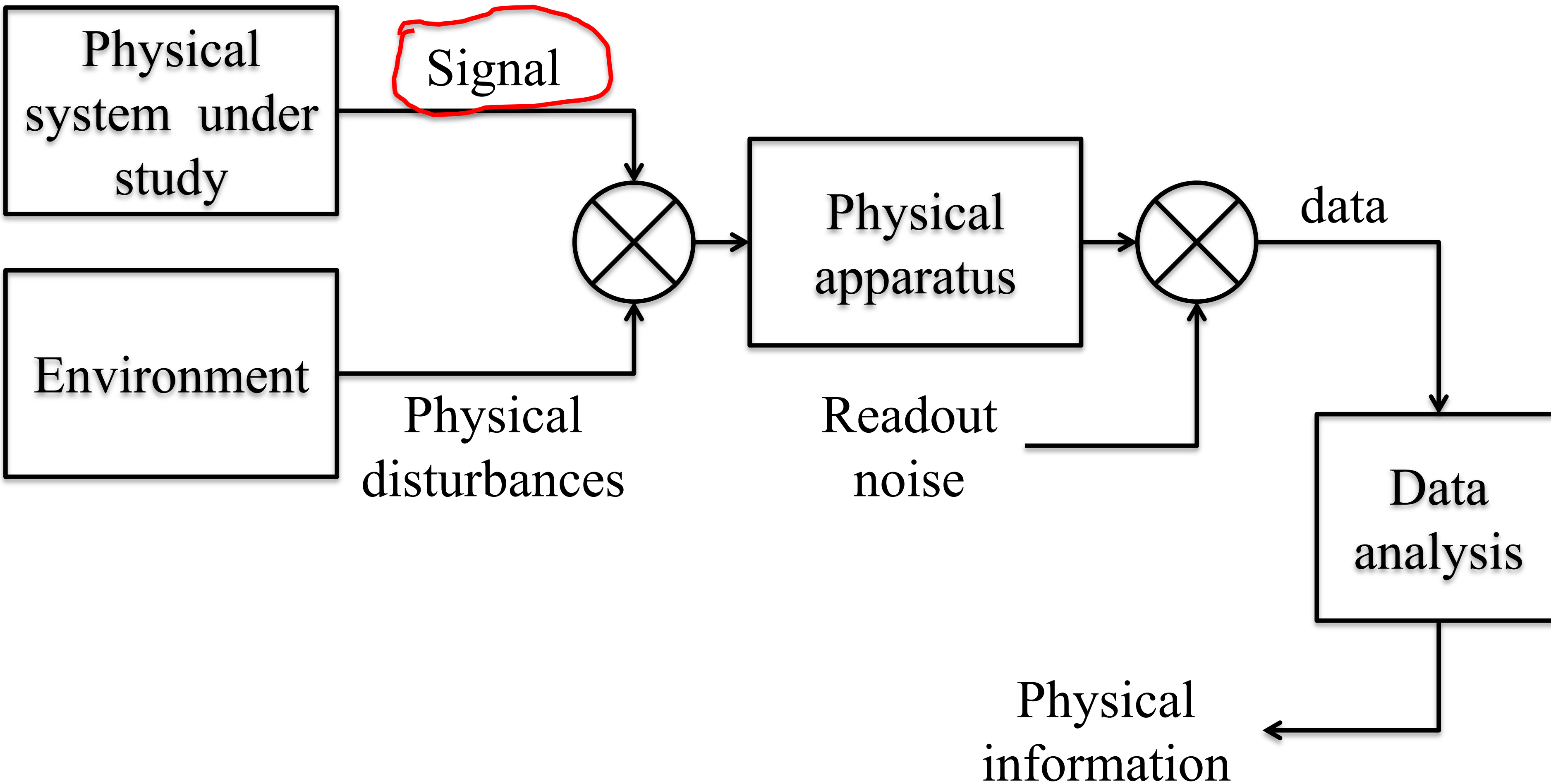
# Experimental Methods
# Lecture 3

September 24th, 2020

# My personal concept for a physical experiment



# Signals

- A measurable quantity that depends on one or more measurable parameters

$$s\left(t, \theta_1, \theta_2 .......\right)$$

Measured quantities

# Fourier Transform ( a quick primer)

- if

$$\int_{-\infty}^{\infty} |s(t)| dt < \infty$$

- (always true for real signals)
- (BTW: find some examples of "mathematical" signals for which this is not true)
- Then: $$s(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt \quad \text{and} \quad s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s(\omega) e^{i\omega t} d\omega$$
- By the way, s(ω) is a signal!
- Very important transformation. Reasons will be clear later

| $f(t)$ | $f(\omega)$ | $\int_{-\infty}^{\infty} \lvert f(t) \rvert^2 dt$ | $\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert f(\omega) \rvert^2 d\omega$ |
|---|---|---|---|
| $e^{-\frac{t^2}{2\Delta T^2}}$ | $e^{-\frac{1}{2}\Delta T^2 \omega^2}\sqrt{2\pi}\,\Delta T$ | $\sqrt{\pi}\,\Delta T$ | $\sqrt{\pi}\,\Delta T$ |
| $\delta[t]$ | $1$ | $\infty$ | $\infty$ |
| $\frac{\mathrm{UnitBox}\left[\frac{t}{\Delta T}\right]}{\Delta T}$ | $\mathrm{Sinc}\left[\frac{\Delta T\,\omega}{2}\right]$ | $\frac{1}{\Delta T}$ | $\frac{1}{\Delta T}$ |
| $\frac{e^{-\frac{\mathrm{Abs}[t]}{\Delta T}}}{\Delta T}$ | $\frac{2}{1+\Delta T^2\omega^2}$ | $\frac{1}{\Delta T}$ | $\frac{1}{\Delta T}$ |
| $\mathrm{Sin}[t\,\omega o]$ | $-i\pi\left(\delta[-\omega+\omega o]-\delta[\omega+\omega o]\right)$ | $\infty$ | $\infty$ |
| $\mathrm{Cos}[t\,\omega o]$ | $\pi\left(\delta[-\omega+\omega o]+\delta[\omega+\omega o]\right)$ | $\infty$ | $\infty$ |

## ***Fourier transforms distribute information in a complemetary way***

*Signal energy density*

Assume we have a signal s(t). Let's define its "energy density" in the time domain as

$$f[t] = \frac{|s[t]|^2}{\int_{-\infty}^{\infty} |s[t]|^2 \, dt}$$

This energy (that has nothing to do with physical energy) has the same properties of a probability density:

$$f[t] \geq 0 \qquad \int_{-\infty}^{\infty} f[t] \, dt = 1$$

***Fourier transforms distribute information in a complemetary way***

One can define the signal “barycenter”

$$\bar{t}=\int_{-\infty}^{\infty} t f[t]\, dt$$

By shifting the time origin

$$t\to t-\bar{t} \qquad s[t]\to s[t-\bar{t}]$$

one can always center the signal at zero.

We can now define a signal width for the zero-centred signal

$$\Delta t=\sqrt{\int_{-\infty}^{\infty} t^2 f[t]\, dt}$$

***Fourier transforms distribute information in a complemetary way***

Same can be done in the frequency domain if s[$\omega$] is the signal Fourier transform. The frequency domain energy density is

$$f[\omega] = \frac{|s[\omega]|^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty}|s[\omega]|^2\, d\omega}$$

As |s[$\omega$]| is an even function of frequency

$$\overline{\omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega f[\omega]\, d\omega = 0$$

Thus the width is

$$\Delta\omega = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2 f[\omega]\, d\omega}$$

## *Principle of uncertainty*

There is a minimal value for the product $\Delta\omega\Delta t$

$$\Delta\omega\Delta t \geq 1/2$$

So that function that are short in time are “wide band” and viceversa.

This is called the uncertainty principle and its a key concept in harmonic analysis

## *Principle of uncertainty*

The demonstration of the uncertainty principle is tedious but straightforward. First expand the definition of $\Delta\omega$

$$\Delta\omega^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2 f[\omega]\, d\omega = \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|\omega s[\omega]|^2 d\omega}{\frac{1}{2\pi}\int_{-\infty}^{\infty}|s[\omega]|^2 d\omega}$$

Using Parseval relations:

$$\frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|\omega s[\omega]|^2 d\omega}{\frac{1}{2\pi}\int_{-\infty}^{\infty}|s[\omega]|^2 d\omega} = \frac{\int_{-\infty}^{\infty}\left(\frac{ds[t]}{dt}\right)^2 dt}{\int_{-\infty}^{\infty}s[t]^2 dt}$$

## *Principle of uncertainty*

Moving to product of widths

$$\Delta\omega^2\,\Delta t^2 = \frac{\int_{-\infty}^{\infty}\left(\frac{ds[t]}{dt}\right)^2 dt}{\int_{-\infty}^{\infty} s[t]^2\, dt}\;\frac{\int_{-\infty}^{\infty}(t\, s[t])^2\, dt}{\int_{-\infty}^{\infty} s[t]^2\, dt}$$

Now you may use the Cauchy–Schwarz inequality

$$\int_a^b h[x]^2\, dx \int_a^b g[x]^2\, dx \geq \left(\int_a^b h[x]\, g[x]\, dx\right)^2$$

So that

$$\frac{\int_{-\infty}^{\infty}\left(\frac{ds[t]}{dt}\right)^2 dt}{\int_{-\infty}^{\infty} s[t]^2\, dt}\;\frac{\int_{-\infty}^{\infty}(t\, s[t])^2\, dt}{\int_{-\infty}^{\infty} s[t]^2\, dt} \geq \frac{\left(\int_{-\infty}^{\infty} t s[t]\,\frac{ds[t]}{dt}\, dt\right)^2}{\left(\int_{-\infty}^{\infty} s[t]^2\, dt\right)^2}$$

## *Principle of uncertainty*

With the following steps we get

$$\Delta\omega^2\,\Delta t^2 \geq \frac{\left(\int_{-\infty}^{\infty} t\,s[t]\,\frac{ds[t]}{dt}\,dt\right)^2}{\left(\int_{-\infty}^{\infty} s[t]^2\,dt\right)^2} =$$

$$\frac{\left(\cancel{\frac{t s^2[t]}{2}\Big|_{-\infty}^{\infty}} - \frac{1}{2}\int_{-\infty}^{\infty} s[t]^2\,dt\right)^2}{\left(\int_{-\infty}^{\infty} s[t]^2\,dt\right)^2} = \frac{1}{4}$$

Then finally

$$\Delta\omega\Delta t \geq \frac{1}{2}$$

Notice that

$$\frac{\frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega s[\omega]|^2\,d\omega}{\frac{1}{2\pi}\int_{-\infty}^{\infty} |s[\omega]|^2\,d\omega} = \frac{\int_{-\infty}^{\infty}\left(\frac{ds[t]}{dt}\right)^2 dt}{\int_{-\infty}^{\infty} s[t]^2\,dt}$$

Indicates that the frequency width originates from fast variation of the time signal. Thus short signals have large frequency width

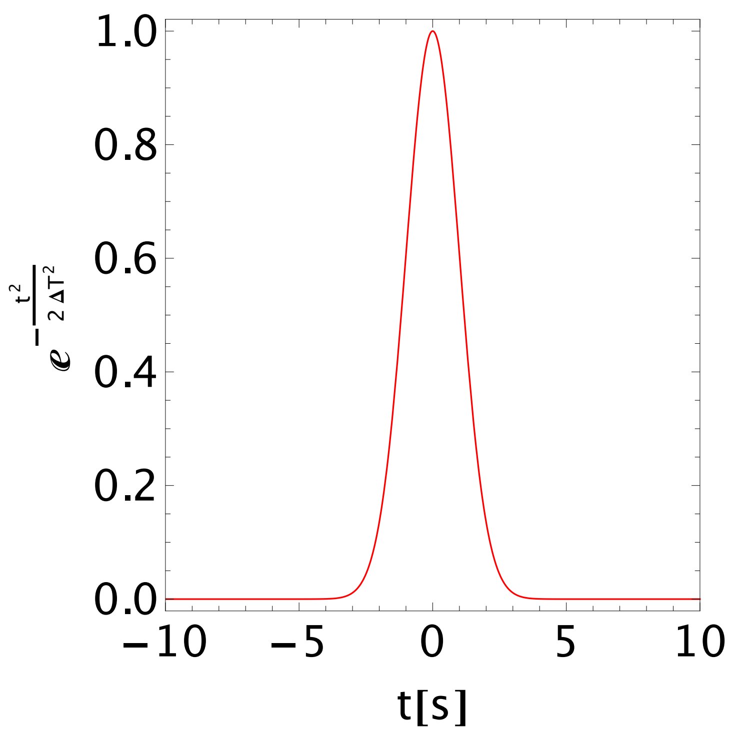
## *The minimum case: Gaussian impulse*

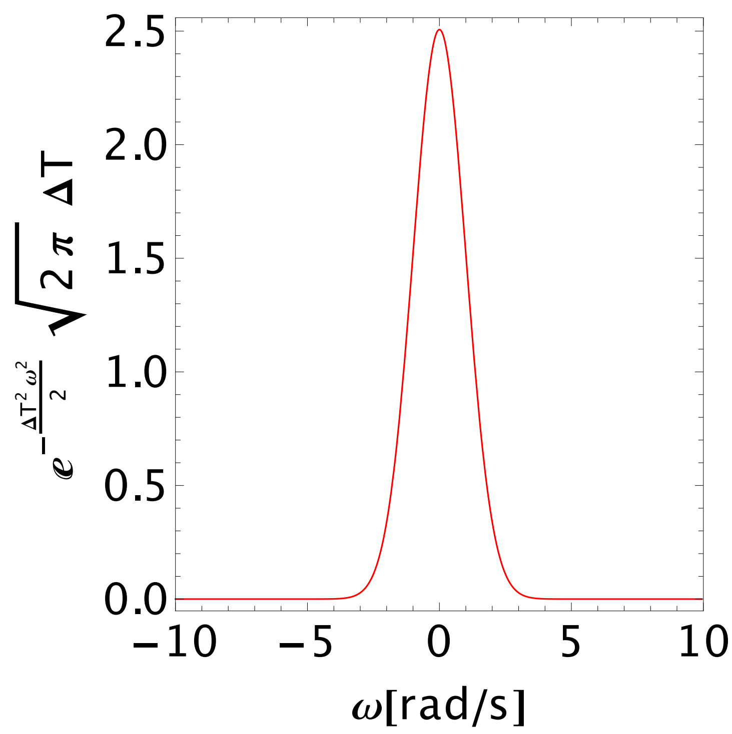
The impulse and its Fourier Transform

$$s[t] = e^{-\frac{t^2}{2\,\Delta T^2}} \qquad s[\omega] = \sqrt{2\,\pi}\,\Delta T\,e^{-\frac{\Delta T^2\,\omega^2}{2}}$$

A plot with ΔT=1 s

Out[68]=



# *The minimum case: Gaussian impulse*

Energy density in the time domain

$$f[t] = \frac{\left(e^{-\frac{t^2}{2\,\Delta T^2}}\right)^2}{\int_{-\infty}^{\infty}\left(e^{-\frac{t^2}{2\,\Delta T^2}}\right)^2 dt} = \frac{e^{-\frac{t^2}{\Delta T^2}}}{\int_{-\infty}^{\infty} e^{-\frac{t^2}{\Delta T^2}}\, dt}$$

Consider that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\,\pi}$$

Then

$$f[t] = \frac{e^{-\frac{t^2}{\Delta T^2}}}{\sqrt{\pi}\,\Delta T}$$

A Gaussian with "standard deviation" $\Delta T / \sqrt{2}$. Then the width is

$$\Delta t = \sqrt{\int_{-\infty}^{\infty} \frac{t^2\, e^{-\frac{t^2}{\Delta T^2}}}{\sqrt{\pi}\,\Delta T}\, dt} = \Delta T / \sqrt{2}$$

# *The minimum case: Gaussian impulse*

In the frequency domain

$$\mathrm{f}[\omega] = \frac{\left(\sqrt{2\pi}\,\Delta T\, e^{-\frac{\Delta T^2 \omega^2}{2}}\right)^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sqrt{2\pi}\,\Delta T\, e^{-\frac{\Delta T^2 \omega^2}{2}}\right)^2 d\omega} = \frac{e^{-\Delta T^2 \omega^2}}{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\Delta T^2 \omega^2}\, d\omega}$$

Using the same rule as above

$$\mathrm{f}[\omega] = \sqrt{2\pi}\,\sqrt{2}\,\Delta T\, e^{-\Delta T^2 \omega^2}$$

Then the width is

$$\Delta\omega = \sqrt{\frac{\sqrt{2\pi}\,\sqrt{2}\,\Delta T}{2\pi}\int_{-\infty}^{\infty}\omega^2\, e^{-\Delta T^2 \omega^2}\, d\omega} =$$

$$\sqrt{\frac{1}{\sqrt{2\pi\left(\frac{1}{\sqrt{2}\,\Delta T}\right)^2}}\int_{-\infty}^{\infty}\omega^2\, e^{-\frac{\omega^2}{2\left(\frac{1}{\sqrt{2}\,\Delta T}\right)^2}}\, d\omega} = 1\Big/\left(\Delta T\sqrt{2}\right)$$

# *The minimum case: Gaussian impulse*

In conclusion

$$\Delta t = \frac{\Delta T}{\sqrt{2}} \qquad \Delta\omega = \frac{1}{\sqrt{2}\ \Delta T}$$

and then

$$\Delta t \Delta\omega = \frac{1}{2}$$

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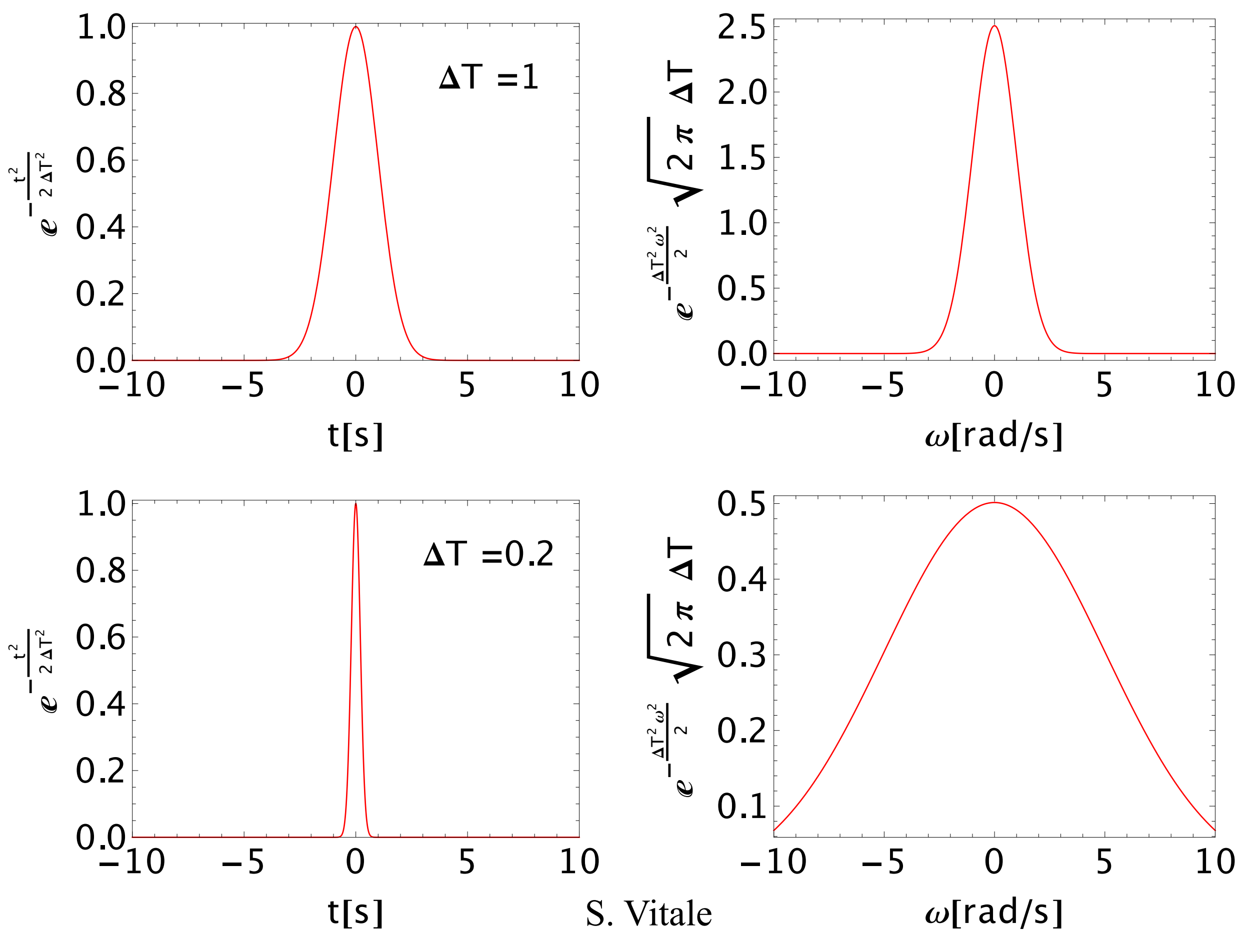


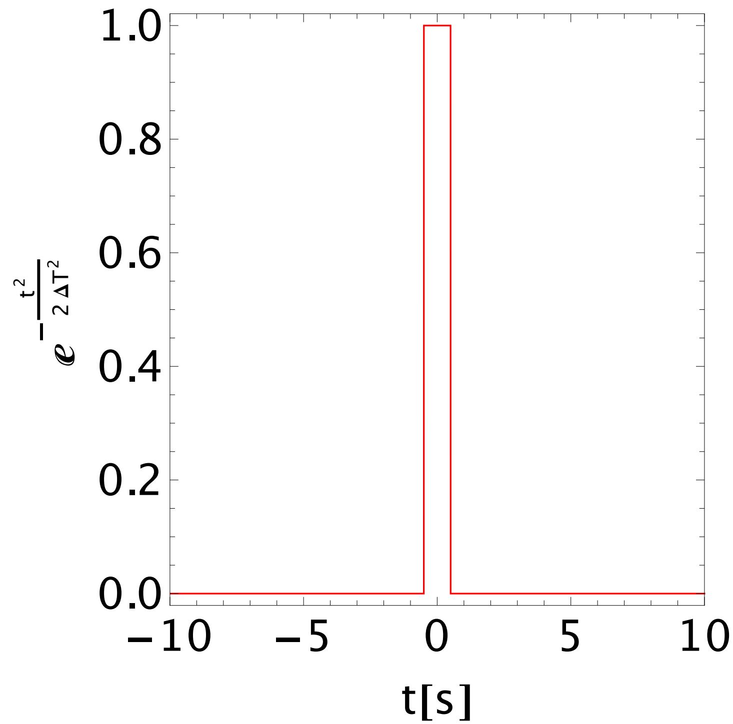
# *A "maximal" case : the box*

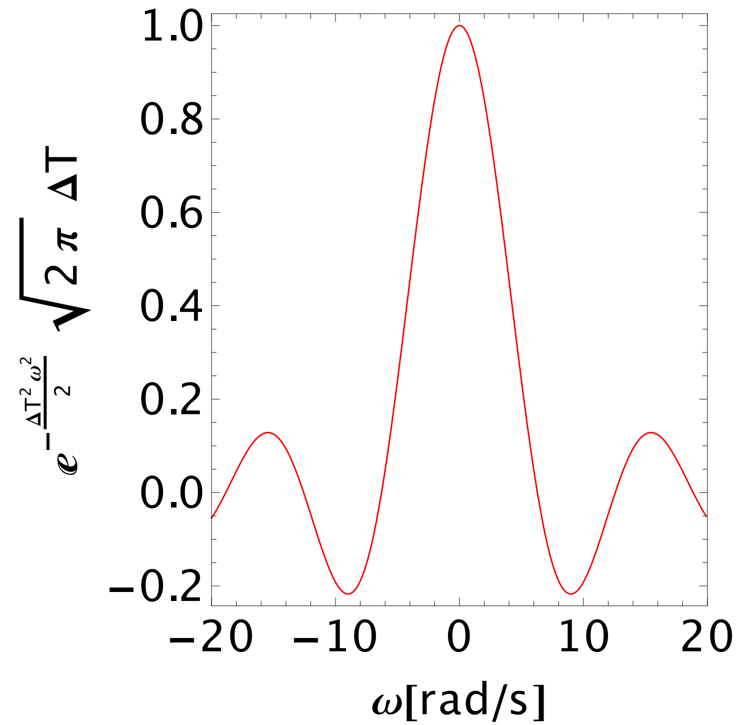
## The box and its Fourier Transform

$$\frac{\Theta\left[t+\frac{\Delta T}{2}\right]-\Theta\left[t-\frac{\Delta T}{2}\right]}{\Delta T} \qquad \frac{2\,\mathrm{Sin}\left[\frac{\Delta T\,\omega}{2}\right]}{\Delta T\,\omega}$$

A plot with ΔT=1 s

Out[97]=



## *A "maximal" case : the box*

Energy density in the time domain

$$f[t] = \frac{\Theta\left[t + \frac{\Delta T}{2}\right] - \Theta\left[t - \frac{\Delta T}{2}\right]}{\int_{-\frac{\Delta T}{2}}^{\frac{\Delta T}{2}} dt} = \frac{\Theta\left[t + \frac{\Delta T}{2}\right] - \Theta\left[t - \frac{\Delta T}{2}\right]}{\Delta T}$$

The width

$$\Delta t = \sqrt{\frac{1}{\Delta T} \int_{-\frac{\Delta T}{2}}^{\frac{\Delta T}{2}} t^2 \, dt} = \Delta T \Big/ \sqrt{12}$$

In the frequency domain

$$f[\omega] = \frac{\left(\frac{2\,\mathrm{Sin}\left[\frac{\Delta T\,\omega}{2}\right]}{\Delta T\,\omega}\right)^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\,\mathrm{Sin}\left[\frac{\Delta T\,\omega}{2}\right]}{\Delta T\,\omega}\right)^2 d\omega}$$

Then the width is

$$\Delta\omega = \sqrt{\frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2\left(\frac{2\,\mathrm{Sin}\left[\frac{\Delta T\,\omega}{2}\right]}{\Delta T\,\omega}\right)^2 d\omega}{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\,\mathrm{Sin}\left[\frac{\Delta T\,\omega}{2}\right]}{\Delta T\,\omega}\right)^2 d\omega}} = \infty$$

# Signal duration

- Suppose the signal has duration T. That is:

$$s\left[t\right]=0 \qquad \left|t\right|\geq T/2$$

- It is easy to show that

$$\Delta t=\sqrt{\int_{-T/2}^{T/2} t^2 s\left[t\right]^2 dt \Bigg/ \int_{-T/2}^{T/2} s\left[t\right]^2 dt} \leq \sqrt{\int_{-T/2}^{T/2}\left(\frac{T}{2}\right)^2 s\left[t\right]^2 dt \Bigg/ \int_{-T}^{T} s\left[t\right]^2 dt}=T/2$$

- And thus

$$\frac{1}{2}\leq\Delta\omega\Delta t\leq\Delta\omega\frac{T}{2}$$

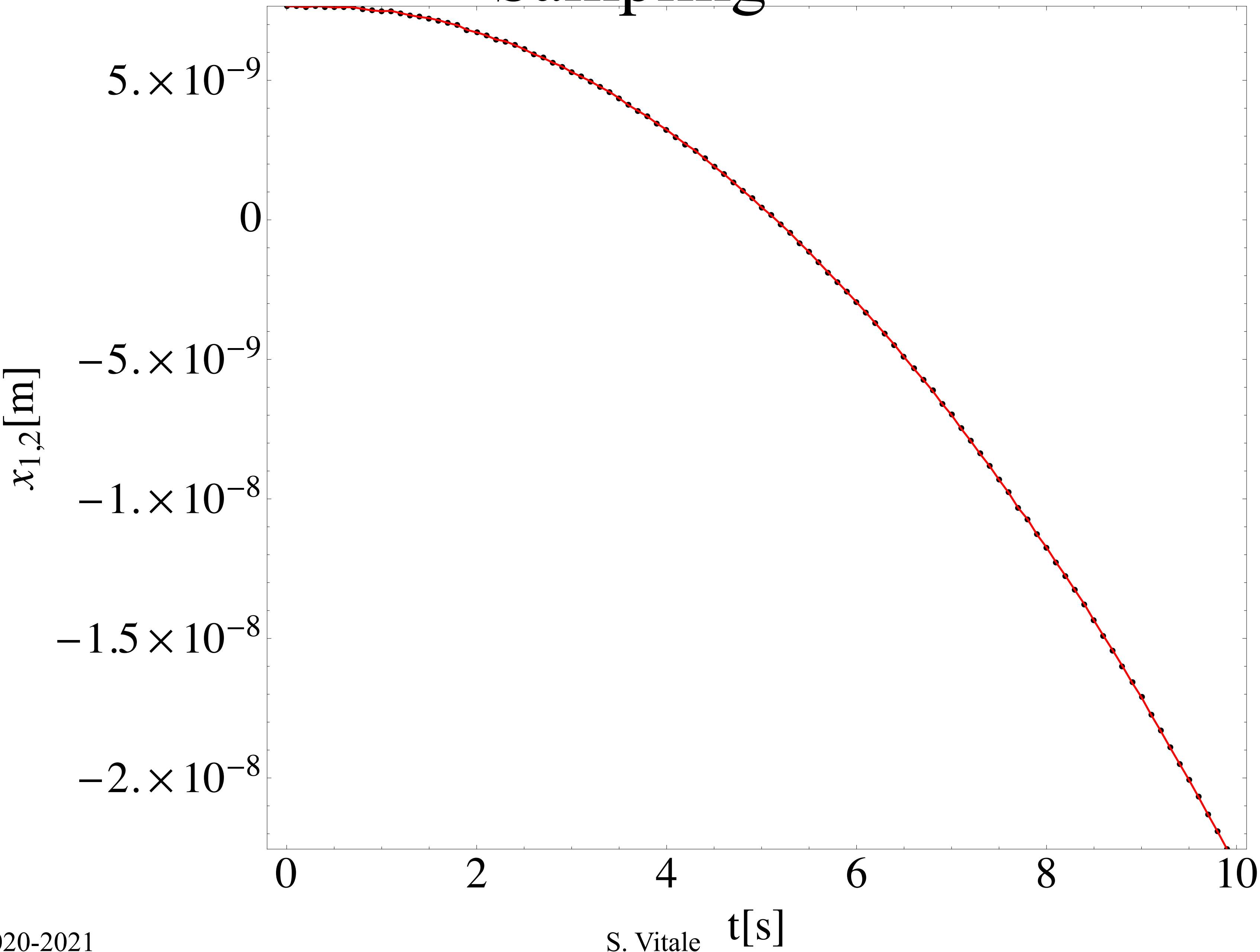
$$\Delta\omega\geq 1/T$$

- A quite relevant result

# A summary table

| $f(t)$ | $f(\omega)$ | $\Delta t^2$ | $\Delta\omega^2$ | $\Delta t^2\Delta\omega^2$ |
|---|---|---|---|---|
| $e^{-\frac{t^2}{2\,\Delta T^2}}$ | $e^{-\frac{1}{2}\Delta T^2\omega^2}\sqrt{2\pi}\,\Delta T$ | $\frac{\Delta T^2}{2}$ | $\frac{1}{2\,\Delta T^2}$ | $\frac{1}{4}$ |
| DiracDelta[t] | 1 | 0 | $\infty$ | $\frac{1}{4}$ |
| $\frac{\text{UnitBox}\left[\frac{t}{\Delta T}\right]}{\Delta T}$ | $\text{Sinc}\left[\frac{\Delta T\,\omega}{2}\right]$ | $\frac{\Delta T^2}{12}$ | $\infty$ | $\infty$ |
| $\frac{e^{-\frac{\text{Abs}[t]}{\Delta T}}}{\Delta T}$ | $\frac{2}{1+\Delta T^2\omega^2}$ | $\frac{\Delta T^2}{2}$ | $\frac{1}{\Delta T^2}$ | $\frac{1}{2}$ |
| $\text{Sin}[t\,\omega o]$ | $-i\pi\,\text{DiracDelta}[-\omega+\omega o]+i\pi\,\text{DiracDelta}[\omega+\omega o]$ | $\infty$ | $\infty$ | $\infty$ |
| $\text{Cos}[t\,\omega o]$ | $\pi\,\text{DiracDelta}[-\omega+\omega o]+\pi\,\text{DiracDelta}[\omega+\omega o]$ | $\infty$ | $\infty$ | $\infty$ |

# Sampling



$5. \times 10^{-9}$
0
$-5. \times 10^{-9}$
$-1. \times 10^{-8}$
$-1.5 \times 10^{-8}$
$-2. \times 10^{-8}$
$x_{1,2}$[m]
0
2
4
6
8
10
t[s]

A.V. **OPPENHEIMER**, R.W. **SHAFER**, “Elaborazione Numerica dei Segnali”, Franco Angeli Editore, 1996

A.V. Oppenheim and R.W. Shafer. In: *Discrete-time Signal Processing* Prentice Hall, Englewood Cliffs, NJ (1989),

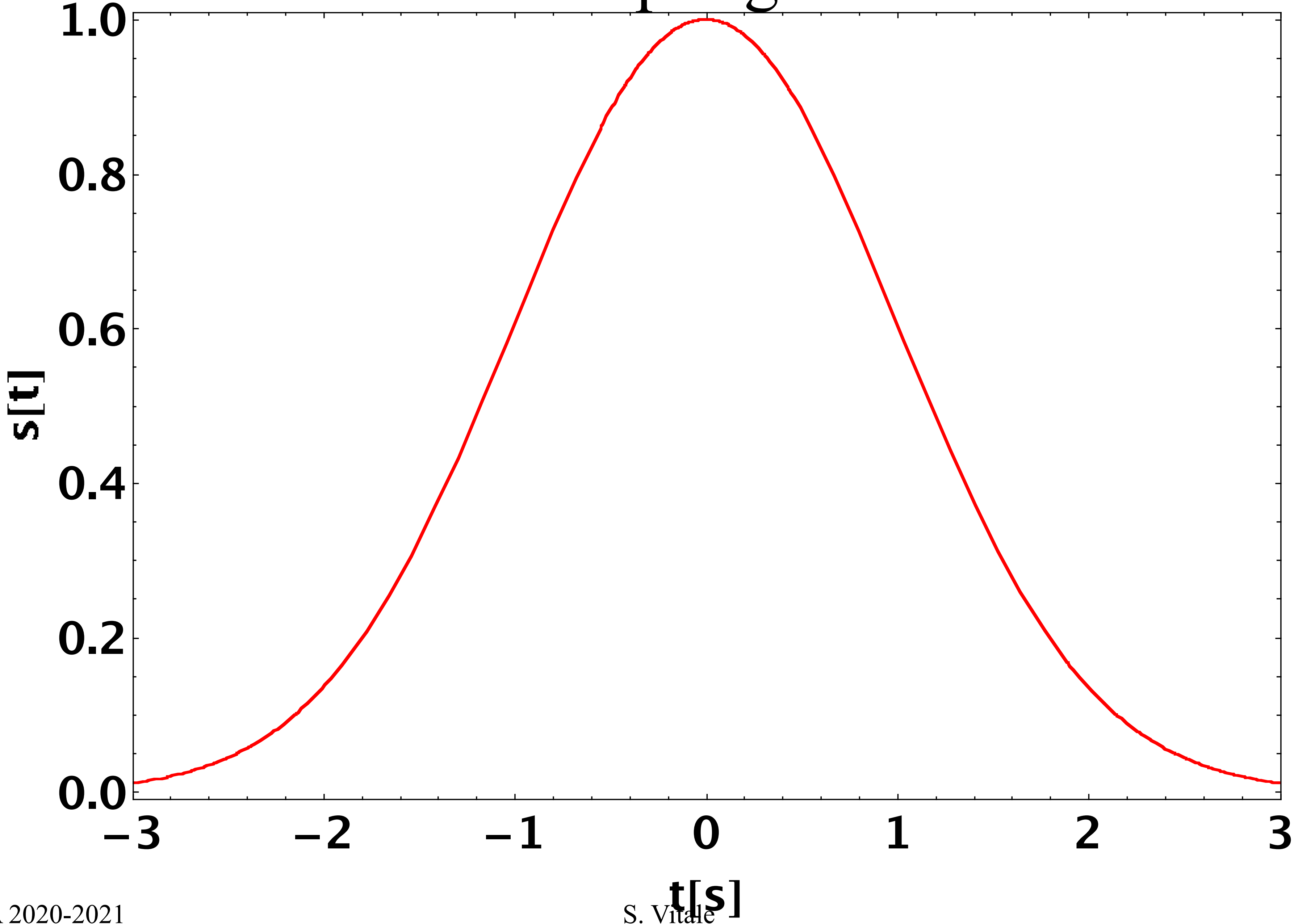
# Sampling

- Signal are mostly recorded as a discrete series of data
- The value of a continuous signal at a given time $t_n$, $s(t_n)$, is called the sample of the signal at time $t_n$
- The series $s[n]=s(t_n)$ is commonly referred to as the sampled signal.
- The case where samples are taken at equal time intervals, $t_n=n\,T$, is often referred to as an evenly sampled signal
- In this case T is called the sampling time, 1/T the sampling frequency and $2\pi/T$ the sampling angular frequency

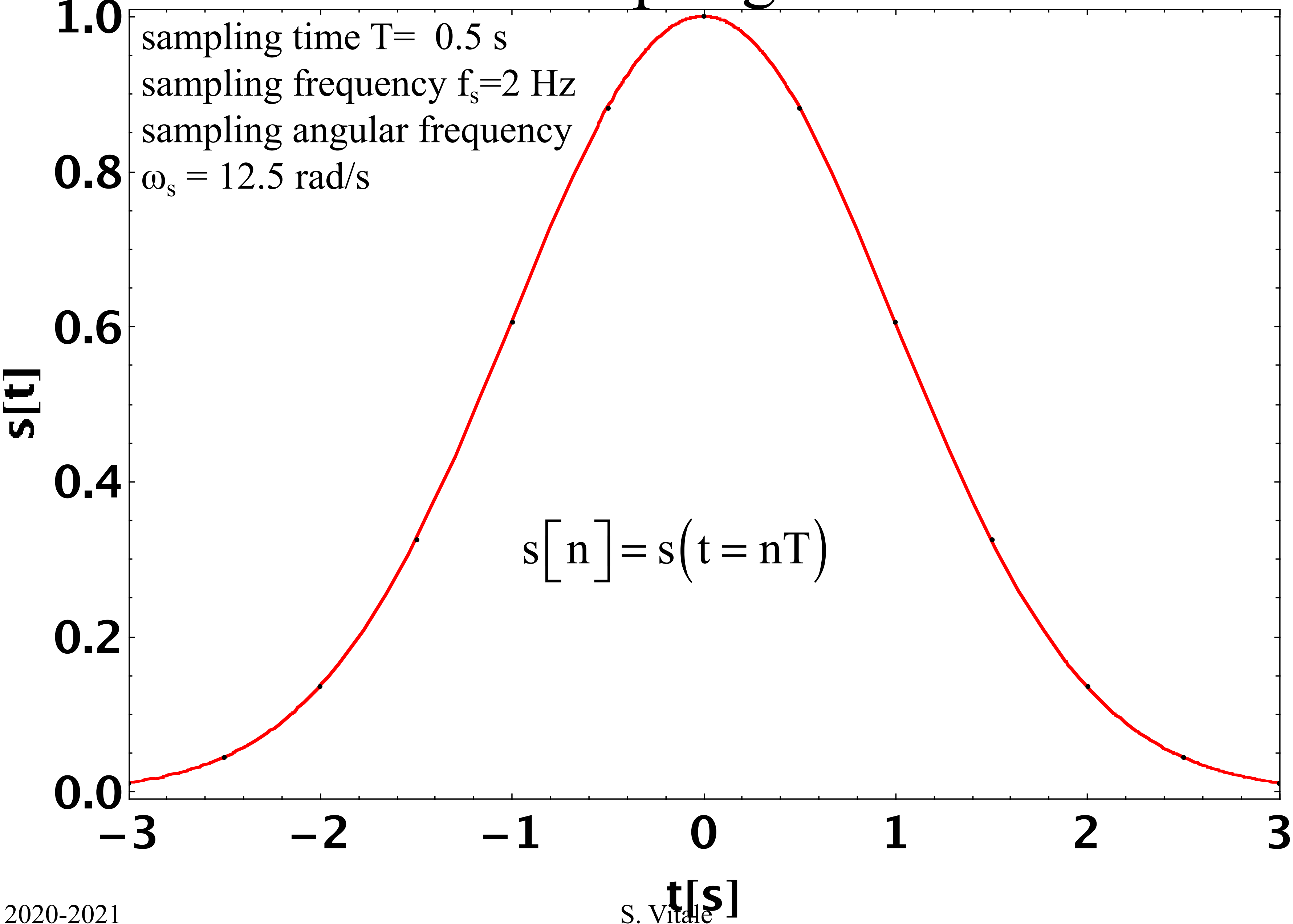
# Reconstructing a continuous signal from its samples

- Is it possible to go back from samples to the original continuous signal?
- That is, is a lossless interpolation method available?
- If yes, under which conditions it gives a faithful reconstruction of the continuous signal?

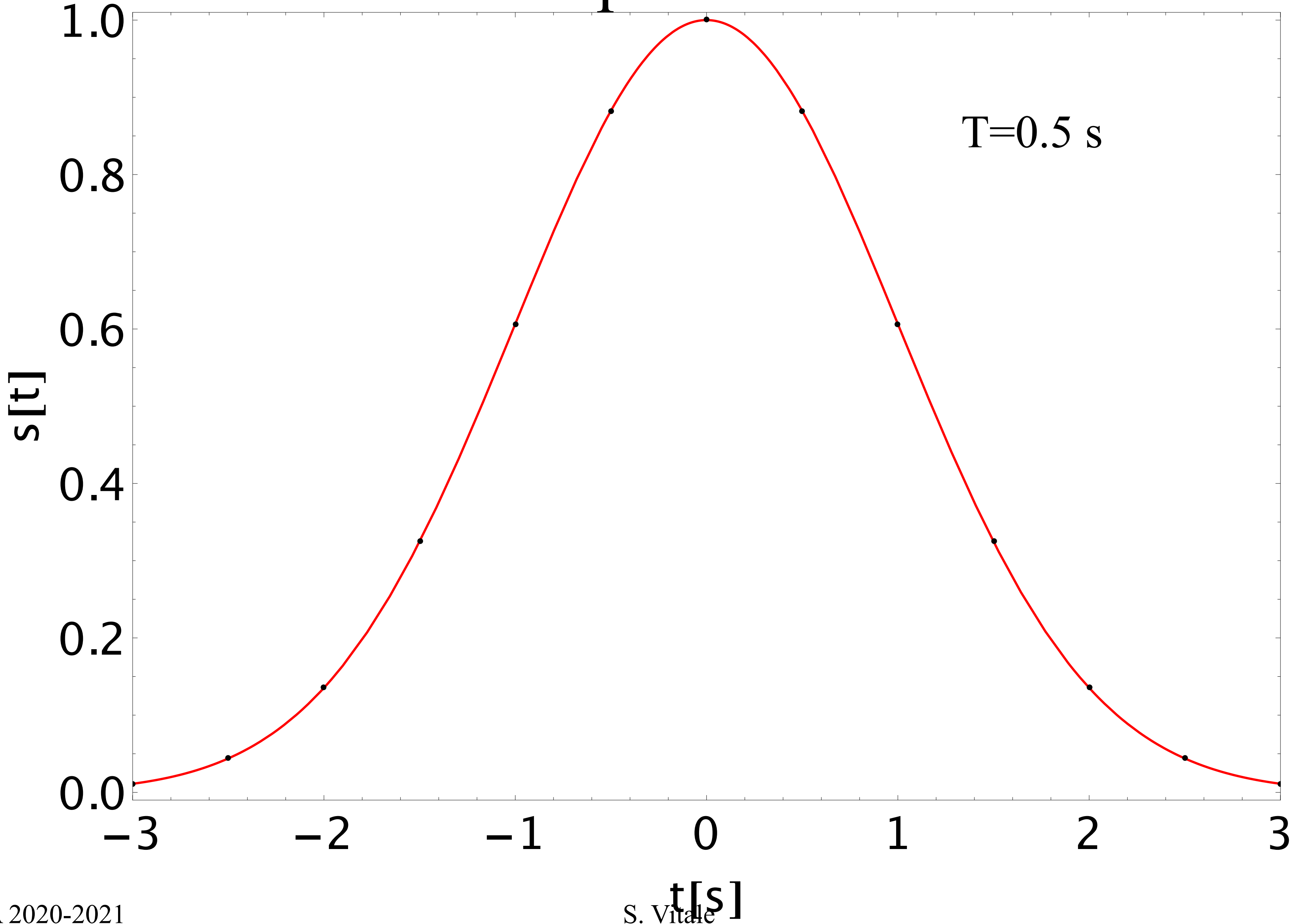
# Sampling



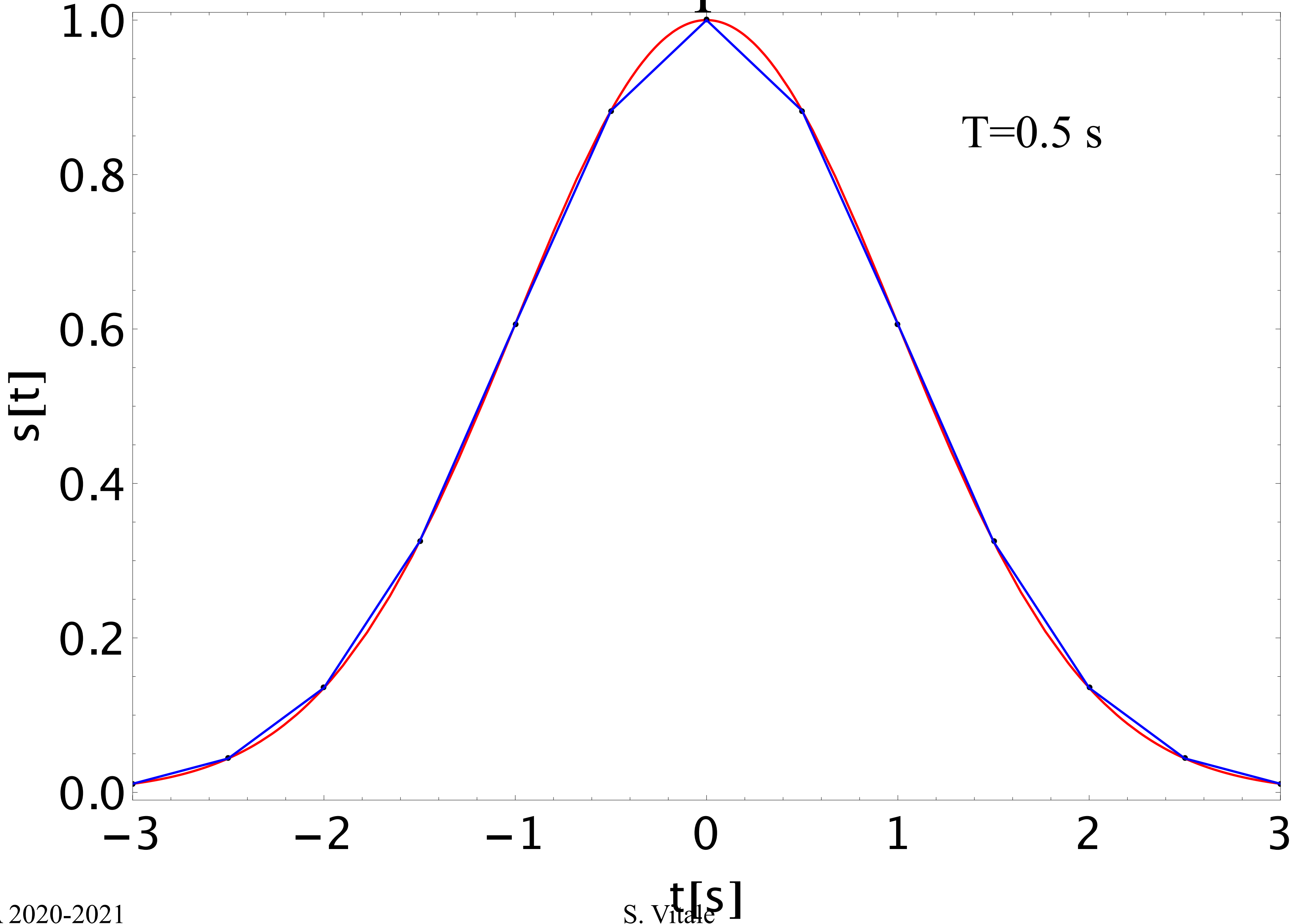
# Sampling


sampling time T= 0.5 s
sampling frequency $f_s$=2 Hz
sampling angular frequency
$\omega_s$ = 12.5 rad/s
$s[n] = s(t = nT)$
1.0
0.8
0.6
0.4
0.2
0.0
s[t]
−3
−2
−1
0
1
2
3
t[s]

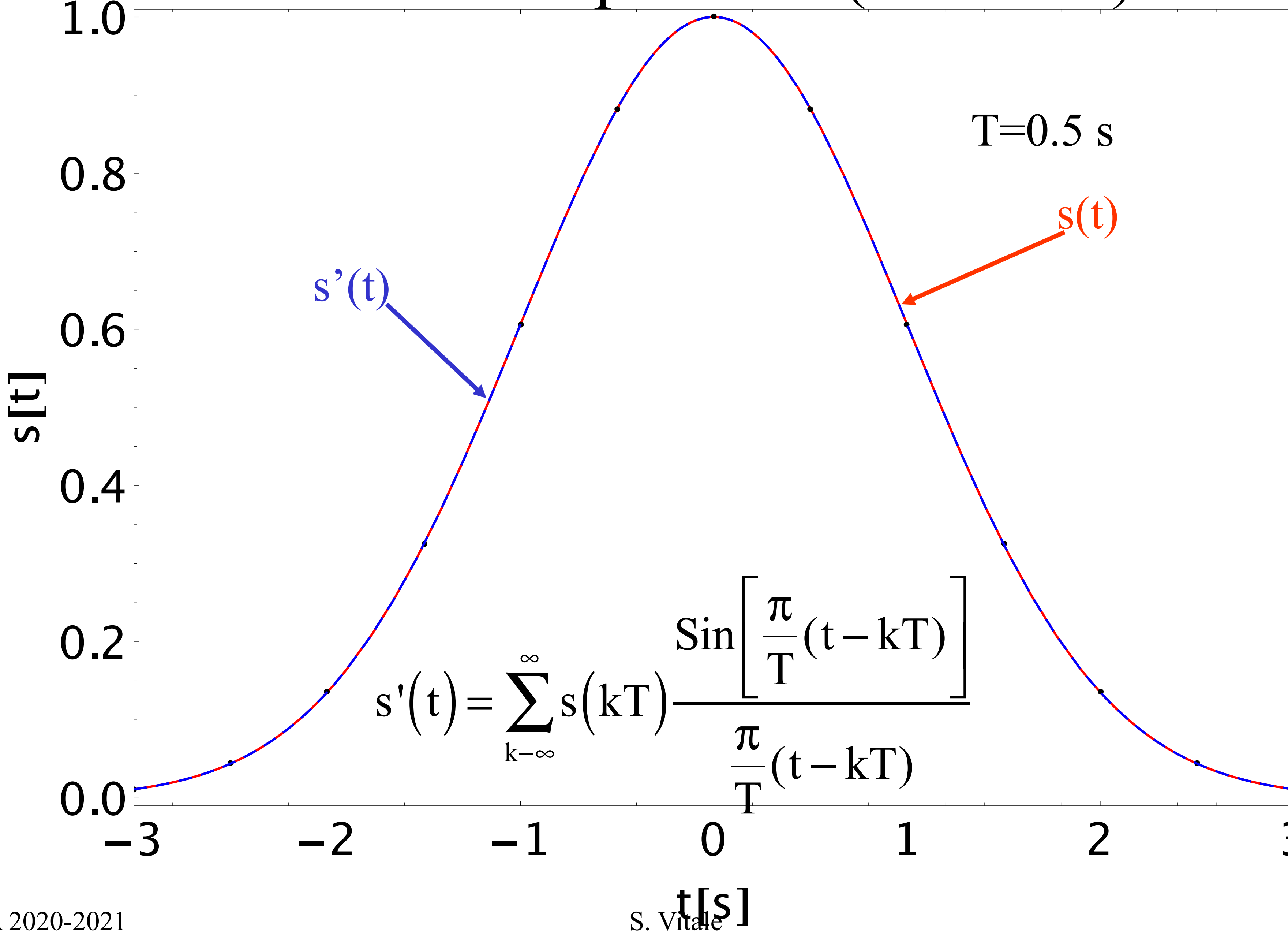
# Interpolation



# Linear interpolation

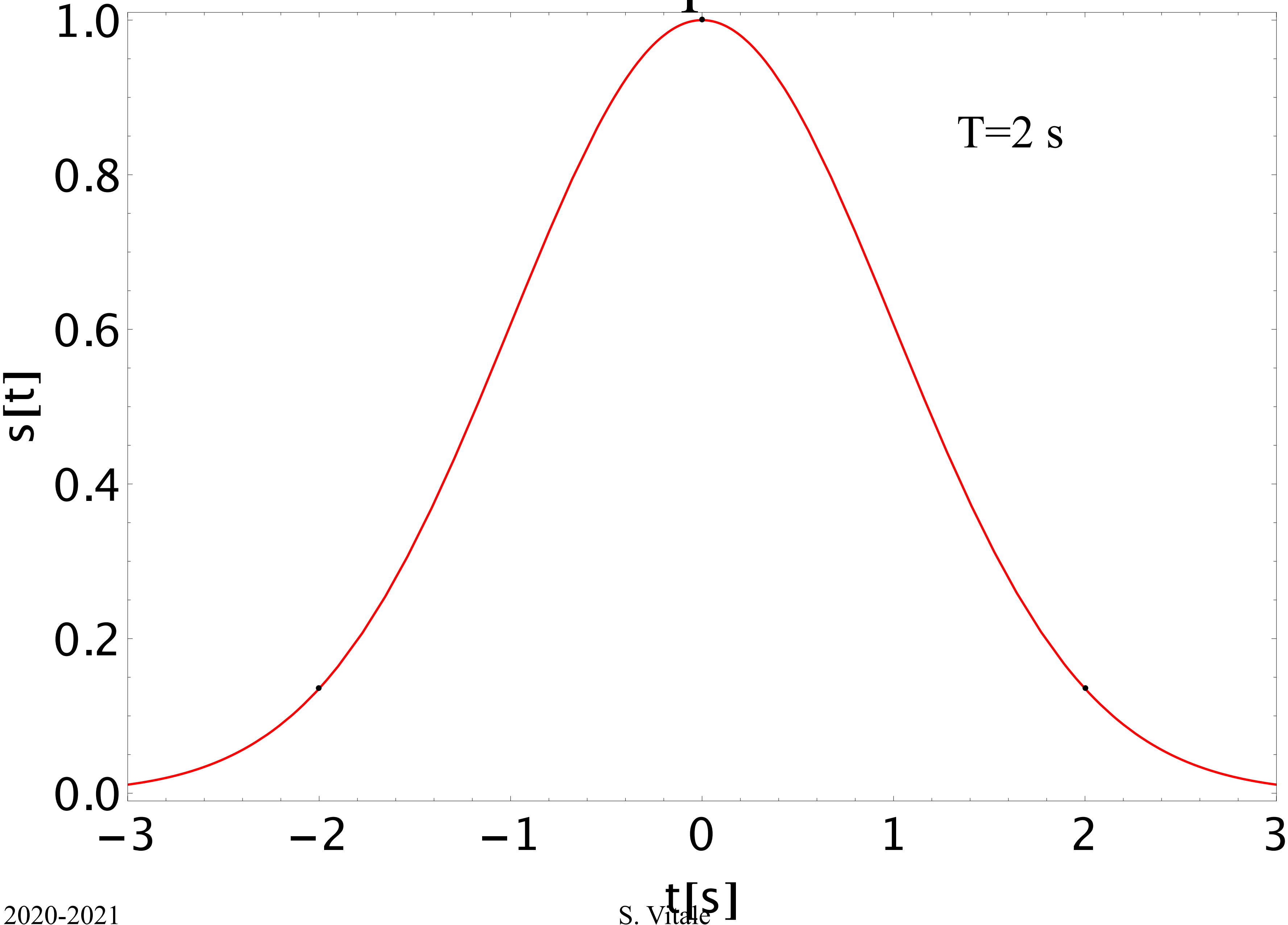


# A smarter interpolation (Shannon)

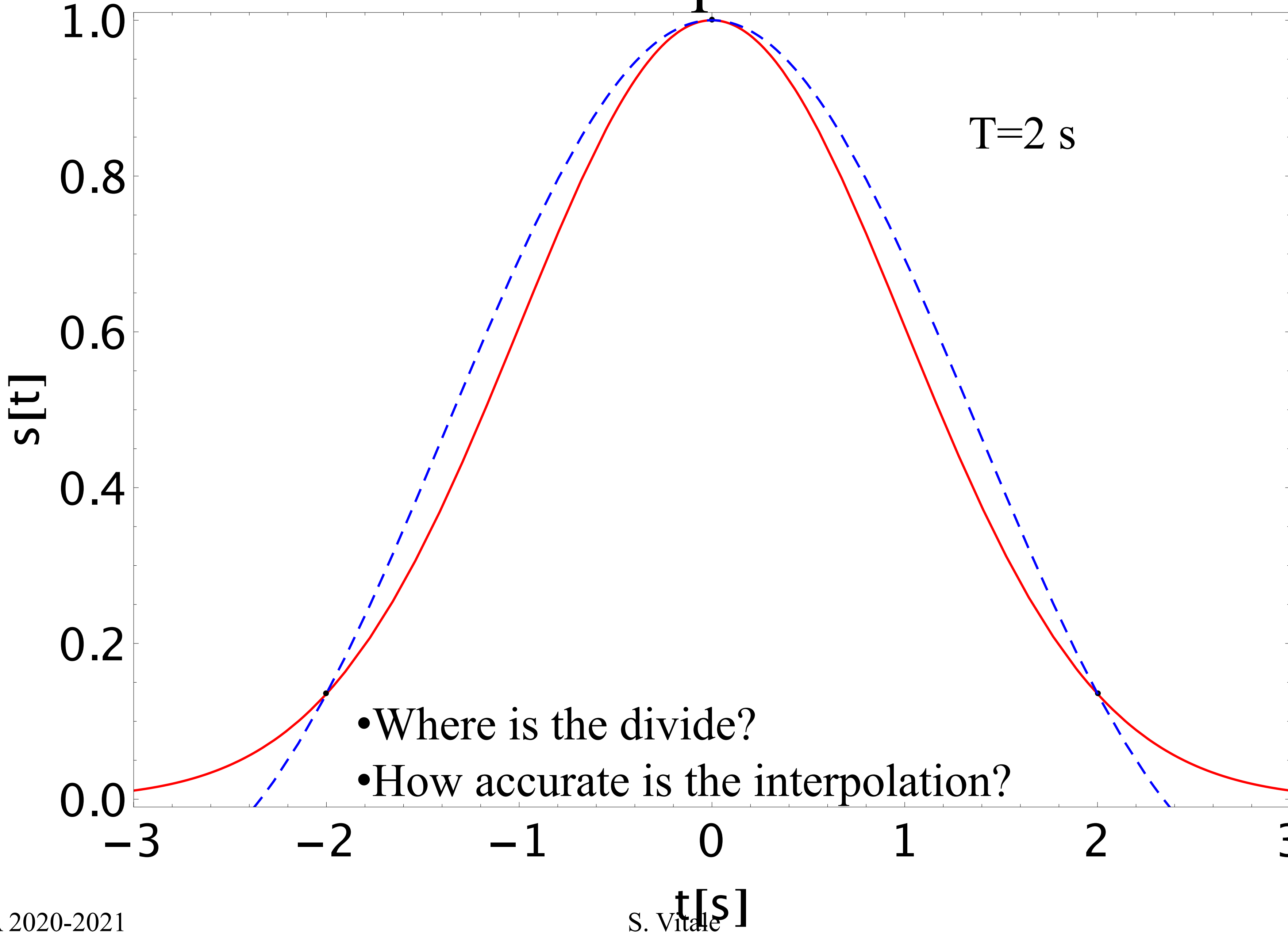



$$s'(t)=\sum_{k-\infty}^{\infty} s(kT)\frac{\mathrm{Sin}\left[\frac{\pi}{T}(t-kT)\right]}{\frac{\pi}{T}(t-kT)}$$

# Too few points



# Too few points



## *Sampling theorem*

A few properties of the reconstructed signal

$$s'[t] = \sum_{k=-\infty}^{\infty} s[k\,T]\,\frac{\mathrm{Sin}\left[\frac{\pi}{T}\,(t-k\,T)\right]}{\frac{\pi}{T}\,(t-k\,T)}$$

Samples

$$s'[n\,T] = \sum_{k=-\infty}^{\infty} s[k\,T]\,\frac{\mathrm{Sin}[\pi\,(n-k)]}{\pi\,(n-k\,)}$$

As

$$\frac{\mathrm{Sin}[\pi\,(n-k)]}{\pi\,(n-k\,)} = \delta_{k,n}$$

s and s’ have same samples

## *Sampling theorem*



Fourier transform

$$s'[\omega] = \sum_{k=-\infty}^{\infty} s[k\,T] \int_{-\infty}^{\infty} \frac{\mathrm{Sin}\left[\frac{\pi}{T}\,(t-k\,T)\right]}{\frac{\pi}{T}\,(t-k\,T)}\; e^{-i\,\omega\,t}\, dt$$

But

$$\int_{-\infty}^{\infty} s[t-t_o]\, e^{-i\,\omega\,t}\, dt = \int_{-\infty}^{\infty} s[t]\, e^{-i\,\omega\,(t+t_o)}\, dt = e^{-i\,\omega\,t_o}\, s[\omega]$$

Then

$$s'[\omega] = \left(\sum_{k=-\infty}^{\infty} s[k\,T]\, e^{-i\,\omega\,k\,T}\right) \int_{-\infty}^{\infty} \frac{\mathrm{Sin}\left[\frac{\pi}{T}\,t\right]}{\frac{\pi}{T}\,t}\; e^{-i\,\omega\,t}\, dt$$

Finally consider that

$$\frac{T}{2\,\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{i\,\omega\,t}\, d\omega = \frac{\mathrm{Sin}\left[\frac{\pi}{T}\,t\right]}{\frac{\pi}{T}\,t}$$

so that, by inverse Fourier transform

$$\int_{-\infty}^{\infty} \frac{\mathrm{Sin}\left[\frac{\pi}{T}\,t\right]}{\frac{\pi}{T}\,t}\; e^{-i\,\omega\,t}\, dt = T\left(\Theta\left[\omega + \frac{\pi}{T}\right] - \Theta\left[\omega - \frac{\pi}{T}\right]\right)$$

## *Sampling theorem*

In conclusion

$$s'[\omega] = \left(\sum_{k=-\infty}^{\infty} s[k\,T]\, e^{-i\,\omega\,k\,T}\right) T\left(\Theta\left[\omega + \frac{\pi}{T}\right] - \Theta\left[\omega - \frac{\pi}{T}\right]\right)$$

We’re not done yet. Consider that the samples s[k T] may be expressed through an inverse Fourier transform as:

$$\mathrm{s}[k\,T] = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \mathrm{s}[\omega']\, e^{i\,\omega'\,k\,T}\, d\omega'$$

Thus

$$s'[\omega] = T\left(\Theta\left[\omega + \frac{\pi}{T}\right] - \Theta\left[\omega - \frac{\pi}{T}\right]\right) \times$$

$$\times \frac{1}{2\,\pi} \int_{-\infty}^{\infty} s[\omega'] \sum_{k=-\infty}^{\infty} e^{-i\,(\omega-\omega')\,k\,T}\, d\omega'$$

Now we are going to demonstrate that:

$$\sum_{k=-\infty}^{\infty} e^{-i(\omega-\omega')kT} = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \omega' + n\frac{2\pi}{T}\right)$$

Indeed

$$\Psi[\omega] = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega + n\frac{2\pi}{T}\right)$$

is a periodic function of $\omega$ with period $2\pi/T$ and thus can be expressed as a Fourier series

$$\Psi[\omega] = \sum_{k=-\infty}^{\infty} c_k\, e^{ik\omega T}$$

With $c_k$

$$c_k = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega + n\frac{2\pi}{T}\right) e^{-ik\omega T}\, d\omega = 1$$

In conclusion

$$\Psi[\omega] = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega + n\frac{2\pi}{T}\right) = \sum_{k=-\infty}^{\infty} e^{ik\omega T} = \sum_{k=-\infty}^{\infty} e^{-ik\omega T}$$

https://en.wikipedia.org/wiki/Dirac_comb

# Sampling theorem

Going back to

$$s'[\omega] = T\left(\Theta\left[\omega + \frac{\pi}{T}\right] - \Theta\left[\omega - \frac{\pi}{T}\right]\right)\times$$

$$\times \frac{1}{2\pi}\int_{-\infty}^{\infty} s[\omega'] \sum_{k=-\infty}^{\infty} e^{-i(\omega-\omega')kT} d\omega'$$

And substituting

$$s'[\omega] = T\left(\Theta\left[\omega + \frac{\pi}{T}\right] - \Theta\left[\omega - \frac{\pi}{T}\right]\right)\times$$

$$\times \frac{1}{2\pi}\int_{-\infty}^{\infty} s[\omega'] \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \omega' + n\frac{2\pi}{T}\right) d\omega'$$

we get

$$s'[\omega] = T\left(\Theta\left[\omega + \frac{\pi}{T}\right] - \Theta\left[\omega - \frac{\pi}{T}\right]\right)\left(\sum_{n=-\infty}^{\infty} \frac{1}{T} s\left[\omega + n\frac{2\pi}{T}\right]\right) =$$

$$\left(\Theta\left[\omega + \frac{\pi}{T}\right] - \Theta\left[\omega - \frac{\pi}{T}\right]\right) \sum_{n=-\infty}^{\infty} s\left[\omega + n\frac{2\pi}{T}\right]$$

# *Sampling theorem*

Let's recap

$$s'[t] = \sum_{k=-\infty}^{\infty} s[k\,T]\,\frac{\mathrm{Sin}\left[\frac{\pi}{T}\,(t-k\,T)\right]}{\frac{\pi}{T}\,(t-k\,T)}$$

and

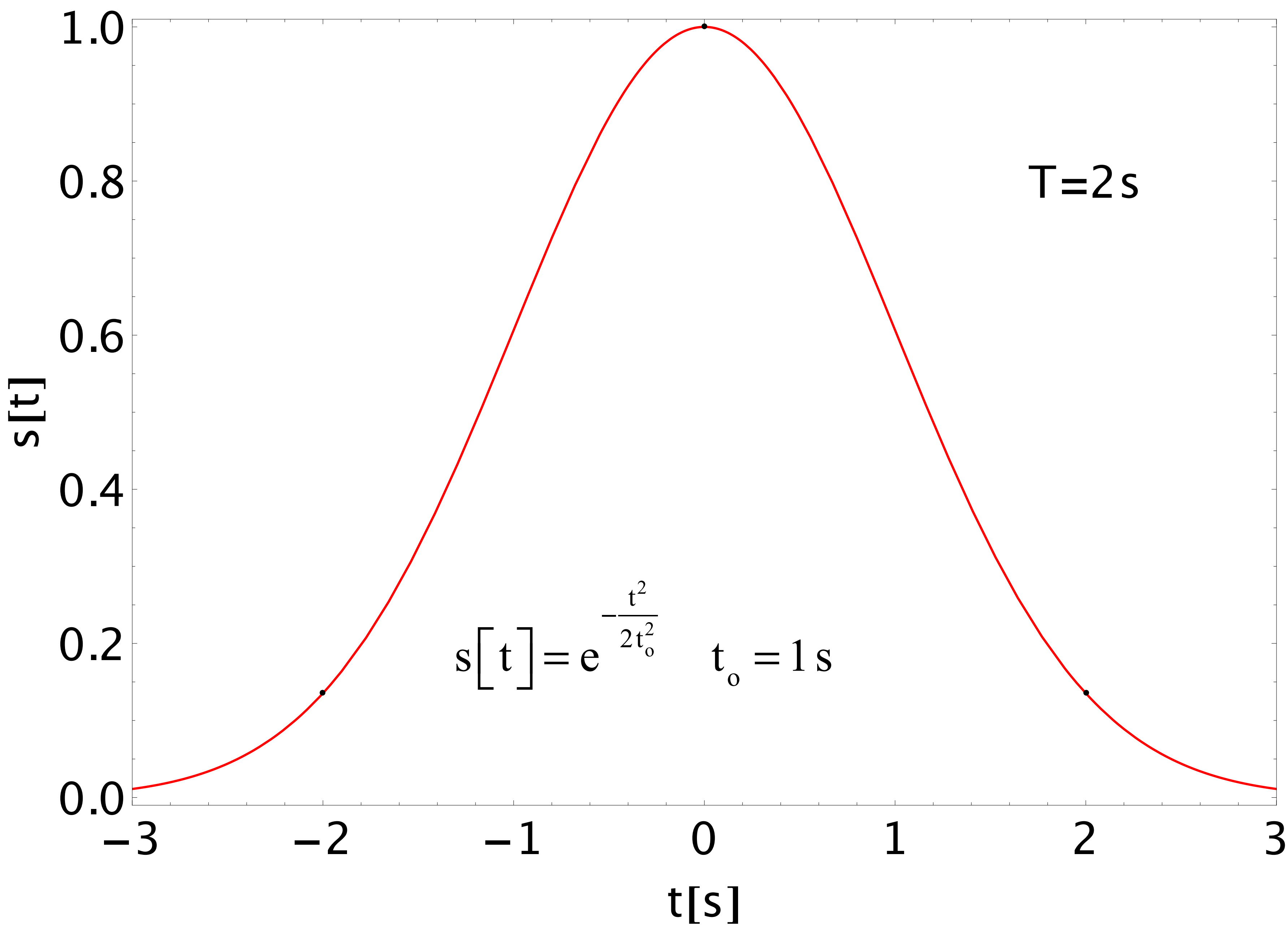
$$s'[\omega] = \left(\Theta\left[\omega + \frac{\pi}{T}\right] - \Theta\left[\omega - \frac{\pi}{T}\right]\right) \sum_{n=-\infty}^{\infty} s\left[\omega + n\,\frac{2\,\pi}{T}\right]$$

Notice that:

$$s'[\omega] \neq 0 \;\; \text{only for} \;\; -\frac{\pi}{T} \leq \omega \leq \frac{\pi}{T}$$

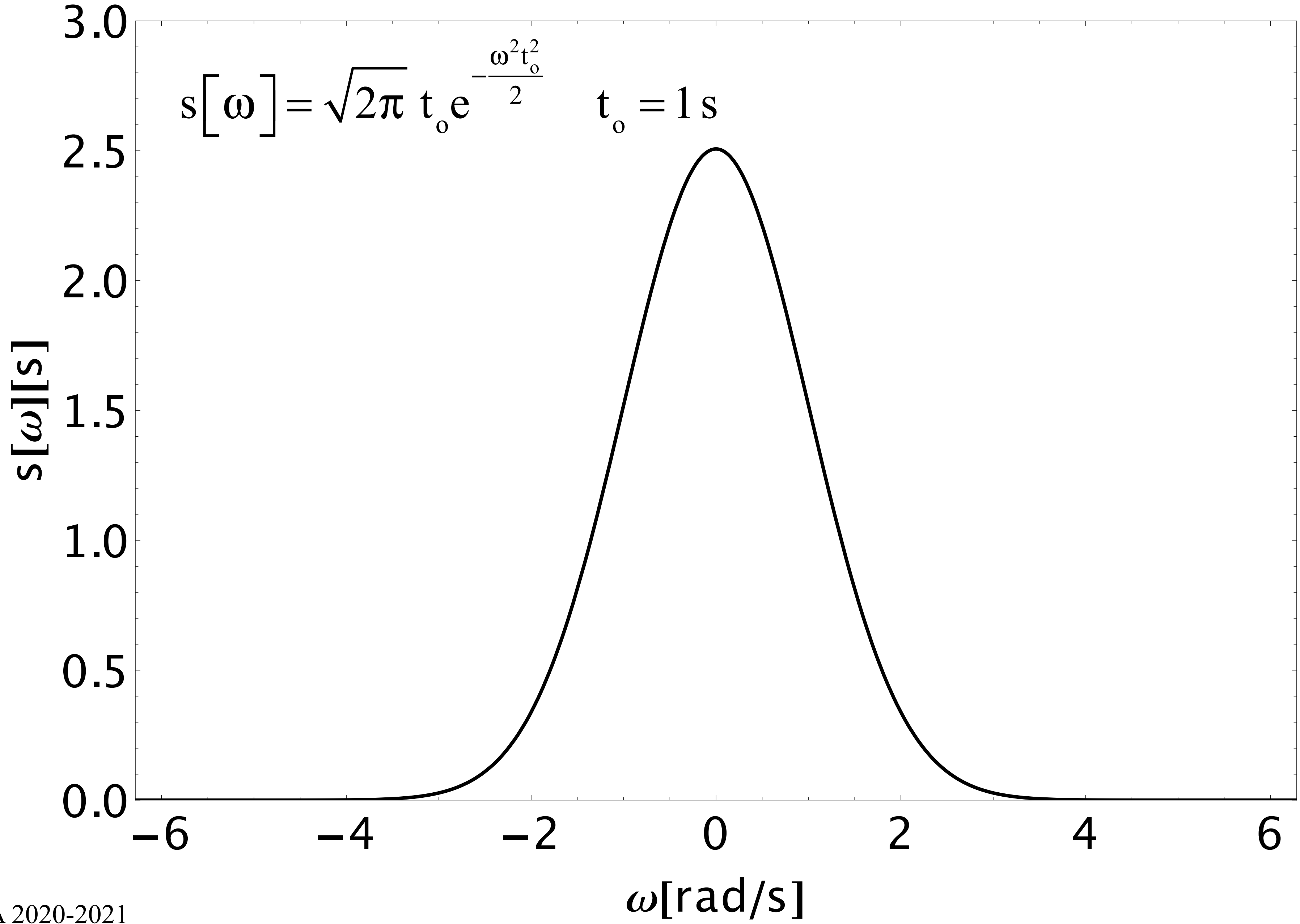
Within this range the "spectrum" of s' is the sum of infinite many "alias" of s[$\omega$] shifted by integer multiples of the sampling angular frequency $2\pi$/T

# An example of aliasing: Gaussian pulse

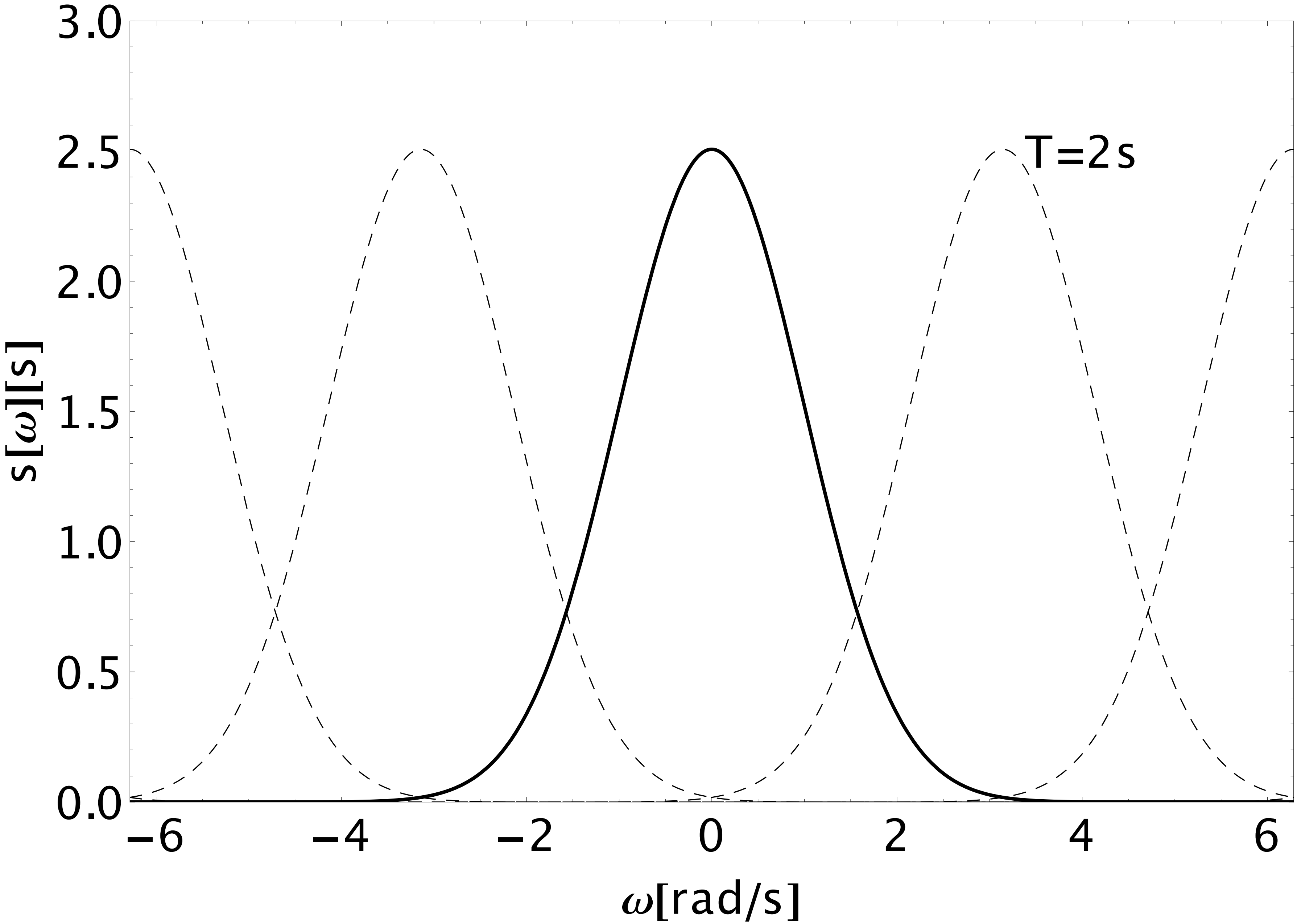


# Fourier transform

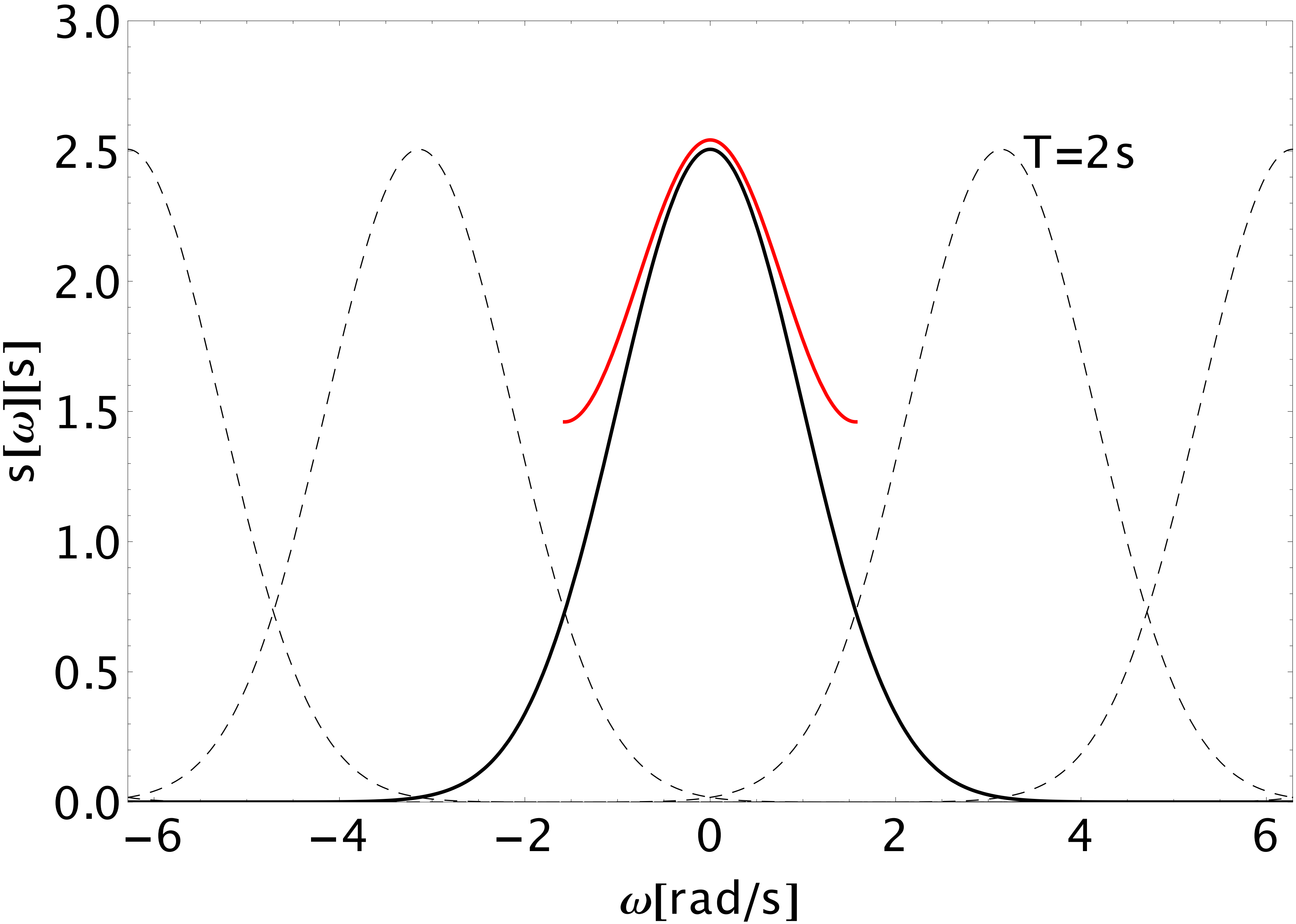




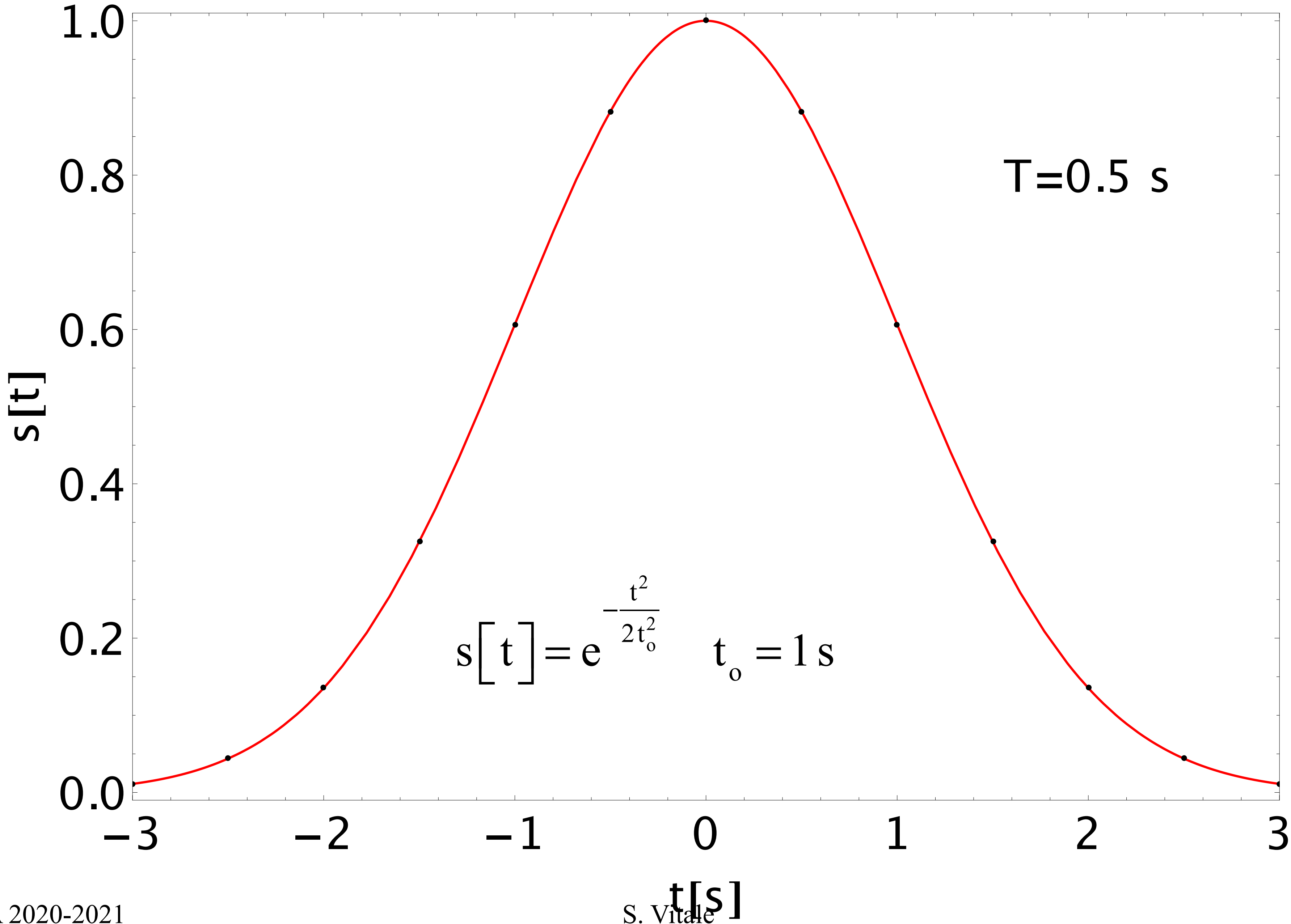
# Aliases



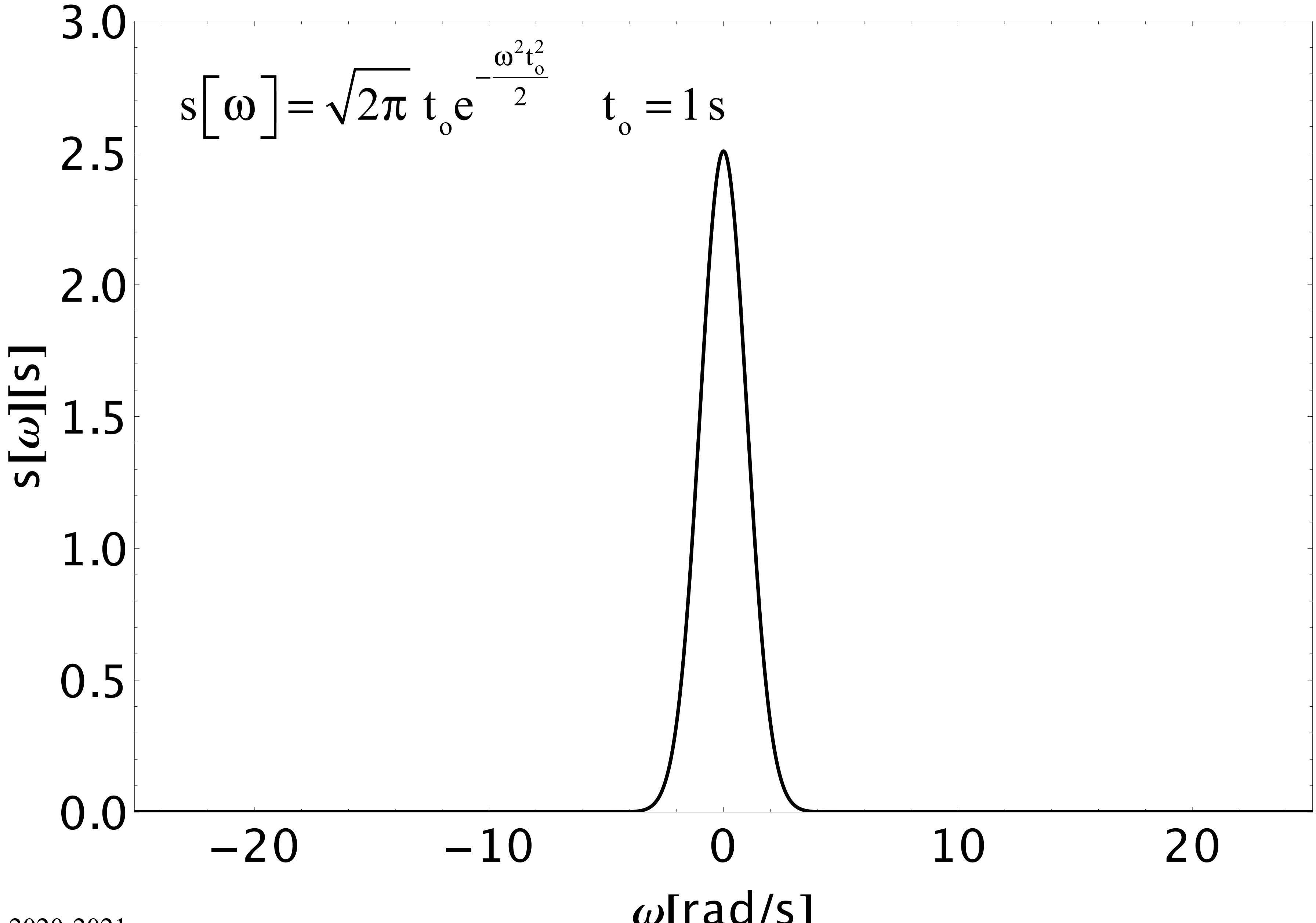
# Fourier transform of s’



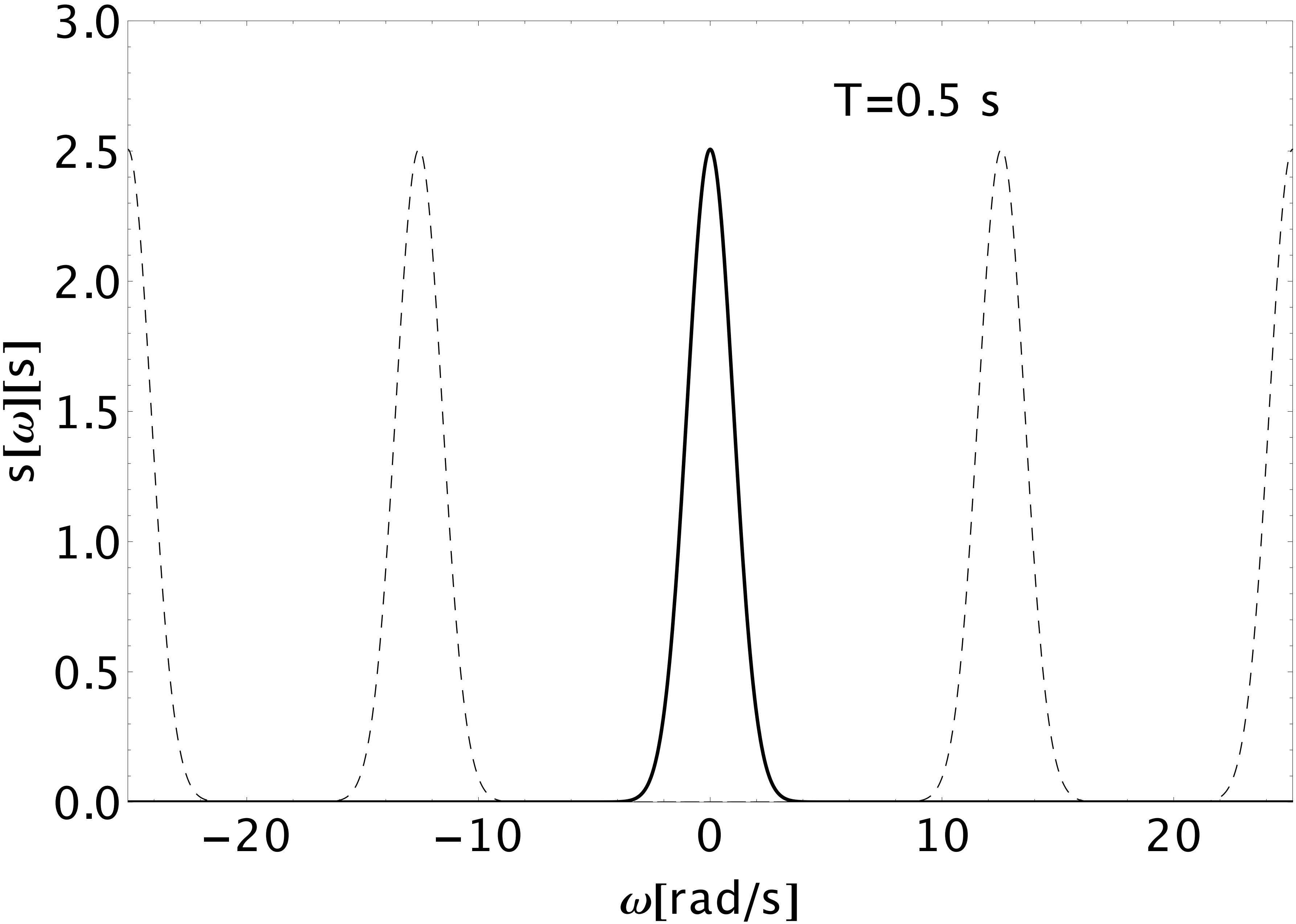
# A better sampling



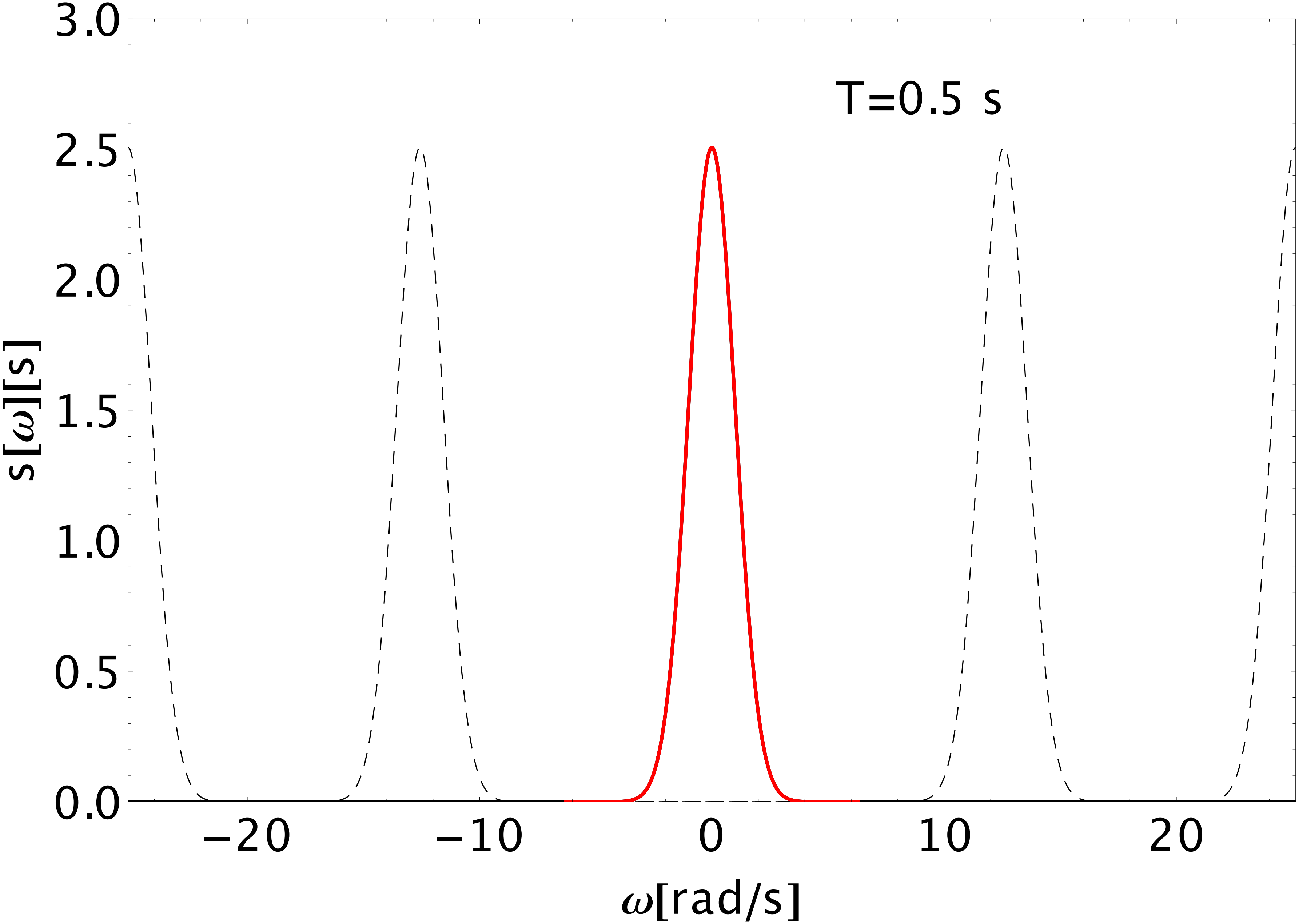
# Fourier transform



# Aliases



# Fourier transform of s’



# *Sampling theorem*

Notice that from

$$s'[\omega] = \left(\Theta\left[\omega + \frac{\pi}{T}\right] - \Theta\left[\omega - \frac{\pi}{T}\right]\right) \sum_{n=-\infty}^{\infty} s\left[\omega + n\frac{2\pi}{T}\right]$$

it follows that if also

$$\boxed{s[\omega] \neq 0 \;\; \text{only for} \;\; |\omega| < \frac{\pi}{T}}$$

Then

$$\left(\Theta\left[\omega + \frac{\pi}{T}\right] - \Theta\left[\omega - \frac{\pi}{T}\right]\right) s\left[\omega + n\frac{2\pi}{T}\right] \neq 0$$

$$\text{only for} \left|\omega + n\frac{2\pi}{T}\right| < \frac{\pi}{T} \;\; \text{and} \; |\omega| \leq \frac{\pi}{T}$$

That is only for n=0. In this case

$$\boxed{s'[\omega] = s[\omega] \;\; \rightarrow \;\; s'[t] = s[t]}$$

# The sampling theorem

- If a signal is "band-limited" such that

$$s\left[\omega\right]=0 \quad \text{for } \left|\omega\right| \geq 2\pi f_n$$

with $f_n$ the Nyquist frequency

- Then if the signal is sampled at a sampling frequency:

$$f_s > 2f_n$$

- The entire information is contained in its discrete samples from which the continuous signal can (in principle) be reconstructed through Shannon interpolation