UNIVERSITÀ DEGLI STUDI DI TRENTO

Experimental Methods
Lecture 25

November 18th, 2020

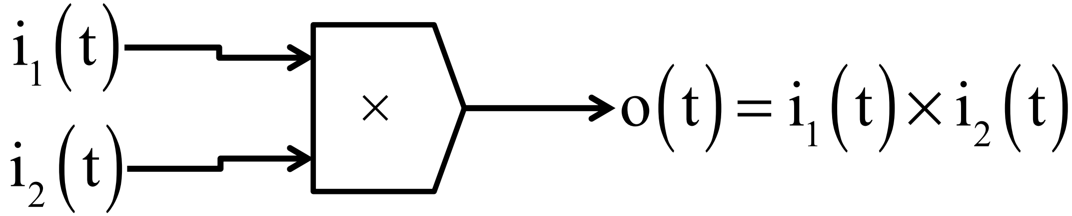
Frequency conversion and narrow-band signals

Introduction: frequency conversion

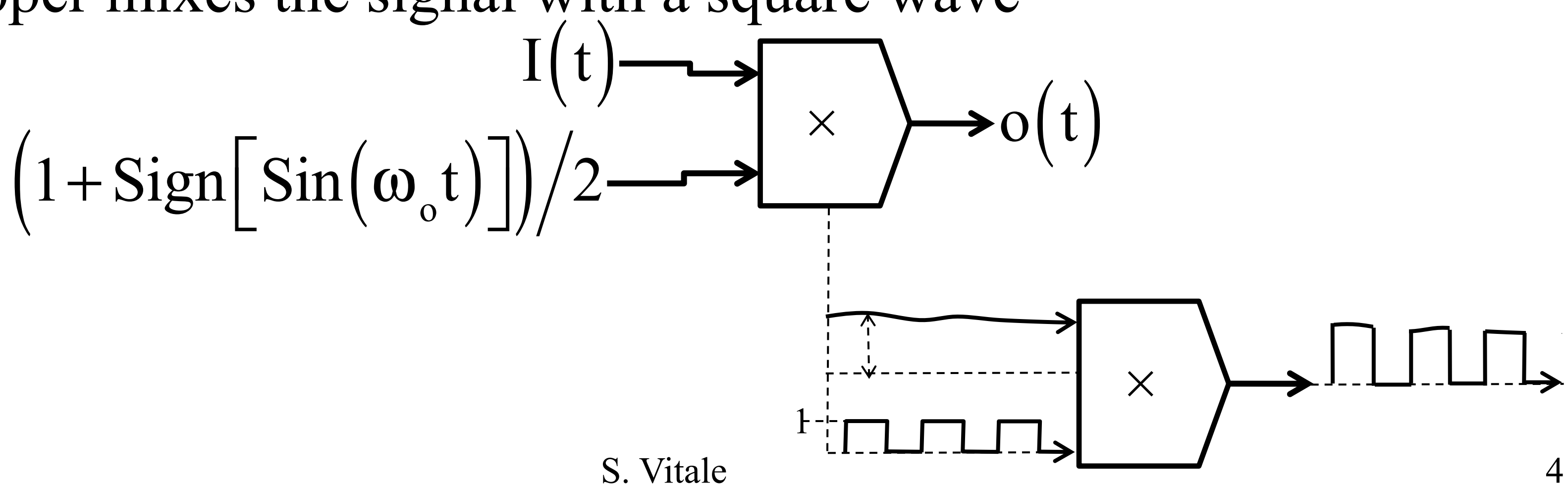
- PSD of most disturbances is not white, thus frequency ranges where the noise is lower may allow high precision measurements otherwise made impossible by low SNR.
- Thus in many experiments, physicists need changing the frequency of a signal and moving it to frequencies where PSD of disturbances is lower. This is called frequency conversion.

Frequency conversion and the mixer

- Linear systems *cannot* change signal frequency. To achieve frequency conversion you need some non-linear element. The most important example of such an element is the multiplier or *mixer*.
- The mixer is an ideal memory-less element that performs the product of two input signals like in the following scheme.

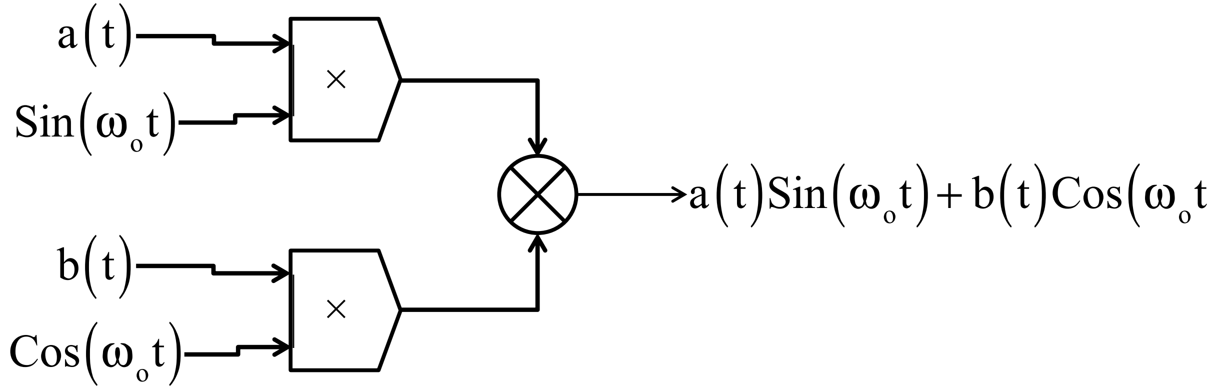


- The chopper mixes the signal with a square wave



The more general narrow band signal

- Suppose you have two (slowly varying signals) a(t) and b(t). You may use two mixers to multiply them by the two phase of a "carrier" at frequency ω_o, according to the following scheme:



- And generate a "narrow-band signal" at the "carrier frequency" ω_o
- a(t) is called the "in-phase" component of the signal and b(t) the "quadrature" component.
- Notice that

$$a(t)Sin(\omega_o t) + b(t)Cos(\omega_o t) = \sqrt{a^2(t) + b^2(t)}Sin\left(\omega_o t + ArcTan\left(\frac{b(t)}{a(t)}\right)\right)$$

$$\sqrt{a^2(t) + b^2(t)} = \text{amplitude} \quad ArcTan\left(\frac{b(t)}{a(t)}\right) = \text{phase}$$

$$\frac{dArcTan\left(\frac{b(t)}{a(t)}\right)}{dt} = \text{frequency modulation}$$

The Fourier transform of a narrow-band signal

- The signal $s(t) = a(t)\mathrm{Sin}(\omega_o t) + b(t)\mathrm{Cos}(\omega_o t)$
- Its Fourier transform

$$s(\omega) = (1/2\pi)\int_{-\infty}^{\infty} a(\omega')(\pi/i)\left[\delta(\omega - \omega_o - \omega') - \delta(\omega + \omega_o - \omega')\right]d\omega'$$
$$+(1/2\pi)\int_{-\infty}^{\infty} b(\omega')\pi\left[\delta(\omega - \omega_o - \omega') + \delta(\omega + \omega_o - \omega')\right]d\omega'$$

that is

$$s(\omega) = (1/2i)\left[a(\omega - \omega_o) - a(\omega + \omega_o)\right] + (1/2)\left[b(\omega - \omega_o) + b(\omega + \omega_o)\right]$$

- As a(t) and b(t) are low frequency signals:

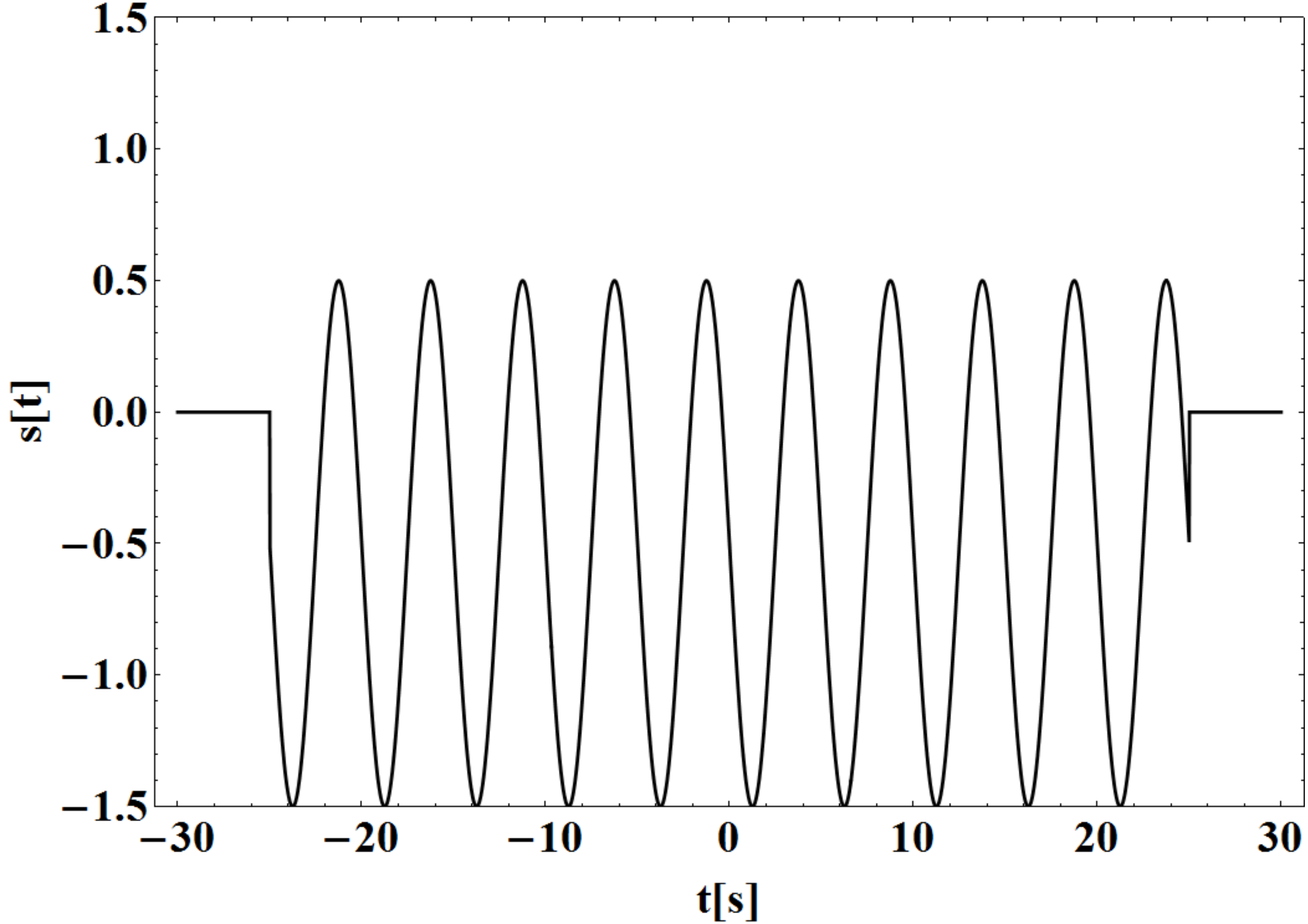
$$|a(\omega)| = |b(\omega)| = 0 \quad |\omega| \geq \omega_{max} << \omega_o$$

then s(ω) consists of two lines, of width $\approx\omega_{max}$, at $\pm\omega_o$

- A numerical example follows

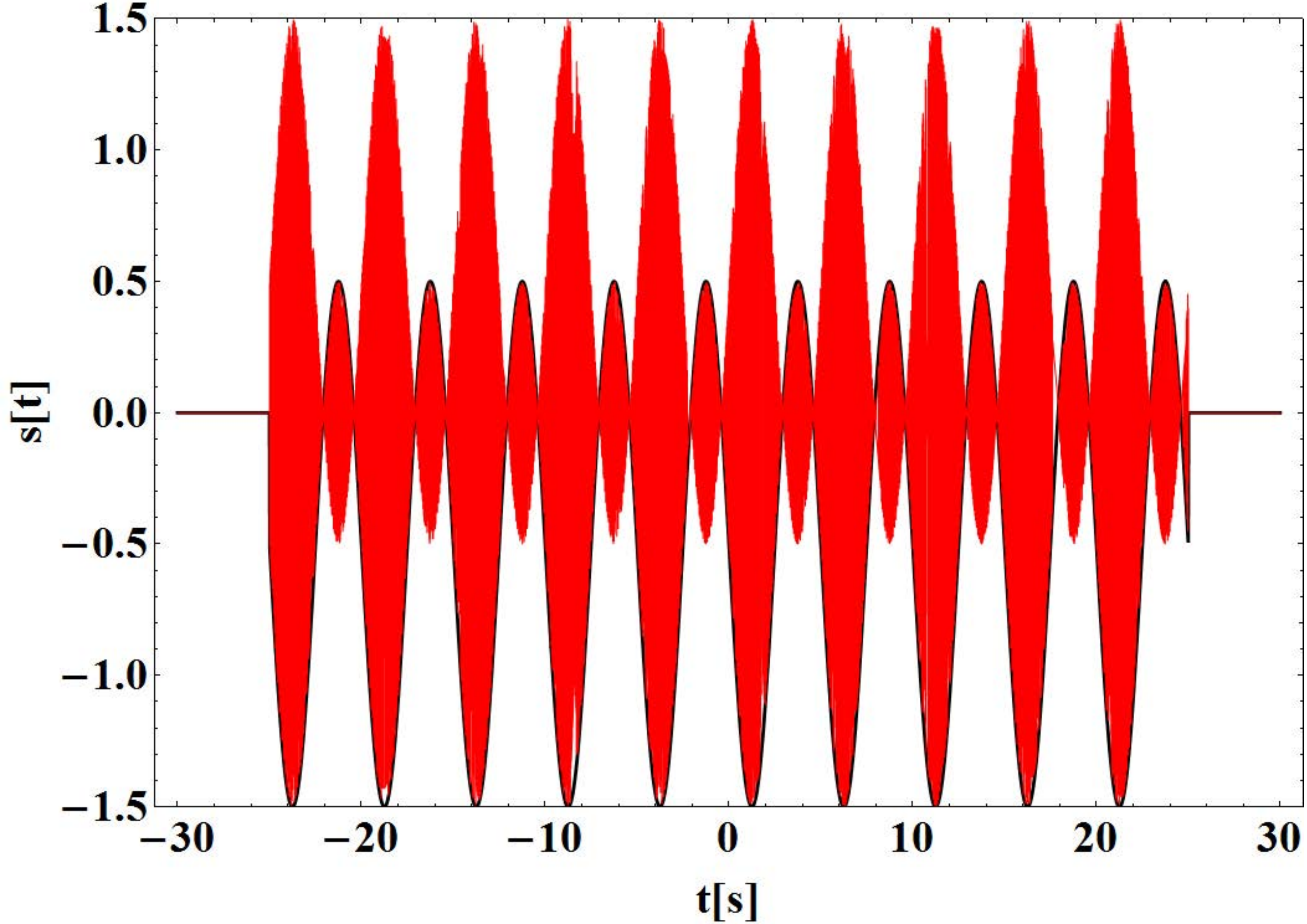
A numerical example. 1 low frequency signal

$$a(t) = \Pi(t/T)\left\{c + \mathrm{Sin}(\omega_m t)\right\}; c = 0.5; T = 50\,\mathrm{s}; \omega_m = 2\pi\, 0.2\,\mathrm{Hz}$$

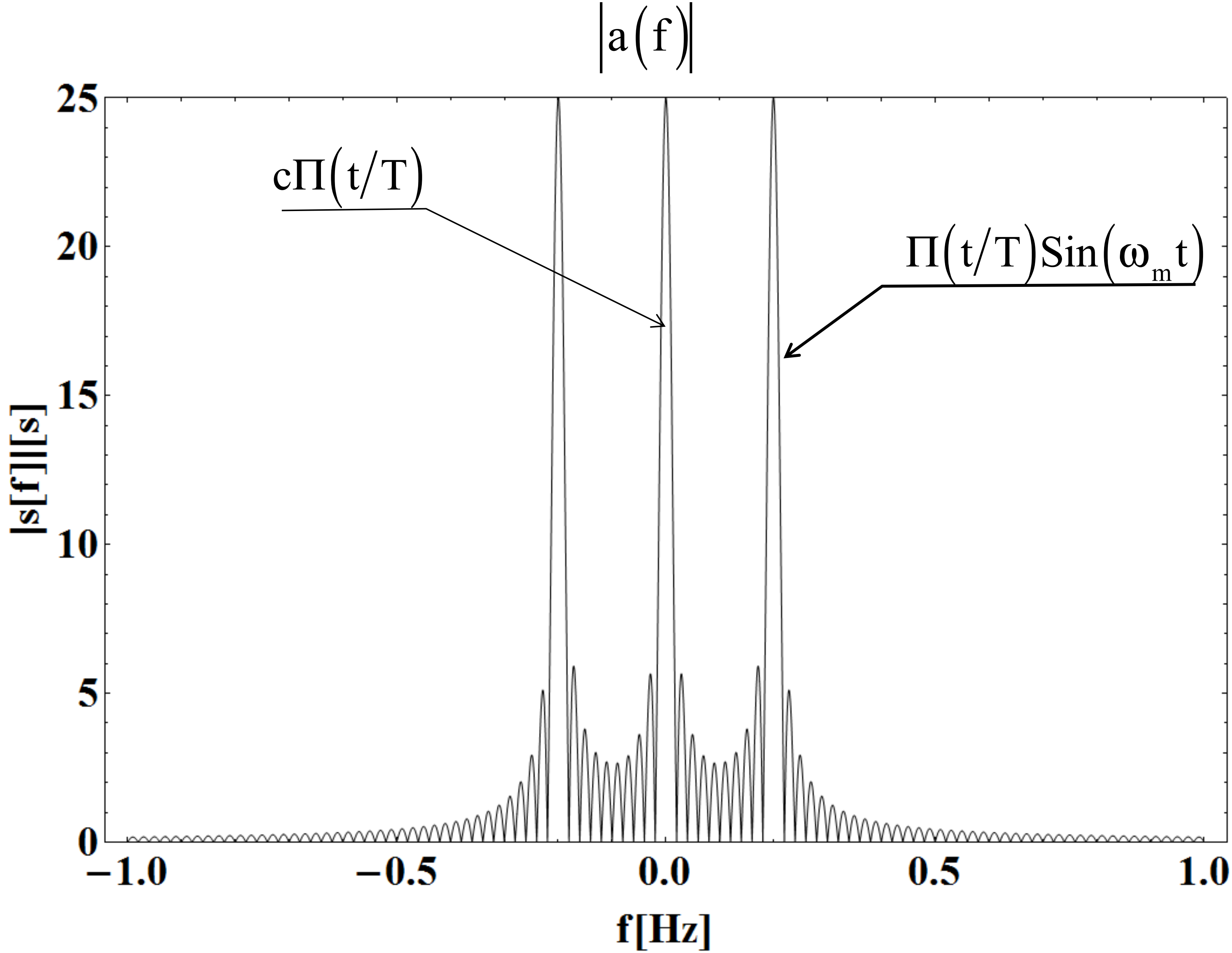


Modulating the carrier

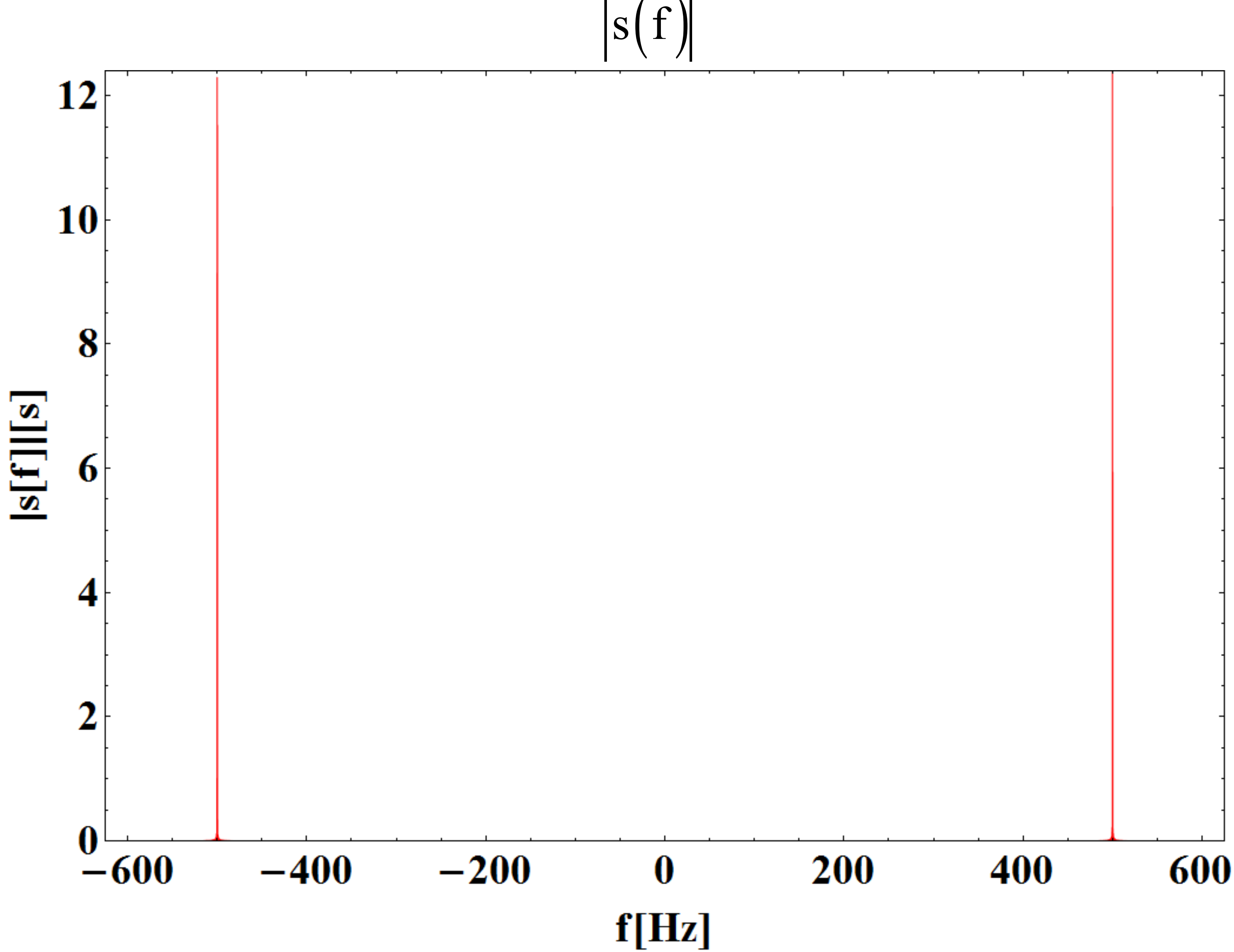
$$s(t) = a(t)\,\mathrm{Sin}(\omega_c t) \quad \omega_c = 2\pi\, 0.5\,\mathrm{kHz}$$



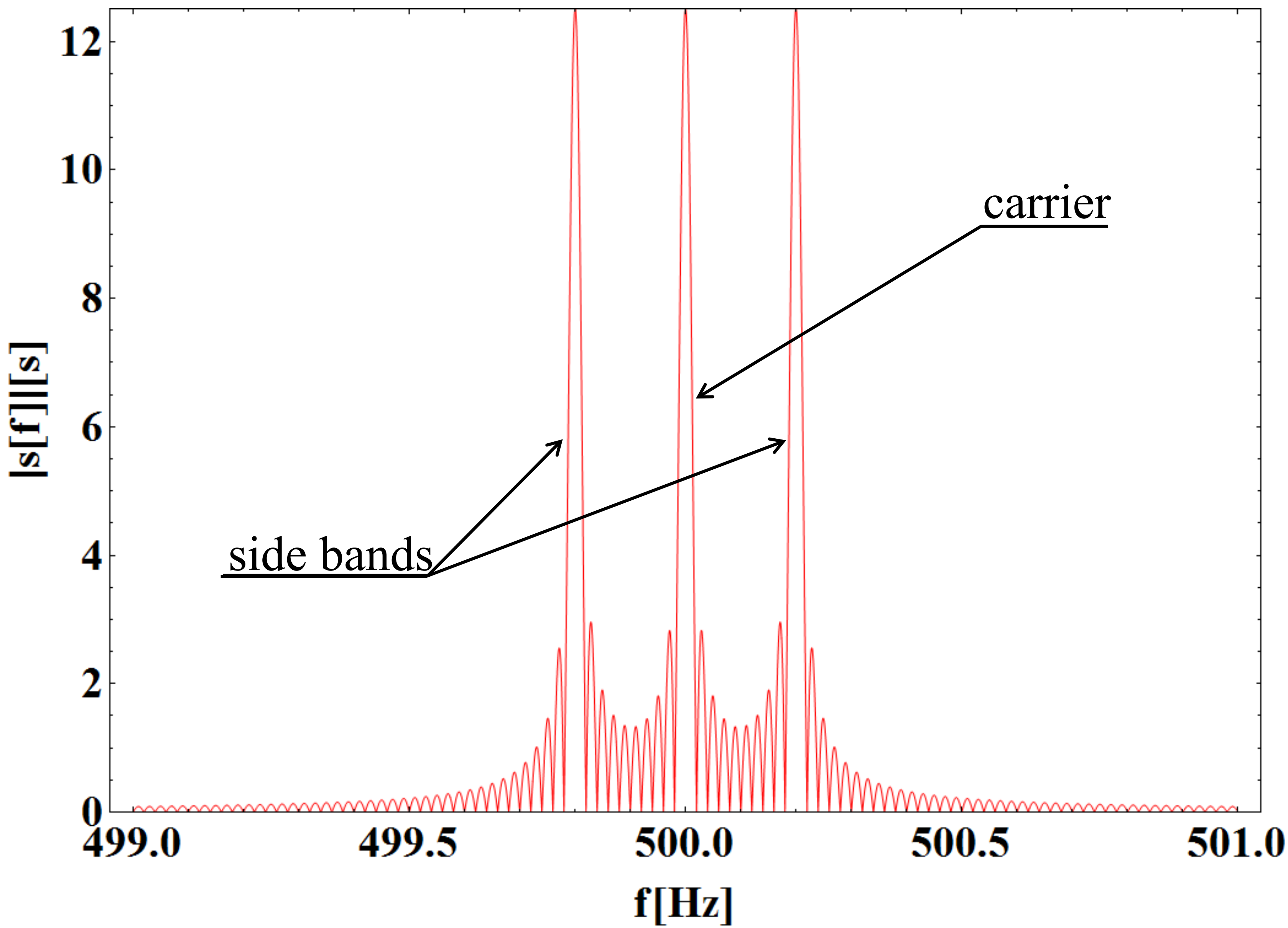
Fourier transform of modulating signal



Fourier transform of signal



A blow-up around carrier frequency

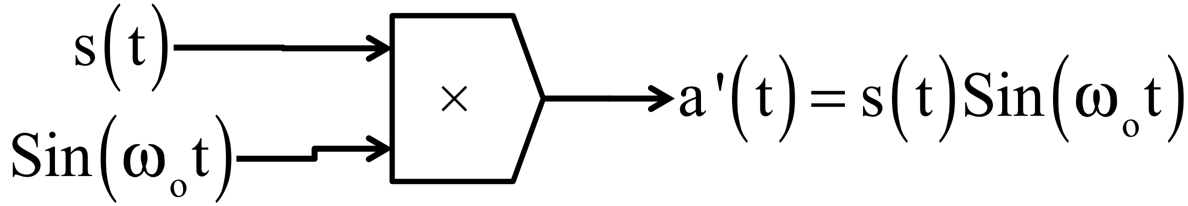


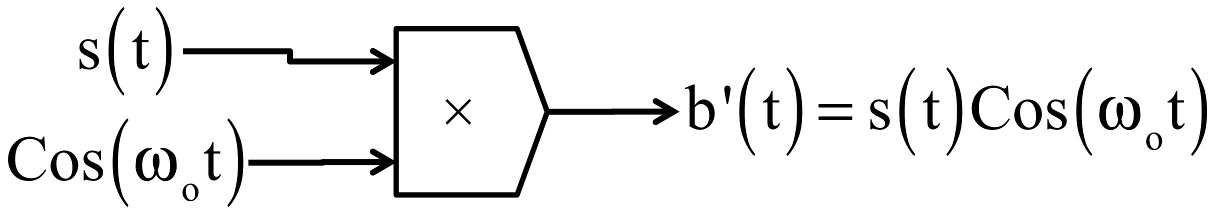
Frequency down-conversion

- In many applications, converting a signal to a higher frequency is not the final step. In these applications the signal is converted to higher frequency in order to transmit it throughout an environment/instrument that is noisy at low-frequency.
- On the other side of the instrument however, the original signal needs to be restored. An example of this is the radio, wherein one wants to recover the sound signal at the receiver.
- In the following we discuss :
  - How to restore the low frequency signal.
  - What happens to noise while performing these operations.

Recovering the components of a narrow band signal

- Take our narrow-band signal $s(t) = a(t)\mathrm{Sin}(\omega_o t) + b(t)\mathrm{Cos}(\omega_o t)$
- Our aim is to recover a(t) and b(t). Let's set up two mixers





- Let's expand a'(t) $a'(t) = a(t)\mathrm{Sin}^2(\omega_o t) + b(t)\mathrm{Cos}(\omega_o t)\mathrm{Sin}(\omega_o t)$

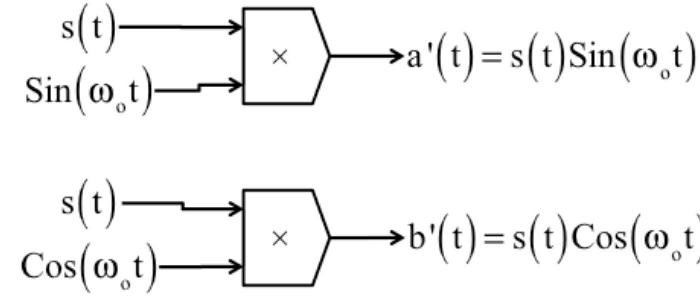
$$= a(t)\left[1 - \mathrm{Cos}(2\omega_o t)\right]/2 + b(t)\mathrm{Sin}(2\omega_o t)/2$$

- Its Fourier transform is $a'(\omega) = a(\omega)/2 -$

$$-(1/4)\left[a(\omega - 2\omega_o) + a(\omega + 2\omega_o)\right] + (1/4i)\left[b(\omega - 2\omega_o) - b(\omega + 2\omega_o)\right]$$

- as $|a(\omega)| = |b(\omega)| = 0 \quad |\omega| \geq \omega_{max} << \omega_o$
- Then a'(ω) is a copy of a(ω) plus two lines at $\pm 2\omega_o$

Recovering the components of a narrow band signal



- Thus at the output of our mixers there a two signal a'(t) and b'(t) whose Fourier transforms are (check for b')

$$a'(\omega)=\frac{a(\omega)}{2}-\frac{1}{4}\left[a(\omega-2\omega_o)+a(\omega+2\omega_o)\right]+\frac{1}{4i}\left[b(\omega-2\omega_o)-b(\omega+2\omega_o)\right]$$

$$b'(\omega)=\frac{b(\omega)}{2}+\frac{1}{4}\left[b(\omega-2\omega_o)+b(\omega+2\omega_o)\right]+\frac{1}{4i}\left[a(\omega-2\omega_o)-a(\omega+2\omega_o)\right]$$

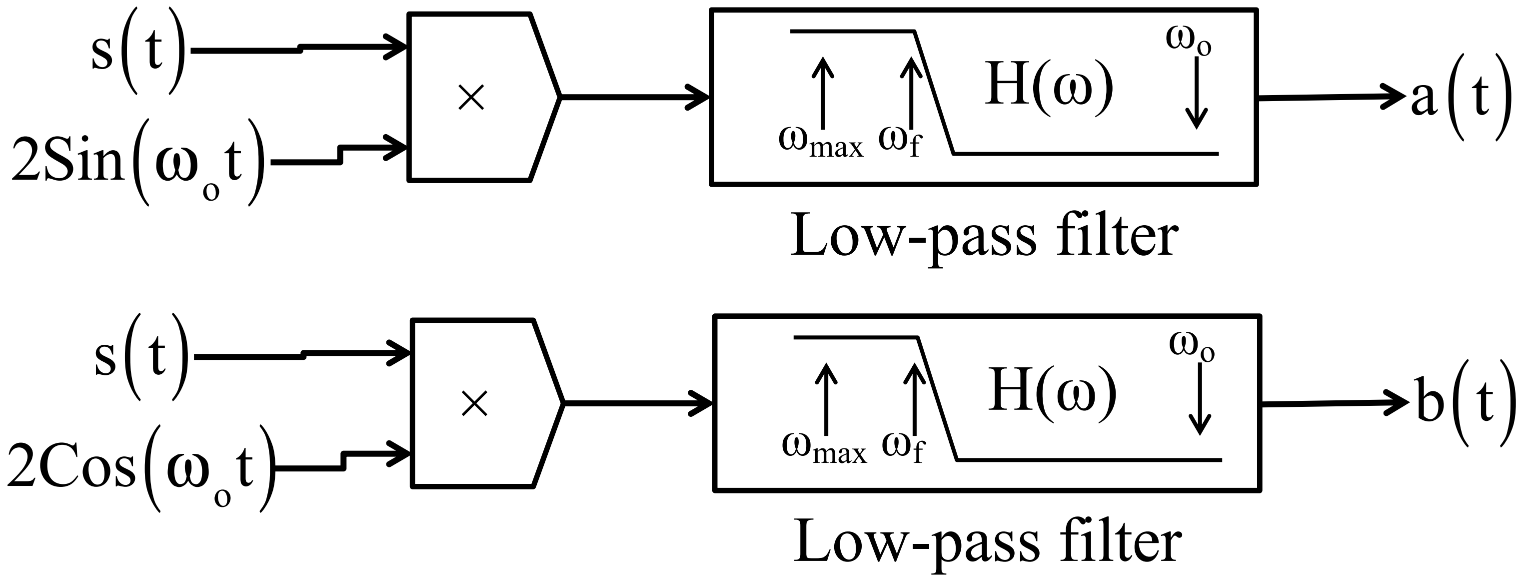
that are copies of $a(\omega)$ and $b(\omega)$ respectively plus two lines at $\pm 2\omega_o$

- As $a(\omega)$ and $b(\omega)$ are zero for $|\omega| \geq \omega_{max} << \omega_o$, these signals may be filtered with a low pass filter with transfer function $H(\omega)=1$ up to a roll-off frequency ω_f such that $\omega_{max} < \omega_f < \omega_o$. Then

$$a'(\omega)\simeq H(\omega)\frac{a(\omega)}{2}\simeq\frac{a(\omega)}{2};\quad b'(\omega)\simeq H(\omega)\frac{b(\omega)}{2}\simeq\frac{b(\omega)}{2}$$

Recovering the components of a narrow band signal

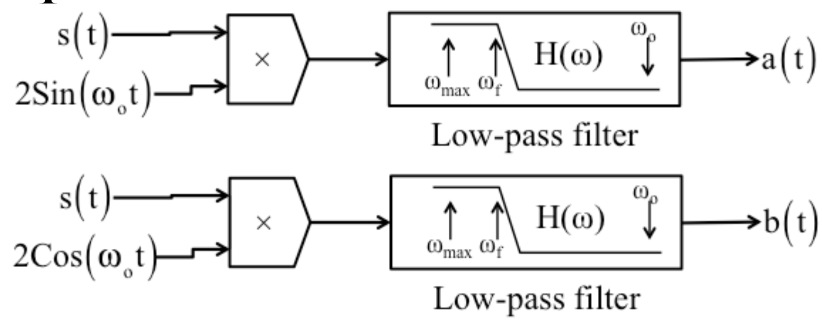
- In conclusion the following scheme



Recovers the “phases” a(t) and b(t) and is called a *phase sensitive detector*. The two oscillating inputs are called the *reference channels*.

- Warning. It is important that $\omega_o >> \omega_{max}$ as you need to accommodate the filter frequency ω_f in between. A typical rule is $\omega_o > 10\ \omega_{max}$. To achieve this result, a low pass filter with a roll-off at ω_{max} it is often used *before up-conversion*.

Recovering amplitude an frequency modulation from the phase sensitive detector



- Amplitude modulation $s(t) = s_o\left[1 + m(t)\right]\mathrm{Sin}(\omega_o t + \phi)$
- The information is contained within m(t) that can be obtained from

$$m(t) = \sqrt{a^2(t) + b^2(t)}\Big/ s_o - 1$$

- Frequency modulation

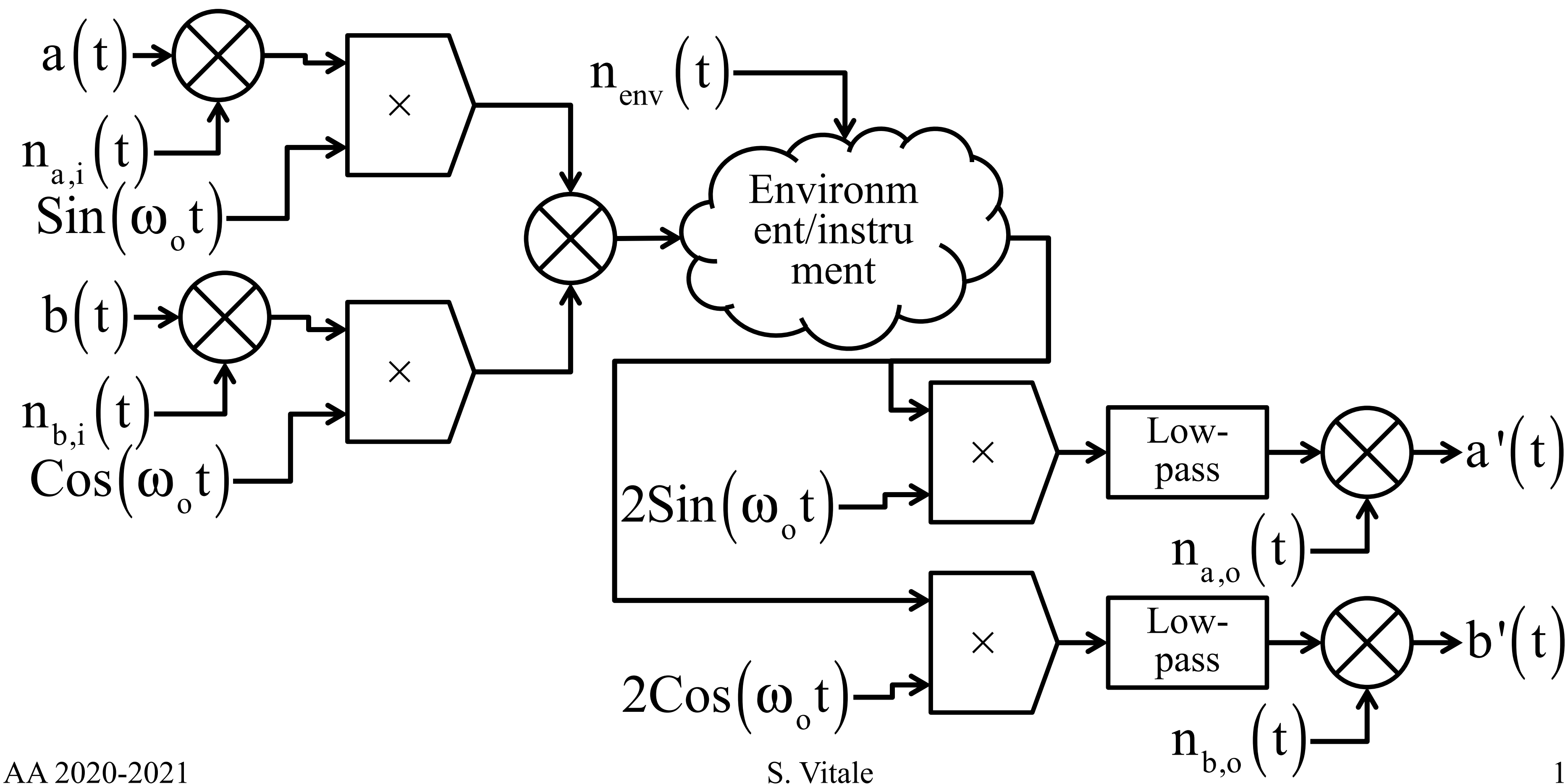
$$s(t) = s_o\mathrm{Sin}\left[\omega_o t + \phi(t)\right] = s_o\mathrm{Cos}\left[\phi(t)\right]\mathrm{Sin}(\omega_o t) + s_o\mathrm{Sin}\left[\phi(t)\right]\mathrm{Cos}(\omega_o t)$$

- The frequency modulation signal can be obtained as

$$\dot{\phi}(t) = \frac{d\mathrm{ArcTan}\left[b(t)/a(t)\right]}{dt} = \frac{a(t)\dot{b}(t) - \dot{a}(t)b(t)}{a(t)^2 + b(t)^2} = \frac{a(t)\dot{b}(t) - \dot{a}(t)b(t)}{s_o^2}$$

Noise and frequency conversion

- As illustrated in the scheme below, noise can enter in many points of the up-conversion/down conversion chain:at input, together with a(t) or b(t) ($n_{a,i}$, $n_{b,i}$) , after up-conversion (n_{env}), after down-conversion ($n_{o,i}$, $n_{o,i}$) . See next slide for discussion

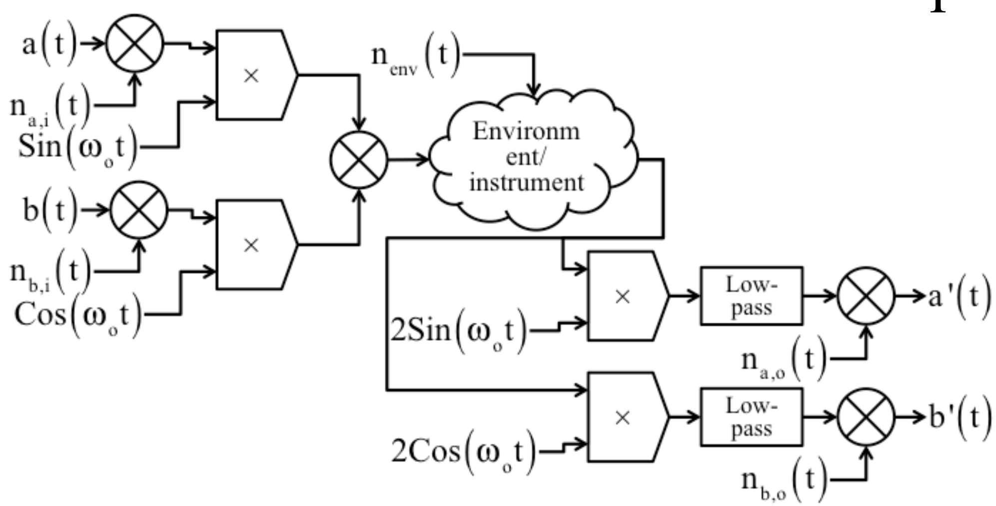


Noise and frequency conversion

- We will now discuss what is the effect of the various sources of noise on the chain outputs. We will first discuss the role of the input and of the output noise sources. However remember that this chain has been set up to beat the low-frequency environmental noise $n_{env}(t)$. This contribution will be discussed in a somewhat richer detail later on.
- Input noise is transferred at output as signal is. Thus the in-phase component will carry a term

$$a_{n_i}(t)=\int_0^\infty dt' H(t')\left[n_{a,i}(t-t')\left(1-Cos\left(2\omega_o(t-t')\right)\right)+n_{b,i}(t-t')Sin\left(2\omega_o(t-t')\right)\right]$$

with H(t) the inverse Fourier transform of the low-pass filter transfer function



Noise and frequency conversion

- The in-phase component carries a term

$$a_{n_i}(t)=\int_0^\infty dt' H(t')\left[n_{a,i}(t-t')\left(1-Cos\left(2\omega_o(t-t')\right)\right)+n_{b,i}(t-t')Sin\left(2\omega_o(t-t')\right)\right]$$

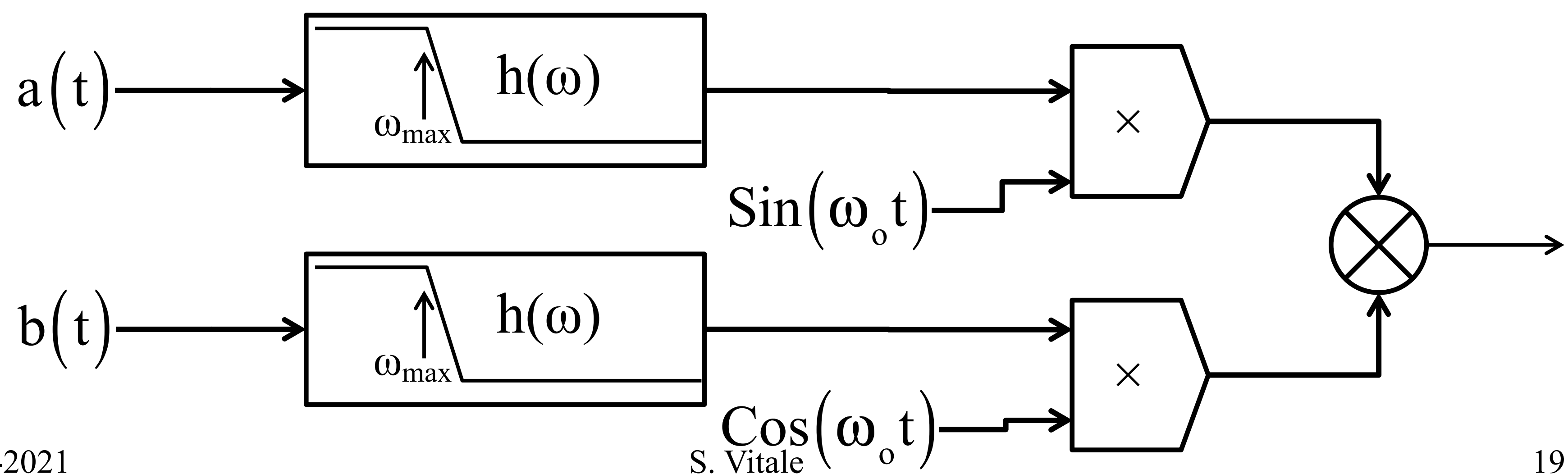
- Symmetrically the quadrature component carries

$$b_{n_i}(t)=\int_0^\infty dt' H(t')\left[n_{b,i}(t-t')\left(1-Cos\left(2\omega_o(t-t')\right)\right)+n_{a,i}(t-t')Sin\left(2\omega_o(t-t')\right)\right]$$

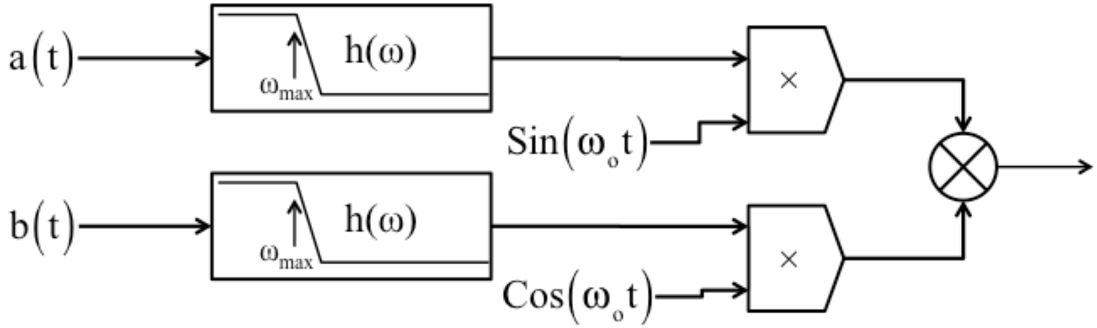
- We assume here that both $n_{a,i}$ and $n_{b,i}$ have spectra such that

$$S_{n_{a,i}n_{a,i}}(\omega),S_{n_{b,i}n_{b,i}}(\omega)\simeq 0 \quad |\omega|\geq\omega_{max}$$

- This condition sometimes may be achieved in practice by using a low-pass filter with a roll-off at ω_{max}, before up-conversion:



Noise and frequency conversion



- With the precaution above it is easy to convince yourself that, as the noise is low frequency, the components in

$$a_{n_i}(t)=\int_0^\infty dt' H(t')\Big[n_{a,i}(t-t')\big(1-\mathrm{Cos}\big(2\omega_o(t-t')\big)\big)+n_{b,i}(t-t')\mathrm{Sin}\big(2\omega_o(t-t')\big)\Big]$$

that are multiplied by the function oscillating at $2\omega_o$ are suppressed for noise as for the signal so that in reality

$$a_{n_i}(t)=\int_0^\infty H(t')n_{a,i}(t-t')dt';\ b_{n_i}(t)=\int_0^\infty H(t')n_{b,i}(t-t')dt'$$

- One can calculate then that these are two stationary processes with PSDs given by

$$S_{a_{n_i}a_{n_i}}(\omega)=S_{n_{a,i}n_{a,i}}(\omega)\left|H(\omega)\right|^2\simeq S_{n_{a,i}n_{a,i}}(\omega);\ S_{b_{n_i}b_{n_i}}(\omega)\simeq S_{n_{b,i}n_{b,i}}(\omega)$$

- The following pages contain a more formal derivation.

This is the the noise contribution at output

$$a_{n_i}[t] = \int_0^\infty H[t'] \left(n_{a,i}[t-t'] \left(1-\mathrm{Cos}[2\,\omega_o\,(t-t')]\right) + n_{b,i}[t-t']\,\mathrm{Sin}[2\,\omega_o\,(t-t')]\right) dt'$$

Its autocorrelation is

$$\langle a_{n_i}[t_1]\,a_{n_i}[t_2]\rangle = \left\langle \int_0^\infty \int_0^\infty H[t']\,H[t''] \left(n_{a,i}[t_2-t''] \left(1-\mathrm{Cos}[2\,\omega_o\,(t_2-t'')]\right) + n_{b,i}[t_2-t'']\,\mathrm{Sin}[2\,\omega_o\,(t_2-t'')]\right) \left(n_{a,i}[t_1-t'] \left(1-\mathrm{Cos}[2\,\omega_o\,(t_1-t')]\right) + n_{b,i}[t_1-t']\,\mathrm{Sin}[2\,\omega_o\,(t_1-t')]\right) dt''\,dt' \right\rangle$$

Evaluating the mean value and assuming that $n_{a,i}$ and $n_{b,i}$ are uncorrelated

$$R_{a_{n_i} a_{n_i}}[t_1, t_2] = \int_0^\infty \int_0^\infty H[t']\,H[t'']\,R_{n_{a,i} n_{a,i}}[t_1-t_2-t'+t''] \left(1-\mathrm{Cos}[2\,\omega_o\,(t_2-t'')]\right) \left(1-\mathrm{Cos}[2\,\omega_o\,(t_1-t')]\right) dt''\,dt' + \int_0^\infty \int_0^\infty H[t']\,H[t'']\,R_{n_{b,i} n_{b,i}}[t_1-t_2-t'+t'']\,\mathrm{Sin}[2\,\omega_o\,(t_2-t'')]\,\mathrm{Sin}[2\,\omega_o\,(t_1-t')]\,dt''\,dt'$$

Now let's express the autocorrelation in terms of the PSD

$$R_{a_{n_i}a_{n_i}}[t_1,t_2]=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{n_{a,i}n_{a,i}}(\omega)\left(\int_0^{\infty}\int_0^{\infty}H[t']\,H[t'']\,e^{i\omega(t_1-t_2-t'+t'')}\,(1-\mathrm{Cos}[2\omega_o(t_2-t'')])\,(1-\mathrm{Cos}[2\omega_o(t_1-t')])\,dt''\,dt'\right)d\omega+$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{n_{b,i}n_{b,i}}(\omega)\left(\int_0^{\infty}\int_0^{\infty}H[t']\,H[t'']\,e^{i\omega(t_1-t_2-t'+t'')}\,\mathrm{Sin}[2\omega_o(t_2-t'')]\,\mathrm{Sin}[2\omega_o(t_1-t')]\,dt''\,dt'\right)d\omega$$

This can be reshuffled as

$$R_{a_{n_i}a_{n_i}}[t_1,t_2]=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{n_{a,i}n_{a,i}}(\omega)\,e^{i\omega(t_1-t_2)}\left(\int_0^{\infty}\int_0^{\infty}H[t']\,H[t'']\,e^{i\omega(t''-t')}\,(1-\mathrm{Cos}[2\omega_o(t_2-t'')])\,(1-\mathrm{Cos}[2\omega_o(t_1-t')])\,dt''\,dt'\right)d\omega+$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}S_{n_{b,i}n_{b,i}}(\omega)\,e^{i\omega(t_1-t_2)}\left(\int_0^{\infty}\int_0^{\infty}H[t']\,H[t'']\,e^{i\omega(t''-t')}\,\mathrm{Sin}[2\omega_o(t_2-t'')]\,\mathrm{Sin}[2\omega_o(t_1-t')]\,dt''\,dt'\right)d\omega$$

and furthermore as

$$R_{a_{n_i}\,a_{n_i}}[t_1, t_2] =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{n_{a,i}\,n_{a,i}}(\omega)\, e^{i\omega(t_1-t_2)} \left(\int_0^\infty H[t']\, e^{-i\omega t'}\,(1-\mathrm{Cos}[2\,\omega_o\,(t_1-t')])\, dt'\right) \left(\int_0^\infty H[t'']\, e^{i\omega t''}\,(1-\mathrm{Cos}[2\,\omega_o\,(t_2-t'')])\, dt''\right) d\omega +$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{n_{b,i}\,n_{b,i}}(\omega)\, e^{i\omega(t_1-t_2)} \left(\int_0^\infty H[t']\, e^{-i\omega t'}\,\mathrm{Sin}[2\,\omega_o\,(t_1-t')]\, dt'\right) \left(\int_0^\infty H[t'']\, e^{i\omega t''}\,\mathrm{Sin}[2\,\omega_o\,(t_2-t'')]\, dt''\right) d\omega$$

Let's now eveluate the following integral

$$\int_0^\infty H[t']\, e^{-i\omega t'}\,(1-\mathrm{Cos}[2\,\omega_o\,(t_1-t')])\, dt' =$$

$$\int_0^\infty H[t']\, e^{-i\omega t'}\, dt' + \mathrm{Cos}[2\,\omega_o\,t_1]\int_0^\infty H[t']\, e^{-i\omega t'}\,\mathrm{Cos}[2\,\omega_o\,t']\, dt' +$$

$$\mathrm{Sin}[2\,\omega_o\,t_1]\int_0^\infty H[t']\, e^{-i\omega t'}\,\mathrm{Sin}[2\,\omega_o\,t']\, dt'$$

You may recognize then that

$$\int_0^\infty H[t']\, e^{-i\omega t'}\,(1-\mathrm{Cos}[2\,\omega_o\,(t_1-t')])\, dt' = H[\omega] +$$

$$\mathrm{Cos}[2\,\omega_o\,t_1]\,\frac{H[\omega-2\,\omega_o]+H[\omega+2\,\omega_o]}{2} + \mathrm{Sin}[2\,\omega_o\,t_1]\,\frac{H[\omega-2\,\omega_o]-H[\omega+2\,\omega_o]}{2\,i}$$

Similarly

$$\int_0^{\infty} H[t'] \, e^{-i \omega t'} \, \mathrm{Sin}[2 \omega_o (t_1 - t')] \, dt' = H[\omega] +$$

$$\mathrm{Sin}[2 \omega_o t_1] \, \frac{H[\omega - 2 \omega_o] + H[\omega + 2 \omega_o]}{2} + \mathrm{Cos}[2 \omega_o t_1] \, \frac{H[\omega - 2 \omega_o] - H[\omega + 2 \omega_o]}{2 i}$$

Now the function of ω above must be used within an integral over ω where they are multiplied by $S_{n_{a,i} n_{a,i}}(\omega)$ or $S_{n_{b,i} n_{b,i}}(\omega)$. Both these functions die out at $\omega_{max} << \omega_o$. Thus $|2 \omega_o \pm \omega| \simeq |2 \omega_o|$. As $H[\omega]$ is zero when $|\omega| \geq \omega_{max}$ the terms multiplied by $\mathrm{Sin}[2 \omega_o t_1]$ or by $\mathrm{Cos}[2 \omega_o t_1]$ are negligible and the same is true for those where t_2 replaces t_1.
Thus in conclusion

$$R_{a_{n_i} a_{n_i}}[t_1, t_2] = \frac{1}{2 \pi} \int_{-\infty}^{\infty} S_{n_{a,i} n_{a,i}}(\omega) \, |H[\omega]|^2 \, e^{i \omega (t_1 - t_2)} \, d\omega$$

Thus the autocorreation depends only on delay, a_{n_i} is stationary and has a PSD of

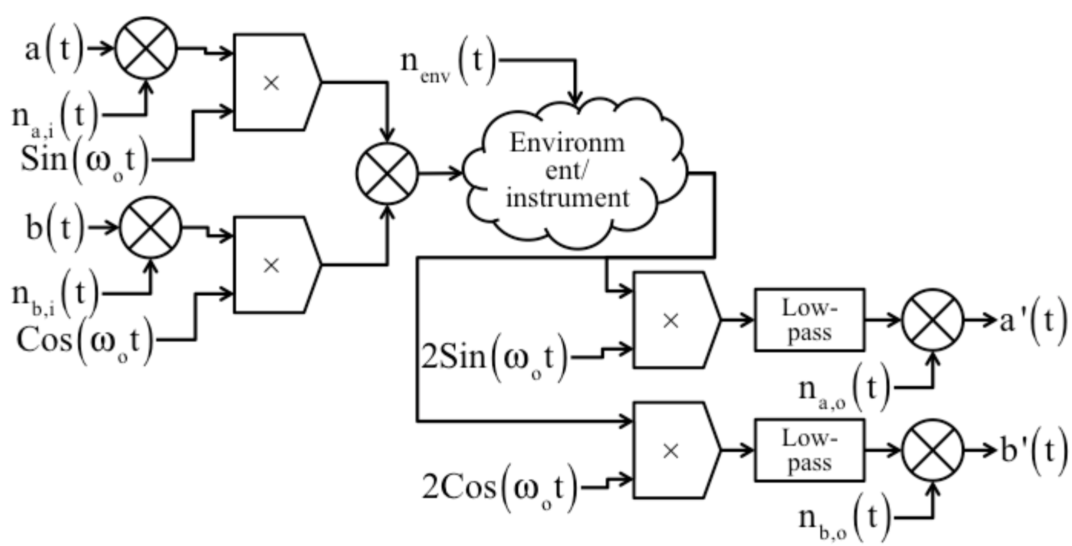
$$S_{a_{n_i} a_{n_i}}[\omega] = S_{n_{a,i} n_{a,i}}(\omega) \, |H[\omega]|^2 \simeq S_{n_{a,i} n_{a,i}}(\omega)$$

Output noise

- Output noise does not require any discussion. As $n_{a,o}$ and $n_{b,o}$ are added at the end of the chain, their PSDs are unaffected by the chain itself.
- As $n_{a,o}$ and $n_{b,o}$ are independent of $n_{a,i}$ and $n_{b,i}$, they will add incoherently such that at output the total PSD due to these contributions will be

$$S_{a_{n_{i,o}} a_{n_{i,o}}}(\omega) = S_{n_{a,i} n_{a,i}}(\omega) + S_{n_{a,o} n_{a,o}}(\omega); \; S_{b_{n_i} b_{n_i}}(\omega) = S_{n_{b,i} n_{b,i}}(\omega) + S_{n_{b,o} n_{b,o}}(\omega)$$

- The chain does not affect the noise added at the input nor that added at the output

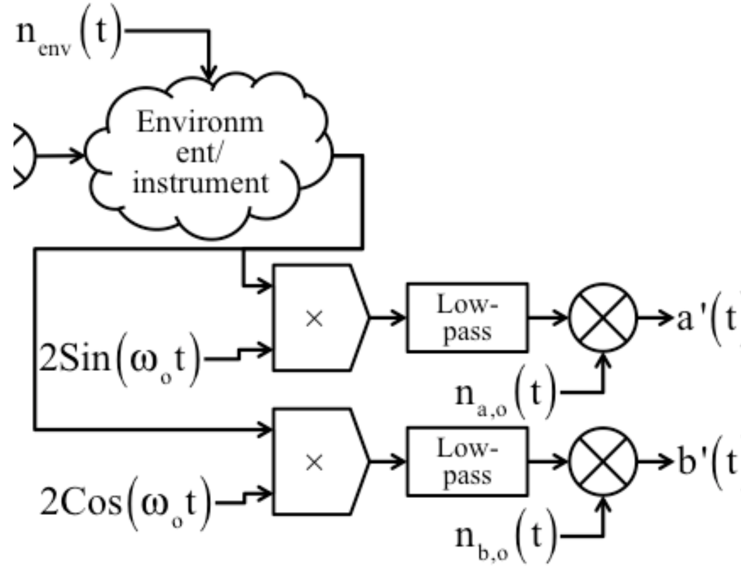


Environmental noise

- The environmental noise adds up at the input of the final Phase Sensitive Detector contributing a term at the output of both phases.

$$a_{n_e}(t) = \int_0^\infty dt' H(t') n_{env}(t-t') 2Sin(\omega_o(t-t'))$$

$$b_{n_e}(t) = \int_0^\infty dt' H(t') n_{env}(t-t') 2Cos(\omega_o(t-t'))$$



- Consider now the following. If an ordinary, low frequency signal enters the Phase Sensitive Detector it gets converted to a frequency around ω_o and filtered away by the low pass.
- If instead the signal is *at frequency ω_o* the multiplication by the carrier produces a signal at dc and a signal at $2\omega_o$

$$ASin(\omega_o t) \times 2Sin(\omega_o t) = A(1 - Cos(2\omega_o t))$$

- While the signal at $2\omega_o$ is suppressed by the low-pass filter the one at dc will be transmitted

Environmental noise

- The consequence of that is that while the low-frequency environmental noise is indeed suppressed by the up-conversion-down-conversion scheme, the noise around the carrier frequency is transmitted.
- We will show that in practical condition the PSD and cross spectral densities of $a_{n_e}(t)$ and $b_{n_e}(t)$ are

$$S_{a_{n,e}a_{n,e}}(\omega) = S_{b_{n,e}b_{n,e}}(\omega) \simeq 2\left|H(\omega)\right|^2 S_{n_{env}n_{env}}(\omega_o) \qquad S_{a_{n,e}b_{n,e}}(\omega) \simeq 0$$

Environmental noise

The two noise-contributed phase signals

$$a_{n_e}(t)=\int_0^\infty dt' H(t') n_{env}(t-t') 2\mathrm{Sin}\left[\omega_o(t-t')\right]$$

$$b_{n_e}(t)=\int_0^\infty dt' H(t') n_{env}(t-t') 2\mathrm{Cos}\left[\omega_o(t-t')\right]$$

are stochastic processes. However, due to the presence of the oscillating function they are not stationary.

We now want to calculate the correlation properties of these processes. We start with the autocorrelation of a:

$$R_{a_{n_e}a_{n_e}}(t_1,t_2)=\left\langle a_{n_e}(t_1) a_{n_e}(t_2)\right\rangle=$$

$$=4\int_0^\infty\int_0^\infty dt'dt'' H(t')H(t'')\left\langle n_{env}(t_1-t') n_{env}(t_2-t'')\right\rangle \mathrm{Sin}\left[\omega_o(t_1-t')\right]\mathrm{Sin}\left[\omega_o(t_2-t'')\right]$$

$$=4\int_0^\infty\int_0^\infty dt'dt'' H(t')H(t'')R_{n_{env},n_{env}}(t_2-t_1+t'-t'') \mathrm{Sin}\left[\omega_o(t_1-t')\right]\mathrm{Sin}\left[\omega_o(t_2-t'')\right]$$

many pages of calculation follow ☺

Environmental noise

Autocorrelation

$$R_{a_{n_e}a_{n_e}}\left(t_1,t_2\right)=$$

$$=4\int_0^\infty\int_0^\infty dt'dt''H\left(t'\right)H\left(t''\right)R_{n_{env},n_{env}}\left(t_2-t_1+t'-t''\right)\mathrm{Sin}\left[\omega_o\left(t_1-t'\right)\right]\mathrm{Sin}\left[\omega_o\left(t_2-t''\right)\right]$$

We can use the definition of PSD to write

$$R_{a_{n_e}a_{n_e}}\left(t_1,t_2\right)=\frac{4}{2\pi}\int_{-\infty}^{\infty}d\omega\, S_{n_{env}n_{env}}\left(\omega\right)\times$$

$$\times\int_0^\infty\int_0^\infty dt'dt''H\left(t'\right)H\left(t''\right)e^{i\omega\left(t_2-t_1-t''+t'\right)}\mathrm{Sin}\left[\omega_o\left(t_1-t'\right)\right]\mathrm{Sin}\left[\omega_o\left(t_2-t''\right)\right]$$

so that

$$R_{a_{n_e}a_{n_e}}\left(t_1,t_2\right)=\frac{4}{2\pi}\int_{-\infty}^{\infty}d\omega\, S_{n_{env}n_{env}}\left(\omega\right)e^{i\omega\left(t_2-t_1\right)}\times$$

$$\int_0^\infty dt''H\left(t'\right)e^{+i\omega t'}\mathrm{Sin}\left[\omega_o\left(t_1-t'\right)\right]\times\int_0^\infty dt'H\left(t'\right)e^{-i\omega t'}\mathrm{Sin}\left[\omega_o\left(t_2-t'\right)\right]$$

Environmental noise

Continuing from

$$R_{a_{n_e} a_{n_e}}(t_1,t_2)=\frac{4}{2\pi}\int_{-\infty}^{\infty} d\omega\, S_{n_{env} n_{env}}(\omega) e^{i\omega(t_2-t_1)}\times$$

$$\int_0^{\infty} dt'' H(t') e^{+i\omega t'} \mathrm{Sin}\left[\omega_o(t_1-t')\right]\times\int_0^{\infty} dt' H(t') e^{-i\omega t'} \mathrm{Sin}\left[\omega_o(t_2-t')\right]$$

Consider the following Fourier Transform

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\mathrm{Sin}\left[\omega_o(t_2-t)\right]dt=\frac{\pi}{i}\left[e^{i\omega_o t_2}\delta(\omega+\omega_o)-e^{-i\omega_o t_2}\delta(\omega-\omega_o)\right]$$

so that,

$$\int_0^{\infty} dt'' H(t') e^{+i\omega t'} \mathrm{Sin}\left[\omega_o(t_1-t')\right]=\frac{e^{i\omega_o t_1}H^*(\omega-\omega_o)-e^{-i\omega_o t_1}H^*(\omega+\omega_o)}{2i}$$

$$\int_0^{\infty} dt' H(t') e^{-i\omega t'} \mathrm{Sin}\left[\omega_o(t_2-t')\right]=\frac{e^{i\omega_o t_2}H(\omega+\omega_o)-e^{-i\omega_o t_2}H(\omega-\omega_o)}{2i}$$

Environmental noise

Continuing ….

$$R_{a_{n_e}a_{n_e}}\left(t_1,t_2\right)=\frac{4}{2\pi}\int_{-\infty}^{\infty}d\omega\, S_{n_{env}n_{env}}\left(\omega\right)e^{i\omega\left(t_2-t_1\right)}\times$$

$$\int_0^{\infty}dt''H\left(t'\right)e^{+i\omega t'}\mathrm{Sin}\left[\omega_o\left(t_1-t'\right)\right]\times\int_0^{\infty}dt'H\left(t'\right)e^{-i\omega t'}\mathrm{Sin}\left[\omega_o\left(t_2-t'\right)\right]$$

and

$$\int_0^{\infty}dt''H\left(t'\right)e^{+i\omega t'}\mathrm{Sin}\left[\omega_o\left(t_1-t'\right)\right]=\frac{e^{i\omega_o t_1}H^*\left(\omega-\omega_o\right)-e^{-i\omega_o t_1}H^*\left(\omega+\omega_o\right)}{2i}$$

$$\int_0^{\infty}dt'H\left(t'\right)e^{-i\omega t'}\mathrm{Sin}\left[\omega_o\left(t_2-t'\right)\right]=\frac{e^{i\omega_o t_2}H\left(\omega+\omega_o\right)-e^{-i\omega_o t_2}H\left(\omega-\omega_o\right)}{2i}$$

Substituting

$$R_{a_{n_e}a_{n_e}}\left(t_1,t_2\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, S_{n_{env}n_{env}}\left(\omega\right)e^{i\omega\left(t_2-t_1\right)}\times\left(\left|H\left(\omega-\omega_o\right)\right|^2e^{-i\omega_o\left(t_2-t_1\right)}+\left|H\left(\omega+\omega_o\right)\right|^2e^{i\omega_o\left(t_2-t_1\right)}\right.$$

$$\left.+H^*\left(\omega-\omega_o\right)H\left(\omega+\omega_o\right)e^{i\omega_o\left(t_2+t_1\right)}+H^*\left(\omega+\omega_o\right)H\left(\omega-\omega_o\right)e^{-i\omega_o\left(t_2+t_1\right)}\right)$$

Environmental noise

$$R_{a_{n_e}a_{n_e}}(t_1,t_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, S_{n_{env}n_{env}}(\omega)e^{i\omega(t_2-t_1)}\times\Big(\left|H(\omega-\omega_o)\right|^2 e^{-i\omega_o(t_2-t_1)}+\left|H(\omega+\omega_o)\right|^2 e^{i\omega_o(t_2-t_1)}$$

$$+H^*(\omega-\omega_o)H(\omega+\omega_o)e^{i\omega_o(t_2+t_1)}+H^*(\omega+\omega_o)H(\omega-\omega_o)e^{-i\omega_o(t_2+t_1)}\Big)$$

There is no frequency at which $\left|H(\omega-\omega_o)H(\omega+\omega_o)\right|\neq 0$ then

$$R_{a_{n_e}a_{n_e}}(t_1,t_2)=$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, S_{n_{env}n_{env}}(\omega)e^{i(\omega-\omega_o)(t_2-t_1)}\left|H(\omega-\omega_o)\right|^2+\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, S_{n_{env}n_{env}}(\omega)e^{i(\omega+\omega_o)(t_2-t_1)}\left|H(\omega+\omega_o)\right|^2$$

That is $R_{a_{n_e}a_{n_e}}(t_1,t_2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|H(\omega)\right|^2\left[S_{n_{env}n_{env}}(\omega+\omega_o)+S_{n_{env}n_{env}}(\omega-\omega_o)\right]e^{i\omega(t_2-t_1)}d\omega$

It follows that the autocorrelation depends on the delay t_2-t_1 only. From the definition of PSD

$$S_{a_{n_e}a_{n_e}}(\omega)=\left|H(\omega)\right|^2\left[S_{n_{env}n_{env}}(\omega+\omega_o)+S_{n_{env}n_{env}}(\omega-\omega_o)\right]$$

Environmental noise

Thus indeed both phases due to the environmental noise are two almost stationary processes both with PSD

$$S_{a_{n_e}a_{n_e}}(\omega)=S_{b_{n_e}b_{n_e}}(\omega)=\left|H(\omega)\right|^2\left[S_{n_{env}n_{env}}(\omega-\omega_o)+S_{n_{env}n_{env}}(\omega+\omega_o)\right]$$

The cross correlations should be evaluated from

$$R_{a_{n_e}b_{n_e}}(t_1,t_2)=\left\langle a_{n_e}(t_1)b_{n_e}(t_2)\right\rangle=$$

$$=4\int_0^\infty\int_0^\infty dt'dt''H(t')H(t'')\left\langle n_{env}(t_1-t')n_{env}(t_2-t'')\right\rangle Sin\left(\omega_o(t_1-t')\right)Cos\left(\omega_o(t_2-t'')\right)$$

The results is

$$S_{a_{n_e}b_{n_e}}(\omega)=i\left|H(\omega)\right|^2\left[S_{n_{env}n_{env}}(\omega-\omega_o)-S_{n_{env}n_{env}}(\omega+\omega_o)\right]$$

Environmental noise

In conclusion both a and be carry a contribution from environmental noise with PSD

$$S_{a_{n_e}a_{n_e}}(\omega)=S_{b_{n_e}b_{n_e}}(\omega)=\left|H(\omega)\right|^2\left[S_{n_{env}n_{env}}(\omega-\omega_o)+S_{n_{env}n_{env}}(\omega+\omega_o)\right]$$

Again consider that $H(\omega)\simeq 0$ If $|\omega|\geq\omega_{max}\ll\omega_o$

If the environmental noise is reasonably flat around ω_o then

$$S_{a_{n_e}a_{n_e}}(\omega)=S_{b_{n_e}b_{n_e}}(\omega)\simeq\left|H(\omega)\right|^2\left[S_{n_{env}n_{env}}(-\omega_o)+S_{n_{env}n_{env}}(\omega_o)\right]=\boxed{2\left|H(\omega)\right|^2 S_{n_{env}n_{env}}(\omega_o)}$$

Thus the spectral shape is determined by the transfer function of the low-pass filter while the intensity is that of the environmental noise *at the carrier frequency*

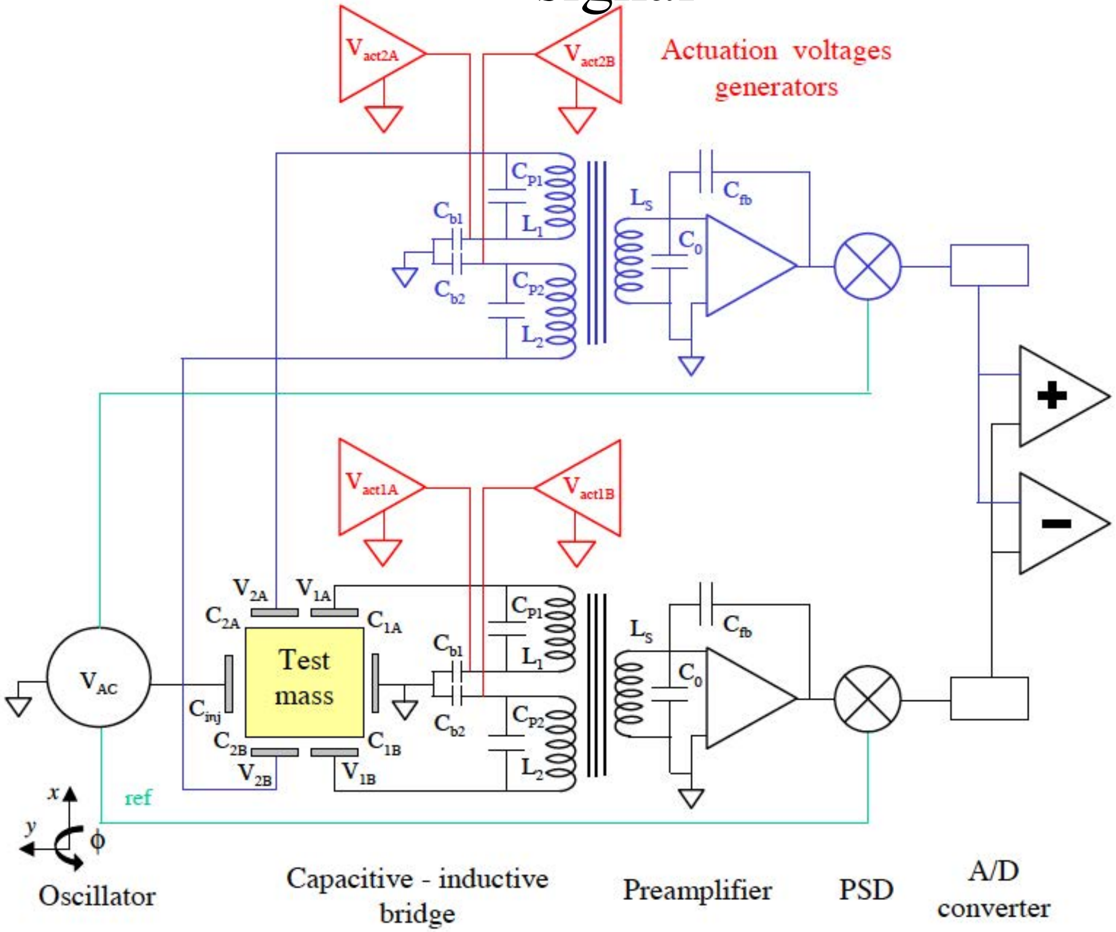
Cross-correlation between the phases is

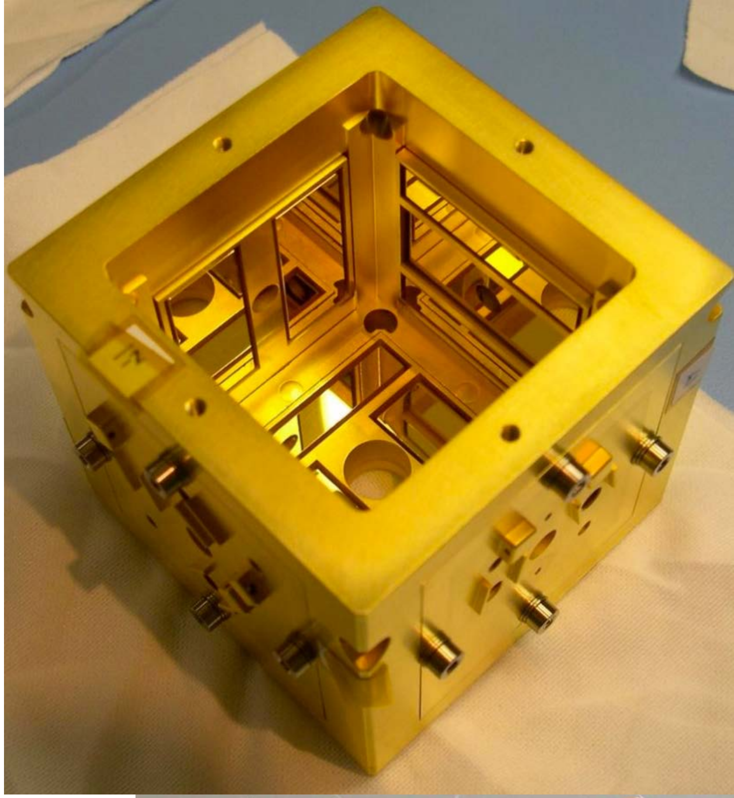
$$S_{a_{n_e}b_{n_e}}(\omega)=i\left|H(\omega)\right|^2\left[S_{n_{env}n_{env}}(\omega-\omega_o)-S_{n_{env}n_{env}}(\omega+\omega_o)\right]$$

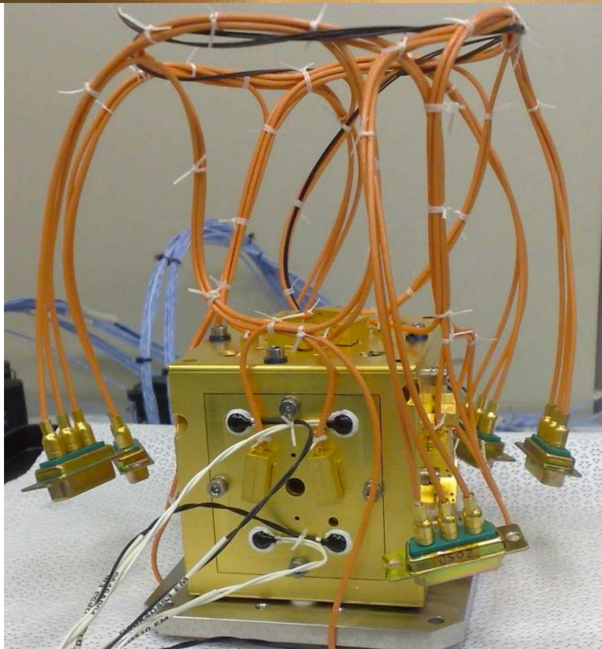
In the same limit as before

$$S_{a_{n_e}b_{n_e}}(\omega)\approx i\left|H(\omega)\right|^2\left[S_{n_{env}n_{env}}(-\omega_o)-S_{n_{env}n_{env}}(\omega_o)\right]=0$$

Some application: capacitive motion sensor:
the test-mass motion modulates the amplitude of a 100 kHz signal
A PhSD is used to recover the motion signal

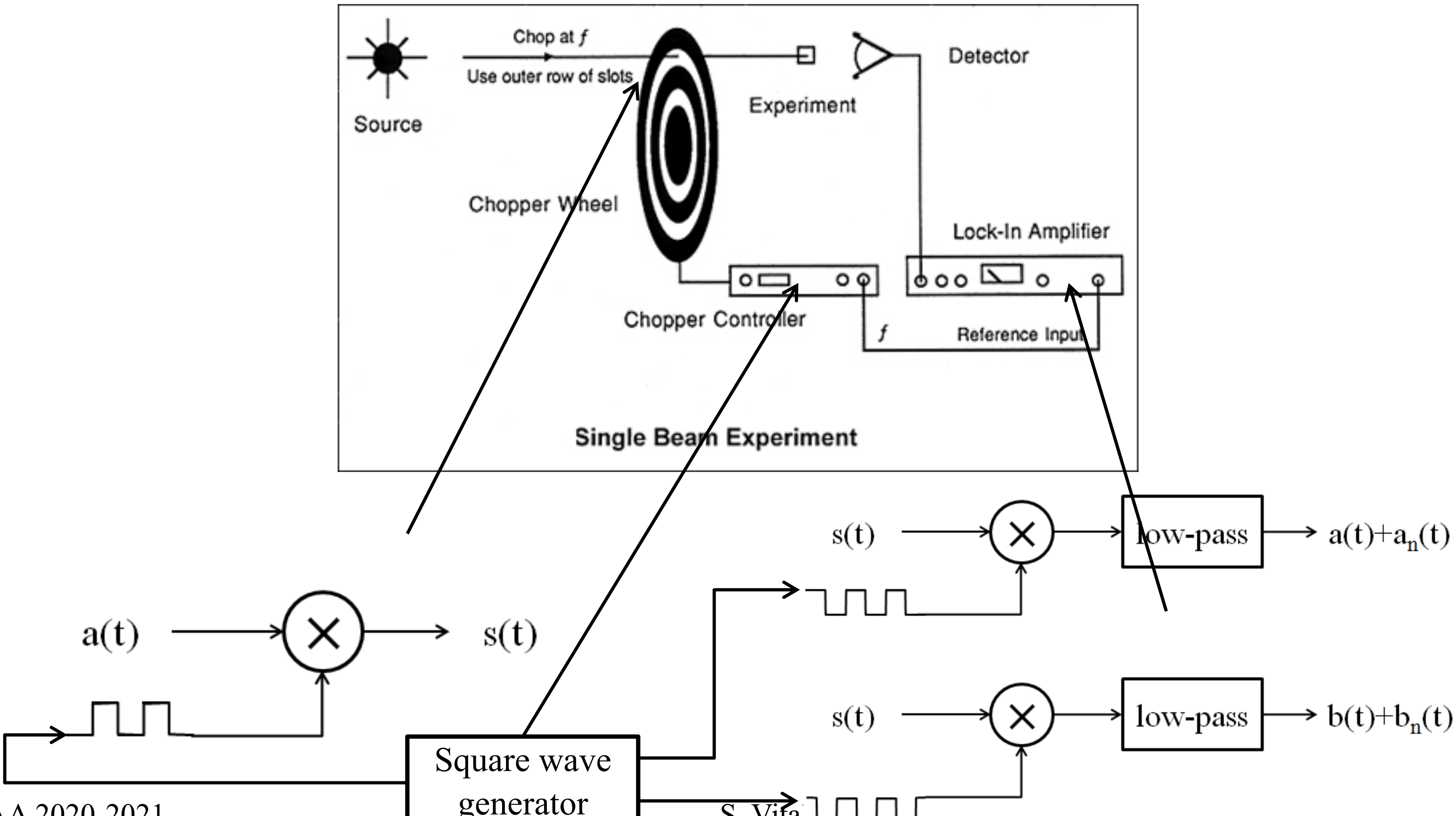






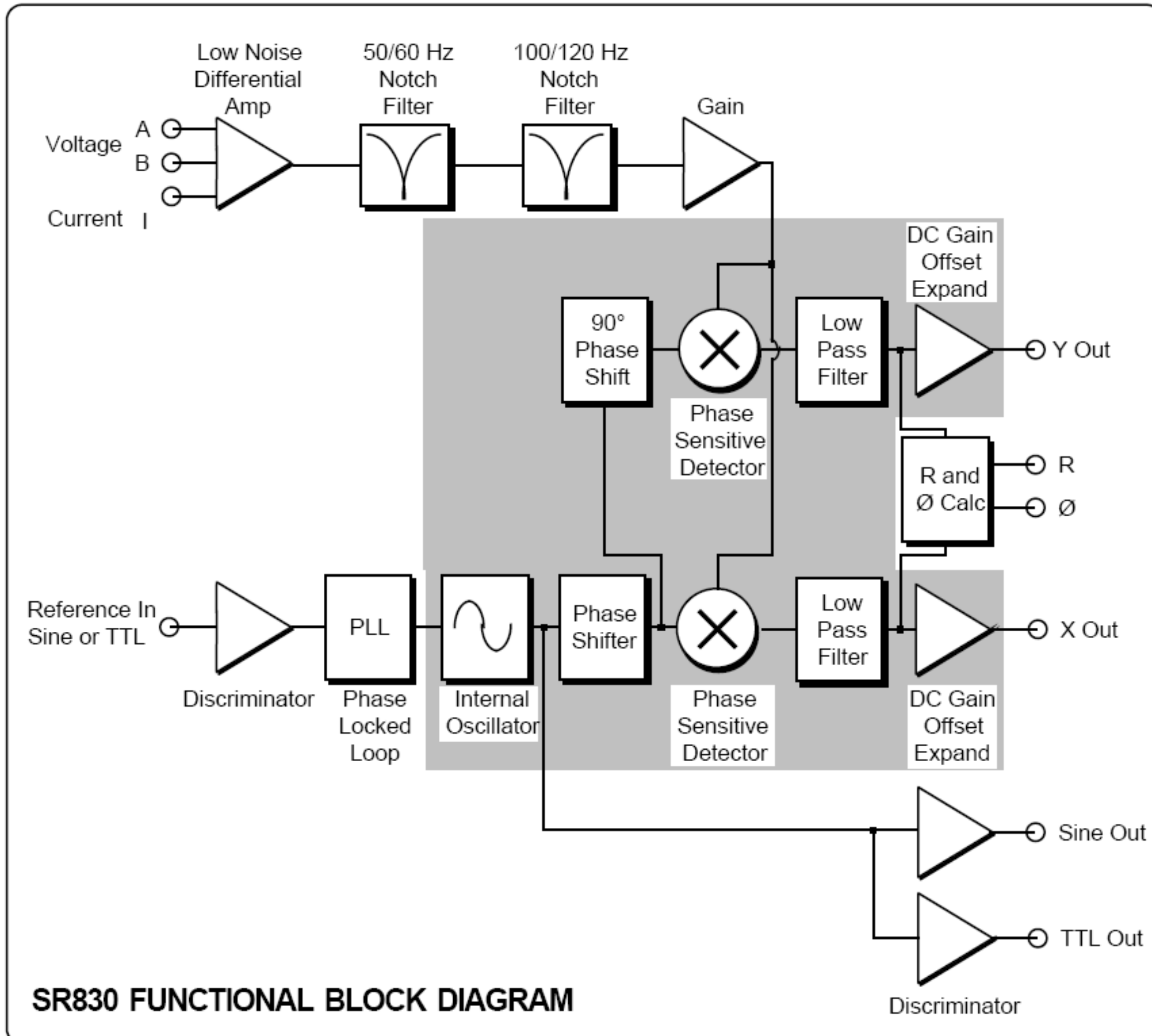
Optical chopper and lock-in amplifier

- A chopper multiplies the signal by a square wave (switching between 0 and 1)
- A lock-in amplifier collects the transmitted signal



Optical chopper and lock-in amplifier

- A lock-in amplifier is (nowadays) a PhSD that utilizes an external reference to generate two adjustable-phase $\pi/2$ shifted sinusoidal reference signal



Lock-in amplifier

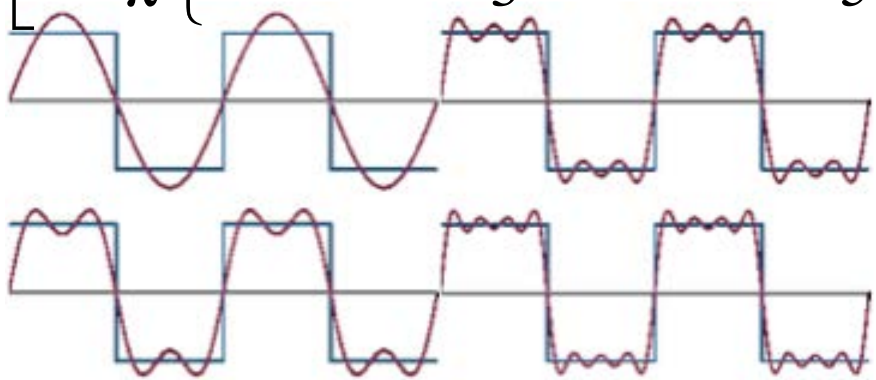
- Conventional: working up to 100 kHz (You can do with a computer after A/D conversion)

- At higher frequencies (up to GHz) PhSD may be implemented with dedicated circuits

Square Wave modulation

- The chopper multiplies the signal by $c(t) = \left[1 + \mathrm{Sign}\left(\mathrm{Sin}\left(\omega_o t\right)\right)\right]/2$
- This is a periodic signal and can be expanded in Fourier series

$$c(t) = \frac{1}{2}\left[1 + \frac{4}{\pi}\left\{\mathrm{Sin}\left(\omega_o t\right) + \frac{1}{3}\mathrm{Sin}\left(3\omega_o t\right) + \frac{1}{5}\mathrm{Sin}\left(5\omega_o t\right)\ldots\right.\right.$$



- Thus if the original signal is a(t), the chopped signal is:

$$s(t) = (1/2)\left[a(t) + (4/\pi)a(t)\left\{\mathrm{Sin}\left(\omega_o t\right) + \mathrm{Sin}\left(3\omega_o t\right)/3 + \mathrm{Sin}\left(5\omega_o t\right)/5\ldots\right.\right.$$

- Multiplication by the sine reference gives

$$s(t)\mathrm{Sin}\left(\omega_o t\right) = (1/2)a(t)\mathrm{Sin}\left(\omega_o t\right) +$$

$$\frac{a(t)}{\pi}\left\{1 - \frac{2}{3}\mathrm{Cos}\left(2\omega_o t\right) - \frac{2}{15}\mathrm{Cos}\left(4\omega_o t\right) - \frac{1}{5}\mathrm{Cos}\left(6\omega_o t\right) + \frac{\pi}{2}\mathrm{Sin}\left(\omega_o t\right)\right\}$$

- Low pass filter will then pick up $a(t)/\pi$

Square Wave modulation

- Thus the signal is $a'(t)=a(t)/\pi$
- Noise goes through the standard PhSD and thus noise at output (we're just multiplying by the sine, no factors 2) has PSD

$$S_{a_{n_e}a_{n_e}}(\omega)=\left|H(\omega)\right|^2 S_{n_{env}n_{env}}(\omega_o)$$

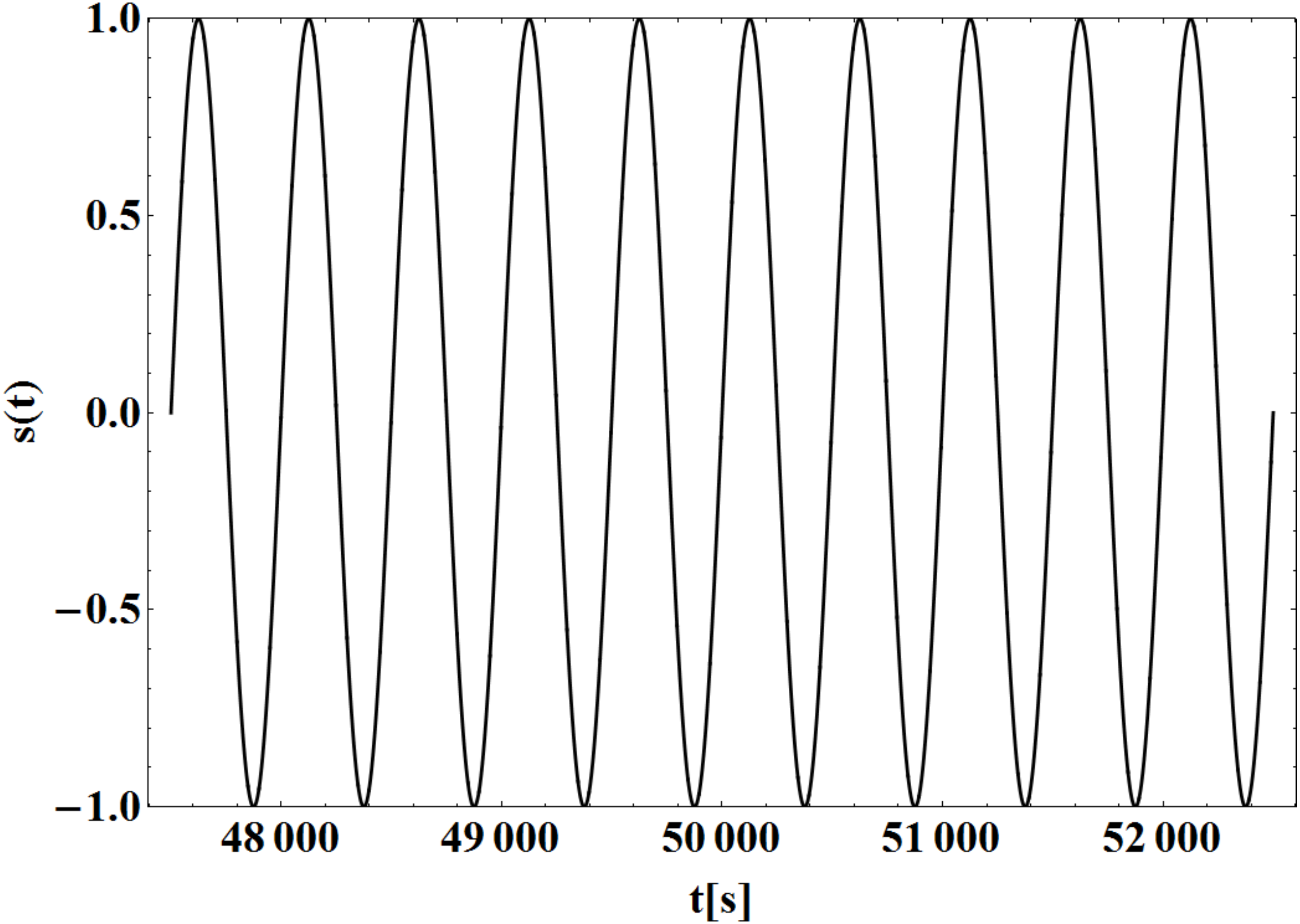
- In terms of equivalent signal, this formula must be multiplied by π^2

$$S^{eq}_{a_{n_e}a_{n_e}}(\omega)=\pi^2\left|H(\omega)\right|^2 S_{n_{env}n_{env}}(\omega_o)$$

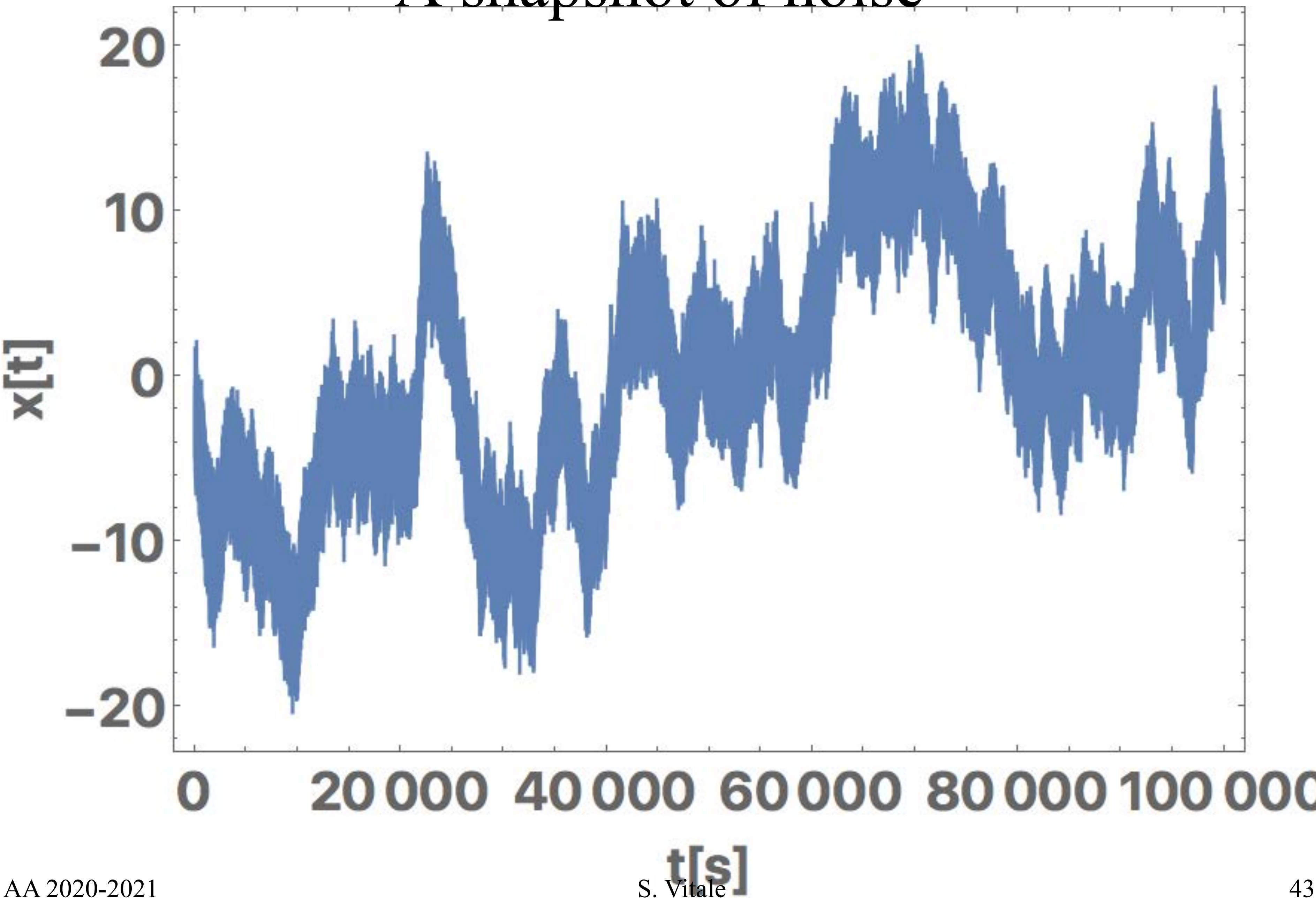
An exercise

- Generating a slow signal
- Adding red noise
- Looking at data
- Up-converting signal
- Adding same red noise
- Using a phase sensitive detector
- Looking at the data

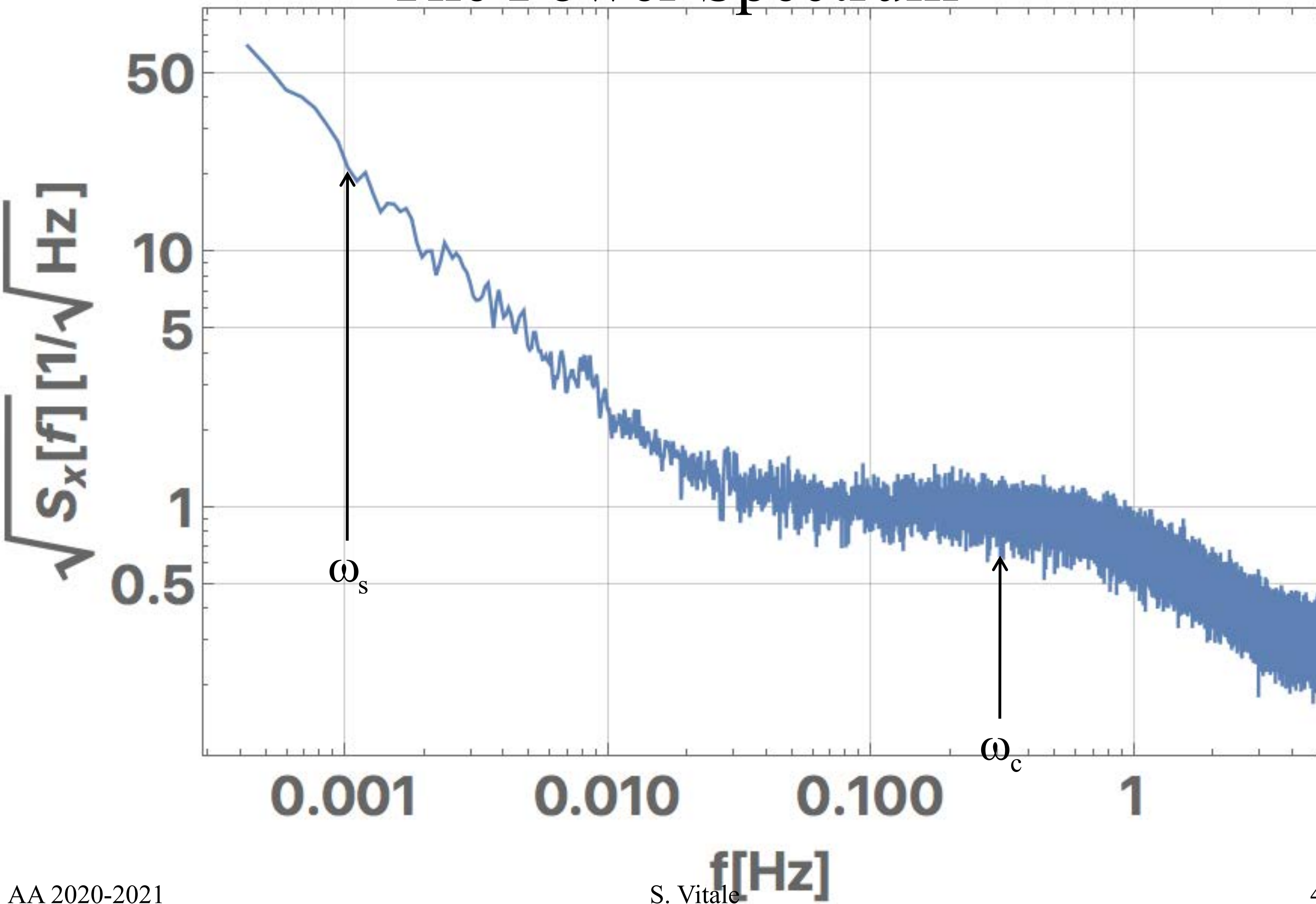
The low frequency signal



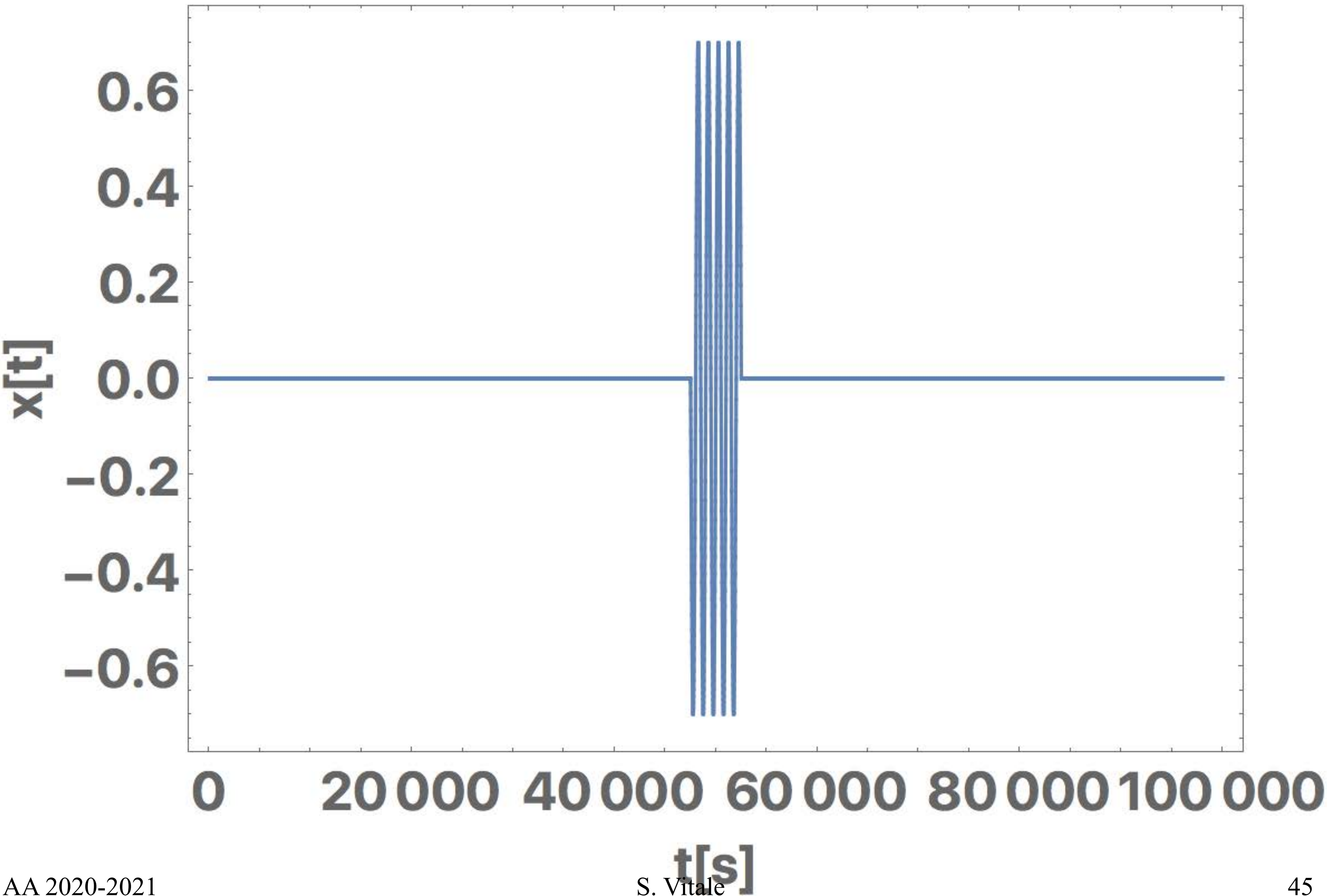
A snapshot of noise



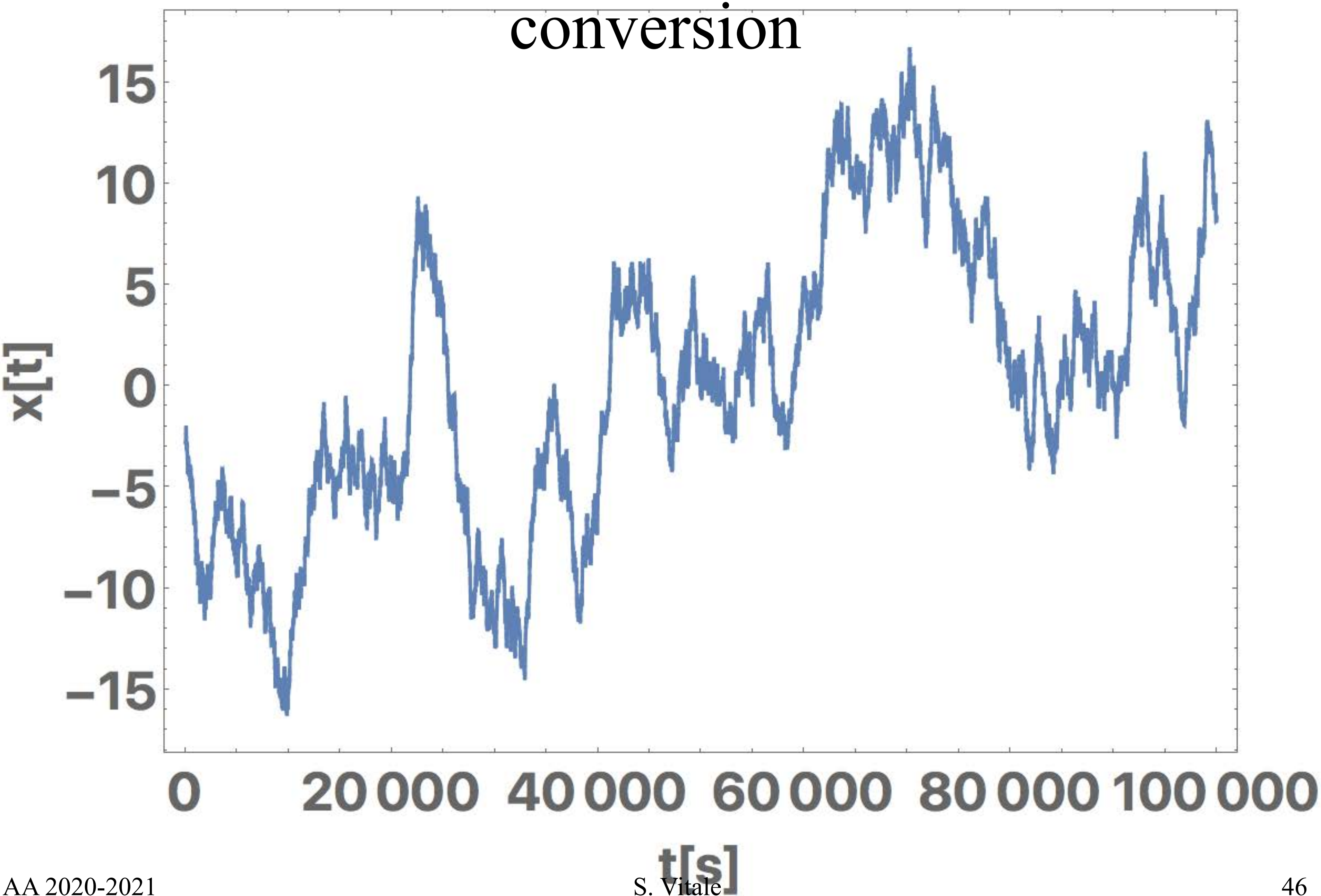
The Power Spectrum



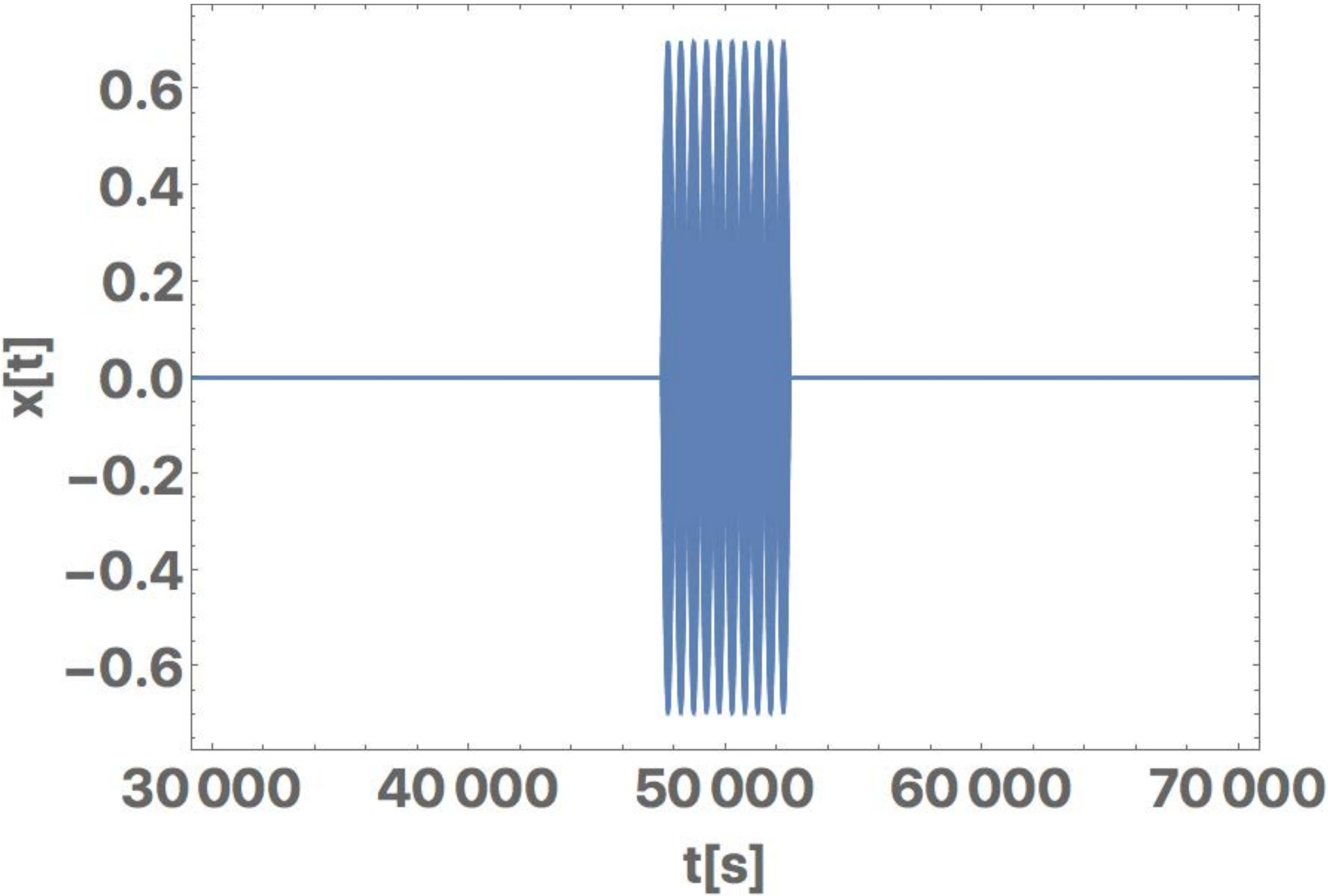
The low frequency signal (filtered ~0.03 Hz)



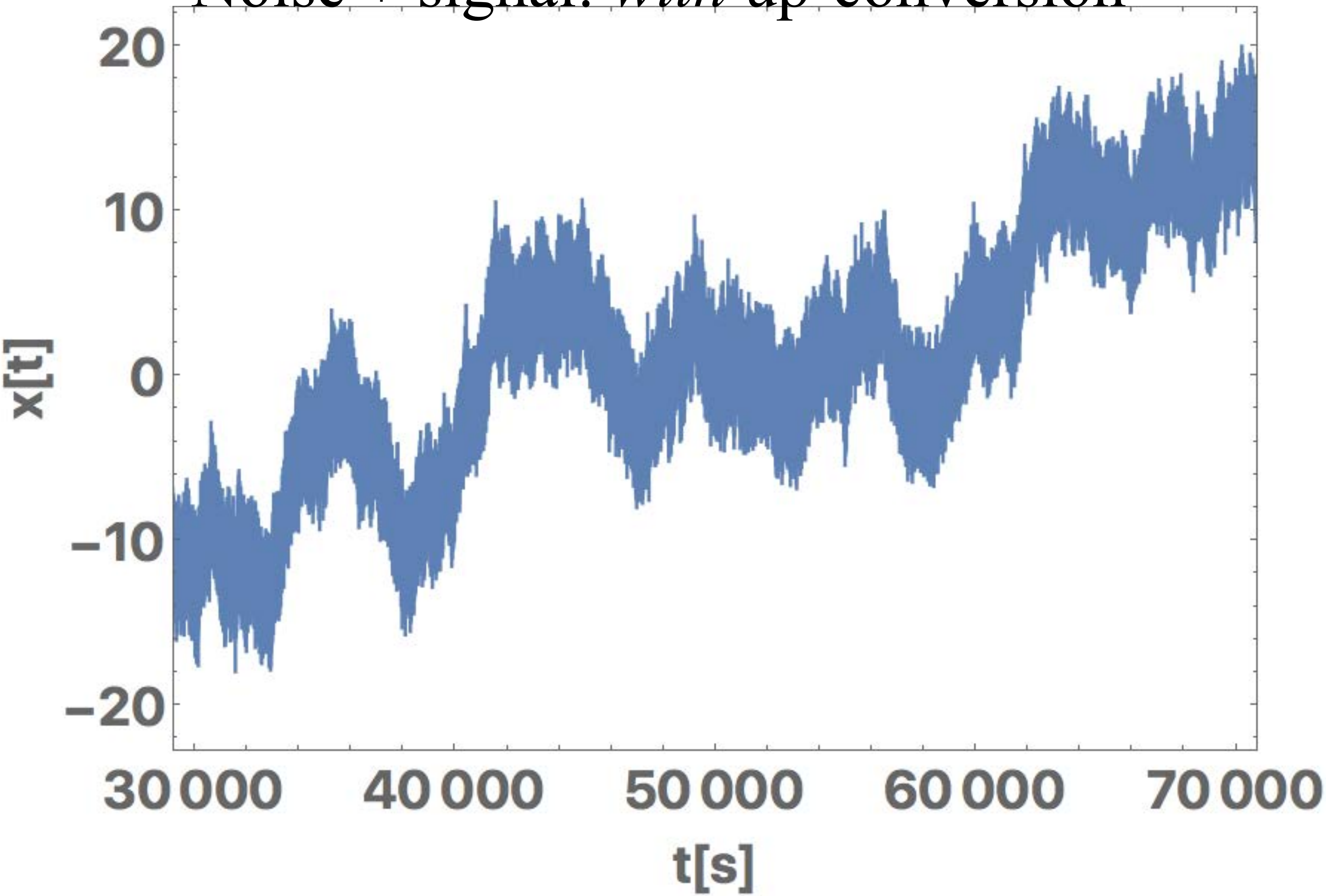
Noise + signal (filtered 0.03 Hz): *no* up-conversion



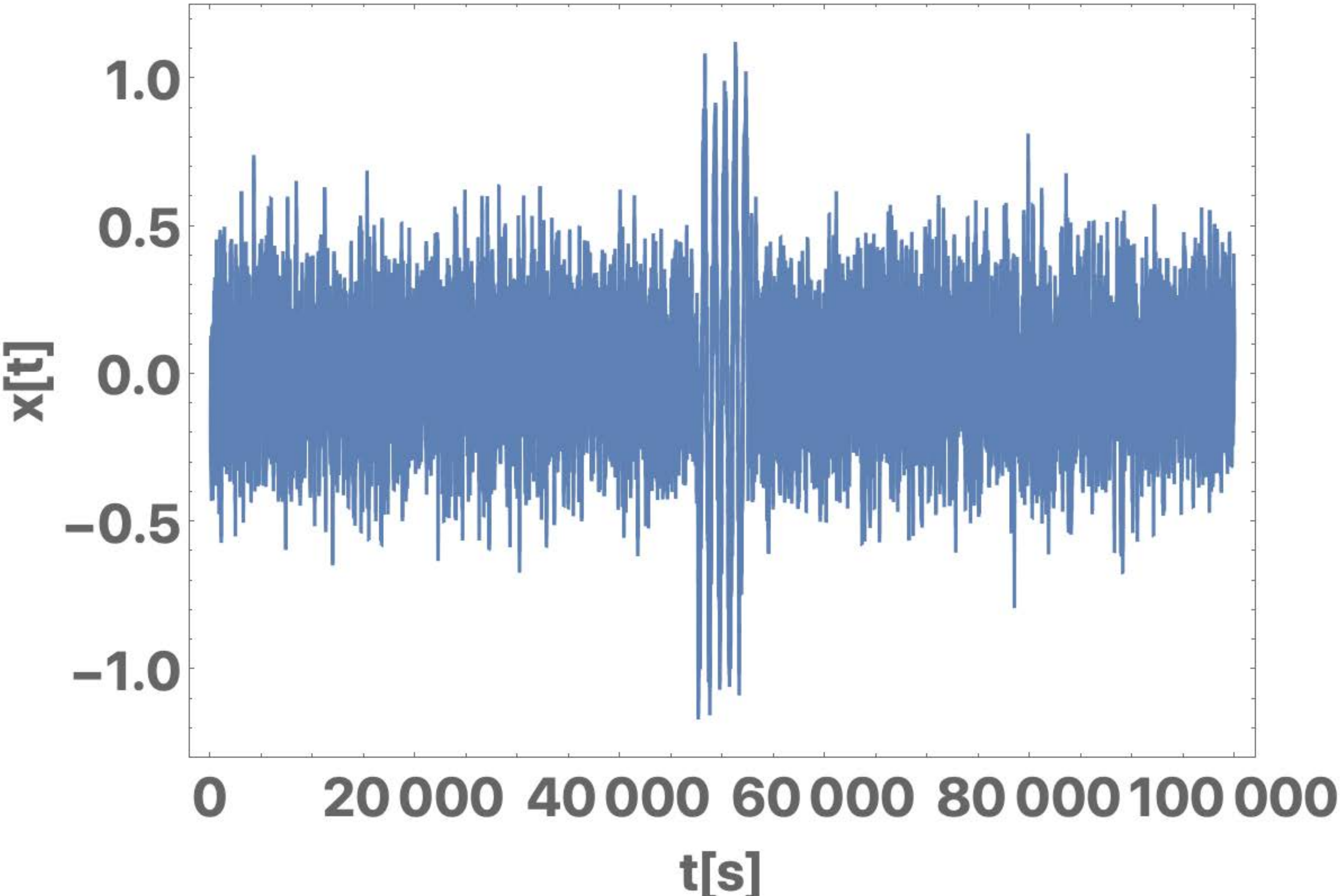
Signal up-converted



Noise + signal: *with* up-conversion



Noise + signal: up-conversion + phase sensitive detector



Noise + signal: up-conversion + phase sensitive detector