Experimental Methods
Lecture 7

October 5th, 2020

Exercise

Apply sampling principles to a “wave packet”

$$s(t) = e^{-\frac{t}{\Delta t}} Sin(2\,\pi \nu_o t)\Theta(t)$$

Take $\Delta t = 10\ s$ and $\nu_o = 10\ Hz$

Calculate continuous Fourier Transform

Sample and estimate alias for $\nu_s = 20, 21, 50, 100\ Hz$

Truncate at $t = [-1, +20]s$ and $[-1, +50]s$ and estimate error within the data range

A physical instrument as a “system”

- A system transforms an input signal into an output one

$$i(t) \xrightarrow{\text{Input}} \boxed{\text{System}} \xrightarrow{\text{Output}} o(t)$$

- The output at any time may depend on the input at all times
- A system is then an operator or a functional in the vector space of signals

$$o(t) = \Im[i(t)]$$

- Example:

$$m\ddot{x} = F$$

$$F(Input) \rightarrow x(output)$$

$$x(t) = \int_0^\infty t' F(t - t') dt'$$

Special properties

- Causality

$$o(t) = \Im[i(t' \leq t)]$$

- Linearity

$$\Im[a_1 i_1(t) + a_2 i_2(t)] = a_1 \Im[i_1(t)] + a_2 \Im[i_2(t)]$$

- Linear systems obey

$$o(t) = \int_{-\infty}^{\infty} h(t,t') i(t') dt'$$

- Causal linear system obey

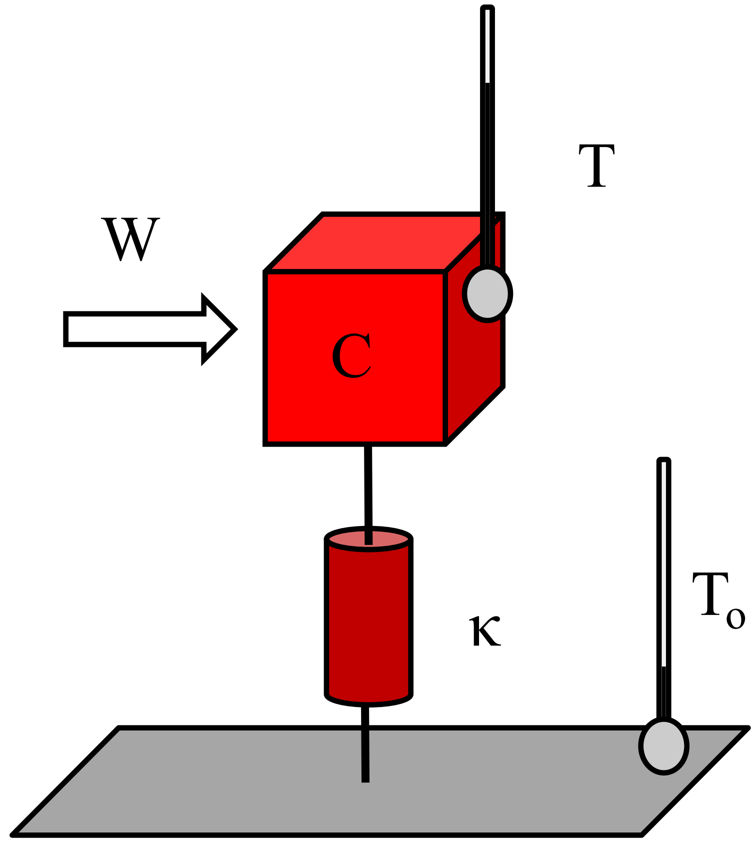
$$o(t) = \int_{-\infty}^{t} h(t,t') i(t') dt'$$

- Causal linear system with free evolution

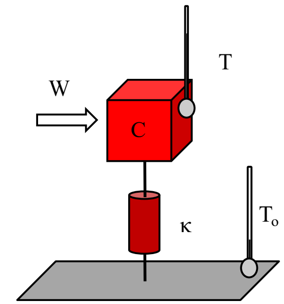
$$o(t) = o_o(t) + \int_{-\infty}^{t} h(t,t') i(t') dt'$$

An example: the calorimeter

- A body of (temperature independent) heat capacitance C, temperature T, receiving a heat input with power W, and with a loss path to the thermostat (at T_o) of (temperature independent) conductance κ.
- Input: $\frac{W}{\kappa}$ Output: $\Delta T = T - T_o$
- First principle of thermodynamics
$$C\frac{d\Delta T}{dt} + \kappa\Delta T = W$$
- Defining $\tau = C/\kappa$
$$\frac{d\Delta T}{dt} + \frac{\Delta T}{\tau} = \frac{1}{\tau}\frac{W}{\kappa}$$
- Linear differential equation: the system is linear



The solution



- Solution to

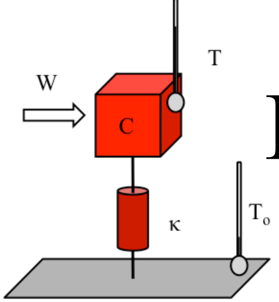
$$\frac{d\Delta T}{dt}+\frac{\Delta T}{\tau}=\frac{1}{\tau}\frac{W}{\kappa}$$

- In the range $0 \leq t \leq \infty$

$$\Delta T=\Delta T_o e^{-\frac{t}{\tau}}+\int_0^{\infty}\frac{W(t')}{\kappa}\frac{e^{\frac{t-t'}{\tau}}}{\tau}\Theta(t-t')dt'$$

- Notice

$$\Delta T(0)=\Delta T_o$$



Impulse response and free evolution

- Input-output relation

$$\Delta T(t) = \Delta T_o e^{-\frac{t}{\tau}} + \int_0^{\infty} \frac{W(t')}{\kappa} \frac{e^{-\frac{t-t'}{\tau}}}{\tau} \Theta(t-t')dt' =$$

- or, equivalently,

$$= \Delta T_o e^{-\frac{t}{\tau}} + \int_0^{t} \frac{W(t-t'')}{\kappa} \frac{e^{-\frac{t''}{\tau}}}{\tau} dt''$$

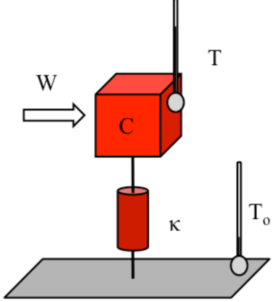
- In the language of the previous slides, taking W/κ as the input, the impulse response of the system is:

$$h(t,t') = \frac{e^{-\frac{t-t'}{\tau}}}{\tau} \Theta(t-t')$$

- While its free evolution is

$$\Delta T_o e^{-\frac{t}{\tau}}$$

- Notice that at t=0, only the free evolution term is different from zero



Impulse response

- Let's check that the impulse response of the system is

$$h(t,t') = \frac{e^{-\frac{t-t'}{\tau}}}{\tau}\Theta(t-t')$$

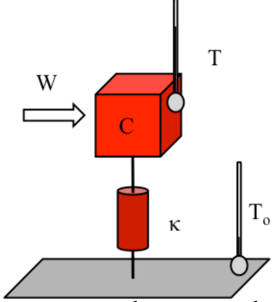
- Assume that

$$\frac{W(t)}{\kappa} = \delta(t-t')$$

that is a delta-pulse arriving at t' (dimensions would require a little discussion)

- Then

$$\Delta T(t) = \int_0^\infty \delta(t'-t'')\frac{e^{-\frac{t-t''}{\tau}}}{\tau}\Theta(t-t'')dt'' = \frac{e^{-\frac{t-t'}{\tau}}}{\tau}\Theta(t-t')$$



Suppressing free evolution

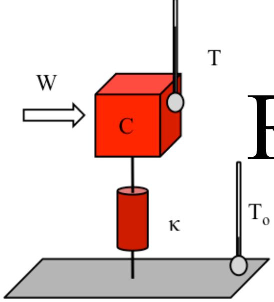
- The role of free evolution and the choice of the integration limits depend on the choice of the interval of validity of the equation.
- The formula below holds for 0≤t≤∞

$$\Delta T(t) = \Delta T_o e^{-\frac{t}{\tau}} + \int_0^t \frac{W(t-t')}{\kappa} \frac{e^{-\frac{t'}{\tau}}}{\tau} \Theta(t') dt'$$

- For -∞≤t≤∞
  – the free evolution has decayed to zero at any finite time t, and for any finite value of τ
  – the response to W becomes

$$\Delta T(t) = \int_0^\infty \frac{e^{-\frac{t'}{\tau}}}{\tau} \Theta(t') \frac{W(t-t')}{\kappa} dt''$$

- A formula that will become very familiar



Response to simple signals: heat step

A step at t=0. $W(t) = W_o\Theta(t)$

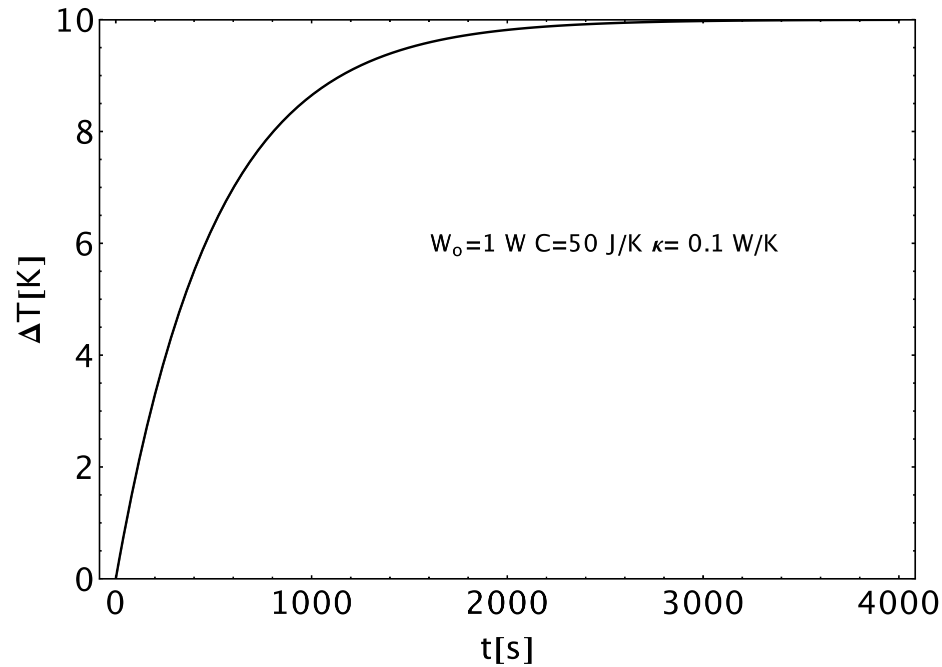
$$\Delta T(t) = \int_0^t \frac{e^{-\frac{t'}{\tau}}}{\tau}\frac{W(t-t')}{\kappa}dt'$$

$$= \frac{W_o}{\kappa}\int_0^t \frac{e^{-\frac{t'}{\tau}}}{\tau}\Theta(t-t')dt'$$

$$= \frac{W_o}{\kappa}\int_0^t \frac{e^{-\frac{t'}{\tau}}}{\tau}dt'$$

That is

$$\Delta T(t) = \frac{W_o}{\kappa}\left(1 - e^{-\frac{t}{\tau}}\right)\theta(t)$$



Response to simple signals: oscillating power

A sinusoid running forever $W_o Sin\left(\frac{2\pi}{T} t\right)$

$$\Delta T(t) = \frac{W_o}{\kappa} \int_0^\infty \frac{e^{-\frac{t'}{\tau}}}{\tau} Sin\left(\frac{2\pi}{T}(t - t')\right) dt'$$

Expanding the sine function

$$\Delta T(t) = \frac{W_o}{\kappa}\frac{1}{2i}\left(e^{i\frac{2\pi}{T}t}\int_0^\infty \frac{e^{-t'\left(\frac{1}{\tau}+i\frac{2\pi}{T}\right)}}{\tau} dt' - e^{-i\frac{2\pi}{T}t}\int_0^\infty \frac{e^{-t'\left(\frac{1}{\tau}-i\frac{2\pi}{T}\right)}}{\tau} dt'\right)$$

Performing the integrals

$$\Delta T(t) = \frac{W_o}{\kappa}\frac{1}{2i}\left(\frac{e^{i\frac{2\pi}{T}t}}{\tau\left(\frac{1}{\tau} + i\,\frac{2\pi}{T}\right)} - \frac{e^{-i\frac{2\pi}{T}t}}{\tau\left(\frac{1}{\tau} - i\,\frac{2\pi}{T}\right)}\right)$$

Response to simple signals: oscillating power

From

$$\Delta T(t) = \frac{W_o}{\kappa}\frac{1}{2i}\left(\frac{e^{i\frac{2\pi}{T}t}}{\tau\left(\frac{1}{\tau}+i\,\frac{2\pi}{T}\right)} - \frac{e^{-i\frac{2\pi}{T}t}}{\tau\left(\frac{1}{\tau}-i\,\frac{2\pi}{T}\right)}\right)$$

Simplifying

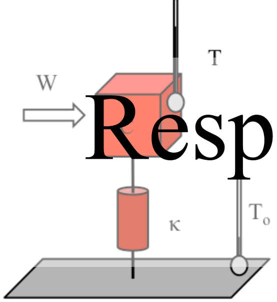
$$\Delta T(t) = \frac{W_o}{\kappa}\frac{1}{2i}\left(\frac{e^{i\frac{2\pi}{T}t}-e^{-i\frac{2\pi}{T}t}}{\tau^2\left(\frac{1}{\tau^2}+\left(\frac{2\pi}{T}\right)^2\right)} - i\,\frac{2\pi}{T}\frac{e^{i\frac{2\pi}{T}t}+e^{-i\frac{2\pi}{T}t}}{\tau\left(\frac{1}{\tau^2}+\left(\frac{2\pi}{T}\right)^2\right)}\right)$$

That is

$$\Delta T(t) = \frac{W_o}{\kappa}\frac{\frac{1}{\tau}}{\frac{1}{\tau^2}+\left(\frac{2\pi}{T}\right)^2}\left(\frac{1}{\tau}Sin\left(\frac{2\pi}{T}t\right) - \frac{2\pi}{T}Cos\left(\frac{2\pi}{T}t\right)\right)$$

Or

$$\Delta T(t) = \frac{W_o}{\kappa}\frac{1}{\sqrt{1+\left(\frac{2\pi}{T}\tau\right)^2}}Sin\left(\frac{2\pi}{T}t - Arctan\left(\frac{2\pi}{T}\tau\right)\right)$$



Response to simple signals: oscillating power

Input: a sinusoid $W_o \mathrm{Sin}[(2\pi/T)t]$

The response: a sinusoid $\Delta T(t) = \Delta T_o \mathrm{Sin}\left[\left(2\pi/T\right)t + \phi\right]$

With amplitude and phase given by

$$\Delta T_o = \frac{\frac{W_o}{\kappa}}{\sqrt{1+\left(\frac{2\pi}{T}\tau\right)^2}}$$

$$\phi = -Arctan\left(\frac{2\pi}{T}\tau\right)$$