Experimental Methods
Lecture 15

October 22th, 2020

Types of stochastic processes: 1) stationary process

- A process is stationary if all its statistical properties are not affected by a translation of the time origin, that is, if for any N and T

$$f_{x(t_1+T)\,x(t_2+T)\ldots x(t_N+T)}(\chi_1,\chi_2\ldots\chi_N)=f_{x(t_1)\,x(t_2)\ldots x(t_N)}(\chi_1,\chi_2\ldots\chi_N)$$

- Immediate consequences: first order density is independent of time

$$f_{x(t+T)}(\chi)=f_{x(t)}(\chi)=f_x(\chi)$$

- Mean value

$$\eta(t)=\langle x(t)\rangle=\int_{-\infty}^{\infty}\chi f_x(\chi)d\chi=\text{Constant}\equiv\eta$$

- Same with standard deviation
- Two points density may only depend on Δt

$$f_{x(t+T)\,x(t+T+\Delta t)}(\chi_1,\chi_2)=f_{x(t)\,x(t+\Delta t)}(\chi_1,\chi_2)$$

- Autocorrelation $R_{x,x}(t,t+\Delta t)=\langle x(t)x(t+\Delta t)\rangle=R_{x,x}(\Delta t)$
- Auto-covariance $C_{x,x}(t,t+\Delta t)=R_{x,x}(\Delta t)-\eta_o^2=C_{x,x}(\Delta t)$

Types of stochastic processes: 2) normal process

- A process is called normal if for any N the joint probability densities of the samples of the process at any t_1, t_2, …..t_N is joint normal

$$f_{x(t_1),x(t_2),\dots x(t_N)}\left(\chi_1,\chi_2,\dots\chi_N\right)=\frac{\sqrt{\|\mu\|}}{\left(2\pi\right)^{\frac{N}{2}}}e^{-\frac{1}{2}\sum_{i,j=1}^{N}\mu_{i,j}(\chi_i-\eta_i)(\chi_j-\eta_j)}$$

- with $\mu_{i,j}$ a positive definite matrix and η_i a real number. One can calculate that

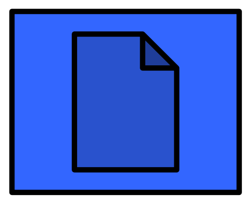
$$\left\langle x\left(t_i\right)\right\rangle=\eta_i$$

- And that $$\left(\mu^{-1}\right)_{i,j}=C\left(t_i,t_j\right)=\left\langle\left[x\left(t_i\right)-\eta_i\right]\left[x\left(t_j\right)-\eta_j\right]\right\rangle$$

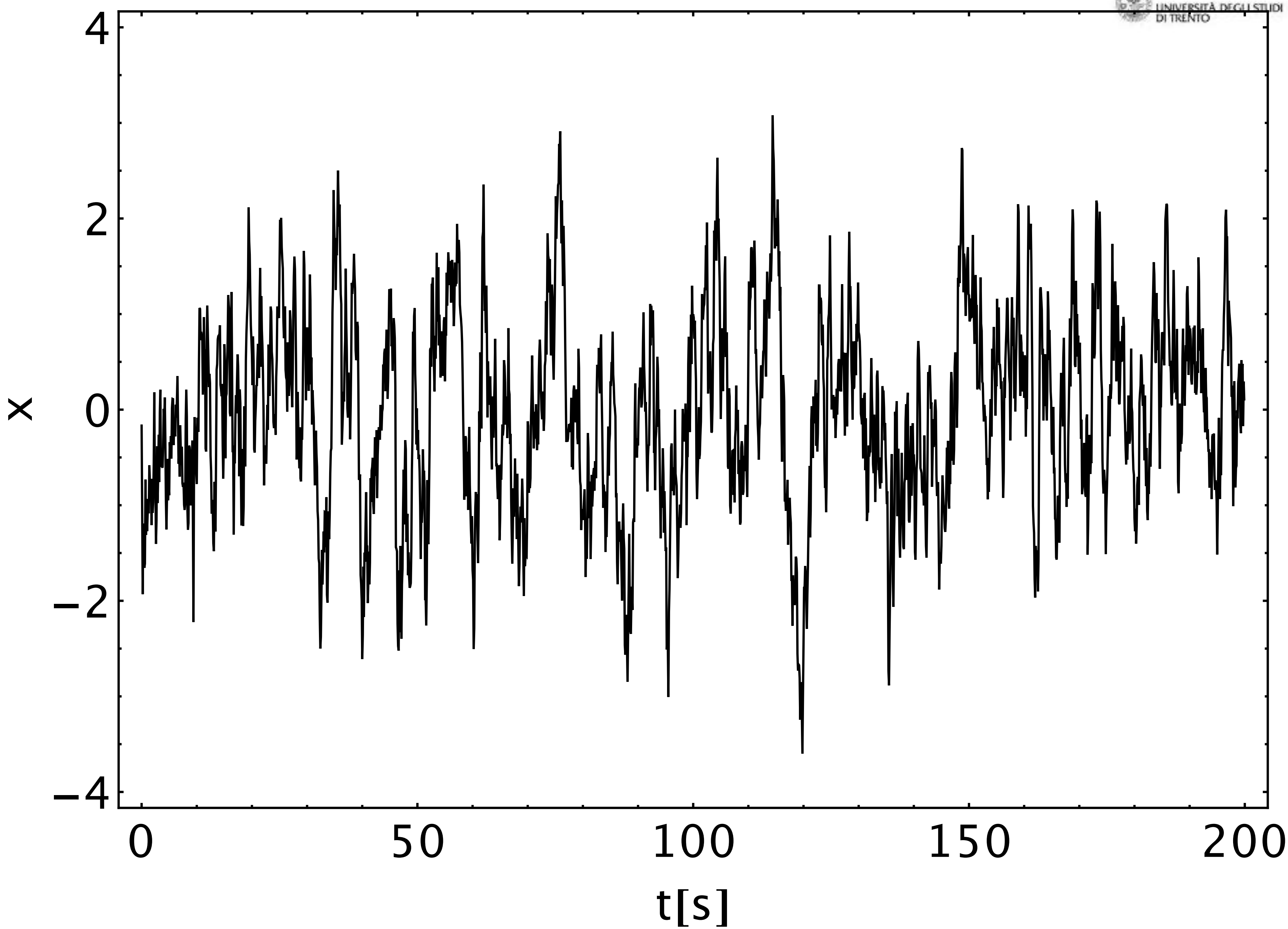
- Thus, for normal processes the entire information is contained within $\eta(t)$ and $C(t,t')$. All other moments may be derived from these functions

Examples of stationary Gaussian noise

- White noise $\eta=0$, $R_{x,x}(\Delta t)=S_o\delta(\Delta t)$
  - Correlation is lost immediately
  - $\sigma^2 = R(0) = \infty$
  - Simulation cannot be performed. Useful in calculations.
- Low-pass noise $\eta=0$, $R_{x,x}(\Delta t)=\sigma^2 e^{-|\Delta t|/\tau}$
  - Correlation decays over τ



  - Next page: one example of simulation with $\sigma,\tau=1$ sampled with sampling time of 100 ms


4
2
0
−2
−4
x
0
50
100
150
200
t[s]

The meaning of autocorrelation

- 1) a remainder from probability theory. Conditional probability of an event A given the event B: probability that also A occurs if B occurs.

$$P(A\,|\,B)=\frac{P(A\cap B)}{P(B)}$$

- Now take two random variables x and y and calculate

$$P(\chi_o \le x \le \chi_o + d\chi \,|\, \psi_o \le y \le \psi_o + d\psi)$$

- From the definition of probability density

$$P(\chi_o \le x \le \chi_o + d\chi \,|\, \psi_o \le y \le \psi_o + d\psi)=\frac{f_{x,y}(\chi_o,\psi_o)\,d\chi\,d\psi}{f_y(\psi_o)\,d\psi}=\frac{f_{x,y}(\chi_o,\psi_o)\,d\chi}{\int_{-\infty}^{\infty} f_{x,y}(\chi,\psi_o)\,d\chi}$$

- Taking the limit for $d\chi \to 0$

$$f_{x|y}(\chi_o\,|\,\psi_o)=\lim_{d\chi\to 0}\frac{P(\chi_o \le x \le \chi_o + d\chi \,|\, \psi_o \le y \le \psi_o + d\psi)}{d\chi}=\frac{f_{x,y}(\chi_o,\psi_o)}{\int_{-\infty}^{\infty} f_{x,y}(\chi,\psi_o)\,d\chi}$$

The case of a zero-mean normal random process

$$f_{x(t_1),x(t_2),\ldots x(t_N)}(\chi_1,\chi_2,\ldots\chi_N)=\frac{\sqrt{\|\mu\|}}{(2\pi)^{\frac{N}{2}}}e^{-\frac{1}{2}\sum_{i,j=1}^{N}\mu_{i,j}(\chi_i-\eta_i)(\chi_j-\eta_j)} \qquad (\mu^{-1})_{i,j}=C(t_i,t_j)=\left\langle\left[x(t_i)-\eta_i\right]\left[x(t_j)-\eta_j\right]\right\rangle$$

- 2-point joint probability density

$$f_{x(t)x(t+\Delta t)}(\chi,\psi)=\frac{e^{-\frac{1}{2}\frac{1}{R(0)^2-R(\Delta t)^2}\left[R(0)\chi^2-2R(\Delta t)\chi\psi+R(0)\psi^2\right]}}{2\pi\sqrt{R(0)^2-R(\Delta t)^2}}=\frac{e^{-\frac{1}{2}\frac{R(0)}{R(0)^2-R(\Delta t)^2}\left[\chi^2-2\frac{R(\Delta t)}{R(0)}\chi\psi+\psi^2\right]}}{2\pi\sqrt{R(0)^2-R(\Delta t)^2}}$$

- 1-point probability density (remember: $R(0)=\sigma^2$)

$$f_x(\chi)=\left(1/\sqrt{2\pi R(0)}\right)e^{-\frac{1}{2}\frac{\chi^2}{R(0)}}$$

- Conditional probability of $x(t+\Delta t)$ given $x(t)\equiv x_o$

$$f_{x(t+\Delta t)|x(t)}(\chi,x_o)=\frac{\sqrt{2\pi R(0)}}{2\pi\sqrt{R(0)^2-R(\Delta t)^2}}\frac{e^{-\frac{1}{2}\frac{R(0)}{R(0)^2-R(\Delta t)^2}\left[\chi^2-2\frac{R(\Delta t)}{R(0)}\chi x_o+x_o^2\right]}}{e^{-\frac{1}{2}\frac{x_o^2}{R(0)}}}$$

The case of a zero-mean normal random process

- Conditional probability of $x(t+\Delta t)$ given $x(t)\equiv x_o$

$$f_{x(t+\Delta t)|x(t)}(\chi, x_o) = \frac{\sqrt{2\pi R(0)}}{2\pi\sqrt{R(0)^2 - R(\Delta t)^2}} \frac{e^{-\frac{1}{2}\frac{R(0)}{R(0)^2-R(\Delta t)^2}\left[\chi^2 - 2\frac{R(\Delta t)}{R(0)}\chi x_o + x_o^2\right]}}{e^{-\frac{1}{2}\frac{x_o^2}{R(0)}}}$$

- Rewrite:

$$= \frac{\sqrt{2\pi R(0)}e^{-\frac{1}{2}\frac{R(0)}{R(0)^2-R(\Delta t)^2}\left[-\left(\frac{R(\Delta t)}{R(0)}x_o\right)^2 + x_o^2\right]}}{2\pi\sqrt{R(0)^2 - R(\Delta t)^2}e^{-\frac{1}{2}\frac{x_o^2}{R(0)}}} e^{-\frac{1}{2}\frac{R(0)}{R(0)^2-R(\Delta t)^2}\left[\chi^2 - 2\frac{R(\Delta t)}{R(0)}\chi x_o + \left(\frac{R(\Delta t)}{R(0)}x_o\right)^2\right]}$$

- Finally

$$f_{x(t+\Delta t)|x(t)}(\chi, x_o) = \frac{e^{-\frac{1}{2}\frac{1}{R(0)\left(1-\frac{R(\Delta t)^2}{R(0)^2}\right)}\left[\chi - \frac{R(\Delta t)}{R(0)}x_o\right]^2}}{\sqrt{2\pi R(0)}\sqrt{1 - R(\Delta t)^2 / R(0)^2}}$$

The case of a zero-mean normal random process

- 1-point density $f_{x(t)}(\chi)=\left(1/\sqrt{2\pi R(0)}\right)e^{-\frac{1}{2}\frac{\chi^2}{R(0)}}$
  - Mean value $\langle x(t)\rangle = 0$
  - Variance $\sigma^2_{x(t)} = R(0)$
- Conditional probability of x(t+Δt) given x(t)≡x$_o$

$$f_{x(t+\Delta t)|x(t)}(\chi, x_o)=\frac{1}{\sqrt{2\pi R(0)}\sqrt{1-R(\Delta t)^2/R(0)^2}}e^{-\frac{1}{2}\frac{1}{R(0)\left(1-\frac{R(\Delta t)^2}{R(0)^2}\right)}\left[\chi-\frac{R(\Delta t)}{R(0)}x_o\right]^2}$$

  - Mean value $\langle x(t+\Delta t)\,|\,x(t)=x_o\rangle=\frac{R(\Delta t)}{R(0)}x_o$
  - Variance $\sigma^2_{x(t+\Delta t)|x(t)=x_o}=R(0)\left(1-\frac{R(\Delta t)^2}{R(0)^2}\right)$

The case of a zero-mean normal random process

- In summary the conditional probability of $x(t+\Delta t)$ given $x(t)\equiv x_o$ is a normal distribution

$$f_{x(t+\Delta t)|x(t)}(\chi, x_o) = \left(1/\sqrt{2\pi\sigma_c^2}\right)e^{-\frac{1}{2}\frac{(\chi-\eta_c)^2}{\sigma_c^2}}$$

  – With mean value $\eta_c = x_o R(\Delta t)/R(0)$

  – And variance $\sigma_c^2 = \sigma^2\left[1-\left(R(\Delta t)/R(0)\right)^2\right]$

- Notice

$$\lim_{\Delta t\to 0}\eta_c = x_o \quad \lim_{\Delta t\to 0}\sigma_c = 0$$

- Then for zero delay $x(t+\Delta t)=x(t)$!

The case of a zero-mean normal random process

- One key property of autocorrelation (auto-covariance). As:

$$\left\langle \left(x(t) \pm x(t+\Delta t)\right)^2 \right\rangle = \left\langle x(t)^2 \right\rangle + \left\langle x(t+\Delta t)^2 \right\rangle \pm 2\left\langle x(t)x(t+\Delta t)\right\rangle \geq 0$$

- Then

$$2R(0) \pm 2R(\Delta t) \geq 0$$

- And

$$R(0) \geq \left|R(\Delta t)\right|$$

- The modulus of autocorrelation (auto-covariance) is a decreasing function of delay. It follows that

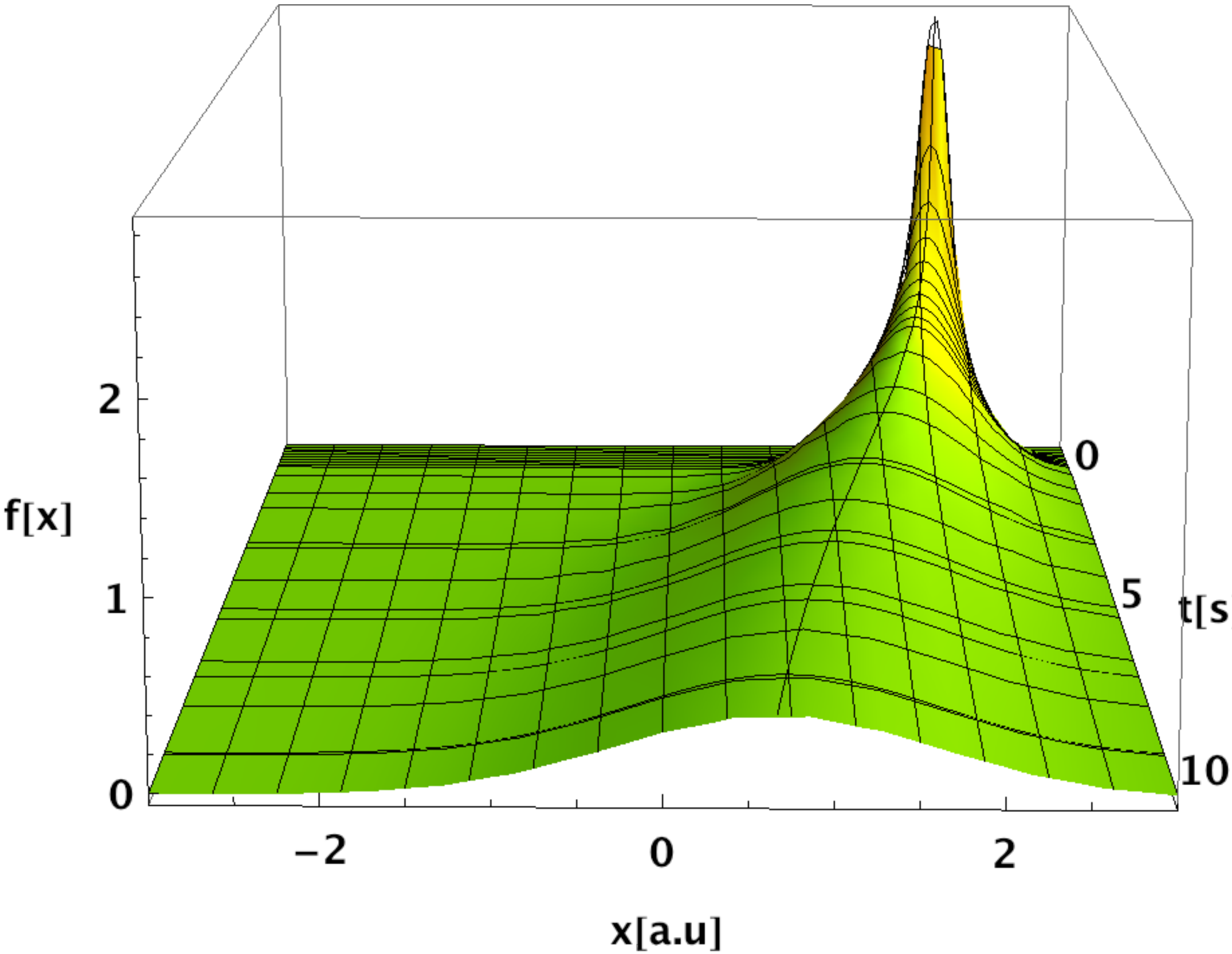
$$\sigma_c^2 = \sigma^2\left[1 - \left(R(\Delta t)/R(0)\right)^2\right] \leq \sigma^2$$

- In addition, except for extraordinary cases $\lim\limits_{\Delta t \to \infty} R(\Delta t) = 0$
- Then

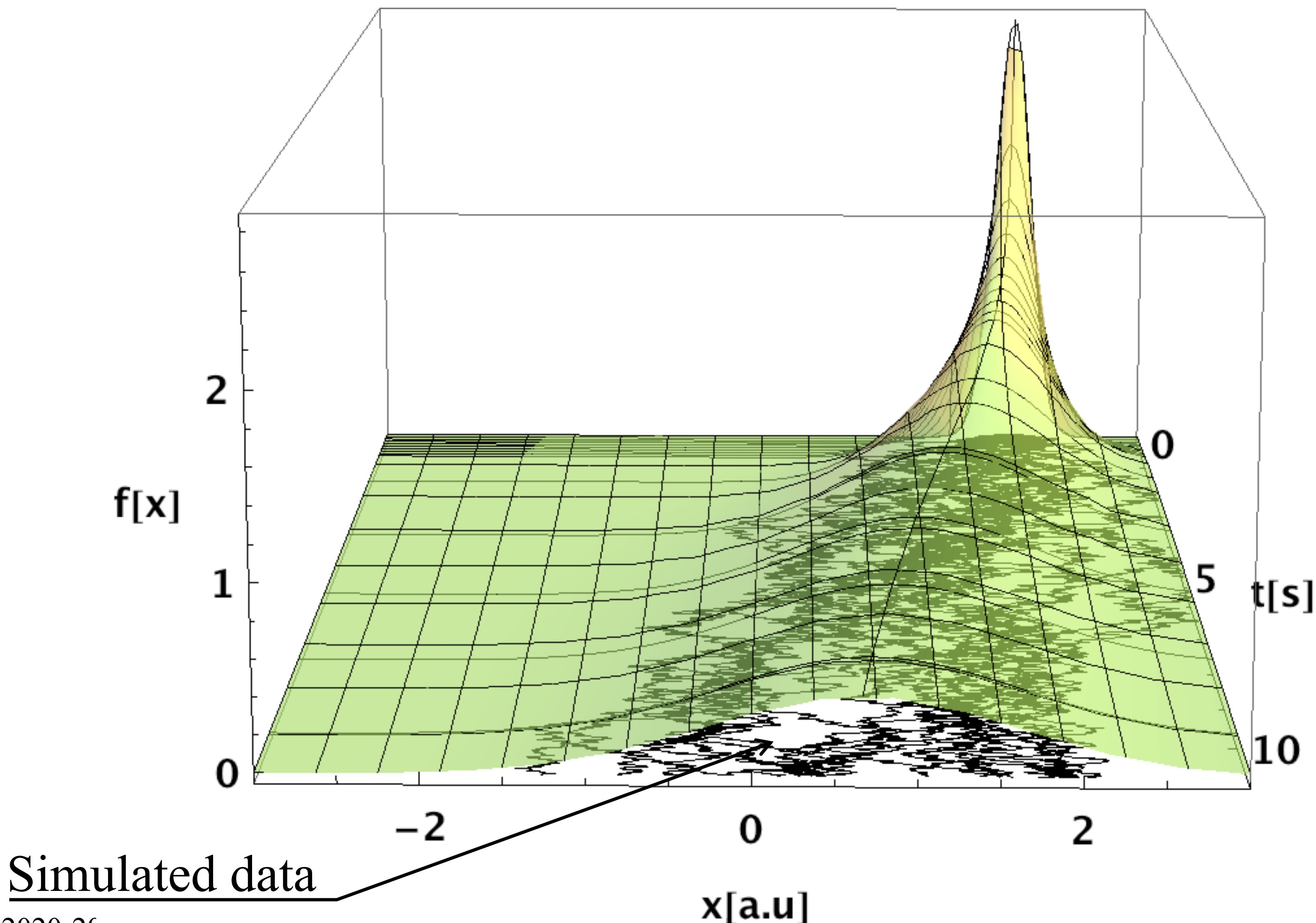
$$\lim_{\Delta t \to \infty} \eta_c = \lim_{\Delta t \to \infty} x_o \frac{R(\Delta t)}{R(0)} = 0 \quad \lim_{\Delta t \to \infty} \sigma_c = \sigma$$

- And memory is lost

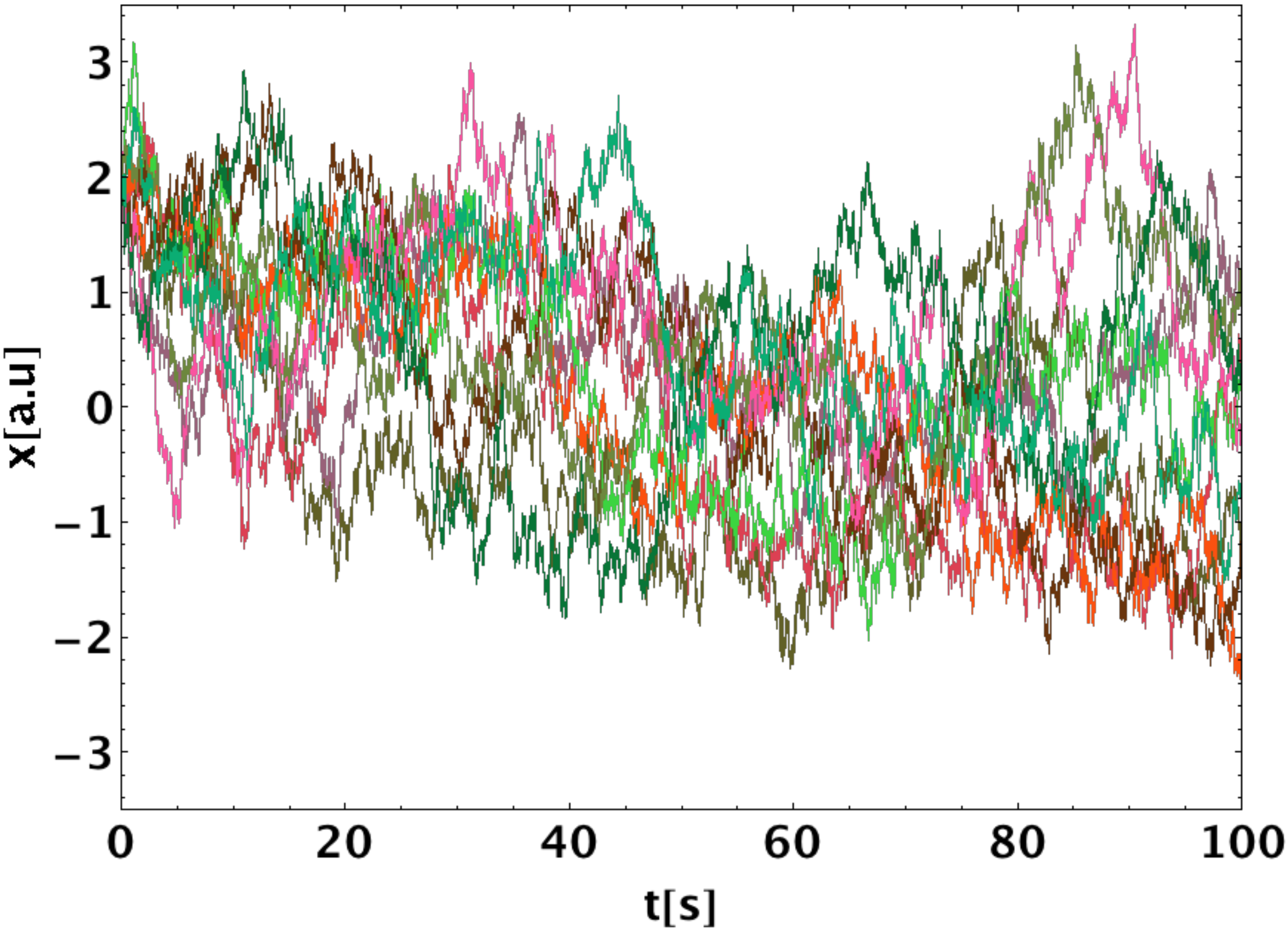
The case for $R_{x,x}(\Delta t)=\sigma^2 e^{-|\Delta t|/\tau}$ $\sigma^2=1$, $x_o=2$, $\tau=10$ s



The case for $R_{x,x}(\Delta t)=\sigma^2 e^{-|\Delta t|/\tau}$ $\sigma^2=1$, $x_o=2$, $\tau=10$ s



A longer simulation



A longer simulation