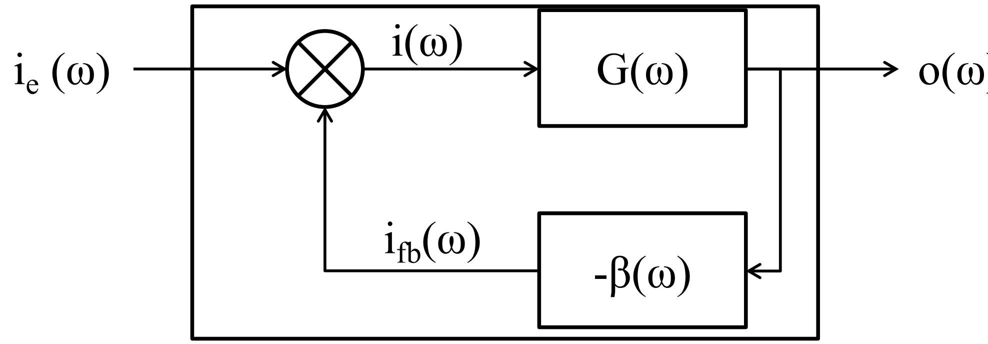
Experimental Methods
Lecture 11

October 14th, 2020

The feedback loop

- Summary



- Closed loop frequency response

$$h_{cl} = \frac{G}{1+\beta G} \rightarrow \lim_{G\rightarrow\infty} \frac{G}{1+\beta G} = \frac{1}{\beta}$$

- Insensitive to G and its fluctuation
- Feedback

$$i_{fb} = -\frac{\beta G}{1+\beta G} i_e \rightarrow \lim_{G\rightarrow\infty} \frac{\beta G}{1+\beta G} i_e = -i_e$$

- A good measurement of i_e
- Total input to G

$$i = \frac{i_e}{1+\beta G} \rightarrow \lim_{G\rightarrow\infty} \frac{i_e}{1+\beta G} = 0$$

- Suppressed input dynamics

The Astrophysical Journal, 883:53 (15pp), 2019 September 20

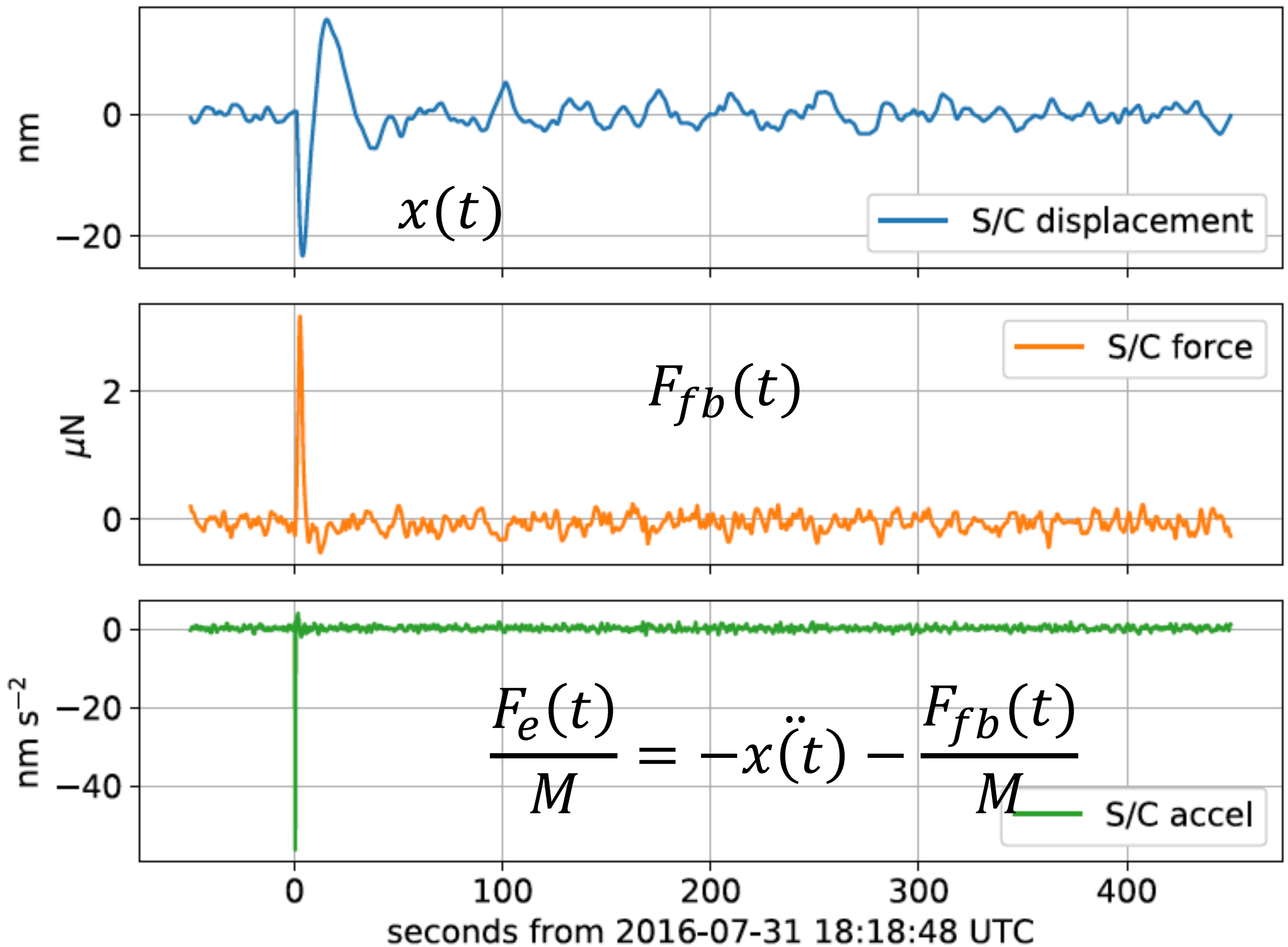


Figure 3. Example of *x*-axis telemetry for impact candidate at GPS time 1154024345.4 (2016 July 31 18:18:48 UTC) and the equivalent free-body acceleration estimated through the calibration procedure. The top panel shows the displacement of the S/C in the *x*-direction. The middle panel shows the commanded force on the S/C in the *x*-direction by the control system. The bottom panel shows the reconstructed external acceleration on the S/C in the *x*-direction using the above data and S/C geometry and mass properties.

The true feedback frequency response

- This lecture: $\beta(\omega) = \omega_o^2 + i\omega/\tau$
- Reality:

Axis	Continuous Transfer Function

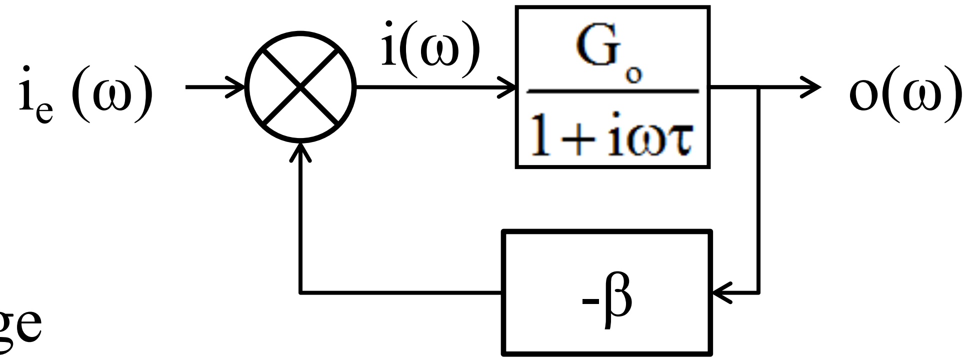
x_1	`0.00095356 (s + 1.286e+04) (s + 1.245) (s + 1.003) (s + 0.1005) (s^2 + 0.1715s + 0.02072)`
	 `--------------------------------------------------------------------------------`
	 `s^2 (s^2 + 1.673s + 1.262) (s^2 + 7.432s + 22.84)`

$$s \equiv i\omega$$

Table 4-4: Drag-Free Controller Transfer Functions.

The gain band-width theorem

- An "amplifier"



- A passive, frequency independent feedback stage
- Closed loop transfer function

$$o(\omega)=i_e(\omega)\frac{G(\omega)}{1+\beta(\omega)G(\omega)}=i_e(\omega)\frac{\dfrac{G_o}{1+i\omega\tau}}{1+\beta\dfrac{G_o}{1+i\omega\tau}}$$

- Further

$$o(\omega)=i_e(\omega)\frac{G_o}{1+i\omega\tau+\beta G_o}=i_e(\omega)\frac{G_o}{1+\beta G_o}\frac{1}{1+i\omega\tau/(1+\beta G_o)}$$

- Define

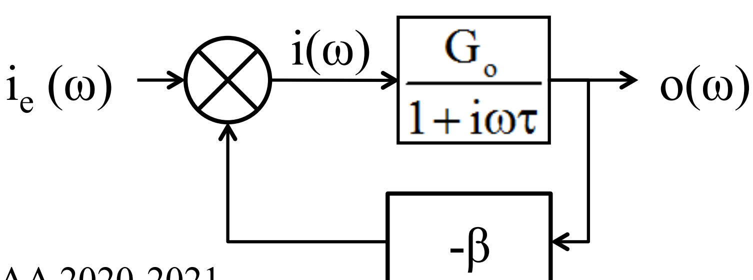
$$G_c=G_o/1+\beta G_o \qquad \tau_c=\tau/(1+\beta G_o)$$

- Then you finally get

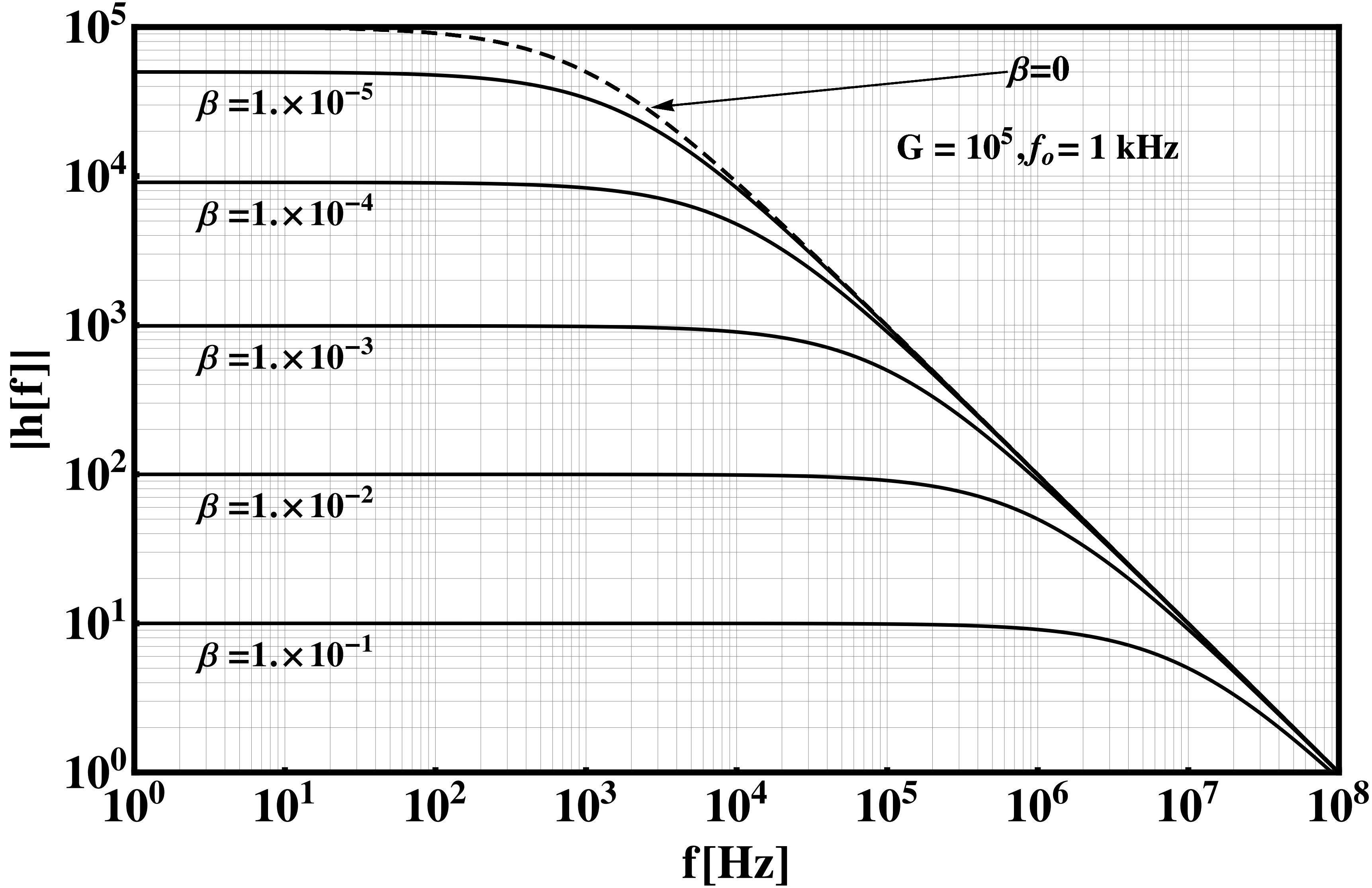
$$o(\omega)=i_e(\omega)\frac{G_c}{1+i\omega\tau_c}$$

The gain band-width theorem

- One interesting consequence
- Open loop transfer function $\quad G_o/(1+i\omega\tau) = G_o/(1+i\,f/f_o)$
  - Gain: G_o
  - "Bandwidth": $f_o = 1/(2\pi\tau)$
- Closed loop transfer function $\quad G_c/(1+i\omega\tau_c)$
  - Gain $\quad G_c = G_o/(1+\beta G_o)$
  - Bandwidth $\quad f_c = 1/2\pi\tau_c = (1+\beta G_o)/2\pi\tau$
- The gain-bandwidth theorem $\quad G_c f_c = G_o f_o$



The gain band-width theorem



Causal systems and Laplace Transforms

- True physical systems are causal. That is, for a linear system, h(t)=0 for t<0, and

$$o(t) = \int_0^\infty h(t')\, i(t-t')\, dt'$$

- For functions such that h(t)=0 for t<0 one can define the Laplace Transform:

$$\mathcal{L}[h](s) \equiv h(s) = \int_0^\infty h(t)\, e^{-st}\, dt$$

- Where s is the complex Laplace frequency
- If the integral exists also for s=iω, then

$$\mathcal{L}[h](s) = \mathcal{F}[h](\omega = s/i)$$

  – ($\boldsymbol{\mathcal{F}}$[h] = Fourier Transform of h)
- The Laplace Transform h(s) of the impulse response of a causal linear system is called the *transfer function*

Causal systems and Laplace Transform

$$h(s)=\int_0^\infty h(t)e^{-st}\,dt$$

- Laplace Transform exists for functions for which the Fourier transform does not exist, that is, the integral may diverge for s=iω, but still be finite for s=iω+s'

$$\left|\int_0^\infty h(t)e^{-s't}e^{-i\omega t}dt\right|<\infty$$

- Example: for h(t)=t the ordinary integral $\int_0^\infty te^{-i\omega t}dt\to\infty$ diverges.

- Laplace transform: $h(s)=\int_0^\infty te^{-st}dt \;=-t\left.\frac{e^{-st}}{s}\right|_0^\infty+\frac{1}{s}\int_0^\infty e^{-st}dt$

- If Re[s]>0 $\;=\frac{1}{s}\int_0^\infty e^{-st}\,dt \;=\frac{1}{s^2}$

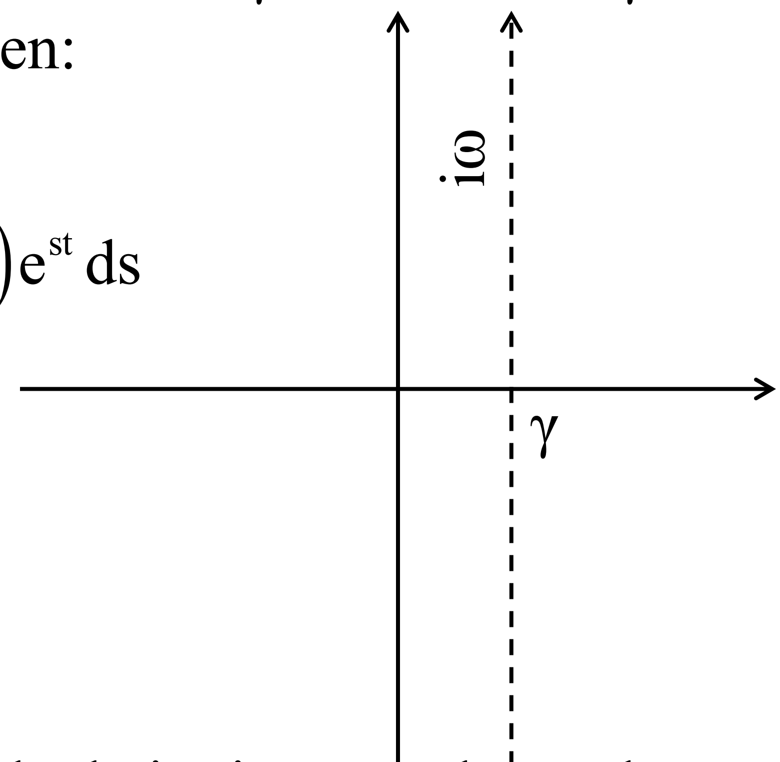
- The transform h(s) exists in the "region of convergence" Re[s]>0

Causal systems and Laplace Transform

$$h(s)=\int_0^{\infty} h(t)e^{-st}\,dt$$

- Inverse Laplace Transform. Pick a real numbers γ such that s= γ+iω is within the region of convergence. Then:

$$\mathcal{L}^{-1}\left[h\right](t)\equiv h(t)=\frac{1}{2\pi i}\lim_{\Omega\to\infty}\int_{\gamma-i\Omega}^{\gamma+i\Omega} h(s)e^{st}\,ds$$



- All possible Laplace transforms and their inverse have been calculated and may be found in tables (see 2 following slides) .

ID	Function	Time domain $x(t) = \mathcal{L}^{-1}\{X(s)\}$	Laplace s-domain $X(s) = \mathcal{L}\{x(t)\}$	Region of convergence
1	ideal delay	$\delta(t-\tau)$	$e^{-\tau s}$	
1a	unit impulse	$\delta(t)$	1	$\text{all } s$
2	delayed *n*th power with frequency shift	$\frac{(t-\tau)^n}{n!} e^{-\alpha(t-\tau)} \cdot u(t-\tau)$	$\frac{e^{-\tau s}}{(s+\alpha)^{n+1}}$	$\mathrm{Re}\{s\} > 0$
2a	*n*th power (for integer *n*)	$\frac{t^n}{n!} \cdot u(t)$	$\frac{1}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$
2a.1	*q*th power (for real *q*)	$\frac{t^q}{\Gamma(q+1)} \cdot u(t)$	$\frac{1}{s^{q+1}}$	$\mathrm{Re}\{s\} > 0$
2a.2	unit step	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
2b	delayed unit step	$u(t-\tau)$	$\frac{e^{-\tau s}}{s}$	$\mathrm{Re}\{s\} > 0$
2c	ramp	$t \cdot u(t)$	$\frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
2d	*n*th power with frequency shift	$\frac{t^n}{n!} e^{-\alpha t} \cdot u(t)$	$\frac{1}{(s+\alpha)^{n+1}}$	$\mathrm{Re}\{s\} > -\alpha$
2d.1	exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$	$\mathrm{Re}\{s\} > -\alpha$
3	exponential approach	$(1-e^{-\alpha t}) \cdot u(t)$	$\frac{\alpha}{s(s+\alpha)}$	$\mathrm{Re}\{s\} > 0$
4	sine	$\sin(\omega t) \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}\{s\} > 0$
5	cosine	$\cos(\omega t) \cdot u(t)$	$\frac{s}{s^2+\omega^2}$	$\mathrm{Re}\{s\} > 0$
6	hyperbolic sine	$\sinh(\alpha t) \cdot u(t)$	$\frac{\alpha}{s^2-\alpha^2}$	$\mathrm{Re}\{s\} > \|\alpha\|$

Wikipedia

7	hyperbolic cosine	$\cosh(\alpha t) \cdot u(t)$	$\frac{s}{s^2 - \alpha^2}$	$\mathrm{Re}\{s\} > \lvert\alpha\rvert$
8	Exponentially-decaying sine wave	$e^{\alpha t} \sin(\omega t) \cdot u(t)$	$\frac{\omega}{(s-\alpha)^2 + \omega^2}$	$\mathrm{Re}\{s\} > \alpha$
9	Exponentially-decaying cosine wave	$e^{\alpha t} \cos(\omega t) \cdot u(t)$	$\frac{s-\alpha}{(s-\alpha)^2 + \omega^2}$	$\mathrm{Re}\{s\} > \alpha$
10	*n*th root	$\sqrt[n]{t} \cdot u(t)$	$s^{-(n+1)/n} \cdot \Gamma\left(1 + \frac{1}{n}\right)$	$\mathrm{Re}\{s\} > 0$
11	natural logarithm	$\ln\left(\frac{t}{t_0}\right) \cdot u(t)$	$-\frac{t_0}{s}\left[\ \ln(t_0 s) + \gamma\ \right]$	$\mathrm{Re}\{s\} > 0$
12	Bessel function of the first kind, of order *n*	$J_n(\omega t) \cdot u(t)$	$\frac{\omega^n \left(s + \sqrt{s^2 + \omega^2}\right)^{-n}}{\sqrt{s^2 + \omega^2}}$	$\mathrm{Re}\{s\} > 0$ $(n > -1)$
13	Modified Bessel function of the first kind, of order *n*	$I_n(\omega t) \cdot u(t)$	$\frac{\omega^n \left(s + \sqrt{s^2 - \omega^2}\right)^{-n}}{\sqrt{s^2 - \omega^2}}$	$\mathrm{Re}\{s\} > \lvert\omega\rvert$
14	Bessel function of the second kind, of order 0	$Y_0(\alpha t) \cdot u(t)$	$-\frac{2 \sinh^{-1}(s/\alpha)}{\pi \sqrt{s^2 + \alpha^2}}$	$\mathrm{Re}\{s\} > 0$
15	Modified Bessel function of the second kind, of order 0	$K_0(\alpha t) \cdot u(t)$		
16	Error function	$\mathrm{erf}(t) \cdot u(t)$	$\frac{e^{s^2/4}\left(1 - \mathrm{erf}(s/2)\right)}{s}$	$\mathrm{Re}\{s\} > 0$

Explanatory notes:

- $u(t)$ represents the Heaviside step function.
- $\delta(t)$ represents the Dirac delta function.
- $\Gamma(z)$ represents the Gamma function.
- γ is the Euler-Mascheroni constant.
- t, a real number, typically represents *time*, although it can represent *any* independent dimension.
- s is the complex angular frequency, and $\mathrm{Re}\{s\}$ is its real part.
- α, β, τ, and ω are real numbers.
- n, is an integer.

Wikipedia

Properties of Laplace Tranforms

- Laplace transform obeys theorems similar to those for Fourier Transforms
  - Convolution
  - Linearity
  - …
- See following pages (from Wikipedia)

Properties of the unilateral Laplace transform

	Time domain	's' domain	Comment
Linearity	$af(t)+bg(t)$	$aF(s)+bG(s)$	Can be proved using basic rules of integration.
Frequency differentiation	$tf(t)$	$-F'(s)$	F' is the first derivative of F.
Frequency differentiation	$t^n f(t)$	$(-1)^n F^{(n)}(s)$	More general form, n^{th} derivative of F(s).
Differentiation	$f'(t)$	$sF(s)-f(0)$	f is assumed to be a differentiable function, and its derivative is assumed to be of exponential type. This can then be obtained by integration by parts
Second Differentiation	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	f is assumed twice differentiable and the second derivative to be of exponential type. Follows by applying the Differentiation property to $f'(t)$.
General Differentiation	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	f is assumed to be n-times differentiable, with n^{th} derivative of exponential type. Follow by mathematical induction.
Frequency integration	$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	
Integration	$\int_0^t f(\tau)\,d\tau=(u*f)(t)$	$\dfrac{1}{s}F(s)$	$u(t)$ is the Heaviside step function. Note $(u * f)(t)$ is the convolution of $u(t)$ and $f(t)$.
Time scaling	$f(at)$	$\dfrac{1}{\|a\|}F\left(\dfrac{s}{a}\right)$	
Frequency shifting	$e^{at}f(t)$	$F(s-a)$	
Time shifting	$f(t-a)u(t-a)$	$e^{-as}F(s)$	$u(t)$ is the Heaviside step function

Multiplication	$f(t)g(t)$	$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} F(\sigma)G(s-\sigma)\,d\sigma$	the integration is done along the vertical line $Re(\sigma) = c$ that lies entirely within the region of convergence of F.[12]
Convolution	$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)\cdot G(s)$	$f(t)$ and $g(t)$ are extended by zero for $t < 0$ in the definition of the convolution.
Complex_conjugation	$f^*(t)$	$F^*(s^*)$	
Cross-correlation	$f(t)\star g(t)$	$F^*(-s^*)\cdot G(s)$	
Periodic Function	$f(t)$	$\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\,dt$	$f(t)$ is a periodic function of period T so that $f(t) = f(t+T),\ \forall t \geq 0$. This is the result of the time shifting property and the geometric series.

- **Initial value theorem**:

$$f(0^+) = \lim_{s\to\infty} sF(s).$$

- **Final value theorem**:

$f(\infty) = \lim_{s\to 0} sF(s)$, if all poles of $sF(s)$ are in the left half-plane.

The final value theorem is useful because it gives the long-term behaviour without having to perform partial fraction decompositions or other difficult algebra. If a function's poles are in the right-hand plane (e.g. e^t or $\sin(t)$) the behaviour of this formula is undefined.

A remarkable property: Laplace Transforms of derivatives

- Let's calculate the Laplace transforms of the time derivative of function with a discontinuity in t=0

$$h(t)=0\ \ t<0;\ \ h(t)\neq 0\ \ t=0$$

- The Laplace transform of the derivative

$$\mathcal{L}\left[\frac{dh}{dt}\right](s)=\int_0^{\infty}\frac{dh(t)}{dt}e^{-st}\,dt$$

- Integrating by part

$$=h(t)e^{-st}\Big|_0^{\infty}+s\int_0^{\infty}h(t)e^{-st}dt=-h(t=0)+sh(s)\equiv -h(0)+s\mathcal{L}\left[h\right](s)$$

- Apply the formula recursively to obtain higher order derivatives

$$\mathcal{L}\left[\frac{d^2h}{dt^2}\right](s)=-\frac{dh}{dt}\bigg|_{t=0}+s\mathcal{L}\left[\frac{dh}{dt}\right](s)=-\dot{h}(0)-sh(0)+s^2\mathcal{L}\left[h\right](s)$$

- In general

$$\mathcal{L}\left[\frac{d^nh}{dt^n}\right](s)=s^n\mathcal{L}\left[h\right](s)-\sum_{m=0}^{n-1}s^{n-m-1}\left(\frac{d^mh}{dt^m}\right)_{t=0}$$

Laplace Transform and differential equation

- Take a system that obeys a linear differential equation. For instance a damped harmonic oscillator

$$m\ddot{x}(t)+\beta\dot{x}(t)+kx(t)=f(t)$$

- We are only interested in x(t) for $t \geq 0$. Then we can Laplace transform the above to get:

$$m\left[s^2x(s)-sx(0)-\dot{x}(0)\right]+\beta\left[sx(s)-x(0)\right]+kx(s)=f(s)$$

- That is

$$x(s)\left[ms^2+\beta s+k\right]=x(0)\left[\beta+ms\right]+m\dot{x}(0)+f(s)$$

- Solving

$$x(s)=\frac{x(0)(1/\tau+s)+\dot{x}(0)+f(s)/m}{s^2+s/\tau+\omega_o^2}$$

Laplace Transform and differential equation

- Full response $$x(s)=\frac{x(0)\left(1/\tau+s\right)+\dot{x}(0)+f(s)/m}{s^2+s/\tau+\omega_o^2}$$
- Free evolution becomes the response to initial conditions
$$x(s)=\underbrace{\frac{x(0)\left(1/\tau+s\right)+\dot{x}(0)}{s^2+s/\tau+\omega_o^2}}_{\text{Free evolution}}$$
- The response to input
$$x(s)=\underbrace{\frac{f(s)/m}{s^2+s/\tau+\omega_o^2}}_{\text{Response to input}}\equiv h(s)f(s)$$
- For these systems past history, $t<0$, is stored entirely into initial conditions! You don't need to know $f(t)$ for $t<0$

Inverse transform

- One can check that

$$f_1(s)=\frac{1}{s^2+s/\tau+\omega_o^2}=\frac{1}{2i\,\omega_1}\left(\frac{1}{s-(-1/2\tau+i\omega_1)}-\frac{1}{s-(-1/2\tau-i\omega_1)}\right)$$

- And that

$$f_2(s)=\frac{s}{s^2+s/\tau+\omega_o^2}=\frac{1}{2}\left(\frac{1+1/(2i\tau\omega_1)}{s-(-1/2\tau-i\omega_1)}+\frac{1-1/(2i\tau\omega_1)}{s-(-1/2\tau+i\omega_1)}\right)$$

- With $\omega_1=\omega_o\sqrt{1-1/(2\tau\omega_o)^2}$
- As

$$\mathcal{L}\left[e^{\alpha t}\right](s)=\int_0^\infty e^{(\alpha-s)t}\,dt=\frac{1}{s-\alpha}\qquad \mathrm{Re}\{s\}>\mathrm{Re}\{\alpha\}$$

- Then

$$f_1(t)=\frac{1}{2i\,\omega_1}\left(e^{(-1/2\tau-i\omega_1)t}-e^{(-1/2\tau+i\omega_1)t}\right)\Theta(t)=\frac{e^{-t/2\tau}}{\omega_1}\mathrm{Sin}(\omega_1 t)\Theta(t)$$

$$f_2(t)=e^{-t/2\tau}\left\{\mathrm{Cos}(\omega_1 t)-1/(2\tau\omega_1)\mathrm{Sin}(\omega_1 t)\right\}\Theta(t)$$

Laplace Transform and differential equation

- Free evolution in the time domain $$x(s)=\frac{x(0)(1/\tau+s)+\dot{x}(0)}{s^2+s/\tau+\omega_o^2}$$

- From

$$f_1(t)=\frac{e^{-t/2\tau}}{\omega_1}\mathrm{Sin}(\omega_1 t)\Theta(t)\quad f_2(t)=e^{-t/2\tau}\left\{\mathrm{Cos}(\omega_1 t)-1/(2\tau\omega_1)\mathrm{Sin}(\omega_1 t)\right\}\Theta(t)$$

- and $$x(t)=\left\{x(0)/\tau+\dot{x}(0)\right\}f_1(t)+x(0)f_2(t)$$

- we get

$$x(t)=\Theta(t)\left\{x(0)e^{-t/2\tau}\left[\mathrm{Cos}(\omega_1 t)+\frac{1}{2\tau\omega_1}\mathrm{Sin}(\omega_1 t)\right]+\frac{\dot{x}(0)}{\omega_1}e^{-t/2\tau}\mathrm{Sin}(\omega_1 t)\right\}$$

Frequency response, transfer function and stability

- A linear system (not necessarily causal) is defined as Bounded-Input-Bounded Output (BIBO) stable if for any limited input, i.e. an input such that $|i(t)| < \infty$, the output is also limited: $|o(t)| < \infty$.
- As, for a generic system

$$o(t) = \int_{-\infty}^{\infty} h(t')\, i(t-t')\, dt'$$

- By applying standard inequalities

$$|o(t)| = \left| \int_{-\infty}^{\infty} h(t')\, i(t-t')\, dt' \right| \leq \int_{-\infty}^{\infty} |h(t')| |i(t-t')|\, dt' \leq |i(t)|_{\max} \int_{-\infty}^{\infty} |h(t')|\, dt'$$

- As $|i(t)| < \infty$, then if

$$\int_{-\infty}^{\infty} |h(t')|\, dt' < \infty$$

- $|o(t)| < \infty$. As this is Dirichelet conditions, we conclude that *the existence of a frequency response implies stability* .

Frequency response, transfer function and stability

- On the other hand suppose that

$$\int_{-\infty}^{\infty} \left| h(t') \right| dt' = \infty$$

- Then form the signal $i(t) = h(-t) / \left| h(-t) \right|$
- For this signal |i(t)|< ∞, and the output is

$$o(t) = \int_{-\infty}^{\infty} \frac{h(t')\, h(t'-t)}{\left| h(t'-t) \right|} dt'$$

- so that

$$o(0) = \int_{-\infty}^{\infty} \left| h(t') \right| dt' = \infty$$

- Thus the system is not BIBO

Laplace transforms and unstable systems

- Thus for a linear time-invariant system, the frequency response exists if and only if the systems is BIBO-stable.
- For a causal system, the transfer function on the contrary exists if

$$\int_0^\infty \left| h(t') e^{-st'} \right| dt' < \infty$$

- Even if

$$\int_0^\infty \left| h(t') \right| dt' = \infty$$

- *Unstable systems can have a transfer function!*

Example of unstable system: Newton law

- The equation of motion

$$m\ddot{x} = F_x$$

- Laplace transform

$$m\left(s^2 x(s) - s\,x(0) - \dot{x}(0)\right) = F_x(s)$$

- Solution

$$x(s) = \frac{s\,x(0) + \dot{x}(0) + F_x(s)/m}{s^2}$$

- That is

$$x(s) = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{F_x(s)}{ms^2}$$

Example of unstable system: Newton law

$$x(s) = \frac{x(0)}{s} + \frac{\dot{x}(0)}{s^2} + \frac{F_x(s)}{ms^2}$$

- Let's go back to time domain

unit step	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
delayed unit step	$u(t-\tau)$	$\frac{1}{s}e^{-\tau s}$	$\mathrm{Re}(s) > 0$
ramp	$t \cdot u(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
Convolution	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s) \cdot G(s)$	

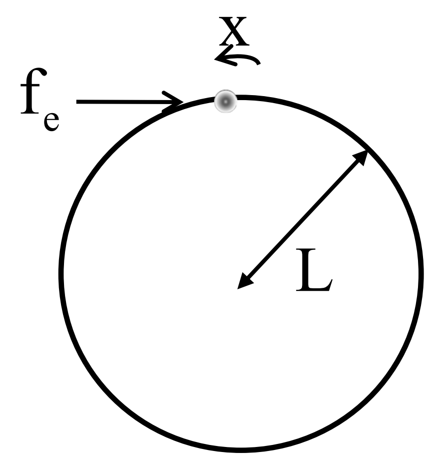
$$x(t) = x(0) + \dot{x}(0)t + \int_0^t t' F_x(t-t')dt'$$

- Which is indeed unstable if $\dot{x}(0) \neq 0$

An example of unstable system: the inverted pendulum

- A particle on top of a vertical circular guide
- Equation of motion without feedback

$$m\ddot{x} = mg\mathrm{Sin}\left(\frac{x}{L}\right)\mathrm{Cos}\left(\frac{x}{L}\right) + f_e$$



- Linearization

$$\ddot{x} - \left(g/L\right)x = \left(f_e/m\right)$$

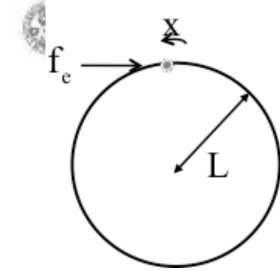
- Laplace

$$\left(s^2 - g/L\right)x\left[s\right] - sx\left(0\right) - \dot{x}\left(0\right) = \left(f_e\left[s\right]/m\right)$$

- Solution

$$x\left[s\right] = \frac{sx\left(0\right) + \dot{x}\left(0\right) + \left(f_e\left[s\right]/m\right)}{s^2 - g/L}$$

Inverted pendulum



- Solution

$$x[s]=\frac{s x(0)+\dot{x}(0)+\left(f_e[s]/m\right)}{s^2-g/L}$$

- Assume f_e=0. You can expand x[s] to make the inverse transform easy

$$x[s]=\frac{x(0)}{2}\left(\frac{1}{s-\sqrt{g/L}}+\frac{1}{s+\sqrt{g/L}}\right)+\frac{\dot{x}(0)}{2\sqrt{g/L}}\left(\frac{1}{s-\sqrt{g/L}}-\frac{1}{s+\sqrt{g/L}}\right)$$

- As

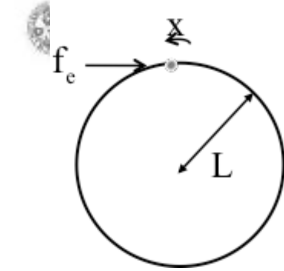
$$\int_0^\infty e^{\alpha t}e^{-st}\,dt=\frac{1}{s-\alpha}$$

- Then, in the time domain:

$$x(t)=\frac{x(0)}{2}\left(e^{\sqrt{g/L}t}+e^{-\sqrt{g/L}t}\right)+\frac{\dot{x}(0)}{2\sqrt{g/L}}\left(e^{\sqrt{g/L}t}-e^{-\sqrt{g/L}t}\right)$$

- For non-zero initial conditions the system is carried away exponentially!

Inverted pendulum



- Numerical example: particle at rest with small initial offset

$$x(t)=\left[x(0)/2\right]\left(e^{\sqrt{g/L}t}+e^{-\sqrt{g/L}t}\right)$$

- Assume L= 1 m and x(0)=10^{-9} m. Then

$$\tau \equiv \sqrt{L/g} \approx 0.3\text{ s}$$

- After t >a few times τ $\quad x(t)\approx e^{\sqrt{g/L}t}x(0)/2$
- Equation has been derived with x/L ≡ δ<<1 that is

$$e^{\sqrt{g/L}t}x(0)/2L \le \delta \qquad t \le \tau\,\mathrm{Log}\left[\frac{2L\delta}{x(0)}\right]$$

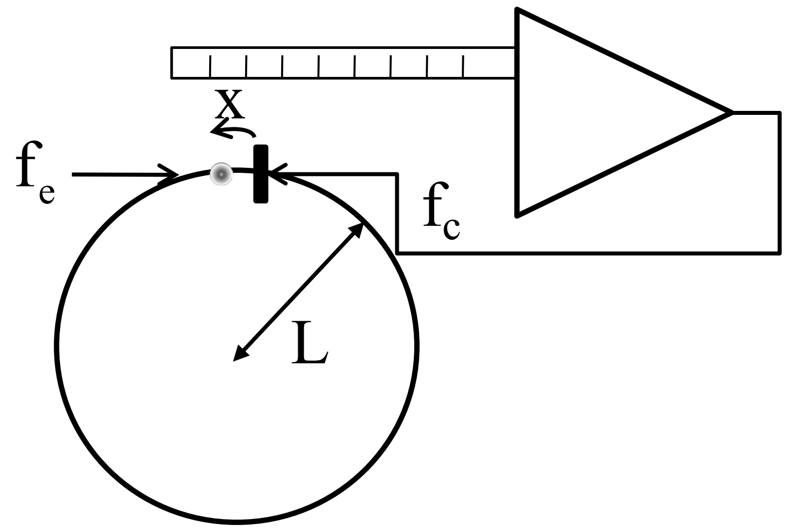
- With δ= 0.05

$$t \le (0.3\text{s})\,\mathrm{Log}\left[\frac{0.1}{10^{-9}}\right]\approx 6\text{ s}$$

- After that the system is already out of the requested linearity

Inverted pendulum

- Stabilization by feedback. A force proportional to the horizontal displacement of the particle



- The usual control law

$$f_c = -m\omega_o^2 x - \left(m/\tau_f\right) dx/dt$$

- Equation of motion

$$\ddot{x} - \left(g/L\right)x = \left(f_e/m\right) - \omega_o^2 x - \left(1/\tau_f\right)dx/dt$$

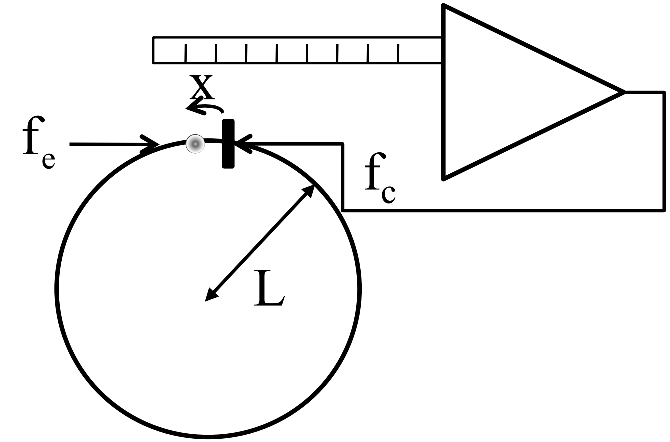
- Laplace transform

$$\left(s^2 - g/L\right)x\left[s\right] - sx\left(0\right) - \dot{x}\left(0\right) = \left(f_e\left[s\right]/m\right) - \left(\omega_o^2 + s/\tau_f\right)x\left[s\right] + x\left(0\right)/\tau_f$$

Solution

$$x\left[s\right] = \frac{f_e\left[s\right]/m + x\left(0\right)\left(1/\tau_f + s\right) + \dot{x}\left(0\right)}{s^2 + \left(\omega_o^2 - g/L\right) + s/\tau_f}$$

Exercise: inverted pendulum



- Solution with feedback

$$x[s]=\frac{f_e[s]/m+x(0)\left(1/\tau_f+s\right)+\dot{x}(0)}{s^2+\left(\omega_o^2-g/L\right)+s/\tau_f}$$

- Now assume

$$\omega_o^2 >> g/L \quad \tau_f\sqrt{\omega_o^2-g/L}<1/2$$

- You can re-write the the response to input or to initial velocity as

$$x[s]=\frac{f_e[s]/m}{(s-s_+)(s-s_-)}=f_e[s]\frac{1}{m(s_+-s_-)}\left(\frac{1}{(s-s_+)}-\frac{1}{(s-s_-)}\right)$$

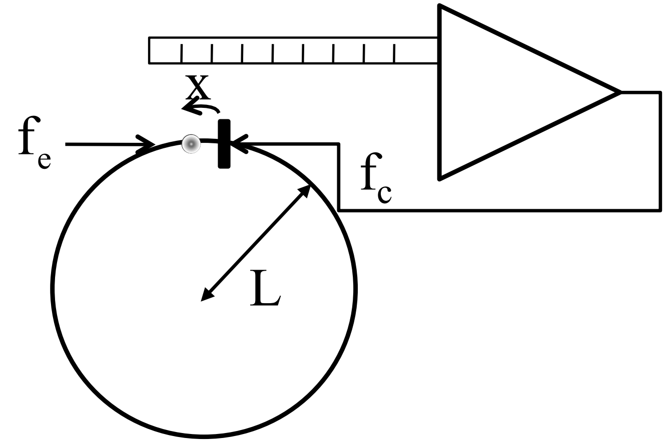
- With

$$s_\pm=-1/(2\tau_f)\left(1\pm\sqrt{1-4\tau_f^2\left(\omega_o^2-g/L\right)}\right)$$

- The impulse response is then

$$h(t)=\frac{1}{m(s_+-s_-)}\left(e^{s_+t}-e^{s_-t}\right)$$

Exercise: inverted pendulum



- The impulse response with feedback

$$h(t)=\frac{1}{m(s_+-s_-)}e^{s_+t}-e^{s_-t}$$

- With

$$s_\pm=-1/(2\tau_f)\left(1\pm\sqrt{1-4\tau_f^2(\omega_o^2-g/L)}\right)$$

- By substituting

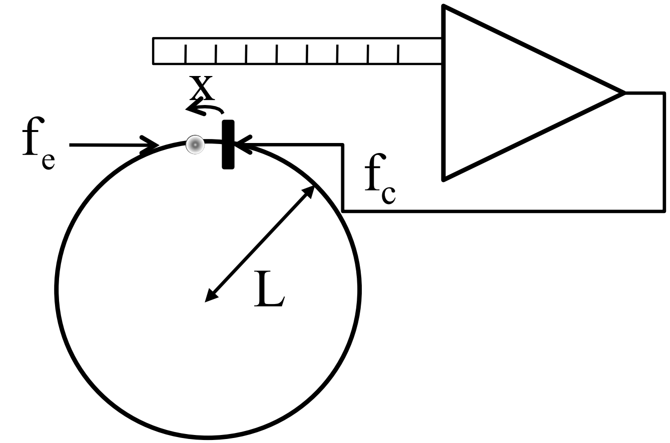
$$h(t)=\frac{\tau_f}{m\sqrt{1-4\tau_f^2(\omega_o^2-g/L)}}\left[e^{-\frac{t}{2\tau_f}\left(1-\sqrt{1-4\tau_f^2(\omega_o^2-g/L)}\right)}-e^{-\frac{t}{2\tau_f}\left(1+\sqrt{1-4\tau_f^2(\omega_o^2-g/L)}\right)}\right]$$

- As time constants are now both positive, the system has been stabilized

Exercise: inverted pendulum

- Notice that the feedback force, once free evolution has decayed to zero is

$$f_c[s] = -\frac{\omega_o^2 + s/\tau_f}{s^2 + \left(\omega_o^2 - g/L\right) + s/\tau_f} f_e[s]$$



- As

$$\omega_o^2 >> g/L$$

- In the limit

$$\left|\omega_o^2 + s/\tau_f\right| >> \left|s^2\right|$$

- We get

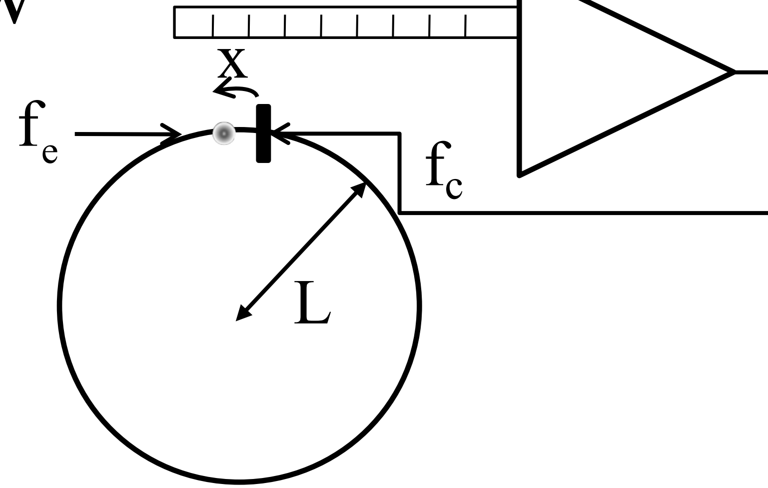
$$f_c[s] \approx -f_e[s]$$

- And then

$$f_c(t) \approx -f_e(t)$$

- Thus for "low frequency" signals the feedback force measures the input force

More common feedback law



- A proportional-derivator-integrator (PID) controller

$$f_c = -\omega_o^2 x - \frac{\dot{x}}{\tau} - \frac{1}{T^3}\int_0^t x(t')dt'$$

- Laplace transform (null initial conditions)

$$f_c(s) = -x(s)\left(\omega_o^2 + \frac{s}{\tau} + \frac{1}{sT^3}\right)$$

- Open loop transfer function

$$m\,\ddot{x} = \frac{g}{L}x + f_e \rightarrow x(s) = \frac{f_e/m}{s^2 - \frac{g}{L}}$$

- Closed loop transfer function

- $$x(s) = \frac{f_e}{m}\frac{\frac{1}{s^2-\frac{g}{L}}}{1+\frac{1}{s^2-\frac{g}{L}}\left(\omega_o^2+\frac{s}{\tau}+\frac{1}{sT^3}\right)} = \frac{f_e}{m}\frac{1}{s^2+\omega_o^2-\frac{g}{L}+\frac{s}{\tau}+\frac{1}{sT^3}} = \frac{f_e}{m}\frac{s}{s^3+(\,\omega_o^2-\frac{g}{L})s+\frac{s^2}{\tau}+\frac{1}{T^3}}$$

More on causal systems: Kramers-Kronig dispersion relation

- For a causal system h(t)=0 for t<0 thus h(t) is one real function different from zero just on one semi-axis of the real parameter t.
- If the system has the frequency response

$$h(\omega)=\int_0^{\infty} h(t)e^{-i\omega t}dt$$

- Thanks to the symmetries of h(ω), Re{h(ω)} and Im{h(ω)} are also independently defined only on the semi-axis $\omega \geq 0$. However these are *two* functions on one semi-axis, not one like in the case of h(t).
- This fact would contradict the one-to-one nature of Fourier transforms.
- Kramers Kronig relations remove this ambiguity establishing a relation between Re{h(ω)} and Im{h(ω)}.

Kramers-Kronig dispersion relations

- Take the frequency response

$$h(\omega)=\int_0^\infty h(t)e^{-i\omega t}dt$$

- Generalize h(ω) to complex frequency ω=ω'+iω''

$$h(\omega)=\int_0^\infty h(t)e^{-i(\omega'+i\omega'')t}dt = \int_0^\infty h(t)e^{(\omega''-i\omega')t}dt$$

- h(ω'+iω'') exists for all ω''<0

- Notice that if ω''→-∞ h(ω)→0

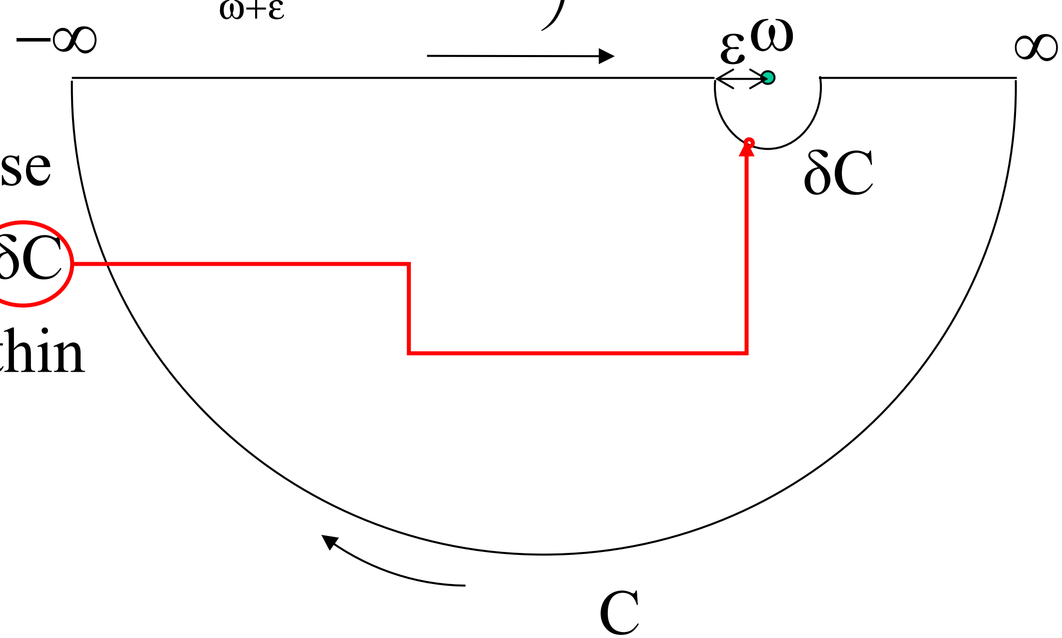
Kramers-Kronig dispersion relations

$$h(\omega'+i\omega'')=\int_0^\infty h(t)e^{(\omega''-i\omega')t}\,dt$$

- Now consider the following principal value integral*

$$P\int_{-\infty}^{\infty}\frac{h(\tilde\omega)}{\omega-\tilde\omega}d\tilde\omega \underset{\text{def}}{\equiv} \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{\omega-\varepsilon}\frac{h(\tilde\omega)}{\omega-\tilde\omega}d\tilde\omega+\int_{\omega+\varepsilon}^{\infty}\frac{h(\tilde\omega)}{\omega-\tilde\omega}d\tilde\omega\right)$$

- As $h(\tilde\omega)\to 0$ when $\mathrm{Im}\{\tilde\omega\}\to -\infty$, we can close the path at $-\infty$, and with δC



- As $h(\tilde\omega)$ has no poles within the domain limited by C

$$\oint_C \frac{h(\tilde\omega)}{\omega-\tilde\omega}d\tilde\omega=0$$

- But

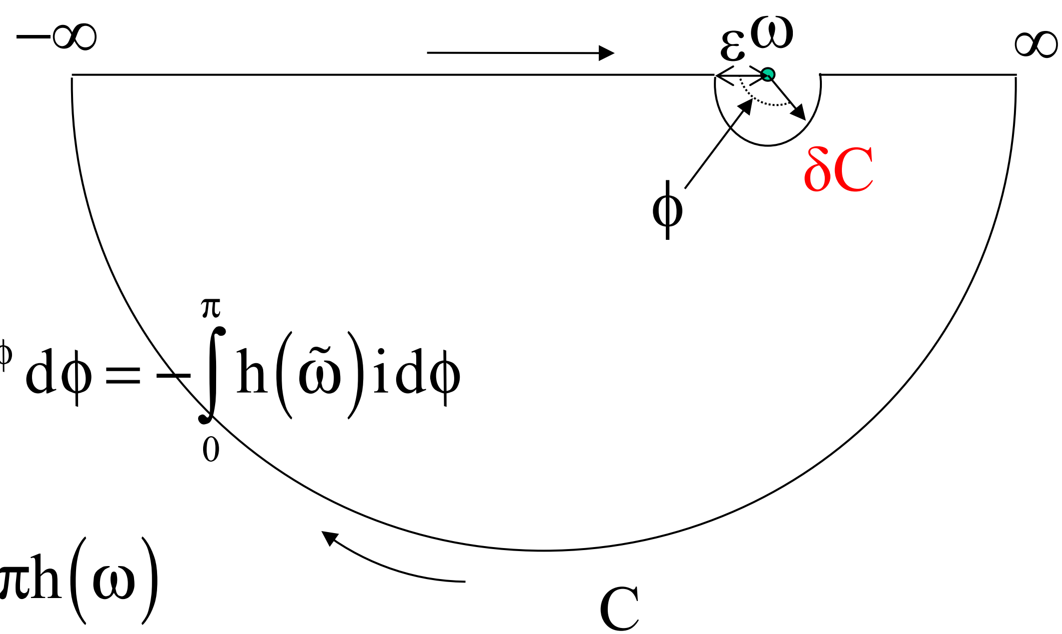
$$\oint_C \frac{h(\tilde\omega)}{\omega-\tilde\omega}d\tilde\omega = P\int_{-\infty}^{\infty}\frac{h(\tilde\omega)}{\omega-\tilde\omega}d\tilde\omega+\int_{\delta C}\frac{h(\tilde\omega)}{\omega-\tilde\omega}d\tilde\omega=0$$

*assume:

$$\lim_{\omega\to\infty}|h(\omega)|\to\frac{1}{|\omega|}$$

Kramers-Kronig dispersion relations

- Thus $$P\int_{-\infty}^{\infty}\frac{h(\tilde{\omega})}{\omega-\tilde{\omega}}d\tilde{\omega}+\int_{\delta C}\frac{h(\tilde{\omega})}{\omega-\tilde{\omega}}d\tilde{\omega}=0$$
- On δC
  $$\tilde{\omega}=\omega+\varepsilon e^{i\phi}$$
- And
  $$d\tilde{\omega}=\varepsilon i e^{i\phi}d\phi$$
- Thus



$$\int_{\delta C}\frac{h(\tilde{\omega})}{\omega-\tilde{\omega}}d\tilde{\omega}=\int_{-\pi}^{0}\frac{h(\tilde{\omega})}{-\varepsilon e^{i\phi}}\varepsilon i e^{i\phi}\,d\phi=-\int_{0}^{\pi}h(\tilde{\omega})\,i\,d\phi$$

- If $\varepsilon\rightarrow 0$ $\quad\int_{0}^{\pi}h(\tilde{\omega})\,i\,d\phi\rightarrow i\pi h(\omega)$
- And finally $$P\int_{-\infty}^{\infty}\frac{h(\tilde{\omega})}{\omega-\tilde{\omega}}d\tilde{\omega}-i\pi h(\omega)=0\rightarrow P\int_{-\infty}^{\infty}\frac{h(\tilde{\omega})}{\omega-\tilde{\omega}}d\tilde{\omega}=i\pi h(\omega)$$

Kramers-Kronig dispersion relations

- In conclusion $$P\int_{-\infty}^{\infty}\frac{h(\tilde{\omega})}{\omega-\tilde{\omega}}d\tilde{\omega}=i\pi h(\omega)$$

- Equating real an imaginary part we finally obtain the dispersion relations

$$\mathrm{Im}\left\{h(\omega)\right\}=-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Re}\left\{h(\tilde{\omega})\right\}}{\omega-\tilde{\omega}}d\tilde{\omega}$$

$$\mathrm{Re}\left\{h(\omega)\right\}=\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Im}\left\{h(\tilde{\omega})\right\}}{\omega-\tilde{\omega}}d\tilde{\omega}$$

Hilbert Transform

The Hilbert transform can be thought of as the convolution of $u(t)$ with the function $h(t) = 1/(\pi t)$. Because $h(t)$ is not integrable the integrals defining the convolution do not converge. Instead, the Hilbert transform is defined using the Cauchy principal value (denoted here by p.v.) Explicitly, the Hilbert transform of a function (or signal) $u(t)$ is given by

$$H(u)(t) = \text{p.v.} \int_{-\infty}^{\infty} u(\tau) h(t-\tau)\, d\tau$$

- That is

$$H[u](t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(\tau)}{t-\tau} d\tau$$

- With the very important "inverse" relation

$$H[H[u]] = -u$$

- Then Kramers-Kronig relations can be written as

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{h(\tilde{\omega})}{\omega - \tilde{\omega}} d\tilde{\omega} \equiv H[h(\omega)] = ih(\omega)$$

Hilbert Transform

Table of selected Hilbert transforms

Signal $u(t)$	Hilbert transform[fn 1] $H(u)(t)$
$\sin(t)$ [fn 2]	$-\cos(t)$
$\cos(t)$ [fn 2]	$\sin(t)$
$\dfrac{1}{t^2+1}$	$\dfrac{t}{t^2+1}$
Sinc function $\dfrac{\sin(t)}{t}$	$\dfrac{1-\cos(t)}{t}$
Rectangular function $\sqcap(t)$	$\dfrac{1}{\pi}\ln\left\lvert\dfrac{t+\frac{1}{2}}{t-\frac{1}{2}}\right\rvert$
Dirac delta function $\delta(t)$	$\dfrac{1}{\pi t}$
Characteristic Function $\chi_{[a,b]}(x)$	$\dfrac{1}{\pi}\log\left\lvert\dfrac{x-a}{x-b}\right\rvert$

Kramers-Kronig Relations

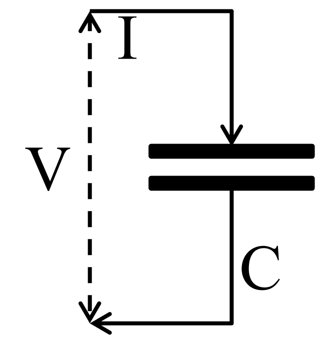
- From $$H\left[h(\omega)\right] = ih(\omega)$$

- We get

$$\mathrm{Re}\left\{h(\omega)\right\} = H\left[\mathrm{Im}\left\{h\right\}\right](\omega) \quad \mathrm{Im}\left\{h(\omega)\right\} = -H\left[\mathrm{Re}\left\{h\right\}\right](\omega)$$

- Causal systems do not have independent real and imaginary parts!

Pure capacitors do not exist



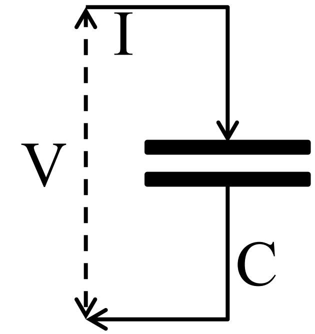
- A pure capacitor

$$V(\omega) = I(\omega)/i\omega C$$

- Transfer function

$$h(\omega) = -i/\omega C$$

Pure capacitors do not exist



- A pure capacitor

$$V(\omega)=I(\omega)/i\omega C$$

- Transfer function

$$h(\omega)=-i/\omega C$$

- According to Kramers-Kronig

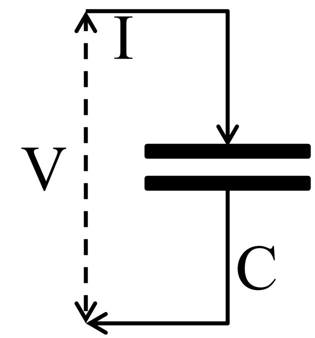
$$\mathrm{Re}\{h(\omega)\}=H[-1/\omega C]=-H[1/\omega]/C$$

- From tables $\quad H[1/\omega]=-\pi\delta(\omega)$

Signal $u(t)$	Hilbert transform[fn 1] $H(u)(t)$
Dirac delta function $\delta(t)$	$\frac{1}{\pi t}$

- A pure capacitor would violate causality!!!

Pure capacitors do not exist



- A pure capacitor: true transfer function

$$h(\omega) = -\frac{i}{\omega C} + \frac{\pi\delta(\omega)}{C}$$

- Calculate inverse (distributions)

$$\text{InverseFourierTransform}\left[\frac{1}{\mathbb{i}\,\omega\,C} + \frac{\pi\,\text{DiracDelta}[\omega]}{C},\ \omega,\ t\right] = \frac{1 + \text{Sign}[t]}{2\,C}$$

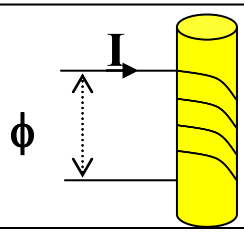
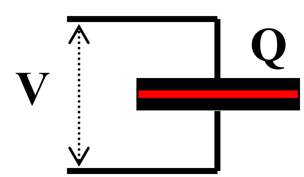
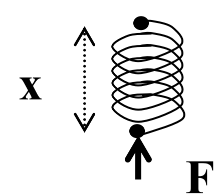
$$\frac{1+\text{Sign}\left[t\right]}{2} = \Theta(t)$$

- Physical meaning:

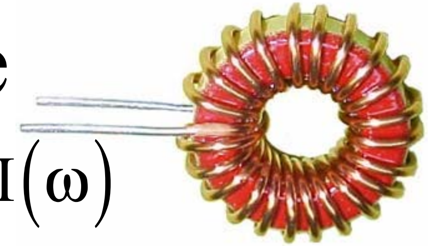
$$V(t) = \frac{1}{C}\int_{-\infty}^{\infty}\Theta(t')I(t-t')dt' = \frac{1}{C}\int_{0}^{\infty}I(t-t')dt' = \frac{1}{C}\int_{-\infty}^{t}I(t')dt' = \frac{Q(t)}{C}$$

One example from condensed matter

A general phenomenon in condensed matter
a frequency independent imaginary part in the "response" $\chi(\omega)$

	Magnetism	Dielectrics	Mechanics
Generalized response $\chi(\omega)$	**Inductance $L(\omega)$** $\phi(\omega) = L(\omega)I(\omega)$ $L(\omega) = L' - iL_o$	**Capacitance $C(\omega)$** $Q(\omega) = C(\omega)V(\omega)$ $C(\omega) = C'(\omega) - iC_o$	**Spring constant $k(\omega)$** $F(\omega) = -k(\omega)x(\omega)$ $k(\omega) = k'(\omega) + ik_o$
Nyquist formula	$S_\phi(\omega) = 4k_B T \frac{L_o}{\omega}$	$S_Q(\omega) = 4k_B T \frac{C_o}{\omega}$	$S_F(\omega) = 4k_B T \frac{k_o}{\omega}$
			

Consider the magnetic case



the magnetic flux-current relation: $\phi(\omega) = L(\omega) I(\omega)$

- The magnetic susceptibility χ of the core material is connected to inductance thru:

$$L(\omega) = \chi(\omega) L_o$$

- Magnetic susceptibility of real material is complex:

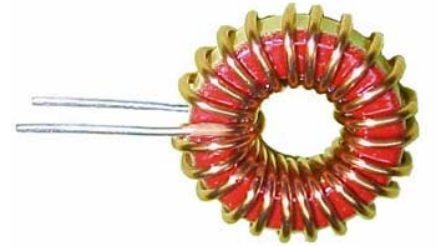
$$\chi(\omega) = \chi'(\omega) - i\chi''(\omega)$$

- Notice that impedance $Z(\omega)=i\omega L(\omega)$ acquires a real part, i.e. a resistive component

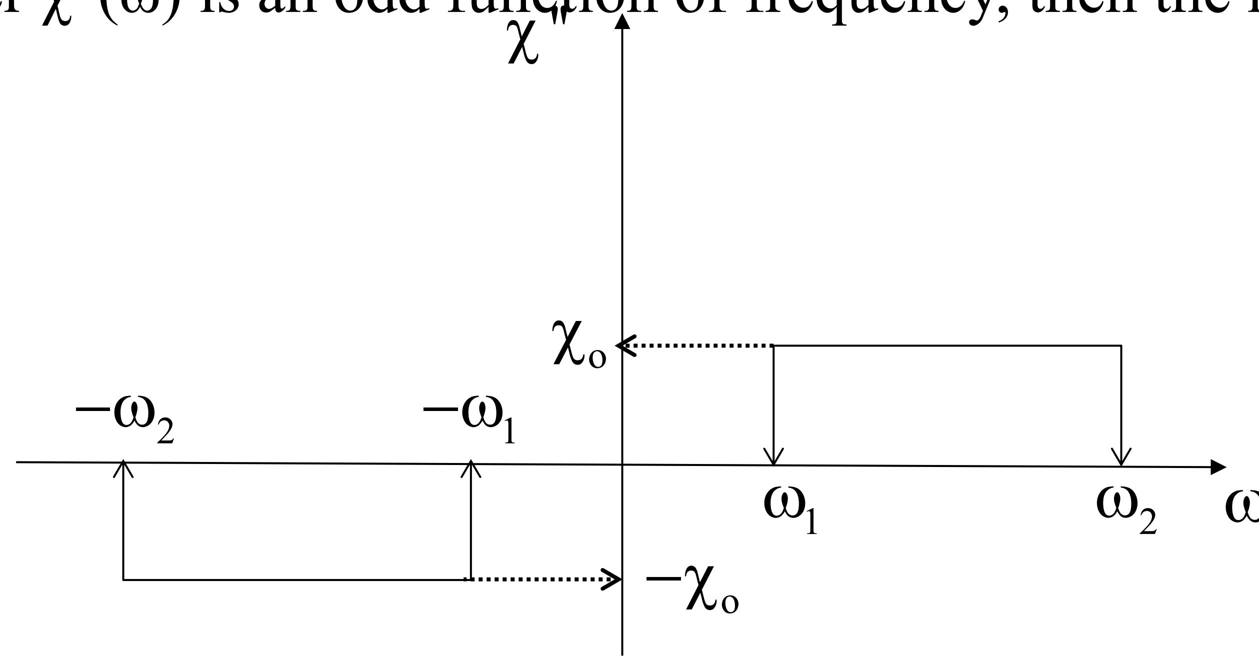
$$\mathrm{Re}\{Z(\omega)\} = \omega\chi''(\omega) L_o$$

- At low frequency $\chi''(\omega) \simeq \chi_o$ in many materials

A popular model



- $\chi''(\omega) \simeq \chi_o$. Assume this is true in some decade-wide frequency interval, outside which $\chi''(\omega)=0$. (Can't be infinite otherwise there is no Fourier transform in strict sense)
- Remember $\chi''(\omega)$ is an odd function of frequency, then the model is:



Real part from Kramers Kronig

- An alternative formula

$$\chi'(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{-\chi''(\omega')}{\omega-\omega'} d\omega' = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\omega+\omega'}{\omega+\omega'} \times \frac{-\chi''(\omega')}{\omega-\omega'} d\omega'$$

- Further

$$\chi'(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \omega \frac{-\chi''(\omega')}{\omega^2-\omega'^2} d\omega' + \frac{1}{\pi} P \int_{-\infty}^{\infty} \omega' \frac{-\chi''(\omega')}{\omega^2-\omega'^2} d\omega'$$

- But $\chi''(\omega)$ is odd. Then

$$\chi'(\omega) = \frac{2}{\pi} P \int_{0}^{\infty} \omega' \frac{-\chi''(\omega')}{\omega^2-\omega'^2} d\omega'$$

- In our case

$$\chi'(\omega) = \frac{2}{\pi} P \int_{\omega_1}^{\omega_2} \omega' \frac{-\chi_o}{\omega^2-\omega'^2} d\omega' = -\chi_o \frac{1}{\pi} P \int_{\omega_1^2}^{\omega_2^2} \frac{1}{\omega^2-x} dx$$

Real part from Kramers Kronig

$$\chi'(\omega) = -\frac{\chi_o}{\pi} P \int_{\omega_1^2}^{\omega_2^2} \frac{1}{\omega^2 - x} dx$$

- The interesting case $\omega_1 \ll \omega \ll \omega_2$

```
In[ ]:= Integrate[1/(a - x), {x, b, c}, PrincipalValue → True, Assumptions → {b > 0, c > 0, b < a < c}]

Out[ ]= Log[(-a + b)/(a - c)]
```

$$\chi'(\omega) = \frac{\chi_o}{\pi} Ln\left[\frac{\omega_2^2 - \omega^2}{\omega^2 - \omega_1^2}\right] \simeq \frac{\chi_o}{\pi}\left(Ln[\omega_2^2] - 2Ln[\omega]\right)$$

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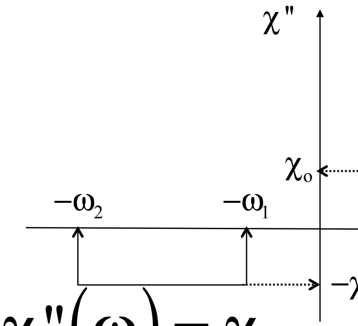
A popular model



- In summary, or $\omega_1 \ll \omega \ll \omega_2$

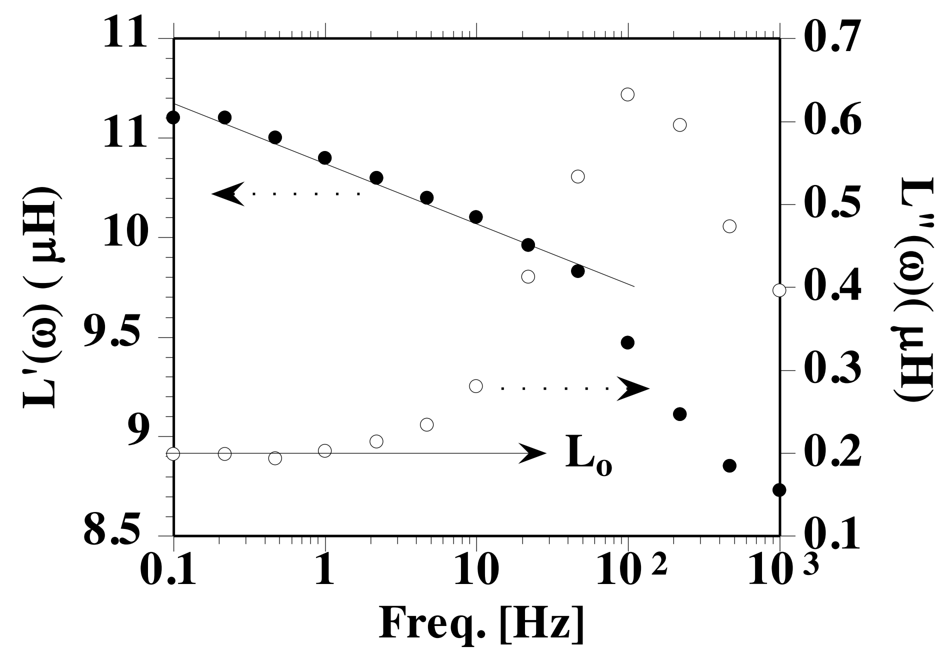
$$\chi'(\omega) \approx \frac{\chi_o}{\pi}\mathrm{Log}\left(\omega_2^2\right) - \frac{2\chi_o}{\pi}\mathrm{Log}(\omega) \quad \chi''(\omega) = \chi_o$$

- Switching to inductance

$$L'(\omega) \approx \left\{\frac{\chi_o}{\pi}\mathrm{Log}\left(\omega_2^2\right) - \frac{2\chi_o}{\pi}\mathrm{Log}(\omega)\right\} L_o \quad L''(\omega) = \chi_o L_o$$

- Notice

$$\frac{1}{L''(\omega)}\frac{dL'(\omega)}{d\mathrm{Log}(\omega)} \approx -\frac{2}{\pi}$$



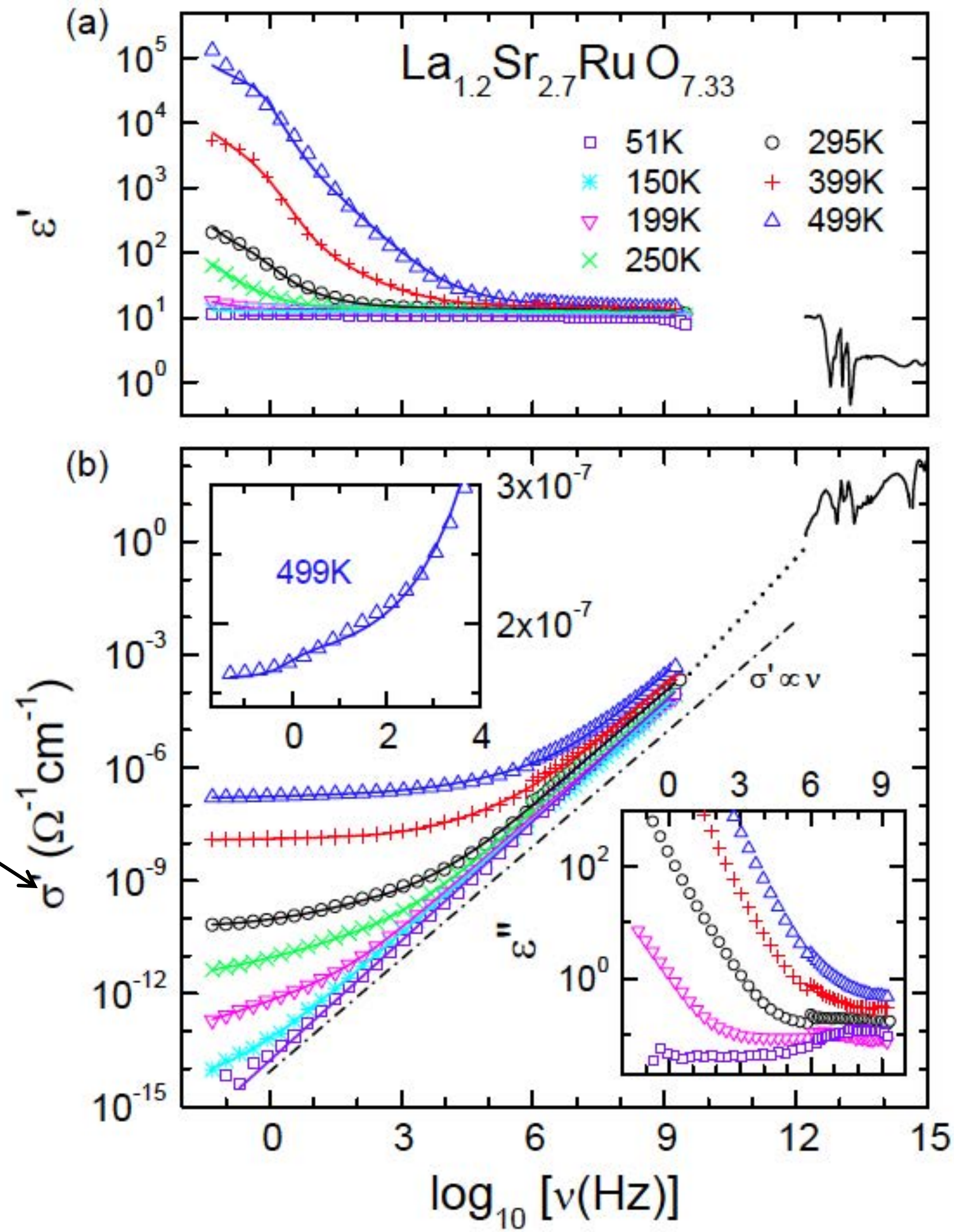
(Vitrovac @ 4,2 K)

G.A. Prodi & al. J. Appl. Phys. **66** 5984 (1989)

Dielectrics: “googling” for 5’

Apparent giant dielectric constants, dielectric relaxation, and ac-conductivity of hexagonal perovskites $La_{1.2}Sr_{2.7}BO_{7.33}$ (B = Ru, Ir)

P. Lunkenheimer[a,*], T. Götzfried[b], R. Fichtl[a], S. Weber[a], T. Rudolf[a], A. Loidl[a], A. Reller[b], and S.G. Ebbinghaus[b]



Proportional to ε”

double emission at 589.29 nm wavelength.

The variation of refractive index with wavelength for various glasses.

Complex index of refraction and absorption [edit]

See also: Mathematical descriptions of opacity

When light passes through a medium, some part of it will always be absorbed. This can be conveniently taken into account by defining a complex index of refraction,

$$\tilde{n} = n + i\kappa.$$

Here, the real part of the refractive index n indicates the phase speed, while the imaginary part κ indicates the amount of absorption loss when the electromagnetic wave propagates through the material.

That κ corresponds to absorption can be seen by inserting this refractive index into the expression for electric field of a plane electromagnetic wave traveling in the z-direction. We can do this by relating the wave number to the refractive index through $k = \frac{2\pi n}{\lambda_0}$, with λ_0 being the vacuum wavelength. With complex wave number $\tilde{k}$ and refractive index $n + i\kappa$ this can be inserted into the plane wave expression as

$$\mathbf{E}(z,t) = \mathrm{Re}(\mathbf{E}_0 e^{i(\tilde{k}z-\omega t)}) = \mathrm{Re}(\mathbf{E}_0 e^{i(2\pi(n+i\kappa)z/\lambda_0-\omega t)}) = e^{-2\pi\kappa z/\lambda_0}\mathrm{Re}(\mathbf{E}_0 e^{i(kz-\omega t)}).$$

Here we see that κ gives an exponential decay, as expected from Beer–Lambert law.

κ is often called the **extinction coefficient** in physics although this has a different definition within chemistry. Both n and κ are dependent on the frequency. In most circumstances $\kappa > 0$ (light is absorbed) or $\kappa = 0$ (light travels forever without loss). In special situations, especially in the gain medium of lasers, it is also possible that $\kappa < 0$, corresponding to an amplification of the light.

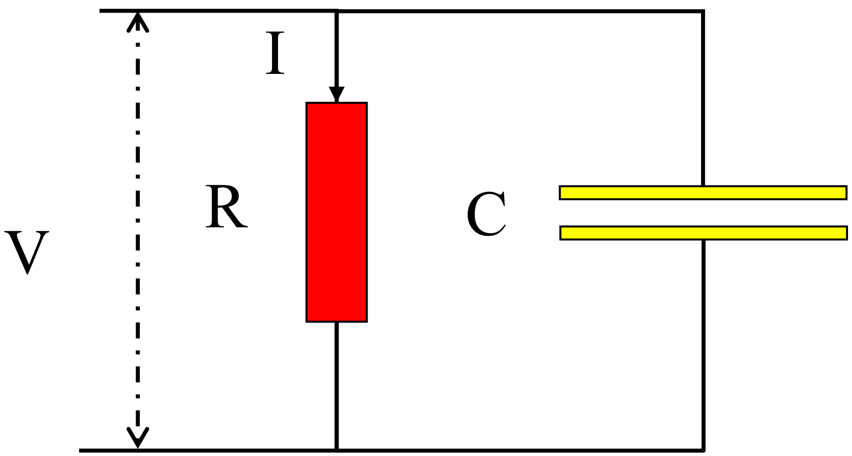
An alternative convention uses $\tilde{n} = n - i\kappa$ instead of $\tilde{n} = n + i\kappa$, but where $\kappa > 0$ still corresponds to loss. Therefore these two conventions are inconsistent and should not be confused. The difference is related to defining sinusoidal time dependence as $\mathrm{Re}(e^{-i\omega t})$ versus $\mathrm{Re}(e^{+i\omega t})$. See Mathematical descriptions of opacity.

Dielectric loss and non-zero DC conductivity in materials cause absorption. Good dielectric materials such as glass have extremely low DC conductivity, and at low frequencies the dielectric loss is also negligible, resulting in almost no absorption ($\kappa \approx 0$). However, at higher frequencies (such as visible light), dielectric loss may increase absorption significantly, reducing the material's transparency to these frequencies.

The real and imaginary parts of the complex refractive index are related through the Kramers–Kronig relations. For example, one can determine a material's full complex refractive index as a function of wavelength from an absorption spectrum of the material.

For X-ray and extreme ultraviolet radiation the complex refractive index deviates only slightly from unity and usually has a real part smaller than 1. It is therefore normally written as $\tilde{n} = 1 - \delta + i\beta$ (or $\tilde{n} = 1 - \delta - i\beta$).[14]

Exercise: verify that



$$Z(\omega) = \frac{V(\omega)}{I(\omega)}$$

$$Z(\omega) = \frac{R/i\omega C}{R + 1/i\omega C} = \frac{R}{1 + i\omega\tau} \qquad \tau = RC$$

Fulfils Kramers-Kronig

What should you know on systems

- Impulse response
- Step response
- Frequency response and transfer function
- Role of free evolution
- Linear damped oscillator, simple pole (low pass) and archetypal transfer functions
- Feedback, stability and linearization
- Dispersion relations