# Experimental Methods
# Lecture 23

November 11th, 2020

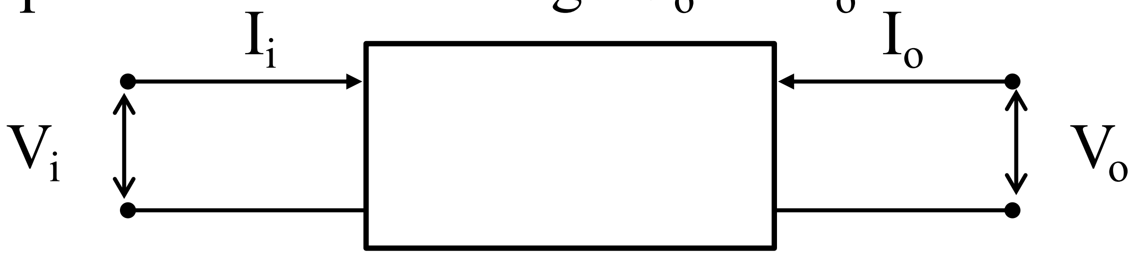
# Two-port networks

- A linear two-port network is a system with multiple inputs and multiple outputs.


- This system is linear and time invariant. It constitutes a more accurate scheme for a measurement instrument, *as it also includes the perturbation that a measurement device exerts on the physical system at its input.*
- two-port networks need not to be passive, and may actively increase the physical energy of signals going through them. The most classical example are electrical active two-port devices, like voltage and/or current amplifiers.

# The voltage amplifier

- The device is constituted by two ports. In each port we find two signals:
  - The input current and voltage $I_i$ and $V_i$
  - The output current and voltage $V_o$ and $I_o$



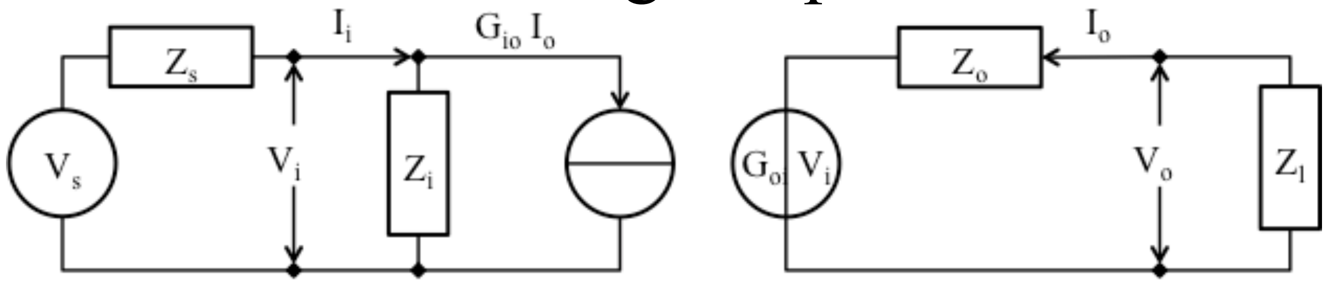

- In the frequency domain, as the system is linear, we can write, (for instance)

$$\begin{aligned} I_i &= V_i/Z_i + G_{io} I_o \\ V_o &= G_{oi} V_i + Z_o I_o \end{aligned} \rightarrow \begin{pmatrix} I_i \\ V_o \end{pmatrix} = \begin{pmatrix} 1/Z_i & G_{io} \\ G_{oi} & Z_o \end{pmatrix} \bullet \begin{pmatrix} V_i \\ I_o \end{pmatrix}$$

- $Z_i$ is called input impedance, $Z_o$ output impedance, and non diagonal coefficients are often called gains.

# The voltage amplifier

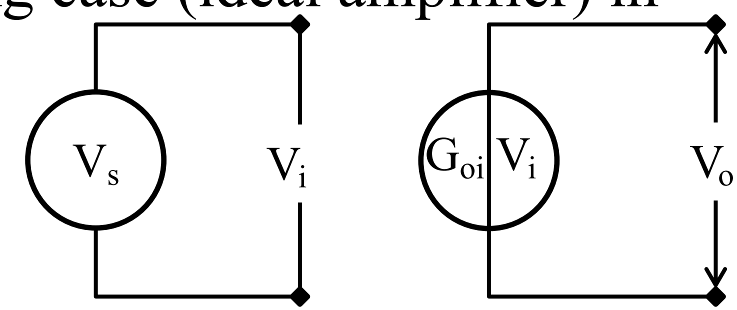





- Adding the new branches, these are now the circuit equations:

$$I_i = \frac{V_i}{Z_i} + G_{io} I_o \quad V_o = G_{oi} V_i + Z_o I_o \quad I_i = \frac{V_s - V_i}{Z_s} \quad V_o = -Z_l I_o$$

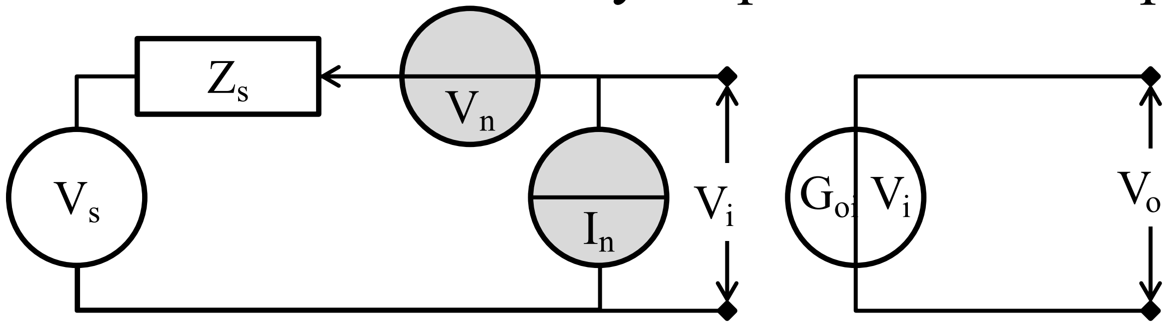
- These can now be solved in the general case.
- We are however interested in the limiting case (ideal amplifier) in which, $Z_i \to \infty$ and $Z_L \to \infty$. Then



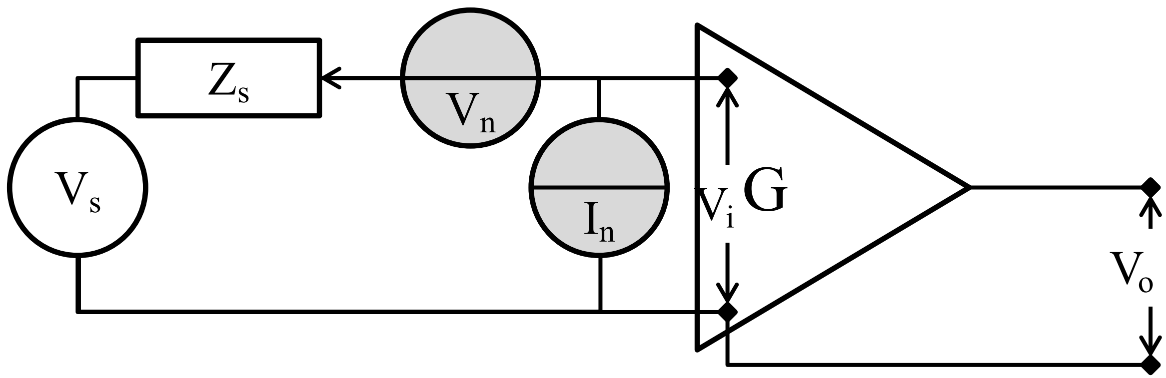

- $I_o = -\frac{V_o}{Z_l} \to 0 \quad I_i = \frac{V_i}{Z_i} - \frac{G_{io} V_o}{Z_l} \to 0$

- $V_i = V_s - Z_s I_i \to V_s \qquad V_o = G_{oi} V_i - \frac{Z_o}{Z_l} V_o \to G_{oi} V_i = G_{oi} V_s$

- Thus, in conclusion the ideal noisy amplifier can be represented as

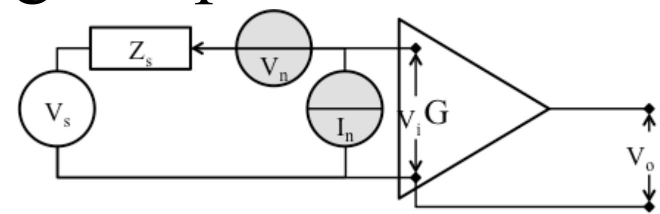



- Or, with a different graphics, and the simpler symbol G for the gain, as




- As the output voltage is $V_o = G\left(V_s + V_n + I_n Z_s\right)$
  noise can be expressed at input as $V_n^{in} = V_n + I_n Z_s$
- Notice that $I_n$ truly circulate within $Z_s$. Thus the amplifier perturbs the source. This perturbation is often called the amplifier *back-action*

# Noise in two port systems: voltage amplifier




- In summary the noise at input of an ideal, noisy amplifier is $V_n^{in} = V_n + I_n Z_s$
- Now remember
  - If $z(t) = \alpha x(t) + \beta y(t)$
  - Then $S_{z,z}(\omega) = \alpha^2 S_{x,x}(\omega) + \beta^2 S_{y,y}(\omega) + 2\alpha\beta \mathrm{Re}\{S_{x,y}(\omega)\}$
- Then

$$S_{V_n^{in},V_n^{in}}(\omega) = S_{V_n,V_n}(\omega) + |Z_s(\omega)|^2 S_{I_n,I_n} + 2\mathrm{Re}\{S_{V_n,Z_s I_n}(\omega)\}$$

- Furthermore

$$R_{V_n,Z_s I_n}(\tau) = \left\langle \int_0^\infty Z_s(t') I_n(t+\tau-t') V_n(t) dt' \right\rangle =$$

$$= \int_0^\infty Z_s(t') \langle I_n(t+\tau-t') V_n(t) \rangle \, dt' = \int_0^\infty Z_s(t') R_{V_n,I_n}(\tau - t') dt'$$

- Then

$$S_{V_n,Z_s I_n}(\omega) = Z_s(\omega) S_{V_n,I_n}(\omega)$$

- And

$$S_{V_n^{in},V_n^{in}}(\omega) = S_{V_n,V_n}(\omega) + |Z_s(\omega)|^2 S_{I_n,I_n} + 2\mathrm{Re}\{Z_s(\omega) S_{V_n,I_n}(\omega)\}$$

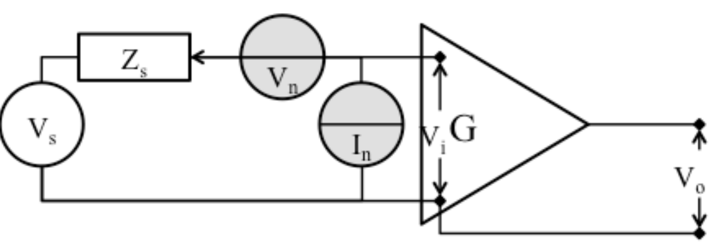
# Critical noise parameters for a 2-port device

- Noise at input $V_n^{in} = V_n + I_n Z_s$
- Before proceeding further let's give a few important definitions:

1. Noise energy or temperature $E_n(\omega) = k_B T_n(\omega) \equiv \sqrt{S_{V_nV_n}(\omega) S_{I_nI_n}(\omega)}$
2. Noise Resistance $R_n(\omega) \equiv \sqrt{S_{V_nV_n}(\omega) / S_{I_nI_n}(\omega)}$
3. Noise cross-coherence $\rho_n(\omega) = S_{V_nI_n}(\omega) \Big/ \sqrt{S_{V_nV_n}(\omega) S_{I_nI_n}(\omega)}$

- Inverting these definitions

1. Voltage PSD: $S_{V_nV_n}(\omega) = k_B T_n(\omega) R_n(\omega)$
2. Current PSD $S_{I_nI_n}(\omega) = k_B T_n(\omega) / R_n(\omega)$
3. Voltage-current cross spectrum $S_{V_nI_n}(\omega) = k_B T_n(\omega) \rho_n(\omega)$



# Noise in a voltage amplifier

- From previous page

$$S_{V_nV_n}(\omega)=k_BT_n(\omega)R_n(\omega);\ S_{I_nI_n}(\omega)=\frac{k_BT_n(\omega)}{R_n(\omega)};\ S_{V_nI_n}(\omega)=k_BT_n(\omega)\rho_n(\omega)$$

- Total PSD at input, is given by

$$S_{V_n^{in}V_n^{in}}(\omega)=S_{V_nV_n}(\omega)+\left|Z_s(\omega)\right|^2S_{I_nI_n}(\omega)+2\operatorname{Re}\left\{Z_s(\omega)S_{V_nI_n}(\omega)\right\}$$

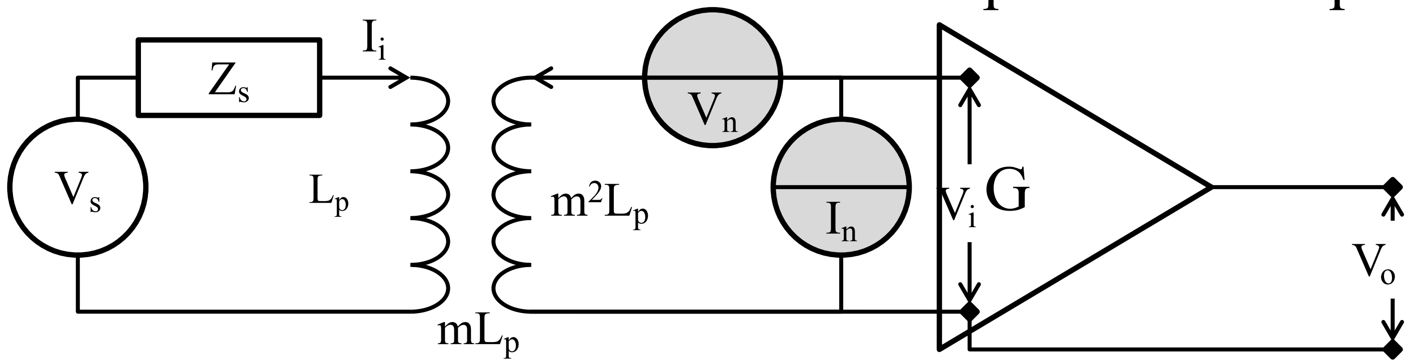
becomes, by substituting the definition above:

$$S_{V_n^{in}V_n^{in}}(\omega)=k_BT_n(\omega)\left[R_n(\omega)+\left|Z_s(\omega)\right|^2/R_n(\omega)+2\operatorname{Re}\left\{Z_s(\omega)\rho_n(\omega)\right\}\right]$$

Voltage noise depends on the impedance of the source and on that of noise

# Adjusting noise resistance

- Now we want to show that, at least to some approximation, the noise resistance can be adjusted without affecting the noise temperature. To do this, put an m-turns ideal transformer at the input of our amplifier:




- You can calculate that :

$$V_s = I_i(Z_s + sL_p) + m\,sL_p I_n = 0 \qquad V_i = s\,mL_p I_i + s\,m^2 L_p I_n + V_n$$

$$I_i = \frac{V_s - m\,sL_p I_n}{Z_s + sL_p} \qquad V_i = s\,mL_p \frac{V_s - m\,sL_p I_n}{Z_s + sL_p} + s\,m^2 L_p I_n + V_n$$

$$V_i = \frac{s\,mL_p}{Z_s + sL_p}\left(V_s - \left(m\,s\,L_p + Z_s m + m\,s\,L_p\right)I_n + \frac{V_n}{m}\frac{Z_s + sL_p}{sL_p}\right)$$

# Adjusting noise resistance



- From

$$I_i = \frac{V_s - m\, sL_p I_n}{Z_s + sL_p}$$

$$V_i = \frac{s\, mL_p}{Z_s + sL_p}\left(V_s - \left(m\, s\, L_p + Z_s m + m\, s\, L_p\right) I_n + \frac{V_n}{m}\frac{Z_s + sL_p}{sL_p}\right)$$

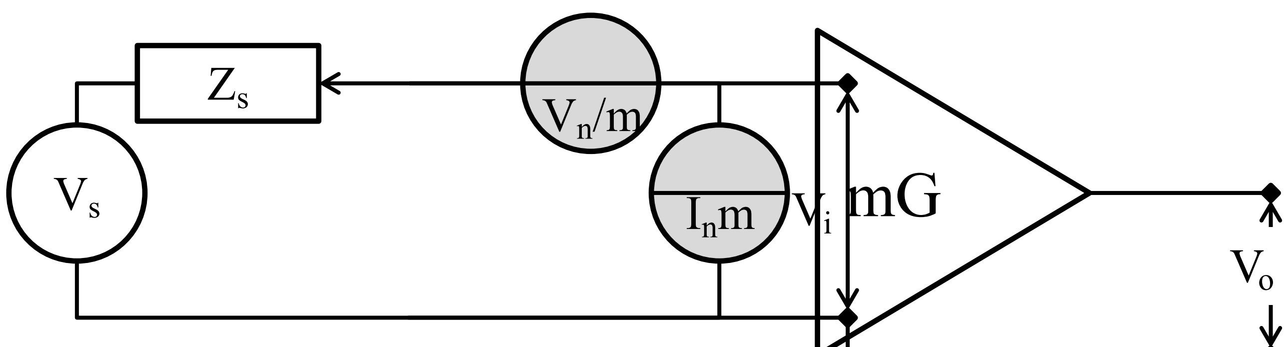
- Finally

$$I_i = \frac{V_s - m\, sL_p I_n}{Z_s + sL_p} V_i = \frac{s\, mL_p}{Z_s + sL_p}\left(V_s + Z_s(-m\, I_n) + \frac{V_n}{m}\frac{Z_s + sL_p}{sL_p}\right)$$

- Now make $|sL_p| \gg |Z_s|$

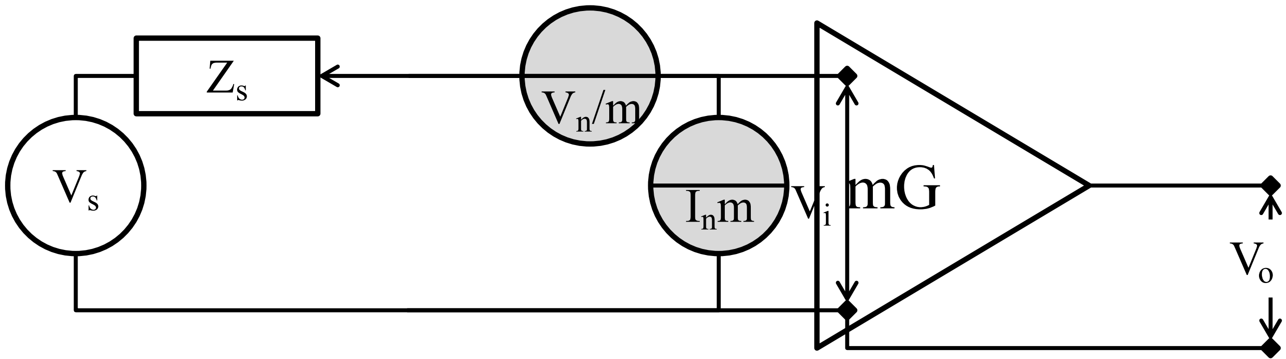
$$I_i = -mI_n \qquad V_i = m\left(V_s + Z_s(-m\, I_n) + \frac{V_n}{m}\right)$$

- Then the situation looks like this:



# Adjusting noise resistance

- The equivalent noise at input consists of a current noise $I_n' = I_n m$ and voltage noise $V_n' = V_n/m$.




- The new noise parameters are then

$$k_B T_n'(\omega) \equiv \sqrt{\frac{S_{V_nV_n}(\omega)}{m^2} m^2 S_{I_nI_n}(\omega)} = k_B T_n(\omega)$$

$$R_n'(\omega) \equiv \sqrt{\frac{S_{V_nV_n}(\omega)}{m^2} \Big/ S_{I_nI_n}(\omega) m^2} = \frac{R_n(\omega)}{m^2}$$

$$\rho_n'(\omega) = \rho_n(\omega)$$

# Minimizing noise by impedance matching

- Total noise PSD at input, at a given frequency

$$S_{V_n^{in}V_n^{in}}(\omega)=k_B T_n(\omega)\left[R_n(\omega)+\left|Z_s(\omega)\right|^2\Big/R_n(\omega)+2\,\mathrm{Re}\left\{Z_s(\omega)\rho_n(\omega)\right\}\right]$$

- Can be minimized by adjusting $R_n$

$$\partial S_{V_n^{in}V_n^{in}}(\omega)\Big/\partial R_n(\omega)=k_B T_n(\omega)\left[1-\left|Z_s(\omega)\right|^2\Big/R_n^2(\omega)\right]=0$$

- That gives $R_n(\omega)=\left|Z_s(\omega)\right|$
- When this condition is fulfilled the contribution of voltage noise and of current noise become equal and the PSD becomes

$$S_{V_n^{in}V_n^{in}}(\omega)=2k_B T_n(\omega)\left[\left|Z_s(\omega)\right|+\mathrm{Re}\left\{Z_s(\omega)\rho_n(\omega)\right\}\right]$$

- Or, for uncorrelated generators

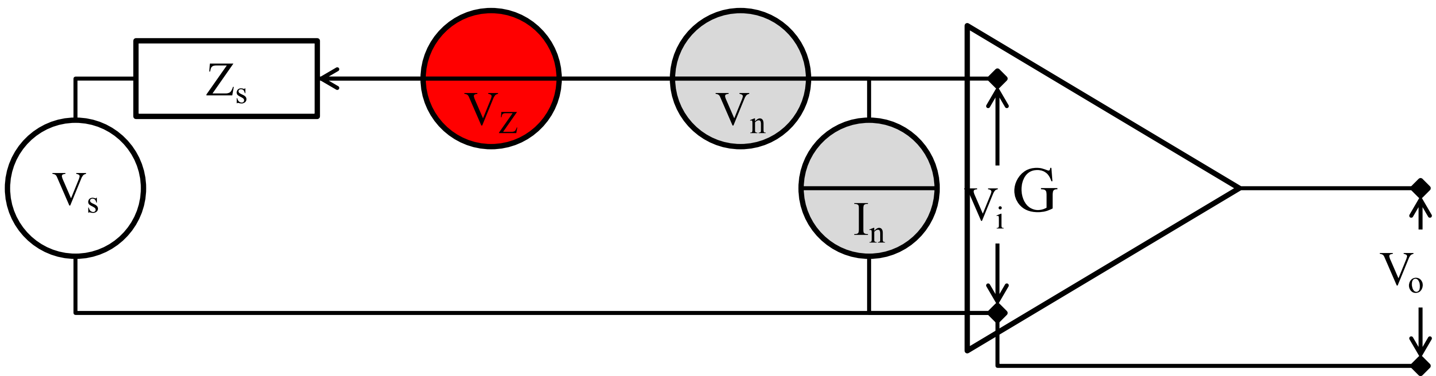
$$S_{V_n^{in}V_n^{in}}(\omega)=2k_B T_n(\omega)\left|Z_s(\omega)\right|$$

- Watch out! This is not Nyquist law, and $T_n$ is not T

# Noise figure and matched source



- We want to discuss now another noise minimization that achieves a similar result, but  in term of source impedance.
- If the source impedance has a real part, at thermal equilibrium it will be noisy, with a noise PSD given by Nyquist formula.




- Total noise at input will then have a PSD (we omit the frequency dependence in order to simplify the writing), given by:

$$S_{V_n^{in}V_n^{in}} = k_B T_n \left[ R_n + \left| Z_s \right|^2 / R_n + 2\,\mathrm{Re}\left\{ Z_s \rho_n \right\} \right] + 2 k_B T\, \mathrm{Re}\left\{ Z_s \right\}$$

# Noise figure and matched source

- From the PSD

$$S_{V_n^{in}V_n^{in}} = k_B T_n \left[ R_n + \left|Z_s\right|^2 \big/ R_n + 2\,\mathrm{Re}\left\{Z_s \rho_n\right\} \right] + 2k_B T\,\mathrm{Re}\left\{Z_s\right\}$$

- We define the noise figure as the ratio, *measured in decibels,* between the total noise PSD and that of the thermal noise alone

$$F = 20\mathrm{Log}_{10}\left\{ \frac{k_B T_n \left[ R_n + \left|Z_s\right|^2 \big/ R_n + 2\,\mathrm{Re}\left\{Z_s \rho_n\right\} \right] + 2k_B T\,\mathrm{Re}\left\{Z_s\right\}}{2k_B T\,\mathrm{Re}\left\{Z_s\right\}} \right\}$$

- That is $F = 20\mathrm{Log}_{10}\left\{ 1 + \frac{T_n}{T}\left[ \frac{R_n}{2\,\mathrm{Re}\left\{Z_s\right\}} + \frac{\left|Z_s\right|^2}{2R_n\,\mathrm{Re}\left\{Z_s\right\}} + \frac{\mathrm{Re}\left\{Z_s\rho_n\right\}}{\mathrm{Re}\left\{Z_s\right\}} \right] \right\}$

- For uncorrelated noise sources this figure has a minimum when

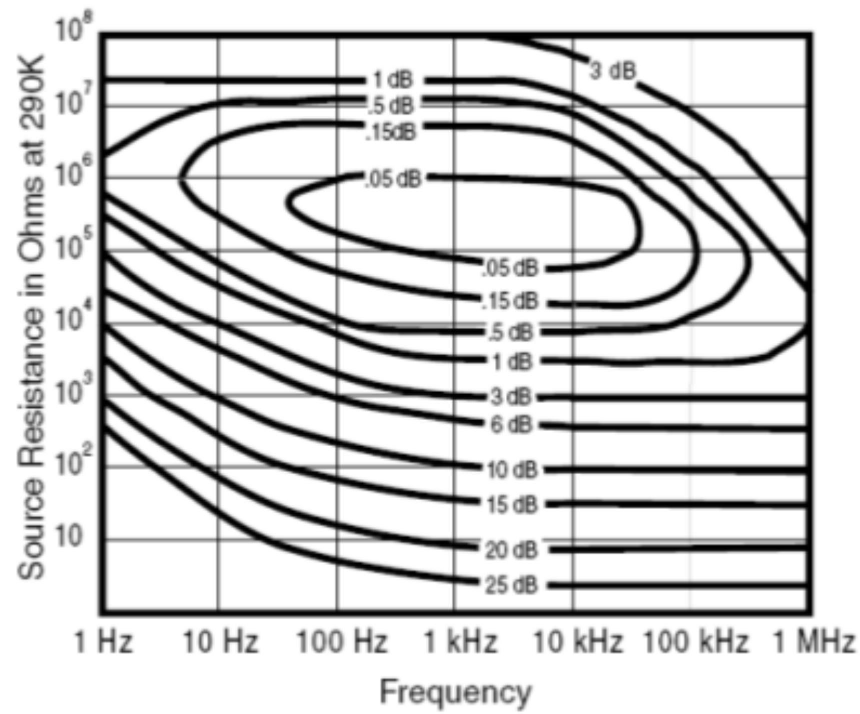
$$Z_s = R_n$$

- Whose value is $F_{min} = 20\mathrm{Log}_{10}\left\{ 1 + \frac{T_n}{T} \right\}$

# A practical example: a popular low-noise instrument amplifier

- From the attached noise figure plot, let's derive noise temperature and noise resistence at 100 Hz, 1 kHz and 10 kHz
- Calculate also the voltage and current PSD
- Notice: one sided PSD are used here


**SR560 Noise Figure**



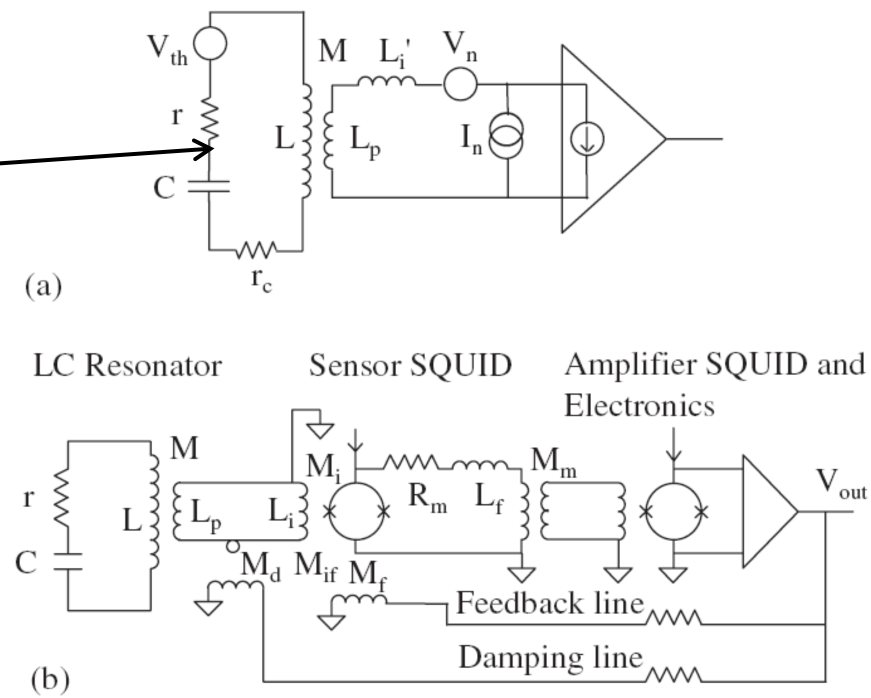
# An experimental example

APPLIED PHYSICS LETTERS **93**, 172506 (2008)

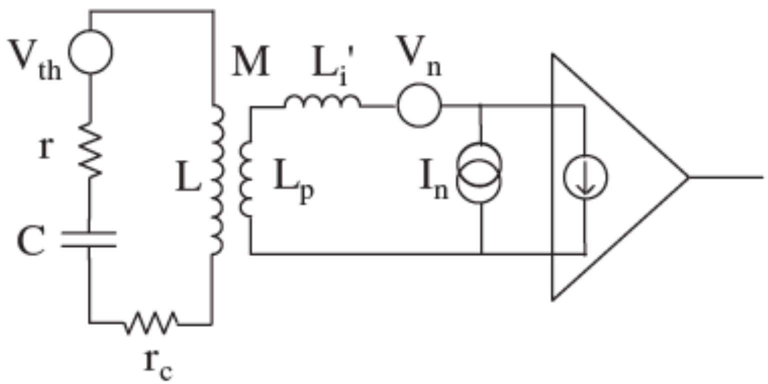
## 10 $\hbar$ superconducting quantum interference device amplifier for acoustic gravitational wave detectors

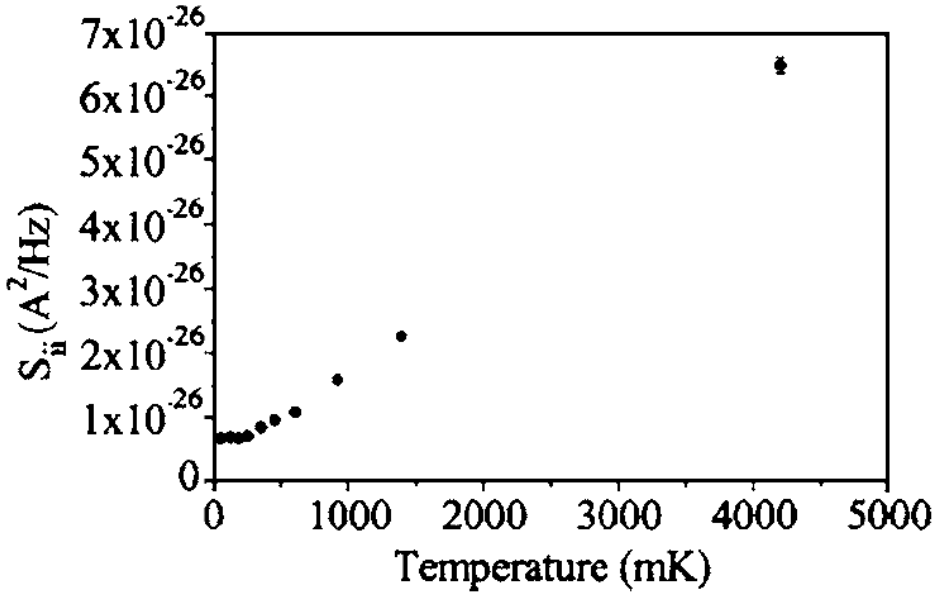
P. Falferi,[1,2,a)] M. Bonaldi,[1,2] M. Cerdonio,[3,4] R. Mezzena,[5,2] G. A. Prodi,[5,2] A. Vinante,[4] and S. Vitale[5,2]

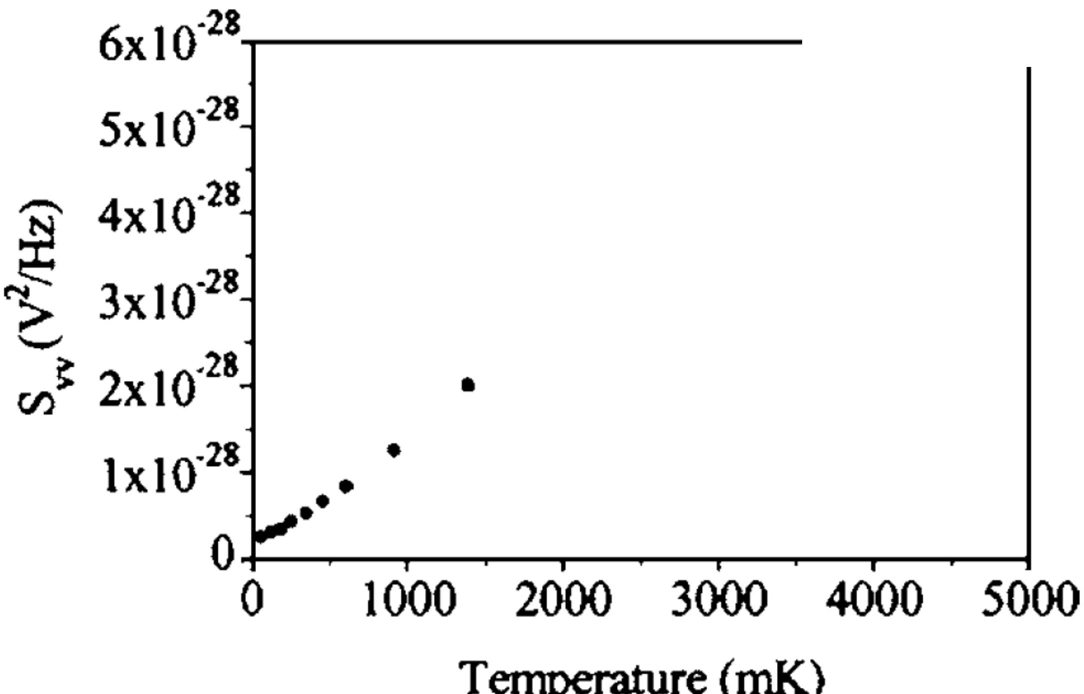
- Current noise excites the resonant circuit (9 kHz).
- The PSD peak gives a measurement of the current noise
- Outside the resonance range the noise is dominated by voltage noise



# 27 $\hbar$ SQUID amplifier operating with high-$Q$ resonant input load



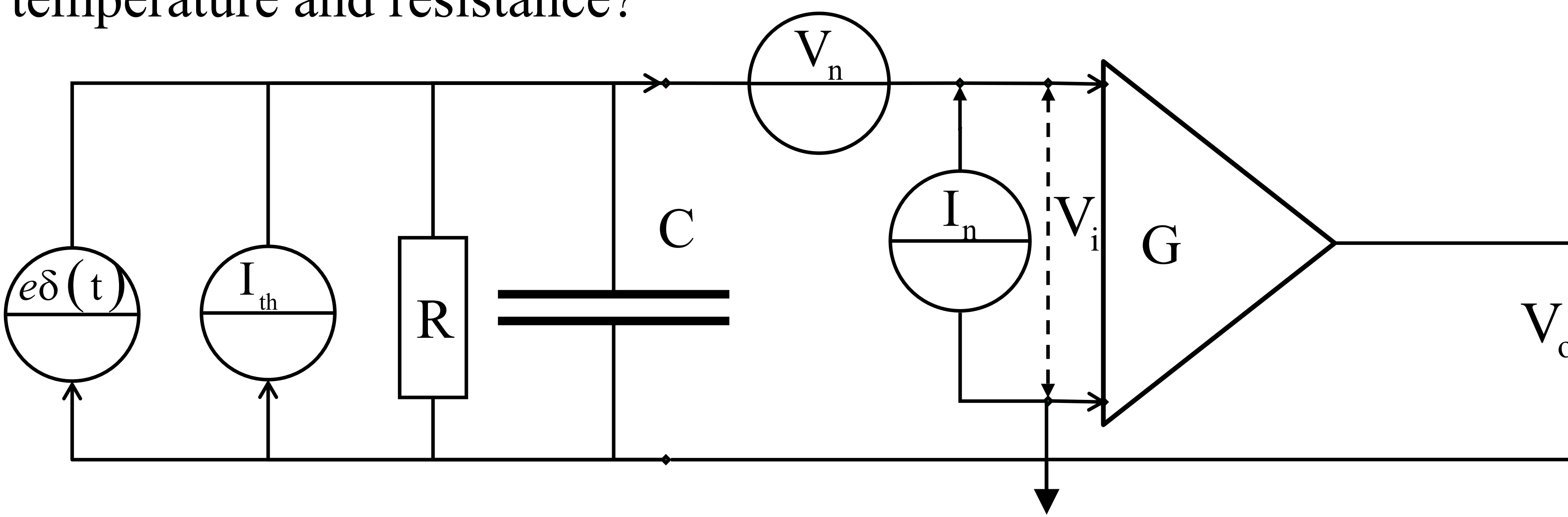




Current noise and voltage noise are then independently measured (pictures from a different experiment)

# The meaning of noise energy. One example: charge detector

- A charge detector consisting of a lossy capacitor read out by a low noise amplifier. $I_{th}$ is the Nyquist current noise generator associated with R. We will assume $V_n$ and $I_n$ to be uncorrelated.
- The signal at input consists of a single charge *e*. That is, the signal consists of a current impulse $I(t)=e\delta(t)$
- What is the minimum measurable value of *e*, given the amplifier noise temperature and resistance?

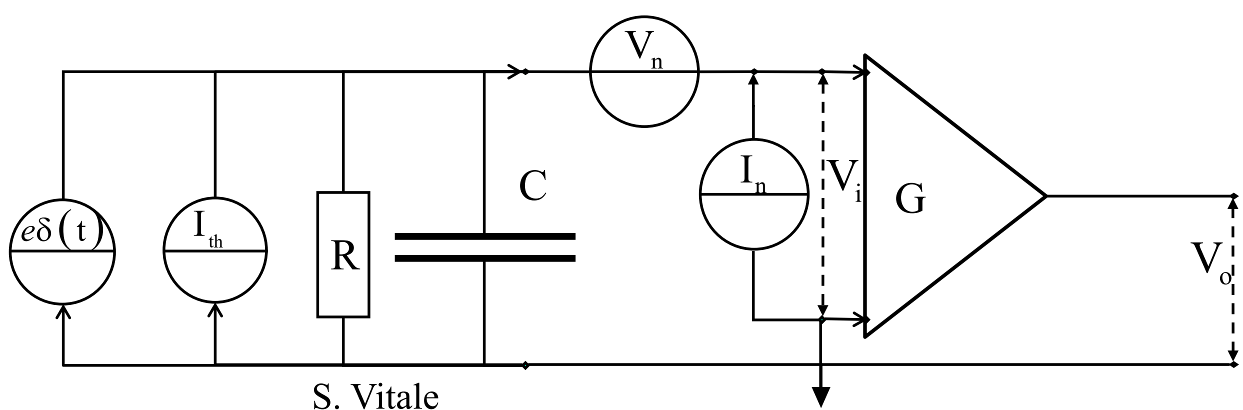


# A short cut

- As the signal is a current, let's work at input, i.e. for current
- Fourier transform of the (unit amplitude) signal $I(\omega)=1$
- Let's call $I_n' = I_n + I_{th}$.
- Total input due to noise is: $V_{i,noise} = V_n + Z I_n'$
- As an equivalent current $I_{n,e} = I_n' + \frac{V_n}{Z}$
- Total equivalent input noise PSD ($I_n$ and $V_n$ white and uncorrelated)

$$S_{I_{n,e}I_{n,e}}(\omega) = S_{I_n' I_n'} + \frac{S_{V_nV_n}(1+\omega^2R^2C^2)}{R^2}$$

$$S_{I_{n,e}I_{n,e}}(\omega) = S_{I_n' I_n'} + \frac{S_{V_nV_n}}{R^2} + S_{V_nV_n}\omega^2C^2$$



# A short cut



- Furthermore, from

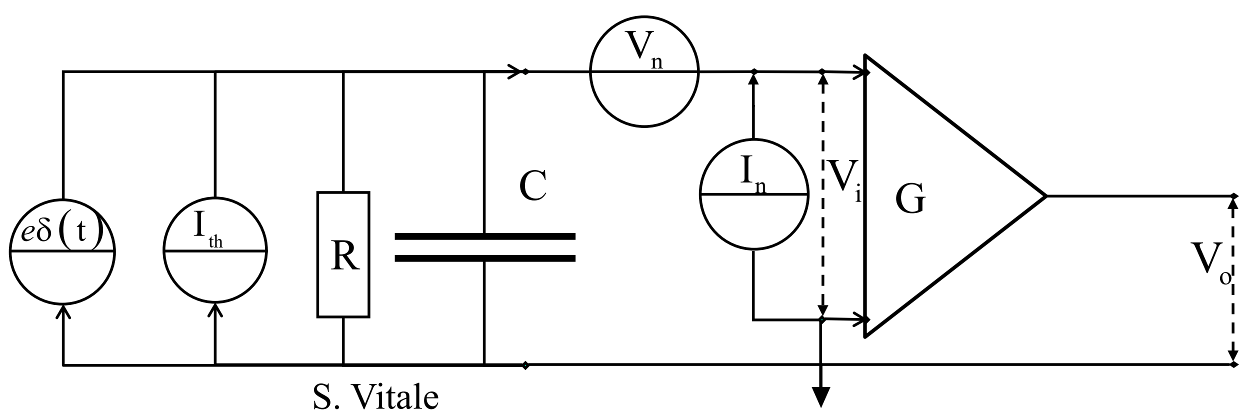
$$S_{I_{n,e}I_{n,e}}(\omega) = S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2} + S_{V_nV_n}\omega^2C^2$$

- We get

$$S_{I_{n,e}I_{n,e}}(\omega) = \left(S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2}\right)\left(1 + \frac{S_{V_nV_n}C^2}{S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2}}\omega^2\right)$$

- or

$$S_{I_{n,e}I_{n,e}}(\omega) = S_{Io}(1 + \tau^2\omega^2)$$



# A short cut



- Thus, signal

$$I(\omega) = 1$$

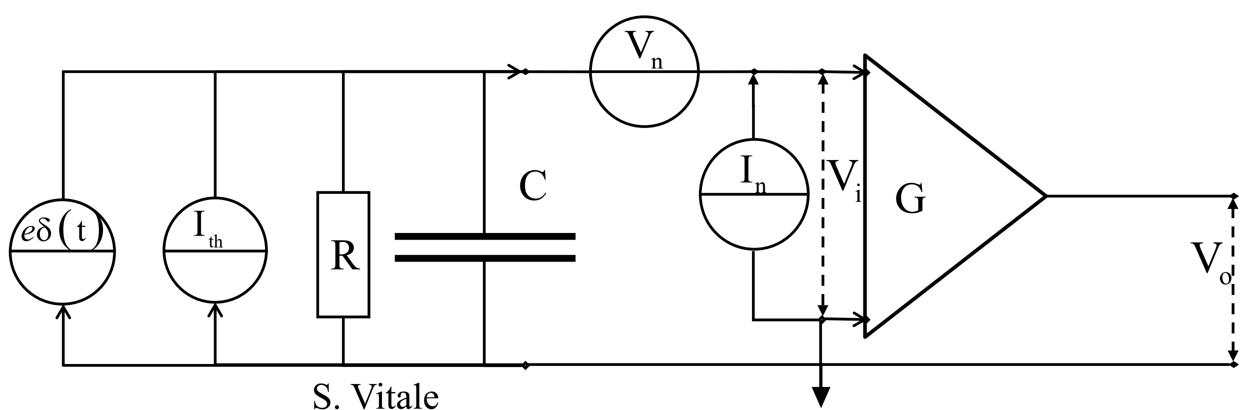
- Noise

$$S_{I_{n,e}I_{n,e}}(\omega) = S_{Io}(1+\tau^2\omega^2)$$

- Wiener filter error

$$\sigma_e = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{S_{Io}(1+\omega^2\tau^2)}d\omega\right)^{-1/2} = \frac{\sqrt{S_{Io}\tau}}{\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx}}$$

$$= \sqrt{2S_{Io}\tau}$$



# A short cut



- Thus,

$$\sigma_e = \sqrt{2S_{Io}\tau}$$

- And

$$S_{I_{n,e}I_{n,e}}(\omega) = \left(S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2}\right)\left(1 + \frac{S_{V_nV_n}C^2}{S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2}}\omega^2\right)$$

- with

$$S_{I_{n,e}I_{n,e}}(\omega) = S_{Io}(1 + \tau^2\omega^2)$$

- Then

$$\sigma_e = \sqrt{2\left(S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2}\right)\sqrt{\frac{S_{V_nV_n}C^2}{S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2}}}} = \sqrt{2C}\left(S_{V_nV_n}\left(S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2}\right)\right)^{\frac{1}{4}}$$

- Now take the limit for $R \to \infty$ (low dissipations)
- Then

$$\sigma_e \to \sqrt{2C\sqrt{S_{V_nV_n}S_{I'_nI'_n}}} = \sqrt{2\,Ck_BT'_n}$$

# A short cut

- In conclusion

$$\sigma_e = \sqrt{2\ C k_B T_n'}$$

- Now suppose that a charge $\sigma_e$ is deposited on the capacitor. This will get an energy

$$E = \frac{1}{2} C \sigma_e^2 = k_B T_n'$$

- This is the crucial result: the minimum detectable energy is the noise energy of the amplifier
- Notice that if $T \to 0$ $T_n' \to T_n$ as the thermal noise becomes negligible

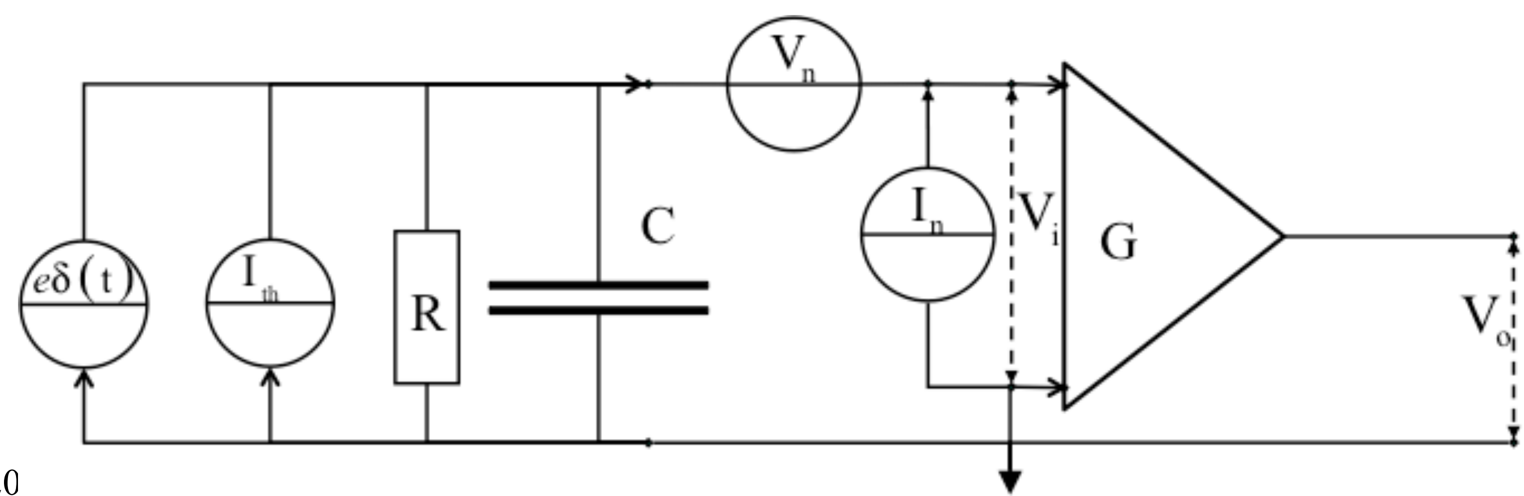
# Example: charge detector



- We answer the question by using optimal filter theory. We are looking in the data for a signal of known shape δ(t) and unknown signal amplitude *e*.
- Let's then calculate in the frequency domain the signal contribution to $V_i$. The Fourier transform of I(t)=δ(t) is just I(ω)=1. Then

$$V_i(\omega) = I(\omega) R(1/i\omega C)/(R + 1/i\omega C) = I(\omega) R/(1 + i\omega CR)$$

- Notice (to be used later) $V_i(t) = (1/C) e^{-t/RC} \Theta(t)$



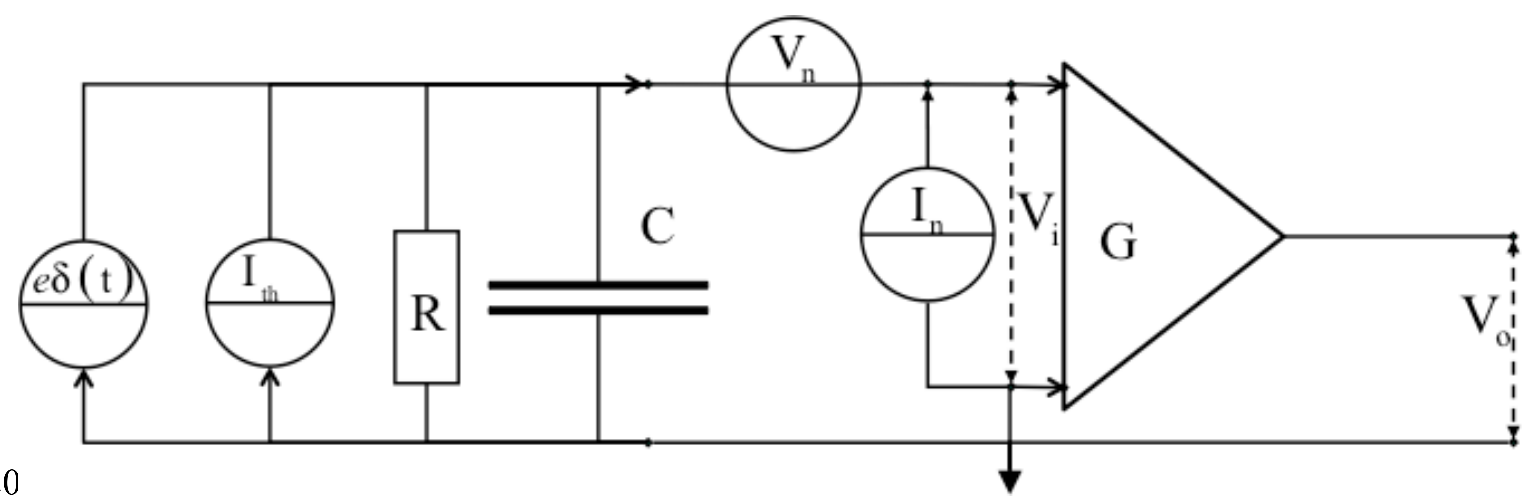
# Example: charge detector

- Noise. Let's first calculate transfer functions from generators to $V_i$. As usual we do that by considering noise as standard signals. Then

$$V_i = \left(I_{th} + I_n\right)R\left(1/i\omega C\right)/\left(R + 1/i\omega C\right) + V_n = \left(I_{th} + I_n\right)R/\left(1 + i\omega CR\right) + V_n$$

- As $I_{th}$ and $I_n$ are uncorrelated, we can write the PSD as (we omit the frequency dependence of $T_n$ and $R_n$)

$$S_{V_n^{in}V_n^{in}} = \left(2k_B T/R + k_B T_n/R_n\right)\left[R^2/\left(1 + \omega^2 C^2 R^2\right)\right] + k_B T_n R_n$$



# Example: charge detector



- From signal $V_i(\omega) = R/(1+i\omega CR)$
- And PSD $S_{V_n^{in}V_n^{in}} = \left(2k_B T/R + k_B T_n/R_n\right)\left[R^2/\left(1+\omega^2C^2R^2\right)\right] + k_B T_n R_n$
- We calculate the SNR as

$$SNR = \left[R^2/\left(1+\omega^2C^2R^2\right)\right] / \left\{\left(2k_B T/R + k_B T_n/R_n\right)\left[R^2/\left(1+\omega^2C^2R^2\right)\right] + k_B T_n R_n\right\}$$

- That can be simplified to

$$SNR = 1/\left\{2k_B T/R + k_B T_n/R_n + k_B T_n R_n\left(1+\omega^2C^2R^2\right)/R^2\right\}$$

- Notice the denominator is the PSD of the noise converted into an input noise current. At this same point the signal is just $I(\omega)=1$
- The SNR can be put in a more simple form

$$SNR = \frac{1}{2k_B T/R + k_B T_n/R_n + k_B T_n R_n/R^2 + \left(k_B T_n/R_n\right)\omega^2 R_n^2 C^2} = \frac{1}{S_{Io}\left(1+\tau_e^2\omega^2\right)}$$

- Where

$$S_{Io} = \left(k_B T_n/R_n\right)\left(1+\left(R_n/R\right)^2 + \left(2T/T_n\right)\left(R_n/R\right)\right); \tau_e = R_n C/\sqrt{1+\left(R_n/R\right)^2 + \left(2T/T_n\right)\left(R_n/R\right)}$$

# Example: charge detector

- According to optimal filter theory, the error on the estimate of the charge $e$ is given by $\sigma_e = 1\Big/\sqrt{(1/2\pi)\int_{-\infty}^{\infty} \mathrm{SNR}(\omega)\,d\omega}$
- with

$$\mathrm{SNR} = 1\Big/\left[S_{Io}\left(1+\tau_e^2\omega^2\right)\right]$$

- To perform the calculation, we assume that both $T_n$ and $R_n$ are frequency independent then

$$\sigma_e = \sqrt{S_{Io}}\Big/\sqrt{(1/2\pi)\int_{-\infty}^{\infty}\left(1+\tau_e^2\omega^2\right)^{-1} d\omega} = \sqrt{2S_{Io}\tau_e}$$

- Now restore the meaning of symbols:

$$S_{Io} = k_B T_n/R_n\left(1+(R_n/R)^2+(2T/T_n)(R_n/R)\right);\ \tau_e = R_n C\Big/\sqrt{1+(R_n/R)^2+(2T/T_n)(R_n/R)}$$

- Then $\sigma_e = \sqrt{2S_{Io}\tau_e} = \sqrt{2k_B T_n C}\times\sqrt{\sqrt{1+(R_n/R)^2+(2T/T_n)(R_n/R)}}$
- Best you can do is when the detector is cold enough and not too lossy

$$(2T/T_n)(R_n/R) \ll 1 \text{ and } (R_n/R) \ll 1$$

- Then

$$\sigma_e = \sqrt{2k_B T_n C}$$

# Example: charge detector



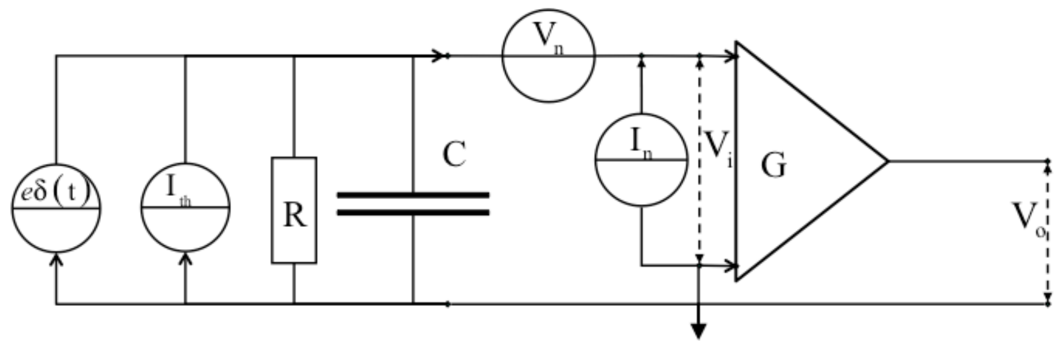
- In summary the minimum error on the estimate of the charge is

$$\sigma_e = \sqrt{2k_B T_n C}$$

- Now suppose that a charge equal to $\sigma_e$ is deposited onto the capacitor. This acquires an electrostatic energy.

$$E = \frac{1}{2}\frac{e^2}{C} = \frac{1}{2}\frac{2k_B T_n C}{C} = k_B T_n = E_n$$

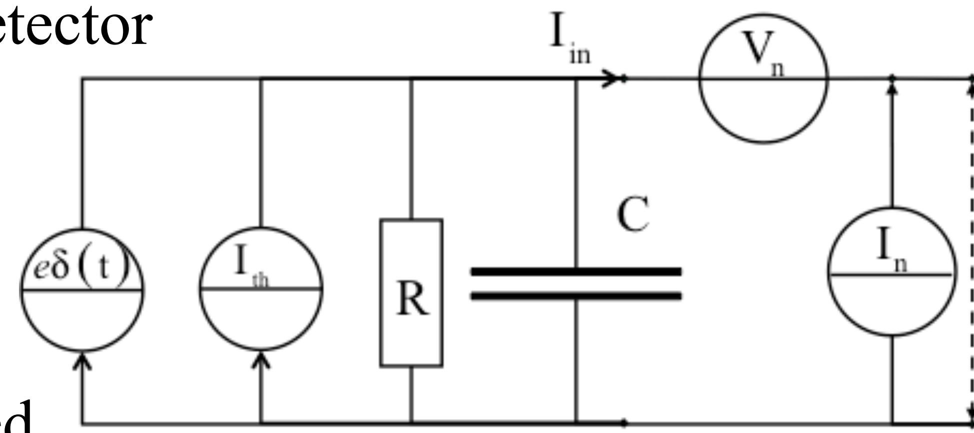
- *Thus the "minimum detectable charge" (with 100% error) is that depositing an amount of energy equal to the noise energy of the amplifier.*
- This result is valid in general not just for charge detection.



# Further on the example of charge detector



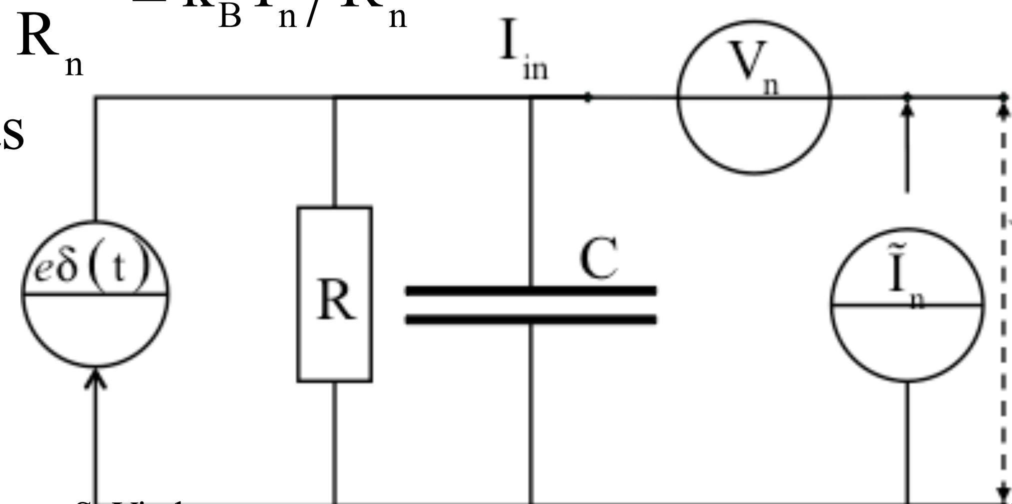
- Equivalent circuit for charge detector




- $V_n$ and $I_n$ white and uncorrelated
- We can define an effective total current noise $\tilde{I}_n = I_n + I_{th}$
- With PSD

$$\tilde{S}_{I_n I_n} = \frac{2k_B T}{R} + \frac{k_B T_n}{R_n} \equiv k_B \tilde{T}_n / \tilde{R}_n$$

- and then simplify the circuit as




$$\tilde{T}_n = \sqrt{\left(\frac{2T}{R} + \frac{T_n}{R_n}\right) T_n R_n} = T_n \sqrt{1 + 2\frac{R_n T}{R T_n}}$$

$$\tilde{R}_n = \sqrt{\frac{T_n R_n}{2T/R + T_n/R_n}} = \frac{R_n}{\sqrt{1 + 2\left(R_n T / R T_n\right)}}$$

# Total noise at input

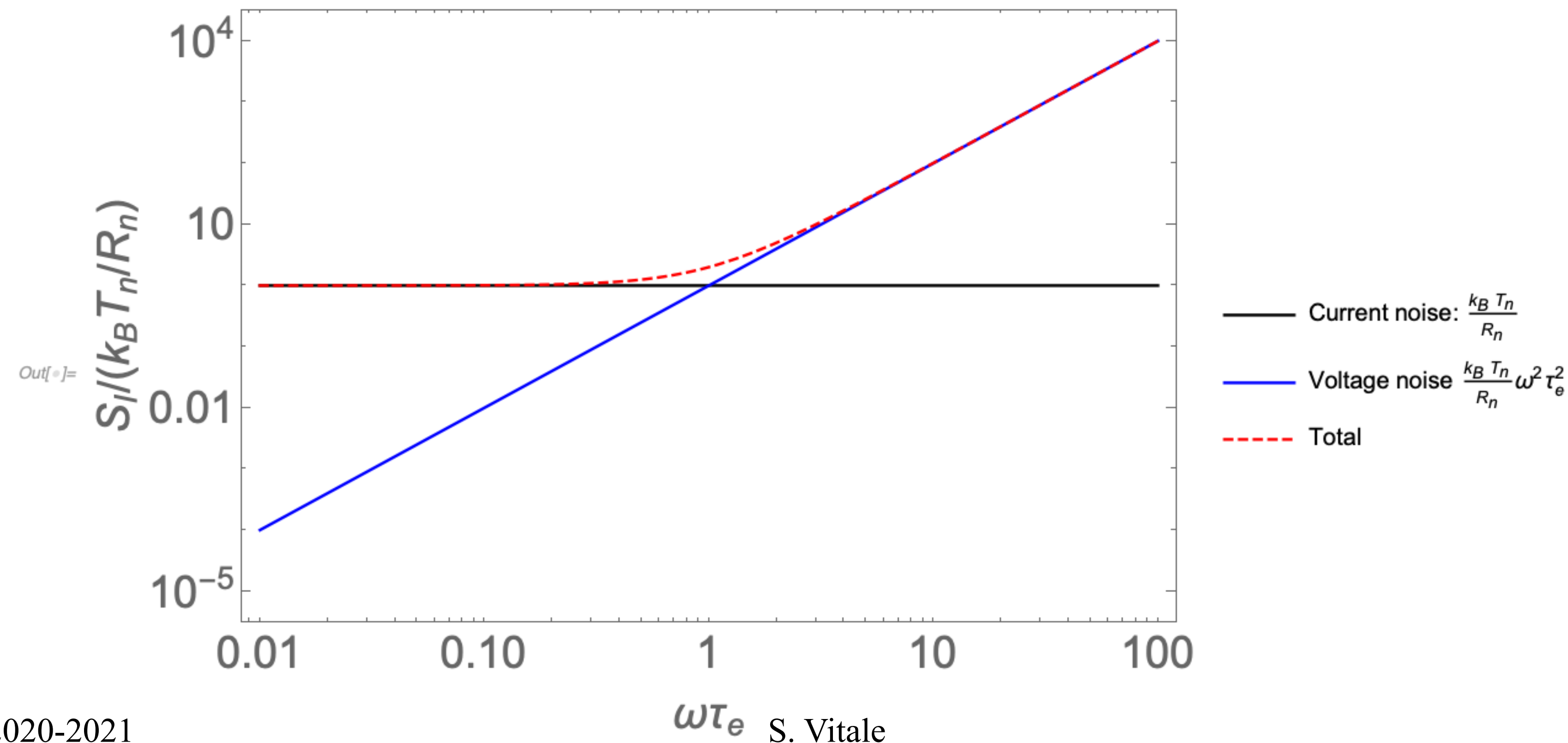


- Total equivalent current noise through Z

$$S_{I_nI_n}=\frac{2k_BT}{R}+\frac{k_BT_n}{R_n}+\frac{k_BT_nR_n}{\frac{R^2}{1+\omega^2R^2C^2}}=\frac{2k_BT}{R}+\frac{k_BT_n}{R_n}+\frac{k_BT_nR_n}{R^2}+\frac{k_BT_nR_n}{R^2}\omega^2R^2C^2=S_{Io}(1+\omega^2\tau_e^2)$$

$$S_{Io}=\frac{2k_BT}{R}+\frac{k_BT_n}{R_n}+\frac{k_BT_nR_n}{R^2}\qquad \tau_e^2=C^2R_n^2\frac{T_n/R_n}{\frac{2T}{R}+\frac{T_n}{R_n}+\frac{T_nR_n}{R^2}}$$

- For $R\to\infty$ $\;S_{Io}\to\frac{k_BT_n}{R_n}\quad \tau_e\to CR_n$



# The optimum template

- A few additional notes on the exercise

2. The filter template.

According to theory, the best estimate is obtained by integrating the data multiplied by a template the Fourier transform of which is

$$h(\omega)=\sigma_e^2\,V_i(\omega)\Big/S_{V_n^{in}V_n^{in}}(\omega)$$

From previous calculations

$$h(\omega)=\sigma_e^2\frac{V_{in}(\omega)}{S_{V_n^{in}V_n^{in}}(\omega)}=\sigma_e^2\frac{\frac{R}{1+i\omega RC}}{\frac{S_{II}R^2}{1+\omega^2R^2C^2}+S_{VV}}=\frac{\sigma_e^2}{R}\frac{1-i\omega RC}{S_{II}+\frac{S_{VV}}{R^2}(1+\omega^2R^2C^2)}$$

That can be rewritten as

$$h(\omega)=\frac{\sigma_e^2}{R}\frac{1-i\omega RC}{S_{Io}\left(1+\omega^2\tau_e^2\right)}$$