LINEAR ALGEBRA

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→ Linear algebra : the study of linear maps on finite dimensional vector spaces.

→ Problems of Type 1 :

$$a_{11}x_1 + a_{12}x_2 \cdots a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 \cdots a_{nn}x_n = b_n$$

Find a list of numbers $(s_1, s_2 \ldots s_n)$ that satisfy this system of equations.

n-tuple

• Gaussian elimination

→ Get the echelon form of the equations i.e.

$$c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + c_{14}x_4 = b_1$$
$$c_{22}x_2 + c_{23}x_3 + c_{24}x_4 = b_2$$
$$c_{33}x_3 + c_{34}x_4 = b_3$$
$$c_{44}x_4 = b_4$$

$m = 4, n = 4$ (rows, columns)

Thus, going bottom to top, we can get values of x_1, x_2, x_3, x_4

→ solutions can also be derived graphically i.e.

each equation is converted to a coordinate structure, and solutions are found at the places where these structures intersect

(A) 2-variable equations = lines that are not parallel to x / y axis
(B) 1-variable equations: line parallel to axes
(C) 3-variable equations = plane
⋮

✦ Linear Algebra: Hoffman - Kunze

→ Some notations :

matrix of dimension $m \times n$: $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

vector of dimension n : $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

→ Linear combination of vectors:

$a_1 \underbrace{V_1}_{\text{vector}} + a_2V_2 + a_3V_3 \ldots a_nV_n = b$

→ Problems of Type - II:

"Given $\vec{V_1}, \vec{V_2} \ldots \vec{V_n}$, is $\vec{b}$ a linear combination of these vectors?

→ terms from the definition:

(i) Linear Map: literally just a fn

(ii) finite dimension: a dimension to which more elements can be added

→ vector operation properties:

1) $\forall V_1, V_2 \in \underbrace{V}_{\text{vectors}}$ $\lambda_1 V_1 + \lambda_2 V_2 \in V$, $\lambda_1, \lambda_2 \in \mathbb{R}$

2) $V_1 + V_2 = V_2 + V_1$ (commutative)

$V_1 + (V_2 + V_3) = (V_1 + V_2) + V_3$ (associative)

3) $0 + \vec{V} = \vec{V} = \vec{V} + 0$, $\forall \vec{V} \in V$

4) $\vec{V} + (-\vec{V}) = 0$

5) $\lambda(\vec{V_1} + \vec{V_2}) = \lambda\vec{V_1} + \lambda\vec{V_2}$

} forms a vector field

———— X ————

07/01/25

* Assignments: ~20%
Quiz 1: 10%
Mid-Sem: 15-20%
Tutorial quizzes: ~%
50%

· Fields

→ A set F, with two binary operations, addition (+) and multiplication (·) satisfying the following rules:

closure is inherent and present by definition as binary operations cannot give results outside the set.

i) addition is commutative
i.e. $a+b = b+a$ $\forall a, b \in F$

ii) addition is associative
$a + (b+c) = (a+b) + c$
$\forall a, b, c \in F$

iii) $\exists$ a unique element '0' s.t. $a + 0 = 0$. $\forall a \in F$
not the number 0
additive identity

iv) $\forall a \in F$, $\exists (-a)$ s.t. $a + (-a) = 0$.
additive inverse
additive identity

} 4 rules of addition

v) Multiplication is commutative

$a \cdot b = b \cdot a \quad \forall\, a,b \in F$

vi) Multiplication is associative

$a(b \cdot c) = (a \cdot b) \cdot c \quad \forall\, a,b,c \in F$

vii) $\exists$ a unique, non-zero element, represented by '1', s.t. multiplicative identity

$a \cdot 1 = a \quad \forall\, a \in F$

viii) $\forall\, x \in F, x \neq 0, \exists\ x'$ s.t. $x \cdot x' = 1$.

multiplicative inverse

ix) multiplication is distributive over addition:

$a \cdot (b + c) = a \cdot b + a \cdot c \quad \forall\, a,b,c \in F$

5 rules of multiplication

i.e. the Field $= (F, +, \cdot)$, if the set F follows the above rules w.r.t. the binary operators

if F is an empty set, it cannot be a field as points 3 and 7 state that the set must contain a particular element.

$\therefore$ The set must contain at least 2 elements.

→ can the set contain exactly 2 elements?

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

$\{0,1\}$ satisfies all the rules w.r.t + and ·

$\therefore$ $(\{0,1\}, +, \cdot)$ is a field

$\therefore$ a field can contain exactly two elements.

→ prf: $(-1) \cdot x = -x$

A: $x + (-1)x$

$= 1 \cdot x + (-1) \cdot x$ (multiplicative identity)

$x(1 + (-1))$ (distributive)

$= x \cdot 0 = x \cdot 0 + 0 = x \cdot 0 + (x + (-x)) = (x \cdot 0 + x) + (-x)$

$x(0+1) = x \cdot 1 = x$

$= x + (-x)$

$= 0$

$= x + (-x)$

$= -x = (-1)x$

> ✦ Q1] For a field, $a, b, c \in F$,
> prove:
> a) if $ab = bc$, then $a = c$.
> b) $a + b = b + c$, then $a = c$
> ↳ deadline: Sunday midnight.

10/01/25.

• Subfield:

A set S is a subfield of a field $(F, +, \cdot)$ is $S \subset F$ and $(S, +, \cdot)$ is a field.

Eg. Real numbers are subfield of complex numbers w.r.t. + and ·

> ✦ Q2] Any subfield of a complex field must contain every rational number. Prove.

• System of linear equations

→ unknown scalars have degree '1'.

→ linear equations = coefficients + unknown scalars

↳ all coeffs must be from same field as each other and scalar.

↳ $x_1, x_2 \dots x_n$

i.e. consider

$$\left.\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \dots A_{1n}x_n &= y_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \dots A_{2n}x_n &= y_2 \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + A_{m3}x_3 \dots A_{mn}x_n &= y_m \end{aligned}\right\} \text{system of linear eqs.}$$

(→ linear eq)

A system in which $y_i = 0$ is called a homogenous. A system with even 1 $y_i \neq 0$ is called a non-homogenous.

Ex. $2x_1 + 3x_2 - 4x_3 = 0$ → 2 equations, 3 unknowns

$x_1 + x_2 + x_3 = 0$

$\times(-2)$ → $-2x_1 - 2x_2 - 2x_3$ +

$0 + x_2 - 6x_3 = 0$

$x_2 = 6x_3$ $\quad\therefore (x_1, x_2, x_3) = (-7x_3, 6x_3, x_3)$

$x_1 = -7x_3$

multiply the system w/ $C_1, C_2 \dots C_m$

$\therefore \cancel{(C_1A_{11} + C_2A_{21} + C_3A_{31} \dots C_mA_{m1})x_1 +}$

$\cancel{(C_1A_{1m} + C_2A_{2m} + \dots C_nA_{nn})}$

$\cancel{= C_1y_1 + C_2y_2 + C_3y_2 \dots C_ny_n}$

$C_1A_{11}x_1 + C_1A_{12}x_2 + \dots + C_1A_{1n}x_n = C_1y_1$

$C_2A_{21}x_2 + C_2A_{22}x_2 + \dots + C_2A_{2n}x_n = C_2y_2$

$\vdots$

$C_mA_{m1}x_1 + C_mA_{m2}x_2 + \dots = C_my_m$

↳ all solutions of original system = solution of this system but not all feasible solutions of this system = solution of original system.

→ How many linear equations can we form from a system of linear equation?:

by changing the value of the coefficient, we can form infinite linear equations.

i.e. Let us form:

$B_{11}x_1 + B_{12}x_2 + B_{13}x_3 \dots B_{1n}x_n = z_1$

$B_{21}x_1 + B_{22}x_2 + B_{23}x_3 \dots B_{2n}x_n = z_2$

$\vdots$

$B_{m1}x_1 + B_{m2}x_2 + B_{m3}x_3 \dots B_{mn}x_n = z_m$

↳ All the solutions of the original system are solutions of this system, but not vice-versa.

• Matrices and elementary row operations

→ original system of eq. can be written as: $AX = Y$,

where $A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ \vdots & & & \vdots \\ A_{m1} & \dots & & A_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$.

→ A matrix is always defined over a field i.e. the entries in a matrix must come from the same field.

A matrix is a function that maps pairs of integers (or any other countable) to a scalar.

i.e. $A(i,j) \in F$

$1 \le i \le m,\ 1 \le j \le n.$

→ To solve $Ax = Y$, we will try to reduce the matrices such that they form a system of equations where each equation contains only one unknown scalar.

→ 3 elementary row operations:

1) multiplication of row by a non-zero scalar $(c \in R, c \neq 0)$
2) Replacing a row (say 'r') with a row 'R' that is of the form '$r + c \cdot s$' i.e. row + scalar × another row s
3) Interchanging two rows. $r \leftrightarrow r$ and s

i.e. $M = [M_{ij}]$

1) $e(M) = \begin{bmatrix} c \cdot M_{rj}, & \text{if } i = r \\ M_{ij}, & \text{if } i \neq r \end{bmatrix}$ ($c \neq 0$) → doing operations only on row 'r'

elementary function

2) $e(M) = \begin{bmatrix} M_{rj} + c \cdot M_{sj}, & \text{if } i = r \\ M_{ij}, & \text{if } i \neq r \end{bmatrix}$

3) $e(M) = \begin{bmatrix} M_{sj}, & \text{if } i = r \\ M_{rj}, & \text{if } i = s \\ M_{ij}, & \text{if } i \neq r \text{ and } i \neq s \end{bmatrix}.$

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17/01/25

Th/. For each elementary row operation e', there exists a corresponding elementary row operation 'e_1', s.t.

$$e_1(e(A)) = e(e_1(A)) = A,$$

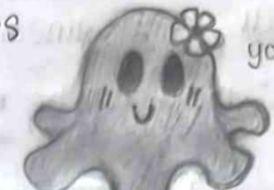
and, e_1 is of the same type as e.

↳ Let A and B be two m×n matrices defined over a field F. A is row equivalent to B if A can be obtained by performing a finite sequence of elementary row operations on B.

↳ if A is row equivalent to B, B is row equivalent to A.

i.e. $A = e_1(e_2(\cdots e_n(B)))$

Th/. if A and B are two row-equivalent m×n matrices over F, then, homogenous systems $Ax=0$ and $Bx=0$ have the same solutions.



you have pending assignments ¨ U

Prf. Elementary row operations of matrix result in linear combination of that matrix

i.e. since A and B are row equivalent,

A is a linear combination of B, and

B is a linear combination of A

∴ They are equivalent systems, and have the same set of solutions.

> Q. Prove that row equivalence is an equivalence relation
> ↳ i) define eq. relation + give properties of

→ An m×n matrix A over F is called a row-reduced matrix if:

i) the first non-zero entry of each non-zero row is 1.

ii) Each column of A which contains leading non-zero entry of a row has all other entries '0'.

→ An m×n matrix R is row-reduced echelon matrix if:

i) R is row-reduced

ii) All the non-zero rows occur together before all the zero rows

i.e. $\left[\begin{array}{c} \text{———} \\ \text{———} \\ \text{———} \\ \\ \end{array}\right]$ } → Non-zero rows } → Zero rows

iii) If the non-zero rows are rows $1, 2, \dots r$, where the leading entry of row 'i' occurs in column K_i,

$$K_1 < K_2 < \dots < K_r.$$

> Q. Every m×n matrix over field F is row equivalent to a row-reduced matrix. Prove.
> → Also row-reduced echelon matrix.

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21/01/25

→ Thm/. Every m×n matrix is row equivalent to a row-reduced echelon matrix.

Prf: Every m×n matrix is row equivalent to a row-reduced matrix, which is row equivalent to a row reduced echelon matrix.

→ Homogenous systems always have a solution → trivial solution.
↳ check by making all scalars of X 0.

Ex) $\underset{m\times n}{A}\ \underset{n\times 1}{X} = \underset{m\times 1}{0}$

$(x_1, x_2 \dots x_n)$

consider $\underset{m\times n}{R}\ \underset{n\times 1}{X} = \underset{m\times 1}{0}$ (R → row-reduced echelon, r non-zero rows)

There will be m linear equations, with n variables

∴ there will be m-r trivial equations
↳ 0 = 0 form

and r non-trivial equations.

Let the leading non-zero entry of a non-zero row 'i' be K_i
(i is from 1 to r)

∴ x_{K_i} is a non-zero scalar with coefficient '1' occurring <u>only</u> in the 'i'th linear equation.

↳ this is because acc. to the conditions of a matrix being row-reduced echelon, only the i^{th} row (corresponding to i^{th} equation) will have a non-zero number in the K_i^{th} column

∴ after matrix multiplication, only the i^{th} eq contains a non-zero coeff. for the K_i^{th} scalar.

Each of the r equations contains a unique x_{K_i}.

∴ the remaining n-r scalars are present in any combination in the linear equations
(i.e. there are no constraints on them).

∴ the equations are of the form:

$$x_{K_1} + \sum_{j=1}^{n-r} C_{1j} \times U_j = 0$$

↓ coefficients, take values from R

↳ free n-r unknown scalars x

⋮

$$x_{K_r} + \sum_{j=1}^{n-r} C_{rj} + U_j = 0$$

∴ if $r<m$, ∃ trivial equations,
if $r<n$, ∃ at least one non-trivial soln

↳ If there were no free scalars $(n-r=0)$, all the r linear eqs would be $x_{K_i}=0$ form i.e. the value of every scalar must be 0. i.e. a non-trivial soln cannot exist.

→ Non-homogenous systems : $AX = B$ form → need not have a solution

we can find solutions using elementary row operations → perform on both sides!!

$$A' = [A_{m\times n} \mid Y_{m\times 1}]_{m\times(n+1)}$$

↓ after performing elementary row operations

$$R' = [R_{m\times n} \mid Z_{m\times 1}]_{m\times(n+1)}$$

↓ row reduced echelon matrix

consider $R_{m\times n}$ has 'r' non-zero rows

∴ it has $m-r$ zero rows

⇓

we can cross-check this w/ the last $m-r$ rows of Z.

i.e. all coeffs of all scalar in that eq is 0

∴ $0 = z_i$

i.e. z_i must be 0 for solution to exist.

if they are not all zero, eq is not consistent and there is no solution.

if they are all zero, eq is consistent and there is a solution.

✦ Prove: If A is an $m\times n$ matrix, $m<n$, Then, $AX=0$ always has a non-trivial solution.

Q. A is a square matrix ($n\times n$). $AX=0$ has only a trivial solution iff A is row-equivalent to an identity matrix.

Ex) $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$; $AX = Y$.

$$A' = \left[\begin{array}{ccc|c} 1 & 0 & 3/5 & \frac{1}{5}(y_1+2y_2) \\ 0 & 1 & -1/5 & y_2+2y_1 \\ 0 & 0 & 0 & y_3+2y_1-y_2 \end{array}\right]$$

∴ soln. exists if $y_3+2y_1-y_2 = 0$.

eq: $x_1 + \frac{3}{5}x_3 = \frac{1}{5}(y_1+2y_2)$ → free scalar, can be given any value.

$x_2 - \frac{x_3}{5} = y_2 - 2y_1$

$0 = y_3 + 2y_1 - y_2$

24/01/25

• matrix multiplication:

let $C = AB$

$B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$ the i^{th} row of C would be:

$\gamma_i = A_{i1}\beta_1 + A_{i2}\beta_2 \cdots + A_{in}\beta_n$

$$\boxed{\gamma_r = \sum_{j=1}^{n} A_{rj} B_j}$$

$$\boxed{C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}}$$

Ex: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{bmatrix}$

→ Exercise, not assignment

Q) $B = [B_1\ B_2 \ldots B_p]$

$B_i = \begin{bmatrix} B_{1i} \\ B_{2i} \\ \vdots \\ B_{ni} \end{bmatrix}$ prove that $AB = [AB_1\ AB_2 \ldots AB_p]$

$n \times p$ matrix

→ Th: A, B, C are matrices over F. AB and (AB)C are defined. Then, BC is defined and $A(BC) = (AB)C$.

Prf. (AB)C is defined, ∴ no. of columns of AB = no. of rows of C.

since no. of columns of AB is determined by no. of columns of B,

we can say no. of columns of B = no. of rows of C

∴ BC is defined.

to prove $(AB)C = A(BC)$

$$[A(BC)]_{ij} = \sum_{r=1}^{n} A_{ir}(BC)_{rj}$$

$$= \sum_{r=1}^{n} A_{ir} \sum_{k=1}^{n} B_{rk} C_{kj}$$

$$= \sum_{r=1}^{n} \sum_{k=1}^{n} A_{ir} B_{rk} C_{kj}$$

we can do this since they are all scalars, so, the summations are associative;

$$= \sum_{r=1}^{n} \left(\sum_{k=1}^{n} A_{ir} B_{rk} \right) C_{kj}$$

also, since A is independent of k, it can be taken into the $\sum_{k=1}^{n}$ summation

$$= \sum_{r=1}^{n} (AB)_{ik} C_{kj}$$

$$= [(AB)C]_{ij}$$

doodle :)

$\therefore [A(BC)]_{ij} = [(AB)C]_{ij} \quad \forall \; i, j$

$\therefore$ we can say $A(BC) = (AB)C$

$\therefore$ Hence proved.

→ A matrix can only be multipied with itself when it is a square matrix

i.e. A^n is well defined only if A is a square matrix.

i.e. $A^p A^q A^r = A^b A^a A^c$ implies $a+b+c = p+q+r$.

Elementary matrix

→ A square matrix A (m×m) is an elementary matrix if it can be obtained by performing a single elementary row operation on an identity matrix

i.e. $A = e(I)$

$\boxed{\text{Prove that theorem} \downarrow}$

↳ do case-by-case

→ Th. Let e be an elementary row operation and E be an elementary matrix s.t. $E = e(I)$. Then $e(A) = EA$.

E: m×m; $e(I)$: m×m; $e(A)$: m×n

⇒ Corollary: consider matrices A and B, of dimensions m×n

A and B are row equivalent iff

$B = PA$, where P is the product of elementary matrices.

Prf. $B = e_n(\ldots e_2(e_1(A)))$

i.e. $B = e_n(\ldots e_2(E_1 A)) \ldots$ $\quad B = E_n \cdots E_2 E_1 A$

let $P = E_n \cdots E_2 E_1$

$\therefore B = PA.$

similarly we can prove the reverse, using $E_i A = e_i(A)$

$\therefore$ Hence proved

→ **invertible matrices**: A square matrix $A_{m\times m}$ is called a invertible matrix if $\exists$ P and Q s.t
(we will only consider square matrices)

$PA = I_{m\times m}$
↳ called left inverse of A

and

$AQ = I_{m\times m}$
↳ called right inverse of A

i.e. if both left and right inverse exists for the matrix.

Th/ if A is an invertible square matrix and $PA = I = AQ$, then $P = Q$.

i.e. left and right inverses of an invertible square matrix are same.

Prf. we know $AQ = I$

and $P \cdot AQ = P \cdot I$

i.e. $(PA)Q = PI$

$IQ = PI$

$\therefore Q = P$

$\therefore$ Hence proved.

we call $P = Q = A^{-1}$

Th/ A and B are $m\times m$ matrices over the same field F.

(a) if A is invertible, A^{-1} is invertible, and $(A^{-1})^{-1} = A$.

(b) if AB exists (i.e. is defined) and A and B are invertible, then AB is also invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

Prf. a) $AA^{-1} = I$ and $A^{-1}A = I$

$\therefore$ A is the left + right inverse of A^{-1}

$\therefore (A^{-1})^{-1} = A$

b) Let $A\bar{A}^{1} = I$

$B\bar{B}^{1} = I$

∴ Let $XAB = I$

$XAB \cdot \bar{B}^{1} = \bar{B}^{1}$

$XAI = \bar{B}^{1}$

$XA\bar{A}^{1} = \bar{B}^{1}\bar{A}^{1}$

$\therefore X = \bar{B}^{1}\bar{A}^{1}$

∴ if $ABX = I$, similarly, we get $X = \bar{B}^{1}\bar{A}^{1}$

∴ $\bar{B}^{1}\bar{A}^{1}$ is left + right inverse of AB

∴ AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Th, For a square matrix $A_{n\times n}$, the following are equivalent:

i) A is invertible → row eq. to I

ii) Homogenous system $AX=0$ has only trivial solution

iii) Non-homogenous system $AX=Y$ has a solution X for every $Y_{n\times 1}$.

→ use (i) to prove (ii) and (iii)

$\boxed{\text{Prove this}}$

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28/01/25

- Vector spaces → learn on your own as well!

→ A vector space, also called a linear space, consists of the following:

i) A field F of scalars

ii) A set V of objects called vectors

iii) A rule called vector addition that associates a vector for any pair of vectors $\vec{\alpha}, \vec{\beta} \in V$, $\vec{\alpha} \neq \vec{\beta}$ s.t:

⇒ add^n is commutative:

$\vec{\alpha} + \vec{\beta} = \vec{\beta} + \vec{\alpha} \quad \forall \vec{\alpha}, \vec{\beta} \in V$

⇒ add^n is associative:

$\forall \vec{\alpha}, \vec{\beta}, \vec{\gamma} \in V$,

$(\vec{\alpha} + \vec{\beta}) + \vec{\gamma} = \vec{\alpha} + (\vec{\beta} + \vec{\gamma})$.

⇒ There exists a unique vector called zero vector s.t.

$\forall \vec{\alpha} \in V$

$\vec{\alpha} + \vec{0} = \vec{\alpha}$ → additive identity

⇒ for each $\vec{\alpha} \in V$, ∃ a unique $-\vec{\alpha} \in V$ s.t.

$\vec{\alpha} + (-\vec{\alpha}) = \vec{0}$ → additive inverse i.e. vector addⁿ is analogous to scalar addⁿ.

(iv) $\exists$ a rule called scalar multiplication that maps every pair of a scalar $c \in F$ of a vector $\vec{\alpha} \in V$, a vector $c\vec{\alpha} \in V$ s.t.

$\Rightarrow$ $1 \cdot \vec{\alpha} = \vec{\alpha}$, $\forall \vec{\alpha} \in V$

$\Rightarrow$ $c_1(c_2\vec{\alpha}) = c_2(c_1\vec{\alpha})$

$\Rightarrow$ $c(\vec{\alpha} + \vec{\beta}) = c\vec{\alpha} + c\vec{\beta}$ $\forall$ $c \in F$ and $\forall$ $\vec{\alpha}, \vec{\beta} \in V$

$\Rightarrow$ $(c_1 + c_2)\vec{\alpha} = c_1\vec{\alpha} + c_2\vec{\alpha}$ $\forall$ $c_1, c_2 \in F$, $\vec{\alpha} \in V$

→ A vector space cannot be empty. it must contain at least one element, the $\vec{0}$ vector.

~~→ Th/ $c(\vec{0}) = \vec{0}$~~

~~$c(\vec{0}) = c(\vec{0} + \vec{0} + \vec{0})$~~

~~i.e $c(\vec{0}) = c(\vec{0} + \vec{0} + \ldots$~~

~~→ Th/ :~~

~~Prf:~~

→ Th/ $c(\vec{0}) = \vec{0}$

Prf :

$c(\vec{0}) = c(\vec{0} + \vec{0})$

$c(\vec{0}) = c(\vec{0}) + c(\vec{0})$

$\therefore c(\vec{0}) = \vec{0}$.

→ Th/ $-\vec{\alpha} = (-1)\vec{\alpha}$

Prf: $0 \cdot \vec{\alpha} = \vec{0}$

$(1-1)\vec{\alpha} = \vec{0}$

$\vec{\alpha} + (-1)\vec{\alpha} = \vec{0}$

$\therefore (-1)\vec{\alpha} = (-\vec{\alpha})$.

$\therefore$ Hence proved.

→ Th/ if $c\vec{\alpha} = \vec{0}$, either $c = 0$ or $\vec{\alpha}$ is 0

Prf: $c\vec{\alpha} = \vec{0}$

prf somewhere :D

→ Examples of vectors + vector spaces: → any vector space contains a zero vector

1) set of complex numbers, over $\mathbb{R}$ → field

→ vector space

→ Linear combination of vectors:

A vector $\bar{\alpha} \in V$ is called a linear combination of vector $\beta_1, \beta_2 \dots \beta_n \in V$ if

$$\vec{\alpha} = c_1\vec{\beta_1} + c_2\vec{\beta_2} \dots c_n\vec{\beta_n},$$

for some $c_1, \dots c_n \in F$ → any field

04/02/25

→ Vector subspace (or just subspace):

$S \subset V$ is a vector subspace of V if S is also a vector space over the same field as V and has the same addition & multiplication rules as V.

Ex. $\{(0, x_2, x_3)\} \subset \{(x_1, x_2, x_3)\}$ over the field $\mathbb{R}$

→ $\bar{x} + \bar{y} = \{(x_1+y_1, x_2+y_2 \dots x_n+y_n)\}$

$c\bar{x} = \{(cx_1, cx_2 \dots cx_n)\}$

~~Q. A non empty subset $W \subset V$ is a subspace of V iff ∀ scalars $c \in F$~~

$\boxed{\text{Q. Solve this} \leftarrow}$

Th) A non-empty subset $W \subset V$ (→ vector space) is a subspace iff:

a) for all scalars $c \in F$ and for all scalars and

b) for each pair of vectors $\bar{\alpha}, \bar{\beta} \in W$,

the vector $c\bar{\alpha} + \bar{\beta} \in W$

Q. An $m \times m$ matrix A is Hermitian if $A_{ij} = (A_{ji})^c$, over complex. Any 2×2 Hermitian matrix is of the form,

$$\begin{bmatrix} z & x+iy \\ x-iy & \omega \end{bmatrix}, \quad x, y, z, \omega \in \mathbb{R}.$$

Th) Let V be a vector space over field F. The intersection of any collection of subspaces of V is a subspace of V.

Prf: Let W_α be a collection of subspaces

∴ to prove: $\bigcap_{\alpha=1}^{n} W_\alpha$ = subspace of V.

∴ $\forall\ \bar{\gamma}, \bar{\beta} \in \bigcap_{\alpha=1}^{n} W_\alpha$ and $\forall\ c \in F$,

→ i.e. $\bar{\gamma}, \bar{\beta}$ also $\in W_\alpha$

$c\bar{\gamma} + \bar{\beta} \in \bigcap_{\alpha=1}^{n} W_\alpha$

∴ $c\bar{\gamma} + \bar{\beta} \in W_\alpha\ \forall\ \alpha$

∴ Hence proved.

→ consider $S = \{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 \cdots \bar{\alpha}_n\} \subset V$.

∴ There will be a subspace that contain the vectors. ∴ we cannot always find the smallest subspace.

↳ subspaces cannot always be compared (they may different data in them).

→ Let S be a set of vectors in a vector space V. The subspace spanned by S is the intersection of all the subspaces of V contain S.

Q. The subset spanned by a non-empty subset $S \subset V$ is the set of all linear combinations of vectors in S

~~Q. The subspace spanned by a non-empty subset or~~

07/02/25

→ If $S_1, S_2 \dots S_K$ are subsets of a vector space V, then the set of all sums of $\vec{\alpha}_1 + \vec{\alpha}_2 + \cdots + \vec{\alpha}_K$ st $\vec{\alpha}_i \in S_i$ is the called the sum of subsets S_i and is denoted by : $\sum_{i=1}^{K} S_i$.

If $W_1, W_2 \dots W_K$ are subspaces of V, $\underbrace{W_1 + W_2 + W_3 \cdots + W_K}$ is also a subspace of V.

↳ spanned by $W_1 \cup W_2 \cdots \cup W_K$.

The sum of all subspace of a vector space is the vector space.

✦ 1. Prove / disprove that $W_1 \cup W_2 \cup \cdots \cup W_K$ is a subspace of V, and that it spans $W_1 + W_2 + \cdots + W_K$

→ In $AX = 0$,

the vector space containing X is the one containing all $n \times 1$ matrices, and all of its solutions form a subspace.

↳ take $\vec{\alpha} = Y$, $\vec{\beta} = Z$

∴ i.e $AY = 0$, $AZ = 0$

∴ take $c\vec{\alpha} + \vec{\beta} = cY + Z$

if $A(CY+Z)=0$, that means that $CY+Z$ = solution of x

i.e if $\vec{\alpha}, \vec{\beta} \in$ solution set, $C\vec{\alpha}+\vec{\beta} \in$ solution set

$\therefore$ solution set = subspace

$\therefore$ $\underbrace{C \cdot AY}_{} + AZ = C(0)+0 = 0$

↳ have to prove $A(CY) = C(AY)$

$\therefore A(CY+Z)=0$

$\therefore$ Hence proved.

- Basis

→ A set of vectors 'S', where $S \subset$ vector space V, $\vec{\alpha_1}, \vec{\alpha_2} \cdots \vec{\alpha_r}$ is linearly dependent if $\exists$ scalars $C_1, C_2, \cdots C_n$ in the field such that not all $C_i = 0$ and $C_1\vec{\alpha_1} + C_2\vec{\alpha_2} \cdots C_n\vec{\alpha_n} = \vec{0}$

↳ i.e. at least one non-zero coeff.

if S is not linearly dependent, it is linearly independent.

Ex) $\{(0,0,1), (0,0,2)\} \subset F^3$ → linearly independent because, we can choose -ve C_1 and C_2, and zero C_3.

i.e. C_1, C_2 = non-zero

C_3 = zero.

$\{(0,0,1), (0,1,0)\}$ ⟶ C_1 : non-zero $\therefore$ C_2, C_3 = zero vector

$\therefore$ C_1 ; C_2, C_3

C_1 ↓ non-zero; C_2, C_3 ↓ zero

① $\therefore \vec{0}$ is linearly dependent

② any subset of linearly independent set of vector is linearly independent.

③ Any set with linearly dependant subset is linearly dependent

④ A set S is linearly independent iff each finite subset of S is linearly independent.

→ A basis B is a subset of vector space V that is linearly dependent and spans V. ↳ if B is a finite set, then V has finite dimensions

There can be multiple basis sets?

Ex) F^3 defined over $\mathbb{C}$ complex no.s:

$$S = \{ \vec{\alpha_1} = (3, 0, -3), \quad \vec{\alpha_2} = (-1, 1, 2), \quad \vec{\alpha_3} = (4, 2, -2), \quad \vec{\alpha_4} = (2, 1, 1)$$

Here S is ~~dependency~~ linearly dependeant.

~~[illegible]~~ ~~[illegible]~~?

✦ Q. How many elements are in the basis set of ~~linearly independent vectors of~~ X, the solution space of $A_{m \times n} X = 0$?

Cross check this

↳ given, there are r non-zero rows in the row-reduced échelon form of A.

→ Th) Let V be a vector space, spanned by a finite set of vectors (m vectors): $\{\vec{\beta_1}, \vec{\beta_2} \dots \vec{\beta_m}\}$. Then, any linearly independent set of vectors in V is finite and contains no more than m elements.

✦ Prove this ↑

11/02/24

Corollary: All bases of a vector space contain the same number of elements/vectors

→ The dimension of a vector space is the cardinality of any basis of the vector space.

Corollary (a) Any subset of a vector space that contains $>n$ vectors is linearly dependent

(b) no subset of the vector space V containing $<n$ elements/vectors can span V

Lemma. Let S be a linearly independent subset of V. Suppose $\vec{\beta} \in V$ and is not in the subspace spanned by S. Then, $S \cup \{\vec{\beta}\}$. i.e. subset of vectors formed by adjoining $\vec{\beta}$ to S, is linearly independent.

→ i.e dimension of $V \geq n$

Prf: $S = \{\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \cdots \vec{\alpha}_n\} \subset V$,

$\vec{\beta} \notin \text{span}(S)$, $\vec{\beta} \in V$

i.e. $c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 \cdots\cdots c_n\vec{\alpha}_n + b\vec{\beta} = \vec{0}$

$$\vec{\beta} = \sum_{i=1}^{n} -(c_i/b)\,\vec{\alpha}_i$$

→ if $\vec{\beta}$ can be written in this form, that would mean $\vec{\beta} \in \text{span}(S)$, which is not possible.

if $\nexists$ a single c_i that is non-zero, $\vec{\beta}$ will not exist in given form

i.e. if even one $c_i \neq 0$, $\vec{\beta} \in \text{span}(S)$.

$\therefore$ $c_i \; \forall \; i \in [1, n]$ is 0

→ $\vec{\beta}$ cannot be $\vec{0}$ either.

Corollary: if W is a proper subspace of V and dimension of V is finite, then, dimension of W is finite and less than dimension of V.

will give artist details

- Hands
- Hey Lover!
- make you mine
- maneskin? idk if you'll like it :/
- Blue - Yung Kai
- That's my life (?) - (?)
- Superposition - Young the Giant
- Deviltown - cavetown (?)
- struck by lightning
- heart b/w my teeth

pretty popular, so you've prolly heard it but wtv

will give more details :thumbsup:

absolutely not the name of the song, I'll give details later.

also will give details.

Q. → Standard basis of a field F^n:

$$\begin{bmatrix} \{1,0,0\ldots\} \\ \{0,1,0\ldots\} \\ \{0,0,1\ldots 0\} \\ \{0,0\ldots\ldots 1\} \end{bmatrix}$$

* Let $A_{n \times n}$ be defined over field F. Consider that the rows of A are linearly independent set of vectors in F. Then, show that A is invertible

↳ $A_{n \times n} : \begin{bmatrix} \vec{\alpha_1} \\ \vdots \\ \vec{\alpha_n} \end{bmatrix}$

collection of n vectors

Th, W_1 and W_2 are finite dimensional spaces of V. Then, (W_1+W_2) is a finite dimensional space, and:

a) $\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$

* Solve this.

• Coordinates

→ V is a vector plane, B is the ordered basis $= \{\vec{\alpha_1}, \vec{\alpha_2} \cdots \vec{\alpha_n}\}$,

where any $\vec{\alpha} \in V = \sum_{i=1}^{n} x_i \vec{\alpha_i}$.

root coordinates do not make sense if basis is not ordered.

→ An ordered basis of a vector space is a sequence of linearly independent vectors that spans the space.

→ Both coordinates of a basis cannot be distinct i.e.

$x_i = y_i \ \forall$ bases.

18/02/2025

→ consider an ordered basis $B = \{\bar{\alpha}_1, \bar{\alpha}_2 \cdots \bar{\alpha}_n\}$ that spans V

∴ for any $\bar{\alpha} \in V$, $\bar{\alpha}$ can be represented as $\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha}_i$

the n tuple $(x_1, x_2 \dots x_n)$ are called the coordinates of $\bar{\alpha}$, and is found in F^n.

it is denoted as $[\bar{\alpha}]_\beta$

Ex) for $\vec{0}$, $x_i = 0 \ \forall i$.

if the basis is changed i.e. basis is now $B' = \{\overline{\alpha_1'}, \dots \overline{\alpha_n'}\}$,

$$\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha_i} = \sum_{i=1}^{n} x_i' \overline{\alpha_i'} \quad \text{— (i)}$$

consider $\bar{\alpha} = (x_1, x_2 \dots x_n)$
$\bar{\beta} = (y_1, y_2 \dots y_n)$

∴ coordinates of $\bar{\alpha} + \bar{\beta} = \sum_{i=1}^{n} (x_i + y_i)$

i.e. they get added

consider: $[c\bar{\alpha}]_\beta = c[\bar{\alpha}]_\beta$

we can say: ∃ scalars P_{ij} s.t.:

$$\overline{\alpha_j'} = \sum_{i=1}^{n} P_{ij} \bar{\alpha_i} \quad \forall i,j \in \{1, 2 \dots n\}$$

from (i),

$$\bar{\alpha} = \sum_{j=1}^{n} x_j' \overline{\alpha_j'}$$

$$= \bar{\alpha} = \sum_{j=1}^{n} x_j' \left(\sum_{i=1}^{n} P_{ij} \bar{\alpha_i} \right)$$

$$\bar{\alpha} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} P_{ij} \bar{x_j'} \right) \bar{\alpha_i}$$

$Px = x'$
↳ is an n×n square matrix and is invertible

∴ $[\bar{\alpha}]_B = P[\alpha']_{B'}$

since P is invertible,

$$\boxed{[\bar{\alpha}]_{B'} = P^{-1}[\alpha]_B}$$

Th/ V is a vector space over F, and its dimensions are n. Let B and B' be two ordered basis. Then, ∃ a unique $n \times n$ matrix P over F, and P is necessarily invertible, s.t.:

(1) $[\bar{\alpha}]_B = P[\bar{\alpha}]_{B'}$

(2) $P^{-1}[\bar{\alpha}]_{B} = [\bar{\alpha}]_{B'}$,

and columns $P_j = [\bar{\alpha}_j']_B \ \forall j$

Ex) Consider P over $\mathbb{R}^2$,

let $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix}$

$P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

• Linear transformations

→ Consider vector spaces V and W over F. A function $T: V \longrightarrow W$ is a linear tranformations if it maps each $\bar{\alpha} \in V$

$\forall c \in F$, and $\forall \alpha, \beta \in V$,

$T(\bar{\alpha} + \bar{\beta}) = T\bar{\alpha} + T\bar{\beta}$

$\therefore T(\bar{0} + \bar{0}) = T(\bar{0}) + T(\bar{0}) = T(\bar{0}) = \bar{0}$

→ no matter which vector spaces you choose, ∃ a linear transformation between them.

$$\therefore T\left(\sum_{i=1}^{n} x_i \bar{\alpha_i}\right) = \sum_{i=1}^{n} x_i \left(T\bar{\alpha_i}\right)$$

Th/ Consider a vector space V, W of dimension m over the field F

Let $\{\alpha_1, \alpha_2 \dots \alpha_n\}$ be an ordered basis of V.

↳ linearly independent

Let $\bar{\beta_1}, \bar{\beta_2} \dots \bar{\beta_n} \in W$ — these can be linearly dependent or independent. Then, there is precisely one linear transformation $T \cdot V \to W$ s.t.

$$T\bar{\alpha_i} = \bar{\beta_i} \;\forall\; i = \{1, 2 \dots n\}.$$

Ex) Define ($T(x_1\bar{\alpha_1} + x_2\bar{\alpha_2} \dots \dots x_n\bar{\alpha_n})$)

$$\sum_{i=1}^{n} x_i \beta_i$$

go

Q. Thm/ consider V, a vector space with finite dimensions. The, rank of T + nullity of T = dim(V).

↳ show that null space is subspace + image of T = subspace.

→ image of T, lies in W. — or range of T, it is a subspace.

→ To know T, it is enough to know image of basis elements of V, rather than the image of V.