LINEAR ALGEBRA

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→ Linear algebra : the study of linear maps on finite dimensional vector spaces.

→ Problems of Type 1 :

$$a_{11}x_1 + a_{12}x_2 \cdots\cdots a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 \cdots\cdots a_{nn}x_n = b_n$$

Find a list of numbers ($s_1, s_2 \ldots s_n$) that satisfy this system of equations.

→ n-tuple

- Gaussian elimination

→ Get the echelon form of the equations i.e.

$$C_{11}x_1 + C_{12}x_2 + C_{13}x_3 + C_{14}x_4 = b_1$$
$$C_{22}x_2 + C_{23}x_3 + C_{24}x_4 = b_2$$
$$C_{33}x_3 + C_{34}x_4 = b_3$$
$$C_{44}x_4 = b_4$$

$m = 4$, $n = 4$ (m → rows, n → columns)

Thus, going bottom to top, we can get values of x_1, x_2, x_3, x_4

→ solutions can also be derived graphically i.e.

each equation is converted to a coordinate structure, and solutions are found at the places where these structures intersect

(A) 2-variable equations = lines that are not parallel to X/Y axis
(B) 1-variable equations: line parallel to axes
(C) 3-variable equations = plane
⋮

✦ Linear Algebra : Hoffman - Kunze

→ Some notations :

matrix of dimension $m \times n$: $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

vector of dimension n : $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

→ Linear combination of vectors:

$a_1 V_1 + a_2 V_2 + a_3 V_3 \dots a_n V_n = b$

V_1 → vector

→ **Problems of Type - II:**

"Given $\vec{V_1}, \vec{V_2} \dots \vec{V_n}$, is $\vec{b}$ a linear combination of these vectors?

→ terms from the definition:

(i) Linear Map: literally just a fn

(ii) [redacted] finite dimension: a dimension to which more elements can be added

→ vector operation properties:

1) $\forall\, V_1, V_2 \in V$ $\lambda_1 V_1 + \lambda_2 V_2 \in V$, $\lambda_1, \lambda_2 \in \mathbb{R}$

V → vectors

2) $V_1 + V_2 = V_2 + V_1$ (commutative)

$V_1 + (V_2 + V_3) = (V_1 + V_2) + V_3$ (associative)

3) $0 + \vec{V} = \vec{V} = \vec{V} + 0, \ \forall\, \vec{V} \in V$

4) $\vec{V} + (-\vec{V}) = 0$

5) $\lambda(\vec{V_1} + \vec{V_2}) = \lambda\vec{V_1} + \lambda\vec{V_2}$

} forms a vector field

———— X ————

07/01/25

* Assignments: ~20%
  Quiz 1: 10%
  Mid-Sem: 15-20%
  Tutorial quizzes: ~%
  50%

• **Fields**

→ A set F, with two binary operations, addition (+) and multiplication (·) satisfying the following rules:

i) addition is commutative
i.e. $a+b = b+a \ \forall\, a,b \in F$

↳ closure is inherent and present by definition as binary operations cannot give results outside the set.

ii) addition is associative
$a + (b+c) = (a+b) + c \ \forall\, a,b,c \in F$

iii) $\exists$ a unique element '0' s.t. $a + 0 = 0$. $\forall a \in F$

→ not the number 0

↳ additive identity

iv) $\forall\, a \in F$, $\exists\, (-a)$ s.t. $a + (-a) = 0$.

→ additive inverse.

↳ additive identity

} 4 rules of addition

v) Multiplication is commutative

$a \cdot b = b \cdot a \quad \forall a, b \in F$

vi) Multiplication is associative

$a(b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in F$

vii) $\exists$ a unique, non-zero element, represented by '1', s.t. multiplicative identity

$a \cdot 1 = a \quad \forall a \in F$

viii) $\forall x \in F, x \neq 0, \exists \; x'$ s.t. $x \cdot x' = 1$.

↳ multiplicative inverse

ix) multiplication is distributive over addition:

$a \cdot (b + c) = a \cdot b + a \cdot c \;, \forall a, b, c \in F$

5 rules of multiplication

i.e. the field $= (F, +, \cdot)$, if the set F follows the above rules w.r.t. the binary operators

if F is an empty set, it cannot be a field as points 3 and 7 state that the set must contain a particular element.

∴ The set must contain at least 2 elements.

→ can the set contain exactly 2 elements?

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

$\{0, 1\}$ satisfies all the rules w.r.t + and ·

∴ $(\{0,1\}, +, \cdot)$ is a field

↳ (mod 2)

∴ a field can contain exactly two elements.

→ prf: $(-1) \cdot x = -x$

A: $x + (-1)x$

$= 1 \cdot x + (-1) \cdot x$ (multiplicative identity)

$x(1 + (-1))$ (distributive)

$= x \cdot 0 = x \cdot 0 + 0 = x \cdot 0 + (x + (-x)) = (x \cdot 0 + x) + (-x)$

$x(0+1) = x \cdot 1 = x$

$= x + (-x)$

$= 0$

$= x + (-x)$

$= -x = (-1)x$

Q1] For a field.
$a, b, c \in F$,
prove:
a) if $ab = bc$, then $a = c$.
b) $a + b = b + c$,
then $a = c$
↳ deadline:
Sunday midnight.

10/01/25.

• Subfield:

A set S is a subfield of a field $(F, +, \cdot)$ is $S \subset F$ and $(S, +, \cdot)$ is a field.

Eg. Real numbers are subfield of complex numbers w.r.t. + and ·

Q2] Any subfield of a complex field must contain every rational number. Prove.

• System of linear equations

→ unknown scalars have degree '1'.

→ linear equations = coefficients + unknown scalars

↳ all coeffs must be from same field as each other and scalar.

↳ $x_1, x_2 \ldots x_n$

i.e. consider

$$\left.\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \cdots A_{1n}x_n &= y_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \cdots A_{2n}x_n &= y_2 \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + A_{m3}x_3 \cdots A_{mn}x_n &= y_m \end{aligned}\right\} \text{system of linear eqs.}$$

→ linear eq

A system in which $y_i = 0$ is called a homogenous. A system with even 1 $y_i \neq 0$ is called a non-homogenous.

Ex. $2x_1 + 3x_2 - 4x_3 = 0$ → 2 equations, 3 unknowns

$x_1 + x_2 + x_3 = 0$

$\times(-2)$ → $-2x_1 - 2x_2 - 2x_3$ +

$0 + x_2 - 6x_3 = 0$

$x_2 = 6x_3$ ∴ $(x_1, x_2, x_3) = (-7x_3, 6x_3, x_3)$

$x_1 = -7x_3$

multiply the system w/ $C_1, C_2 \dots C_m$

$\therefore (C_1A_{11} + C_2A_{21} + C_3A_{31} \dots C_mA_{m1})x_1 +$

$\vdots$

$(C_1A_{1m} + C_2A_{2m} + \dots C_nA_{nn})$

$= C_1y_1 + C_2y_2 + C_3y_2 \dots C_ny_n$

$C_1A_{11}x_1 + C_1A_{12}x_2 + \dots + C_1A_{1n}x_n = C_1y_1$

$C_2A_{21}x_2 + C_2A_{22}x_2 + \dots + C_2A_{2n}x_n = C_2y_2$

$\vdots$

$C_mA_{m1}x_1 + C_mA_{m2}x_2 + \dots = C_my_m$

↳ all solutions of original system = solution of this system but not all feasible solutions of this system = solution of original system.

→ How many linear equations can we form from a system of linear equation?:

by changing the value of the coefficient, we can form infinite linear equations.

i.e. Let us form:

$B_{11}x_1 + B_{12}x_2 + B_{13}x_3 \dots B_{1n}x_n = z_1$

$B_{21}x_1 + B_{22}x_2 + B_{23}x_3 \dots B_{2n}x_n = z_2$

$\vdots$

$B_{m1}x_1 + B_{m2}x_2 + B_{m3}x_3 \dots B_{mn}x_n = z_m$

↳ All the solutions of the original system are solutions of this system, but not vice-versa.

• Matrices and elementary row operations

→ original system of eq. can be written as: $AX = Y$,

where $A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ \vdots & & & \vdots \\ A_{m1} & \dots & & A_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$.

→ A matrix is always defined over a field i.e. the entries in a matrix must come from the same field.

A matrix is a function that maps pairs of integers (or any other countable) to a scalar.

i.e. $A(i,j) \in F$
$1 \le i \le m,\ 1 \le j \le n$.

→ To solve $AX = Y$, we will try to reduce the matrices such that they form a system of equations where each equation contains only one unknown scalar.

→ 3 elementary row operations:

1) multiplication of row by a non-zero scalar ($c \in R, c \neq 0$)
2) Replacing a row (say 'r') with a row 'R' that is of the form '$r + c \cdot s$' i.e. row + scalar × another row s
3) Interchanging two rows. $r \leftrightarrow r$ and s

i.e. $M = [M_{ij}]$

1) $e(M) = \begin{bmatrix} c \cdot M_{rj}, & \text{if } i = r \\ M_{ij}, & \text{if } i \neq r \end{bmatrix}$ $c \neq 0$ → doing operations only on row 'r'.
elementary function

2) $e(M) = \begin{bmatrix} M_{rj} + c \cdot M_{sj}, & \text{if } i = r \\ M_{ij}, & \text{if } i \neq r \end{bmatrix}$

3) $e(M) = \begin{bmatrix} M_{sj}, & \text{if } i = r \\ M_{rj}, & \text{if } i = s \\ M_{ij}, & \text{if } i \neq r \text{ and } i \neq s \end{bmatrix}$.

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Th/. For each elementary row operation e, there exists a corresponding elementary row operation 'e_1', s.t.

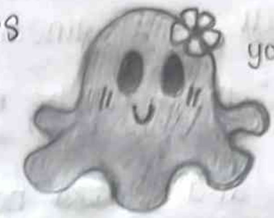
$$e_1(e(A)) = e(e_1(A)) = A,$$

and, e_1 is of the same type as e.

↳ Let A and B be two $m \times n$ matrices defined over a field F. A is row equivalent to B if A can be obtained by performing a finite sequence of elementary row operations on B.
↳ if A is row equivalent to B, B is row equivalent to A.

i.e. $A = e_1(e_2(\cdots e_n(B)))$

Th/. if A and B are two row-equivalent $m \times n$ matrices over F, then, homogenous systems $Ax = 0$ and $Bx = 0$ have the same solutions.



Prf. Elementary row operations of matrix result in linear combination of that matrix

i.e. since A and B are row equivalent,

A is a linear combination of B, and

B is a linear combination of A

Q. Prove that row equivalence is an equivalence relation
↳ i) define eq. relation + give properties of

∴ They are equivalent systems, and have the same set of solutions.

→ An $m \times n$ matrix A over F is called a row-reduced matrix if:

i) the first non-zero entry of each non-zero row is 1.

ii) Each column of A which contains leading non-zero entry of a row has all other entries '0'.

→ An $m \times n$ matrix R is row-reduced echelon matrix if:

i) R is row-reduced

ii) All the non-zero rows occur together before all the zero rows

i.e. $\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix}$ }→ Non-zero rows }→ Zero rows

Q. Every $m \times n$ matrix over field F is row equivalent to a row-reduced matrix. Prove.
→ Also row-reduced echelon matrix.

iii) If the non-zero rows are rows $1, 2, \ldots r$, where the leading entry of row 'i' occurs in column K_i,

$$K_1 < K_2 < \cdots < K_r.$$

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→ Thm/. Every $m \times n$ matrix is row equivalent to a row-reduced echelon matrix.

Prf: Every $m \times n$ matrix is row equivalent to a row-reduced matrix, which is row equivalent to a row reduced echelon matrix.

→ Homogenous systems always have a solution →
↳ check by making all scalars of x 0. } → trivial solution.

Ex) $\underset{m \times n}{A}\ \underset{n \times 1}{X} = \underset{m \times 1}{0}$

$(x_1, x_2 \ldots x_n)$

consider $\underset{m \times n}{R}\ \underset{n \times 1}{X} = \underset{m \times 1}{0}$ → row-reduced echelon, r non-zero rows

There will be m linear equations, with n variables

$\therefore$ there will be m-r trivial equations
↳ $0=0$ form

and r non-trivial equations.

Let the leading non-zero entry of a non-zero row 'i' be K_i (i is from 1 to r)

$\therefore x_{K_i}$ is a non-zero scalar with coefficient '1' occurring <u>only</u> in the 'i'th linear equation.

↳ this is because acc. to the conditions of a matrix being row-reduced echelon, only the i^{th} row (corresponding to i^{th} equation) will have a non-zero number in the K_i^{th} column

$\therefore$ after matrix multiplication, only the i^{th} eq contains a non-zero coeff. for the K_i^{th} scalar.

Each of the r equations contains a unique x_{K_i}.

$\therefore$ the remaining n-r scalars are present in any combination in the linear equations
(i.e. there are no constraints on them).

$\therefore$ the equations are of the form:

$$x_{K_1} + \sum_{j=1}^{n-r} C_{1j} \times U_j = 0$$

↳ free n-r unknown scalars x

coefficients, take values from R

$\vdots$

$$x_{K_r} + \sum_{j=1}^{n-r} C_{rj} + U_j = 0$$

$\therefore$ if $r<m$, $\exists$ trivial equations,

if $r<n$, $\exists$ at least one non-trivial soln

↳ If there were no free scalars. $(n-r=0)$, all the r linear eqs would be $x_{K_i}=0$ form i.e. the value of every scalar must be 0. i.e. a non-trivial soln. cannot exist.

→ Non-homogenous systems: $AX = B$ form → need not have a solution

we can find solutions using elementary row operations → perform on both sides!!

$$A' = [A_{m\times n} \mid Y_{m\times 1}]_{m\times(n+1)}$$

↓ after performing elementary row operations

$$R' = [R_{m\times n} \mid Z_{m\times 1}]_{m\times(n+1)}$$

↓

row reduced echelon matrix

> ✦ Prove: If A is an $m\times n$ matrix, $m<n$, Then, $AX=0$ always has a non-trivial solution.
>
> Q. A is a square matrix $(n\times n)$. $AX = 0$ has only a trivial solution iff A is row-equivalent to an identity matrix.

consider $R_{m\times n}$ has 'r' non-zero rows

∴ it has $m-r$ zero rows

⇓

we can cross-check this w/ the last $m-r$ rows of Z.

i.e. all coeffs of all scalar in that eq is 0

∴ $0 = z_i$

i.e. z_i must be 0 for solution to exist.

if they are not all zero, eq is not consistent and there is no solution.

if they are all zero, eq is consistent and there is a solution.

Ex) $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix}$ $\quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$; $AX = Y$.

$$A' = \left[\begin{array}{ccc|c} 1 & 0 & 3/5 & \frac{1}{5}(y_1+2y_2) \\ 0 & 1 & -1/5 & y_2-2y_1 \\ 0 & 0 & 0 & y_3+2y_1-y_2 \end{array}\right]$$

∴ soln. exists if $y_3+2y_1-y_2=0$.

eq: $x_1 + \frac{3}{5}x_3 = \frac{1}{5}(y_1+2y_2)$ → free scalar, can be given any value.

$x_2 - \frac{x_3}{5} = y_2 - 2y_1$

$0 = y_3 + 2y_1 - y_2$

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• Matrix multiplication:

let $C = AB$

$$B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$ the i^{th} row of C would be:

$$\gamma_i = A_{i1}\beta_1 + A_{i2}\beta_2 \cdots + A_{in}\beta_n$$

$$\boxed{\gamma_r = \sum_{j=1}^{n} A_{rj} B_j}$$

$$\boxed{C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}}$$

Ex: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{bmatrix}$

→ Exercise, not assignment

Q) $B = [B_1 \; B_2 \; \dots \; B_p]$

$B_i = \begin{bmatrix} B_{1i} \\ B_{2i} \\ \vdots \\ B_{ni} \end{bmatrix}$ prove that $AB = [AB_1 \; AB_2 \; \dots \; AB_p]$

$n \times p$ matrix

→ Th: A, B, C are matrices over F. AB and (AB)C are defined. Then, BC is defined and $A(BC) = (AB)C$.

Prf. (AB)C is defined, ∴ no. of columns of AB = no. of rows of C.

since no. of columns of AB is determined by no. of columns of B,

we can say no. of columns of B = no. of rows of C

∴ BC is defined.

to prove $(AB)C = A(BC)$

$$[A(BC)]_{ij} = \sum_{r=1}^{n} A_{ir}(BC)_{rj}$$

$$= \sum_{r=1}^{n} A_{ir} \sum_{k=1}^{n} B_{rk} C_{kj}$$

$$= \sum_{r=1}^{n} \sum_{k=1}^{n} A_{ir} B_{rk} C_{kj}$$

we can do this since they are all scalars, so, the summations are associative;

$$= \sum_{r=1}^{n} \left(\sum_{k=1}^{n} A_{ir} B_{rk} \right) C_{kj}$$

also, since A is independent of k, it can be taken into the $\sum_{k=1}^{n}$ summation

$$= \sum_{r=1}^{n} (AB)_{ik} C_{kj}$$

$$[(AB)C]_{ij}$$

doodle :)

$\therefore [A(BC)]_{ij} = [(AB)C]_{ij} \quad \forall \ i, j$

$\therefore$ we can say $A(BC) = (AB)C$

$\therefore$ Hence proved.

→ A matrix can only be multipied with itself when it is a square matrix

i.e. A^n is well defined only if A is a square matrix.

i.e. $A^p A^q A^r = A^b A^a A^c$ implies $a+b+c = p+q+r$.

Elementary matrix

→ A square matrix A (m×m) is an elementary matrix if it can be obtained by performing a single elementary row operation on an identity matrix

i.e. $A = e(I)$

$\boxed{\text{Prove that theorem} \downarrow}$

↳ do case-by-case

→ Th. Let e be an elementary row operation and E be an elementary matrix s.t. $E = e(I)$. Then $e(A) = EA$.

E: $m \times m$; I: $m \times m$; A: $m \times n$

⇒ Corollary : consider matrices A and B, of dimensions $m \times n$

A and B are row equivalent iff

$B = PA$, where P is the product of elementary matrices.

Prf. $B = e_n(\dots e_2(e_1(A)))$

i.e. $B = e_n(\dots e_2(E_1 A)) \dots \quad B = E_n \cdots E_2 E_1 A$

let $P = E_n \cdots E_2 E_1$

$\therefore B = PA.$

similarly we can prove the reverse, using $E_i A = e_i(A)$

$\therefore$ Hence proved

→ invertible matrices: A square matrix $A_{m \times m}$ is called a invertible matrix if $\exists$ P and Q s.t.

(we will only consider square matrices)

$PA = I_{m \times m}$
↳ called left inverse of A

and

$AQ = I_{m \times m}$
↳ called right inverse of A

i.e. if both left and right inverse exists for the matrix.

Th/ if A is an invertible square matrix and $PA = I = AQ$, then $P = Q$.

i.e. left and right inverses of an invertible square matrix are same.

Prf. we know $AQ = I$

and $P \cdot AQ = P \cdot I$

i.e. $(PA)Q = PI$

$IQ = PI$

$\therefore Q = P$

$\therefore$ Hence proved.

we call $P = Q = A^{-1}$

Th/ A and B are $m \times m$ matrices over the same field F.

(a) if A is invertible, A^{-1} is invertible, and $(A^{-1})^{-1} = A$.

(b) if AB exists (i.e. is defined) and A and B are invertible,

then AB is also invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

Prf. a) $AA^{-1} = I$ and $A^{-1}A = I$

$\therefore$ A is the left + right inverse of A^{-1}

$\therefore (A^{-1})^{-1} = A$

b) Let $A\bar{A}^{1} = I$

$B\bar{B}^{1} = I$

$\therefore$ Let $XAB = I$

$XAB \cdot \bar{B}^{1} = \bar{B}^{1}$

$XAI = \bar{B}^{1}$

$XA\bar{A}^{1} = \bar{B}^{1}\bar{A}^{1}$

$\therefore X = \bar{B}^{1}\bar{A}^{1}$

$\therefore$ if $ABX = I$, similarly, we get $X = \bar{B}^{1}\bar{A}^{1}$

$\therefore \bar{B}^{1}\bar{A}^{1}$ is left + right inverse of AB

$\therefore AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Th, For a square matrix $A_{n\times n}$, the following are equivalent:

i) A is invertible → row eq. to I

ii) Homogenous system $AX=0$ has only trivial solution

iii) Non-homogenous system $AX=Y$ has a solution X for every $Y_{n\times 1}$.

→ use (i) to prove (ii) and (iii)

$\boxed{\text{Prove this}}$

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- Vector spaces → learn on your own as well!

→ A vector space, also called a linear space, consists of the following:

i) A field F of scalars

ii) A set V of objects called vectors

iii) A rule called vector addition that associates a vector for any pair of vectors $\vec{\alpha}, \vec{\beta} \in V$, $\vec{\alpha} \neq \vec{\beta}$ s.t:

⇒ add^n is commutative:

$\vec{\alpha} + \vec{\beta} = \vec{\beta} + \vec{\alpha} \quad \forall \vec{\alpha}, \vec{\beta} \in V$

⇒ add^n is associative:

$\forall \vec{\alpha}, \vec{\beta}, \vec{\gamma} \in V$,

$(\vec{\alpha} + \vec{\beta}) + \vec{\gamma} = \vec{\alpha} + (\vec{\beta} + \vec{\gamma})$.

⇒ There exists a unique vector called zero vector s.t.

$\forall \vec{\alpha} \in V$

$\vec{\alpha} + \vec{0} = \vec{\alpha}$ → additive identity

⇒ for each $\vec{\alpha} \in V$, $\exists$ a unique $-\vec{\alpha} \in V$ s.t.

$\vec{\alpha} + (-\vec{\alpha}) = \vec{0}$ → additive inverse i.e. vector addⁿ is analogous to scalar addⁿ.

(iv) $\exists$ a rule called scalar multiplication that maps every pair of a scalar $c \in F$ of a vector $\vec{\alpha} \in V$, a vector $c\vec{\alpha} \in V$ s.t.

$\Rightarrow$ $1 \cdot \vec{\alpha} = \vec{\alpha}, \ \forall \vec{\alpha} \in V$

$\Rightarrow$ $c_1(c_2\vec{\alpha}) = c_2(c_1\vec{\alpha})$

$\Rightarrow$ $c(\vec{\alpha} + \vec{\beta}) = c\vec{\alpha} + c\vec{\beta} \ \forall c \in F$ and $\forall \vec{\alpha}, \vec{\beta} \in V$

$\Rightarrow$ $(c_1 + c_2)\vec{\alpha} = c_1\vec{\alpha} + c_2\vec{\alpha} \ \forall c_1, c_2 \in F, \vec{\alpha} \in V$

→ A vector space cannot be empty. it must contain at least one element, the $\vec{0}$ vector.

→ Th/ $c(\vec{0}) = \vec{0}$

Prf:

$c(\vec{0}) = c(\vec{0} + \vec{0})$

$c(\vec{0}) = c(\vec{0}) + c(\vec{0})$

$\therefore c(\vec{0}) = \vec{0}$.

→ Th/ $-\vec{\alpha} = (-1)\vec{\alpha}$

Prf: $\vec{0} \cdot \vec{\alpha} = \vec{0}$

$(1-1)\vec{\alpha} = \vec{0}$

$\vec{\alpha} + (-1)\vec{\alpha} = \vec{0}$

$\therefore (-1)\vec{\alpha} = (-\vec{\alpha})$.

$\therefore$ Hence proved.

→ Th/ if $c\vec{\alpha} = \vec{0}$, either $c = 0$ or $\vec{\alpha}$ is 0.

Prf: $c\vec{\alpha} = \vec{0}$ prf somewhere :D

→ Examples of vectors + vector spaces:

1) set of complex numbers, over $\mathbb{R}$ → field

$\underbrace{\text{set of complex numbers}}_{\text{vector space}}$

→ linear combination of vectors:

A vector $\bar{\alpha} \in V$ is called a linear combination of vector $\beta_1, \beta_2 \dots \beta_n \in V$ if

$$\vec{\alpha} = c_1\vec{\beta_1} + \vec{c_2\beta_2} \dots C_n\vec{\beta_n},$$

for some $C_1, \dots C_n \in F$ → any field.