University of Massachusetts Lowell
Name of School

Homework I

Introductory Quantum Mechanics I

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Question I

Problem 1.2 Five cards are dealt face-down from a 52-card deck.

(a) How many possible sets of five cards are there? How much information do we lack about the cards?

The system's entropy, representing its lack of information, is calculated as:

$$H = log_2 M \tag{1}$$

where M is the number of possible states accessible. The formula for finding possible states is

$$P_r^n = \frac{n!}{(n-r)!}$$

with n as the set size and r as the number selected. In our case,

$$M = \frac{52!}{(52-5)!}$$

$$H = log_2\left(\frac{52!}{(52-5)!}\right) \approx 3.12$$

(b) The first three are turned over and revealed. Knowing these, how many possibilities remain?

In this scenario, the distinction from the previous case lies in the revelation of the first three cards, excluding them from our potential permutations. This still leaves us with a deck containing three fewer cards, resulting in a new

$$M' = \frac{(52-3)!}{(52-5)!} = 2352$$

(c) How much information was conveyed when the three cards were revealed? Is this 3/5 of the total? Why or why not?

The information gained from revealing the three cards is determined by the difference between the original entropy and the current entropy:

$$M - M' = log_2\left(\frac{M}{M'}\right) \approx 17.02$$

This information gain exceeds $\frac{3}{5}$ of the original missing information by approximately 0.087 **bits.**

(d) Repeat parts (a)–(c) if the five cards are dealt from five independent decks.

In this case, the only difference in potential states is that cards are chosen with replacement. This allows us to simply multiply the size of the deck we choose from as many times as we choose cards.

(a) $M = 52^5$

Applying the same logic as before, we eliminate the first three cards chosen to find our new potential states.

(b) $M' = 52^2$

Now, using our previous equation, we determine the information gained in this process

$M - M' = log_2\left(\frac{M}{M'}\right) \approx 17.10$

The information gained in this situation is precisely $\frac{3}{5}$ of the original information.

Question II

Problem 1.5 The kinetic energy K of a particle is related to its momentum p by $K = \frac{p^2}{\mu}$, where μ is the particle's mass. In a gas at absolute temperature T, the molecules have a typical kinetic energy of $\frac{3}{2}k_BT$. Derive an expression for the thermal de Broglie wavelength, a typical value for the de Broglie wavelength λ of a molecule in a gas. For helium atoms ($\mu = 6.7 \times 10 - 27kg$), calculate the thermal de Broglie wavelength at room temperature ($T = 300K$) and at the boiling point of helium ($T = 4K$). Quantum effects become most significant in matter when the thermal de Broglie wave-length of the particles is greater than their separation. At atmospheric pressure, gas molecules are about $1 - 2nm$ apart; in a condensed phase (liquid, solid) they are about ten times closer. How do these compare with the thermal de Broglie wavelengths you calculated for helium?

(a) **Equation for the thermal de Broglie wavelength?**

To obtain the thermal de Broglie wavelength equation, we utilize the de Broglie wavelength:

$$\lambda = \frac{h}{p} \tag{2}$$

the kinetic energy equation:

$$K = \frac{p^2}{2\mu} \tag{3}$$

and the given thermal energy equation for a molecule:

$$\frac{3}{2}k_BT \tag{4}$$

Starting by solving for momentum in terms of general kinetic energy, Then we substitute the thermal energy equation for kinetic energy.

$$p = \sqrt{2K\mu} = \sqrt{3k_BT\mu}$$

A straightforward substitution into the de Broglie wavelength equation yields our thermal de Broglie wavelength equation.

$$\lambda = \frac{h}{\sqrt{3k_BT\mu}}$$

(b) What is the de Broglie wavelength of helium at $T = 300K$

Entering our known values into the equation and calculating the wavelength.

$$\lambda = \frac{h}{\sqrt{3k_B 300\mu}} \approx 7.26 \times 10^{-2} nm$$

(c) What is the de Broglie wavelength at $T = 4K$?

The same process as above.

$$\lambda = \frac{h}{\sqrt{3k_B 300\mu}} \approx 6.29 \times 10^{-1} nm$$

(d) How does the de Broglie wavelength compare to the $1 - 2nm$ separation of the molecules at atmospheric pressure.

The de Broglie wavelength at each temperature is significantly smaller, by one to two orders of magnitude, than the separation between molecules. This indicates that the quantum effects of these molecules are negligible in comparison to the motion of the molecules.

Question III

A single photon passes through a barrier with four slits and strikes a screen some distance away. Consider a point X on the screen. The probability amplitudes for reaching X via the four slits are ϕ_1, ϕ_2, ϕ_3, and ϕ_4.

(a) What is the net probability P that the photon is found at X if no measurement is made of which slit the photon passed through?

The probability equation is given by:

$$P_\alpha = |\phi_\alpha|^2 \tag{5}$$

In this case, the system is informational isolated. Instead of classical probabilities, we need to sum their probability amplitudes to determine the probability of finding the photon at X.

$$P_x = |\phi_1 + \phi_2 + \phi_3 + \phi_4|^2$$

(b) A detector is placed by slit #4, which can register whether or not the photon passes that slit (but does not absorb the photon or deflect it). What is P in this case?

In this case we now know whether or not the particle passes through slit four which makes slit four no longer informational isolated. this does not change the state of slits one through three they are all still informational isolated and so there probabilities amplitudes get summed while slit four just gets treated as our classical probability.

$$P_x = |\phi_1 + \phi_2 + \phi_3|^2 + |\phi_4|^2$$

(c) The detector is now moved to a point between slits #3 and #4 and registers whether or not the photon passes through one of these slits. However, the detector does not record which of these two slits the photon passes. What is P in this case?

In this scenario, slits three and four are informational isolated from each other but not from slits one and two. This implies that we must sum the probability amplitudes of one and two, the probability amplitudes of three and four, and then sum these two respective probabilities.

$$P_x = |\phi_1 + \phi_2|^2 + |\phi_3 + \phi_4|^2$$

Question IV

Find expressions for the probabilities P_0 and P_1 for the Mach–Zehnder interferometer of the figure bellow for arbitrary values of ϕ. From your results, show that $P_0 + P_1 = 1$ for any ϕ, as expected

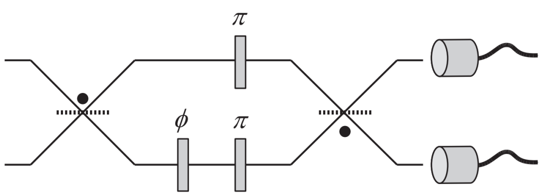

Figure 1: The Mach–Zehnder interferometer

(a) Finding the transformation matrix

We can express the transformation from the initial state to the final state as:

$$|\psi'\rangle = C\,|\psi\rangle \tag{6}$$

where C is a matrix representing the applied transformations on our initial state. Examining the diagram reveals that C equals.

$$C = B_l M_l M_u P_l B_u$$

Calculating this matrix in standard bases provides us with:

$$C = \frac{1}{2}exp\,(i\pi)\begin{bmatrix}(exp\,(i\phi)-1) & (exp\,(i\phi)+1)\\(-exp\,(i\phi)-1) & (-exp\,(i\phi)+1)\end{bmatrix}$$

(b) Calculating the Probability

Using Equation 8, we can compute the probability amplitudes for states α' and β' as follows.

$$\begin{bmatrix}\alpha'\\\beta'\end{bmatrix} = \frac{1}{2}exp\,(i\pi)\begin{bmatrix}(exp\,(i\phi)-1) & (exp\,(i\phi)+1)\\(-exp\,(i\phi)-1) & (-exp\,(i\phi)+1)\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}$$

Performing standard vector algebra and converting the vector equation into a system of equations, we extract the expressions for α' and β'.

$$\alpha' = -\frac{1}{2}\left[\left(exp\left(i\phi\right)-1\right)\alpha + \left(exp\left(i\phi\right)+1\right)\beta\right]$$

$$\beta' = -\frac{1}{2}\left[\left(-exp\left(i\phi\right)-1\right)\alpha + \left(-exp\left(i\phi\right)+1\right)\beta\right]$$

Now, using Equation 7 we can calculate the probability of obtaining state α' or β'. We simplify our result by assuming $|\alpha|^2+|\beta|^2=1$. as this assumption is necessary since the total probability of our starting states must equal one.

$$P_{\alpha'} = |\alpha'|^2 = \alpha'\left(\alpha'\right)^*$$

$$= \frac{1}{4}\left[2 + e^{i\phi}\left(|\beta|^2 - |\alpha|^2 + \alpha\beta^* - \beta\alpha^*\right) + e^{-i\phi}\left(|\beta|^2 - |\alpha|^2 - \alpha\beta^* + \beta\alpha^*\right)\right]$$

$$P_{\beta'} = |\beta'|^2 = \beta'\left(\beta'\right)^*$$

$$= \frac{1}{4}\left[2 + e^{i\phi}\left(|\alpha|^2 - |\beta|^2 - \alpha\beta^* + \beta\alpha^*\right) + e^{-i\phi}\left(|\alpha|^2 - |\beta|^2 + \alpha\beta^* - \beta\alpha^*\right)\right]$$

(c) proving that $P_0 + P_1 = 1$

Having obtained the probabilities for states α' and β' , we simply need to add these probabilities together to demonstrate their sum equals one. Upon inspection, it is evident that $e^{i\phi}$ and $e^{-i\phi}$ sum to 0, leaving us with only the contribution of 2 from each probability.

$$P_{\alpha'} + P_{\beta'} = \frac{1}{4}\left[4 + e^{i\phi}\left(0\right) + e^{-i\phi}\left(0\right)\right] = 1$$

Question V

Read and summarize the subsection "Testing bombs" towards the end of section 2.1. You do not need to solve the exercises within the subsection.

(a) Summery

In the thought experiment proposed by Avshalon Elitzur and Lev Vaidman in 1993, they demonstrated the observer effect in quantum mechanics using a quantum interferometer. The experiment highlights how the act of measuring (aka observing) whether or not a photon passed through a specified path influenced the outcome, shifting the probabilities of finding said photon. In the setup used in the thought experiment it increased the likely hood of finding the photon in the lower path from 0 to $\frac{1}{4}$ a dramatic increase.

Question VI

Adapt the derivation of

$$P_u(t) = \frac{1}{2}\left[1 + cos\left(\omega_1 t - \omega_0 t\right)\right] \tag{7}$$

to derive $P_+(t) = \frac{1}{2}(1 + cos\Omega t)$

(a) defining our initial conditions

In deriving equation 9**, we first define** $|u\rangle$ **as our initial state. In this case, our initial state is** $|+\rangle$**, represented in the** S_z **basis as:**

$$|+\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] \tag{8}$$

Given that the initial state of our system is $|\psi\rangle = |+\rangle$ **, we can express** $|\psi\rangle$ **in the same basis:**

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

Now, we require a matrix $U(t)$ **that describes how** $|\psi\rangle$ **is modified over time. The book defines such a matrix, multiplying each basis by** $e^{i\omega_k t}$**, where** ω_k **represents the frequency associated with the energy state. This gives us:**

$$U(t)\,|\psi(0)\rangle = |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\omega_0 t}\,|0\rangle + e^{-i\omega_1 t}\,|1\rangle\right]$$

(b) finding the associated frequency's

Because we are utilizing spin in the z-direction as the basis for this system, we need to determine the associated energies of these spins. We can do this through the magnetic moment and a uniform magnetic field:

$$E = -\vec{\mu} \cdot \vec{B} \tag{9}$$

The magnetic moment is linked to the spin by:

$$\vec{\mu} = \gamma \vec{S} \tag{10}$$

In this case $\vec{S}$ is in the z-direction and can have values of $S_{z\pm} = \pm\frac{\hbar}{2}$. Assuming parallel alignment of spin and magnetic field, we find the associated energies.

$$E_{z\pm} = \mp\frac{\gamma\hbar B}{2}$$

Introducing a new term $\Omega = \gamma B$ we express E in terms of frequencies.

$$E_{z\pm} = \mp\frac{\Omega}{2}\hbar$$

Now, using the general energy equation, we can find the frequency in terms of our energy bases.

$$E = \omega_k\hbar \tag{11}$$

$$\omega_{\pm} = \mp\frac{\Omega}{2}$$

(c) finding the probability

To determine the probability of a state occurring, we use:

$$P_u = |\langle u|\psi\rangle|^2 \tag{12}$$

Substituting our newly found frequency into the equation for. $|\psi(t)\rangle$ and replacing the initial state, we obtain.

$$P_+ = |\langle +|\psi(t)\rangle|^2 = \langle +|\psi(t)\rangle\left(\langle +|\psi(t)\rangle\right)^*$$

Let's first find $\langle +|\psi(t)\rangle$.

$$\langle +|\psi(t)\rangle = \frac{1}{2}\left[\langle 0| + \langle 1|\right]\left[exp\left(i\frac{\Omega}{2}t\right)|0\rangle + exp\left(-i\frac{\Omega}{2}t\right)|1\rangle\right]$$

because $|0\rangle$ and $|1\rangle$ are basis states of our space $\langle 0|1\rangle = \langle 1|0\rangle = 0$ applying this yields.

$$\frac{1}{2}\left[exp\left(i\frac{\Omega}{2}t\right) + exp\left(-i\frac{\Omega}{2}t\right)\right] = cos\left(\frac{\Omega}{2}t\right) = \sqrt{\frac{cos\left(\Omega t\right)}{2}}$$

Since the probability is the square of the expression above, we have derived the desired expression. This expression indicates that the system's probability oscillates at some frequency Ω.

Question VII

If we represent the state $|\psi\rangle$ of a two-level atom as a complex column vector with respect to the $|E_0\rangle$ and $|E_1\rangle$ basis states, then the time evolution operator $U(t)$ will be a $2X2$ complex matrix $U(t)$. Find this matrix and show that it is unitary.

(a) what is $U(t)$

To determine $U(t)$, we observe its effect on the some general initial state $|\psi(0)\rangle$:

$$U(t)|\psi(0)\rangle = |\psi(t)\rangle = \alpha e^{-i\omega_0 t}|E_0\rangle + \beta e^{-i\omega_1 t}|E_1\rangle \tag{13}$$

using $|\psi(t)\rangle$, we find

$$|\psi(t)\rangle = e^{-i\omega_0 t}|E_0\rangle = U(t)|E_0\rangle$$

when $(\alpha,\beta) = (1,0)$

$$|\psi(t)\rangle = e^{-i\omega_1 t}|E_1\rangle = U(t)|E_1\rangle$$

when $(\alpha,\beta) = (0,1)$

Next, we determine the matrix entries U_{mn} by applying $\langle m|U|n\rangle = U_{mn}$

$$U_{00} = \langle E_0|U|E_0\rangle = e^{-i\omega_0 t}\langle E_0|E_0\rangle = e^{-i\omega_0 t}$$

$$U_{11} = \langle E_1|U|E_1\rangle = e^{-i\omega_1 t}\langle E_1|E_1\rangle = e^{-i\omega_1 t}$$

$$U_{10} = U_{01} = \langle E_1|U|E_0\rangle = e^{-i\omega_0 t}\langle E_1|E_0\rangle = 0$$

Thus, expressing $U(t)$ in terms of the E bases

$$U(t) = e^{-i\omega_0 t}|E_0\rangle\langle E_0| + e^{-i\omega_1 t}|E_1\rangle\langle E_1|$$

(b) is $U(t)$ unitary

$U(t)$ **is unitary when the following equation holds:**

$$UU^\dagger = I \tag{14}$$

Expanding this equation while considering $\langle E_0|E_1\rangle = \langle E_1|E_0\rangle = 0$ **gives us**

$$UU^\dagger = e^0 |E_0\rangle\langle E_0| + e^0 |E_1\rangle\langle E_1| = I$$

implying $U(t)$ **is unitary.**

Question VIII

Two boxes each produce a stream of qubits. Box A produces the qubits all in the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Box B randomly produces qubits in states $|0\rangle$ and $|1\rangle$, each with probability $\frac{1}{2}$. We have one of the boxes, but it is unmarked and so we do not know which kind it is. Describe an experiment on the qubits that can tell the difference between box A and box B. Can you reliably tell the difference between the boxes by examining only one of the qubits?

(a) What should we do

Examining the two boxes reveals that they are indistinguishable in the z bases. However, in the x bases, one of the boxes has encoded information. This implies that if we measure the box with respect to this axis, they should be distinguishable.

(b) calculating the probability of box 1

Equation 12 allows us to compute the probability of box 1 being in the state $|+\rangle$:

$$P_+ = |\langle +|+\rangle|^2 = 1$$

This states that we will always find this box in the positive x spin.

(c) calculating the probability of box 2

Since $|0\rangle$ and $|1\rangle$ are not interacting with one another, we sum up their probabilities classically with a weighted likelihood of observing one or the other:

$$P_+ = P_0|\langle +|0\rangle|^2 + P_1|\langle +|1\rangle|^2$$

We can calculate this by converting from the z spin bases to the x spin bases:

$$|0\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle + |-\rangle\right] \tag{15}$$

$$|1\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle - |-\rangle\right] \tag{16}$$

substituting equation 15 and 16 into the given equation and substituting in the probabilities of $|0\rangle$ and $|1\rangle$, while also considering $\langle +|-\rangle = \langle -|+\rangle = 0$ we get.

$$P_+ = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

(d) Can you reliably tell the difference between the boxes by examining only one of the qubits?

It's evident that observing just one qubit doesn't allow us to determine which box we have. After the initial measurement, box B has a $50/50$chance of exhibiting a positive x spin. The distribution depicting box B and the probability of obtaining a negative x spin after n number of qubits examined follows a geometric distribution, with an expected value of $\frac{1}{p}$. Utilizing this information and considering the probability of obtaining a negative x spin, we anticipate identifying the box, on average, after 2.