Experimental Methods
Lecture 19

November 2nd, 2020

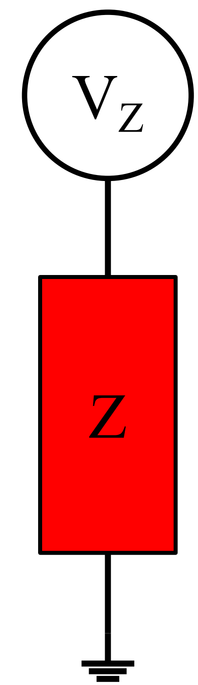
Nyquist law

- Thermal equilibrium voltage noise across passive, lumped circuit elements, can be described by a voltage noise generator in series to a voltage-free element.
- Noise voltage is Gaussian, zero-mean, and has power spectral density:

$$S_{V_Z V_z}(\omega) = 2\, k_B T\, \mathrm{Re}\{Z(\omega)\}$$

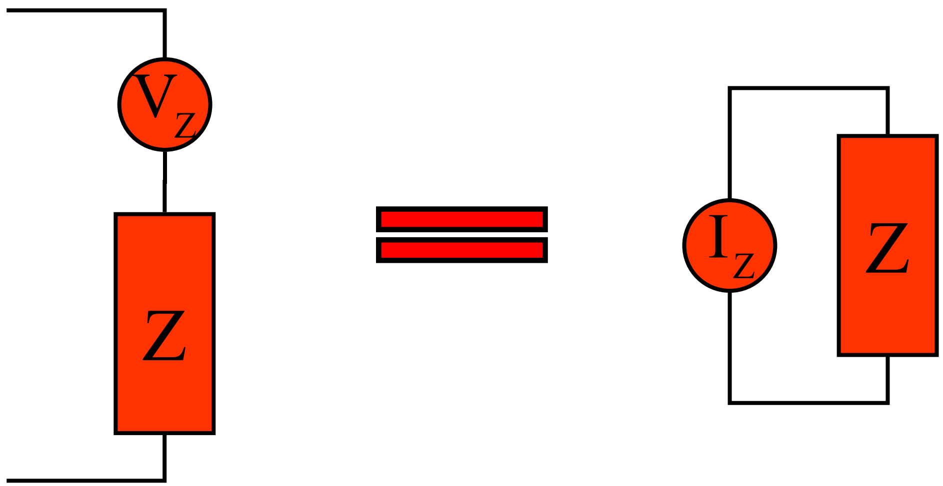
- In the special case of a pure resistor with resistance R, the noise is white, with power spectral density:

$$S_{V_Z V_z} = 2\, k_B T\, R$$



Nyquist +Norton

- Notice that circuit theory (Norton theorem) states that the noisy impedance can also be represented as a the noiseless device *with a parallel current generator*.



- And that the current I_z is related to V_z by $I_Z = V_Z / Z(\omega)$
- It follows that the PSD of I_Z is

$$S_{I_Z I_Z}(\omega) = S_{VV}(\omega) \Big/ |Z(\omega)|^2 = 2k_B T \,\mathrm{Re}\{Z(\omega)\} \Big/ |Z(\omega)|^2 = 2k_B T \,\mathrm{Re}\{1/Z(\omega)\}$$

- Notice that if Z=R, then $S_{II}(\omega) = 2k_B T / R$

One-sided vs two-sided

- Experimental physics and electronics often use the following convention:

$$\left\langle x^2 \right\rangle = \left(1/2\pi\right)\int_0^\infty S_{xx}^{\text{exp. phys.}}\left(\omega\right)d\omega$$

- Comparing with our

$$\left\langle x^2 \right\rangle = \left(1/2\pi\right)\int_{-\infty}^\infty S_{xx}\left(\omega\right)d\omega = \left(1/2\pi\right)\int_0^\infty 2S_{xx}\left(\omega\right)d\omega$$

- We get

$$S_{xx}^{\text{exp. phys.}}\left(\omega\right) = 2S_{xx}\left(\omega\right)$$

- With this convention Nyquist law becomes

$$S_{V_Z V_Z}\left(\omega\right) = 4k_B T\,\mathrm{Re}\left\{Z\left(\omega\right)\right\}$$

Autocorrelation and PSD of linear combination of processes

- Assume n zero-mean, stationary stochastic processes x(t), y(t)…..
- Form the linear combination: $w(t) = \alpha\, x(t) + \beta\, y(t) + ...$
- Where α, β…, are numerical coefficients. The mean value is

$$\langle w(t)\rangle = \alpha\langle x(t)\rangle + \beta\langle y(t)\rangle + ... = 0$$

- While the autocorrelation is: $\langle w(t)w(t+\tau)\rangle = \alpha^2\langle x(t)x(t+\tau)\rangle +$

$$+\beta^2\langle y(t)y(t+\tau)\rangle + \alpha\beta\left[\langle x(t)y(t+\tau)\rangle + \langle y(t)x(t+\tau)\rangle\right] + ...$$

- As a consequence the PSD becomes

$$S_{w,w}(\omega) = \alpha^2 S_{x,x}(\omega) + \beta^2 S_{y,y}(\omega) + \alpha\beta\left[S_{x,y}(\omega) + S_{y,x}(\omega)\right] + ...$$

- Now consider that

$$S_{x,y}(\omega) = \int_{-\infty}^{\infty}\langle x(t)y(t+\tau)\rangle e^{-i\omega\tau}\,d\tau = \int_{-\infty}^{\infty}\langle x(t-\tau)y(t)\rangle e^{-i\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty}\langle y(t)x(t+\tau)\rangle e^{i\omega\tau}\,d\tau = S^*_{y,x}(\omega)$$

- So finally $S_{w,w}(\omega) = \alpha^2 S_{x,x}(\omega) + \beta^2 S_{y,y}(\omega) + 2\alpha\beta\,\mathrm{Re}\left[S_{x,y}(\omega)\right] + ...$
- For independent processes $S_{w,w}(\omega) = \alpha^2 S_{x,x}(\omega) + \beta^2 S_{y,y}(\omega) + ...$

Exercise 1/2

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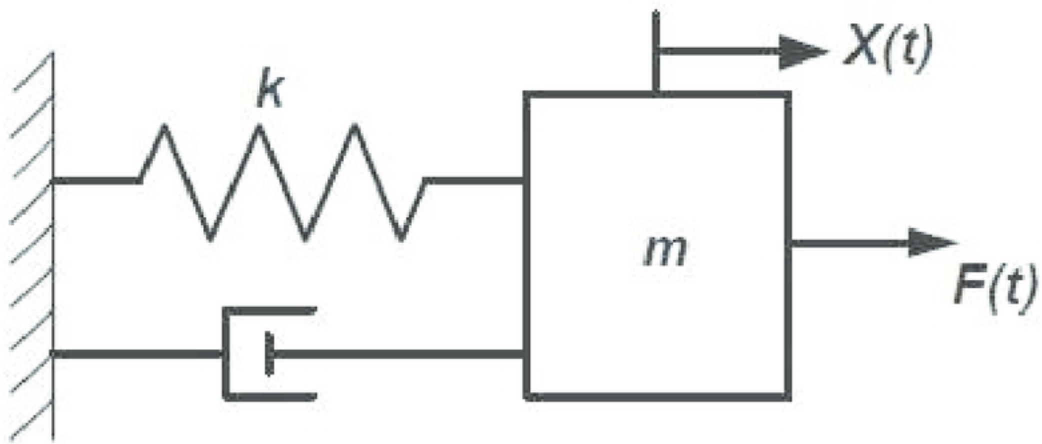
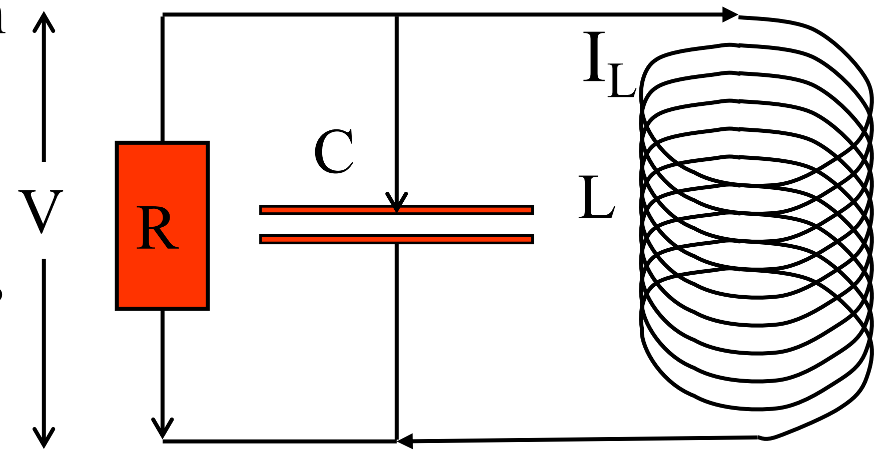
- The fundamental mode of the resonant gravitational wave detector Auriga can me simplified to the following model



- We know that $m \simeq 1.3 \times 10^3 kg$. (The effective mass of the mode is approximately equal to half the true mass of the antenna).
- The resonant frequency is about $f_o \simeq 990\ Hz$, the Q factor is $Q \simeq 10^6$ and the temperature is $T = 2\ K$
- Calculate PSD and autocorrelation of $X(t)$ due to Brownian noise.
- Does the system obeys equipartition?

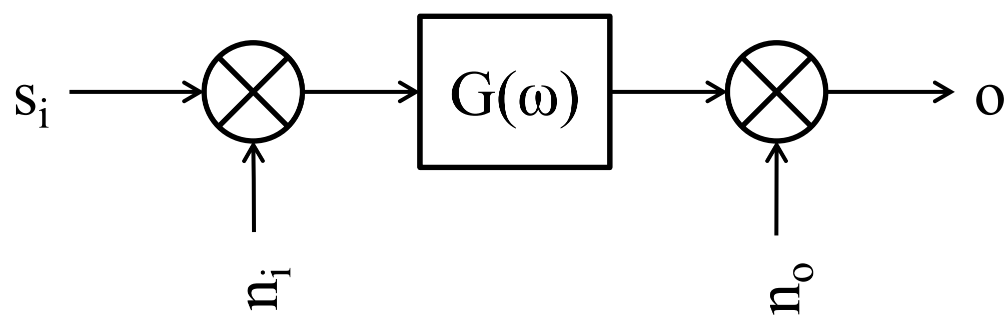
Exercise 2/2

- Consider the following circuit and assume it at thermal equilibrium.
- Calculate the PSD of V.
- Calculate the PSD of I_L
- Calculate the autocorrelation of both quantities by using inverse Fourier transforms
- Check that equipartition of energy is obeyed
- Calculate the conditional probability of V(t) conditioned to $V(0)=V_o$
- Use L= 1mH, C= 0.25 μF, R=50 MΩ and T=293 K.



Stationary noise in a linear measurement apparatus

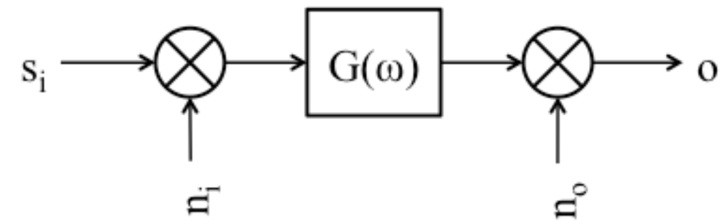
- We will discuss now the relation between random noise and the measurability of signals in noisy instruments.
- Let's consider a linear time invariant system as measurement instrument.
- As already stated, the experimental noise properties of most instruments are well described by assuming that stationary stochastic processes are linearly superimposed both, at input, to the signal to be measured, and to the output according to the following scheme



- Here s stands for "signal", n for "noise", i for "input", o for "output". Noise is assumed to be stationary, and so is assumed the noiseless system (instrument), with transfer function $G(\omega)$.

Stationary noise in a linear measurement apparatus

- In order to derive the noise properties, we first treat noise as any other signal. With this we can calculate transfer functions to be used to propagate PSD



$$o(\omega) = n_o(\omega) + G(\omega)\left(s_i(\omega) + n_i(\omega)\right)$$

- A very convenient way of writing the expression above, is that of re-writing each term as a signal acting at input:

$$o(\omega) = G(\omega)\left(n_o(\omega)/G(\omega) + s_i(\omega) + n_i(\omega)\right) \equiv G(\omega)\left(n_e(\omega) + s_i(\omega)\right)$$

- The PSD of the noise part of the output o(t) is

$$S_{oo}(\omega) = S_{n_o n_o}(\omega) + S_{n_i n_i}(\omega)\left|G(\omega)\right|^2$$

- The PSD of the equivalent noise $n_e(t)$ is

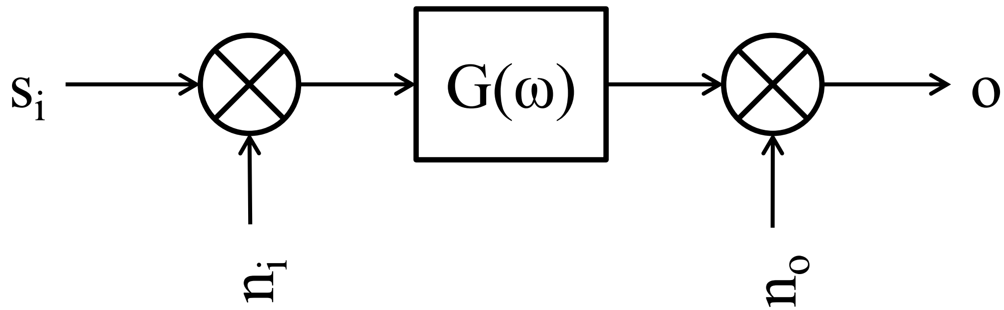
$$S_{n_e n_e}(\omega) = S_{n_o n_o}(\omega)\Big/\left|G(\omega)\right|^2 + S_{n_i n_i}(\omega)$$

- You can check that

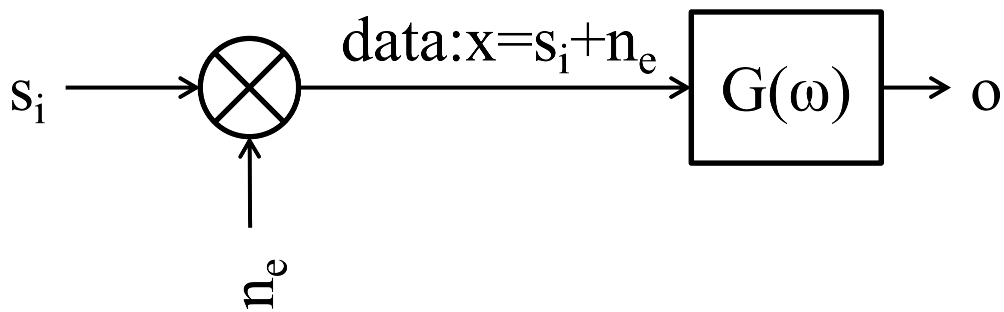
$$S_{oo}(\omega) = \left|G(\omega)\right|^2 S_{n_e n_e}(\omega)$$

Stationary noise in a linear measurement apparatus

- Thus the system



- becomes

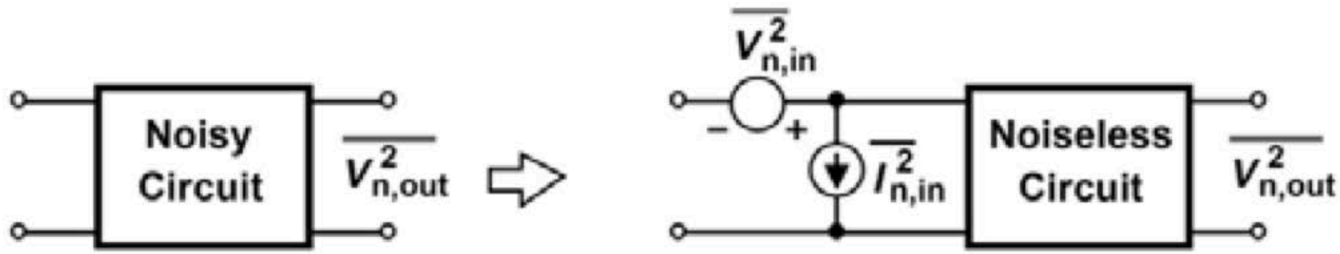


- n_e directly competes with s_i, and the apparatus becomes, in a sense, irrelevant.
- The apparatus only enters in setting the PSD of n_e

$$S_{n_e n_e}(\omega)=S_{n_o n_o}(\omega)\Big/\left|G(\omega)\right|^2+S_{n_i n_i}(\omega)$$

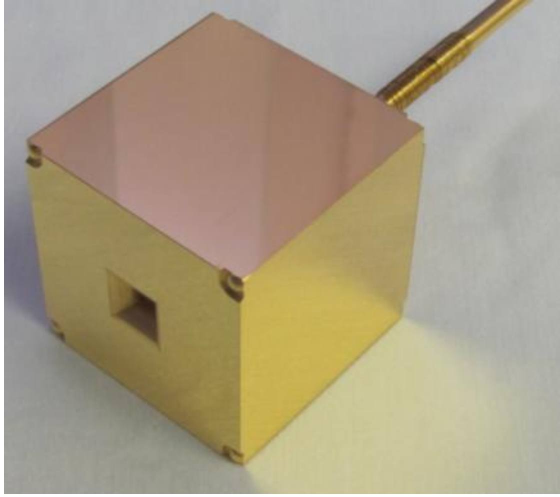
For instance

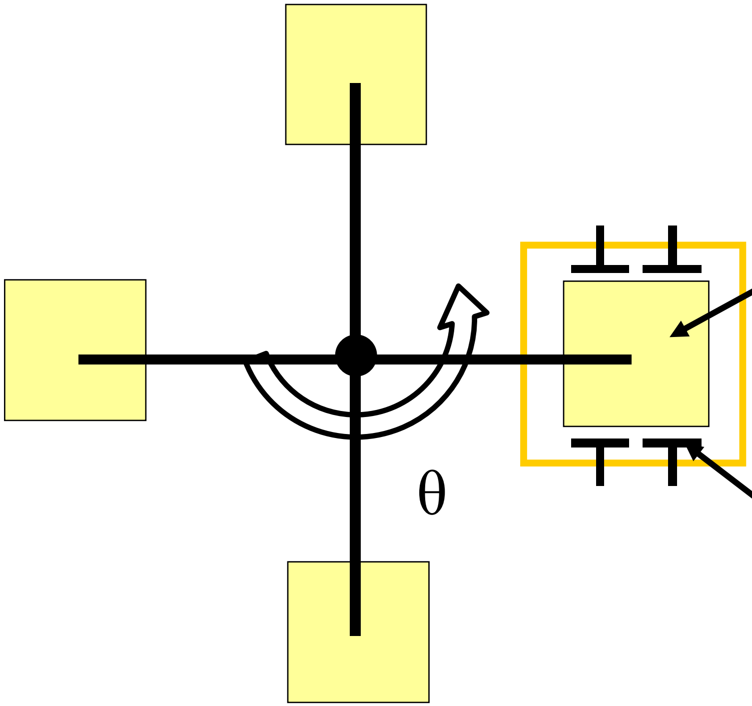
- Amplifiers (a few lectures to come on two ports systems….)



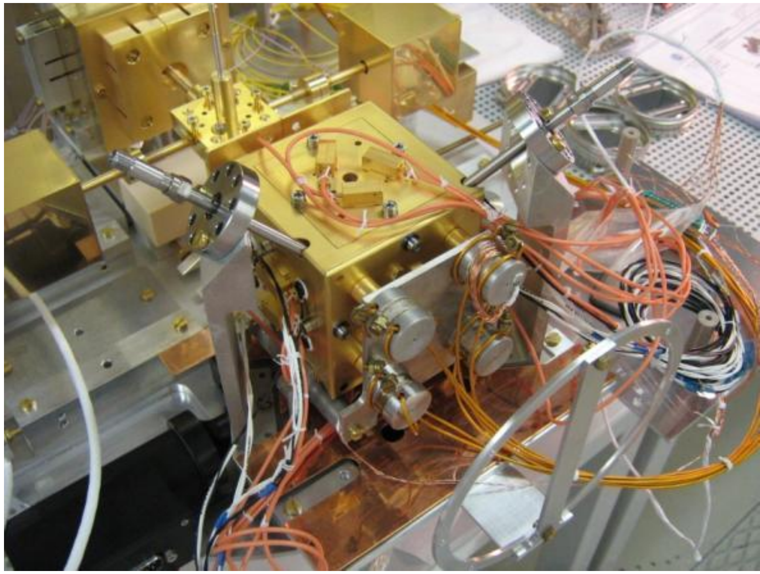
(b)

Example for noise translated at input: torsion pendulum



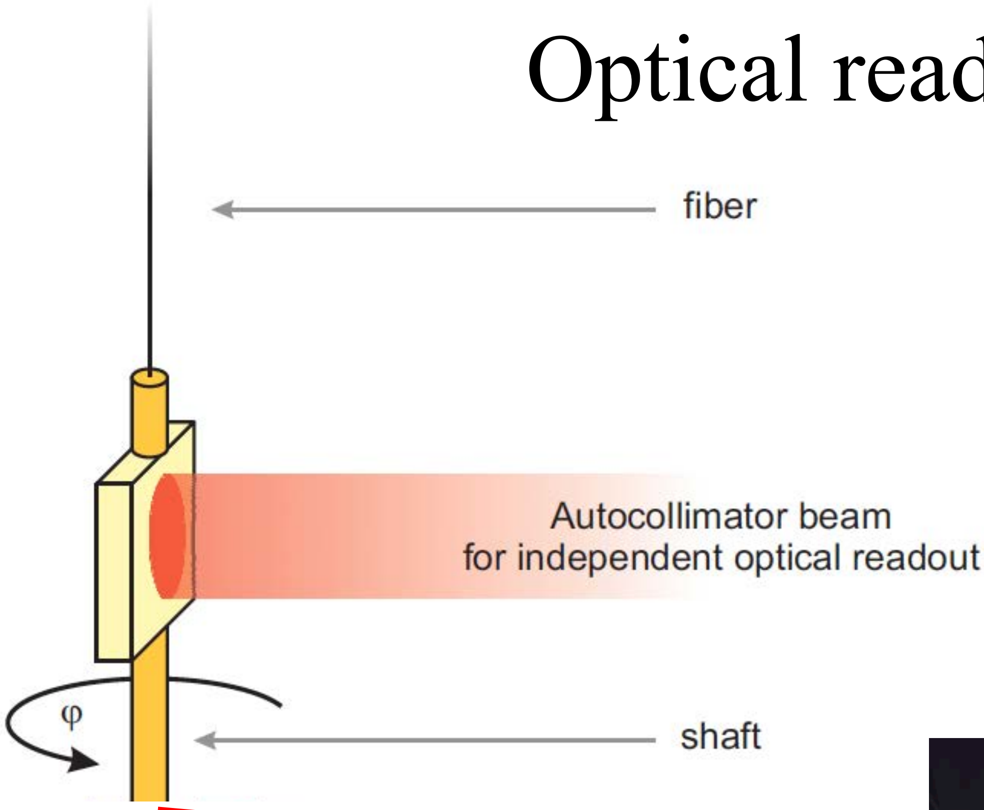


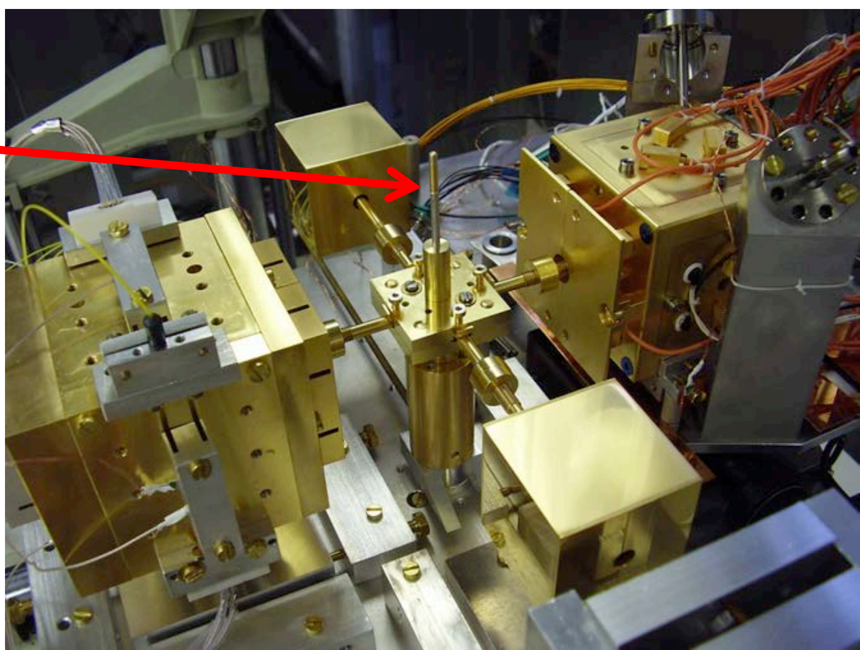
Test-mass (hollow)



Disturbing surroundings (GRS)

Optical readout





Equation of motion and transfer function

- Equation of motion (I moment of inertia, β viscous damping factor, k torsional spring constant, γ_z torque along the torsion fiber)

$$I\ddot{\theta}(t)+\beta\dot{\theta}(t)+k\theta(t)=\gamma_z(t)$$

- In the frequency domain

$$\theta(\omega)=\left[\gamma_z(\omega)/I\right]\left(k/I-\omega^2+i\omega(\beta/I)\right)^{-1}\equiv\left(\omega_o^2-\omega^2+i\omega/\tau\right)^{-1}\left[\gamma_z(\omega)/I\right]$$

- The transfer function is that of a damped harmonic oscillator

$$h(\omega)=(1/I)\left[\omega_o^2-\omega^2+i\omega/\tau\right]^{-1}$$

- Note: in reality damping is not due to viscous drag. However key results are independent of such an assumption

Noise

$$h(\omega)=(1/I)\left[\omega_o^2-\omega^2+i\omega/\tau\right]^{-1}$$

1. True torque noise. PSD = $S_{\gamma\gamma}(\omega)$, a smooth function of frequency
2. Readout noise. PSD: $S_{\theta\theta}(\omega)$. Also a smooth function of frequency

- Noise at output $\quad S_{\theta\theta}^{tot}(\omega)=\left|h(\omega)\right|^2 S_{\gamma\gamma}(\omega)+S_{\theta\theta}(\omega)$
- substituting

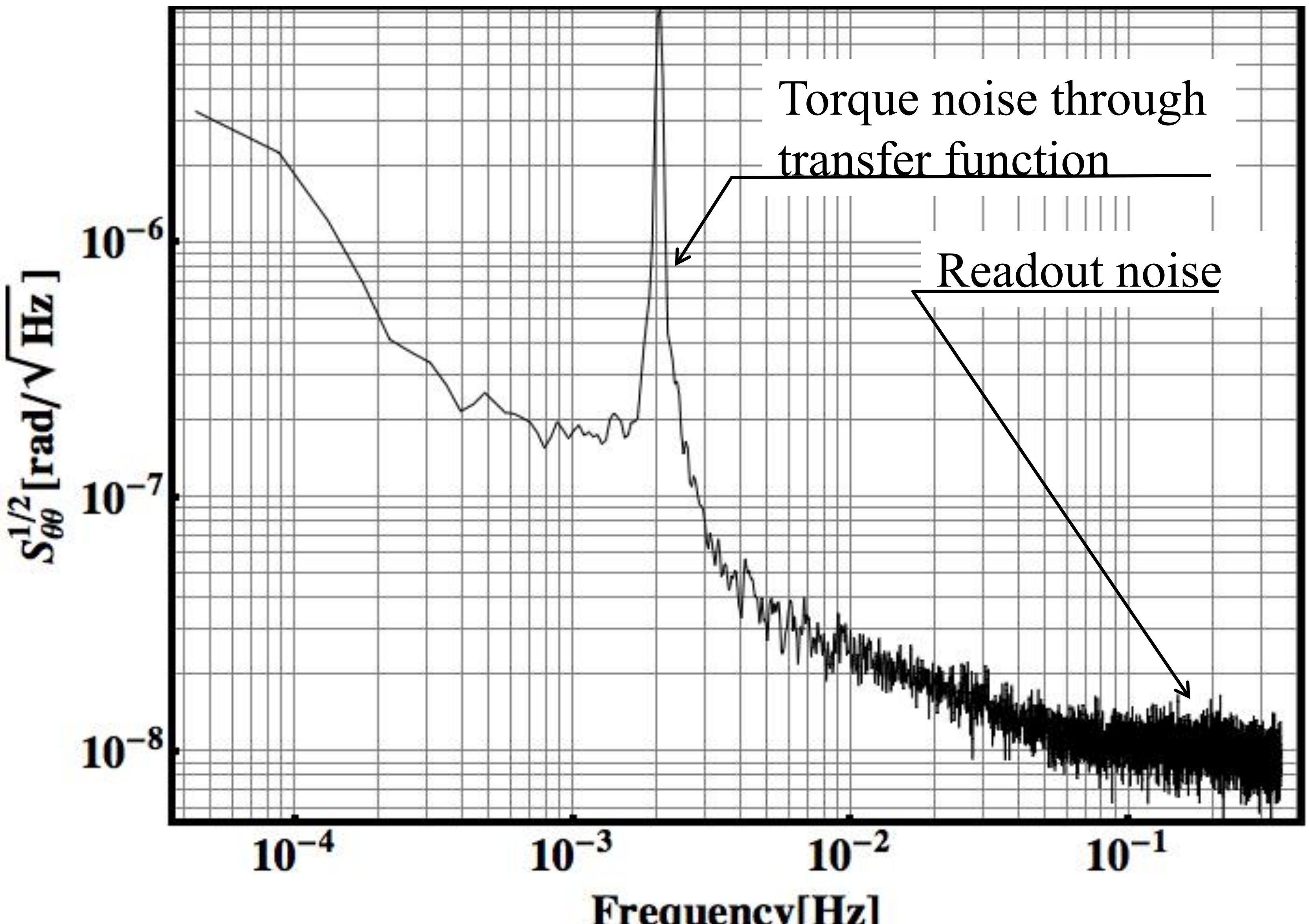
$$S_{\theta\theta}^{tot}(\omega)=\left(S_{\gamma\gamma}(\omega)/I^2\right)\Big/\left[\left(\omega_o^2-\omega^2\right)^2+\omega^2/\tau^2\right]+S_{\theta\theta}(\omega)$$

- Torque noise contribution peaks up at the resonance $\omega=\omega_o$, while readout noise stays smooth.
- At input

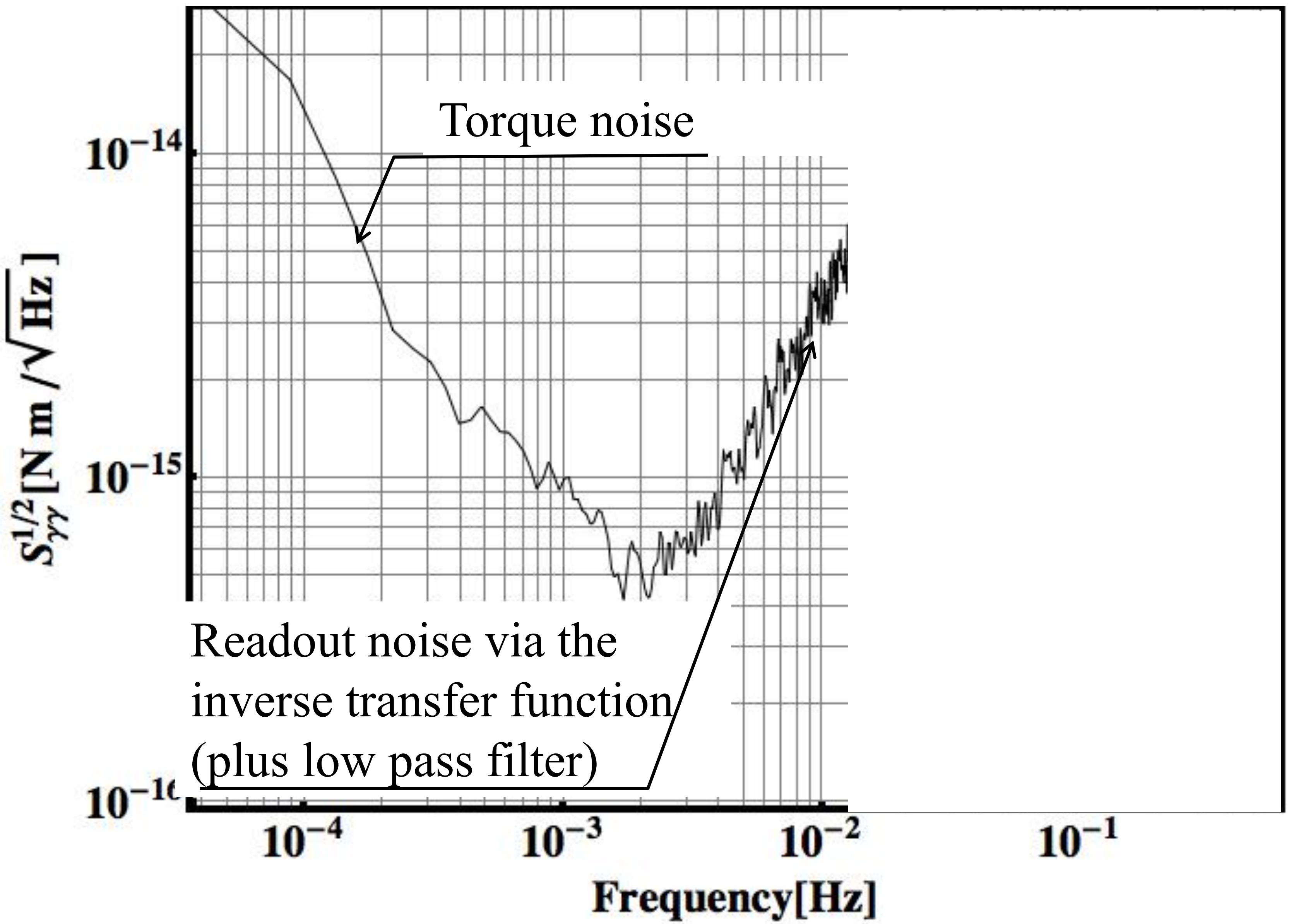
$$S_{\gamma\gamma}^{tot}(\omega)=S_{\gamma\gamma}(\omega)+S_{\theta\theta}(\omega)\Big/\left|h(\omega)\right|^2=S_{\gamma\gamma}(\omega)+S_{\theta\theta}(\omega)I^2\left[\left(\omega_o^2-\omega^2\right)^2+\omega^2/\tau^2\right]$$

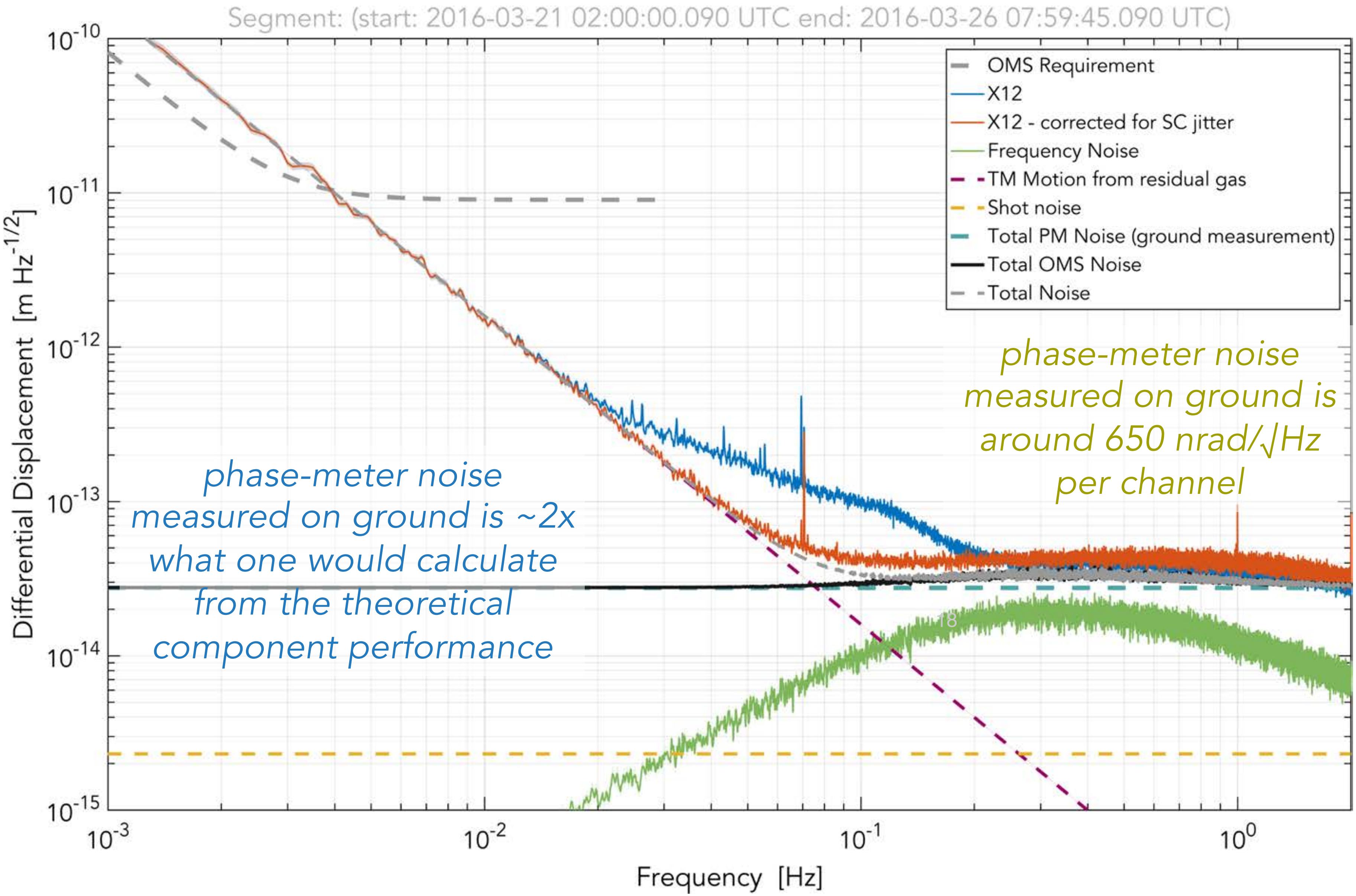
- Readout noise would shows a dip at resonance, but the dip is partly hidden beneath $S_{\gamma\gamma}$. For high ω the readout contribution $\propto\omega^4$.

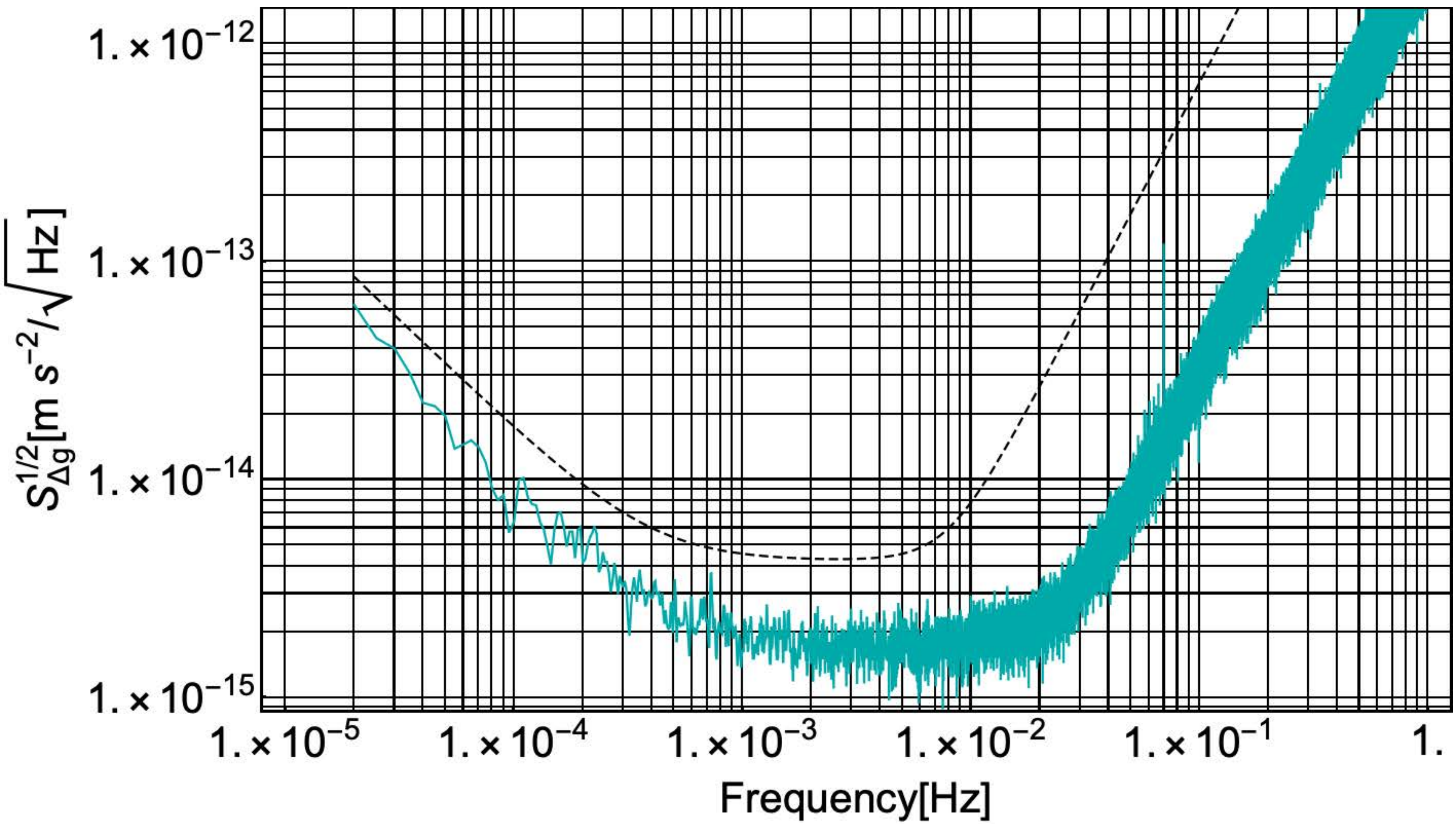
Estimated PSD: Angle



Estimated PSD: Torque (low-pass at ≃0.04 Hz)




Segment: (start: 2016-03-21 02:00:00.090 UTC end: 2016-03-26 07:59:45.090 UTC)
Differential Displacement [m Hz$^{-1/2}$]
Frequency [Hz]
10^{-10}
10^{-11}
10^{-12}
10^{-13}
10^{-14}
10^{-15}
10^{-3}
10^{-2}
10^{-1}
10^{0}
OMS Requirement
X12
X12 - corrected for SC jitter
Frequency Noise
TM Motion from residual gas
Shot noise
Total PM Noise (ground measurement)
Total OMS Noise
Total Noise
phase-meter noise measured on ground is ~2x what one would calculate from the theoretical component performance
phase-meter noise measured on ground is around 650 nrad/√Hz per channel


$1.\times10^{-12}$
$1.\times10^{-13}$
$1.\times10^{-14}$
$1.\times10^{-15}$
$S_{\Delta g}^{1/2}[\mathrm{m\ s^{-2}}/\sqrt{\mathrm{Hz}}]$
$1.\times10^{-5}$
$1.\times10^{-4}$
$1.\times10^{-3}$
$1.\times10^{-2}$
$1.\times10^{-1}$
1.
Frequency[Hz]

Signal detection

- We now briefly consider the *theory of optimal signal detection*.
- We will show that the results of this theory only depend on the so called signal to noise ratio. In particular for a sinusoidal signal, uncertainty only depends on the noise PSD at the frequency of the signal.
- Theory assumes that we observe "data" x(t) and that these are the sum of a zero-mean stationary Gaussian noise n(t) and of a deterministic signal s(t)

$$x(t) = \underbrace{s(t)}_{\substack{\text{Signal} \\ (\text{deterministic})}} + \underbrace{n(t)}_{\substack{\text{noise} \\ (\text{zero-mean}, \\ \text{random})}}$$

- The signal is known up to an undetermined amplitude A. That is the shape f(t) of the signal is known but not its amplitude:

$$s(t) = Af(t)$$

- The optimal filter theory addresses the question: "does it exists an optimal linear operator that, acting on the data, gives an estimate of A?"

Extracting a signal from noise; Wiener's optimal filter

- The data: $$x(t) = s(t) + n(t)$$
- Data are known between T_1 and T_2
- The signal: a function of known shape and unknown amplitude A
$$s(t) = Af(t)$$
- Is there a linear combination of data $\hat{A}$ that gives an estimate for A
$$\hat{A} = \int_{T_1}^{T_2} h(t)x(t)dt$$
- Such that:
  1. $\hat{A}$ is not affected by systematic errors (unbiased), i.e. the mean value of $\hat{A}$ is equal to the true value to be estimated:
  $$\langle \hat{A} \rangle = A$$
  2. Among all possible linear estimators that fulfill 1. $\hat{A}$ has the minimum possible value for the variance ?
  $$\sigma_{\hat{A}}^2 = \left\langle \left(\hat{A} - A \right)^2 \right\rangle$$

Extracting a signal from noise; Wiener's optimal filter

- Requirement n.1. Mean value of $\langle\hat{A}\rangle$ =A
- Use the definition of x(t) (the mean value of number is just the number itself!)

$$\left\langle\hat{A}\right\rangle=\int_{T_1}^{T_2}h(t)\left\langle x(t)\right\rangle dt \quad =A\int_{T_1}^{T_2}h(t)f(t)dt+\int_{T_1}^{T_2}h(t)\left\langle n(t)\right\rangle dt$$

- As n(t) has zero mean-value $\left\langle\hat{A}\right\rangle=A\int_{T_1}^{T_2}h(t)f(t)dt$

- Thus requirement 1. is equivalent to $\int_{T_1}^{T_2}h(t)f(t)dt=1$

Extracting a signal from noise; Wiener's optimal filter

- Requirement n.2. Variance is minimum
- Variance
$$\sigma_{\hat{A}}^2 = \left\langle \left(\hat{A} - A \right)^2 \right\rangle = \left\langle \left(A \int_{T_1}^{T_2} h(t) f(t) dt + \int_{T_1}^{T_2} h(t) n(t) dt - A \right)^2 \right\rangle$$
- As $\int_{T_1}^{T_2} h(t) f(t) dt = 1$
- The variance is $\sigma_{\hat{A}}^2 = \left\langle \left(\int_{T_1}^{T_2} h(t) n(t) dt \right)^2 \right\rangle$
- Which can be expanded as

$$\sigma_{\hat{A}}^2 = \left\langle \left(\int_{T_1}^{T_2} h(t) n(t) dt \right)^2 \right\rangle = \left\langle \int_{T_1}^{T_2} \int_{T_1}^{T_2} h(t) h(t') n(t) n(t') dt\, dt' \right\rangle$$

- And finally

$$\sigma_{\hat{A}}^2 = \int_{T_1}^{T_2} \int_{T_1}^{T_2} h(t) h(t') \left\langle n(t) n(t') \right\rangle dt\, dt' = \int_{T_1}^{T_2} \int_{T_1}^{T_2} h(t) h(t') R_{nn}(t - t') dt\, dt'$$

Extracting a signal from noise; Wiener's optimal filter

- In summary
  - From requirement 1. $\int_{T_1}^{T_2} h(t)f(t)dt = 1$
  - From requirement 2. $\sigma_{\hat{A}}^2 = \int_{T_1}^{T_2}\int_{T_1}^{T_2} h(t)h(t')R_{nn}(t-t')dt\,dt'$

  must be minimum
- Minimizing the integral as a function of h(t), under the condition that requirement 1. holds. Conditional minimum with Lagrange multipliers: find the minimum of

$$a = \int_{T_1}^{T_2}\int_{T_1}^{T_2} h(t)h(t')R_{nn}(t-t')dt\,dt' + \lambda\int_{T_1}^{T_2} h(t)f(t)dt$$

- And then find the multiplier λ by imposing the condition $\int h(t)f(t)dt=1$.
- Finding the minimum: the variation of a for an infinitesimal variation $\delta h(t)$ of h(t) around the minimum must be 0.

$$\delta a = \int_{T_1}^{T_2}\int_{T_1}^{T_2} \left[h(t)+\delta h(t)\right]\left[h(t')+\delta h(t')\right]R_{nn}(t-t')dt\,dt' + \lambda\int_{T_1}^{T_2}\left[h(t)+\delta h(t)\right]f(t)dt -$$

$$\int_{T_1}^{T_2} h(t)h(t')R_{nn}(t-t')dt\,dt' - \lambda\int_{T_1}^{T_2} h(t)f(t)dt = 0$$

Extracting a signal from noise; Wiener's optimal filter

- Variation

$$\delta a = \int_{T_1}^{T_2}\int_{T_1}^{T_2}\left[h(t)+\delta h(t)\right]\left[h(t')+\delta h(t')\right]R_{nn}(t-t')\,dt\,dt' + \lambda\int_{T_1}^{T_2}\left[h(t)+\delta h(t)\right]f(t)\,dt -$$

$$\int_{T_1}^{T_2} h(t)h(t')R_{nn}(t-t')\,dt\,dt' - \lambda\int_{T_1}^{T_2} h(t)f(t)\,dt = 0$$

- To first order in $\delta h(t)$

$$\delta a = 2\int_{T_1}^{T_2}\int_{T_1}^{T_2}\delta h(t)h(t')R_{nn}(t-t')\,dt\,dt' + \lambda\int_{T_1}^{T_2}\delta h(t)f(t)\,dt$$

- That is

$$\delta a = \int_{T_1}^{T_2} dt\,\delta h(t)\left[2\int_{T_1}^{T_2} h(t')R_{nn}(t-t')\,dt' + \lambda f(t)\right]$$

- In order for δa to be zero for any small δh, we need

$$\int_{T_1}^{T_2} h(t')R_{nn}(t-t')\,dt' = -(\lambda/2)f(t) \quad T_1 \le t \le T_2$$

- Notice that the minimum variance is

$$\sigma_{\hat{A}}^2 = \int_{T_1}^{T_2}\int_{T_1}^{T_2} h(t)h(t')R_{nn}(t-t')\,dt\,dt' = -(\lambda/2)\int_{T_1}^{T_2} h(t)f(t)\,dt = -(\lambda/2)$$

Extracting a signal from noise; Wiener's optimal filter

- In summary
- The best linear estimator of A $\hat{A} = \int_{T_1}^{T_2} h(t)x(t)dt$
- is that for which the function h(t) fulfills the integral equation

$$\int_{T_1}^{T_2} h(t')R_{nn}(t-t')dt' = -(\lambda/2)f(t) \quad T_1 \le t \le T_2$$

- $\hat{A}$ is unbiased $\qquad \langle \hat{A} \rangle = A$

- And has the minimum variance $\sigma_{\hat{A}}^2 = -(\lambda/2)$

- The solution can be found explicitly in some remarkable cases (see next pages). In other cases it must be found numerically.

1^{st} Case: $T_1 = -\infty$ $T_2 = +\infty$

- Estimator: $\hat{A} = \int_{-\infty}^{\infty} h(t)\,x(t)\,dt$
- Condition $\int_{-\infty}^{\infty} h(t)\,f(t)\,dt = 1$
- Variance $\sigma_{\hat{A}}^2 = -(\lambda/2)$
- Integral equation to be solved

$$\int_{-\infty}^{\infty} h(t')\,R_{nn}(t-t')\,dt' = -(\lambda/2)\,f(t) \qquad -\infty \le t \le \infty$$

- As the interval is now the entire t axis, we can switch to Fourier Transforms, and equality still holds $h(\omega)S_{nn}(\omega) = -(\lambda/2)f(\omega)$
- The solution being $h(\omega) = -(\lambda/2)f(\omega)/S_{nn}(\omega)$
- Let's now find the value of $-(\lambda/2)$. Use Parseval relations:

$$1 = \int_{-\infty}^{\infty} h(t)\,f(t)\,dt = (1/2\pi)\int_{-\infty}^{\infty} h(\omega)\,f^*(\omega)\,d\omega$$

- Substituting $-(\lambda/2)(1/2\pi)\int_{-\infty}^{\infty} f^*(\omega)f(\omega)/S_{nn}(\omega)\,d\omega = 1$
- That is

$$-(\lambda/2) = \sigma_{\hat{A}}^2 = \left[(1/2\pi)\int_{-\infty}^{\infty}\left(\left|f(\omega)\right|^2 / S_{nn}(\omega)\right)d\omega\right]^{-1}$$

1^{st} Case: $T_1 = -\infty \;\; T_2 = +\infty$

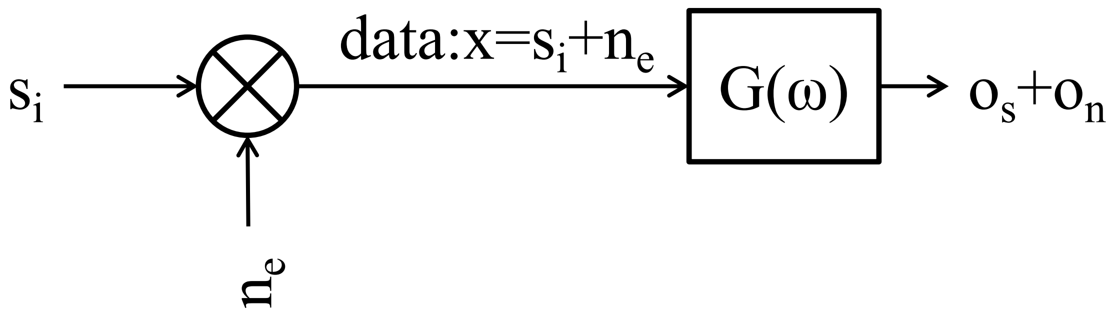
- In conclusion the function h $$h(\omega) = -(\lambda/2) f(\omega) / S_{nn}(\omega)$$
- the multiplier (and the variance)
$$-(\lambda/2) = \sigma^2_{\hat{A}} = \left[(1/2\pi) \int_{-\infty}^{\infty} \left(\left| f(\omega) \right|^2 / S_{nn}(\omega) \right) d\omega \right]^{-1}$$
- Putting both together
$$h(\omega) = \sigma^2_{\hat{A}} \frac{f(\omega)}{S_{nn}(\omega)}$$
- With
$$\sigma^2_{\hat{A}} = \frac{1}{(1/2\pi) \int_{-\infty}^{\infty} \left(\left| f(\omega) \right|^2 / S_{nn}(\omega) \right) d\omega}$$
- Notice:
  1. The variance is inversely proportional to the integral of the *Signal to Noise Ratio* (SNR):
$$\left| f(\omega) \right|^2 / S_{nn}(\omega)$$
  1. The "template" h(t) is the inverse Fourier transform of the signal Fourier transform, "weighted", at each frequency, with the inverse of the noise PSD.

1st Case: $T_1 = -\infty \quad T_2 = +\infty$

- Notice that the signal to noise ratio

$$\mathrm{SNR}(\omega) = \left|f(\omega)\right|^2 / S_{nn}(\omega)$$

- stays the same if data are further filtered like in



- Indeed upon filtering

$$f_{output}(\omega) = G(\omega) f(\omega) \qquad S_{nn,output}(\omega) = \left|G(\omega)\right|^2 S_{nn}(\omega)$$

- that is

$$\left|f_{output}(\omega)\right|^2 / S_{nn,output}(\omega) = \left|f(\omega)\right|^2 \left|G(\omega)\right|^2 / \left|G(\omega)\right|^2 S_{nn}(\omega) = \left|f(\omega)\right|^2 / S_{nn}(\omega)$$