, /n\neq -1$$

This means that the exponent of x will increase by one degree and then the term will be divided by the new degree, $n+1$. Finally, since this is an indefinite integral, the constant of integration $+C$ is required. Notice how the exponent, n, cannot be -1 because then the denominator of the fraction created by the rule would be equal to zero. A fraction with a denominator of zero is $undefined$. When $n=-1$, the power rule may not be used.

Using the power rule, here is how to integrate $\int 4x\,dx$ step by step. Add a degree to the exponent of x, then divide the term by that new exponent.

$$\begin{aligned}\int 4x\,dx &= \frac{4x^{1+1}}{1+1}+C\\ &= \frac{4x^2}{2}+C\\ &= 2x^2+C\end{aligned}$$

Along with the power rule is the **constant rule**. The constant rules allows for a constant in an integral to be moved to the outer part of the integral to make it easier to solve the integral. For example, take the function from before $f(x) = 4x$ when solving $\int 4xdx$, the integral can be rewritten as $4\int xdx$. The integral would be solved with the power rule and the exponent would increase by one and the term x would still be divided by the new exponent which is 2 but the 4 is multiplied back in after the power rule is used. The result is the same. The constant rule is helpful but not always necessary.

$$\begin{aligned}\int 4x\,dx &= 4\int x\,dx \\ &= 4\frac{x^{1+1}}{1+1}+C \\ &= \frac{4x^2}{2}+C \\ &= 2x^2+C\end{aligned}$$

A monomial integral problem could also look like this: $\int 5\,dx$. Whenever the integral problem is just a constant, then the solution will be $5x + C$. Since no variable exists in this problem, imagine that x simply has a power of zero, like this:

$$\begin{aligned}\int 5\,dx &= \int 5x^0\,dx \\ &= \frac{5x^{0+1}}{0+1}+C \\ &= 5x+C\end{aligned}$$

Here is another example of using the power rule to solve an indefinite integral involving a monomial:

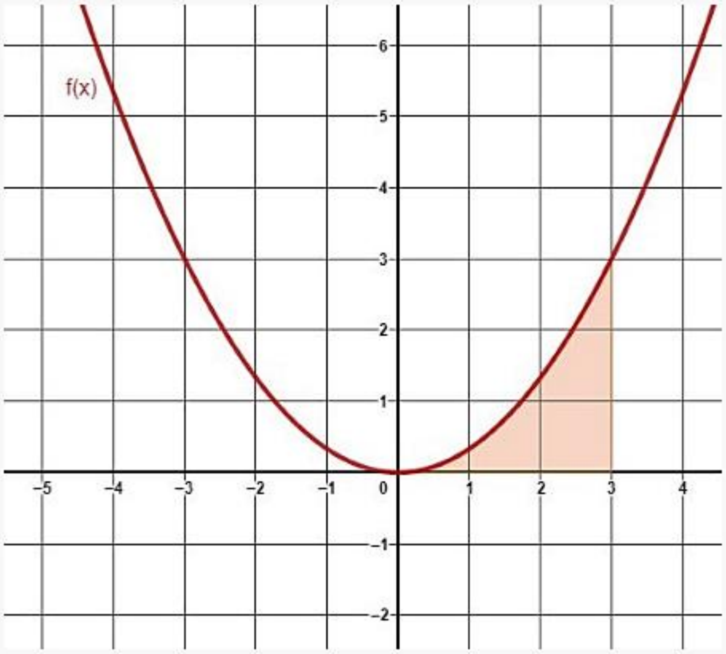
$\int 12x^3\,dx$, the solution to this integral is

$$\begin{aligned}\int 12x^3\,dx &= \frac{12x^{3+1}}{3+1}+C \\ &= \frac{12x^4}{4}+C \\ &= \frac{12}{4}x^4+C \\ &= 3x^4+C\end{aligned}$$

As mentioned, definite integrals can also be solved algebraically but the result is a numerical value instead of an equation. Using the power rule, the integral $\int_0^6 \frac{1}{3}x^2\,dx$ would be solved by evaluating the integral at the proposed limits of integration. First, use the power rule to find the new function, then take turns plugging in the limits of integration. Then finally subtract those values. Here is the step by step:

$$\begin{aligned}\int_0^3 \frac{1}{3}x^2\,dx &= \frac{1}{3}\int_0^3 x^2\,dx \\ &= \frac{1}{3}\cdot\frac{x^{2+1}}{2+1} \\ &= \frac{1}{3}\cdot\frac{x^3}{3} \\ &= \frac{1}{3}\cdot[(\frac{3^3}{3})-(\frac{0^3}{3})] \\ &= \frac{1}{3}\cdot[(\frac{27}{3})-(\frac{0}{3})] \\ &= \frac{1}{3}\cdot[9-0] \\ &= \frac{9}{3} \\ &= 3\end{aligned}$$

This is equivalent to graphing $f(x) = \frac{1}{3}x^2$ and finding the area under the graph of f(x), between $x = 0$ and $x = 3$.



The graph of f(x)=1/3 x^2 with the area under the graph between x=0 and x=3 shaded in.

Reciprocals

A reciprocal function looks like this $f(x) = \frac{1}{x}$. This function, in particular, can be written as a negative exponent instead of as a fraction like so: $f(x) = x^{-1}$. The power rule can not be used with this function because when the exponent is raised by one degree it becomes x^0 and a fraction with a denominator of zero.

$$\begin{aligned} \int \frac{1}{x}\,dx &= \int x^{-1}\,dx \\ &= \frac{x^{-1+1}}{-1+1} + C \\ &= \frac{x^0}{0} + C \end{aligned}$$

This is a problem because fractions can not have a denominator of zero. This integral problem cannot be solved using the power rule is because it is related to the natural log function. Remember that derivatives are the prerequisite to integrals. The derivative of $ln\ x$ is $\frac{1}{x}$.

$\frac{d}{dx}[ln\ x] = \frac{1}{x}$

Which means that the integral of $\frac{1}{x}$ is $ln\ x$

$$\int \frac{1}{x}\,dx = ln\ |x| + C$$

The absolute value bars are necessary because $ln\ x$ is not defined for $x < 0$.

The integral rule for $ln\ x$ is:

$$\int ln\ x\ dx = x(ln\ x - 1) + C$$

Example: $\int 2\ ln\ x\ dx = 2\int ln\ x\ dx$

$2x \cdot (ln\ x - 1) + C$

Here is an example of a reciprocal function that can be solved using the power rule:

$$\begin{aligned}\int \frac{6}{x^3}\,dx &= \int 6x^{-3}\,dx\\ &= 6\int x^{-3}\,dx\\ &= 6\frac{x^{-3+1}}{-3+1}+C\\ &= 6\frac{x^{-2}}{-2}+C\\ &= \frac{6}{-2}x^{-2}+C\\ &= -3x^{-2}+C\\ &= \frac{-3}{x^2}+C\end{aligned}$$

Exponentials

An exponential function is a function of the form $f(x) = a^x$ where a is a constant and $a > 0$ but $a \neq 1$. The most popular exponential function is e^x.

The rules for integrating exponential functions are:

$\int a^x\,dx = \frac{a^x}{ln\ a} + C$ and $\int e^x\,dx = e^x + C$.

Example: evaluate $\int 5^x\,dx$.

$$\int 5^x\,dx = \frac{5^x}{ln\ 5} + C$$

Example: evaluate $\int 2e^x\,dx$.

$$\begin{aligned}\int 2e^x\,dx &= 2\int e^x\,dx\\ &= 2e^x + C\end{aligned}$$

Example: evaluate $\int 7e^{x+2}\,dx$.

$$\begin{aligned}\int 7e^{x+2}\,dx &= 7\int e^{x+2}\,dx\\ &= 7e^{x+2} + C\end{aligned}$$

Trigonometric Functions

Trigonometry is the branch of mathematics dealing with the relationship between the sides and angles of triangles. The ratios used to study this relationship are called trigonometric ratios. The six ratios are sine, cosine, tangent, cotangent, secant, and cosecant. These ratios are used in trigonometric functions. The integration rules for trigonometric functions are complicated due to them all being different. Once again, background knowledge in derivatives is helpful. By knowing the derivative rules for these trigonometric functions, it is easier to learn the integral rules.

$$\begin{aligned}\int sin\ x\,dx &= -cos\ x + C\\ \int cos\ x\,dx &= sin\ x + C\\ \int sec^2\ x\,dx &= tan\ x + C\\ \int sec\ x\ tan\ x\,dx &= sec\ x + C\\ \int csc\ x\ cot\ x\,dx &= -csc\ x + C\\ \int csc^2\ x\,dx &= -cot\ x + C\end{aligned}$$

Integral Practice Problems

Now here are some calculus example problems. Put all of these rules to work, solve the following indefinite integral problems.

1.

$$\int 45x^4\,dx$$

2.

$$\int 3\,cos\,x\,dx$$

3.

$$\int 2\,ln\,x\,dx$$

4.

$$\int 7\,dx$$

5.

$$\int \frac{3}{x+6}\,dx$$

Solutions

1.

$$\begin{aligned}\int 45x^4\,dx &= \frac{45x^{4+1}}{4+1} + C\\ &= \frac{45x^5}{5} + C\\ &= \frac{45}{5}x^5 + C\\ &= 9x^5 + C\end{aligned}$$

2.

$$\begin{aligned}\int 3\,cos\,x\,dx &= 3\int cos\,x\,dx\\ &= 3\,sin\,x + C\end{aligned}$$

3.

$$\begin{aligned}\int 2\,ln\,x\,dx &= 2\int ln\,x\,dx\\ &= 2x\cdot(ln\,x - 1) + C\end{aligned}$$

4.

$$\begin{aligned}\int 7\,dx &= \int 7x^0\,dx\\ &= \frac{7x^{0+1}}{0+1} + C\\ &= 7x + C\end{aligned}$$

5.

$$\begin{aligned}\int \frac{3}{x+6}\,dx &= 3\int (x+6)^-1\,dx\\ &= 3\,ln\,|x+6| + C\end{aligned}$$

Integration Sums

The **sum rule of integration** problems is used when the integral problem is a made up of adding more than one function. It also looks like a bunch of monomials or a polynomial. To solve an integral problem with the sum rule, split the functions into separate integrals. For example:

$$\begin{aligned}\int 5x^4 + 3/dx &= \int 5x^4\,dx + \int 3/dx \\ &= \frac{5x^{4+1}}{4+1} + \frac{3x^{0+1}}{0+1} + C \\ &= \frac{5x^5}{5} + \frac{3x^1}{1} + C \\ &= \frac{5}{5}\cdot x^5 + \frac{3}{1}\cdot x + C \\ &= x^5 + 3x + C\end{aligned}$$

Lesson Summary

What is integral calculus? While **differential calculus** focuses on rates of change, **integral calculus** deals with totals of lengths, areas, and volumes. An integral is a numerical value equal to the area under a graph, but it could also be the equation of an original function that was derived. Recall that a **function** is an equation where every input has exactly one unique output. Integrals are the branch of calculus that deals with totals. There are two types of integrals, **definite** and **indefinite**. Definite integrals result in a numerical value. Indefinite integrals result in a new equation of a function. Definite integrals can be evaluated graphically or algebraically, whereas indefinite integrals are solved algebraically. To evaluate integrals algebraically, a specific set of rules are followed. The rule used depends on the function type. The rules covered in this lesson were the **power rule** and the **constant rule**. There are also specific rules that need to be followed for certain types of functions such as **reciprocal functions**, **exponential functions**, and **trigonometric functions**. Finally, the **sum rule of integration** is really solving a string of different functions all at once and can involve any of the rules, just working with one term at a time but integrating one function at a time.

For indefinite integrals, a **constant of integration** must always be added to the solution because it is a part of the solution that is unknown. To solve an indefinite integral using the power rule, each term has to increase by one degree (or exponent has to increase by one power). Then divide the term by the new power. For example, the function $f(x) = 6x^2$ has a degree (or exponent) of 2. Here is step by step of how the integral is solved:

$\int 6x^2\,dx = \frac{6x^{2+1}}{2+1} + C = \frac{6x^3}{3} + C = 2x^3 + C.$

To solve a definite integral, use the power rule to find the new equation but without the constant of integration since the **limits of integration** will be used to find the value of the integral, like so:

$$\begin{aligned}&\int_1^2 6x^2\,dx = 6\int_1^2 x^2\,dx \\ &= 6\cdot\frac{x^{2+1}}{2+1} \\ &= 6\cdot\frac{x^3}{3} \\ &= \frac{6}{3}\cdot x^3 \\ &= 2\cdot x^3\end{aligned}$$

plug in the limits of integration and subtract.

$$\begin{aligned}&= 2\cdot[(2^3) - (1^3)] \\ &= 2\cdot[8-1] \\ &= 2\cdot 7 \\ &= 14\end{aligned}$$