Experimental Methods
Lecture 29

November 26th, 2020

Fluctuation dissipation-theorem

Fluctuation-dissipation theorem

We are going now to enunciate the fluctuation dissipation theorem, that we had anticipated when discussing thermal noise.

Take a system at thermodynamic equilibrium described by an Hamiltonian $H_o(p,x)$, and assume that, upon the action of an external field f(t), the Hamiltonian is perturbed as $H(p,x)=H_o(p,x)+f(t)x$, with x a thermodynamic variable.

If the system would be a mechanical one, it would obey

$$\dot{p} = -\partial H/\partial x = -\partial H_o/\partial x - f(t)$$

Thus -f(t) is a generalized force acting on a systems that otherwise will only feel the conservative force $f_c = -\partial H_o/\partial x$

The variation of the energy E under the effect of the perturbation is well known to be $dE/dt = d\langle H\rangle/dt = \partial\langle H\rangle/\partial t = \langle x\rangle \dot{f}(t)$

Fluctuation-dissipation theorem

Thus the presence of perturbation implies a dissipation $\langle x \rangle \dot{f}(t)$

One can show that (see demonstration), to first order, the system responds linearly to the perturbation, according to:

$$\langle x(t) \rangle = \int_0^\infty \chi(t')f(t-t')dt'$$

The dissipation-fluctuation theorem states that x(t) in equilibrium is a stationary random process with PSD given by:

$$S_{xx}(\omega) = 2k_B T\left(\chi''(\omega)/\omega\right)$$

where $\chi(\omega)$ is the Fourier transform of $\chi(t)$.

This is one of the formulations of the theorem. What dissipation has to do with this is discussed a bit later.

Input and output fluctuations

The PSD of x in equilibrium

$$S_{xx}(\omega) = 2k_B T\left(\chi''(\omega)/\omega\right)$$

It is useful to reformulate thermodynamical fluctuations "at input" as the force that would cause x to have the PSD above

Such a force would have a PSD

$$S_{ff}(\omega) = S_{xx}(\omega)\Big/\left|\chi(\omega)\right|^2 = 2k_B T\left(\chi''(\omega)\Big/\left(\left|\chi(\omega)\right|^2 \omega\right)\right)$$

using the properties of complex number

$$S_{ff}(\omega) = -2k_B T\left(\mathrm{Im}\left\{1/\chi(\omega)\right\}/\omega\right)$$

dissipation

Let's assume now that f is a periodic function of time $f(t)=f_o \mathrm{Sin}(\omega_o t)$

The mean power dissipated per cycle by the external force is $(T=2\pi/\omega_o)$

$$P=(1/T)\int_0^T(\partial H/\partial t)dt=(1/T)\int_0^T \dot{f}(t)x(t)dt$$

Which can be re-written as

$$P=(1/T)\int_0^T \dot{f}(t)\int_0^\infty \chi(t')f(t-t')dt'dt=(1/T)\int_0^\infty \chi(t')\left[\int_0^T \dot{f}(t)f(t-t')dt\right]dt'$$

Substituting $P(\omega_o)=(1/T)f_o^2\omega_o\int_0^\infty \chi(t')\left[\int_0^T \mathrm{Cos}(\omega_o t)\mathrm{Sin}(\omega_o(t-t'))dt\right]dt'$

Using trigonometry

$$P(\omega_o)=\left(f_o^2\omega_o/2T\right)\int_0^\infty dt'\chi(t')\left\{\mathrm{Cos}(\omega_o t')\left[\int_0^T \mathrm{Sin}(2\omega_o t)dt\right]\right.$$

$$\left.-\mathrm{Sin}(\omega_o t')\left[\int_0^T(1+\mathrm{Cos}(2\omega_o t))dt\right]\right\}$$

Performing the integral

$$P(\omega_o)=\frac{f_o^2\omega_o}{2}\int_0^\infty \chi(t')\left\{\mathrm{Cos}(\omega_o t')\frac{\mathrm{Sin}^2(\omega_o T)}{\omega_o T}-\mathrm{Sin}(\omega_o t')\left[1+\frac{\mathrm{Sin}(2\omega_o T)}{2\omega_o T}\right]\right\}dt'$$

dissipation

From previous slide, substituting $\omega_o \to \omega$

$$P(\omega)=\frac{f_o^2\omega}{2}\int_0^\infty \chi(t')\left\{Cos(\omega t')\frac{Sin^2(\omega T)}{\omega T}-Sin(\omega t')\left[1+\frac{Sin(2\omega T)}{2\omega T}\right]\right\}dt'$$

Remembering that $T\omega=2\pi$

$$P=-\left(f_o^2\omega/2\right)\int_0^\infty \chi(t')Sin(\omega t')dt'$$

One can easily realize that

$$-\int_0^\infty \chi(t)Sin(\omega t)dt=\mathrm{Im}\left\{\int_0^\infty \chi(t)e^{-i\omega t}\,dt\right\}$$

Then $$P(\omega)=\chi''(\omega)\left(f_o^2\omega/2\right)$$

Compare with $$S_{xx}(\omega)=2k_BT\left(\chi''(\omega)/\omega\right)$$

To obtain the fluctuation-dissipation formulation of the theorem

$$S_{xx}(\omega)=2k_BT\frac{P(\omega)/\left(f_o^2/2\right)}{\omega^2}$$

Fluctuation dissipation-theorem

Example 1: Mechanical motion

- Suppose you have a particle driven by an external force F(t). In the language of the fluctuation dissipation theorem f(t)=-F(t)
- Thus susceptibility is defined as $\chi(\omega)=x(\omega)/f(\omega)=-x(\omega)/F(\omega)$
- In mechanics if a particle is subject to the action of a spring with spring constant k, and of an external force, equilibrium is reached when $F-kx=0\rightarrow x=F/k$
- It is natural then to define a generalized spring constant as
$$x(\omega)=F(\omega)/k(\omega)$$
- The susceptibility is then $\chi(\omega)=-1/k(\omega)$
- Then x will fluctuate at equilibrium with PSD
$$S_{xx}(\omega)=2k_BT\left(k''(\omega)/\left|k(\omega)\right|^2\omega\right)$$
- Corresponding to a drive force with PSD
$$S_{FF}(\omega)=\left|k(\omega)\right|^2 S_{xx}(\omega)=2k_BT\left(k''(\omega)/\omega\right)$$

Example 1: Mechanical motion

Take for instance a damped harmonic oscillator subject to an external force F(t). The input output relation is $x(\omega)=\left(F(\omega)/m\right)/\left[\left(k/m-\omega^2\right)+i\beta\omega/m\right]$

Thus susceptibility is $\chi(\omega)=-(1/m)/\left[\left(k/m-\omega^2\right)+i\beta\omega/m\right]$

While the spring constant is $k(\omega)=m\left[\left(k/m-\omega^2\right)+i\beta\omega/m\right]$

The imaginary part is $k''(\omega)=\beta\omega$

So that $S_{FF}(\omega)=2k_BT\left(k''(\omega)/\omega\right)=2\beta k_BT$

Which is the Brownian formula

Example 1: Mechanical motion

Real solid states springs have indeed an imaginary part related to viscous flow of dislocation.

This imaginary part is found to be frequency independent at low frequency

$$k = k' + i k'' \mathrm{Sign}(\omega)$$

The force PSD is then

$$S_{FF}(\omega) = 2k_B T\left(k'' \mathrm{Sign}(\omega)/\omega\right)$$

$$= 2k_B T\left(k''/|\omega|\right)$$

That is, a thermodynamic equilibrium 1/f noise

From pendulum data →

PHYSICAL REVIEW D **75**, 042001 (2007)

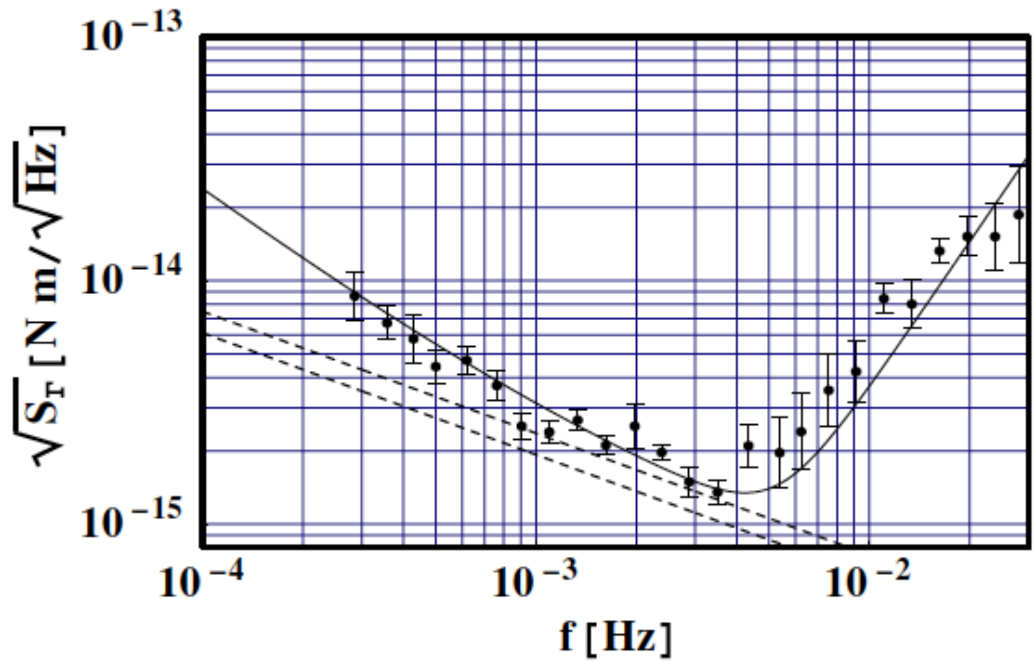

FIG. 2 (color online). Total torque noise. Data points: square root of torque noise PSD estimated from autocollimator and capacitive sensor cross correlation after time domain subtraction of the effect of PM motion along *y*. Values and error bars are estimated as the mean and its standard deviation of PSD data falling within the corresponding frequency bins. Bins have equal logarithmic width. Dashed lines delimit the error band for the estimate of thermal noise. The continuous smooth line is a linear least square fit to data as described in the text.

One minute google

"Characterizing many-body systems by observing density fluctuations"

Wolfgang Ketterle
Massachusetts Institute of Technology
MIT-Harvard Center for Ultracold Atoms

8/7/2010

QFS 2010 Satellite Workshop
Grenoble

New methods to detect interesting new phases of matter

Density fluctuations

$\boxed{\text{fluctuation-dissipation theorem}}$

$$\frac{\Delta N^2}{N} = n k_B T \kappa_T$$

n atomic density
N atom number in probe volume V
κ_T isothermal compressibility

- ideal classical gas $\kappa_T = \dfrac{1}{n k_B T}$ $\dfrac{\Delta N^2}{N} = 1$

Poissonian fluctuations

Selected for a Viewpoint in *Physics*
PRL **105**, 040402 (2010) PHYSICAL REVIEW LETTERS week ending 23 JULY 2010

Suppression of Density Fluctuations in a Quantum Degenerate Fermi Gas

Christian Sanner, Edward J. Su, Aviv Keshet, Ralf Gommers, Yong-il Shin, Wujie Huang, and Wolfgang Ketterle
MIT-Harvard Center for Ultracold Atoms, Research Laboratory of Electronics, and Department of Physics, Massachusetts Institute of Technology, Cambridge Massachusetts 02139, USA
(Received 7 May 2010; published 19 July 2010)

The development of a technique to sensitively measure density fluctuations was motivated by the connection between density fluctuations and compressibility through the fluctuation-dissipation theorem. In this Letter, we validate our technique for determining the compressibility by applying it to the ideal Fermi gas. In future work, it could be

$$\frac{{}^2}{} = \frac{3}{2}\frac{k_B T}{E_F}$$

sub-Poissonian
Pauli suppression
of fluctuations

Demonstration of theorem (rough)

First the formula for susceptibility. Suppose that the perturbation due to the force f(t) is small. Suppose that a constant force f_o has been applied for $-\infty \leq t \leq 0$ and the suddenly turned off.

One can calculate the mean value of <x> using Boltzmann distribution ($\beta=1/k_B T$) and transition probabilities

$$\langle x(t)\rangle = \int\int x' P(x'|x'') e^{-\beta\left[H_o(x'')+f_o x''\right]} dx' dx'' \Big/ \int e^{-\beta\left[H_o(x'')+f_o x''\right]} dx''$$

where $P(x'|x'') \equiv P\left(x(t)=x' | x(0)=x''\right)$

is the conditional probability of finding x=x' at time t, having found x=x'' at time zero.

As the perturbation is small we can expand the formula to first order in f the result is illustrated on next page

Demonstration of theorem (rough)

Linear expansion of mean value as a function of f_o

$$\langle x(t)\rangle = \int\int x'P(x'|x'')e^{-\beta\left[H_o(x'')+f_o x''\right]}dx'dx''\Big/\int e^{-\beta\left[H_o(x'')+f_o x''\right]}dx''$$

$$\approx \int\int x'P(x'|x'')e^{-\beta H_o(x'')}dx'dx''\Big/\int e^{-\beta H_o(x'')}dx''$$

$$-\beta f_o\Bigg\{\int\int x'x''P(x'|x'')e^{-\beta H_o(x'')}dx'dx''\Big/\int e^{-\beta H_o(x'')}dx''$$

$$-\int x''e^{-\beta H_o(x'')}dx''\int\int x'P(x'|x'')e^{-\beta H_o(x'')}dx'dx''\Big/\left[\int e^{-\beta H_o(x'')}dx''\right]^2\Bigg\}$$

We now assume that the equilibrium value in the absence of applied force is zero $\langle x(\infty)\rangle = \int\int x'P(x'|x'')e^{-\beta H_o(x'')}dx'dx''\Big/\int e^{-\beta H_o(x'')}dx''$

$$= \int x''e^{-\beta H_o(x'')}dx''\Big/\int e^{-\beta H_o(x'')}dx'' = 0$$

Then

$$\langle x(t)\rangle = -\beta f_o\Big\{\int\int x'x''P(x'|x'')e^{-\beta H_o(x'')}dx'dx''\Big/\int e^{-\beta H_o(x'')}dx'' = -\beta f_o R_{xx}(t)$$

where we have introduced the autocorrelation of the process x(t) at equilibrium $R_{xx}(t)$

Demonstration of theorem (rough)

Thus the step response of the system (we applied a step of amplitude $-f_o$) is

$$h_{-1}(t) = \beta R_{xx}(t)$$

That is the susceptibility is $\chi(t) = dh_{-1}(t)/dt = \beta\Theta(t) dR_{xx}(t)/dt$

where the $\Theta(t)$ comes from the fact that the calculation has been done only for a positive delay

Transforming to the frequency domain the imaginary part of susceptibility is

$$\chi''(\omega) = \beta \,\mathrm{Im}\left\{\int_0^\infty \left(dR_{xx}/dt\right) e^{-i\omega t}\, dt\right\} = -\beta \int_0^\infty \left(dR_{xx}/dt\right) \mathrm{Sin}(\omega t)\, dt$$

R(t) is an even function of t, so dR/dt is odd and dR/dt Sin(ωt) is even.

Then $\chi''(\omega) = -(\beta/2)\int_{-\infty}^{\infty} \left(dR_{xx}/dt\right) \mathrm{Sin}(\omega t)\, dt = (\beta/2)\,\mathrm{Im}\left\{i\omega S_{xx}(\omega)\right\}$

As a PSD is a real number, the theorem follows:

$$\chi''(\omega) = (\beta/2)\,\omega S_{xx}(\omega) \rightarrow S_{xx}(\omega) = 2k_B T \chi''(\omega)/\omega$$

Fluctuation dissipation theorem

- Hamiltonian

$$H(p,x) = H_o(p,x) + x\,f(t)$$

- Linear, causal response

$$x(t) = \int_0^{\infty} \chi(t')f(t-t')dt'$$

- Fluctuation of x around equilibrium: zero-mean Gaussian process with Power Spectral Density

$$S_{xx}(\omega) = 2\,k_B T\,\frac{\chi''(\omega)}{\omega}$$

- Equivalent to a force noise with PSD

$$S_{ff}(\omega) = 2\,k_B T\,\frac{\chi''(\omega)}{|\chi(\omega)|^2\omega}$$