Neural stochastic differential equations

Part 2

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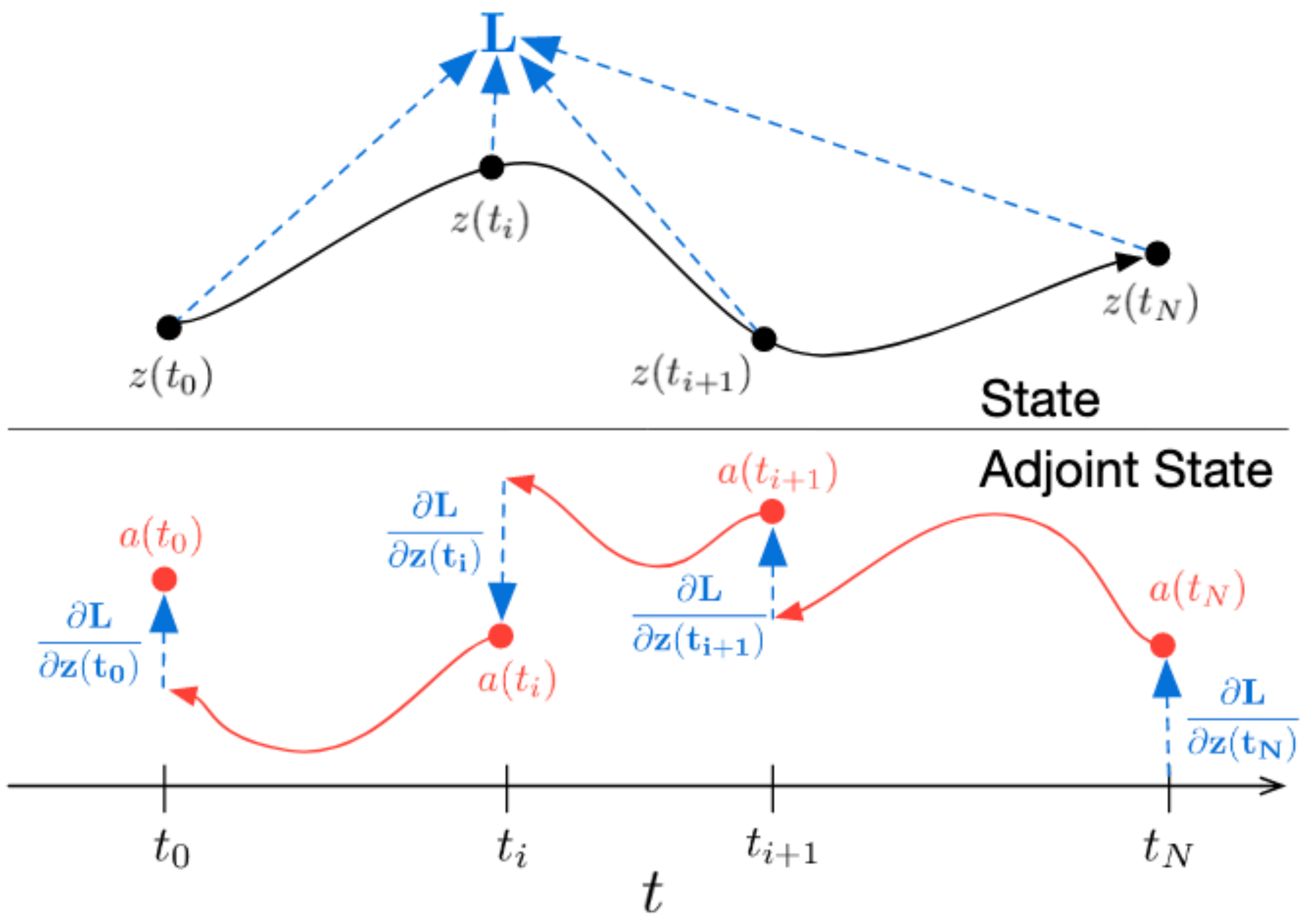
Neural ODE

Adjoint method

$\frac{d\mathbf{z}(t)}{dt} = f(\mathbf{z}(t), t, \theta)$, where θ are the parameters.

$$\mathbf{a}(t) = \frac{dL}{d\mathbf{z}(t)}$$

$$\frac{d\mathbf{a}(t)}{dt} = -\mathbf{a}(t)\frac{\partial f(\mathbf{z}(t), t, \theta)}{\partial \mathbf{z}(t)}$$

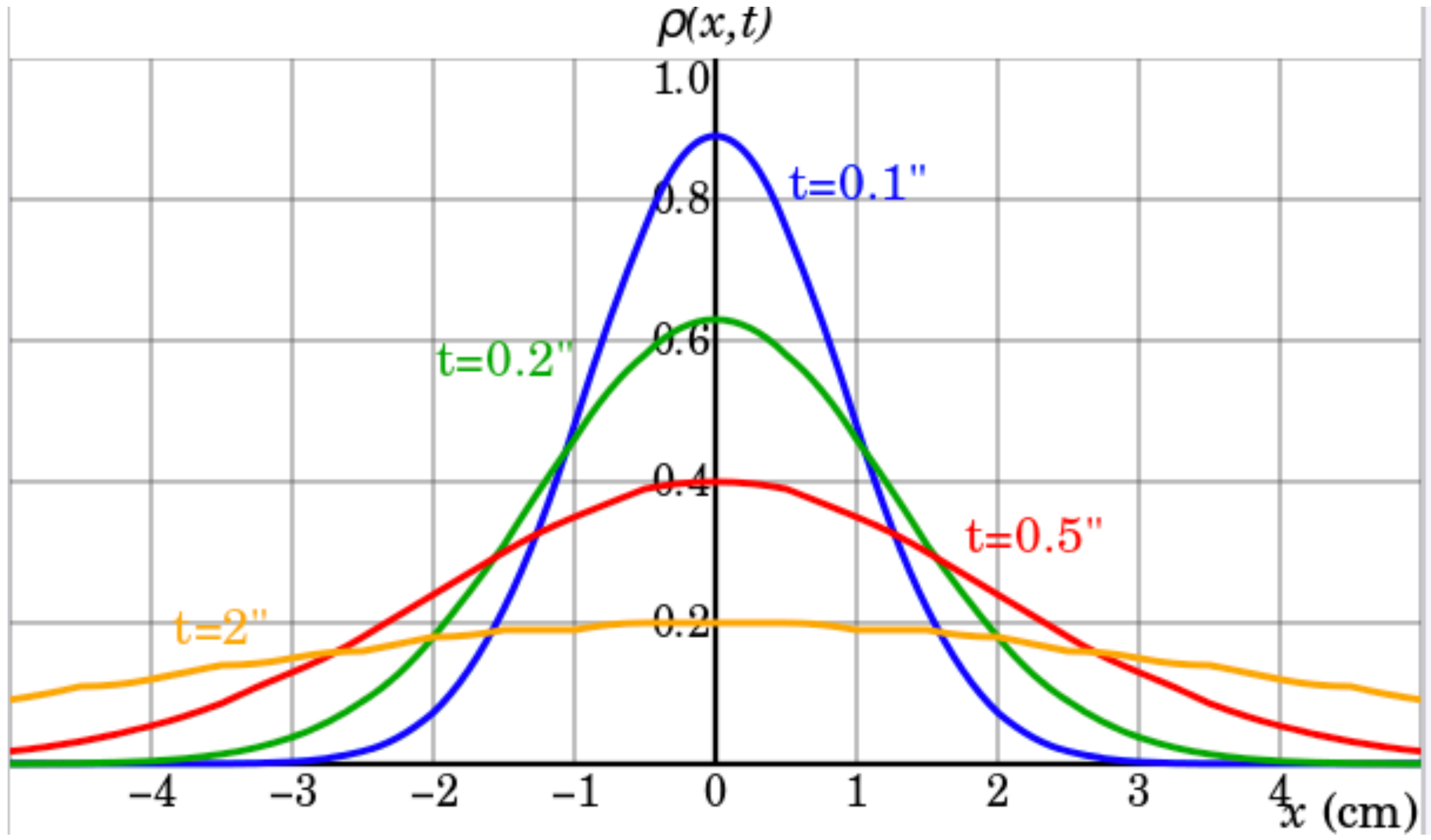


Neural SDE

Problems with adjoint method

$$Z_T = z_0 + \int_0^T b(Z_t, t)\ \mathrm{d}t + \sum_{i=1}^{m} \int_0^T \sigma_i(Z_t, t)\ \mathrm{d}W_t^{(i)}$$

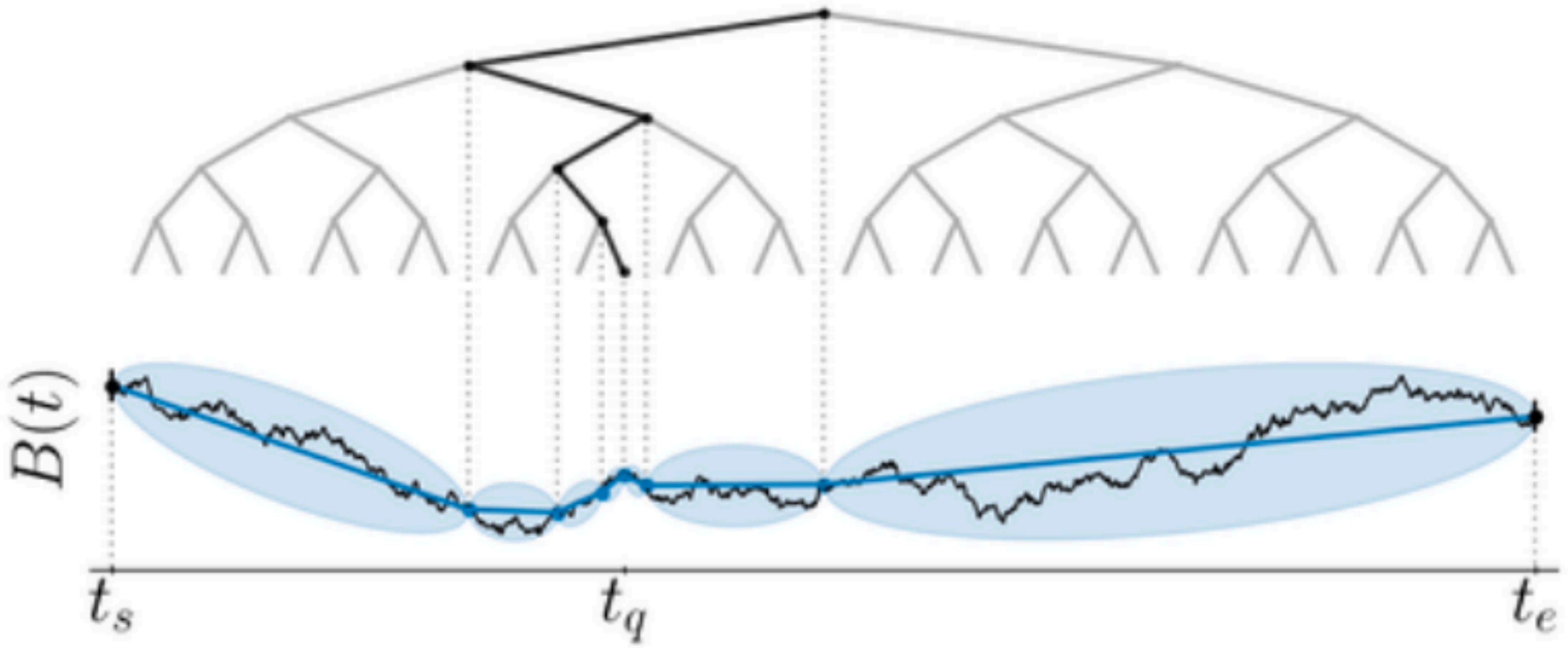
$$\mathcal{N}\left(\frac{(t_e - t)w_s + (t - t_s)w_e}{t_e - t_s}, \frac{(t_e - t)(t - t_s)}{t_e - t_s} I_d\right)$$



$$\breve{A}_{s,t}(z) = \nabla\mathcal{L}(z) + \int_s^t \nabla b(\breve{\Psi}_{r,t}(z), r)^\top \breve{A}_{r,t}(z)\ \mathrm{d}r + \int_s^t \nabla\sigma(\breve{\Psi}_{r,t}(r), r)^\top \breve{A}_{r,t}(z) \circ \mathrm{d}\widetilde{W}_r$$

Neural SDE

Brownian tree



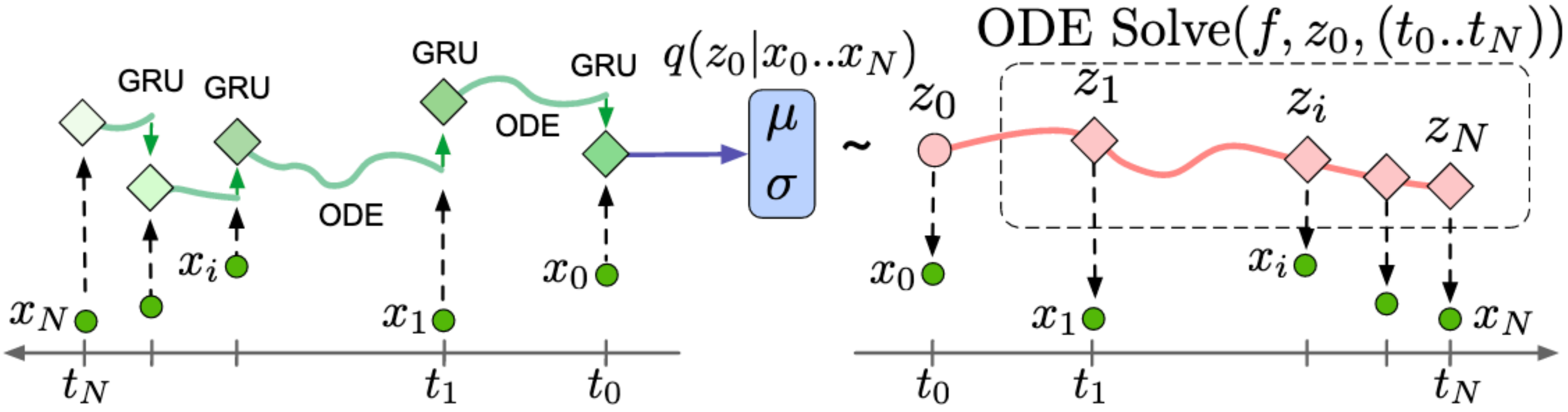
Algorithm 3 Virtual Brownian Tree

Input: Seed s, query time t, error tolerance ϵ, start time t_s, start state w_s, end time t_e, end state w_e.

$t_{\text{mid}} = (t_s + t_e)/2$
$w_{\text{mid}} = \texttt{BrownianBridge}(t_s, w_s, t_e, w_e, t_{\text{mid}}, s)$
while $|t - t_{\text{mid}}| > \epsilon$ **do**
  $s_l, s_r = \texttt{split}(s)$
  if $t < t_{\text{mid}}$ **then** $t_e, x_e, s = t_{\text{mid}}, w_{\text{mid}}, s_l$
  else $t_s, x_s, s = t_{\text{mid}}, w_{\text{mid}}, s_r$
  end if
  $t_{\text{mid}} = (t_s + t_e)/2$
  $w_{\text{mid}} = \texttt{BrownianBridge}(t_s, w_s, t_e, w_e, t_{\text{mid}}, s)$
end while
return w_{mid}

Dealing with irregular time series

Latent ODE



$$z_0 \sim p(z_0)$$

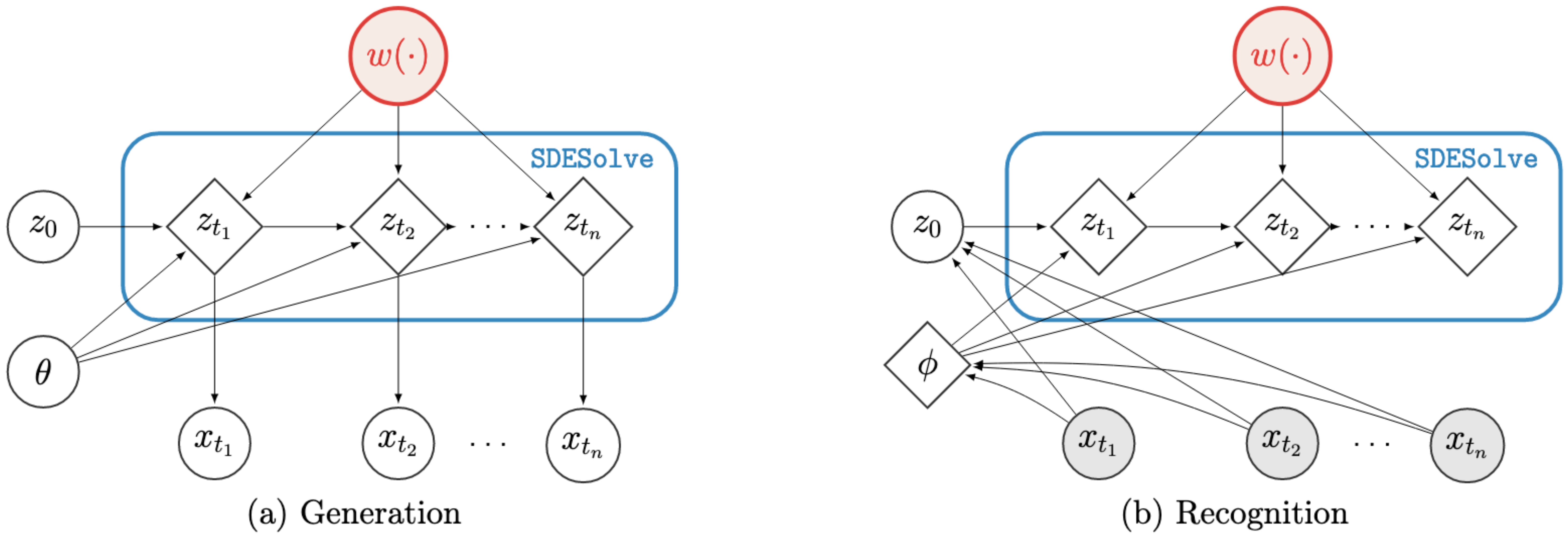
$$z_0, z_1, \ldots, z_N = \mathrm{ODESolve}(f_\theta, z_0, (t_0, t_1, \ldots, t_N))$$

$$\text{each} \quad x_i \overset{indep.}{\sim} p(x_i|z_i) \quad i = 0, 1, \ldots, N$$

$$\mathrm{ELBO}(\theta, \phi) = \mathbb{E}_{z_0 \sim q_\phi(z_0|\{x_i, t_i\}_{i=0}^N)} [\log p_\theta(x_0, \ldots, x_N))] - \mathrm{KL}\left[q_\phi(z_0|\{x_i, t_i\}_{i=0}^N)||p(z_0)\right]$$

Dealing with irregular time series

Latent SDE



$$d\tilde{Z}_t = h_\theta(\tilde{Z}_t, t)\ dt + \sigma(\tilde{Z}_t, t)\, dW_t, \qquad \text{(prior)}$$
$$dZ_t = h_\phi(Z_t, t)\ dt + \sigma(Z_t, t)\ dW_t, \qquad \text{(approx. post.)}$$

$$\log p(x_1, x_2, \ldots, x_N|\theta) \geq \mathbb{E}_{Z_t}\left[\sum_{i=1}^{N} \log p(x_{t_i}|z_{t_i}) - \int_0^T \frac{1}{2}|u(z_t, t)|^2\ dt\right]$$

$$\sigma(z,t)u(z,t) = h_\phi(z,t) - h_\theta(z,t)$$

Dealing with irregular time series

Latent SDE

$$
\begin{aligned}
\mathrm{d}X_t &= \sigma\left(Y_t - X_t\right)\,\mathrm{d}t + \alpha_x\,\mathrm{d}W_t, \quad X_0 = x_0,\\
\mathrm{d}Y_t &= \left(X_t\left(\rho - Z_t\right) - Y_t\right)\,\mathrm{d}t + \alpha_y\,\mathrm{d}W_t, \quad Y_0 = y_0,\\
\mathrm{d}Z_t &= \left(X_tY_t - \beta Z_t\right)\,\mathrm{d}t + \alpha_z\,\mathrm{d}W_t, \quad Z_0 = z_0.
\end{aligned}
$$