Experimental Methods
Lecture 30

November 30th, 2020

Fluctuation dissipation theorem

- Hamiltonian

$$H(p,x) = H_o(p,x) + x\, f(t)$$

- Linear, causal response

$$x(t) = \int_0^\infty \chi(t')f(t-t')dt'$$

- Fluctuation of x around equilibrium: zero-mean Gaussian process with Power Spectral Density

$$S_{xx}(\omega) = 2\,k_B T\,\frac{\chi''(\omega)}{\omega}$$

- Equivalent to a force noise with PSD

$$S_{ff}(\omega) = 2\,k_B T\,\frac{\chi''(\omega)}{|\chi(\omega)|^2\omega}$$

The quantum limit

A quantum calculation

PHYSICAL REVIEW VOLUME 83, NUMBER 1 JULY 1, 1951

Irreversibility and Generalized Noise*

HERBERT B. CALLEN AND THEODORE A. WELTON†
Randal Morgan Laboratory of Physics, University of Pennsylvania, Philadelphia, Pennsylvania

(Received January 11, 1951)

A relation is obtained between the generalized resistance and the fluctuations of the generalized forces in linear dissipative systems. This relation forms the extension of the Nyquist relation for the voltage fluctuations in electrical impedances. The general formalism is illustrated by applications to several particular types of systems, including Brownian motion, electric field fluctuations in the vacuum, and pressure fluctuations in a gas.

Gives instead $S_{xx}(\omega)=\left(\chi''(\omega)/\omega\right)\left\{\hbar\omega+2\hbar\omega\big/\left(e^{\hbar\omega/k_B T}-1\right)\right\}$

that has the two limits

$$S_{xx}(\omega)=\begin{array}{ll} 2k_B T\chi''(\omega)/\omega & \hbar\omega \ll k_B T \\ \hbar\chi''(\omega) & \hbar\omega \gg k_B T \end{array}$$

the divide being the thermal frequency that, at room temperature is

$$f_T = k_B/h \times 300\ \mathrm{K} \approx 6\times 10^{12}\ \mathrm{Hz} \equiv 6\ \mathrm{THz}$$

corresponding to a wavelength

$$\lambda_T = c/f_T \approx 50\ \mu\mathrm{m}$$

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Comparison of these equations yields directly our fundamental theorem:

$$\langle V^2 \rangle = (2/\pi) \int_0^\infty R(\omega) E(\omega, T) d\omega, \qquad (4.8)$$

where

$$E(\omega, T) = \tfrac{1}{2}\hbar\omega + \hbar\omega[\exp(\hbar\omega/kT) - 1]^{-1}. \qquad (4.9)$$

It may be recognized that $E(\omega, T)$ is, formally, the expression for the mean energy at the temperature T of an oscillator of natural frequency ω.

The original paper contains a demonstration that fluctuation of a dipole are due to black-body radiation

VI. ELECTRIC DIPOLE RADIATION RESISTANCE AND ELECTRIC FIELD FLUCTUATIONS IN THE VACUUM

An oscillating electric charge radiates energy, leading to a radiation resistance. We shall see that this radiation resistance implies a fluctuating electric field as given by the Planck radiation law.

A parenthesis to fill up a hole

We discussed 1/f thermal noise

In the electronics world, 1/f or “flicker” noise is a different phenomenon.

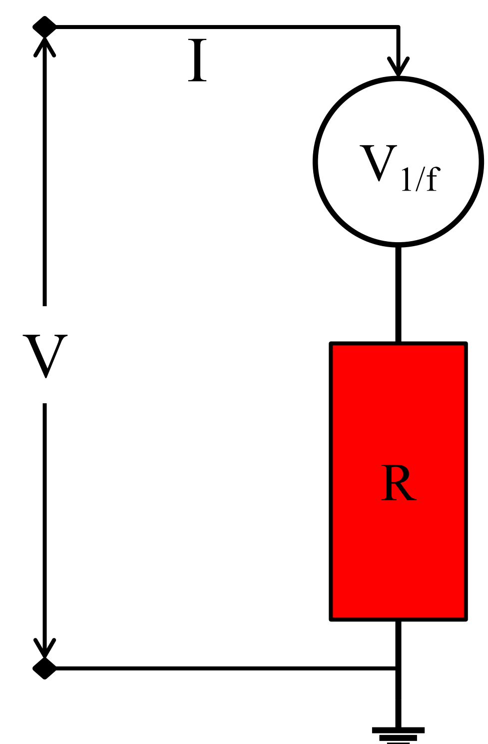
When a dc-current is passed through a resistor a voltage V=RI is established.

Even for perfectly stable current, the voltage fluctuates with PSD

$$S_{VV}(\omega) = \alpha R^2 I^2 / |\omega|$$

here α is a constant, that depend mostly on the resistor technology

It was experimentally demonstrated that the phenomenon is due to fluctuations of the resistance over time



PHYSICAL REVIEW B VOLUME 13, NUMBER 2 15 JANUARY 1976

Flicker ($1/f$) noise: Equilibrium temperature and resistance fluctuations*

Richard F. Voss† and John Clarke

Department of Physics, University of California, Berkeley, California 94720
and Inorganic Materials Research Division, Lawerence Berkeley Laboratory, Berkeley, California 94720

(Received 20 August 1975)

We have measured the $1/f$ voltage noise in continuous metal films. At room temperature, samples of pure metals and bismuth (with a carrier density smaller by 10^5) of similar volume had comparable noise. The power spectrum $S_V(f)$ was proportional to $\bar{V}^2/\Omega f^\gamma$, where $\bar{V}$ is the mean voltage across the sample, Ω is the sample volume, and $1.0 \lesssim \gamma \lesssim 1.4$. $S_V(f)/\bar{V}^2$ was reduced as the temperature was lowered. Manganin, with a temperature coefficient of resistance (β) close to zero, had no discernible noise. These results suggest that the noise arises from equilibrium temperature fluctuations modulating the resistance to give $S_V(f) \propto \bar{V}^2\beta^2 k_B T^2/C_V$, where C_V is the total heat capacity of the sample. The noise was spatially correlated over a length $\lambda(f) \approx (D/f)^{1/2}$, where D is the thermal diffusivity, implying that the fluctuations obey a diffusion equation. The usual theoretical treatment of spatially uncorrelated temperature fluctuations gives a spectrum that flattens at low frequencies in contradiction to the observed spectrum. However, the empirical inclusion of an explicit $1/f$ region and appropriate normalization lead to $S_V(f)/\bar{V}^2 \propto \beta^2 k_B T^2/C_V[3 + 2\ln(l/w)]f$, where l is the length and w is the width of the film, in excellent agreement with the measured noise. If the fluctuations are assumed to be spatially correlated, the diffusion equation can yield an extended $1/f$ region in the power spectrum. We show that the temperature response of a sample to δ- and step-function power inputs has the same shape as the autocorrelation function for uncorrelated and correlated temperature fluctuations, respectively. The spectrum obtained from the cosine transform of the measured step-function response is in excellent agreement with the measured $1/f$ voltage noise spectrum. Spatially correlated equilibrium temperature fluctuations are not the dominant source of $1/f$ noise in semiconductors and discontinuous metal films. However, the agreement between the low-frequency spectrum of fluctuations in the mean-square Johnson-noise voltage and the resistance fluctuation spectrum measured in the presence of a current demonstrates that in these systems the $1/f$ noise is also due to equilibrium resistance fluctuations.

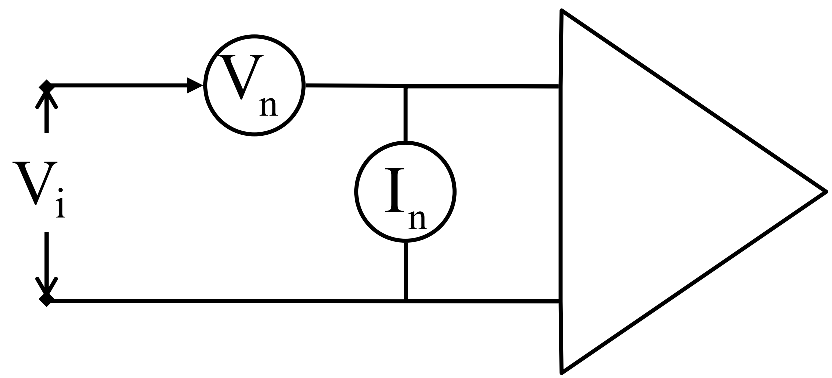
Suggested exercise: optional

- Work out the electrical case and deduce Nyquist law from fluctuation-dissipation thorem:
  - Write the Hamiltonian
  - Identify susceptibility
  - Get both the output and the input PSD

The uncertainty principle and the standard quantum limit for amplifiers

The noise energy of an amplifier

- Remember that the noise properties of an amplifier with high enough input impedance and negligible back-action, are described by the following scheme:



- The two noise generators have PSD S_V and S_I respectively. These are equivalently described by the amplifier noise energy and noise resistance E_n and R_n.

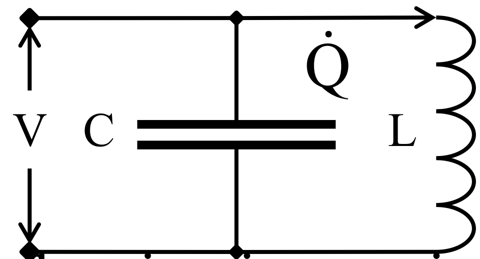
$$E_n(\omega) = \sqrt{S_V(\omega) S_I(\omega)} \qquad R_n(\omega) = \sqrt{S_V(\omega)/S_I(\omega)}$$

- We are going to show that Heisemberg's uncertainty principle implies, for uncorrelated generators, that: $E_n(\omega) \geq \hbar\omega$

The basic electrical oscillator

In order to discuss this result, we will show that if the limit above is not obeyed, one would violate Heisemberg principle on any device attached to the above amplifier.

The simplest of these devices is an electrical harmonic oscillator



For the sake of determining the circuit equations, in analytical mechanics one would first write the Hamiltonian

$$H = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}\frac{\phi^2}{L}$$

where Q is the charge on the capacitor and ϕ the flux through the inductor. Threating Q as a coordinate and ϕ as its associated canonical momentum, Hamilton equations give indeed

$$V = -\dot{\phi} = \partial H/\partial Q = Q/C \qquad \dot{Q} = \partial H/\partial \phi = \phi/L$$

Conjugate variables

In order to make a transition to quantum mechanics, let's calculate the Poisson brackets of Q and ϕ

$$\left[Q,\phi\right] \equiv \frac{\partial Q}{\partial Q}\frac{\partial \phi}{\partial \phi} - \frac{\partial Q}{\partial \phi}\frac{\partial \phi}{\partial Q} = 1$$

The transition to quantum mechanics is obtained identifying the Poisson brackets with $i\hbar$ times the commutator thus $Q\phi - \phi Q = i\hbar$

Quantum mechanics establishes that a simultaneous measurement of Q and ϕ must be affected by uncertainties $\sigma_Q \sigma_\phi \geq \hbar$

However Q and ϕ in a harmonic oscillator are not conserved quantities. Thus in principle they cannot be measured separately with arbitrary precision. Indeed the general solution of the equations of motion is well known to be

$$Q(t)/C = V_p \mathrm{Cos}(\omega_o t) + V_q \mathrm{Sin}(\omega_o t)$$

$$\omega_o \phi(t) = \dot{Q}(t)/\omega_o C = -V_p \mathrm{Sin}(\omega_o t) + V_q \mathrm{Cos}(\omega_o t)$$

Where V_p (p="phase") and V_q (q="quadrature) are arbitrary constants and where $\omega_o=1/\sqrt{LC}$.

Conjugate variables

V_p and V_q have many interesting properties. First they are two constants of motion, i.e. they are conserved. Second they are given by:

$$V_p = \left(Q(t)/C\right)\mathrm{Cos}(\omega_o t) - \omega_o \phi(t)\mathrm{Sin}(\omega_o t)$$

$$V_q = \left(Q(t)/C\right)\mathrm{Sin}(\omega_o t) + \omega_o \phi(t)\mathrm{Cos}(\omega_o t)$$

Thus their Poisson brackets $\left[V_p, V_q\right] \equiv \frac{\partial V_p}{\partial Q}\frac{\partial V_q}{\partial \phi} - \frac{\partial V_p}{\partial \phi}\frac{\partial V_q}{\partial Q} = \frac{\omega_o}{C}$

Thus V_p and V_q can separately be measured with arbitrary precision but in a simultaneous measurement

$$\sigma_{V_p}\sigma_{V_q} \geq \hbar\omega_o/C$$

We are now going to implement the following program:

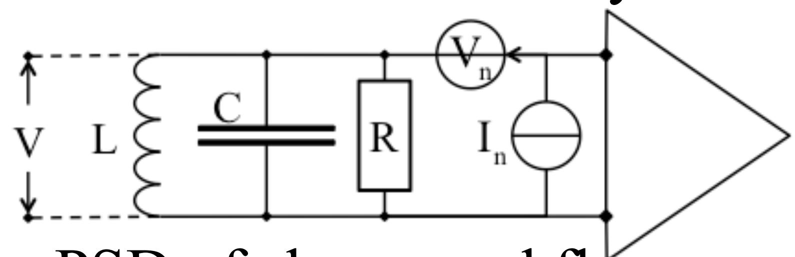
We set up a classical measurement of V_p and V_q

We calculate the measurement uncertainties that come from the amplifier noise.

We apply Heisemberg principle and get a lower limit to the noise energy

Uncertainty and noise generators

Let's now go back to our amplifier and connect it to the LC circuit (with a resistor R in parallel that we will eventually set to infinite)



Let's now calculate the PSD of charge and flux as a consequence of the current noise. From circuit theory we obtain the transfer function from current to input voltage:

$$V(\omega)=\frac{I_n(\omega)}{1/R+i\omega C+1/i\omega L}\equiv I_n(\omega)R\frac{i\omega/\tau}{\omega_o^2-\omega^2+i\omega/\tau}$$

where, as usual, τ=RC and ω_o=1/√LC, with $\tau\omega_o \gg 1$. It follows that current noise generates a voltage noise with spectral density

$$S_{V_I V_I}(\omega)=\frac{k_B T_n(\omega)R^2}{R_n(\omega)}\frac{\omega^2/\tau^2}{\left(\omega_o^2-\omega^2\right)^2+\omega^2/\tau^2}\approx\frac{k_B T_n(\omega_o)R^2}{R_n(\omega_o)}\frac{\omega^2/\tau^2}{\left(\omega_o^2-\omega^2\right)^2+\omega^2/\tau^2}$$

Phase noise

- Now consider the following. Suppose we have two zero-mean random processes $V_p(t)$ and $V_q(t)$ such that

$$R_{V_pV_p}\left(\Delta t\right)=R_{V_qV_q}\left(\Delta t\right)\equiv R\left(\Delta t\right) \qquad R_{V_pV_q}\left(\Delta t\right)=-R_{V_qV_p}\left(\Delta t\right)=-R_{V_pV_q}\left(-\Delta t\right)$$

- And form the following quantity

$$V_I\left(t\right)=V_p\left(t\right)\mathrm{Cos}\left(\omega_1 t\right)+V_q\left(t\right)\mathrm{Sin}\left(\omega_1 t\right)$$

- Let's calculate the autocorrelation:

$$\begin{aligned}
&\left\langle V_I\left(t\right)V_I\left(t+\Delta t\right)\right\rangle=\\
&\left\langle V_p\left(t\right)V_p\left(t+\Delta t\right)\right\rangle \mathrm{Cos}\left(\omega_1 t\right)\mathrm{Cos}\left(\omega_1\left(t+\Delta t\right)\right)+\\
&\left\langle V_q\left(t\right)V_q\left(t+\Delta t\right)\right\rangle \mathrm{Sin}\left(\omega_1 t\right)\mathrm{Sin}\left(\omega_1\left(t+\Delta t\right)\right)+\\
&\left\langle V_p\left(t\right)V_q\left(t+\Delta t\right)\right\rangle \mathrm{Cos}\left(\omega_1 t\right)\mathrm{Sin}\left(\omega_1\left(t+\Delta t\right)\right)+\\
&\left\langle V_q\left(t\right)V_p\left(t+\Delta t\right)\right\rangle \mathrm{Sin}\left(\omega_1 t\right)\mathrm{Cos}\left(\omega_1\left(t+\Delta t\right)\right)
\end{aligned}$$

Phase noise

- As $R_{V_pV_p}(\Delta t) = R_{V_qV_q}(\Delta t) \equiv R(\Delta t)$; $R_{V_pV_q}(\Delta t) = -R_{V_qV_p}(\Delta t) \equiv R_{pq}(\Delta t)$
- Then

$$\begin{aligned}
&\left\langle V_I(t)V_I(t+\Delta t)\right\rangle = \\
&\left\langle V_p(t)V_p(t+\Delta t)\right\rangle \mathrm{Cos}(\omega_1 t)\mathrm{Cos}(\omega_1(t+\Delta t)) + \\
&\left\langle V_q(t)V_q(t+\Delta t)\right\rangle \mathrm{Sin}(\omega_1 t)\mathrm{Sin}(\omega_1(t+\Delta t)) + \\
&\left\langle V_p(t)V_q(t+\Delta t)\right\rangle \mathrm{Cos}(\omega_1 t)\mathrm{Sin}(\omega_1(t+\Delta t)) + \\
&\left\langle V_q(t)V_p(t+\Delta t)\right\rangle \mathrm{Sin}(\omega_1 t)\mathrm{Cos}(\omega_1(t+\Delta t))
\end{aligned}$$

- Simplifies to

$$R_{V_IV_I}(\Delta t) = R(\Delta t)\mathrm{Cos}(\omega_1\Delta t) + R_{pq}(\Delta t)\mathrm{Sin}(\omega_1\Delta t)$$

-

Phase noise

UNIVERSITÀ DEGLI STUDI DI TRENTO

- Now consider the following calculation:

Untitled-1

In[8]:= PSD1 =

$$\text{FourierTransform}\left[\frac{1}{2\,\tau}\,e^{-\frac{\text{Abs}[\Delta t]}{2\,\tau}}\left(\text{Cos}\left[\sqrt{\omega o^2-\frac{1}{4\,\tau^2}}\,\Delta t\right]-\frac{1}{2\,\tau\,\sqrt{\omega o^2-\frac{1}{4\,\tau^2}}}\,\text{Sin}\left[\sqrt{\omega o^2-\frac{1}{4\,\tau^2}}\,\Delta t\right]\text{Sign}[\Delta t]\right),\ \Delta t,\ \omega,\ \text{FourierParameters}\to\{1,\ -1\}\right]$$

Out[8]=
$$\frac{\omega^2}{\tau\left(\frac{\omega^2}{\tau}+\tau\,\omega^4-2\,\tau\,\omega^2\,\omega o^2+\tau\,\omega o^4\right)}$$

In[9]:=
$$\text{FullSimplify}\left[\frac{\frac{\omega^2}{\tau^2}}{\left(\frac{\omega^2}{\tau^2}+\left(\omega^2-\omega o^2\right)^2\right)}-\text{PSD1}\right]$$

Out[9]= 0

- And compare with $S_{VV}^{I}(\omega)=\dfrac{k_B T_n(\omega_o) R^2}{R_n(\omega_o)}\dfrac{\omega^2/\tau^2}{\left(\omega_o^2-\omega^2\right)^2+\omega^2/\tau^2}$

Phase noise

- Putting together

$$\frac{1}{2\tau}e^{-\frac{\mathrm{Abs}[\Delta t]}{2\tau}}\left(\mathrm{Cos}[\omega_1\Delta t]-\frac{1}{2\tau\omega_1}\mathrm{Sin}[\omega_1\Delta t]\mathrm{Sign}[\Delta t]\right)\qquad \omega_1=\sqrt{\omega_o^2-\frac{1}{4\tau^2}}$$

$$R_{V_IV_I}(\Delta t)=R(\Delta t)\mathrm{Cos}(\omega_1\Delta t)+R_{pq}(\Delta t)\mathrm{Sin}(\omega_1\Delta t)$$

$$S_{V_IV_I}(\omega)=\frac{k_BT_n(\omega_o)R^2}{R_n(\omega_o)}\frac{\omega^2/\tau^2}{\left(\omega_o^2-\omega^2\right)^2+\omega^2/\tau^2}$$

- We can state that

$$V_I(t)=V_p(t)\mathrm{Cos}(\omega_1 t)+V_q(t)\mathrm{Sin}(\omega_1 t)$$

- Where $V_p(t)$ and $V_q(t)$ are two stationary random processes with (equal) autocorrelation and cross-correlation respectively

$$R(\Delta t)=\frac{k_BT_n(\omega_o)R^2}{R_n(\omega_o)}\frac{1}{2\tau}e^{-\frac{\mathrm{Abs}[\Delta t]}{2\tau}}\qquad R_{pq}(\Delta t)=\frac{k_BT_n(\omega_o)R^2}{R_n(\omega_o)}\frac{1}{4\tau^2\omega_1}e^{-\frac{\mathrm{Abs}[\Delta t]}{2\tau}}\mathrm{Sign}(\Delta t)$$

Phase noise

- These

$$R(\Delta t)=\frac{k_B T_n(\omega_o) R^2}{R_n(\omega_o)}\frac{1}{2\tau}e^{-\frac{\mathrm{Abs}[\Delta t]}{2\tau}} \qquad R_{pq}(\Delta t)=\frac{k_B T_n(\omega_o) R^2}{R_n(\omega_o)}\frac{1}{4\tau^2\omega_1}e^{-\frac{\mathrm{Abs}[\Delta t]}{2\tau}}\mathrm{Sign}(\Delta t)$$

- Transform to

$$S(\omega)=\frac{k_B T_n(\omega_o) R^2}{R_n(\omega_o)}\frac{1}{1+4\omega^2\tau^2} \qquad S_{pq}(\omega)=-\frac{k_B T_n(\omega_o) R^2}{R_n(\omega_o)}\frac{2i}{2\tau\omega_1}\frac{1}{1+4\omega^2\tau^2}$$

- You may check that the cross coherence is

$$\rho_{pq}(\omega)=-\frac{i}{\tau\omega_1}$$

- And thus vanishes for $\tau\to\infty$ and the phases become independent

White phase noise

- If one attaches a phase sensitive detector at the output of the amplifier, with reference carrier at frequency ω_o, and low-pass filter with a roll-off at $f >> 1/\tau$, at the two outputs one would get V_p and V_q, two almost independent processes, each with PSD

$$S(\omega)=\frac{k_B T_n(\omega_o) R^2}{R_n(\omega_o)}\frac{1}{1+4\omega^2\tau^2}+\ k_B T_n(\omega_o) R_n(\omega_o)$$

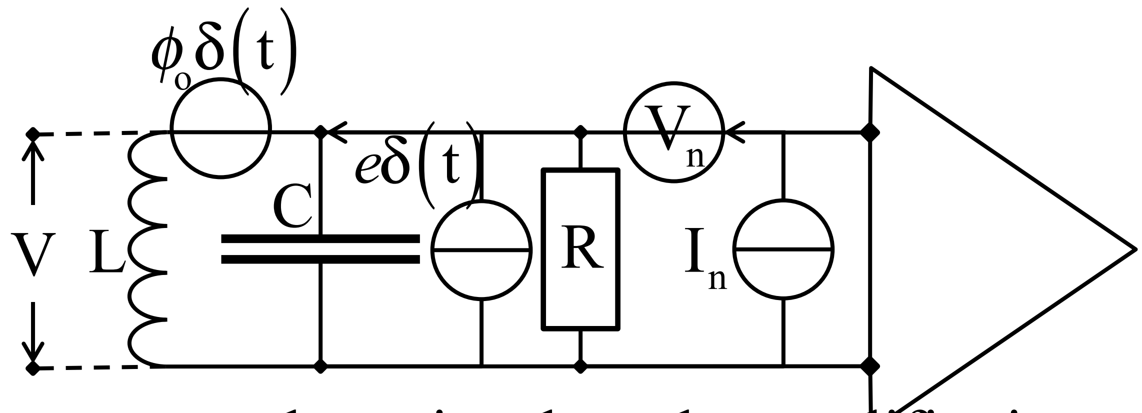
- That is

$$S(\omega)=2k_B T_n(\omega_o) R_n(\omega_o)\left(\frac{R^2}{R_n^2(\omega_o)}\frac{1}{1+4\omega^2\tau^2}+\ 1\right)$$

- Such spectral density should be multiplied by the square modulus of the transfer function of the phase sensitive detector. As this would simplify out in the following calculations, I leave it out.

Uncertainty and noise generators

Assume now that a charge e is instantaneously deposited on the capacitor, and that a static flux ϕ_o is established through the inductor. From a circuit point of view these are equivalent to



These two generators produce signals at the amplifier input given respectively by

$$V_e(\omega) = \frac{e}{C}\frac{i\omega}{\omega_o^2 - \omega^2 + i\omega/\tau}; \quad V_\phi(\omega) = \phi_o \frac{\omega_o^2}{\omega_o^2 - \omega^2 + i\omega/\tau}$$

The inverse transform are calculated on the next page

From Wikipedia

- Laplace transforms

exponentially decaying sine wave	$e^{-\alpha t}\sin(\omega t)\cdot u(t)$	$\dfrac{\omega}{(s+\alpha)^2+\omega^2}$	Re(s) > −α
exponentially decaying cosine wave	$e^{-\alpha t}\cos(\omega t)\cdot u(t)$	$\dfrac{s+\alpha}{(s+\alpha)^2+\omega^2}$	Re(s) > −α

- As $\omega_o^2-\omega^2+i\omega/\tau=\omega_o^2+s^2+s/\tau=$

$$=\omega_o^2-\left(1/4\tau^2\right)+\left(s^2+s/\tau+1/4\tau^2\right)\equiv\omega_1^2+\left(s+1/2\tau\right)^2$$

- Then $V_e(\omega)=\dfrac{e}{C}\dfrac{s}{\omega_1^2+\left(s+1/2\tau\right)^2};\quad V_\phi(\omega)=\phi_o\dfrac{\omega_o^2}{\omega_1^2+\left(s+1/2\tau\right)^2}$
- and transform to

$$V_e(t)=\frac{e}{C}e^{-t/2\tau}\left\{Cos(\omega_1 t)-\left(1/2\tau\omega_1\right)Sin(\omega_1 t)\right\}\Theta(t)\approx\Theta(t)\frac{e}{C}e^{-t/2\tau}Cos(\omega_o t)$$

$$V_\phi(t)=\phi_o\frac{\omega_o^2}{\omega_1}e^{-t/2\tau}Sin(\omega_1 t)\Theta(t)\approx\Theta(t)\phi_o\omega_o e^{-t/2\tau}Sin(\omega_1 t)$$

Detecting phases

- Thus in summary, in the presence of the signal, at the input of the amplifier: $$V(t) = V_p(t)\mathrm{Cos}(\omega_o t) + V_q \mathrm{Sin}(\omega_o t)$$
- With

$$V_p(t) = \Theta(t)\frac{e}{C}e^{-t/2\tau} + V_{p,n}(t) \text{ and } V_q(t) = \Theta(t)\phi_o\omega_o e^{-t/2\tau} + V_{q,n}(t)$$

- We can extract $V_p(t)$ and $V_q(t)$ with a phase sensitive detector and work with those to estimate the precision with which we can measure the amplitudes of the signals
- At the detector output $V_{p,n}(t)$ and $V_{q,n}(t)$ are independent random processes both with PSD

$$S(\omega) = k_B T_n(\omega_o) R_n(\omega_o)\left(\frac{R^2}{R_n^2(\omega_o)}\frac{1}{1+4\omega^2\tau^2} + 1\right)$$

- Notice that $$e^{-t/2\tau} \rightarrow \frac{2\tau}{1+i2\omega\tau}$$

Uncertainty and noise generators

Optimal filter gives:

$$\sigma_{\frac{e}{C}}^{-2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\frac{4\tau^2}{1+4\tau^2\omega^2}}{\mathrm{k_B T_n(\omega_o) R_n(\omega_o)}\left(\frac{\mathrm{R}^2}{\mathrm{R_n^2(\omega_o)}}\frac{1}{1+4\omega^2\tau^2}+1\right)} d\omega$$

an integral that can be reshuffled as:

$$\sigma_{\frac{e}{C}}^{-2} = \frac{1}{2\pi}\frac{4\tau^2}{\mathrm{k_B T_n(\omega_o) R_n(\omega_o)}\left(\frac{\mathrm{R}^2}{\mathrm{R_n^2(\omega_o)}}+1\right)}\int_{-\infty}^{\infty} \frac{1}{1+\omega^2\frac{4\tau^2}{\frac{\mathrm{R}^2}{\mathrm{R_n^2(\omega_o)}}+1}} d\omega$$

That gives

$$\sigma_{\frac{e}{C}}^{-2} = \frac{\tau}{\mathrm{k_B T_n(\omega_o) R_n(\omega_o)}\sqrt{\left(\frac{\mathrm{R}^2}{\mathrm{R_n^2(\omega_o)}}+1\right)}}$$

Uncertainty and noise generators

Thus

$$\sigma_{\frac{e}{C}}^{-2} = \frac{\tau}{k_B T_n(\omega_o) R_n(\omega_o) \sqrt{\left(\frac{R^2}{R_n^2(\omega_o)} + 1\right)}}$$

Similarly

$$\sigma_{\phi_o \omega_o}^{-2} = \frac{\tau}{k_B T_n(\omega_o) R_n(\omega_o) \sqrt{\left(\frac{R^2}{R_n^2(\omega_o)} + 1\right)}}$$

Now remember that τ=RC

$$\sigma_{\frac{e}{C}} \sigma_{\phi_o \omega_o} = \frac{k_B T_n(\omega_o)}{C} \frac{R_n(\omega_o)}{R} \sqrt{\left(\frac{R^2}{R_n^2(\omega_o)} + 1\right)} \rightarrow_{R \to \infty} \rightarrow \frac{k_B T_n(\omega_o)}{C}$$

Uncertainty and noise generators

In conclusion

$$\sigma_{V_p}\sigma_{V_q} = \frac{k_B T_n(\omega_o)}{C}$$

But Heisemberg prescribes

$$\sigma_{V_p}\sigma_{V_q} \geq \frac{\hbar\omega_o}{C}$$

Then

$$k_B T_n(\omega_o) = E(\omega_o) \geq \hbar\omega_o$$

Which set the lower limit to the noise energy of an amplifier

Numerically

$$\frac{k_B}{h} = 21\,\frac{GHz}{K}$$

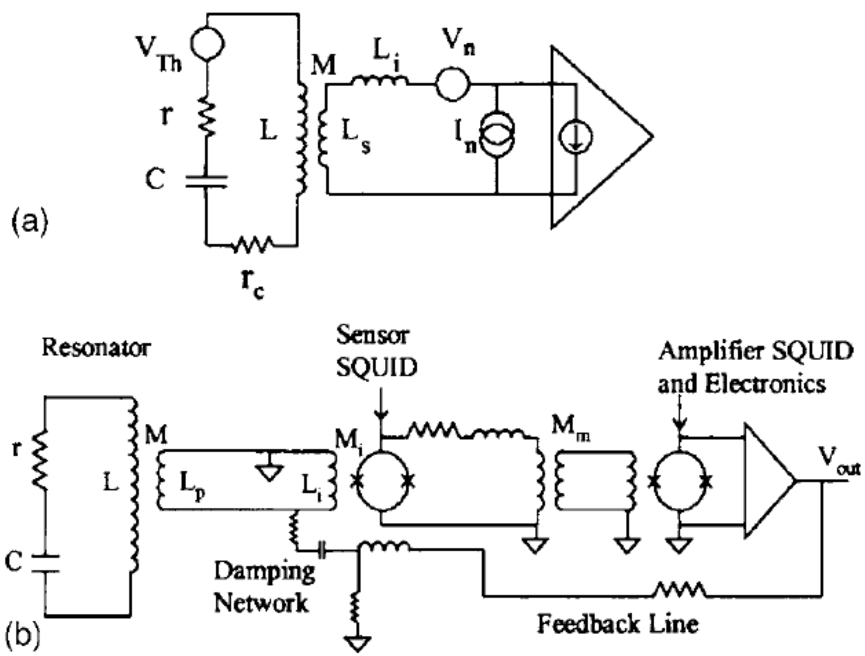
SQUID: audio amplifier

APPLIED PHYSICS LETTERS **93**, 172506 (2008)

10 $\hbar$ superconducting quantum interference device amplifier for acoustic gravitational wave detectors

P. Falferi,[1,2,a)] M. Bonaldi,[1,2] M. Cerdonio,[3,4] R. Mezzena,[5,2] G. A. Prodi,[5,2] A. Vinante,[4] and S. Vitale[5,2]

[1]Istituto di Fotonica e Nanotecnologie, CNR-Fondazione Bruno Kessler, 38100 Povo, Trento, Italy
[2]INFN, Gruppo Collegato di Trento, Sezione di Padova, 38100 Povo, Trento, Italy
[3]Dipartimento di Fisica, Università di Padova, 35131 Padova, Italy
[4]INFN, Sezione di Padova, 35131 Padova, Italy
[5]Dipartimento di Fisica, Università di Trento, 38100 Povo, Trento, Italy



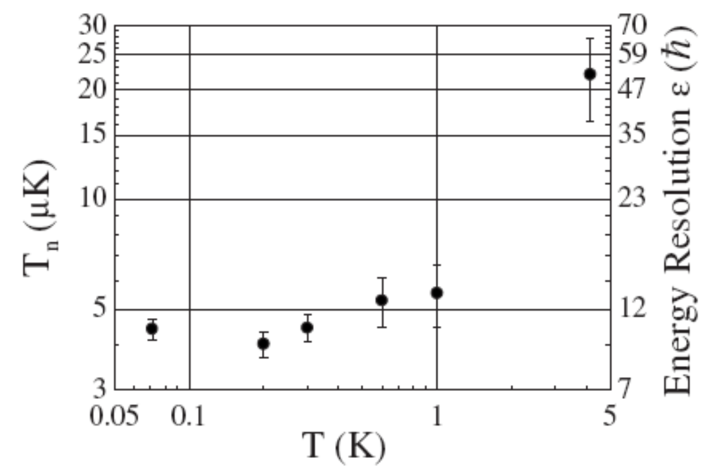


FIG. 2. Noise temperature and corresponding energy resolution in number of quanta of the two-stage SQUID amplifier as a function of the operating temperature.

Formal Theory

REVIEWS OF MODERN PHYSICS, VOLUME 82, APRIL–JUNE 2010

Introduction to quantum noise, measurement, and amplification

A. A. Clerk*

Department of Physics, McGill University, 3600 rue University Montréal, Quebec, Canada H3A 2T8

M. H. Devoret

Department of Applied Physics, Yale University, P.O. Box 208284, New Haven, Connecticut 06520-8284, USA

S. M. Girvin

Department of Physics, Yale University, P.O. Box 208120, New Haven, Connecticut 06520-8120, USA

Florian Marquardt

Department of Physics, Center for NanoScience, and Arnold Sommerfeld Center for Theoretical Physics, Ludwig-Maximilians-Universität München, Theresienstrasse 37, D-80333 München, Germany

R. J. Schoelkopf

Department of Applied Physics, Yale University, P.O. Box 208284, New Haven, Connecticut 06520-8284, USA

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