Experimental Methods
Lecture 10

October 12th, 2020

Linear system in the frequency domain

- Output is the convolution between input and impulse response

$$o(t) = \int_{-\infty}^{\infty} h(t')i(t-t')dt'$$

- Using convolution theorem

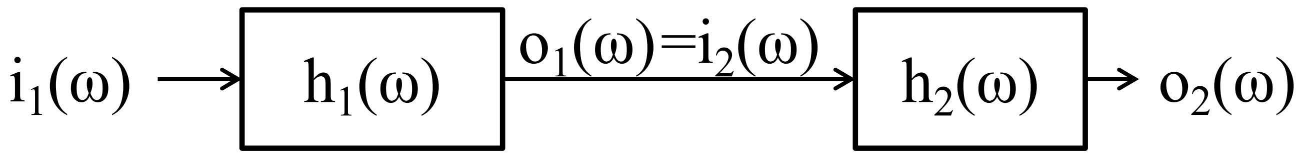
$$o(\omega) = h(\omega)i(\omega)$$

- h(t): impulse response h(ω): ***frequency response***
- Linear response does not mix frequency
- Response of linear system to sinusoidal signal

$$i(t) = i_o \mathrm{Sin}(\omega_o t) \rightarrow o(t) = i_o |h(\omega_o)| \mathrm{Sin}(\omega_o t + \mathrm{Arg}(h(\omega_o)))$$

Linear systems in series

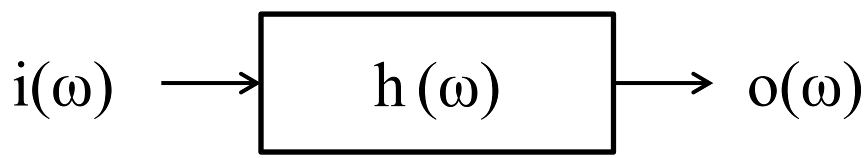
- Two systems in series: the output of the first system is the input to the second one



- It follows that

$$o_2(\omega)=h_2(\omega)i_2(\omega)=h_2(\omega)o_1(\omega)=h_2(\omega)h_1(\omega)i_1(\omega)$$

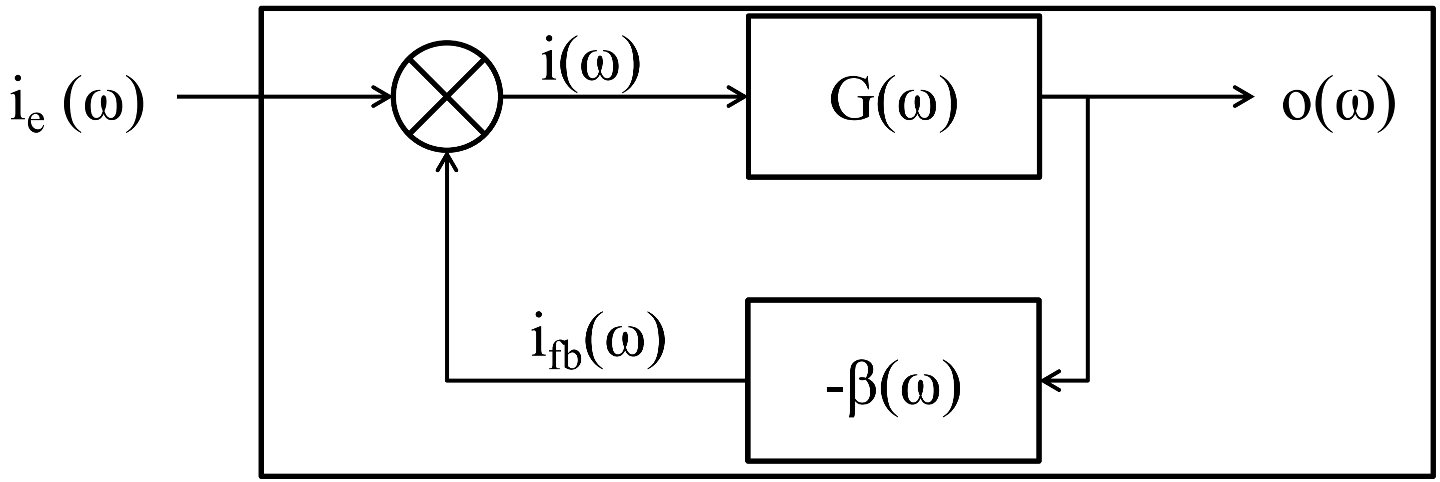
- Thus the system series is equivalent to



- with $$h(\omega)=h_1(\omega)h_2(\omega)$$

A remarkable example: the feedback loop

- The output of a system is fed back and added to external input via another linear system:



- Deriving the input output relations of the full system $i_e \rightarrow o$

$$o(\omega) = G(\omega) i(\omega) = G(\omega)\left[i_e(\omega) + i_{fb}(\omega)\right]$$

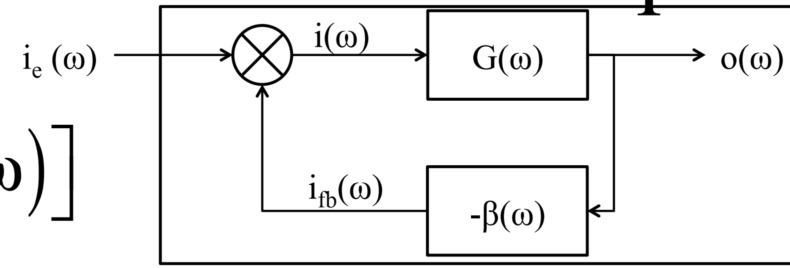
- But:

$$i_{fb}(\omega) = -\beta(\omega) o(\omega)$$

- Thus

$$o(\omega) = G(\omega)\left[i_e(\omega) - \beta(\omega) o(\omega)\right]$$

A remarkable example: the feedback loop



- Input output relations

$$o(\omega)=G(\omega)\left[i_e(\omega)-\beta(\omega)o(\omega)\right]$$

- Solving

$$o(\omega)\left[1+\beta(\omega)G(\omega)\right]=G(\omega)i_e(\omega)$$

- In conclusion ($h_{cl}\equiv$ closed loop frequency response)

$$o(\omega)=\frac{G(\omega)i_e(\omega)}{1+\beta(\omega)G(\omega)}\equiv h_{cl}(\omega)i_e(\omega)$$

- The feedback

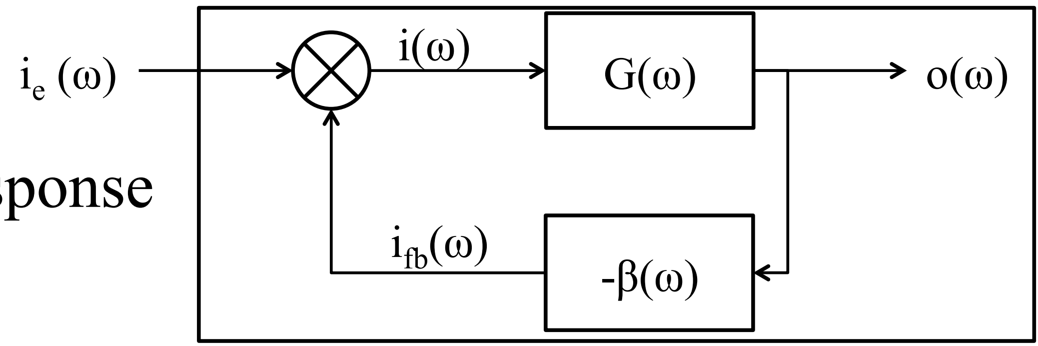
$$i_{fb}(\omega)=-\beta(\omega)o(\omega)\;=-\frac{\beta(\omega)G(\omega)i_e(\omega)}{1+\beta(\omega)G(\omega)}$$

- Total input to G

$$i(\omega)=i_e(\omega)+i_{fb}(\omega)\;=i_e(\omega)-\frac{\beta(\omega)G(\omega)i_e(\omega)}{1+\beta(\omega)G(\omega)}\;=\frac{i_e(\omega)}{1+\beta(\omega)G(\omega)}$$

A remarkable example: the feedback loop

- Summary



- $G(\omega) \equiv$ open loop frequency response

- Closed loop frequency response

$$h_{cl}(\omega) = \frac{o(\omega)}{i_e(\omega)} = \frac{G(\omega)}{1+\beta(\omega)G(\omega)}$$

- Feedback

$$i_{fb}(\omega) = -\frac{\beta(\omega)G(\omega)i_e(\omega)}{1+\beta(\omega)G(\omega)}$$

- Total input

$$i(\omega) = \frac{i_e(\omega)}{1+\beta(\omega)G(\omega)}$$

Feedback key properties for $|G|\rightarrow\infty$ and $|\beta G|>>1$

- Let's take all transfer functions around $(\beta G)\rightarrow\infty$
- Closed loop transfer function (from now on h_{cl})

$$h_{cl}(\omega)=\frac{G(\omega)}{1+\beta(\omega)G(\omega)}\rightarrow\frac{G(\omega)}{\beta(\omega)G(\omega)}=\frac{1}{\beta(\omega)}$$

- Feedback

$$\frac{i_{fb}(\omega)}{i_e(\omega)}=-\frac{\beta(\omega)G(\omega)}{1+\beta(\omega)G(\omega)}\rightarrow-\frac{\beta(\omega)G(\omega)}{\beta(\omega)G(\omega)}=-1$$

- Total input

$$\frac{i(\omega)}{i_e(\omega)}=\frac{1}{1+\beta(\omega)G(\omega)}\rightarrow 0$$

Feedback key properties for $|G|\to\infty$ and $|\beta G|>>1$

- 1) resilience to gain variations
  - Suppose the gain $G(\omega)$ undergoes a variation $\delta G(\omega)$
  - Suppose the input is a constant amplitude sinusoid of amplitude i_o
  - At open loop, the variation of the amplitude o_o of the sinusoid at output is

$$\left|\frac{\delta o_o}{o_o}\right|_{\substack{\text{open}\\\text{loop}}} = \left|\frac{\delta G i_o}{G i_o}\right| = \left|\frac{\delta G}{G}\right|$$

Feedback key properties for $|G|\to\infty$ and $|\beta G|>>1$

- 1) resilience to gain variations
  - The closed loop transfer function

$$h_{cl} = \frac{G}{1+\beta G} = \frac{1}{\beta}\frac{x}{1+x}$$

  - With $x = \beta G$
  - Relative variation of h_{cl}

$$\frac{\delta h_{cl}}{h_{cl}} = \frac{\dfrac{x+\delta x}{1+x+\delta x} - \dfrac{x}{1+x}}{\dfrac{x}{1+x}}$$

- That is

$$\frac{\delta h_{cl}}{h_{cl}} = \frac{1+x}{x}\frac{x+\delta x}{1+x+\delta x} - 1 = \frac{(1+x)(x+\delta x) - x(1+x+\delta x)}{x(1+x+\delta x)}$$

- then

$$\frac{\delta h_{cl}}{h_{cl}} = \frac{\delta x}{x(1+x+\delta x)} = \frac{\delta x}{x}\frac{1}{1+x+\delta x}$$

Feedback key properties for |G|→∞ and |βG|>>1

- From

$$\frac{\delta h_{cl}}{h_{cl}} = \frac{\delta x}{x(1+x+\delta x)} = \frac{\delta x}{x}\frac{1}{1+x+\delta x}$$

- Going back to G

$$\left|\frac{\delta h_{cl}}{h_{cl}}\right| = \left|\frac{\delta G}{G}\right| \frac{1}{|1+\beta(G+\delta G)|}$$

- If $|\beta(G+\delta G)| \gg 1$ then

$$\left|\frac{\delta o_o}{o_o}\right|_{\substack{\text{closed}\\ \text{loop}}} = \left|\frac{\delta h_{cl}}{h_{cl}}\right| \ll \left|\frac{\delta G}{G}\right| = \left|\frac{\delta o_o}{o_o}\right|_{\substack{\text{open}\\ \text{loop}}}$$

Feedback key properties for $|G|\to\infty$ and $|\beta G|>>1$

- 2) Suppression of input variation
  - Suppose the input is a sinusoid at frequency ω_o

$$i_e(t)=i_o \mathrm{Sin}(\omega_o t)$$

  - In the absence of feedback the input to the stage “G” $i(t)=i_e(t)$ and varies between $-i_o$ and $+i_o$.
  - In closed loop i(t) is given by

$$i(t)\approx i_o \frac{1}{\left|\beta(\omega_o)G(\omega_o)\right|}\mathrm{Sin}\left[\omega_o t+\mathrm{Arg}\left(\beta(\omega_o)G(\omega_o)\right)\right]$$

  - The amplitude is then suppressed by the large factor $1/|\beta G|$
  - The system can be brought into its range of linear response, even if the amplitude of the original signal would drive it out.

Feedback key properties for $|G|\to\infty$ and $|\beta G|>>1$

- 3) Magnitude of feedback
  - Still for our sinusoidal system

$$i_e(t) = i_o \mathrm{Sin}(\omega_o t)$$

  - The feedback transfer function is

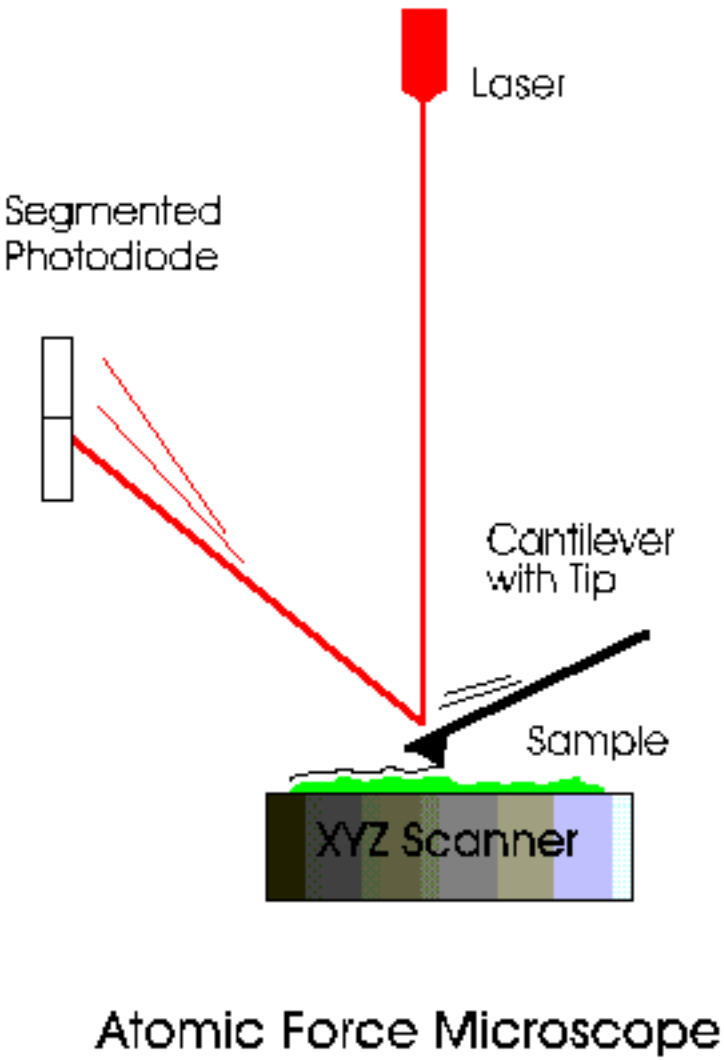
$$\frac{i_{fb}(\omega)}{i_e(\omega)} \approx -1 + \frac{1}{\beta(\omega)G(\omega)} \approx -1$$

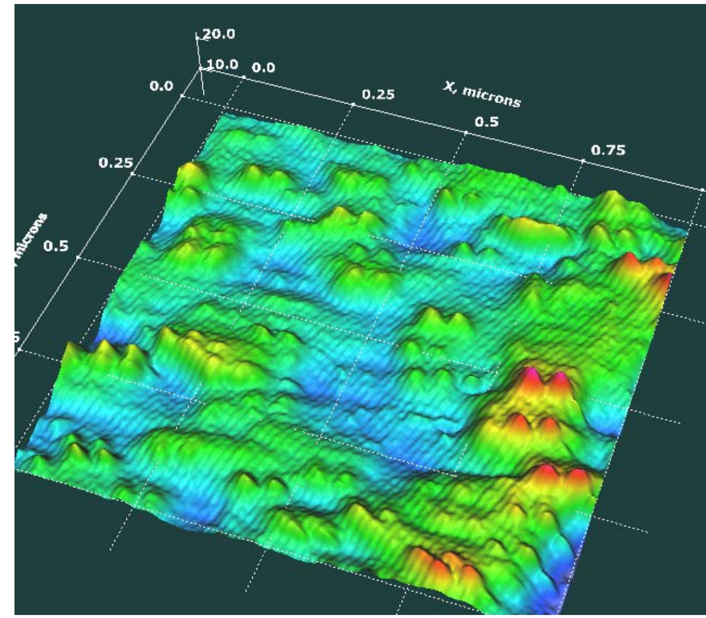
  - Then

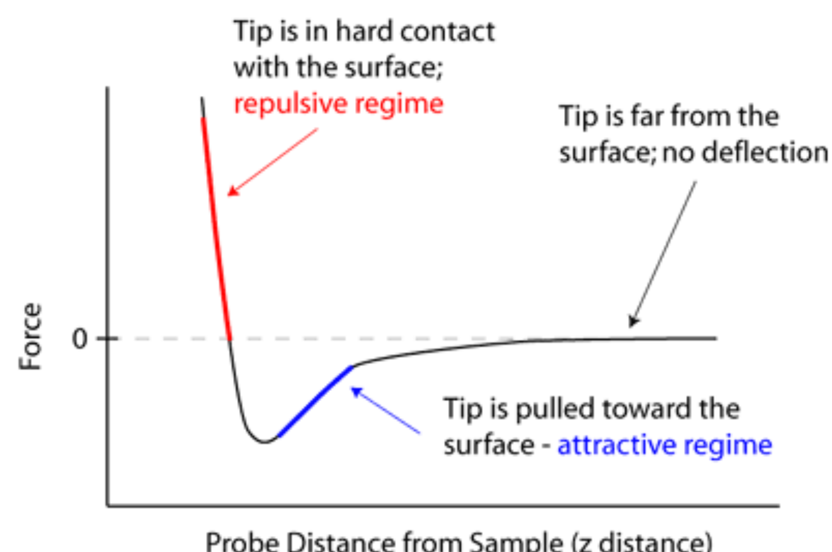
$$i_{fb}(t) \approx -i_e(t) = -i_o \mathrm{Sin}[\omega_o t]$$

  - By measuring the feedback signal one recovers the original input signal!
- A few more remarks
  - linearity requires β to be linear in the range $|i_e/G\beta|$ of variation of its input
  - Phase of β should never convert negative feedback into positive feedback (see you electronics course for details)

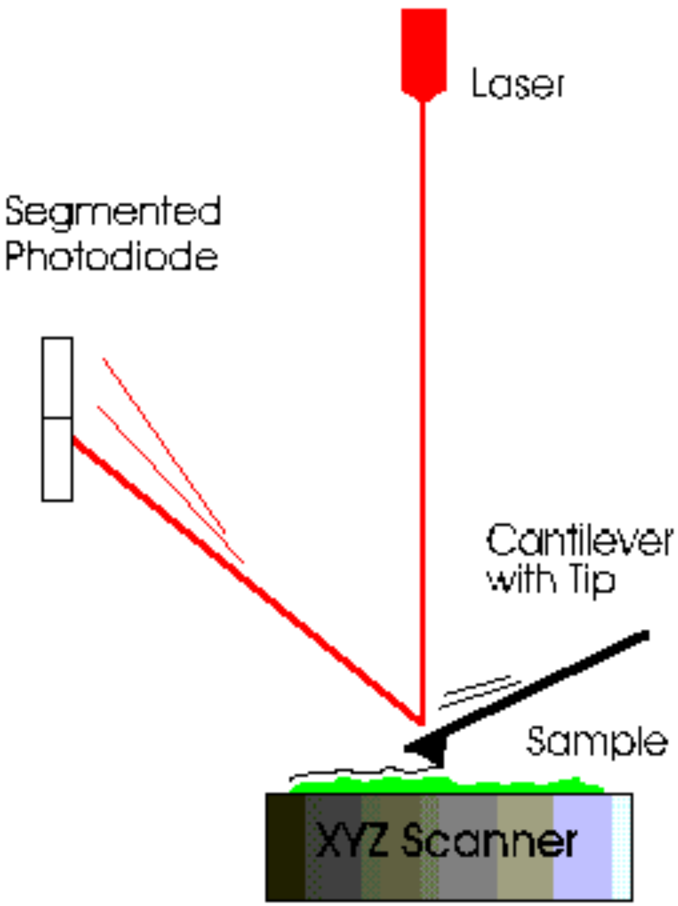
One example atomic force microscope



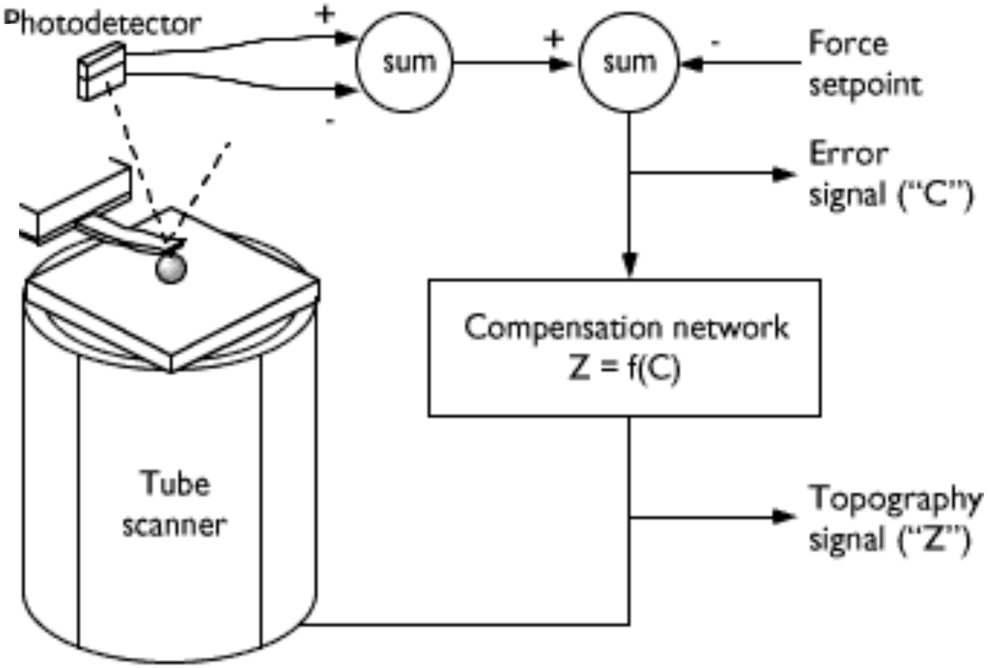


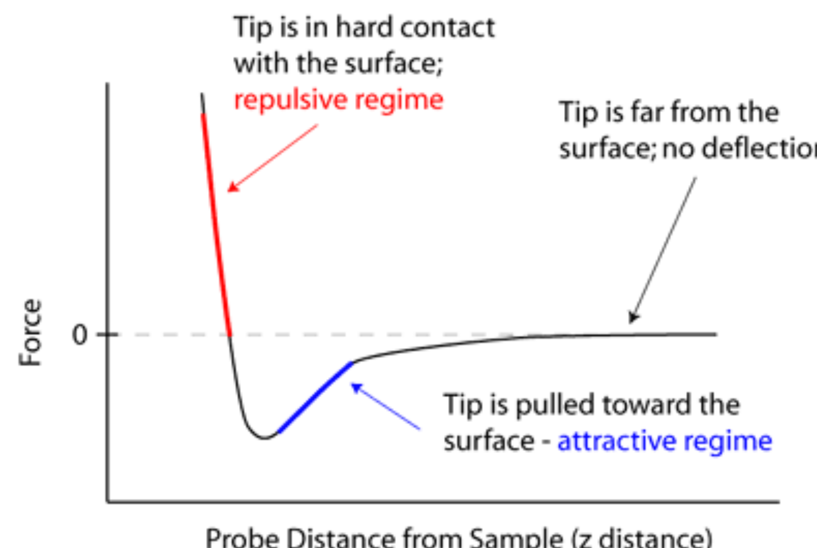


One example atomic force microscope



Atomic Force Microscope





Let's do a model

- The tip is a system with force F_e at input and displacement at output
- Let's approximate it with a particle and a (lossy) spring

$$m\ddot{x} = -k(x)x - \beta\dot{x} + F_e$$

- Let's apply a feedback F_{fb}

$$F_e = F_{sample} + F_{fb}$$

- For small displacement we linearize the spring and, furthermore, assume

$$F_{fb} = -m\omega_o^2 x$$

- Switching to frequency domain

$$\left[\omega_o^2 + k(0)/m - \omega^2 + i\omega(\beta/m)\right]x(\omega) = F_{sample}(\omega)/m$$

- That is

$$x(\omega) = \frac{F_{sample}(\omega)/m}{\omega_o^2 + k(0)/m - \omega^2 + i\omega(\beta/m)}$$

Let's do a model

- Frequency response
- Open loop

$$x(\omega)=\frac{F_{sample}(\omega)/m}{k(0)/m-\omega^2+i\omega(\beta/m)}$$

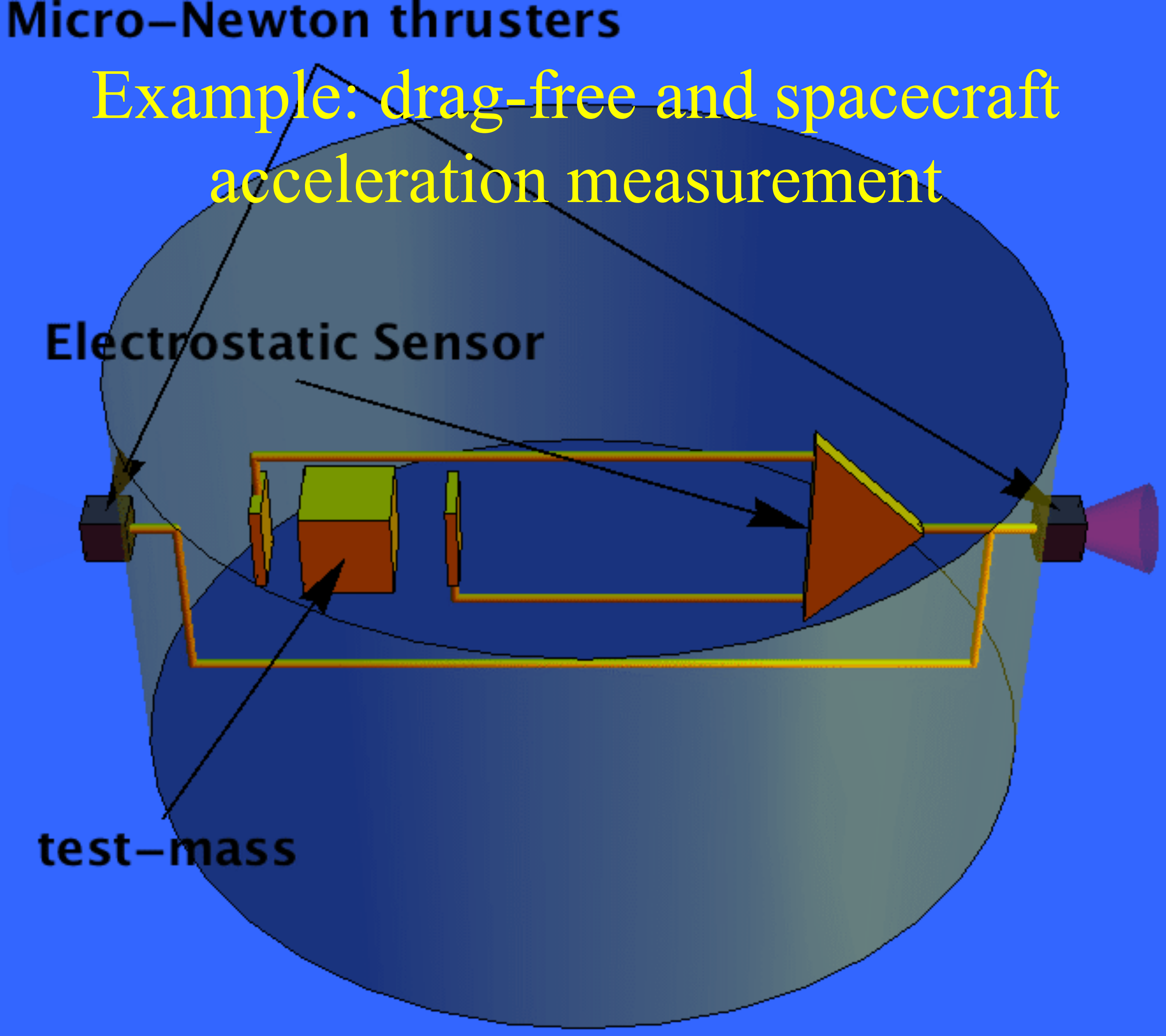
- Closed loop

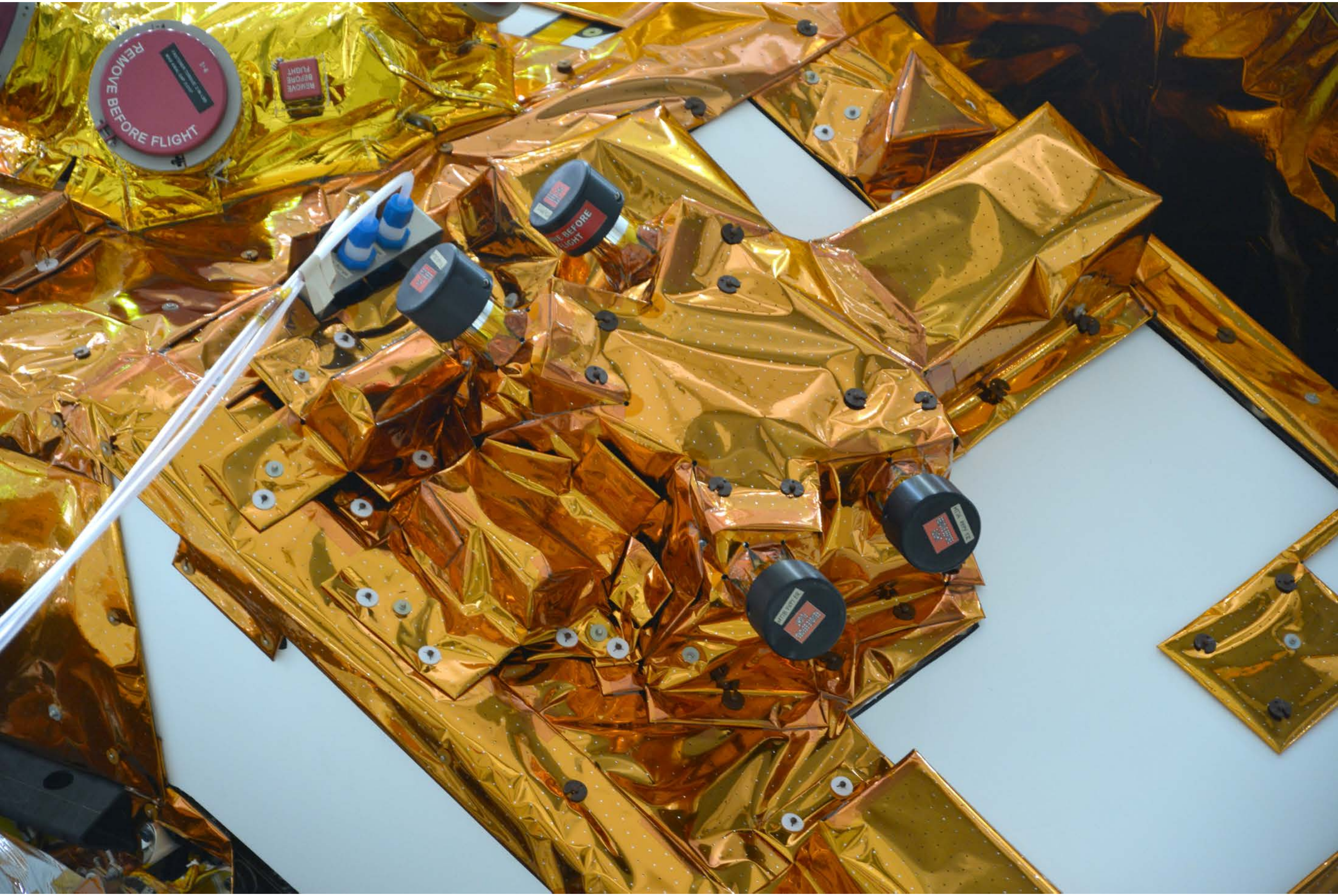
$$x(\omega)=\frac{F_{sample}(\omega)/m}{\omega_o^2+k(0)/m-\omega^2+i\omega(\beta/m)}$$

- Assume F_{sample} a low frequency sinusoid, If $\omega_o{}^2 >> k(0)/m$ $|x|_{cl} << |x|_{ol}$ and motion of the tip around zero is much reduced
- The feedback force $F_{fb}(\omega)=-\dfrac{\omega_o^2 F_{sample}(\omega)}{\omega_o^2+k(0)/m-\omega^2+i\omega(\beta/m)}$
- If $\omega_o^2 \gg \left|k(0)/m-\omega^2+i\omega(\beta/m)\right|$ (true at low enough frequency)

$$F_{fb}(\omega)=-F_{sample}(\omega)$$

Example: drag-free and spacecraft acceleration measurement




REMOVE BEFORE FLIGHT
REMOVE BEFORE FLIGHT

Test-mass + electrostatic sensor as an accelerometer

- In the following exercise we describe these items:
  - Acceleration of test-mass relative to spacecraft gives a measure of acceleration of this one relative to inertial frame
  - Acceleration can be measured from second derivative of displacement
  - Maximum tolerated displacement for linear behavior ≈ 10μm.
  - In the mHz frequency range (minutes to hours), such a range is exceeded for a peak acceleration of $10\mu m \times (2\,\pi\,10^{-3} Hz)^2 \approx 4\times10^{-10} ms^{-2}$
  - Spacecraft acceleration may instead exceed 10^{-7} ms^{-2}
  - Feedback keeps the test-mass centered within required range
  - Force applied by control loop give minus the forces that are acting on the spacecraft
  - Without feedback instead, the system is unstable and the test-mass can irreversibly drift away

Acceleration measurement

- Let's describe the spacecraft and the test-mass as two point particles with masses M and m respectively
- Assume that the forces on the test-mass are negligible (including negative spring and damping).
- The spacecraft is subject instead to external forces F that would accelerate it if not counteracted
- Let's write down Newton laws along one axis. Particle (coordinate x)

$$m\ddot{x} = 0$$

- Spacecraft (coordinate X)

$$M\ddot{X} = F$$

- Change coordinates x-X → x: motion of test-mass relative to spacecraft (measured by motion sensor (actually interferometer))

$$m\ddot{x} + m\ddot{X} = 0$$

A parenthesis: Newton's law impulse response

- With initial conditions and force taken at zero in some remote past

$$m\,x(t) = \int_0^\infty t' F(t-t')dt'$$

- Derivative

$$m\dot{x}(t) = \int_0^\infty t' \frac{dF(t-t')}{dt} dt' = -\int_0^\infty t' \frac{dF(t-t')}{dt'} dt' =$$

$$= -t' F(t-t')\Big|_0^\infty + \int_0^\infty F(t-t')dt' = \int_0^\infty F(t-t')dt'$$

- Second derivative

$$m\ddot{x}(t) = \int_0^\infty \frac{dF(t-t')}{dt} dt' = -\int_0^\infty \frac{dF(t-t')}{dt'} dt' =$$

$$= F(t-t')\Big|_\infty^0 = F(t)$$

Acceleration measurement

$$M\ddot{X} = F \quad m\ddot{x} + m\ddot{X} = 0$$

- Let's calculate the open loop impulse response: considering the force per unit mass as the input and the displacement of test-mass relative to spacecraft as the output

$$\ddot{x} = -\ddot{X} = -F/M$$

- The effect of apparent force! We already know that:

$$x(t) = -\frac{1}{M}\int_0^{\infty} t' F(t-t')\,dt'$$

- Indeed this is the equation of a linear system:

$$F(t)/M \longrightarrow \boxed{h(t)} \longrightarrow x(t)$$

- With impulse response

$$h(t) = -t\Theta(t)$$

- For many forces (e.g. a constant) the response may diverge at $t\rightarrow\infty$

Acceleration measurement

- Open loop impulse response: $x(t) = -\frac{1}{M}\int_0^\infty t' F(t-t')dt'$

- Let's now establish a feedback: x(t) is measured and a force proportional to x is applied to the spacecraft by means of the thrusters. Thus

$$F(t) = F_e(t) + M\int_0^\infty \beta(t')x(t-t')dt'$$

- And

$$\ddot{x}(t) = -F_e(t)/M - \int_0^\infty \beta(t')x(t-t')dt'$$

- Now switch to the frequency domain

$$-\omega^2 x(\omega) = -F_e(\omega)/M - \beta(\omega)x(\omega)$$

- To finally obtain

$$x(\omega) = \frac{-F_e(\omega)/M}{\beta(\omega)-\omega^2}$$

Acceleration measurement

$$x(\omega)=\frac{-F_e(\omega)/M}{\beta(\omega)-\omega^2}$$

- Closed loop frequency response (force to displacement):

$$h_{cl}(\omega)=-\frac{1}{M}\frac{1}{\beta(\omega)-\omega^2}$$

- Now let's pick β. A classical choice is a "proportional" term and one proportional to velocity (remember the rule for the transform of a derivative)

$$\beta(\omega)=\omega_o^2+i\omega/\tau$$

- Then

$$h_{cl}(\omega)=-\frac{1}{M}\frac{1}{\omega_o^2-\omega^2+i\omega/\tau}$$

The simple case of low frequency signals

- Suppose

$$\left|F_e(\omega)\right| = 0 \quad \text{for} \quad |\omega| \geq \omega_c \ll \omega_o$$

- Then the response at closed loop and at open loop have respectively the following Fourier Transform

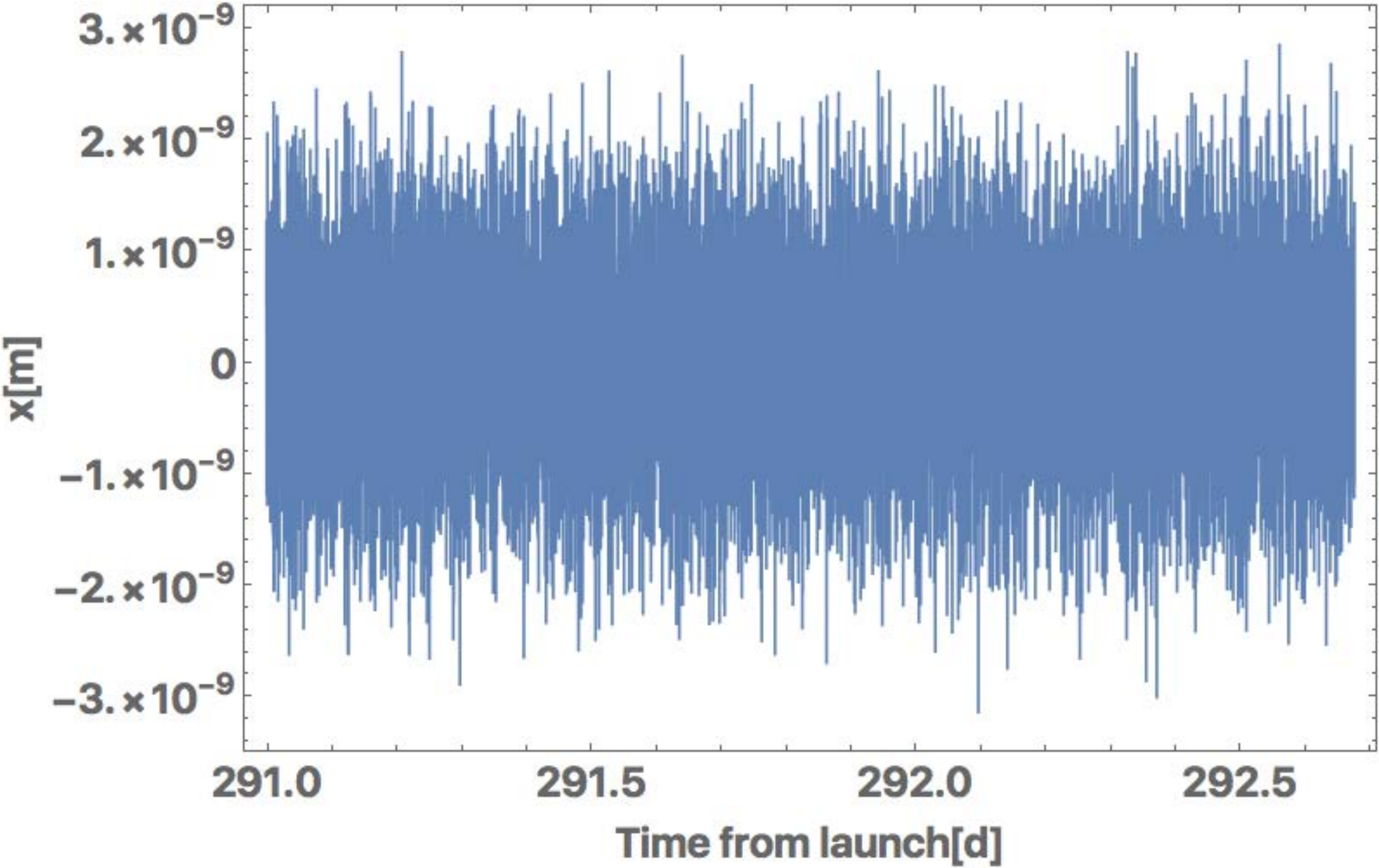
$$x_{ol}(\omega) = \frac{F_e(\omega)}{M\omega^2} \xrightarrow[\omega \to 0]{} \infty \qquad x_{cl}(\omega) = -\frac{1}{M}\frac{F_e(\omega)}{\omega_o^2 - \omega^2 + i\omega/\tau} \approx -\frac{F_e(\omega)}{M\omega_o^2}$$

- Thus in open loop the Fourier transform may not even exist, while in closed loop amplitude is proportional to $1/\omega_o^2$ (check with sinusoidal signal) and can be reduced nominally at will.
- Feedback force

$$\frac{F_{fb}(\omega)}{M} = -\frac{F_e(\omega)}{M}\frac{\omega_o^2 + i\omega/\tau}{\omega_o^2 - \omega^2 + i\omega/\tau} \approx -\frac{F_e(\omega)}{M}$$

- A faithful copy of F_e

Spacecraft residual motion



The Astrophysical Journal, 883:53 (15pp), 2019 September 20

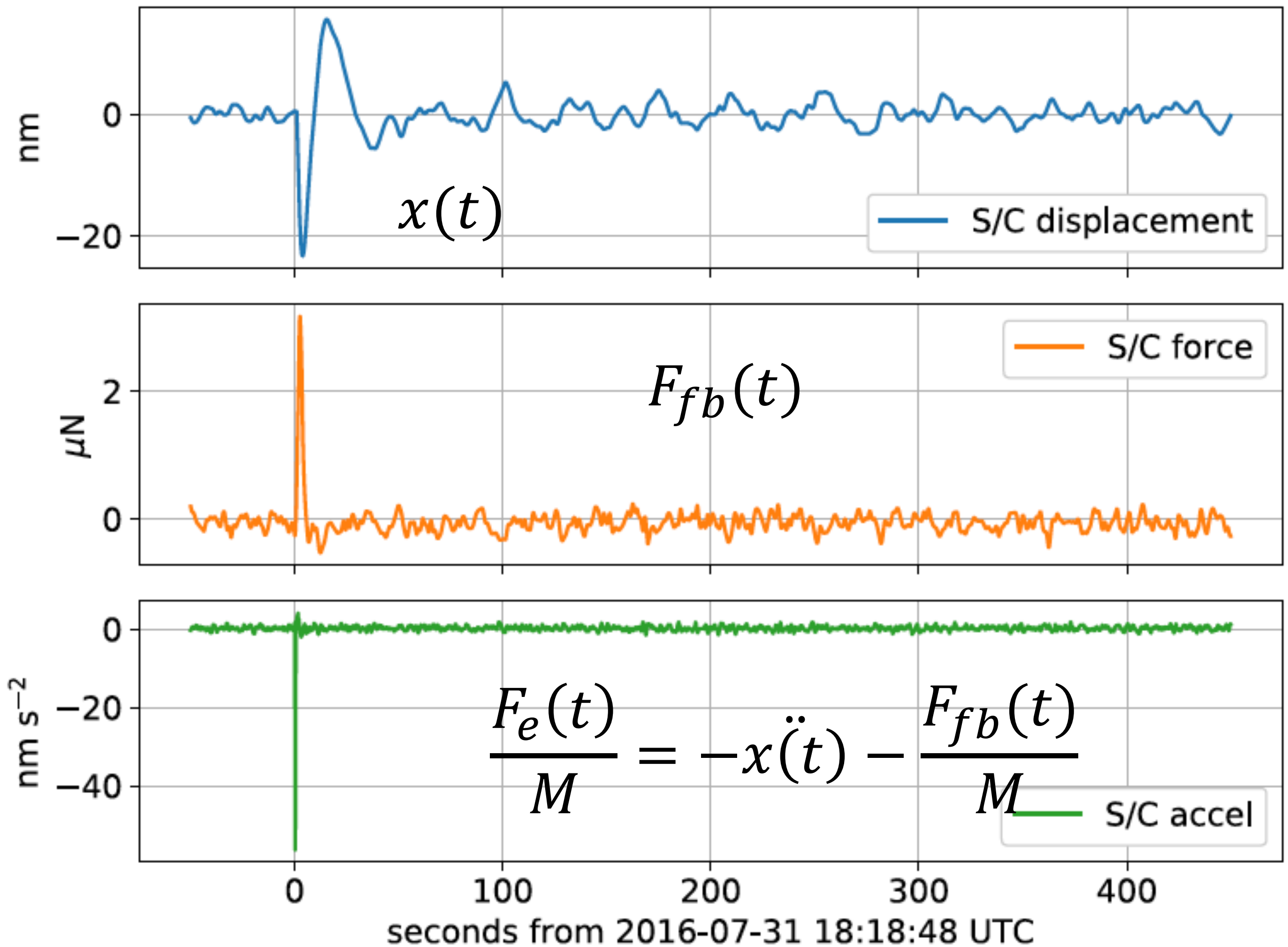


Figure 3. Example of *x*-axis telemetry for impact candidate at GPS time 1154024345.4 (2016 July 31 18:18:48 UTC) and the equivalent free-body acceleration estimated through the calibration procedure. The top panel shows the displacement of the S/C in the *x*-direction. The middle panel shows the commanded force on the S/C in the *x*-direction by the control system. The bottom panel shows the reconstructed external acceleration on the S/C in the *x*-direction using the above data and S/C geometry and mass properties.