Experimental Methods
Lecture 18

October 29th, 2020

Power Spectral Density of Stationary Process

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$$

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega$$

$$\sigma^2 = R(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, d\omega$$

$$S(\omega) = S(-\omega) \geq 0$$

Stationary process across stationary linear system

- If system has transfer function $h(\omega)$ then

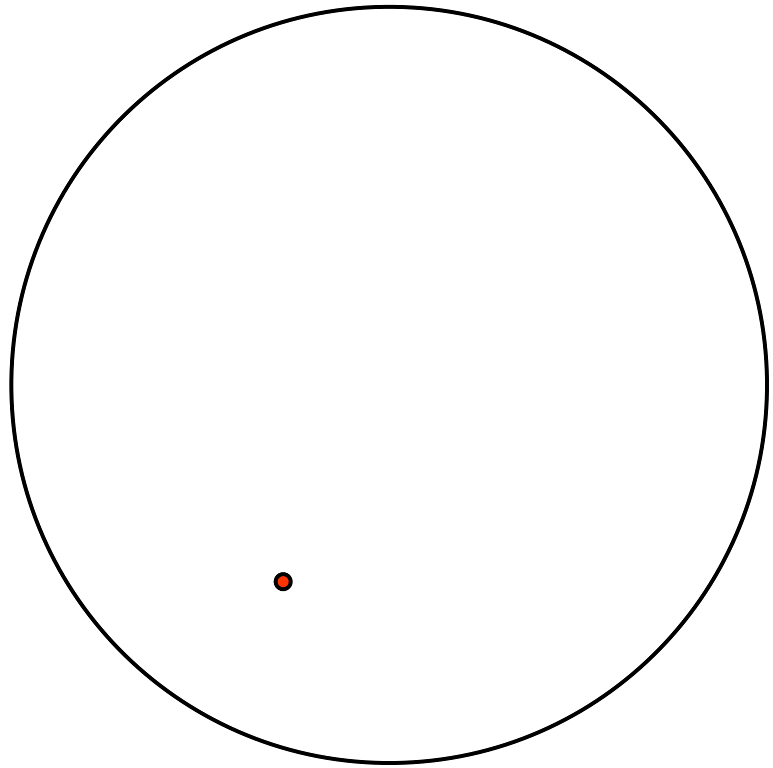
$$S_{xy}(\omega) = h(\omega)S_{xx}(\omega)$$

- And

$$S_{yy}(\omega) = |h(\omega)|^2 S_{xx}(\omega)$$

Another key example: Brownian noise

- Because of collisions with water molecules, micron-size particles undergo random motion



The model

1. Directions of exchanged momentum during collisions are at random
2. Collisions are very frequent and "instantaneous".
3. Collisions are many and independent.
4. On the average there is no net exchange of momentum between water and the molecule.

- Thus the molecule is subject to a stochastic force with the following properties:
  - From1, the Cartesian components of the force, $f_x(t)$, $f_y(t)$, and $f_z(t)$ are independent stochastic processes.
  - From 2, each of these processes has a very rapidly decaying autocorrelation that, on the time scales of interest, may be approximated with a delta.
  - From 3, because of central limit theorem, each component is a Gaussian process.
  - From 4. the mean value of each of these processes is 0
- In summary $\left\langle f_x(t)\right\rangle = 0 \quad R_{f_x f_x}(\tau) = S_o \delta(\tau) \quad S_{f_x f_x}(\omega) = S_o$
- that is, the force is white noise.

Brownian motion summary

- A small particle in a viscous fluid is subject to collisions with fluid molecules.
- The effect of exchange of momentum during these collisions is twofold:
  - If the particle moves on a macroscopic scale, the exchange of momentum is equivalent to a force

$$\vec{f}(t) = -\beta\vec{v}$$

  - A stochastic white force superimposes to the above with PSD

$$S_{ff} = 2\beta k_B T$$

  - Where the coefficient β is the same for both phenomena!
  - The particle is set into motion by this force as by any other force. The resulting velocity has spectrum

$$S_{v_x, v_x}(\omega) = 2k_B T \frac{\beta}{m^2\omega^2 + \beta^2}$$

Noise and dissipation

- There is a remarkable relation between Brownian velocity noise and macroscopic energy dissipation for the Brownian particle.
- Consider the force to velocity frequency response $h(\omega)=(i\omega\, m+\beta)^{-1}$
- According to our results, for a sinusoidal force $f(t)=f_o \mathrm{Sin}[\omega_o t]$
- The velocity response is $v(t)=|h(\omega_o)| f_o \mathrm{Sin}(\omega_o t+\mathrm{Arg}\{h(\omega_o)\})$
- Let's calculate the power dissipated per cycle (because of drag) :

$$\overline{P}=(\omega_o/2\pi)\int_0^{2\pi/\omega_o} v(t) f(t) dt$$

- By substituting

$$\overline{P}=|h(\omega_o)| f_o^2 (\omega_o/2\pi)\int_0^{2\pi/\omega_o} \mathrm{Sin}(\omega_o t+\mathrm{Arg}\{h(\omega_o)\})\mathrm{Sin}(\omega_o t) dt$$

- Calculating the integral $\overline{P}=|h(\omega_o)|(f_o^2/2)\mathrm{Cos}[\mathrm{Arg}\{h(\omega_o)\}]$

In[43]:= $\frac{\omega_o}{2\pi}\int_0^{\frac{2\pi}{\omega_o}} \mathrm{Sin}[\omega_o t+\phi]\,\mathrm{Sin}[\omega_o t]\, dt$

Out[43]= $\frac{\mathrm{Cos}[\phi]}{2}$

Noise and dissipation

- Power per cycle $\bar{P} = \left|h(\omega_o)\right|\left(f_o^2/2\right)\mathrm{Cos}\left[\mathrm{Arg}\left\{h(\omega_o)\right\}\right]$
- From $h(\omega) = \left(i\omega\, m + \beta\right)^{-1}$
- We get

$$\bar{P} = \frac{1}{\sqrt{m^2\omega^2 + \beta^2}}\left(f_o^2/2\right)\mathrm{Cos}\left[\mathrm{Arctan}\left(m\omega/\beta\right)\right] = \left(f_o^2/2\right)\frac{\beta}{m^2\omega^2 + \beta^2}$$

- Notice that $f_{rms} \equiv \sqrt{\left(\omega_o/2\pi\right)\int_0^{2\pi/\omega_o} f^2(t)\,dt} = f_o\big/\sqrt{2}$
- So that $\bar{P} = f_{rms}^2\,\beta\big/\left(m^2\omega^2 + \beta^2\right)$
- Now compare with spectral density of Brownian velocity

$$S_{v_x,v_x}(\omega) = 2k_B T\beta\big/\left(m^2\omega^2 + \beta^2\right)$$

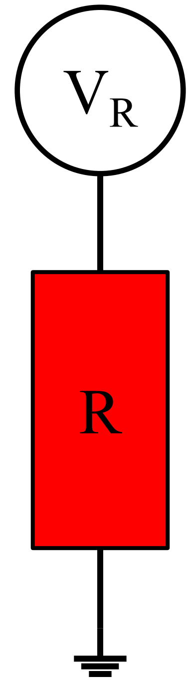
- So that $S_{v_x,v_x}(\omega) = 2k_B T\,\bar{P}\big/f_{rms}^2$
- This is the first manifestation of the fluctuation-dissipation theorem that we will formulated and discuss later

Nyquist-Johnson noise

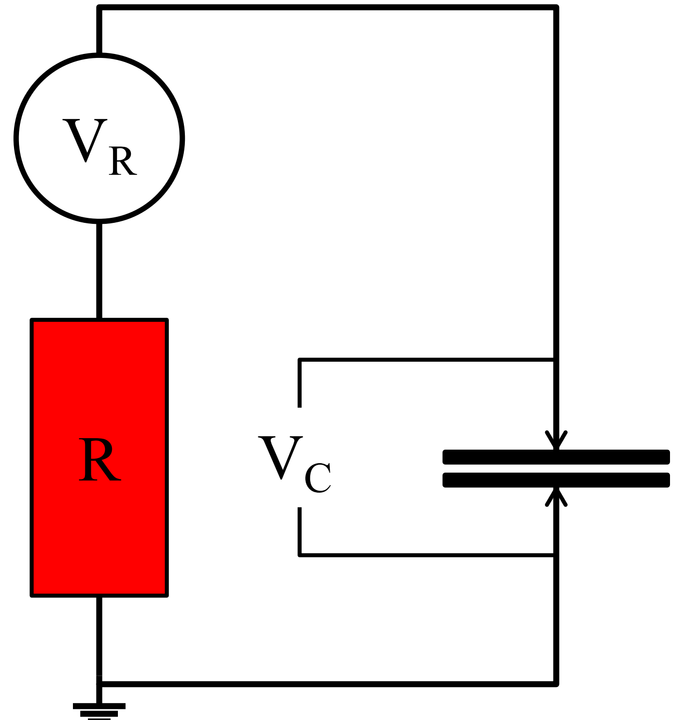
- A resistor
- Voltage can be described by a generator in series to a voltage-free resistor (Thévenin)
- Noise is white

$$\langle V_R \rangle = 0 \quad S_{V_R V_R}(\omega) = S_o$$

- We are going to derive the value of S_o



Nyquist-Johnson noise



- Put a capacitor in parallel to the resistor.
- The transfer function form V_R to V_C is

$$V_C(\omega) = \frac{V_R(\omega)}{i\omega RC + 1}$$

- The power spectrum of V_C is then

$$S_{V_C V_C}(\omega) = \frac{S_o}{1 + \omega^2 (RC)^2}$$

- We can calculate the mean square voltage across the capacitor

$$\langle V_C^2 \rangle = S_o (1/2\pi) \int_{-\infty}^{\infty} \left(1 + \omega^2 (RC)^2\right)^{-1} d\omega = S_o / (2RC)$$

- Using again equipartition law

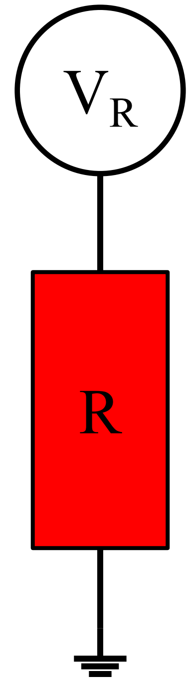
$$(1/2) C \langle V_C^2 \rangle = (1/2) k_B T$$

- We get Nyquist law

$$S_o = 2Rk_BT$$

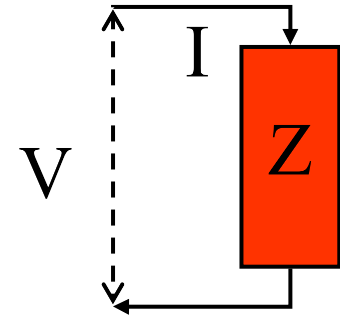
Nyquist law

- Noise voltage can be described by a generator in series to a voltage-free resistor (Thévenin).
- Generator is white noise with power $S_{V_R V_R}(\omega) = 2Rk_B T$



UNIVERSITÀ DEGLI STUDI DI TRENTO

Nyquist law: a more general result



- Power dissipated into Z by a stochastic current I
- Consider the impedance as a linear system with input I and output V
- The frequency response is just the impedance

$$V(\omega)=Z(\omega)I(\omega)$$

- If I(t) is a zero-mean stochastic process V(s) is a zero-mean stochastic process.
- The mean dissipated power is $\langle P(t)\rangle=\langle I(t)V(t)\rangle=R_{IV}(0)$
- We know that

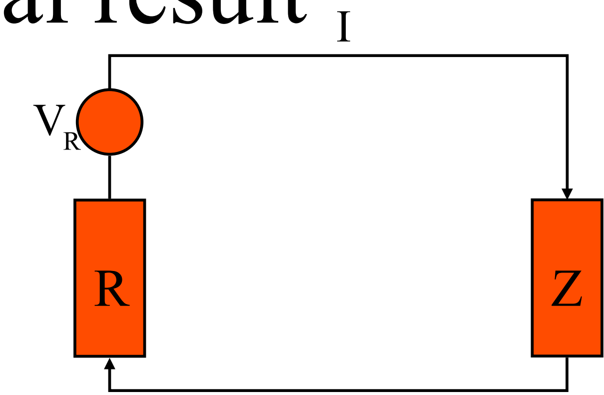
$$R_{I,V}(0)=(1/2\pi)\int_{-\infty}^{\infty}S_{IV}(\omega)d\omega=(1/2\pi)\int_{-\infty}^{\infty}Z(\omega)S_{II}(\omega)d\omega$$

- Remembering that

$$\mathrm{Re}\{Z(\omega)\}=\mathrm{Re}\{Z(-\omega)\}\quad \mathrm{Im}\{Z(\omega)\}=-\mathrm{Im}\{Z(-\omega)\}\quad S_{II}(\omega)=S_{II}(-\omega)$$

- We conclude that $\langle P\rangle=(1/2\pi)\int_{-\infty}^{\infty}\mathrm{Re}\{Z(\omega)\}S_{II}(\omega)d\omega$

Nyquist law: a more general result



- Let's go back to our series circuit
- The power dissipated into Z

$$\langle P\rangle = (1/2\pi)\int_{-\infty}^{\infty} \mathrm{Re}\{Z(\omega)\} S_{II}(\omega)\,d\omega$$

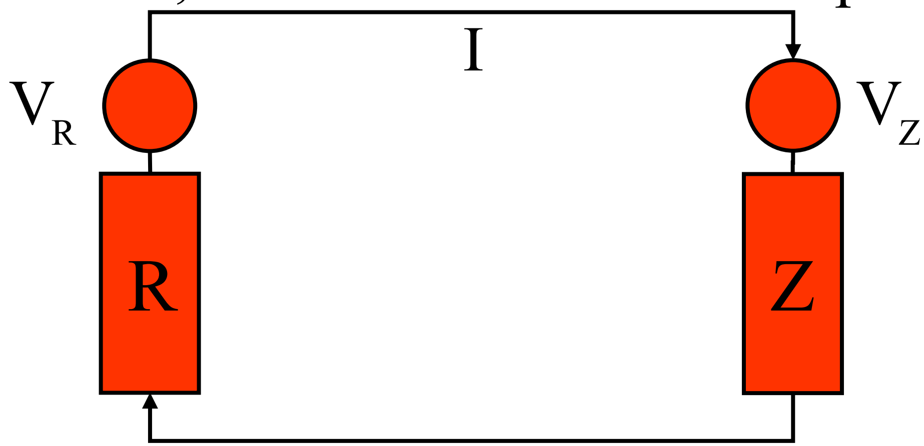
- Let's calculate the PSD of current. As $I(\omega) = V_R(\omega)/(R+Z(\omega))$
- Then $$S_{II}(\omega) = 2k_B TR\Big/\left|R+Z(\omega)\right|^2$$
- And $$\langle P\rangle = 2k_B T(1/2\pi)\int_{-\infty}^{\infty}\left(\mathrm{Re}\{Z(\omega)\}R\Big/\left|R+Z(\omega)\right|^2\right)d\omega$$
- This power is a steady flow of heat form R to Z. At thermal equilibrium it would violate the 2^{nd} principle of thermodynamics
- There must be a noise generator associated with Z to balance the heat flow.

Nyquist law: a more general result

- Mean power dissipated by Nyquist generator into Z

$$\left\langle P_{R\to z}\right\rangle = 2k_BT\left(1/2\pi\right)\int_{-\infty}^{\infty}\left(\mathrm{Re}\left\{Z(\omega)\right\}R\Big/\left|R+Z(\omega)\right|^2\right)d\omega$$

- To balance this power there must be a random voltage across Z
- According to Thévenin, the circuit can then be represented as



- The transfer function from V_z to the current circulating through the resistor is calculated from $I(\omega)=V_Z(\omega)/\left(R+Z(\omega)\right)$
- The mean power dissipated into R by this current is (just exchange R with Z and S_{VV} with $2Rk_BT$) is:

$$\left\langle P_{Z\to R}(t)\right\rangle = \left(1/2\pi\right)\int_{-\infty}^{\infty}S_{V_ZV_Z}(\omega)R\Big/\left|R+Z(\omega)\right|^2 d\omega$$

Nyquist law: a more general result

- Mean power dissipated by V_R into Z

$$\left\langle P_{R\to z}\right\rangle = 2k_B T\left(1/2\pi\right)\int_{-\infty}^{\infty}\left(\mathrm{Re}\left\{Z(\omega)\right\}R\Big/\left|R+Z(\omega)\right|^2\right)d\omega$$

- Mean power dissipated by V_Z into R

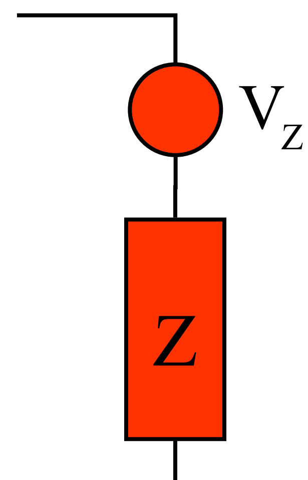
$$\left\langle P_{Z\to R}(t)\right\rangle = \left(1/2\pi\right)\int_{-\infty}^{\infty}S_{V_Z V_Z}(\omega)R\Big/\left|R+Z(\omega)\right|^2 d\omega$$

- Thermodynamic equilibrium requires that, for whatever Z

$$\left\langle P_{R\to Z}(t)\right\rangle = \left\langle P_{Z\to R}(t)\right\rangle$$

- That is, any impedance Z can be considered as the series of a noiseless element and a zero-mean normal stationary voltage noise generator with PSD

$$S_{V_Z V_Z}(\omega) = 2k_B T\,\mathrm{Re}\left\{Z(\omega)\right\}$$



- This is the generalized Nyquist law