Exercise on interferometer

The LISA Pathfinder motion detector (fig. 1) is a Mach-Zender interferometer. Light from the same laser source of wavelength λ, is split in two beams by a beam splitter. The two beams cross two acusto-optic modulators (AOM), the effect of which is to shift their frequency, by an amount $f1$ and $f2$ respectively, relative to the unperturbed frequency of the laser $f0 = c/\lambda$, without spoiling their relative phase coherence.

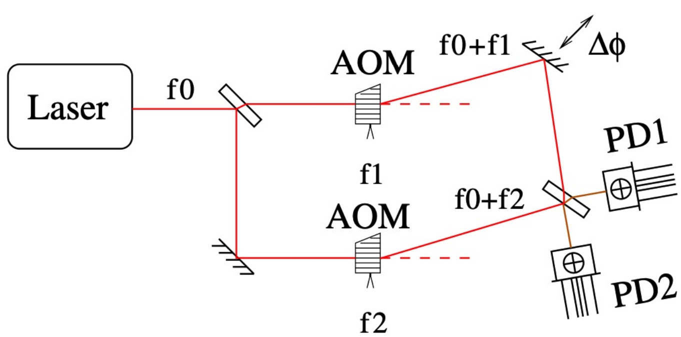

FIG. 1. Schematics of the system.

The upper beam in the figure is reflected off the surface of one of LISA Pathfinder test-masses. Thus when the mass moves by a small amount δx, along the direction of the arrow in the figure, the optical path of the beam is modulated by a length $\delta l = \gamma \delta x$, with γ a dimensionless constant, thus inducing a phase shift in the beam $\Delta\phi = 2\pi\delta l/\lambda \ll 2\pi$.

The basic idea of a Mach-Zender interferometer is that, due to the difference in the sequence of transmissions and reflections, the electrical field hitting photodiode PD1 is the sum of the electrical fields of the two beams, while that hitting PD2 is their difference. These electrical fields, at time t, are given by:

$$\begin{aligned} E_{beam,1} &= (E_o + \delta E(t_1)) Sin\left(2\pi(f0+f1)t_1 + \phi_n(t_1) + \Delta\phi(\tilde{t})\right) \\ E_{beam,2} &= (E_o + \delta E(t_2)) Sin\left(2\pi(f0+f2)t_2 + \phi_n(t_2)\right) \end{aligned} \tag{1}$$

where $t_1 = t - L_1/c$ and $t_2 = t - L_2/c$, with L_1 and L_2 the lengths of the optical paths of the upper and lower beam of the figure respectively. As one tries to match the two optical paths as much as he can, $\delta T \equiv t_2 - t_1 \equiv \delta L/c$ is a small quantity. The time $\tilde{t}$ differs from t by the constant light propagation time between the test-mass and the photodiodes. Finally, in eq. 1 we have also introduced the (small) fluctuation $\delta E(t)$ of the amplitude of the laser beam, whose unperturbed value is E_o, and the (small) phase noise of the laser $\phi_n(t)$. These are zero-mean, stationary and normal stochastic processes, with power spectral densities (PSD) respectively $S_{\delta E}(f) = E_o^2 S_{\mathcal{E}}$ and $S_\phi(f) = S_o(f_o/f)^2$, where $S_{\mathcal{E}}$, S_o, and f_o are constants.

A photodiode is a reverse biased diode, in the depleted region of which, photons create electron-vacancy pairs and thus induce a current. The current is proportional to the intensity of the light beam hitting the diode, i.e. is proportional to the square modulus of the associated electrical field.

With a proper bias circuit, such a current is in turn converted into a proportional voltage signal which, as a consequence, is also proportional to the square of the electric field. The biased photo-diode has a limited bandwidth with a roll-off frequency f_{PD}, never higher than some GHz. We summarise these properties by modelling the response of both photodiodes as

$$V_{PD,i}(f) = \frac{V_o}{1 + i\frac{f}{f_{PD}}} \frac{E_{PD,i}^2(f)}{E_o^2} \tag{2}$$

with V_o a constant voltage, and $1 \leq i \leq 2$. Remember that $E_{PD,1} = E_{beam,1} + E_{beam,2}$, and $E_{PD,2} = E_{beam,1} - E_{beam,2}$ In addition we assume that both photodiodes are affected by an additive noise with white power spectral density $S_V = V_o^2 S_{\mathcal{V}}$

With the purpose of measuring $\Delta\phi(t)$, a slow signal with a Nyqvist frequency f_N, one forms the signal $\Delta V = V_{PD,1} - V_{PD,2}$ and sends such signal to the input of a phase-sensitive detector with a reference frequency f_{ref} and a single pole low-pass filter with roll-off frequency f_{ro}.

Questions

Notice: numerical figures are all in table I below

Q1 Neglecting all sources of noise, and considering $\Delta\phi$ as a constant, calculate the Fourier transform of ΔV. As $\Delta\phi$ is small, linearise the result as a function of $\Delta\phi$

Q2 Based on the answer to the question above, and on the numerical figures in table I, what should the value of f_{ref} and f_{ro} be, in order that the some combination of the two outputs of the phase sensitive detector is proportional to $\Delta\phi$?

Q3 Once you have properly set up the phase sensitive detector, reintroduce the amplitude and phase noise, linearise the formulas as a function of them, and of δt, add the effect of the voltage noise, and calculate the overall noise power spectral density. Express it as an effective displacement noise for the test-mass.

Parameter	Value	Units	Parameter	Value	Units
λ	1.064	μm	$f1$	400	MHz
$f2$	401	MHz	δL	2	cm
$S_{\mathcal{E}}$	10^{-8}	Hz^{-1}	S_o	10^{10}	Hz^{-1}
f_o	10	mHz	f_{PD}	1	GHz
$S_{\mathcal{V}}$	10^{-13}	Hz^{-1}	γ	0.7	

TABLE I. List of parameter values