Experimental Methods
Lecture 20

November 4th, 2020

Extracting a signal from noise; Wiener's optimal filter

- The data: $$x(t)=s(t)+n(t)$$
- Data are known between T_1 and T_2
- The signal: a function of known shape and unknown amplitude A
$$s(t)=Af(t)$$
- Is there a linear combination of data $\hat{A}$ that gives an estimate for A
$$\hat{A}=\int_{T_1}^{T_2} h(t)x(t)dt$$
- Such that:
  1. $\hat{A}$ is not affected by systematic errors (unbiased), i.e. the mean value of $\hat{A}$ is equal to the true value to be estimated:
  $$\langle \hat{A} \rangle = A$$
  2. Among all possible linear estimators that fulfill 1. $\hat{A}$ has the minimum possible value for the variance ?
  $$\sigma_{\hat{A}}^2=\left\langle \left(\hat{A}-A\right)^2 \right\rangle$$

Extracting a signal from noise; Wiener's optimal filter

- In summary
- The best linear estimator of A $\hat{A} = \int_{T_1}^{T_2} h(t)x(t)dt$
- is that for which the function h(t) fulfills the integral equation

$$\int_{T_1}^{T_2} h(t')R_{nn}(t-t')dt' = -(\lambda/2)f(t) \quad T_1 \le t \le T_2$$

- $\hat{A}$ is unbiased $\langle \hat{A} \rangle = A$
- And has the minimum variance $\sigma^2_{\hat{A}} = -(\lambda/2)$
- The solution can be found explicitly in some remarkable cases (see next pages). In other cases it must be found numerically.

1st Case: $T_1 = -\infty \quad T_2 = +\infty$

- Estimator: $\hat{A} = \int_{-\infty}^{\infty} h(t)\,x(t)\,dt$
- Condition $\int_{-\infty}^{\infty} h(t)\,f(t)\,dt = 1$
- Variance $\sigma_{\hat{A}}^2 = -(\lambda/2)$
- Integral equation to be solved

$$\int_{-\infty}^{\infty} h(t')\,R_{nn}(t-t')\,dt' = -(\lambda/2)\,f(t) \qquad -\infty \le t \le \infty$$

- As the interval is now the entire t axis, we can switch to Fourier Transforms, and equality still holds $h(\omega)S_{nn}(\omega) = -(\lambda/2)f(\omega)$
- The solution being $h(\omega) = -(\lambda/2)f(\omega)/S_{nn}(\omega)$
- Let's now find the value of $-(\lambda/2)$. Use Parseval relations:

$$1 = \int_{-\infty}^{\infty} h(t)\,f(t)\,dt = (1/2\pi)\int_{-\infty}^{\infty} h(\omega)\,f^*(\omega)\,d\omega$$

- Substituting $-(\lambda/2)(1/2\pi)\int_{-\infty}^{\infty} f^*(\omega)f(\omega)/S_{nn}(\omega)\,d\omega = 1$
- That is

$$-(\lambda/2) = \sigma_{\hat{A}}^2 = \left[(1/2\pi)\int_{-\infty}^{\infty}\left(\left|f(\omega)\right|^2 \big/ S_{nn}(\omega)\right)d\omega\right]^{-1}$$

1st Case: $T_1 = -\infty \;\; T_2 = +\infty$

- In conclusion the function h $\quad h(\omega) = -(\lambda/2) f(\omega) / S_{nn}(\omega)$
- the multiplier (and the variance)

$$-(\lambda/2) = \sigma^2_{\hat{A}} = \left[(1/2\pi) \int_{-\infty}^{\infty} \left(\left| f(\omega) \right|^2 / S_{nn}(\omega) \right) d\omega \right]^{-1}$$

- Putting both together

$$h(\omega) = \sigma^2_{\hat{A}} \frac{f(\omega)}{S_{nn}(\omega)}$$

- With

$$\sigma^2_{\hat{A}} = \frac{1}{(1/2\pi) \int_{-\infty}^{\infty} \left(\left| f(\omega) \right|^2 / S_{nn}(\omega) \right) d\omega}$$

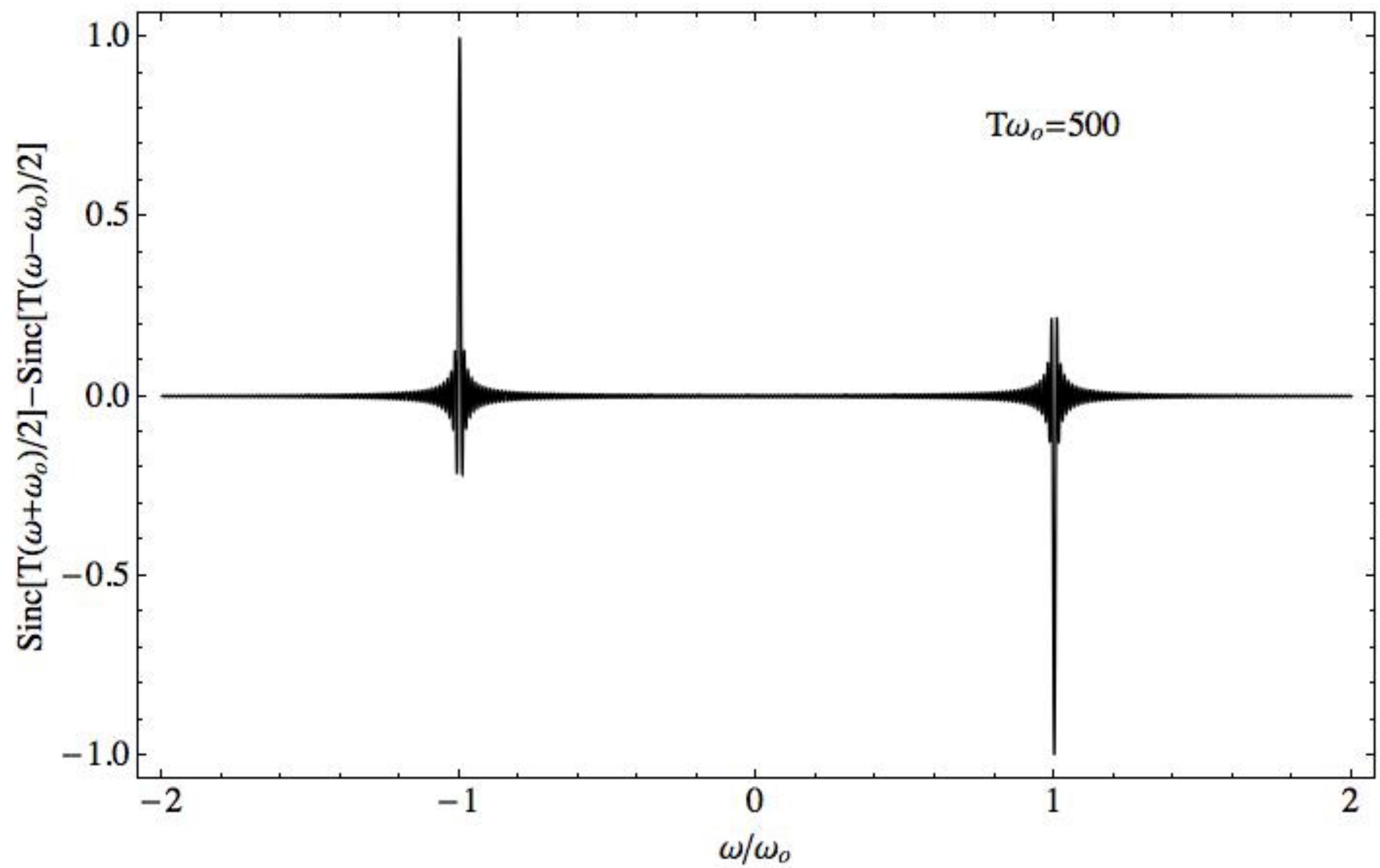
- Notice:
  1. The variance is inversely proportional to the integral of the *Signal to Noise Ratio* (SNR):

$$\left| f(\omega) \right|^2 / S_{nn}(\omega)$$

  1. The "template" h(t) is the inverse Fourier transform of the signal Fourier transform, "weighted", at each frequency, with the inverse of the noise PSD.

1^{st} Case: $T_1 = -\infty \quad T_2 = +\infty$

- Let's apply these results to the important case of a sinusoidal signal at frequency ω_o and finite duration T $f(t) = \Pi(t/T)\,\mathrm{Sin}(\omega_o t)$
- Its Fourier transform f(ω) , for T >>$\omega_{o,}$ consists of two narrow lines at $\pm\omega_o$ $f(\omega) = (iT/2)\left\{\mathrm{Sinc}\left[T(\omega+\omega_o)/2\right] - \mathrm{Sinc}\left[T(\omega-\omega_o)/2\right]\right\}$



1^{st} Case: $T_1 = -\infty \quad T_2 = +\infty$

- As $f(\omega) = (iT/2)\left\{\mathrm{Sinc}\left[T(\omega+\omega_o)/2\right] - \mathrm{Sinc}\left[T(\omega-\omega_o)/2\right]\right\}$ consists of two narrow lines at $\pm\omega_o$ we can write

$$h(\omega) = \sigma^2_{\hat{A}}\, f(\omega)/S_{nn}(\omega) \approx \sigma^2_{\hat{A}}\, f(\omega)/S_{nn}(\omega_o)$$

- And

$$\sigma^2_{\hat{A}} = \left[(1/2\pi)\int_{-\infty}^{\infty}\left(\left|f(\omega)\right|^2 / S_{nn}(\omega)\right)d\omega\right]^{-1} \approx \left[(1/2\pi)\left(1/S_{nn}(\omega_o)\right)\int_{-\infty}^{\infty}\left|f(\omega)\right|^2 d\omega\right]^{-1}$$

- Thus

$$h(\omega) \approx f(\omega)\Big/\left[(1/2\pi)\int_{-\infty}^{\infty}\left|f(\omega)\right|^2 d\omega\right] \quad \sigma^2_{\hat{A}} \approx S_{nn}(\omega_o)\Big/\left[(1/2\pi)\int_{-\infty}^{\infty}\left|f(\omega)\right|^2 d\omega\right]$$

- Considering the definition of signal energy E, and its invariance under Fourier transformation, and by going back to the time domain for h, we get $h(t) \approx f(t)/E \quad \sigma^2_{\hat{A}} \approx S_{nn}(\omega_o)/E$
- The energy can be rapidly calculated to be

$$E = \int_{-T/2}^{T/2}\mathrm{Sin}^2(\omega_o t)\,dt = (T/2)\left(1-\mathrm{Sinc}(\omega_o T)\right) \approx T/2$$

1^{st} Case: $T_1 = -\infty \;\; T_2 = +\infty$

- Thus for a sinusoidal signal of duration T $f(t) = \Pi(t/T)\mathrm{Sin}(\omega_o t)$
- Wiener theory prescribes to estimate the amplitude as

$$\hat{A} = (2/T)\int_{-T/2}^{T/2} \mathrm{Sin}(\omega_o t)\, x(t)\, dt$$

- And predicts that the variance of such an estimate is

$$\sigma_{\hat{A}}^2 \approx 2S_{nn}(\omega_o)/T$$

- That is that the error on the estimate is

$$\sigma_{\hat{A}} \approx \sqrt{2S_{nn}(\omega_o)/T}$$

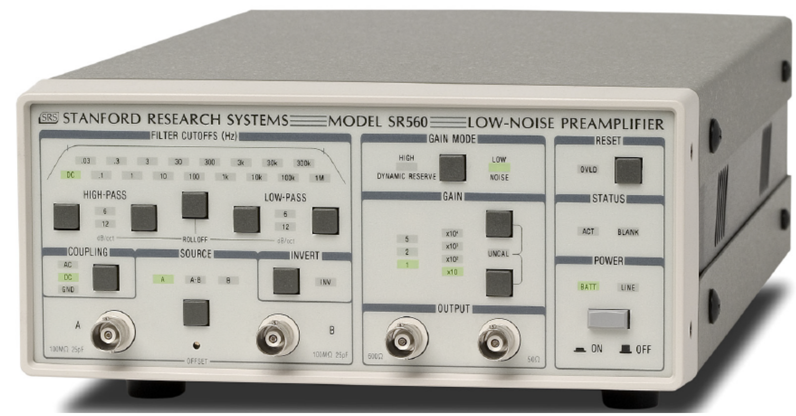
- This last formula is the main message: *the minimum error on the estimate of the amplitude of a sinusoidal signal is the square root of the noise (one-sided) PSD at the frequency of the signal, divided by the square root of the measurement time!*

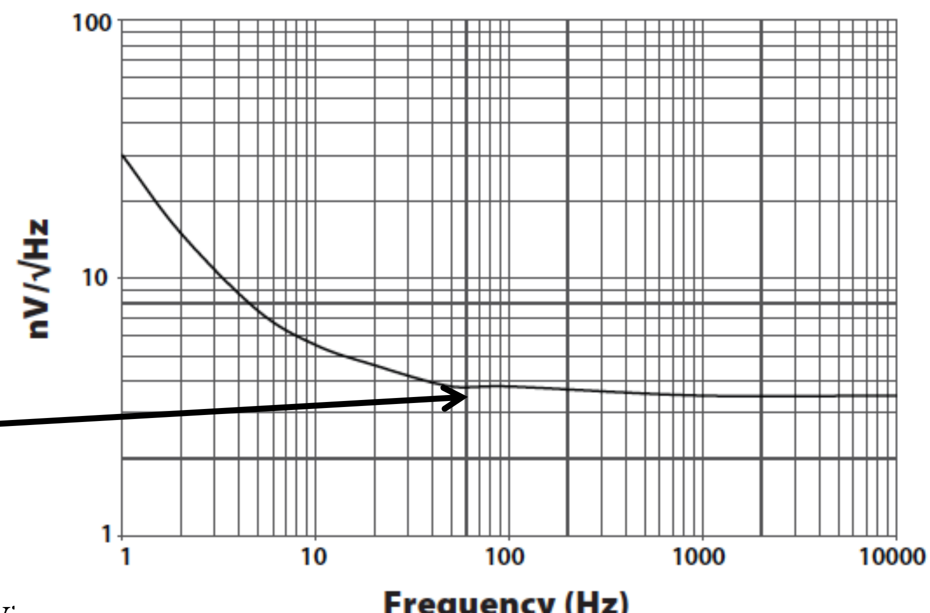
Example a low noise amplifier

Low-Noise Voltage Preamplifier

SR560 — DC to 1 MHz voltage preamplifier

- Amplifier raises the level of signals.
- You can then acquire the signal digitally with negligible error
- What is the minimum measurable amplitude of a sinusoidal signal at, say 100 Hz, in 10 min measurement time if the preamplifier input noise (see picture) is the only source of uncertainty? (this is an oversimplified description of the problem!)





$$\sigma = \frac{4\times10^{-9}\,\mathrm{V}\sqrt{\mathrm{s}}}{\sqrt{600\ \mathrm{s}}} = 1.6\times10^{-10}\,V$$

2^{nd} Case: $T_1 = 0$ $T_2 = T$

- Estimator: $\hat{A} = \int_0^T h(t)x(t)dt$
- Condition $\int_0^T h(t)f(t)dt = 1$
- Variance $\sigma_{\hat{A}}^2 = -(\lambda/2)$
- Integral equation to be solved

$$\int_0^T h(t')R_{nn}(t-t')dt' = -(\lambda/2)f(t) \quad 0 \le t \le T$$

- Can be solved numerically. However there is an analytical solution if noise is white: $R_{nn}(\tau)=S_o\delta(\tau)$. Then

$$S_o\int_0^T h(t')\delta(t-t')dt' = -(\lambda/2)f(t) \quad 0 \le t \le T$$

- That is $h(t) = -(\lambda/2)f(t)/S_o \quad 0 \le t \le T$
- Applying the condition $-(\lambda/2)(1/S_o)\int_0^T f^2(t)dt = 1$
- We get $-(\lambda/2) = \sigma_{\hat{A}}^2 = S_o \Big/ \int_0^T f^2(t)dt$
- and $h(t) = f(t) \Big/ \int_0^T f^2(t)dt$

2^{nd} Case: $T_1 = 0 \;\; T_2 = T$

- Thus the best choice for h in $\hat{A} = \int_0^T h(t)\, x(t)\, dt$
- is $h(t) = f(t) \Big/ \int_0^T f^2(t)\, dt$
- And the minimum error is $\sigma_{\hat{A}} = \sqrt{S_o} \Big/ \sqrt{\int_0^T f^2(t)\, dt} = \sqrt{S_o / E}$
- Thus the best choice is that of multiplying the data *with a template of the expected signal (normalized to have energy = 1/E)*
- Compare with the results of linear least square fitting where one wants to fit the function f to data x $x_k = Af(t_k)$
- Theory prescribes to minimize $\chi^2 = \sum_{k=1}^{N} \left[x_k - Af(t_k) \right]^2$
relative to A.
- This gives $\sum_{k=1}^{N} \left[x_k - Af(t_k) \right] f(t_k) \rightarrow A = \sum_{k=1}^{N} x_k f(t_k) \Big/ \sum_{k=1}^{N} f^2(t_k)$
- Optimal filter theory is equivalent to least square fitting in the case of white noise!
- In the general case is equivalent to maximum likelihood fitting.

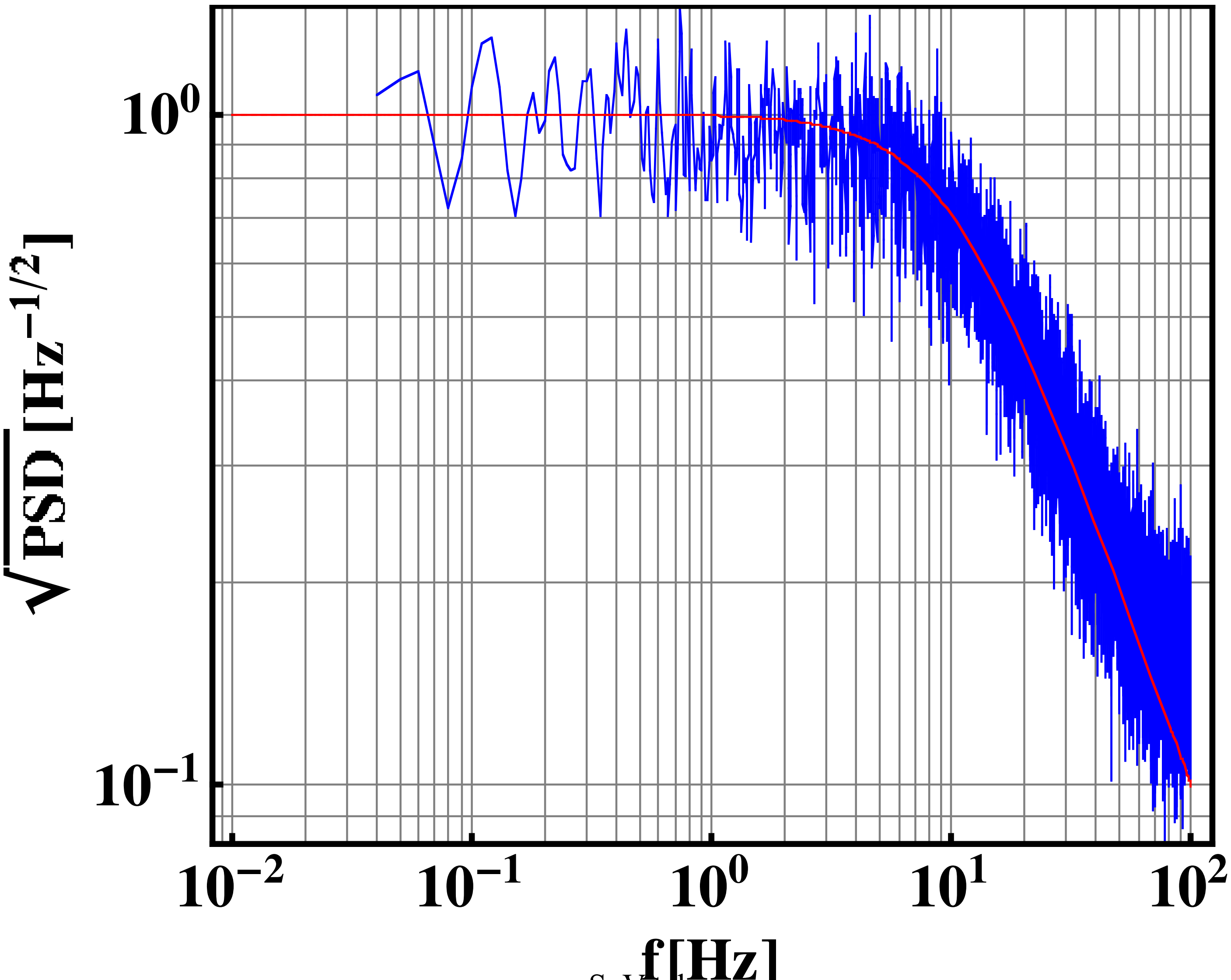
3^{nd} Case: $T_1 = 0$ $T_2 = T$ noise of arbitrary PSD

- Estimator: $$\hat{A} = \int_0^T h(t)x(t)dt$$
- Condition $$\int_0^T h(t)f(t)dt = 1$$
- Variance $$\sigma_{\hat{A}}^2 = -(\lambda/2)$$
- Integral equation to be solved

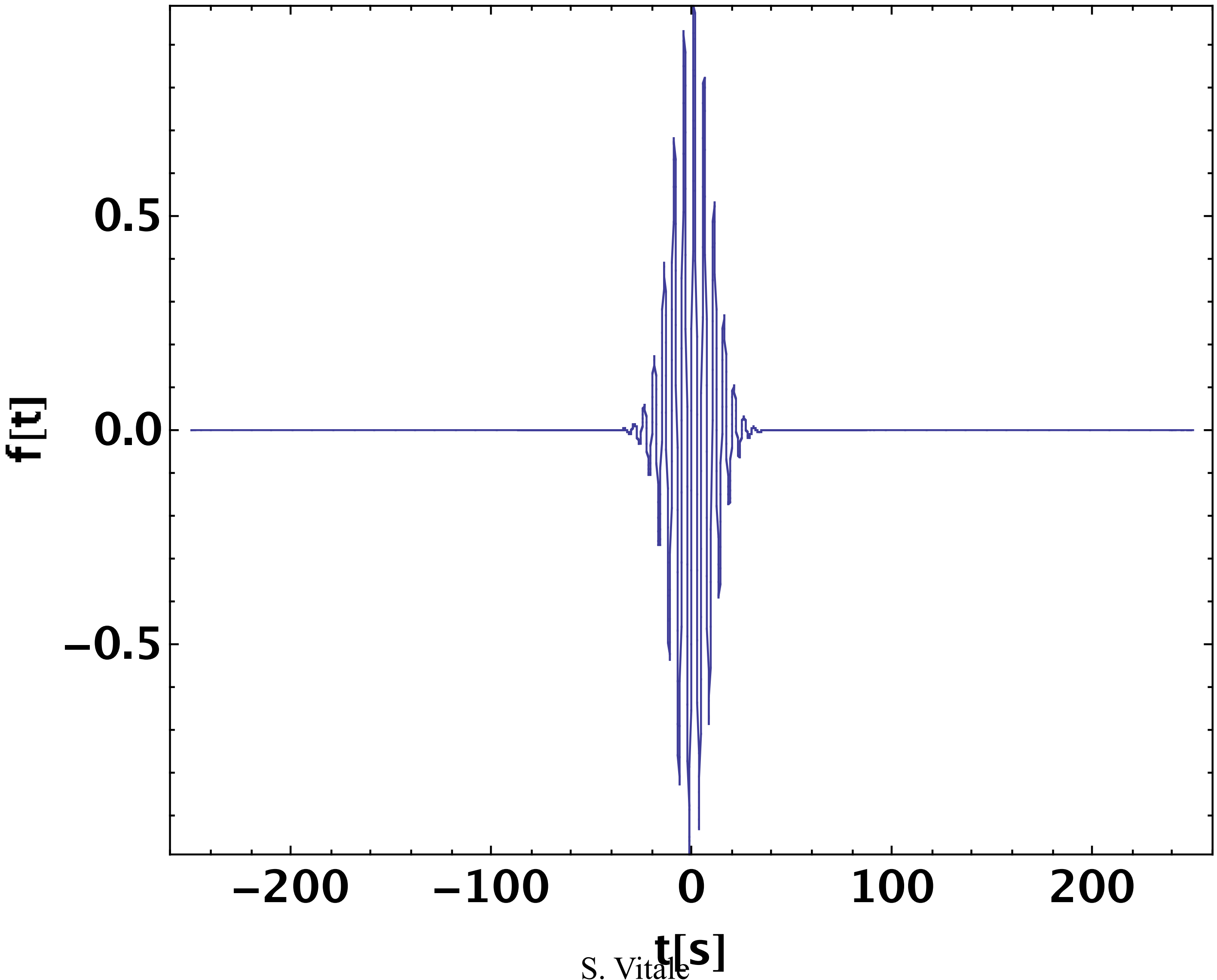
$$\int_0^T h(t')R_{nn}(t-t')dt' = -(\lambda/2)f(t) \quad 0 \le t \le T$$

- Must be solved numerically.
- If signal only different from zero for $-\frac{t_o}{2} \le t \le \frac{t_o}{2}$ and you have data for $-\frac{T}{2} \le t \le \frac{T}{2}$ with $T \gg t_o$, then you can pretend $T = \infty$ and use the theory for infinite time data

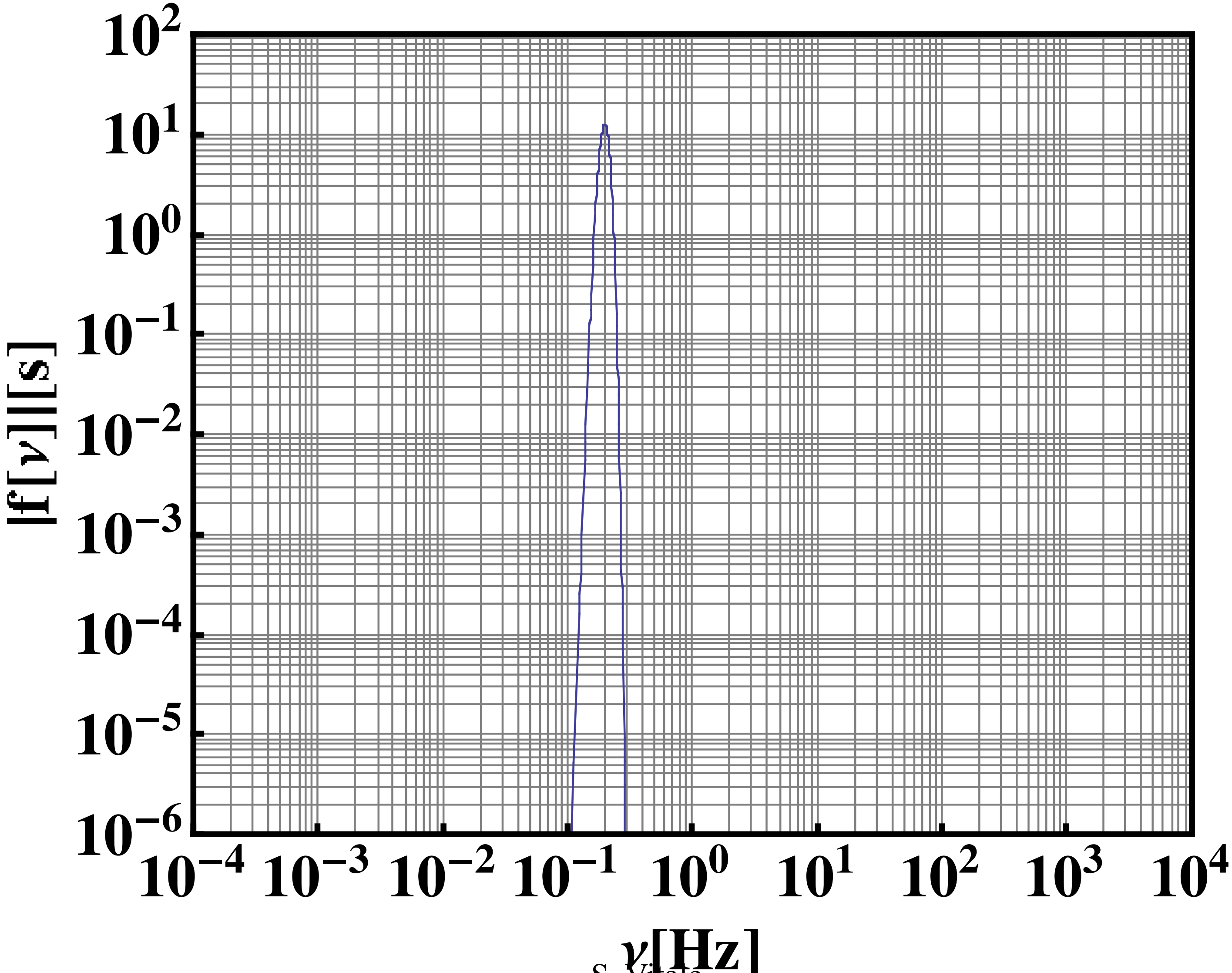
An example: signal in “low-pass” noise
noise



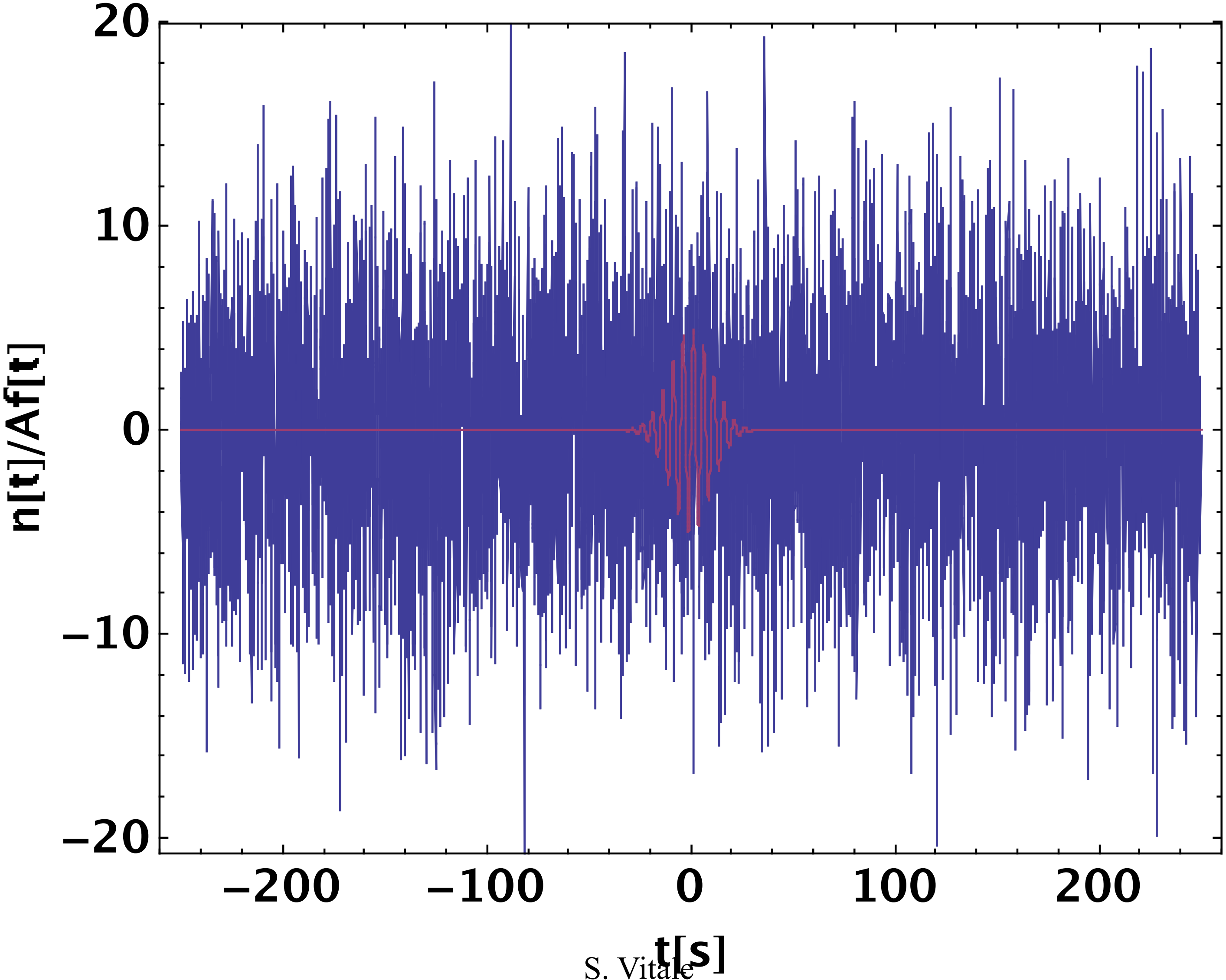
An example: signal in "low-pass" noise
signal



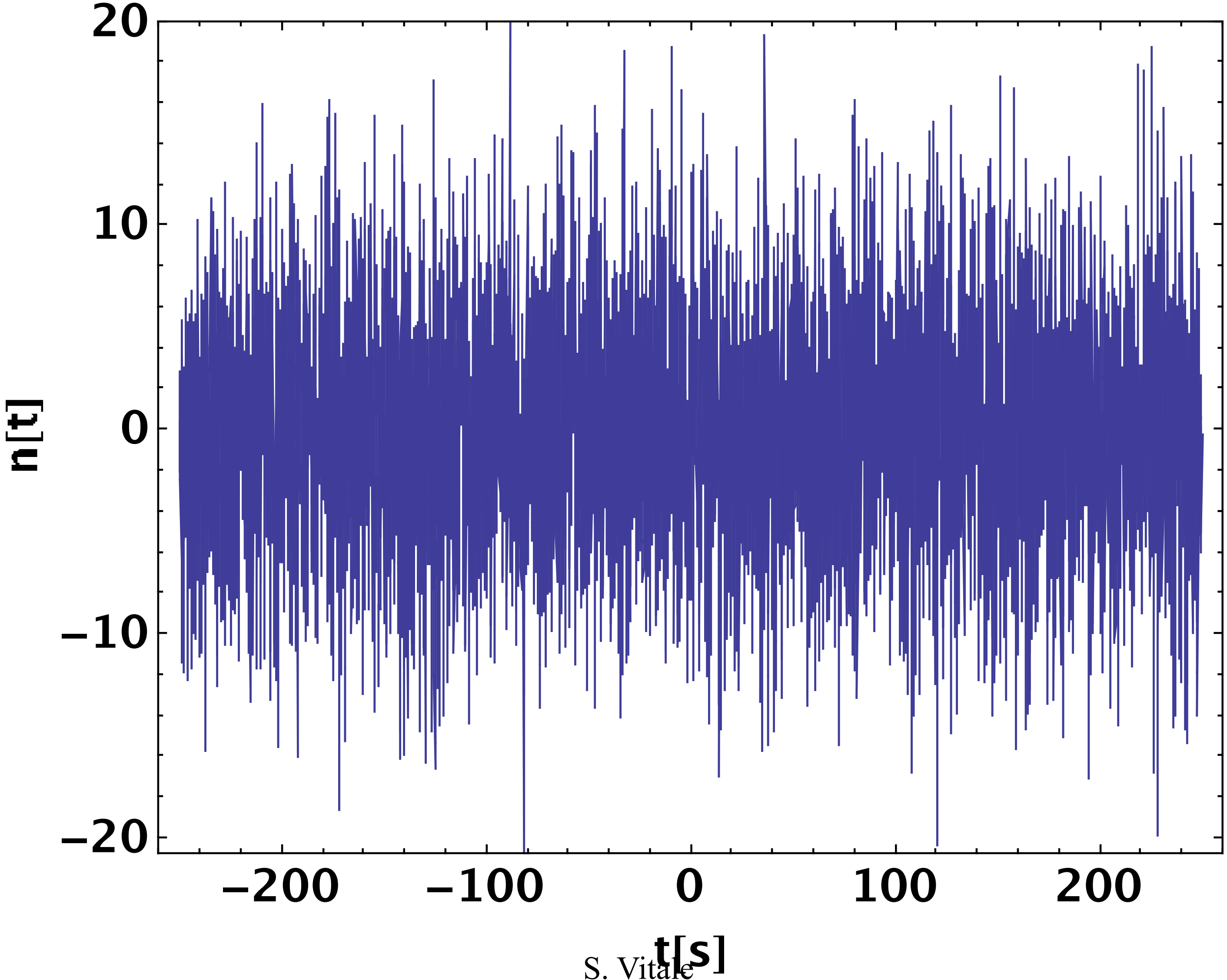
An example: signal in “low-pass” noise
Fourier transform of signal



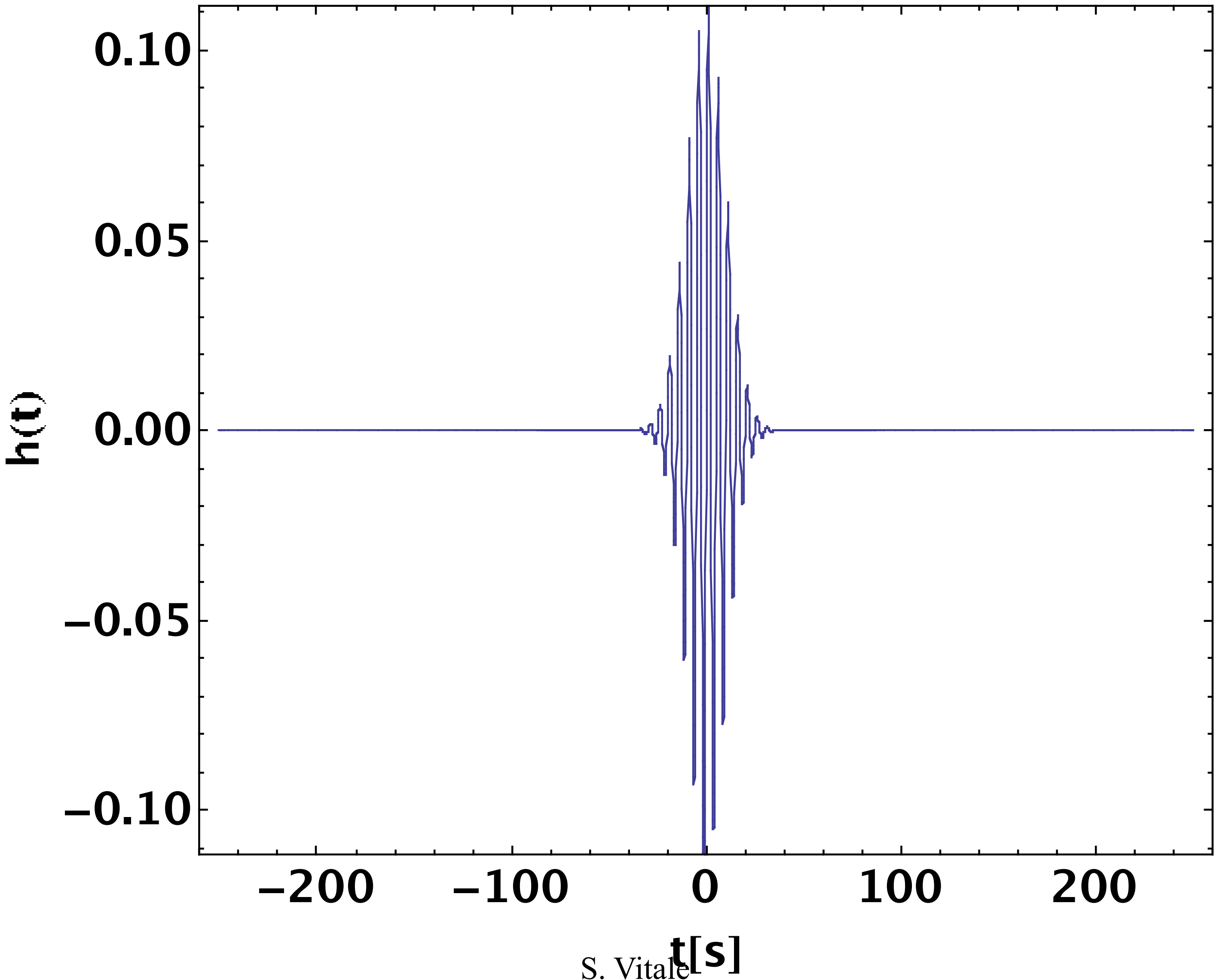
An example: signal in “low-pass” noise signal and noise



An example: signal in "low-pass" noise
signal *plus* noise



An example: signal in “low-pass” noise
the template



An example: signal in “low-pass” noise distribution of estimate for repeated experiments