Experimental Methods
Lecture 9

October 8th, 2020

Invertible operator, the role of input and output, and free evolution 1/3

- Some linear systems are described by operators that have an inverse:

$$o(t)=\Im\left[i(t)\right] \quad i(t)=\Im^{-1}\left[o(t)\right]$$

- That is, the equation

$$o(t)=\Im\left[i(t)\right]$$

  has just one solution

- For these systems the role of input and output can be interchanged
  - Example 1: multiplication by a constant

$$o(t)=ci(t) \quad i(t)=\frac{1}{c}o(t)$$

  - Example 2: Fourier transform

$$o(\omega)=\int_{-\infty}^{\infty} i(t)e^{-i\omega t}\,dt \qquad i(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} o(t)e^{i\omega t}\,dt$$

Invertible operator, the role of input and output, and free evolution 2/3

- Some linear systems (e.g. those described by differential equations) are better described by

$$D\left[o(t)\right]=i(t)$$

- For these system there may be n solutions:

$$D\left[o_j(t)\right]=0$$

- Because the operator is linear

$$D\left[o(t)+\sum_{j=1}^{n}c_j\,o_j(t)\right]=D\left[o(t)\right]+\sum_{j=1}^{n}c_j\,D\left[o_j(t)\right]=i(t)$$

- (c_j arbitrary constants)

Invertible operator, the role of input and output, and free evolution 3/3

$$D\left[o(t)+\sum_{j=1}^{n} c_j\, o_j(t)\right]=D\left[o(t)\right]+\sum_{j=1}^{n} c_j\, D\left[o_j(t)\right]=i(t)$$

- For all functions such that

$$\tilde{o}(t)\neq\sum_{j=1}^{n} c_j\, o_j(t)$$

- $D[\tilde{o}(t)]=i(t)$ has an inverse

$$\tilde{o}(t)=D^{-1}\left[i(t)\right]=\int_{-\infty}^{\infty} h(t,t')\,i(t')\,dt'$$

- Finally

$$o(t)=\sum_{j=1}^{n} c_j\, o_j(t)+\int_{-\infty}^{\infty} h(t,t')\,i(t')\,dt'$$

Free evolution: output without an input

An example: the pendulum

- Equation of motion for the x coordinate

$$m\ddot{x} = -mg\,\mathrm{Sin}(\theta)\,\mathrm{Cos}(\theta) + F_{ex}$$

- Converting angles

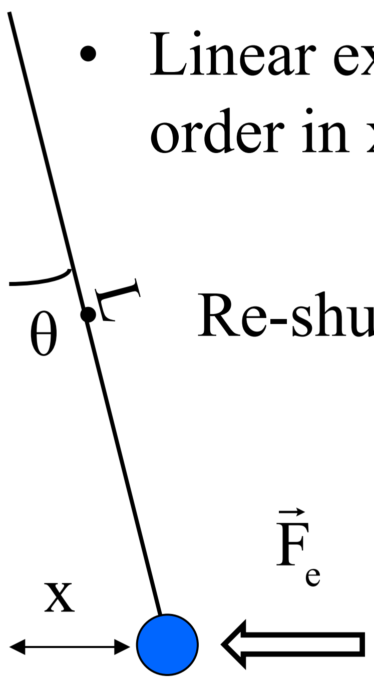
$$m\ddot{x} = -mg\frac{x}{L}\sqrt{1-\left(\frac{x}{L}\right)^{2}} + F_{ex}$$

- Linear expansion to first order in x

$$m\ddot{x} = -mg\frac{x}{L} + F_{ex}$$

Re-shuffling

$$\ddot{x} + \frac{g}{L}x \equiv \ddot{x} + \omega_o^2 x = F_{ex}/m$$



$$F_x \longrightarrow \boxed{\text{Pendulum}} \longrightarrow x$$

A basic example: the pendulum

- Linearized equation for the x coordinate $\ddot{x} + \omega_o^2 x = F_{ex}/m$
- Special solution $\tilde{x}(t) = \frac{1}{m\,\omega_o}\int_0^\infty \mathrm{Sin}(\omega_o t')F_{ex}(t-t')dt'$
- Indeed $\dot{\tilde{x}}(t) = -\frac{1}{m\,\omega_o}\int_0^\infty \mathrm{Sin}(\omega_o t')\,\frac{dF_{ex}(t-t')}{dt'}dt' =$

$$= \frac{1}{m}\int_0^\infty \mathrm{Cos}(\omega_o t')F_{ex}(t-t')dt'$$

- and $\ddot{\tilde{x}}(t) = -\frac{1}{m}\int_0^\infty \mathrm{Cos}(\omega_o t')\,\frac{dF_{ex}(t-t')}{dt'}dt' =$

$$= -\frac{1}{m}\mathrm{Cos}(\omega_o t')F_{ex}(t-t')]_0^\infty - \frac{\omega_o}{m}\int_0^\infty \mathrm{Sin}(\omega_o t')F_{ex}(t-t')dt' =$$

$$= \frac{F_{ex}(t)]}{m} - \omega_o^2\tilde{x}(t)$$

- then:

$$\ddot{\tilde{x}}(t) + \omega_o^2\tilde{x}(t) = \frac{F_{ex}(t)]}{m} - \omega_o^2\tilde{x}(t) + \omega_o^2\tilde{x}(t) = \frac{F_{ex}(t)]}{m}$$

A basic example: the pendulum

- Response to input: $\tilde{x}(t) = \frac{1}{m\,\omega_o}\int_0^\infty \mathrm{Sin}(\omega_o t')F_{ex}(t - t')dt'$
- In addition, for arbitrary values of x_p and x_q:

$$x(t) = x_p \mathrm{Sin}(\omega_o t) + x_q \mathrm{Cos}(\omega_o t) \rightarrow \ddot{x}(t) + \omega_o^2 x(t) = 0$$

- The most general solution is then:

$$x(t) = \underbrace{x_p \mathrm{Sin}(\omega_o t) + x_q \mathrm{Cos}(\omega_o t)}_{\text{Free evolution}} + \underbrace{\frac{1}{m\,\omega_o}\int_0^\infty \mathrm{Sin}(\omega_o t')F_{ex}(t - t')dt'}_{\text{Response}}$$

- Note: for an undamped pendulum, the free evolution never decays.
- Passive undamped pendulums do not exist in nature.

Inverting the relation to obtain the input

- Notice that the equation of motion D[o(t)]=i(t) allows for calculating the value of i(t) from o(t) even if this one includes some free evolution
- For the pendulum for instance by calculating $m\ddot{x}(t)$ at time t and adding up the value of mgx(t)/L at the same time, one obtains $F_{ex}(t)=m\ddot{x}(t)+mgx(t)/L$, no matter what's the value of the free evolution.
- A practical example with a torsion pendulum

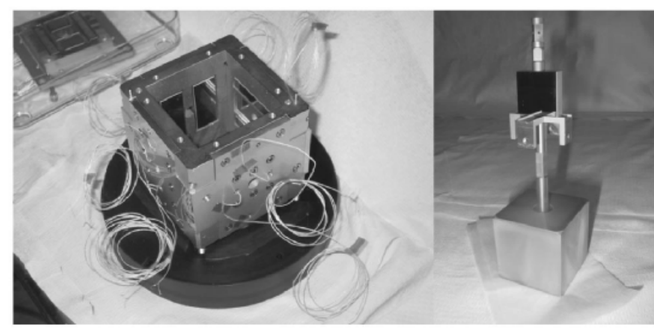
PHYSICAL REVIEW D **75,** 042001 (2007)

B. EVALUATION OF TORQUE AND BACKGROUND SUBTRACTION

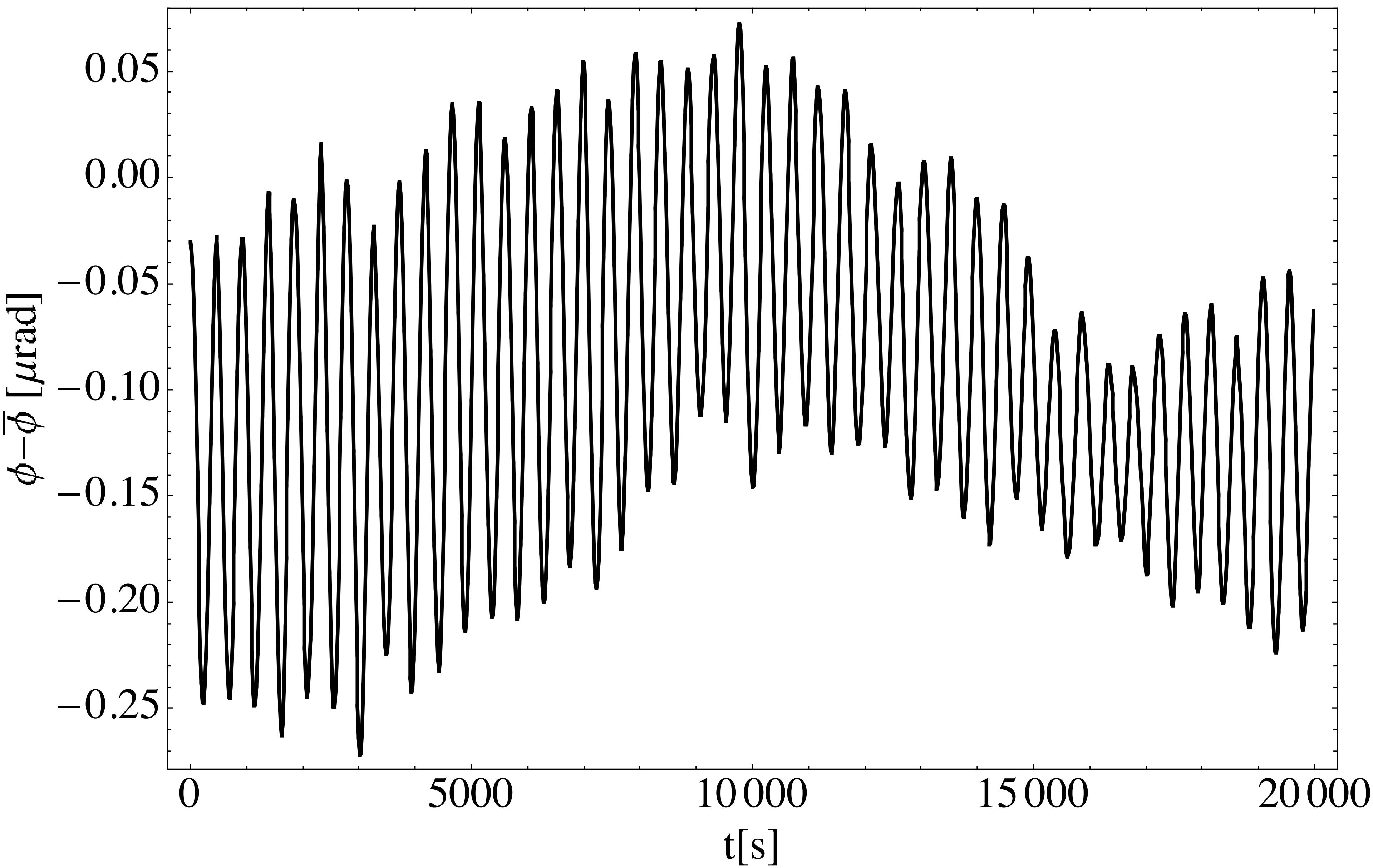
The angular motion of the pendulum $\phi(t)$ is converted into an instantaneous applied torque $N(t)$ as

$$N(t) = I_o\{\ddot{\phi}(t) + (2\pi/T_oQ)\dot{\phi}(t) + (2\pi/T_o)^2\phi(t)\}. \quad (1)$$

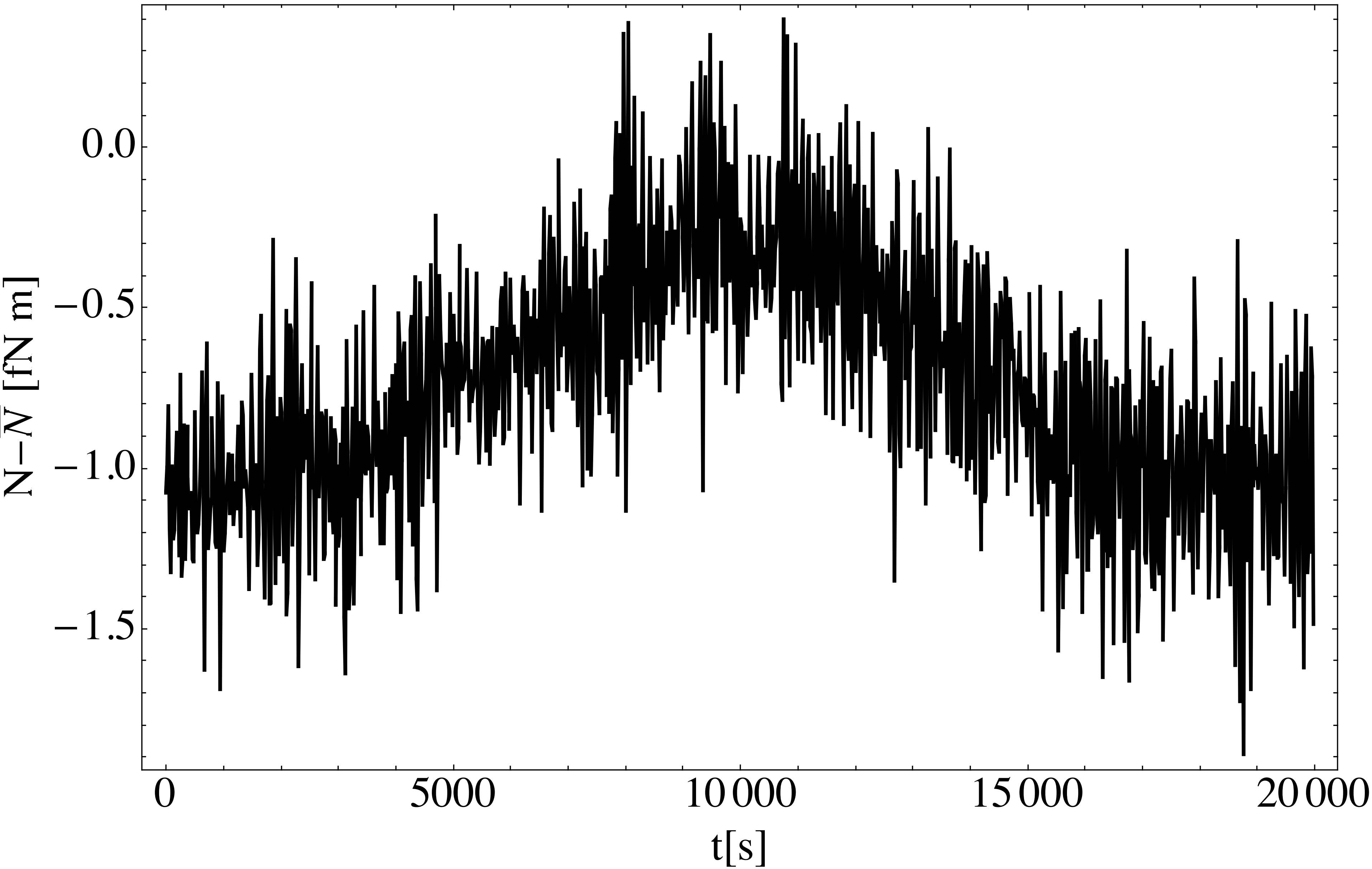
The derivatives are estimated from a sliding second order fit to 5 adjoining data.



Angular data



Torque data



Step response

- From

$$o(t) = \int_{-\infty}^{\infty} h(t')i(t-t')dt'$$

- We derived that h(t) is the impulse response

$$o(t) = \int_{-\infty}^{\infty} h(t')\delta(t-t')dt' = h(t)$$

- Consider now the *step response* defined as

$$o_{-1}(t) = \int_{-\infty}^{\infty} h(t')\Theta(t-t')dt' = \int_{-\infty}^{t} h(t')dt'$$

- We can derive o_{-1}

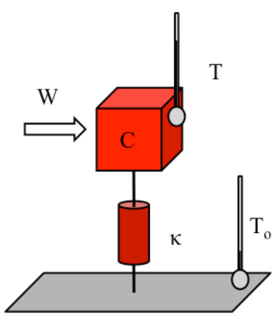
$$do_{-1}(t)/dt = h(t)$$

- Thus h(t) is both the impulse response and the derivative of the step response

The importance of the step response

- As delta impulse cannot be obtained in reality, h(t) cannot be measured as the response of the system to an impulse.
- High but finite amplitude pulses, easily drive the system out of linearity
- On the contrary steps can be made very small and fed to the system under study.
- The derivative of the system response is proportional to h
- Example: the calorimeter. Response to a step at $t=0$

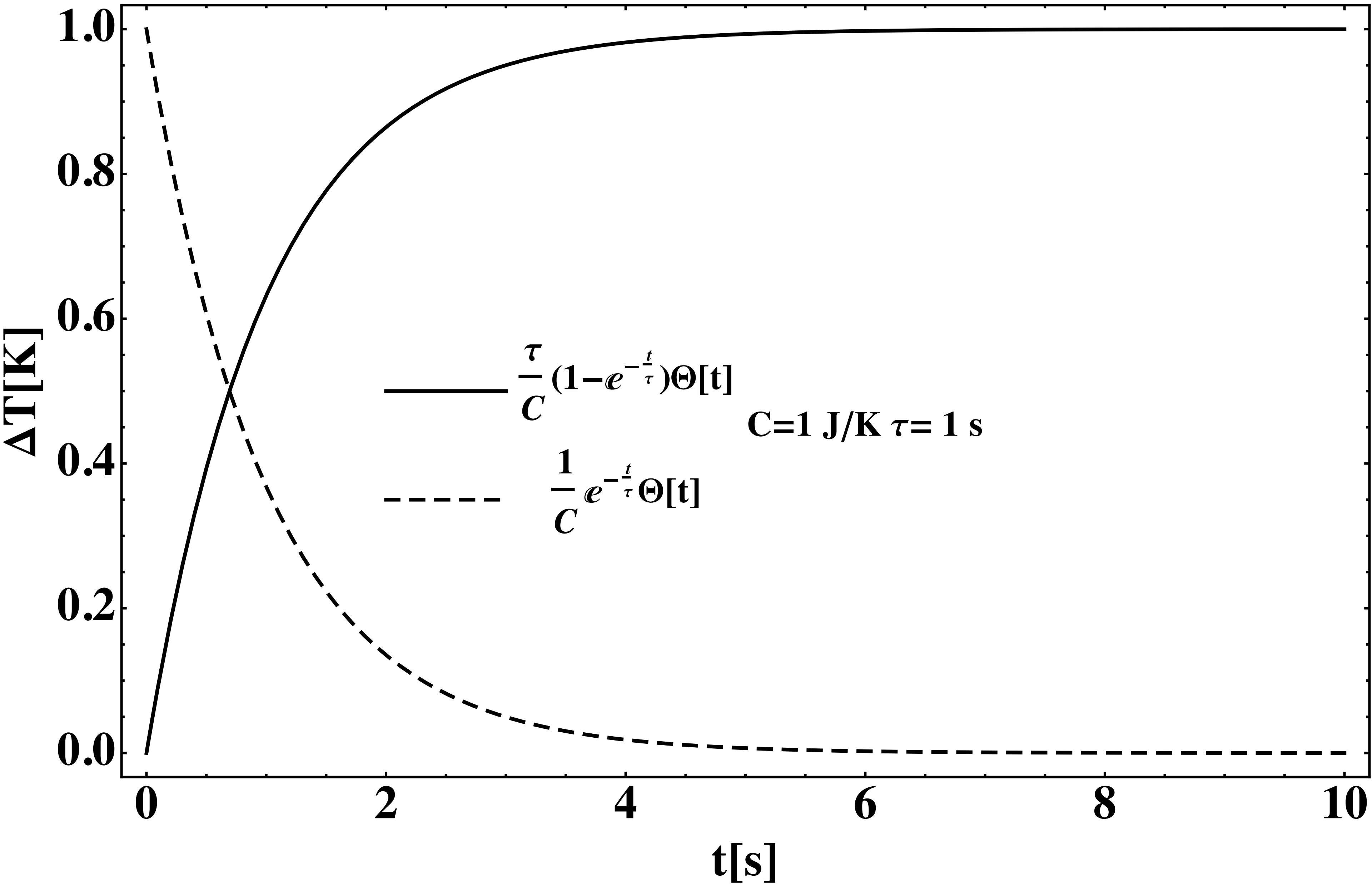
$$\Delta T(t)=\left(W_o/\kappa\right)\left(1-e^{-\frac{t}{\tau}}\right)\Theta(t)$$



- Derivative

$$d\Delta T(t)/dt=\left(W_o/\tau\kappa\right)e^{-\frac{t}{\tau}}\Theta(t)=\left(W_o/C\right)e^{-\frac{t}{\tau}}\Theta(t)=W_o h(t)$$

- Thus h(t) can be measured with W_o as small as the measurement noise allows for.


1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
t[s]
ΔT[K]
$\frac{\tau}{C}(1-e^{-\frac{t}{\tau}})\Theta[t]$
$\frac{1}{C}e^{-\frac{t}{\tau}}\Theta[t]$
C=1 J/K τ= 1 s

Summary: linear stationary systems, continuous

- A linear stationary system $o(t)=\int_{-\infty}^{\infty} i(t')h(t-t')dt' = \int_{-\infty}^{\infty} h(t')i(t-t')dt'$

- A linear, stationary, and causal system

$$o(t)=\int_{0}^{\infty} h(t')i(t-t')dt'$$

- A linear, stationary system with free evolution (not all systems have free evolution)

$$o(t)=\sum_{j=1}^{n} c_j o_j(t)+\int_{-\infty}^{\infty} h(t')i(t-t')dt'$$

Linear response and Fourier Transforms

- Fourier Transforms convert input-output relations into algebraic equations
- Within linear response, frequencies don’t mix!

Linear system in the frequency domain

- Output is the convolution between input and impulse response

$$o(t) = \int_{-\infty}^{\infty} h(t')\, i(t - t')\, dt'$$

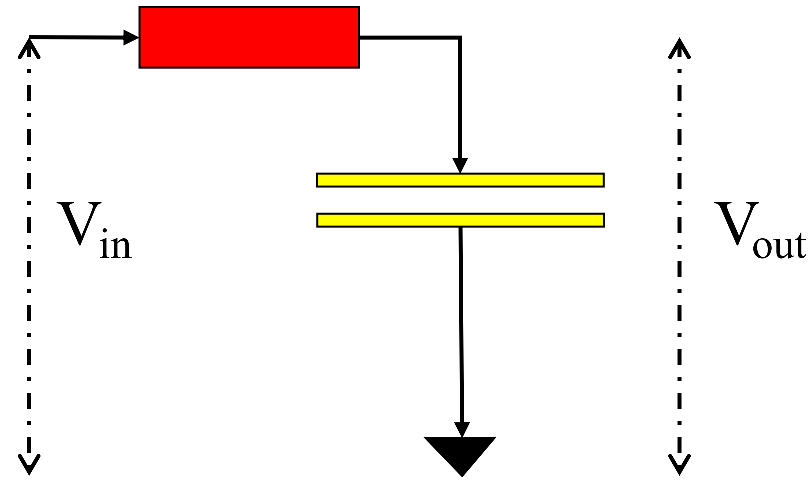
- Using convolution theorem

$$o(\omega) = h(\omega)\, i(\omega)$$

- h(t): impulse response
- h(ω): ***frequency response***

Linear systems do not mix frequencies

A remarkable example: differential equations



- Consider a low pass electrical circuit
- The circuit equation

$$C\,dV_{out}(t)/dt = \left[V_{in}(t) - V_{out}(t)\right]/R$$

- That is (define τ=RC)

$$dV_{out}(t)/dt + V_{out}(t)/\tau = V_{in}(t)/\tau$$

- Take the Fourier Transform

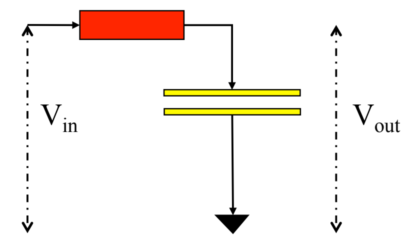
$$i\omega V_{out}(\omega) + V_{out}(\omega)/\tau = V_{in}(\omega)/\tau$$

- That is $V_{out}(\omega) = V_{in}(\omega)/(1 + i\omega\tau)$
- Or

$$h(\omega) = 1/(1 + i\omega\tau)$$

A remarkable example: differential equations

- Input output relation in the frequency domain



$$V_{out}(\omega)=V_{in}(\omega)/(1+i\omega\tau)=h(\omega)V_{in}(\omega)$$

- From convolution theorem ($\boldsymbol{\mathcal{F}}^{-1}$= inverse Fourier transform)

$$V_{out}(t)=\int_0^\infty \mathcal{F}^{-1}\{h(\omega)\}(t')V_{in}(t-t')dt'$$

- Let Mathematica™ calculate the inverse Fourier transform

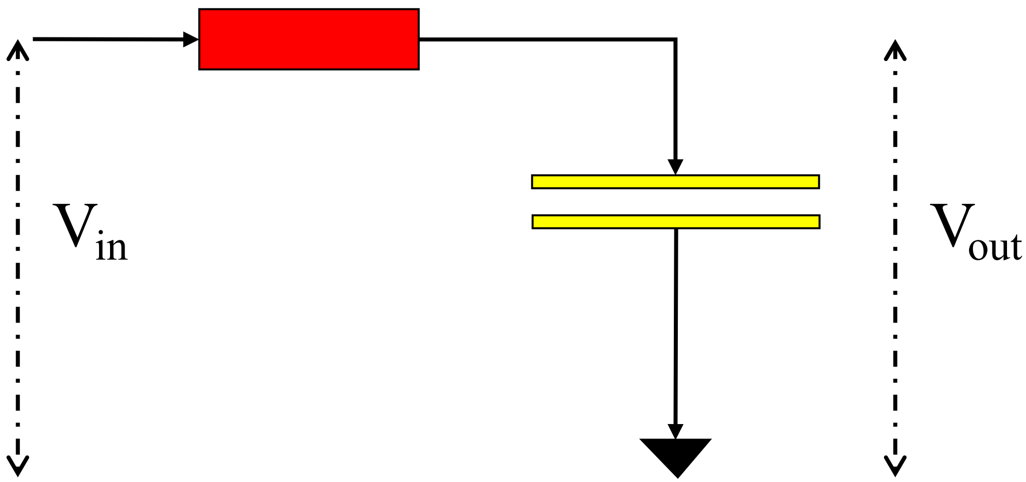
```
Simplify[InverseFourierTransform[1/(1 + ⅈ ω τ), ω, t], τ > 0]
```

```
33]=
ⅇ^(-t/τ) HeavisideTheta[t]
--------------------------
            τ
```

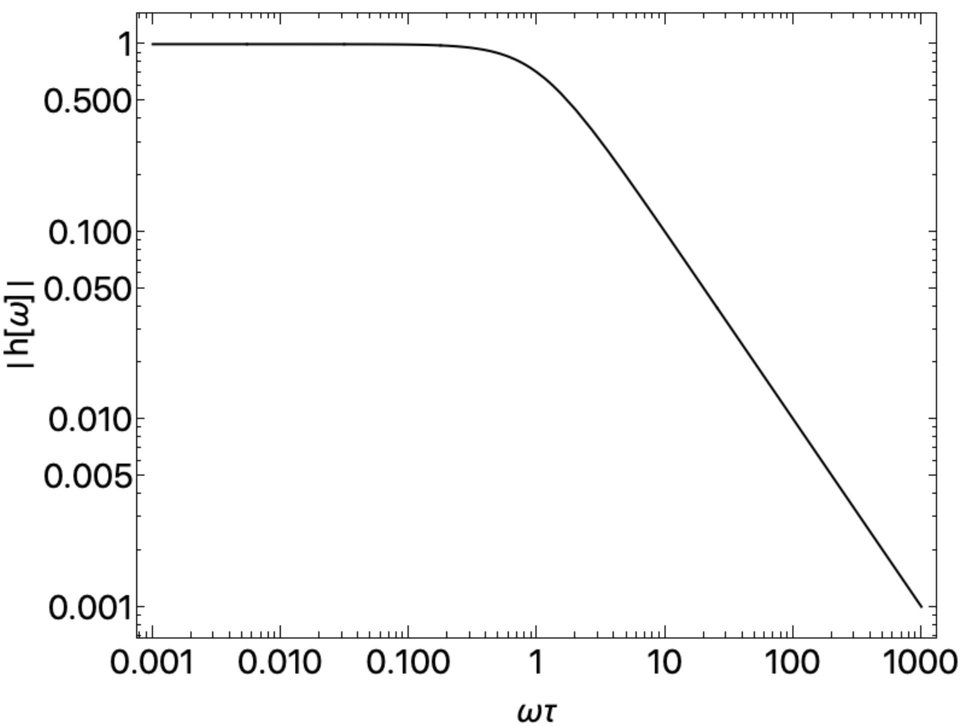
- Then finally

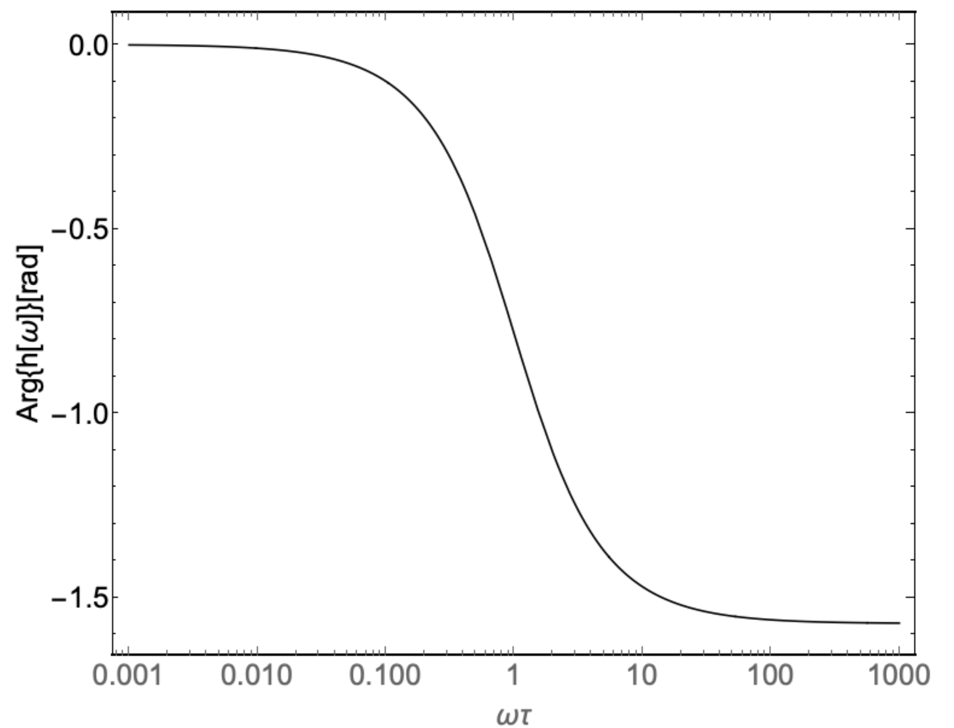
$$V_{out}(t)=\int_0^\infty \frac{1}{\tau}e^{-\frac{t'}{\tau}}\theta(t')V_{in}(t-t')dt'$$

A remarkable example: differential equations



$$h(\omega) = \frac{1}{1 + i\omega\tau}$$





Frequency response and sinusoidal signals

- Take an (ever lasting) sinusoid

$$i(t) = \mathrm{Sin}(\omega_o t)$$

- Its Fourier Transform is

$$i(\omega) = \frac{\pi}{i}\left[\delta(\omega - \omega_o) - \delta(\omega + \omega_o)\right]$$

- Let's feed it to a system

$$i(\omega) \rightarrow \boxed{h(\omega)} \rightarrow o(\omega)$$

- The output is

$$o(\omega) = \frac{\pi}{i}\left[h(\omega_o)\delta(\omega - \omega_o) - h(-\omega_o)\delta(\omega + \omega_o)\right]$$

- Using symmetry of h

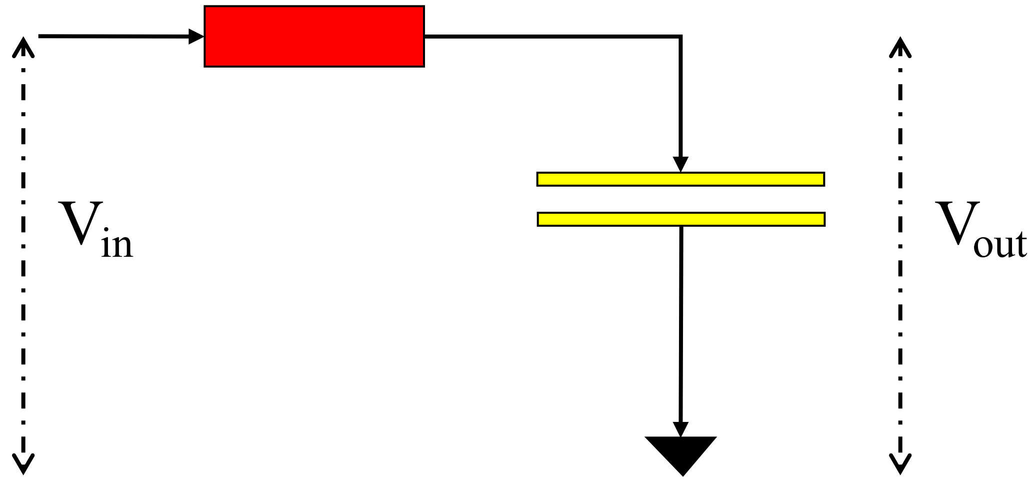
$$= \frac{\pi}{i}h'(\omega_o)\left[\delta(\omega - \omega_o) - \delta(\omega + \omega_o)\right] + \pi h''(\omega_o)\left[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)\right]$$

- In the time domain

$$o(t) = h'(\omega_o)\mathrm{Sin}(\omega_o t) + h''(\omega_o)\mathrm{Cos}(\omega_o t)$$

$$= \left|h(\omega_o)\right|\mathrm{Sin}\left(\omega_o t + \mathrm{Arg}\left\{h(\omega_o)\right\}\right)$$

The low pass



- The frequency response function

$$h(\omega)=\frac{1}{1+i\omega\tau}$$

- $\mathrm{Sin}(\omega_o t)$ at input

$$o(\omega)=\frac{\pi}{i}\left[\frac{\delta(\omega-\omega_o)}{1+i\omega_o\tau}-\frac{\delta(\omega+\omega_o)}{1-i\omega_o\tau}\right]$$

- In the time domain

$$o(t)=\frac{\mathrm{Sin}\left(\omega_o t-\mathrm{ArcTan}[\omega_o\tau]\right)}{\sqrt{1+\omega_o^2\tau^2}}$$

Measuring the transfer function of a system

- Feed a system with an input i(t) which is wideband enough for your purposes. Measure the response o(t). The transfer function is

$$h(\omega)=o(\omega)/i(\omega)=\left|o(\omega)\right|/\left|i(\omega)\right|e^{i\left\{\mathrm{Arg}\left[o(\omega)-\mathrm{Arg}\left[i(\omega)\right]\right]\right\}}$$

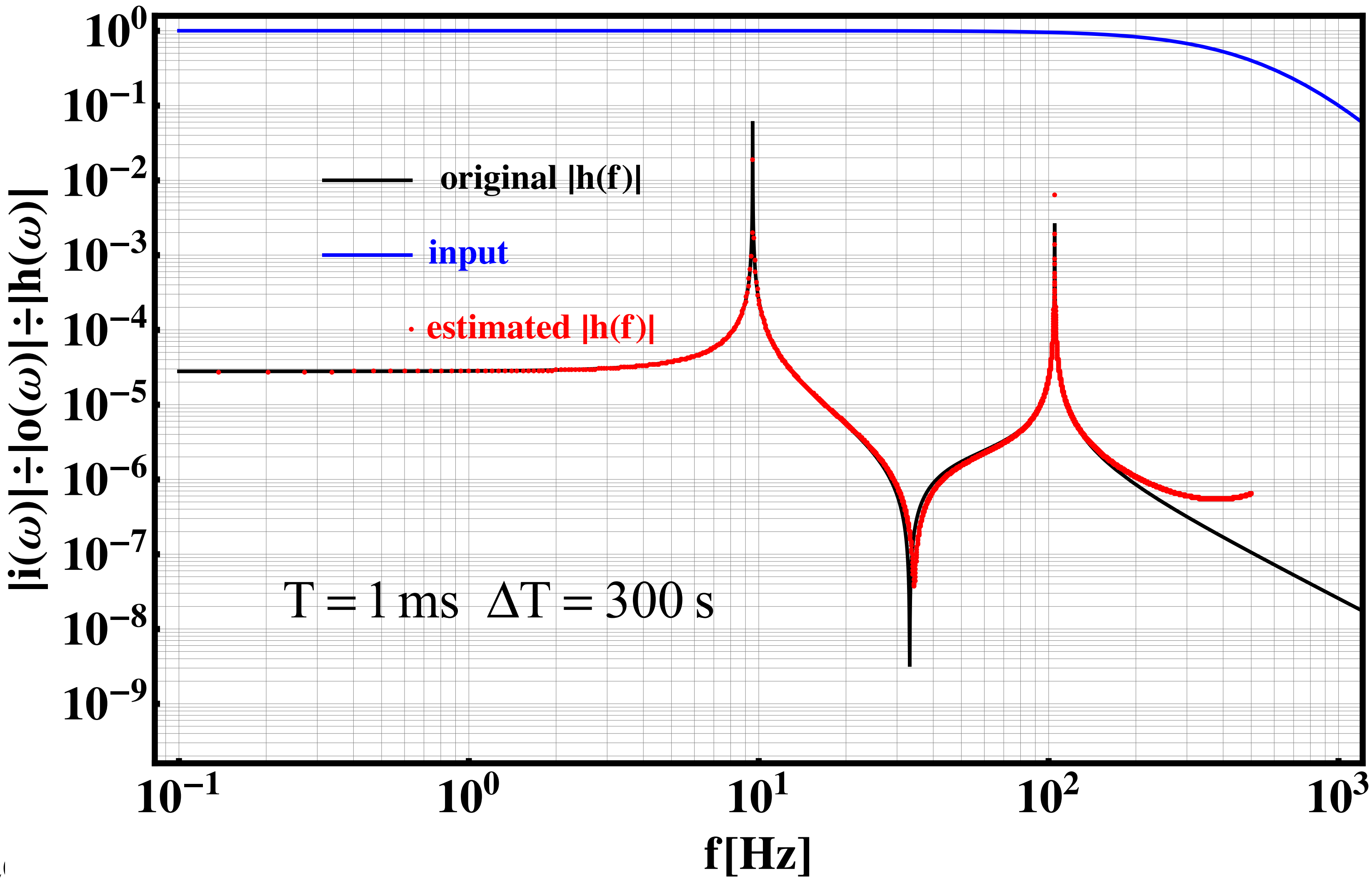
- Notice

$$h(\omega)=o(\omega)/i(\omega)$$

- Holds if i(ω) and o(ω) are the true continuous Fourier transform of the continuous signal
- Use of DFT is allowed if:
  - Sampling of o(t) and i(t) is fast enough to produce negligible aliasing
  - Both i(t) and o(t) die out at t→0 and t→ΔT to make circular convolution equivalent to standard convolution

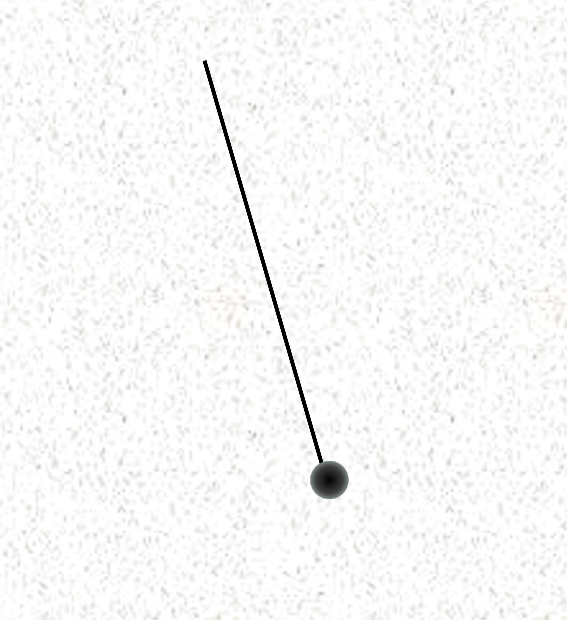
Numerical example

- Input $i(\omega)=\left[1+\left(\omega/2\pi\,1\,\mathrm{kHz}\right)^2\right]^{-1}\left[1+\left(\omega/2\pi\,500\,\mathrm{Hz}\right)^2\right]^{-1}$
- Frequency response $\dfrac{2.17\times10^{7}+62.8i\ f-1.97\times10^{4}f^{2}}{7.79\times10^{11}+2.04\times10^{7}i\ f-8.65\times10^{9}f^{2}-2.11\times10^{3}i\ f^{3}+7.79\times10^{5}f^{4}}$



Exercise: the pendulum

- Take the standard, *viscously damped* pendulum
- Consider torque as input and angle as output
- Write the input-output relation
- Linearize for small angles
- Find the frequency response
- Find the impulse response
- Find the free evolution
- Calculate the response to a Gaussian pulse



Linear systems in series

- Two systems in series: the output of the first system is the input to the second one

$$i_1(\omega) \longrightarrow \boxed{h_1(\omega)} \xrightarrow{o_1(\omega)=i_2(\omega)} \boxed{h_2(\omega)} \rightarrow o_2(\omega)$$

- It follows that

$$o_2(\omega)= h_2(\omega)i_2(\omega)= h_2(\omega)o_1(\omega)= h_2(\omega)h_1(\omega)i_1(\omega)$$

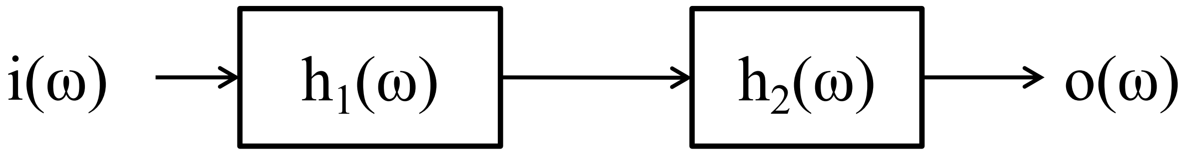
- Thus the system series is equivalent to

$$i(\omega) \longrightarrow \boxed{h(\omega)} \longrightarrow o(\omega)$$

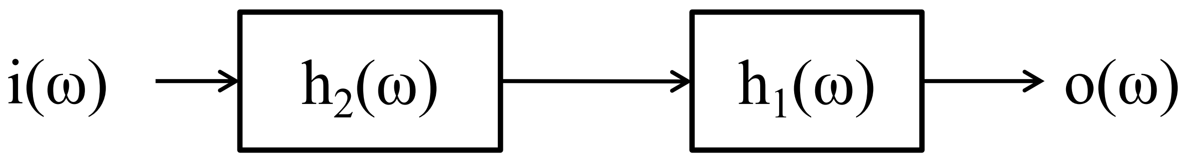
- with $$h(\omega)= h_1(\omega)h_2(\omega)$$

Linear systems in series

- Notice



- Is equivalent to



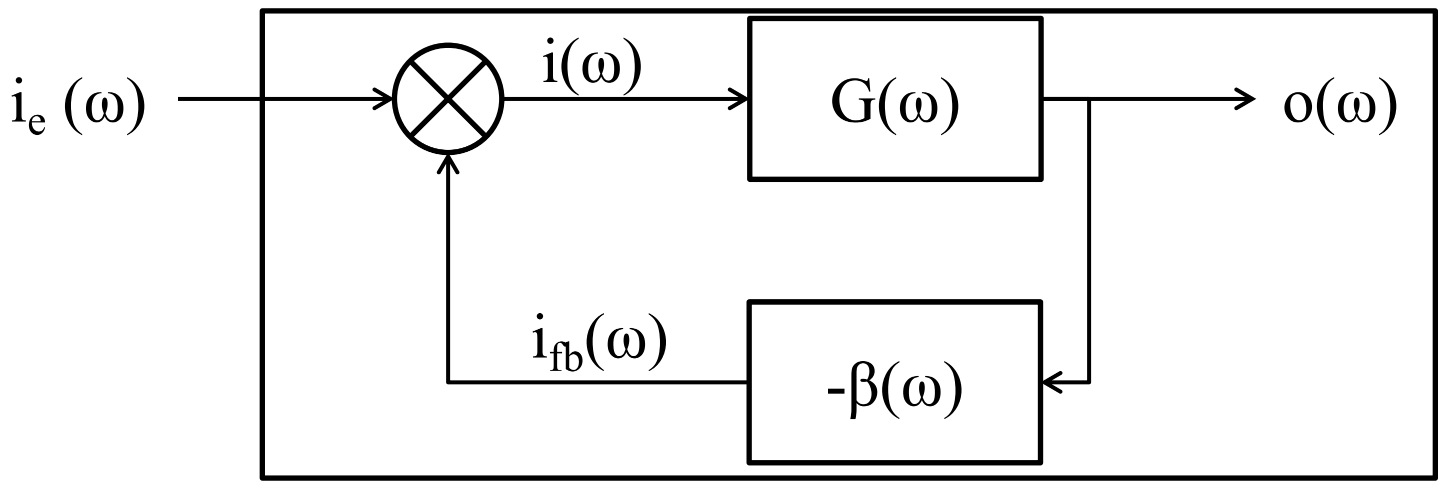
- As product is commutative

$$h_2(\omega)h_1(\omega) = h_2(\omega)h_1(\omega)$$

- Thus the order of the systems within the series is irrelevant
- Warning: physical interchanging two systems may alter their respective frequency response

A remarkable example: the feedback loop

- The output of a system is fed back and added to external input via another linear system:



- Deriving the input output relations of the full system $i_e \rightarrow o$

$$o(\omega) = G(\omega)i(\omega) = G(\omega)\left[i_e(\omega) + i_{fb}(\omega)\right]$$

- But:

$$i_{fb}(\omega) = -\beta(\omega)o(\omega)$$

- Thus

$$o(\omega) = G(\omega)\left[i_e(\omega) - \beta(\omega)o(\omega)\right]$$