Experimental Methods
Lecture 21

November 5th, 2020

A few observation on SNR and optimal filtering

- Suppose that we "filter" the data by passing them through a noiseless filter with transfer function H(ω)

$$x \longrightarrow \boxed{H(\omega)} \longrightarrow x'$$

- The Fourier transform of the signal contribution to the filter output x' is AH(ω)f(ω)
- The spectral density of n' at output is $S_{n'n'}(\omega)=\left|H(\omega)\right|^2 S_{nn}(\omega)$
- If we apply Wiener theory to these new data x' we get

$$h'(\omega)=\sigma'^2_{\hat{A}}\, H(\omega) f(\omega)\Big/\left(\left|H(\omega)\right|^2 S_{nn}(\omega)\right)$$

- And

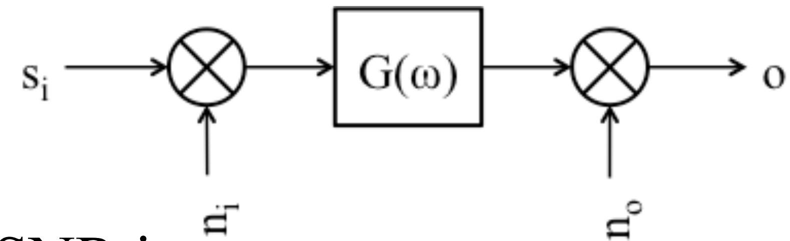
$$\sigma'^2_{\hat{A}}=\left[(1/2\pi)\int_{-\infty}^{\infty}\left(\left|H(\omega)\right|^2\left|f(\omega)\right|^2\Big/\left|H(\omega)\right|^2 S_{nn}(\omega)\right)d\omega\right]^{-1}$$

$$=\left[(1/2\pi)\int_{-\infty}^{\infty}\left(\left|f(\omega)\right|^2\Big/S_{nn}(\omega)\right)d\omega\right]^{-1}=\sigma^2_{\hat{A}}$$

- Thus the presence of the filter does not change the error on the estimate. Optimal filter results and errors depend on SNR only and are then independent of any filter acting on both signal and noise!

SNR at input and at output

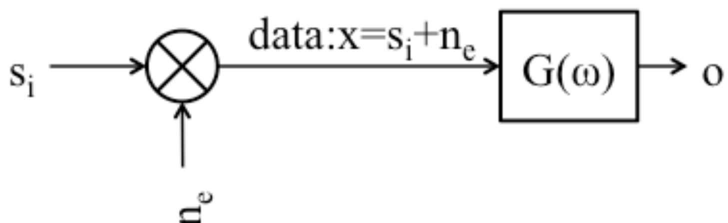
- We have shown that output data can be thought *either as*



- in which case the SNR is

$$SNR_o(\omega)=\left|G(\omega)\right|^2\left|s_i(\omega)\right|^2 \Big/ \left(S_{n_on_o}(\omega)+\left|G(\omega)\right|^2 S_{n_in_i}(\omega)\right)$$

- Or as



- With $S_{n_en_e}(\omega)=S_{n_on_o}(\omega)\Big/\left|G(\omega)\right|^2+S_{n_in_i}(\omega)$
- then

$$SNR_i(\omega)=\left|s_i(\omega)\right|^2 \Big/ S_{n_en_e}(\omega)=\left|s_i(\omega)\right|^2 \Big/ \left(S_{n_on_o}(\omega)\Big/\left|G(\omega)\right|^2+S_{n_in_i}(\omega)\right)$$

- By multiplying numerator and denominator by $\left|G(\omega)\right|^2$

$$SNR_i(\omega)=SNR_o(\omega)$$

Data whitening

- Often you can find a filter such that $H(\omega) = 1 \big/ \sqrt{S_{nn}(\omega)}$
- The spectral density of the noise n' at the output of such a filter is:

$$S_{n'n'}(\omega) = \left|H(\omega)\right|^2 S_{nn}(\omega) = S_{nn}(\omega) \big/ S_{nn}(\omega) = 1$$

- Noise is white with unit PSD! Such a filter is called "whitening filter"
- The signal f' at the output of this filter has now a Fourier transform

$$f'(\omega) = f(\omega) \big/ \sqrt{S_{nn}(\omega)}$$

- As the noise is white, the template by which you have to multiply the data is just $h'(t) = f'(t) \Big/ \int_0^T f'^2(t)\,dt$
- The filter performs just a linear least square fitting of the signal f' to the whitened data.
- Notice that the presence of the filter does not change the SNR and then the estimate error!

Example

- Take the example of a signal in low pass noise. Data are available between $-T_e$ and T_e

$$S_{nn}(\omega) = \frac{S_o}{1+\omega^2\tau^2}$$

- Take the filter

$$h(\omega) = \frac{1}{\sqrt{S_o}}(1+i\omega\tau)$$

- At the output of this filter noise is white

$$S_{n'n'}(\omega) = \frac{\left(1+\omega^2\tau^2\right)}{S_o}\frac{S_o}{1+\omega^2\tau^2} = 1$$

- The filter is described by the following equation in the time domain

$$o(t) = \frac{1}{\sqrt{S_o}}\left[i(t) + \tau\left.\frac{di}{dt}\right|_t\right]$$

Example

- whitening filter
- The signal

$$o(t)=\frac{1}{\sqrt{S_o}}\left[i(t)+\tau\left.\frac{di}{dt}\right|_t\right]$$

$$f(t)=e^{-\frac{1}{2}\left(\frac{t}{2T}\right)^2}\mathrm{Sin}\left(\frac{2\pi}{T}t\right)$$

- Whitening the signal

$$f'(t)=\frac{1}{\sqrt{S_o}}\left[f(t)+\tau\left.\frac{df}{dt}\right|_t\right]=\frac{e^{-\frac{t^2}{8T^2}}}{\sqrt{S_o}}\left\{2\pi\frac{\tau}{T}\mathrm{Cos}\left[\frac{2\pi t}{T}\right]+\left(1-\frac{t\tau}{4T^2}\right)\mathrm{Sin}\left[\frac{2\pi t}{T}\right]\right\}$$

- The optimal filter

$$h(t)=\frac{\sqrt{S_o}}{T}\frac{e^{-\frac{t^2}{8T^2}}\left\{2\pi\frac{\tau}{T}\mathrm{Cos}\left[\frac{2\pi t}{T}\right]+\left(1-\frac{t\tau}{4T^2}\right)\mathrm{Sin}\left[\frac{2\pi t}{T}\right]\right\}}{\sqrt{\pi}\left(1+\frac{\tau^2}{8T^2}\right)\left(1-e^{-16\pi^2}\right)+4\pi^2\frac{\tau^2}{T^2}}$$

- $\tau \ll T$

$$h(t)\approx\frac{\sqrt{S_o}}{T}\frac{e^{-\frac{t^2}{8T^2}}\left\{\mathrm{Sin}\left[\frac{2\pi t}{T}\right]\right\}}{\sqrt{\pi}}$$

Example

- Now the amplitude estimate is just

$$A = \int_{-T_e \ll -T}^{T_e \gg T} h(t)\, x'(t)\, dt$$

- with x' the whitened data

$$x'(t) = \frac{1}{\sqrt{S_o}} \left[x(t) + \tau \left. \frac{dx}{dt} \right|_t \right]$$

- And the error is

$$\sigma_A = \sqrt{\frac{S_o}{T\sqrt{\pi}\left[\left(1+\frac{\tau^2}{8T^2}\right)\left(1-e^{-16\pi^2}\right)+4\pi^2\frac{\tau^2}{T^2}\right]}} \approx \sqrt{\frac{S_o}{T\sqrt{\pi}}}$$

- Notice $\tau \ll T$ means the signal is enough low-frequency that noise is approximately white anyway