1 Edge current calculation

Calculation of edge current, using trajectories from a point $(-L/2, y_i)$ at the left boundary through a fixed point $(x, y) = (0, \pm W/2)$ to a point at the right boundary $(+L/2, y_f)$. Assuming, trajectories are isotropically scattered from there to the right boundary with a transmission coefficient $\mathcal{T}_{\text{edge}}$.

1.1 Upper edge

Starting with the upper edge $(x, y) = (0, +W/2)$. Parametrization of trajectory by angles $\theta_{i,f}$:

$$\tan\theta_i = \frac{W - 2y_i}{L}, \quad \frac{d\theta_i}{dy_i} = -\frac{2L}{L^2 + (W - 2y_i)^2} \tag{1}$$

$$\tan\theta_f = \frac{W - 2y_f}{L}, \quad \frac{d\theta_f}{dy_f} = -\frac{2L}{L^2 + (W - 2y_f)^2}. \tag{2}$$

Effective magnetic phase

$$\begin{aligned}
\frac{2\pi}{\Phi_0}\int d\mathbf{l}\cdot\mathbf{A} &= -\frac{2\pi B}{\Phi_0}\left(\int_{-L/2}^{0} x\tan\theta_i dx + \int_{0}^{+L/2} x\tan\theta_f dx\right) && (3)\\
&= -\frac{\pi B}{\Phi_0}\left(\frac{L}{2}\right)^2(\tan\theta_i + \tan\theta_f) && (4)\\
&= \frac{\pi\phi}{2W}(y_i + y_f) + \frac{\pi\phi}{2} && (5)
\end{aligned}$$

First the critical current for zero flux, where we have no effective phase. Using the parametrization above and the current-phase relation

$$\mathcal{J}(\chi) \simeq \mathcal{T}\sin\chi, \quad \mathcal{T} \ll 1, \tag{6}$$

we find for zero magnetic field:

$$\begin{aligned}
I_{c0}^{\text{edge}} &= \max_\chi \int d\theta_i \cos^2\theta_i \int d\theta_f \cos\theta_f \mathcal{J}(\chi) && (7)\\
&= \int_{-W/2}^{W/2} dy_i \frac{-2L^3}{(L^2 + (W - 2y_i)^2)^2}\int_{-W/2}^{W/2} dy_f \frac{-2L^2}{(L^2 + (W - 2y_f)^2)^{3/2}} && (8)\\
&= \frac{W}{\sqrt{L^2 + 4W^2}}\arctan\left(\frac{2W}{L}\right) + \frac{2LW^2}{(L^2 + 4W^2)^{3/2}}. && (9)
\end{aligned}$$

For comparison the zero field critical current in the QPC setup is

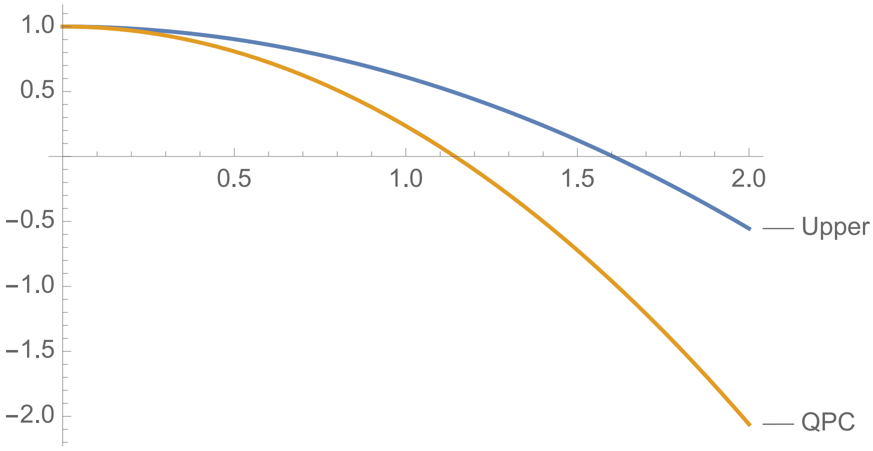
$$I_{c0}^{\text{qpc}} = \frac{L}{\sqrt{L^2 + W^2}}\arctan\frac{W}{L} + \frac{L^2 W}{(L^2 + W^2)^{3/2}} \tag{10}$$

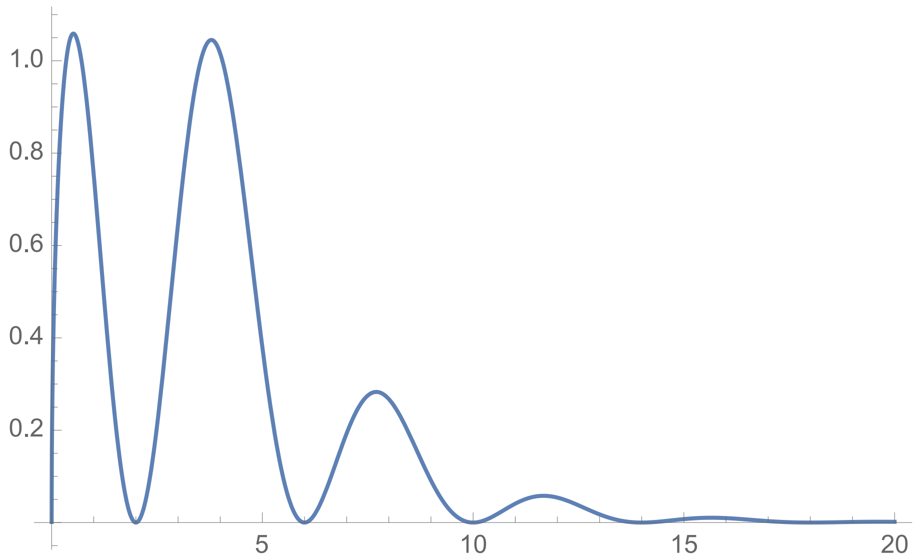
Now with finite, but small magnetic flux ϕ. Using the effective phase from eq. (5) for the Josephson relation in eq.(6) (leads to factor $\cos(\pi\phi/2)$):

$$\begin{aligned}
\frac{I_c(\phi)}{I_{c0}^{\text{edge}}} &= \frac{1}{I_{c0}}\max_\chi \int d\theta_i \cos^2\theta_i \int d\theta_f \cos\theta_f \mathcal{J}(\tilde{\chi}(y_i, y_f)) && (11)\\
&= 1 - \frac{\pi^2\phi^2}{32} f_{\text{egde}}(W/L) && (12)\\
f_{\text{egde}}(x) &= 2 - \frac{\sqrt{4x^2+1} - 1}{x^2} - \frac{4\left(2x^2+1\right)}{2x^2 + x\left(4x^2+1\right)\arctan(2x)} && (13)\\
&\quad - \frac{\sqrt{4x^2+1}\log\left(\sqrt{4x^2+1} - 2x\right)}{x^3} && (14)
\end{aligned}$$

QPC function:

$$\begin{aligned}
\frac{I_c(\phi)}{I_{c0}^{\text{qpc}}} &\simeq 1 - \frac{\pi^2\phi^2}{32} f_0(W/L) && (15)\\
f_0(x) &= \frac{\sqrt{x^2+1}\log\left(\sqrt{x^2+1} + x\right)}{x} - \frac{x}{x + (x^2+1)\arctan(x)} && (16)
\end{aligned}$$





In the limit of high fields, extending the integration over $y_{i,f}$ to infinity, using $\tilde{\phi} = \frac{\pi\phi}{2W}$ we get

$$\frac{1}{16}\pi l\,|\phi|\, e^{-\frac{l|\phi|}{2}}(l\,|\phi| + 2)K_1\left(\frac{l\,|\phi|}{2}\right) \tag{17}$$

$$\begin{aligned}
\frac{I_c(\phi)}{I_{c0}} &= \frac{\cos(\pi\phi/2)}{I_{c0}} \int_{-W/2}^{+W/2} dy_i \frac{2L^3\cos(\tilde{\phi}y_i)}{(L^2+(W-2y_i)^2)^2} \int_{-W/2}^{W/2} dy_f \frac{2L^2\cos(\tilde{\phi}y_f)}{\left(L^2+\tilde{y_f}^2\right)^{3/2}} && (18)\\
&= \frac{1}{I_{c0}} \int_{-\infty}^{\infty} d\tilde{y_i} \frac{L^3\cos\left(\tilde{\phi}(W-\tilde{y_i})\right)}{\left(L^2+\tilde{y_i}^2\right)^2} \int_{-\infty}^{\infty} d\tilde{y_f} \frac{L^2\cos\left(\tilde{\phi}(W-\tilde{y_f})\right)}{\left(L^2+\tilde{y_f}^2\right)^{3/2}} && (19)\\
&= -\frac{1}{8}\exp\left(-\frac{\pi L}{2W}\phi\right)\cos(\pi\phi/4)\sqrt{\frac{2W}{L\phi}}\frac{L}{W}\pi^2\phi\left(2+\frac{\pi L}{2W}\phi\right) && (20)
\end{aligned}$$