# Experimental Methods. Simulated test , 2020

## Solution

### Data

| Parameter | Value | Units |
|---|---|---|
| $\mathcal{E}$ | $7. \times 10^{7}$ | V/m |
| $L_o$ | 8. | H |
| q | $6. \times 10^{6}$ | |
| $\nu$ | 0.9 | kHz |
| $L_s$ | 4. | μH |
| T | 2. | K |
| M | 1300. | kg |
| $\mathcal{N}$ | 50 | |
| $L_n$ | 1.3 | μH |
| $\ell$ | 3. | m |

### Q1

The key step is to linearize the force on the mass due to the capacitor, and the modulation of the capacitor voltage by the mass displacement. First the force

$$F_C = \frac{1}{2} Q_{tot}^2 \frac{\partial (1/C)}{\partial x} \simeq \frac{1}{2} \frac{Q_o^2 + 2\, Q_o\, Q}{\mathcal{C}_T\, d_o} \tag{1}$$

Thus for small signals

$$F_C = \frac{Q_o\, Q}{\mathcal{C}_T\, d_o} = \mathcal{E}\, Q \tag{2}$$

Now the voltage

$$V_C = \frac{Q_{tot}}{C_T}\left(1 - \frac{x}{d_o}\right) = \frac{Q_o + Q}{C_T}\left(1 - \frac{x}{d_o}\right) \simeq \frac{Q}{C_T} - \frac{Q_o}{C_T}\frac{x}{d_o} = \frac{Q}{C_T} - \mathcal{E}\, x \tag{3}$$

The equation of motion in the time domain (including and other force F acting on the oscillator)

$$\begin{gathered} M\,\ddot{x} + M\,\frac{2\pi\nu}{Q}\,\dot{x} + M\,(2\pi\nu)^2\, x = \mathcal{E}Q + M\,\ddot{h}\,\frac{\ell}{2} + F \\ \mathcal{I} = -\frac{dQ}{dt} \\ V_C = \frac{Q}{C_T} - \mathcal{E}\, x = -L_o\,\frac{d\mathcal{I}}{dt} + \sqrt{L_s\, L_i}\,\frac{d\mathcal{I}_n}{dt} \\ \phi = \sqrt{L_o\, L_s}\,\mathcal{I} + L_s\,\mathcal{I}_n \end{gathered} \tag{4}$$

That, in the frequency domain, give

$$M\,x\left(s^2+\frac{s\,(2\pi\nu)}{q}+(2\pi\nu)^2\right)-\mathcal{E}\,Q=M\,\ell\,s^2\,h+F$$
$$Q\,L_o\left((2\pi\nu)^2+s^2\right)-\mathcal{E}\,x=s\,\sqrt{L_o\,L_s}\;\mathcal{I}_n \tag{5}$$
$$\phi=-\,s\,Q\,\sqrt{L_o\,L_s}+\mathcal{I}_n\,L_s$$

This could be simplified by introducing a normalized Laplace frequency y=s/(2 $\pi\,\nu$)=$i$ f/$\nu$

$$M\,(2\pi\nu)^2\,x\left(y^2+\frac{y}{q}+1\right)-\mathcal{E}\,Q=M\,(2\pi\nu)^2\,\ell\,y^2\,h+F$$
$$Q\,(2\pi\nu)^2\,L_o\left(1+y^2\right)-\mathcal{E}\,x=2\pi\nu\;y\,\sqrt{L_o\,L_s}\;\mathcal{I}_n \tag{6}$$
$$\phi=-\,y\,Q\,2\pi\nu\,\sqrt{L_o\,L_s}+\mathcal{I}_n\,L_s$$

# Q2

Solving

$$\phi=\frac{1}{\frac{\mathcal{E}^2}{16\,M\,\pi^4\,\nu^4\,L_o}-\left(1+y^2\right)\left(1+y^2+y/q\right)}\times$$
$$\times\left(h\,\frac{\mathcal{E}\,\ell\,\sqrt{L_o\,L_s}}{2\pi\nu\,L_o}\,y^3+\mathcal{I}_n\,L_s\left(\frac{\mathcal{E}^2}{M\,L_o\,(2\pi\nu)^4}-\left(1+y/q+y^2\right)\right)+F\,\frac{\mathcal{E}\,\sqrt{L_o\,L_s}}{M\,L_o\,(2\pi\nu)^3}\,y\right) \tag{7}$$

So that the transfer function from h to flux is

$$H[y]=\frac{\mathcal{E}\,\ell\,\sqrt{L_o\,L_s}}{2\pi\nu\,L_o}\;\frac{y^3}{Y-\left(1+y^2\right)\left(1+y^2+y/q\right))} \tag{8}$$

where

$$Y=\frac{\mathcal{E}^2}{M\,L_o\,(2\pi\nu)^4}=4.6\times10^{-4} \tag{9}$$

and the prefactor

$$\frac{\mathcal{E}\,\ell\,\sqrt{L_o\,L_s}}{2\pi\nu\,L_o}=26\ \mathrm{Wb} \tag{10}$$

A plot follows

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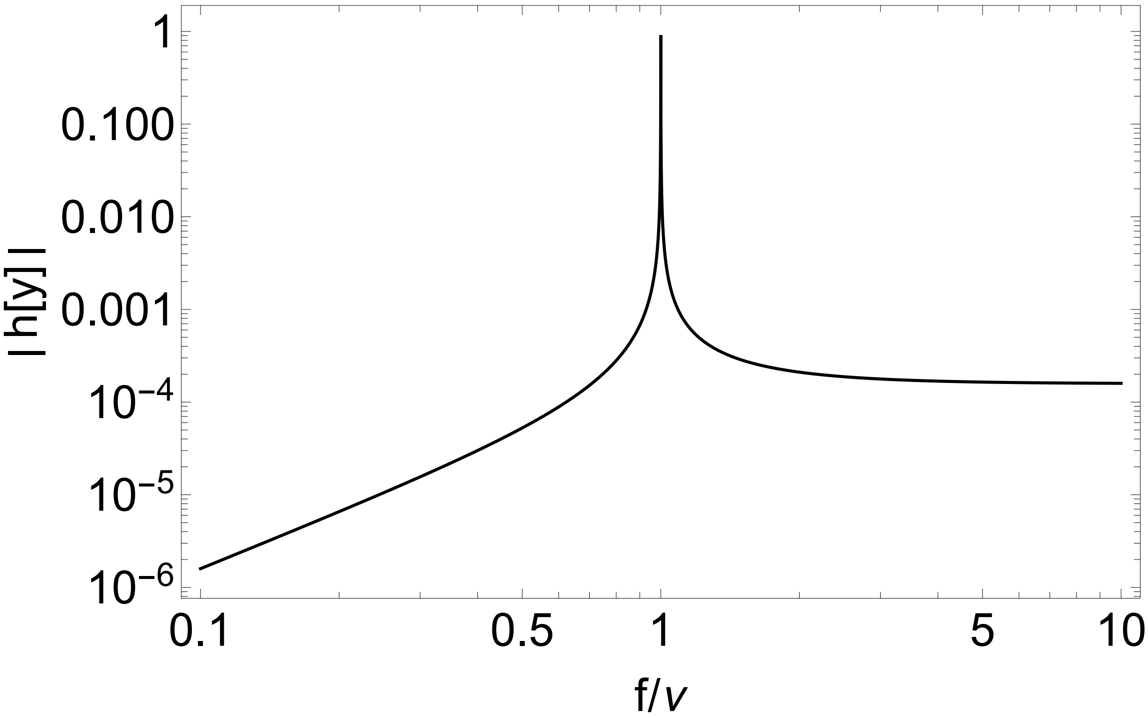


Q3

Noise is contributed by the sources in the sensor and by the thermal noise in the mechanical oscillator

- The force spectral density of thermal noise is

$$S_{th} = 4\, k_B\, T\beta = 4\, k_B\, T\, \frac{M\, 2\, \pi\, \nu}{q} \tag{11}$$

this is transferred to $\phi$ (see above) as

$$S_{\phi,th} = 4\, k_B\, T\, \frac{M\, 2\, \pi\, \nu}{q} \left(\frac{\mathcal{E}\, \sqrt{L_o\, L_s}}{M\, (2\, \pi\, \nu)^3\, L_o}\right)^2 \left|\frac{y}{Y - (1 + y^2)\, (1 + y^2 + y / q)}\right|^2 \tag{12}$$

Numerically

$$4\, k_B\, T\, \frac{M\, 2\, \pi\, \nu}{q} \left(\frac{\mathcal{E}\, \sqrt{L_o\, L_s}}{M\, (2\, \pi\, \nu)^3\, L_o}\right)^2 = 6.0 \times 10^{-42}\ \mathrm{Wb^2/Hz} \tag{13}$$

- From the solution above one also gets that the current is transferred to flux as

$$\phi = \frac{Y - (1 + y / q + y^2)}{Y - (1 + y^2)\, (1 + y / q + y^2)}\, L_s\, \mathcal{I}_n \tag{14}$$

Thus the contribution of the current noise is

$$S_{\phi,\mathcal{I}}[y] = \frac{\mathcal{N}\, \hbar}{L_n}\, L_s^2 \left|\frac{Y - (1 + y / q + y^2)}{Y - (1 + y^2)\, (1 + y / q + y^2)}\right|^2 \tag{15}$$

Numerically

$$\frac{\mathcal{N}\, \hbar}{L_n}\, L_s^2 = 6.5 \times 10^{-38}\ \mathrm{Wb^2/Hz} \tag{16}$$

- flux noise just adds up straightforwardly

$$S_{\phi,\phi}[y] = \mathcal{N}\, \hbar\, L_n = 6.9 \times 10^{-39}\ \mathrm{Wb^2/Hz} \tag{17}$$

Total is then

$$S_{\phi,\text{tot}}[y] = \\ = \left|\frac{1}{Y - (1+y^2)(1+y^2+y/q)}\right|^2 \left(4 k_B T \frac{M 2\pi\nu}{q}\left(\frac{\mathcal{E}\sqrt{L_o L_s}}{M(2\pi\nu)^3 L_o}\right)^2 |y|^2 + \frac{\mathcal{N}\hbar}{L_n} L_s^2 \left|Y - \left(1 + y/q + y^2\right)\right|^2\right) + \mathcal{N}\hbar L_n \tag{18}$$

Or numerically

$$S_{\phi,\text{tot}}[y] = \\ = 6.5\times 10^{-38}\ \text{Wb}^2/\text{Hz} \\ \left(\left|\frac{1}{Y - (1+y^2)(1+y^2+y/q))}\right|^2 \left(9.2\times 10^{-5}|y|^2 + \left|Y - \left(1+y/q+y^2\right)\right|^2\right) + 0.11\right) \tag{19}$$

A plot follows that shows that thermal noise is negligible

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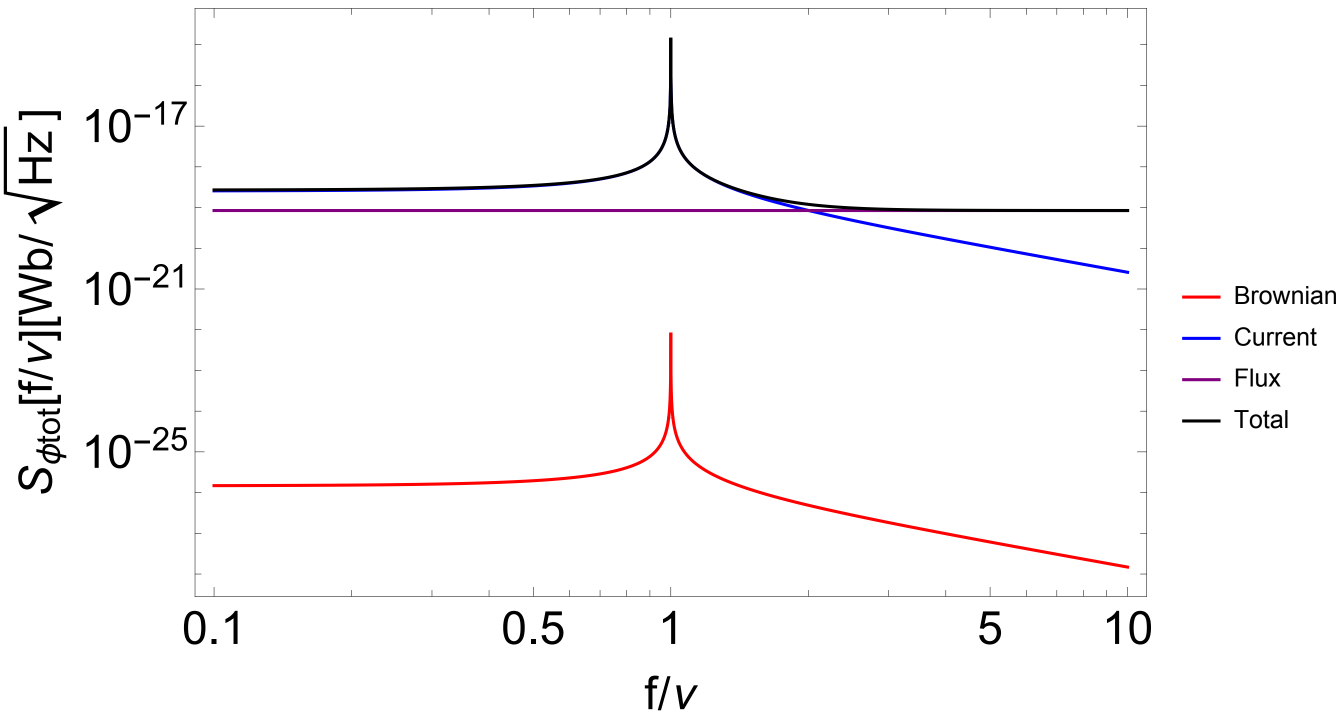



## Q4

Converting to force is straightforward

$$S_{F,\text{tot}}[y] = \\ = \left(4 k_B T \frac{M 2\pi\nu}{q} + \frac{\frac{\mathcal{N}\hbar}{L_n} L_s^2}{\left(\frac{\mathcal{E}\sqrt{L_o L_s}}{M(2\pi\nu)^3 L_o}\right)^2}\left|\frac{Y - (1 + y/q + y^2)}{y}\right|^2\right) + \\ \frac{\mathcal{N}\hbar L_n}{\left(\frac{\mathcal{E}\sqrt{L_o L_s}}{M(2\pi\nu)^3 L_o}\right)^2}\left|\frac{Y - (1+y^2)(1+y^2+y/q)}{y}\right|^2 \tag{20}$$

From that one can convert to h through

$$F_h[y] = M y^2 (2\pi\nu)^2 h[y]\,\ell \tag{21}$$

Thus

$$S_h[y] = \frac{1}{M^2\ (2\,\pi\,\nu)^4\,\ell^2\ |y|^4}$$

$$= \left(4\ k_B\ T\ \frac{M\,2\,\pi\,\nu}{q} + \frac{\frac{\mathcal{N}\,\hbar}{L_n}\ L_s^2}{\left(\frac{\varepsilon\ \sqrt{L_o\,L_s}}{M\ (2\,\pi\,\nu)^3\ L_o}\right)^2}\ \left|\ \frac{Y-\ (1+y\,/\,q+\ y^2)}{y}\right|^2\right) + \tag{22}$$

$$\frac{\mathcal{N}\,\hbar\ L_n}{\left(\frac{\varepsilon\ \sqrt{L_o\,L_s}}{M\ (2\,\pi\,\nu)^3\ L_o}\right)^2}\ \left|\frac{Y-\ (1+y^2)\ (1+y^2+y\,/\,q)}{y}\right|^2$$

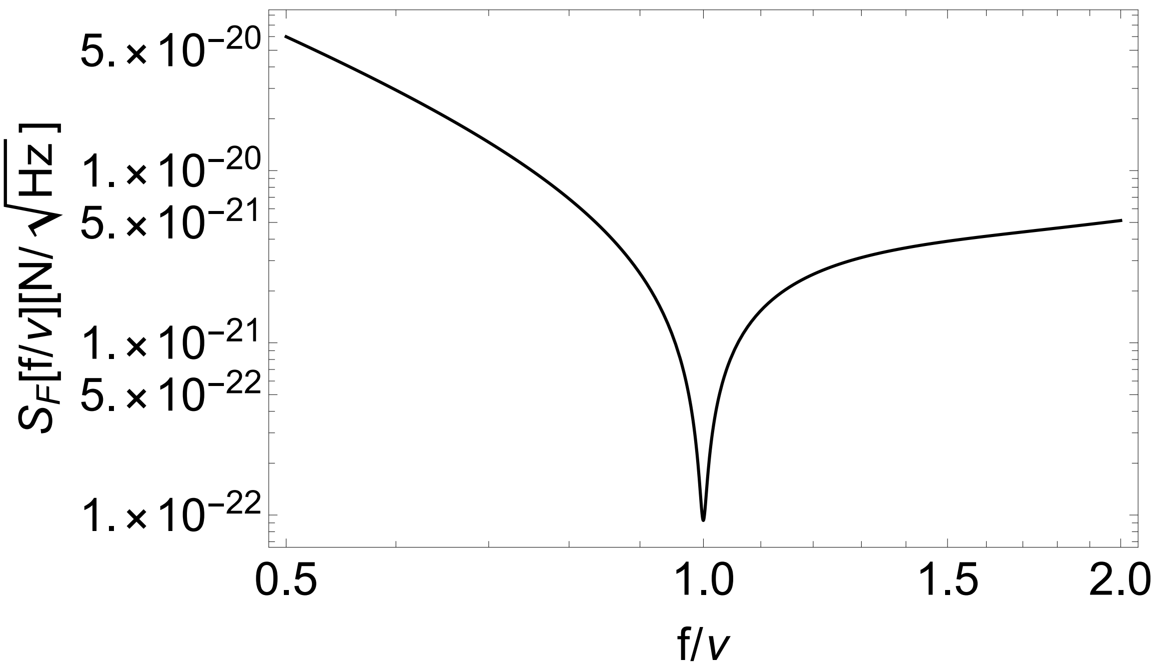
Numerically

$$S_h[y] = 9.4\times10^{-41}\ /\mathrm{Hz}$$

$$= \frac{1}{|y|^4}\ \left(9.2\times10^{-5} + \left|\ \frac{Y-\ (1+y\,/\,q+\ y^2)}{y}\right|^2\right) + .11\ \left|\frac{Y-\ (1+y^2)\ (1+y^2+y\,/\,q)}{y}\right|^2 \tag{23}$$

And a plot of the “sweet spot”

Out[•]=




## Q5

One can change variable in the spectral density z=y/𝕚=f/ν

$$S_h[z] = 9.4\times10^{-41}\ /\mathrm{Hz}$$

$$= \frac{1}{z^4}\ \left(9.2\times10^{-5} + \frac{1}{z^2}\ \left|\ Y-\ \left(1+\mathbb{i}\ z\,/\,q-\ z^2\right)\right|^2\right) + \frac{.11}{z^2}\ \left|Y-\ \left(1-z^2\right)\ \left(1-z^2+\mathbb{i}\,z\,/\,q\right)\right|^2 \tag{24}$$

And then use it in the Wiener formula

$$\sigma_\tau = \left(\frac{1}{2\,\pi}\int_{-\infty}^{\infty}\frac{1}{S_h[\omega]}\,d\omega\right)^{-1/2} = \left(\int_{-\infty}^{\infty}\frac{1}{S_h[f]}\,df\right)^{-1/2} = \left(\nu\int_{0}^{\infty}\frac{2}{S_h[z]}\,dz\right)^{-1/2} \simeq 4.7\times 10^{-19}\ \text{s} \tag{25}$$