Foreword

The final product of the exam must be a short report containing the answers to the various questions (Q) listed in the following section, together with a brief justification for those answers. ~~The report must be in the form of a .pdf file and should be emailed, at the end of the exam, to stefano.vitale@unitn.it, with a copy to mario.scotoni@unitn.it, from your 'studenti' account. Please put 'EM_yourname' in the subject line.~~ You are allowed, during the test, to use notes, books, and lecture slides. You are also allowed to consult Wikipedia and other databases. The use of computing tools like Mathematica, Matlab, or any other tool of your choice is warmly encouraged. You are allowed to use whatever method you find fit to answer the questions: analytical calculations, numerical calculations, simulations etc. You are allowed to approximate and neglect terms at your will, provided such approximations are physically sound and justified. You are not required to answer all the questions to obtain full credit. However, you are supposed to grasp the essence of the exercise. It is absolutely forbidden to exchange any information with third parties, either fellow students or external experts. Good luck!

Test

Please note: all needed numerical values are listed in table I on page 4.

The experiment

Before the era of the large interferometric gravitational wave (GW) detectors, many groups around the world have been operating the so called acoustic or mechanic detectors. These consisted of large (few meter-long) metal bars cooled to cryogenic temperatures. The GW was expected to excite the fundamental compressional mode of the bar, the amplitude of which was monitored by a very high resolution SQUID-based motion transducer (see for intance DOI: 10.1103/PhysRevLett.94.241101. See below for the definition of SQUID).

Classical solid body mechanics shows that the amplitude of fundamental mode of an elongated bar can be treated as the displacement of point mass mechanical oscillator, with an effective mass $M = M_{bar}/2$, subject to a force equal to the total stress on the bar. I thus approximate here an hypothetical acoustic detector as a one-dimensional harmonic oscillator of mass M, resonant frequency ν, q-factor q, at equilibrium at temperature T. The oscillator is subject to a force F, to whom the GW gives an effective contribution $F_{GW}(t) = M\ddot{h}(t)\ell/2$, with $h(t)$ the perturbation of the metric tensor, i.e. the signal to be detected, and ℓ the length of the bar (see figure 1).

The motion of the mass is read out by a transducer consisting of the parallel plate capacitor between one face of the mass and a plate fixed to the laboratory (see fig. 1). The displacement of the mass modulates the capacitance, as the capacitor gap can be written as $d(x) = d_o - x$. Here d_o is the value of the gap when the mass is at equilibrium, and when the capacitor has its nominal capacitance C_T(see fig. 1).

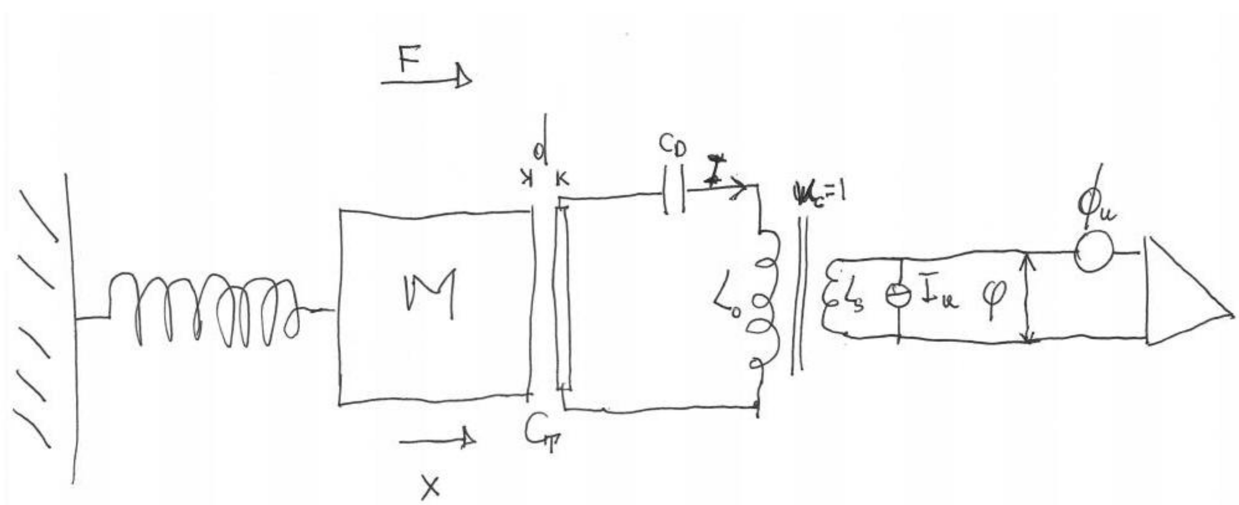

FIG. 1. Schematics of the system

The capacitor is connected an inductor of inductance $L_o = (2\pi\nu)^{-2}C_T^{-1}$ via a large decoupling capacitor of capacitance C_D. As $C_D \gg C_T$, the contribution of the decoupling capacitor to the total capacitance of the loop is negligible, and can be ignored in the calculations. Thanks to the decoupling capacitor, the trasducer capacitor can be charged with a permanent charge Q_o that produces a permanent electric field $\mathcal{E} = \frac{Q_o}{C_T d_o}$. Thus the total capacitor charge is going to be the sum of Q_o and any (small-signal) charge Q due to currents flowing in the circuit. We neglect any dissipation in the electrical circuit.

Finally the magnetic flux through the inductor is measured by a coil, of inductance L_s, which is coupled to the inductor with the 'perfect' mutual inductance coefficient $m_c = \sqrt{L_o L_s}$. A SQUID [1] magnetometer reads the flux through the coil $\phi = m_c \mathcal{I}$ where $\mathcal{I}$ is the current flowing through the inductor.

The SQUID injects a noise current $\mathcal{I}_n$ in the coil and superimposes a flux noise ϕ_n to the magnetometer output. In the kHz range we are interested in, these two noise processes are independent, and have power spectral densities (PSD) S_ϕ and $S_\mathcal{I}$ respectively, with $\sqrt{\frac{S_\phi}{S_\mathcal{I}}} = L_n$ and $\sqrt{S_\phi S_\mathcal{I}} = N\hbar$.

Questions

Q1 Write down, in the Laplace or Fourier domain, the linearised, coupled equations of motion for x, Q and ϕ. Remember that the force on the plates of a capacitor, with charge Q_{tot} and capacitance C, is $-\frac{1}{2}Q_{tot}^2 \frac{\partial \frac{1}{C}}{\partial x}$

Q2 Calculate, as a function of frequency, the transfer function from $h(s)$ to $\phi(s)$

Q3 Calculate the total flux noise PSD at the SQUID input.

Q4 Express the above as and equivalent PSD of $h(t)$

Q5 Calculate numerically the minimum uncertainty of the amplitude of a signal $h(t) = \tau_o \delta(t)$

[1] Superconducting Quantum Interference Device

Parameter	Value	Units	Parameter	Value	Units
M	1.3×10^3	kg	q	6×10^6	
ν	0.9	kHz	T	2	K
ℓ	3	m	$\mathcal{E}$	7×10^7	V/m
L_o	8	H	L_s	4	μH
$\mathcal{N}$	50		L_n	1.3	μH

TABLE I. List of parameter values