Exercise

Apply sampling principles to a “wave packet”

$$s(t) = e^{-\frac{t}{\Delta t}} Sin(2\,\pi \nu_o t)\Theta(t)$$

Take $\Delta t = 10\ s$ and $\nu_o = 10\ Hz$

Calculate continuous Fourier Transform

Sample and estimate alias for $\nu_s = 20, 21, 50, 100\ Hz$

Truncate at $t = [-1, +20]s$ and $[-1, +50]s$ and estimate error within the data range

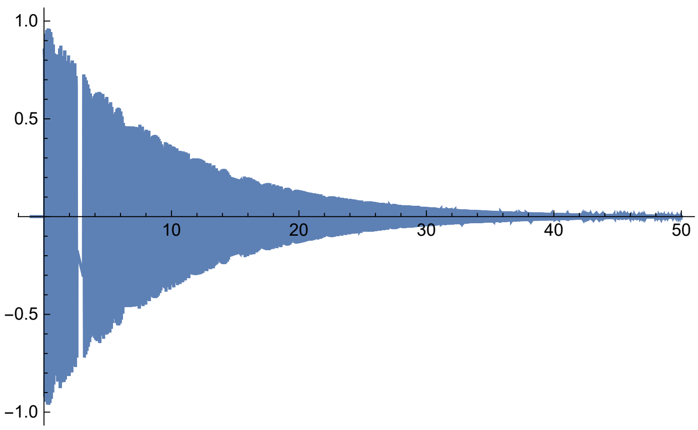
- Guglielmo Grillo

Constants and function definition

```
ΔT = 10; (* seconds *)
ν0 = 10; (* hertz *)
νs = {20, 21, 50, 100} ; (* sample frequencies *)
s[t_] = E^(-t/ΔT) Sin[2 Pi ν0 t] HeavisideTheta[t];
```

```
In[•]:= Plot[s[t], {t, -1, 50}, PlotRange → Full]
```

Out[•]=



A) Continuous Fourier Transform

```
sft[ω_] = FourierTransform[s[t], t, ω]
```

Out[•]=

$$\frac{1000\sqrt{2\pi}}{1 + 40000\,\pi^2 - 20\,i\,\omega - 100\,\omega^2}$$

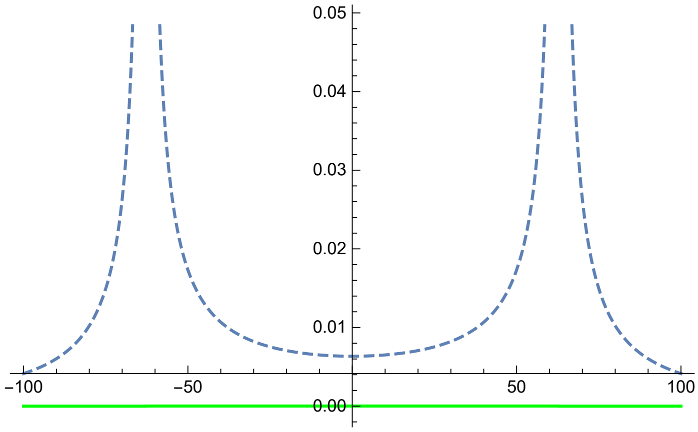
B) Sample and estimation of aliases for the given frequencies

```
(* Reconstruction of transformed from samples*)
sft-rec[ω_, T_] := (HeavisideTheta[ω + Pi/T] - HeavisideTheta[ω - Pi/T])
   Sum[ sft[ω + n 2 Pi/T], {n, -200, +200} ];
```

```
(* 20 Hz *)
Show[ {Plot[ Abs[sft[ω]],                    {ω, -100, +100}, PlotStyle → Dashed ],
       Plot[ Abs[sft-rec[ω, 1/νs[[1]]] ], {ω, -100, +100} , PlotStyle → Green]},
 PlotRange → All]

(* reconstruction error at 20 Hz *)
N[Sum[ (Abs[sft[ω]] - Abs[sft-rec[ω, 1/νs[[1]]] ])^2, {ω, -100, +100} ] , 6]
```

Out[∘]=

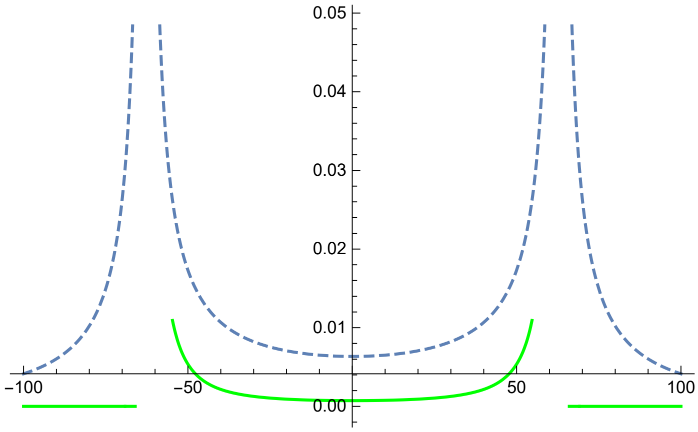


Out[∘]= 2.34838

```
(* 21 Hz *)
Show[ {Plot[ Abs[sft[ω]],                      {ω, -100, +100}, PlotStyle → Dashed ],
       Plot[ Abs[sft-rec[ω, 1/νs[[2]]] ], {ω, -100, +100} , PlotStyle → Green]},
 PlotRange → All]

(* reconstruction error at 21 Hz *)
N[Sum[ (Abs[sft[ω]] - Abs[sft-rec[ω, 1/νs[[2]]] ])^2 , {ω, -100, +100} ] , 6]
```

Out[ ]=

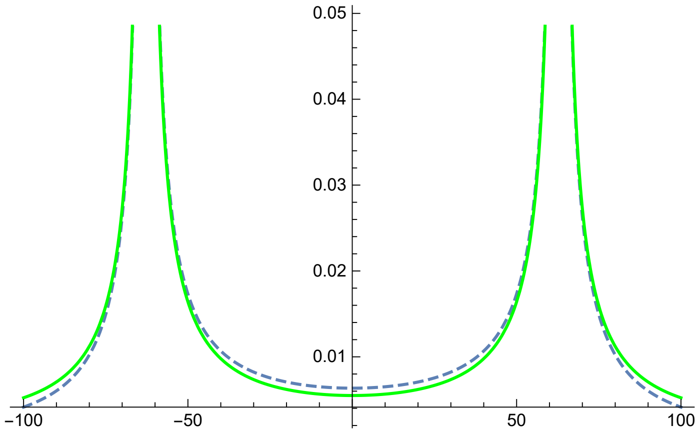


Out[ ]= 0.0487401

```
(* 50 Hz *)
Show[ {Plot[ Abs[sft[ω]],                      {ω, -100, +100}, PlotStyle → Dashed ],
       Plot[ Abs[sft-rec[ω, 1/νs[[3]]] ], {ω, -100, +100} , PlotStyle → Green]},
 PlotRange → All]

(* reconstruction error at 50 Hz *)
N[Sum[ (Abs[sft[ω]] - Abs[sft-rec[ω, 1/νs[[3]]] ])^2 , {ω, -100, +100} ] , 6]
```

Out[ ]=

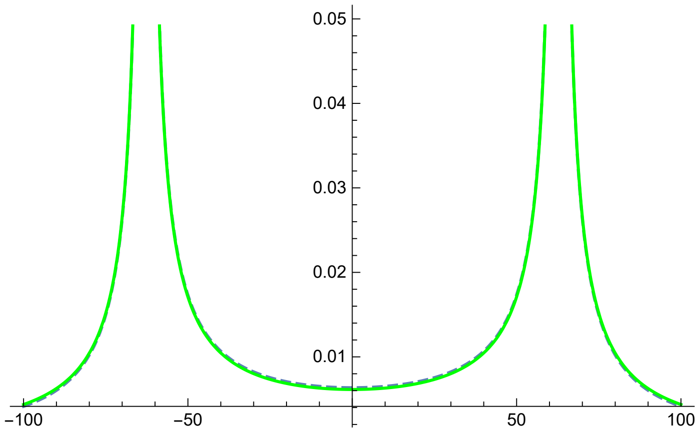


Out[ ]= 0.000173487

```
(* 100 Hz *)
Show[ {Plot[ Abs[sft[ω]],                {ω, -100, +100}, PlotStyle → Dashed ],
       Plot[ Abs[sft-rec[ω, 1/νs[[4]]] ], {ω, -100, +100} , PlotStyle → Green]},
 PlotRange → All]

(* reconstruction error at 100 Hz *)
N[Sum[ (Abs[sft[ω]] - Abs[sft-rec[ω, 1/νs[[4]]] ])^2 , {ω, -100, +100} ] , 6]
```

Out[∘]=



Out[∘]= 9.12534×10^{-6}

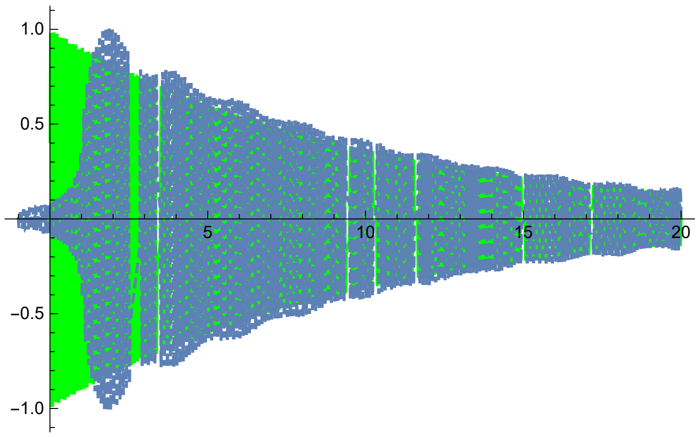
Error estimation withing given ranges

```
(* Reconstruction of the non-trasformed signal from samples *)
srec[t_, samples_, T_] :=
  Sum[ samples[[k]] Sinc[ Pi/T (t - k T)], {k, 1, Length@samples} ];
```

In[∘]:=

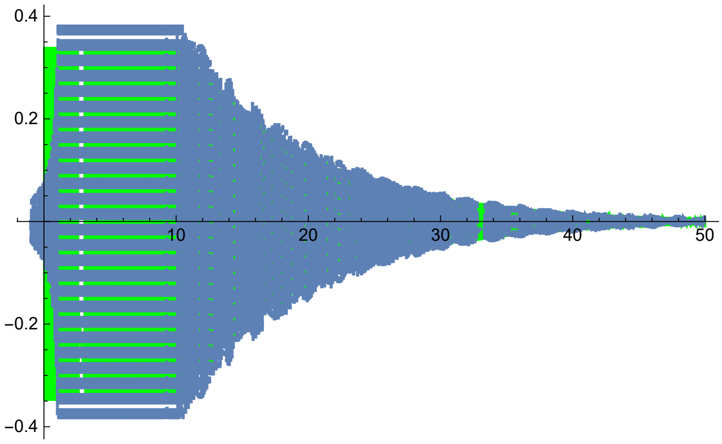
```
(* Vs=21 *)
points = Range[-1, 20, 1/νs[[2]] ];
samples = Map[s, points];
Show[ {Plot[ s[t],                           {t, -1, 20}, PlotStyle → Green ],
       Plot[ srec[t, samples, 1/νs[[2]]], {t, -1, 20}, PlotStyle → Dashed]},
 PlotRange → All]
N[Sum[ (s[t] - srec[t, samples, 1/νs[[2]]])^2 , {t, -1, 20} ] , 6]
```

Out[∘]=



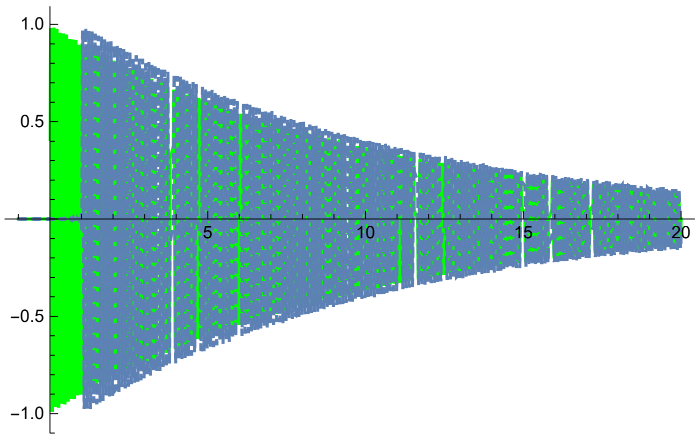
Out[∘]= 0.0990253

In[ ]:=
```
points = Range[-1, 50, 1/νs[[2]] ];
samples = Map[s, points];
Show[ {Plot[ s[t],                                  {t, -1, 50}, PlotStyle → Green ],
       Plot[ srec[t, samples, 1/νs[[2]]], {t, -1, 50}, PlotStyle → Dashed]},
 PlotRange → All]
N[Sum[ (s[t] - srec[t, samples, 1/νs[[2]]])^2, {t, -1, 50} ] , 6]
```

Out[ ]=


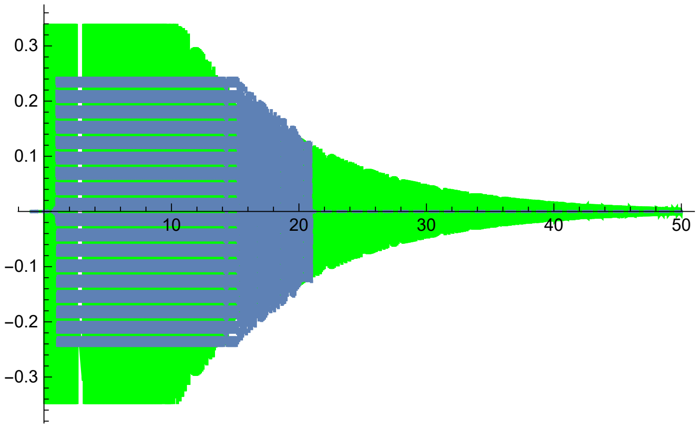
Out[ ]= 0.101286

In[ ]:=
```
(* νs=100 *)
points = Range[-1, 20, 1/νs[[4]] ];
samples = Map[s, points];
Show[ {Plot[ s[t],                                  {t, -1, 20}, PlotStyle → Green ],
       Plot[ srec[t, samples, 1/νs[[4]]], {t, -1, 20}, PlotStyle → Dashed]},
 PlotRange → All]
N[Sum[ (s[t] - srec[t, samples, 1/νs[[4]]])^2, {t, -1, 20} ] , 6]
```

Out[ ]=


Out[ ]= 1.52861

In[∘]:=
```
points = Range[-1, 20, 1/νs[[4]] ];
samples = Map[s, points];
Show[ {Plot[ s[t],                              {t, -1, 50}, PlotStyle → Green ],
       Plot[ srec[t, samples, 1/νs[[4]]], {t, -1, 50}, PlotStyle → Dashed]},
 PlotRange → All]
N[Sum[ (s[t] - srec[t, samples, 1/νs[[4]]])^2 , {t, -1, 50} ] , 6]
```

Out[∘]=


Out[∘]= 1.53495