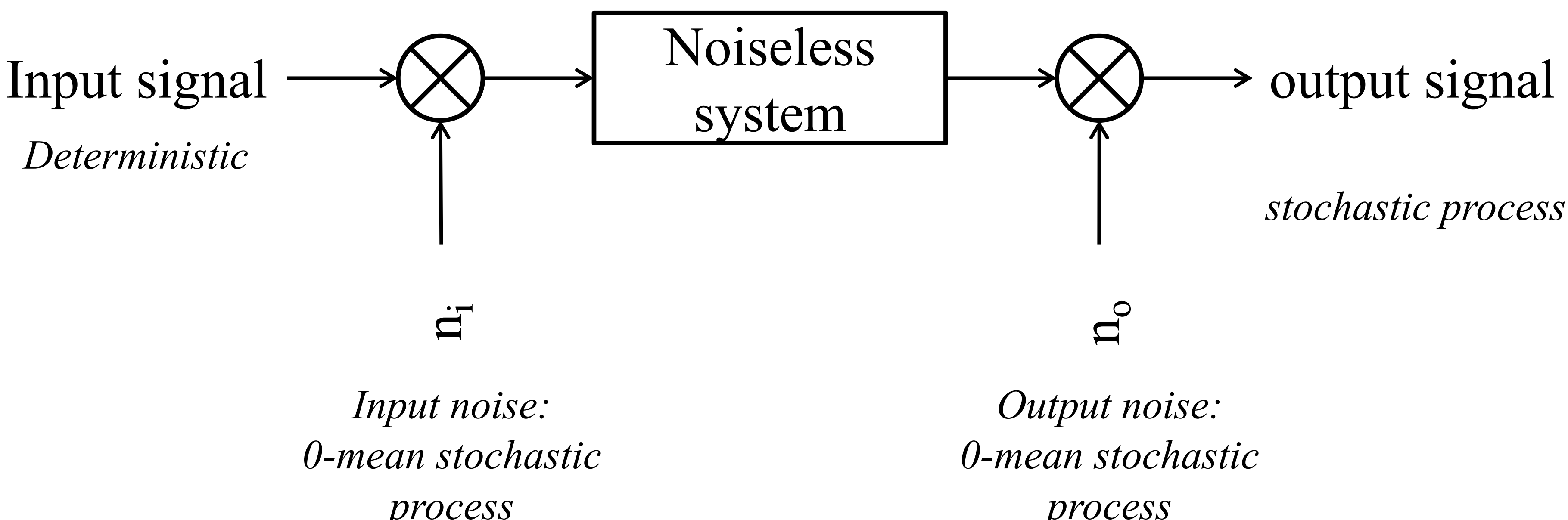
Experimental Methods
Lecture 17

October 28th, 2020

The conceptual scheme for an apparatus in the presence of noise

- Disturbances in physical systems are successfully described as stochastic processes acting at input and at output of an intrinsically noiseless system:
- Notice: this implies that noise is independent of signal and signal levels.
- Not true for parametric noise



Stationary noise and linear stationary systems

- A stationary noise at the input of a linear time-invariant system

1. Mean value

$$\left\langle y(t)\right\rangle = \int_{-\infty}^{\infty} h(t')\eta_x\, dt' = \eta_x \int_{-\infty}^{\infty} h(t')dt' = \text{Constant} \equiv \eta_y$$

2. Input-output cross correlation

$$R_{y,x}(t, t+\Delta t) = \left\langle y(t)x(t+\Delta t)\right\rangle = \int_{-\infty}^{\infty} h(t')\left\langle x(t-t')x(t+\Delta t)\right\rangle dt'$$

Using time-invariance of x

$$R_{y,x}(\Delta t) = \int_{-\infty}^{\infty} h(t')R_{x,x}(\Delta t + t')dt'$$

3. Output auto-correlation

$$R_{y,y}(t, t+\Delta t) = \left\langle y(t)y(t+\Delta t)\right\rangle =$$

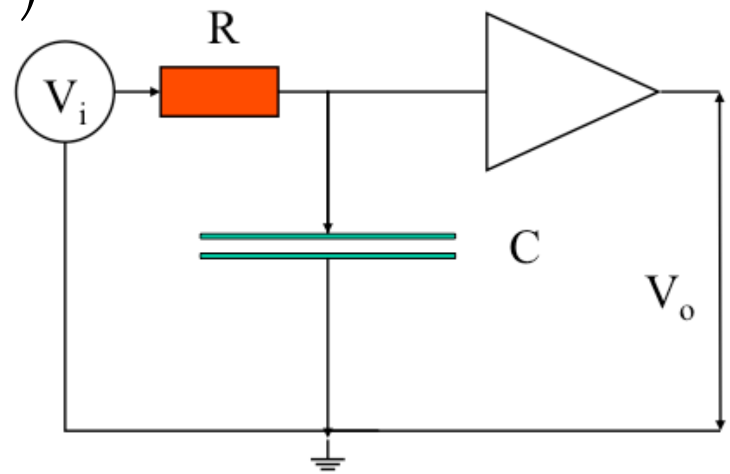
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt'dt''h(t')h(t'')\left\langle x(t-t')x(t+\Delta t - t'')\right\rangle$$

That is

$$R_{y,y}(\Delta t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt'dt''h(t')h(t'')R_{x,x}(\Delta t + t' - t'')$$

Example n. 2: white noise into a low pass filter

- $V_i(t)$ is a white noise voltage at the input of an RC circuit
- White noise:
  - A normal process $V_i(t)$ with zero mean $\langle V_i(t) \rangle = 0$
  - And autocorrelation $R_{V_iV_i}(\tau) = S_V \delta(\tau)$



- Circuit equation: $$\frac{V_i - V_o}{R} = C\frac{dV_o}{dt}$$
- that is $$\frac{dV_o}{dt} + \frac{V_o}{RC} = \frac{V_i}{RC}$$
- Solution (with no free evolution)

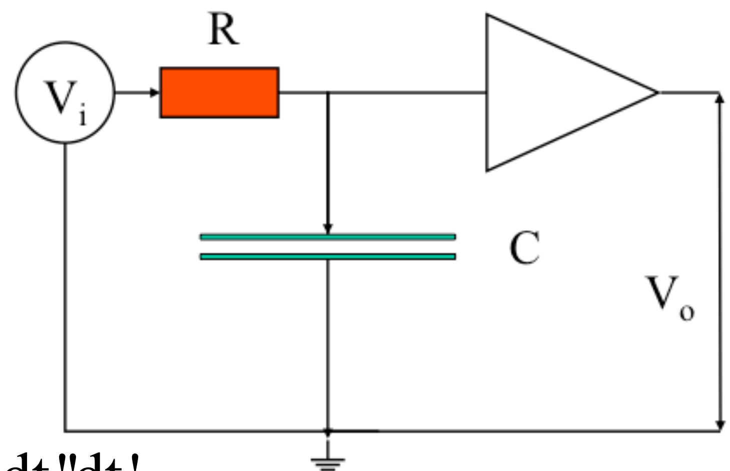
$$V_o(t) = \frac{1}{RC}\int_0^\infty e^{-\frac{t'}{RC}} V_i(t-t')\,dt'$$

Example n. 2: white noise into a low pass filter

- Thus $\quad \langle V_i(t) \rangle = 0 \qquad R_{V_iV_i}(\tau) = S_V \delta(\tau)$

- And $\quad V_o(t) = \dfrac{1}{RC} \displaystyle\int_0^\infty e^{-\frac{t'}{RC}} V_i(t-t')\,dt'$

- Mean value of output



$$\langle V_o(t) \rangle = \frac{1}{RC} \int_0^\infty e^{-\frac{t'}{RC}} \langle V_i(t-t') \rangle\,dt' = 0$$

- Autocorrelation

$$R_{V_oV_o}(\Delta t) = \frac{S_V}{(RC)^2} \int_0^\infty \int_0^\infty e^{-\frac{t'}{RC}} e^{-\frac{t''}{RC}} \delta(\Delta t - t' + t'')\,dt''dt'$$

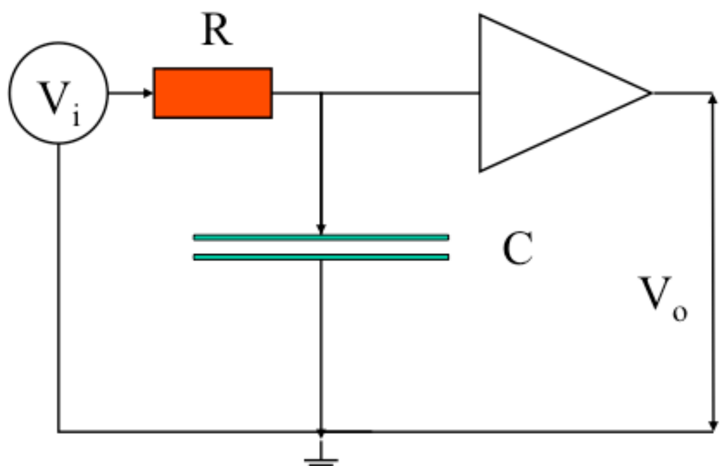
- Same calculation we did for shot noise

$$R_{V_oV_o}(\Delta t) = \frac{S_V}{2RC} e^{-\frac{|\Delta t|}{RC}}$$

Example n. 2: white noise into a low pass filter

- In conclusion $R_{V_iV_i}(\tau) = S_V \delta(\tau)$ $R_{V_oV_o}(\Delta t) = \frac{S_V}{2RC} e^{-\frac{|\Delta t|}{RC}}$

- Input process is memory-less
- The deterministic dynamics of the system introduces correlation (memory) in the output process



Linear systems and normal processes

- Linear combinations of normal (Gaussian) random variables are normal
- Thus if x(t) is normal

$$y(t)=\int_{-\infty}^{\infty} h(t,t')\,x(t')\,dt'$$

is also normal

Power spectral density

- Calculation of output statistical properties of linear, time invariant systems driven by stationary noise becomes easier in the frequency domain.
- We need a way of dealing with stochastic processes in the frequency domain.
- Ordinary Fourier transforms of stationary stochastic processes do not exist as:

$$\int_{-\infty}^{\infty} \left| x(t) \right| dt = \infty$$

- One can instead define Fourier transforms of statistical quantities. The Power Spectral Density (PSD) is the most important of these transforms.

Stationary noise and power spectral density

- Assume we have a stationary process x(t) with autocorrelation $R_{x,x}(\tau)$
  - We define the PSD as the Fourier transform of the autocorrelation

$$S_{xx}(\omega)=\int_{-\infty}^{\infty} R_{xx}(\tau)e^{-i\omega\tau}\,d\tau$$

  - It follows that

$$R_{xx}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{i\omega\tau}\,d\omega$$

- Suppose that you have two processes x(t) and y(t) that are joint stationary with cross correlation $R_{x,y}(\tau)$
  - We define the cross-spectral density as

$$S_{xy}(\omega)=\int_{-\infty}^{\infty} R_{xy}(\tau)e^{-i\omega\tau}\,d\tau$$

  - With an inverse formula

$$R_{xy}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)e^{i\omega\tau}d\omega$$

Examples of PSD

- Definitions $S(\omega)=\int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau \quad R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}d\omega$
- Shot noise (noisy part).
  - Autocorrelation $R(\tau)=e^2\lambda\delta(\tau)$
  - PSD $S(\omega)=e^2\lambda$
- White noise
  - Autocorrelation $R(\tau)=S_o\delta(\tau)$
  - PSD $S(\omega)=S_o$
- Low pass noise
  - Autocorrelation $R(\tau)=\left(S_o/2\Delta t\right)e^{-|\tau|/\Delta t}$
  - PSD

$$S(\omega)=\frac{S_o}{1+\omega^2\Delta t^2}$$

```
In[3]:= FourierTransform[So/(2 Δt) e^(-|τ|/Δt), τ, ω]

Out[3]= So/(1 + Δt^2 ω^2)
```

Properties of power spectrum

- From definitions

$$S(\omega)=\int_{-\infty}^{\infty}R(\tau)e^{-i\omega\tau}\,d\tau \quad R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\omega)e^{i\omega\tau}\,d\omega$$

- Variance and total power for a zero-mean process (pure noise)

$$\sigma^2=R(0)=(1/2\pi)\int_{-\infty}^{\infty}S(\omega)d\omega$$

- Power spectrum is real: as R(τ)=R(-τ) then

$$S(\omega)=\int_{-\infty}^{\infty}R(-\tau)e^{-i\omega\tau}\,d\tau=\int_{-\infty}^{\infty}R(\tau)e^{i\omega\tau}\,d\tau=S^*(\omega)$$

- As all Fourier transforms of real process

$$S(\omega)=S^*(-\omega)$$

- Then

$$S(\omega)=S(-\omega)$$

- Dimensions: [dimensions of process]^2×[time]
- Units: (units of process)^2/Hz

Fourier transforms of stochastic processes

- Is there any possible definition of Fourier transform of a random process, and what is its relation to PSD?
- There is no standard Fourier transform for a stationary random process x(t), because $\int_{-\infty}^{\infty}\left|x(t)\right|dt=\infty$
- However the following quantity exists:

$$\tilde{x}(\omega)=\left(1/\sqrt{T}\right)\int_{-T/2}^{T/2}x(t)e^{-i\omega t}\,dt$$

- Assume x(t) is zero-mean then also $\langle\tilde{x}(\omega)\rangle=0$
- Let's calculate the moment $\langle\tilde{x}(\omega)\tilde{x}^*(\omega)\rangle$

$$\langle\tilde{x}(\omega)\tilde{x}^*(\omega)\rangle=\left(1/T\right)\int_{-T/2}^{T/2}\int_{-T/2}^{T/2}\langle x(t')x(t)\rangle e^{-i\omega(t'-t)}\,dtdt'$$
$$=\left(1/T\right)\int_{-T/2}^{T/2}\int_{-T/2}^{T/2}R(t'-t)e^{-i\omega(t'-t)}\,dtdt'$$

- Let's change variables: from t and t' to t and τ=t'-t. Note |Jacobian|=1

$$\langle\tilde{x}(\omega)\tilde{x}^*(\omega)\rangle=\left(1/T\right)\int_{-T}^{T}R_{xx}(\tau)e^{-i\omega\tau}\,d\tau\int_{\mathrm{Max}[-\tau-T/2,-T/2]}^{\mathrm{Min}[T/2-\tau,T/2]}dt$$

A detail of the calculation

- From

$$< \tilde{x}(\omega)\tilde{x}^*(\omega) > = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\int_{-\frac{T}{2}}^{\frac{T}{2}} R(t'-t)e^{-i\omega(t'-t)}dtdt'$$

- One can change variables: $\tau = t' - t$ and t
- Jacobian $\begin{vmatrix} \partial t/\partial t & \partial\tau/\partial t \\ \partial t/\partial t' & \partial\tau/\partial t' \end{vmatrix} = 1$
- Notice

$$-T \le \tau \le T \text{ but } -\frac{T}{2} \le t \le \frac{T}{2} \text{ and } -\frac{T}{2} \le \tau + t \le \frac{T}{2}$$

- Then

$$Max\left(-\frac{T}{2} - \tau, -\frac{T}{2}\right) \le t \le Min\left(\frac{T}{2}, \frac{T}{2} - \tau\right)$$

- And the integral becomes

$$< \tilde{x}(\omega)\tilde{x}^*(\omega) > = \frac{1}{T}\int_{-T}^{T} R(\tau)e^{-i\omega\tau}d\tau \int_{Max\left(-\frac{T}{2}-\tau,-\frac{T}{2}\right)}^{Min\left(\frac{T}{2},\frac{T}{2}-\tau\right)} dt$$

Fourier transforms of stochastic processes

- Continuing the calculation

$$\left\langle \tilde{x}(\omega)\tilde{x}^*(\omega)\right\rangle = (1/T)\int_{-T}^{T} R_{xx}(\tau)e^{-i\omega\tau}\,d\tau \int_{Max[-\tau-T/2,-T/2]}^{Min[T/2-\tau,T/2]} dt$$

- Performing the second integral

$$= (1/T)\int_{-T}^{T} R_{xx}(\tau)\left(T-|\tau|\right)e^{-i\omega\tau}\,d\tau = \int_{-T}^{T} R_{xx}(\tau)\left(1-|\tau|/T\right)e^{-i\omega\tau}\,d\tau$$

- It follows that

$$\lim_{T\to\infty}\left\langle \tilde{x}(\omega)\tilde{x}^*(\omega)\right\rangle = \lim_{T\to\infty}\int_{-T}^{T} R_{xx}(\tau)\left(1-|\tau|/T\right)e^{-i\omega\tau}\,d\tau = S(\omega)$$

- This is called the Wiener-Kinchine theorem:
  - If $$\tilde{x}(\omega) = \left(1/\sqrt{T}\right)\int_{-T/2}^{T/2} x(t)e^{-i\omega t}\,dt$$
  - Then $$S(\omega) = \lim_{T\to\infty}\left\langle \tilde{x}(\omega)\tilde{x}^*(\omega)\right\rangle$$
- It represents the basis for PSD *estimation* (to be discussed later)

Power spectra and linear stationary systems

- Consider a stationary process x(t) at the input of a linear time-invariant system $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$ $y(t)=\int_{-\infty}^{\infty} h(t')x(t-t')dt'$
- We have calculated that the input output cross correlation is

$$R_{xy}(\tau)=\int_{-\infty}^{\infty} h(t')\langle x(t)x(t+\tau-t')\rangle dt'=\int_{-\infty}^{\infty} h(t')R_{xx}(\tau-t')dt'$$

- From the convolution theorem, the Fourier transforms of R_{xy} and R_{xx} are related by $S_{xy}(\omega)=h(\omega)S_{xx}(\omega)$
- Similarly, for the output autocorrelation

$$R_{yy}(\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t')h(t'')\langle x(t-t')x(t+\tau-t'')\rangle dt'dt''$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t')h(t'')R_{xx}(\tau+t'-t'')dt'dt''$$

- From the definition of PSD and from the convolution theorem we get:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t')h(t'')R_{xx}(\tau+t'-t'')dt'dt''$$
$$=\int_{-\infty}^{\infty} dt'h(t')(1/2\pi)\int_{-\infty}^{\infty} h(\omega)S_{xx}(\omega)e^{i\omega(\tau+t')}d\omega$$

- Calculation continues next page

Power spectra and linear stationary systems

- So the output autocorrelation

$$R_{yy}(\tau) = (1/2\pi)\int_{-\infty}^{\infty} dt' h(t') \int_{-\infty}^{\infty} h(\omega) S_{xx}(\omega) e^{i\omega(\tau+t')} d\omega$$

- We can perform first the integration over t'

$$R_{yy}(\tau) = (1/2\pi)\int_{-\infty}^{\infty} h(\omega) S_{xx}(\omega) \left(\int_{-\infty}^{\infty} dt' h(t') e^{i\omega t'} \right) e^{i\omega\tau} d\omega$$

- And obtain

$$R_{yy}(\tau) = (1/2\pi)\int_{-\infty}^{\infty} h(\omega) S_{xx}(\omega) h^*(\omega) e^{i\omega\tau} d\omega$$

$$= (1/2\pi)\int_{-\infty}^{\infty} |h(\omega)|^2 S_{xx}(\omega) e^{i\omega\tau} d\omega$$

- From the definition of PSD

$$R_{yy}(\tau) = (1/2\pi)\int_{-\infty}^{\infty} S_{yy}(\omega) e^{i\omega\tau} d\omega$$

- It follows that

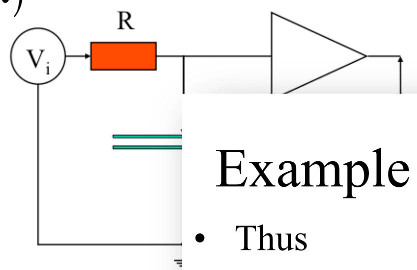
$$S_{yy}(\omega) = |h(\omega)|^2 S_{xx}(\omega)$$

- A key result: output PSD is the product of input PSD times the square modulus of the frequency response!!!

Example: let's perform again the calculation for white noise into a low pass filter

Example n. 2: white noise into a low pass filter

- $V_i(t)$ is a white noise voltage at the input of an RC circuit
- White noise:
  - A normal process $V_i(t)$ with zero mean $\langle V_i(t)\rangle = 0$
  - And autocorrelation $R_{V_iV_i}(\tau) = S_V\delta(\tau)$



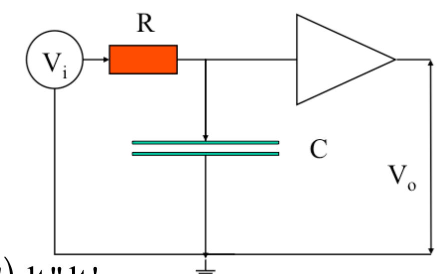
- Circuit equation: $\dfrac{V_i - V_o}{R} = C\dfrac{dV_o}{dt}$
- that is $\dfrac{dV_o}{dt} + \dfrac{V_o}{RC} = \dfrac{V_i}{RC}$
- Solution (with no free evolution)

$$V_o(t) = \frac{1}{RC}\int_0^\infty e^{-\frac{t'}{RC}} V_i(t-t')dt'$$

Example n. 2: white noise into a low pass filter

- Thus $\langle V_i(t)\rangle = 0 \qquad R_{V_iV_i}(\tau) = S_V\delta(\tau)$
- And $V_o(t) = \dfrac{1}{RC}\displaystyle\int_0^\infty e^{-\frac{t'}{RC}} V_i(t-t')dt'$
- Mean value of output

$$\langle V_o(t)\rangle = \frac{1}{RC}\int_0^\infty e^{-\frac{t'}{RC}} \langle V_i(t-t')\rangle dt' = 0$$



- Autocorrelation

$$R_{V_oV_o}(\Delta t) = \frac{S_V}{(RC)^2}\int_0^\infty\int_0^\infty e^{-\frac{t'}{RC}} e^{-\frac{t''}{RC}} \delta(\Delta t - t' + t'')dt''dt'$$

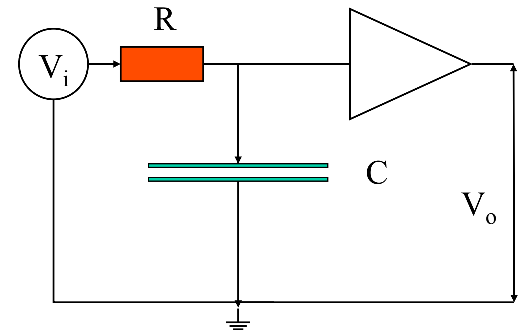
- Same calculation we did for shot noise

$$R_{V_oV_o}(\Delta t) = \frac{S_V}{2RC} e^{-\frac{|\Delta t|}{RC}}$$

Feeding white noise to a low-pass

- Input noise power spectrum from autocorrelation: $R_{V_iV_i}(\tau)=S_V\delta(\tau)$

$$S_{V_iV_i}(\omega)=S_V$$



- From time domain equation $\dfrac{dV_o}{dt}+\dfrac{V_o}{RC}=\dfrac{V_i}{RC}$
- The frequency response is

$$h(\omega)=\frac{1/RC}{i\omega+1/RC}=\frac{1}{1+i\omega RC}$$

- Then the output PSD is

$$S_{V_oV_o}(\omega)=S_{V_iV_i}(\omega)\left|h(\omega)\right|^2=\frac{S_V}{1+\omega^2(RC)^2}$$

- If we are interested in the autocorrelation

```
In[7]:= InverseFourierTransform[1/(1 + ω^2 RC^2), ω, τ] // Simplify[#, RC > 0] &
```

$$\text{Out[7]=}\ \frac{1}{2\,RC}\left(e^{\frac{\tau}{RC}}\,\text{HeavisideTheta}[-\tau]+e^{-\frac{\tau}{RC}}\,\text{HeavisideTheta}[\tau]\right)$$

- Then $R_{V_oV_o}(\tau)=\dfrac{S_V}{2RC}e^{-\frac{|\tau|}{RC}}$

- Same calculation we did for shot noise

$$R_{V_oV_o}(\Delta t)=\frac{S_V}{2RC}e^{-\frac{|\Delta t|}{RC}}$$

AA 2011-2012 S. Vitale

Compare results

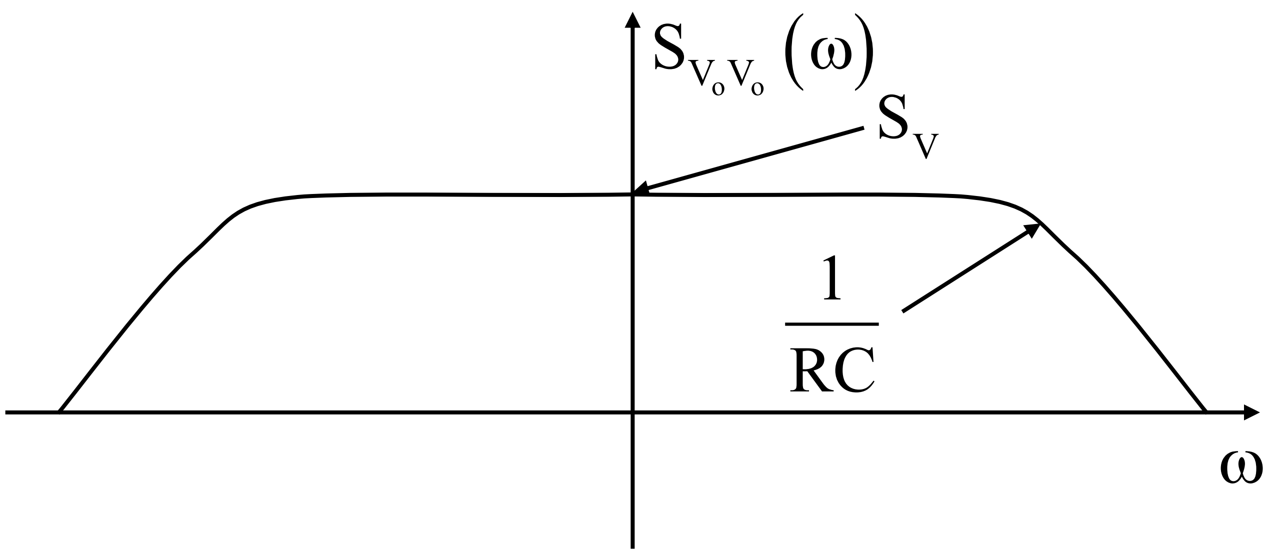
Feeding white noise to a low-pass

- A typical "low-pass PSD"

$$S_{V_oV_o}(\omega) = \frac{S_V}{1+\omega^2 (RC)^2}$$

- PSD is flat and equal to S_V up to a roll-off frequency

$$\omega_{ro} = 1/RC$$



Why noise "spectral density"?

- Total rms of a (zero-mean) stationary process

$$\langle x^2 \rangle = R_{xx}(0) = \frac{1}{2\pi}\int_{\infty}^{\infty} S(\omega) e^{i\omega 0} d\omega = \frac{1}{2\pi}\int_{\infty}^{\infty} S(\omega) d\omega$$

- Thus total "noise" results from the integration of PSD on the whole frequency axis
- Now take a narrow band filter. For instance:

$$h(\omega) = \frac{1}{\sqrt{\Delta\omega}}\left(\Pi\left(\frac{\omega-\omega_o}{\Delta\omega}\right) + \Pi\left(\frac{\omega+\omega_o}{\Delta\omega}\right)\right)$$

- The filter is non-causal, but might be implemented numerically and applied to recorded data.
- Feed the filter with a process $x(t)$ with PSD $S_{xx}(\omega)$
- The output $y(t)$ will have PSD

$$S_{yy}(\omega) = S_{xx}(\omega)\frac{1}{\Delta\omega}\Pi\left(\frac{\omega-\omega_o}{\Delta\omega}\right) + S_{xx}(\omega)\frac{1}{\Delta\omega}\Pi\left(\frac{\omega+\omega_o}{\Delta\omega}\right)$$

Why noise "spectral density" ?

- The output $y(t)$ will have PSD

$$S_{yy}(\omega) = S_{xx}(\omega)\frac{1}{\Delta\omega}\Pi\left(\frac{\omega-\omega_o}{\Delta\omega}\right) + S_{xx}(\omega)\frac{1}{\Delta\omega}\Pi\left(\frac{\omega+\omega_o}{\Delta\omega}\right)$$

- In the limit where $\Delta\omega \to 0$

$$S_{yy}(\omega) \simeq S_{xx}(-\omega_o)\frac{1}{\Delta\omega}\Pi\left(\frac{\omega-\omega_o}{\Delta\omega}\right) + S_{xx}(\omega_o)\frac{1}{\Delta\omega}\Pi\left(\frac{\omega+\omega_o}{\Delta\omega}\right) =$$
$$= S_{xx}(\omega_o)\frac{1}{\Delta\omega}\left(\Pi\left(\frac{\omega-\omega_o}{\Delta\omega}\right) + \Pi\left(\frac{\omega+\omega_o}{\Delta\omega}\right)\right)$$

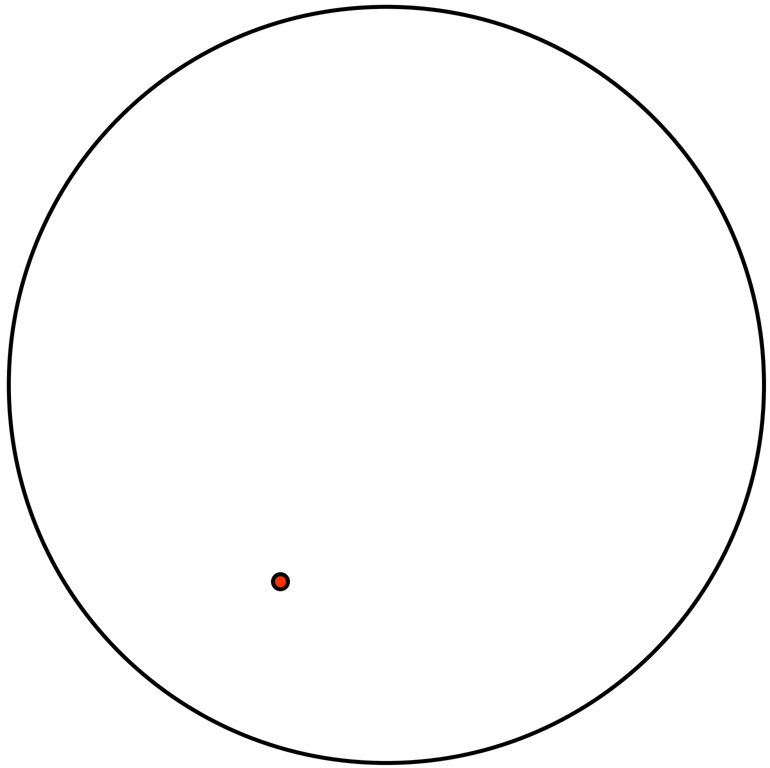
- The rms of y is

$$\langle y^2\rangle = \frac{1}{2\pi}\int_{\infty}^{\infty} S_{yy}(\omega)d\omega = 2S_{xx}(\omega_o)$$

- Thus, modulo the factor 2 to be discussed later, $S_{xx}(\omega_o)$ tells you how dense is the noise at ω_o
- Notice: as $\langle y^2\rangle > 0 \to S_{xx}(\omega_o) > 0!$

Another key example: Brownian noise

- Because of collisions with water molecules, micron-size particles undergo random motion



- Let’s develop a model and use our tools to calculate quantities

The model

1. Directions of exchanged momentum during collisions are at random
2. Collisions are very frequent and "instantaneous".
3. Collisions are many and independent.
4. On the average there is no net exchange of momentum between water and the molecule.

- Thus the molecule is subject to a stochastic force with the following properties:
  - From1, the Cartesian components of the force, $f_x(t)$, $f_y(t)$, and $f_z(t)$ are independent stochastic processes.
  - From 2, each of these processes has a very rapidly decaying autocorrelation that, on the time scales of interest, may be approximated with a delta.
  - From 3, because of central limit theorem, each component is a Gaussian process.
  - From 4. the mean value of each of these processes is 0
- In summary $\left\langle f_x(t)\right\rangle = 0 \quad R_{f_x f_x}(\tau) = S_o \delta(\tau) \quad S_{f_x f_x}(\omega) = S_o$
- that is, the force is white noise.

Macroscopic dynamics

- Let's now consider the motion of the particle at macroscopic level.
- Newton law, particle in viscous fluid $\left(\vec{f}_{drag} = -\beta \vec{v}\right)$ $\quad m\dot{v}_x + \beta v_x = f_x$
- Fourier Transforms $m\, i\omega\, v_x(\omega) + \beta v_x(\omega) = f_x(\omega)$
- Frequency response

$$v_x(\omega) = \frac{f_x(\omega)}{i\omega\, m + \beta} \equiv h(\omega) f_x(\omega)$$

- System is linear. The stochastic force due to collisions with water molecules superimposes to whatever macroscopic force is acting on the molecules. The velocity due to it, is a normal stationary stochastic process with PSD:

$$S_{v_x,v_x}(\omega) = \left|h(\omega)\right|^2 S_{f_x f_x} = \frac{S_o}{m^2\omega^2 + \beta^2}$$

- And autocorrelation

```
InverseFourierTransform[1/(m^2 ω^2 + β^2), ω, τ] // Sim
```

$$\frac{e^{\frac{\beta\tau}{m}}\,\mathrm{HeavisideTheta}[-\tau] + e^{-\frac{\beta\tau}{m}}\,\mathrm{HeavisideTheta}[\tau]}{2\,m\,\beta}$$

$$R_{v_x v_x}(\tau) = \left(S_o/2m\beta\right) e^{-\frac{\beta}{m}|\tau|}$$

Statistics of velocity and the role of temperature

- Velocity autocorrelation

$$R_{v_x v_x}(\tau) = \left[S_o/(2m\beta)\right] e^{-(\beta/m)|\tau|}$$

- The mean square value of each component of velocity is

$$\sigma_{v_x}^2 = R_{v_x v_x}(0) = S_o/2m\beta$$

- Now the physics. From the law of equipartition:

$$\left\langle \frac{1}{2} m v_x^2 \right\rangle = \frac{1}{2} k_B T$$

- Then $$\left\langle \frac{1}{2} m v_x^2 \right\rangle = \frac{1}{2} m \left\langle v_x^2 \right\rangle = \frac{1}{2} m \sigma_{v_x}^2 = \frac{S_o}{4\beta} = \frac{1}{2} k_B T$$

- In conclusion $$S_o = 2\beta k_B T$$

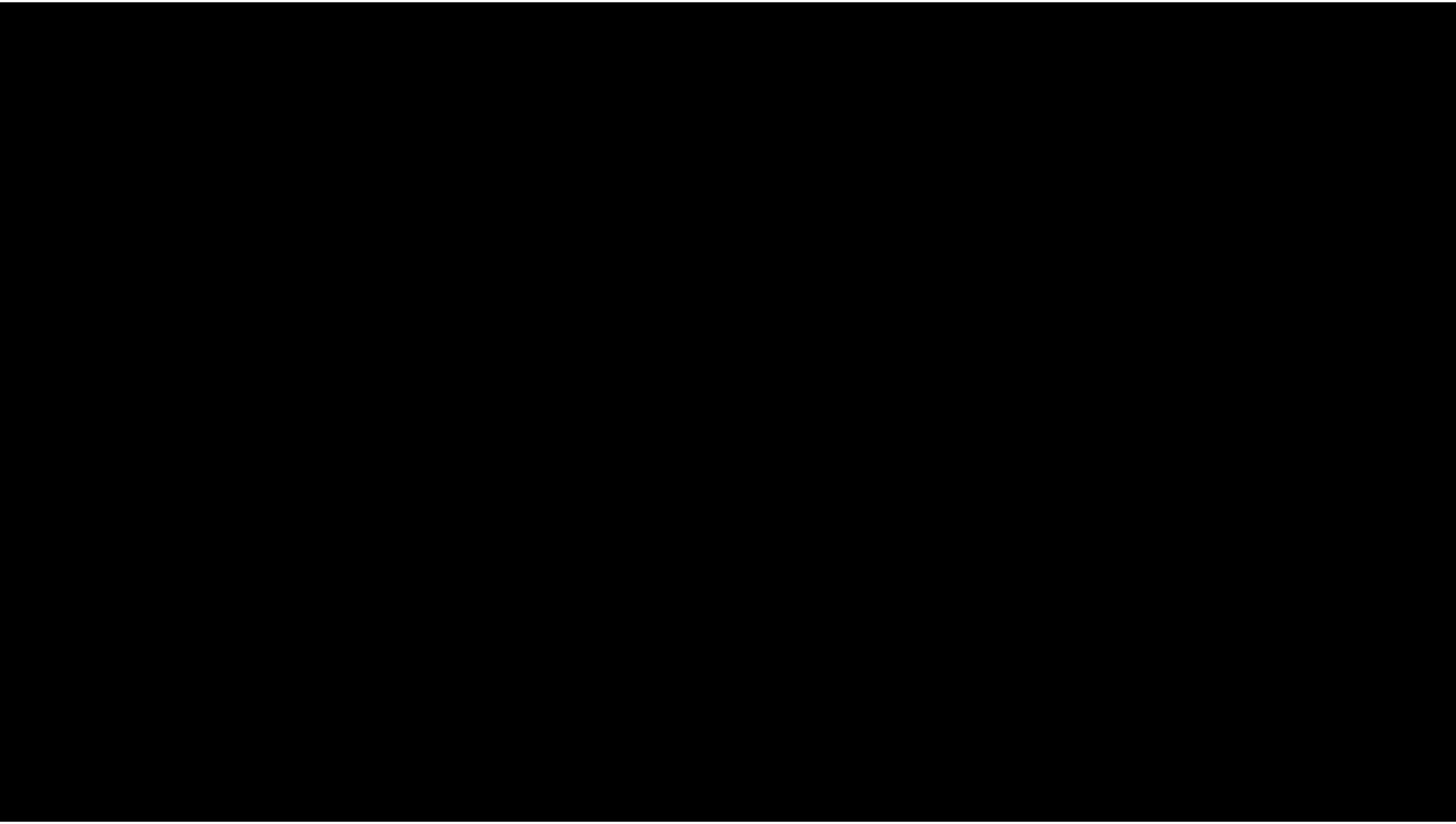
- Thus particle velocity is a normal, zero-mean random process with power spectral density ($\tau = m/\beta$)

$$S_{v_x, v_x}(\omega) = 2 k_B T \frac{\beta}{m^2\omega^2 + \beta^2} = \frac{2 k_B T}{m} \frac{1/\tau}{\omega^2 + 1/\tau^2}$$

Brownian motion summary

- A small particle in a viscous fluid is subject to collisions with fluid molecules.
- The effect of exchange of momentum during these collisions is twofold:
  – If the particle moves on a macroscopic scale, the exchange of momentum is equivalent to a force
  $$\vec{f}(t) = -\beta\vec{v}$$
  – A stochastic white force superimposes to the above with PSD
  $$S_{ff} = 2\beta k_B T$$
  – Where the coefficient β is the same for both phenomena!
  – The particle is set into motion by this force as by any other force. The resulting velocity has spectrum
  $$S_{v_x,v_x}(\omega) = 2k_B T \frac{\beta}{m^2\omega^2 + \beta^2}$$

Lipid droplets in water



In[1015]:=

$$\text{data} = \{r \to 10^{-5}\,\text{m},\ \rho \to 1000\,\text{kg}\,\text{m}^{-3},\ T \to 300\,\text{K},\ k_B \to 1.38\,10^{-23}\,\text{kg}\,\text{m}^2\,\text{s}^{-2}\,\text{K}^{-1},\ \tau \to 0.1\,\text{s}\};$$

In[1016]:=

$$M_p = \frac{4}{3}\pi r^3 \rho;$$

In[1022]:=

$$S_v[f\_] = \frac{2\,k_B\,T}{M_p\,\tau}\,\frac{1}{(2\,\pi\,f)^2 + \frac{1}{\tau^2}};$$

In[1024]:=

$$S_v[1\,\text{s}^{-1}]\ /.\ \text{data}\ //\ \text{Simplify}$$

Out[1024]=

$$\frac{1.41721\times 10^{-10}\,\text{m}^2}{\text{s}}$$

In[1028]:=

$$E_{kin} = \text{Simplify}\left[\frac{1}{2}\,M_p\int_{-\infty}^{\infty} S_v[f]\,df,\ \tau > 0\right]$$

Out[1028]=

$$\frac{T\,k_B}{2}$$

In[1029]:=

$$E_{kin}\ /.\ \text{data}$$

Out[1029]=

$$\frac{2.07\times 10^{-21}\,\text{kg}\,\text{m}^2}{\text{s}^2}$$