Experimental Methods
Lecture 5

September 30th, 2020

Fourier Transforms of Discrete Data

- Two transforms:
  - Discrete-time Fourier Transform (infinite length data series)
  - Discrete Fourier Transform (finite length data series)
- Can they be used to estimate Fourier Transform of original continuous signals?

Discrete-time Fourier Transform

- The transform

$$s(\phi) = \sum_{k=-\infty}^{\infty} s_k e^{-i\,k\phi}$$

- The inversion formula

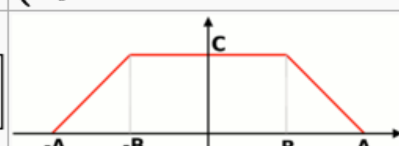
$$s_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} s(\phi) e^{ik\phi} d\phi$$

Sequence transform vs continuous transform

$$s_d(\phi) = \frac{1}{T}\sum_{n=-\infty}^{\infty} s_c\left(\frac{\phi}{T} + n\frac{2\pi}{T}\right) = \frac{1}{T}s'_c\left(\frac{\phi}{T}\right)$$

- $s_d(\phi)$: discrete time transform
- $s_c(\omega)$: continuous transform
- $s'_c(\omega)$:continuous transform of the Shannon interpolation of $s(t)$
- If T fulfils the sampling theorem, then

$$s_d(\phi) = \frac{1}{T}s_c\left(\frac{\phi}{T}\right) = \frac{1}{T}s'_c\left(\frac{\phi}{T}\right)$$

Time domain $x[n]$	**Frequency domain** $X(\omega)$	**Remarks**
$\delta[n]$	1	
$\delta[n-M]$	$e^{-i\omega M}$	integer *M*
$\sum_{m=-\infty}^{\infty} \delta[n-Mm]$	$\sum_{m=-\infty}^{\infty} e^{-i\omega Mm} = \frac{1}{M}\sum_{k=-\infty}^{\infty}\delta\left(\frac{\omega}{2\pi}-\frac{k}{M}\right)$	integer *M*
$u[n]$	$\frac{1}{1-e^{-i\omega}} + \sum_{k=-\infty}^{\infty}\pi\delta(\omega-2\pi k)$	The $1/(1-e^{-i\omega})$ term must be interpreted as a distribution in the sense of a Cauchy principal value around its poles at $\omega = 2\pi k$.
$a^n u[n]$	$\frac{1}{1-ae^{-i\omega}}$	$\|a\| < 1$
e^{-ian}	$2\pi\delta(\omega+a)$	real number *a*
$\cos(an)$	$\pi\left[\delta(\omega-a)+\delta(\omega+a)\right]$	real number *a*
$\sin(an)$	$\frac{\pi}{i}\left[\delta(\omega-a)-\delta(\omega+a)\right]$	real number *a*
$\mathrm{rect}\left[\frac{(n-M/2)}{M}\right]$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}\,e^{-i\omega M/2}$	integer *M*
$\mathrm{sinc}[(a+n)]$	$e^{ia\omega}$	real number *a*
$W\cdot\mathrm{sinc}^2(Wn)$	$\mathrm{tri}\left(\frac{\omega}{2\pi W}\right)$	real number *W* $0 < W \leq 0.5$
$W\cdot\mathrm{sinc}(Wn)$	$\mathrm{rect}\left(\frac{\omega}{2\pi W}\right)$	real numbers *W* $0 < W \leq 1$
$\begin{cases} 0 & n=0 \\ \frac{(-1)^n}{n} & \text{elsewhere}\end{cases}$	$j\omega$	it works as a differentiator filter
$\frac{W}{(n+a)}\left\{\cos[\pi W(n+a)]-\mathrm{sinc}[W(n+a)]\right\}$	$j\omega\cdot\mathrm{rect}\left(\frac{\omega}{\pi W}\right)e^{ja\omega}$	real numbers *W*, *a* $0 < W \leq 1$
$\frac{1}{\pi n^2}[(-1)^n-1]$	$\|\omega\|$	
$\begin{cases} 0; & n \text{ even} \\ \frac{2}{\pi n}; & n \text{ odd}\end{cases}$	$\begin{cases} j & \omega<0 \\ 0 & \omega=0 \\ -j & \omega>0\end{cases}$	Hilbert transform
$\frac{C(A+B)}{2\pi}\cdot\mathrm{sinc}\left[\frac{A-B}{2\pi}n\right]\cdot\mathrm{sinc}\left[\frac{A+B}{2\pi}n\right]$		real numbers *A*, *B* complex *C*

 $\omega \rightarrow \varphi$

Discrete-time transform

- They share almost literally the properties of continuous transform.
- In particular, the transform of the convolution of two sequences

$$x[n] = \sum_{k=-\infty}^{\infty} z[k] y[n-k]$$

- is

$$x[\phi] = z[\phi] y[\phi]$$

Property	Time domain $x[n]$	Frequency domain $X_{2\pi}(\omega)$	Remarks	Reference
Linearity	$a \cdot x[n] + b \cdot y[n]$	$a \cdot X_{2\pi}(\omega) + b \cdot Y_{2\pi}(\omega)$	complex numbers a, b	[16]:p.294
Time reversal / Frequency reversal	$x[-n]$	$X_{2\pi}(-\omega)$		[16]:p.297
Time conjugation	$x[n]^*$	$X_{2\pi}(-\omega)^*$		[16]:p.291
Time reversal & conjugation	$x[-n]^*$	$X_{2\pi}(\omega)^*$		[16]:p.291
Real part in time	$\Re(x[n])$	$\frac{1}{2}(X_{2\pi}(\omega) + X^*_{2\pi}(-\omega))$		[16]:p.291
Imaginary part in time	$\Im(x[n])$	$\frac{1}{2i}(X_{2\pi}(\omega) - X^*_{2\pi}(-\omega))$		[16]:p.291
Real part in frequency	$\frac{1}{2}(x[n] + x^*[-n])$	$\Re(X_{2\pi}(\omega))$		[16]:p.291
Imaginary part in frequency	$\frac{1}{2i}(x[n] - x^*[-n])$	$\Im(X_{2\pi}(\omega))$		[16]:p.291
Shift in time / Modulation in frequency	$x[n-k]$	$X_{2\pi}(\omega) \cdot e^{-i\omega k}$	integer k	[16]:p.296
Shift in frequency / Modulation in time	$x[n] \cdot e^{ian}$	$X_{2\pi}(\omega - a)$	real number a	[16]:p.300
Decimation	$x[nM]$	$\frac{1}{M}\sum_{m=0}^{M-1} X_{2\pi}\left(\frac{\omega - 2\pi m}{M}\right)$ [F]	integer M	
Time Expansion	$\begin{cases} x[n/M] & n\text{=multiple of M} \\ 0 & \text{otherwise} \end{cases}$	$X_{2\pi}(M\omega)$	integer M	[1]:p.172
Derivative in frequency	$\frac{n}{i}x[n]$	$\frac{dX_{2\pi}(\omega)}{d\omega}$		[16]:p.303
Integration in frequency				
Differencing in time	$x[n] - x[n-1]$	$\left(1 - e^{-i\omega}\right) X_{2\pi}(\omega)$		
Summation in time	$\sum_{m=-\infty}^{n} x[m]$	$\frac{1}{(1 - e^{-i\omega})} X_{2\pi}(\omega) + \pi X(0) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$		
Convolution in time / Multiplication in frequency	$x[n] * y[n]$	$X_{2\pi}(\omega) \cdot Y_{2\pi}(\omega)$		[16]:p.297
Multiplication in time / Convolution in frequency	$x[n] \cdot y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_{2\pi}(\nu) \cdot Y_{2\pi}(\omega - \nu) d\nu$	Periodic convolution	[16]:p.302
Cross correlation	$\rho_{xy}[n] = x[-n]^* * y[n]$	$R_{xy}(\omega) = X_{2\pi}(\omega)^* \cdot Y_{2\pi}(\omega)$		
Parseval's theorem	$E_{xy} = \sum_{n=-\infty}^{\infty} x[n] \cdot y[n]^*$	$E_{xy} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{2\pi}(\omega) \cdot Y_{2\pi}(\omega)^* d\omega$		[16]:p.302

Discrete Fourier Transform

- Assume you have a finite-length data sequence s_n with $0 \leq n \leq N\text{-}1$.
- We can define the following function of the integer k

$$\hat{s}_k = \sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}kn}$$

- The function is periodic with period N, thus only the values for $0 \leq k \leq N\text{-}1$ have an independent meaning

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Discrete Fourier Transform

- $\hat{s}_k$ is a transform. Indeed define

$$\breve{s}_m = \frac{1}{N}\sum_{k=0}^{N-1} \hat{s}_k e^{i\frac{2\pi}{N}km}$$

- Substituting $\hat{s}_k$ from its definition

$$\breve{s}_m = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}kn}\right) e^{i\frac{2\pi}{N}km} \qquad = \sum_{n=0}^{N-1} s_n \left(\frac{1}{N}\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}k(m-n)}\right)$$

- A well known rule

$$\sum_{k=0}^{N-1} e^{ika} = \frac{1-e^{iNa}}{1-e^{ia}}$$

Consider the sum of the first n terms of the geometric series

$$s_n = a + ax + ax^2 + ax^3 + \ldots + ax^{n-1} \tag{1}$$

Multiply both sides by x

$$xs_n = ax + ax^2 + ax^3 + ax^4 + \ldots + ax^n \tag{2}$$

Subtract Equation (2) from Equation (1)

$$\begin{aligned}(1-x)s_n &= a - ax^n \\ &= a(1-x^n)\end{aligned}$$

If $x \neq 1$ divide both sides by $1 - x$ to get

$$s_n = \frac{a(1-x^n)}{1-x} \qquad (x \neq 1) \tag{3}$$

Discrete Fourier Transform

- $\hat{s}_k$ is a transform. Indeed define

$$\breve{s}_m = \frac{1}{N}\sum_{k=0}^{N-1} \hat{s}_k e^{i\frac{2\pi}{N}km}$$

- Substituting $\hat{s}_k$ from its definition

$$\breve{s}_m = \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}kn}\right)e^{i\frac{2\pi}{N}km} \qquad = \sum_{n=0}^{N-1} s_n \left(\frac{1}{N}\sum_{k=0}^{N-1} e^{i\frac{2\pi}{N}k(m-n)}\right)$$

- A well known rule

$$\sum_{k=0}^{N-1} e^{ika} = \frac{1-e^{iNa}}{1-e^{ia}}$$

- Then

$$\breve{s}_m = \frac{1}{N}\sum_{n=0}^{N-1} s_n \frac{1-e^{iN\frac{2\pi}{N}(m-n)}}{1-e^{i\frac{2\pi}{N}(m-n)}} = \frac{1}{N}\sum_{n=0}^{N-1} s_n \frac{1-e^{i2\pi(m-n)}}{1-e^{i\frac{2\pi}{N}(m-n)}}$$

Discrete Fourier Transform

- A special formula

$$\frac{1}{N}\frac{1-e^{i2\pi(m-n)}}{1-e^{i\frac{2\pi}{N}(m-n)}}$$

- If m≠n and m,n <N

$$\frac{1}{N}\frac{1-e^{i2\pi(m-n)}}{1-e^{i\frac{2\pi}{N}(m-n)}}=0$$

- Limit for m→n

$$m\rightarrow n:\frac{1}{N}\frac{1-e^{i2\pi(m-n)}}{1-e^{i\frac{2\pi}{N}(m-n)}}\approx\frac{1}{N}\frac{2\pi(m-n)}{\frac{2\pi}{N}(m-n)}\rightarrow 1$$

- Then

$$\frac{1}{N}\frac{1-e^{i2\pi(m-n)}}{1-e^{i\frac{2\pi}{N}(m-n)}}=\delta_{m,n}$$

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Discrete Fourier Transform

- Back to the calculation

$$\breve{s}_m = \frac{1}{N}\sum_{n=0}^{N-1} s_n \frac{1-e^{iN\frac{2\pi}{N}(m-n)}}{1-e^{i\frac{2\pi}{N}(m-n)}} = \frac{1}{N}\sum_{n=0}^{N-1} s_n \frac{1-e^{i2\pi(m-n)}}{1-e^{i\frac{2\pi}{N}(m-n)}} = \sum_{n=0}^{N-1}\delta_{n,m} s_n = s_m$$

- In summary

$$\hat{s}_k = \sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}kn} \qquad s_n = \frac{1}{N}\sum_{k=0}^{N-1}\hat{s}_k e^{i\frac{2\pi}{N}kn}$$

Discrete Fourier Transform of Sampled Signals

- Take a signal of duration

$$\Delta T = (N-1)\,T \approx NT$$

- Sampled with sampling time T. The discrete-time Fourier transform of the signal padded with 0's is

$$s[\phi] = \sum_{n=0}^{N-1} s_n e^{-i\phi n}$$

- Compare with

$$\hat{s}_k = \sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}k n}$$

- It follows that the $\hat{s}_k$ are samples of $s[\phi]$

$$\hat{s}_k = s\left[\phi = \left(k\,2\pi/N\right)\right]$$

- If the signal has been correctly sampled and truncated

$$\hat{s}_k = s_d\left[\phi = k\frac{2\pi}{N}\right] = \frac{1}{T} s_c\left[\omega = k\frac{2\pi}{NT}\right]$$

Discrete Fourier Transform of Sampled Signals

- In conclusion

$$\hat{s}_k = \frac{1}{T} s_c\left[\omega = k\frac{2\pi}{NT}\right]$$

- Thus for a signal that has been correctly sampled and truncated the $\hat{s}_k$, from which the signal can be reconstructed, are proportional to the samples of the Fourier Transform of the original continuous signal at multiple integers of the fundamental frequency:

$$f_o = \frac{1}{NT} = \frac{1}{\Delta T}$$

- The resolution of the "spectrum" is 1/(signal duration)!
- This is also the minimum, non-zero frequency for which the "spectrum" is available

Discrete Fourier Transform

$$\hat{s}_k = \sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}kn}$$

- Some properties
  - A key property

$$\hat{s}_{N-k} = \sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}(N-k)n} = \sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}Nn} e^{+i\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} s_n e^{+i\frac{2\pi}{N}kn} = \hat{s}_k^*$$

  - Thus, of the N coefficients only $\simeq$N/2 have an independent value

$$\hat{s}_0;\ \hat{s}_1 = \hat{s}_{N-1}^*;\ \hat{s}_2 = \hat{s}_{N-2}^* \ldots\ldots$$

$$\text{Odd N} \rightarrow \hat{s}_{(N-1)/2} = \hat{s}_{(N+1)/2}^*$$

$$\text{Even N} \rightarrow \hat{s}_{N/2-1} = \hat{s}_{N/2+1}^*; \hat{s}_{N/2}$$

  - It's a consequence of conservation of information. N real data are transformed into N/2 complex data.

It's a remnant of the symmetry of the continuous transform

$$s_{N-k} = \sum_{n=0}^{N-1} s_n e^{-i\,(N-k)\frac{2\pi}{N}n}$$

- Use periodicity:

$$s_{N-k} = \sum_{n=0}^{N-1} s_n e^{-i\,(N-k-N)\frac{2\pi}{N}n} = s_{-k}$$

- Then

$$s_{-k} = s_k^*$$

Independent coefficients

- An explicit table

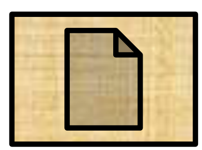
1	s[0]									
2	s[0]	s[1]								
3	s[0]	Re[s[1]]	Im[s[1]]							
4	s[0]	Re[s[1]]	Im[s[1]]	s[2]						
5	s[0]	Re[s[1]]	Im[s[1]]	Re[s[2]]	Im[s[2]]					
6	s[0]	Re[s[1]]	Im[s[1]]	Re[s[2]]	Im[s[2]]	s[3]				
7	s[0]	Re[s[1]]	Im[s[1]]	Re[s[2]]	Im[s[2]]	Re[s[3]]	Im[s[3]]			
8	s[0]	Re[s[1]]	Im[s[1]]	Re[s[2]]	Im[s[2]]	Re[s[3]]	Im[s[3]]	s[4]		
9	s[0]	Re[s[1]]	Im[s[1]]	Re[s[2]]	Im[s[2]]	Re[s[3]]	Im[s[3]]	Re[s[4]]	Im[s[4]]	
10	s[0]	Re[s[1]]	Im[s[1]]	Re[s[2]]	Im[s[2]]	Re[s[3]]	Im[s[3]]	Re[s[4]]	Im[s[4]]	s[5]

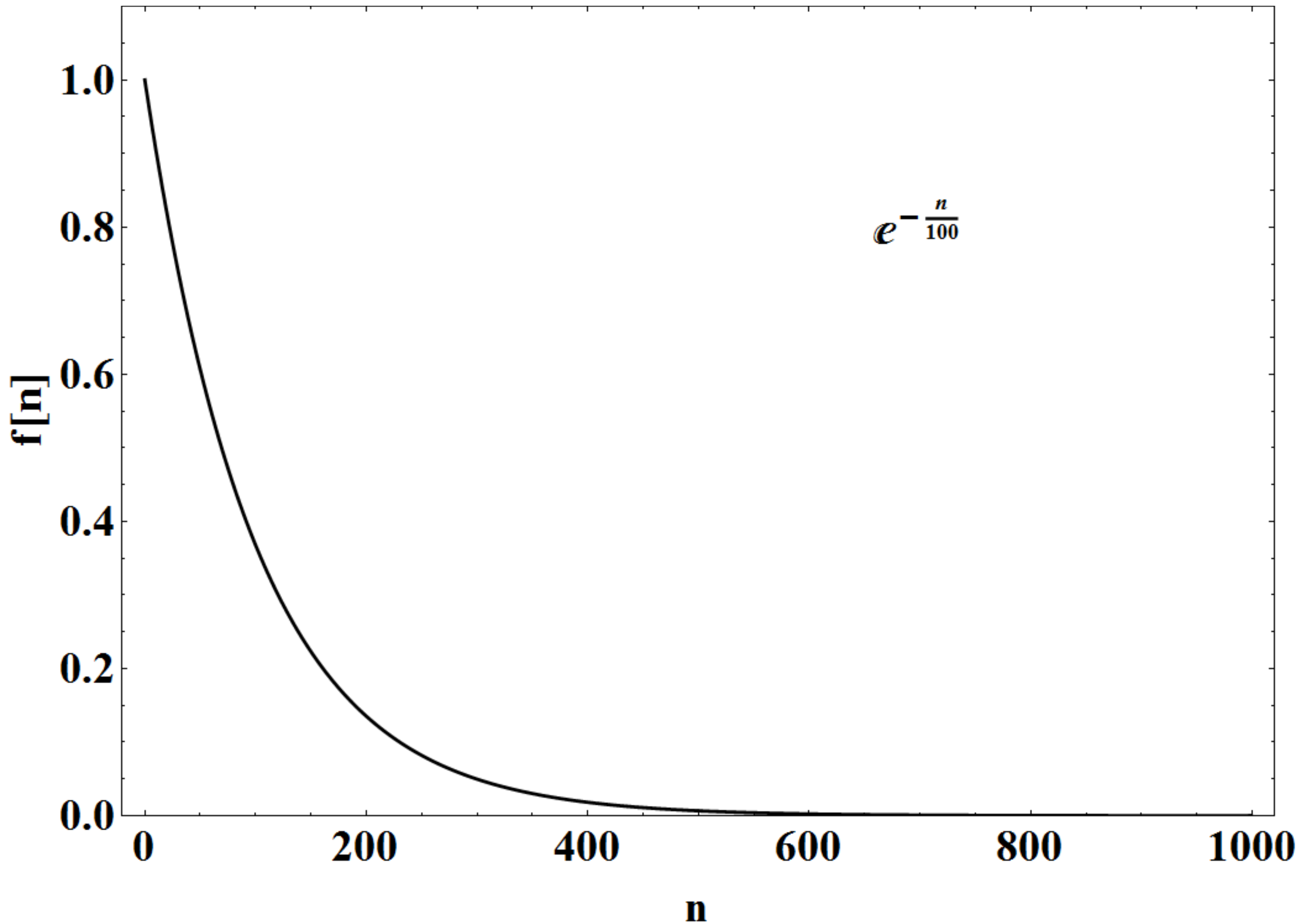
- Maximum frequency: Integer Part(N/2)

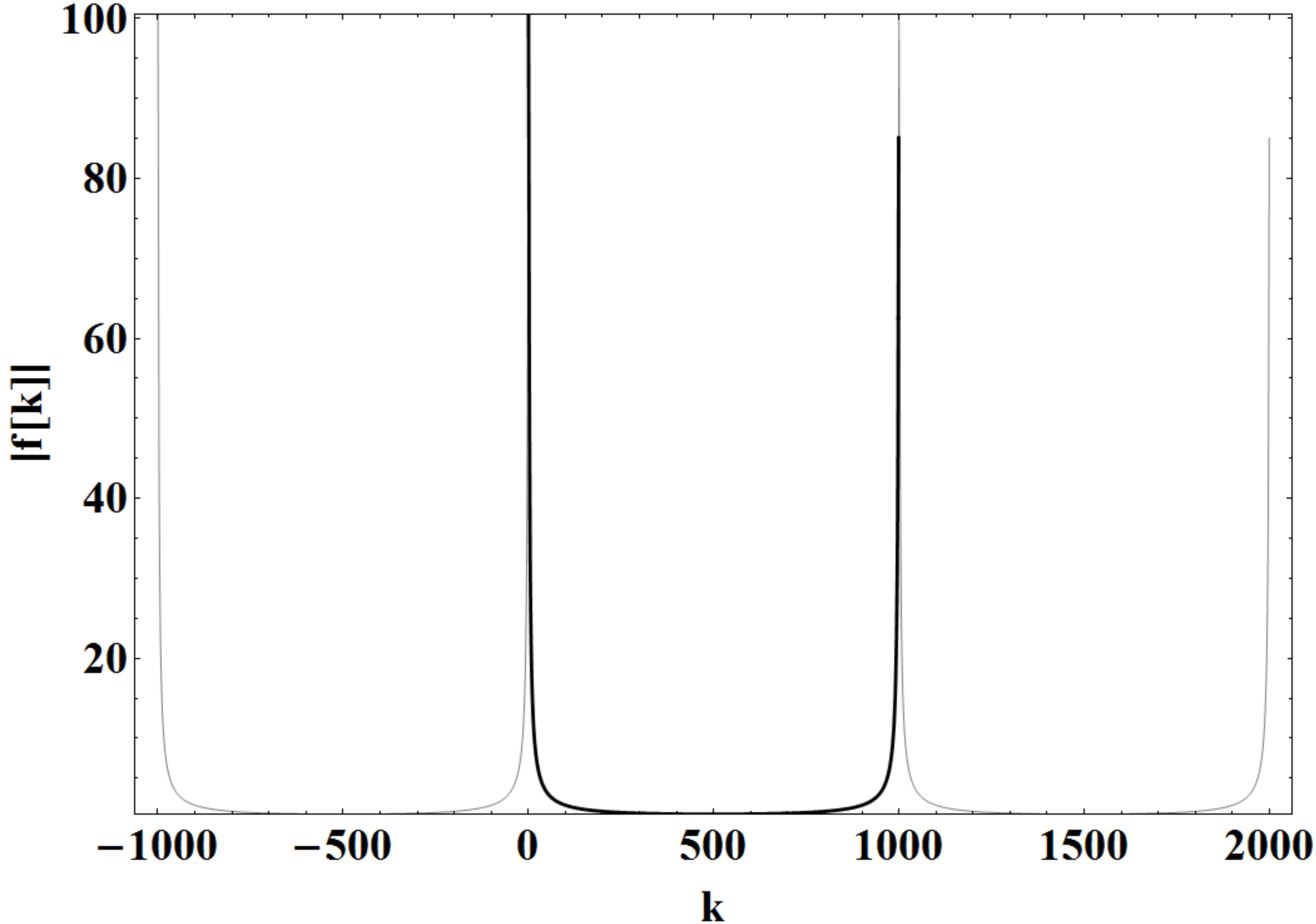
$$\frac{\mathrm{IntegerPart}\left[N/2\right]}{NT} \simeq \frac{1}{2T} = f_{\mathrm{Nyquist}}$$

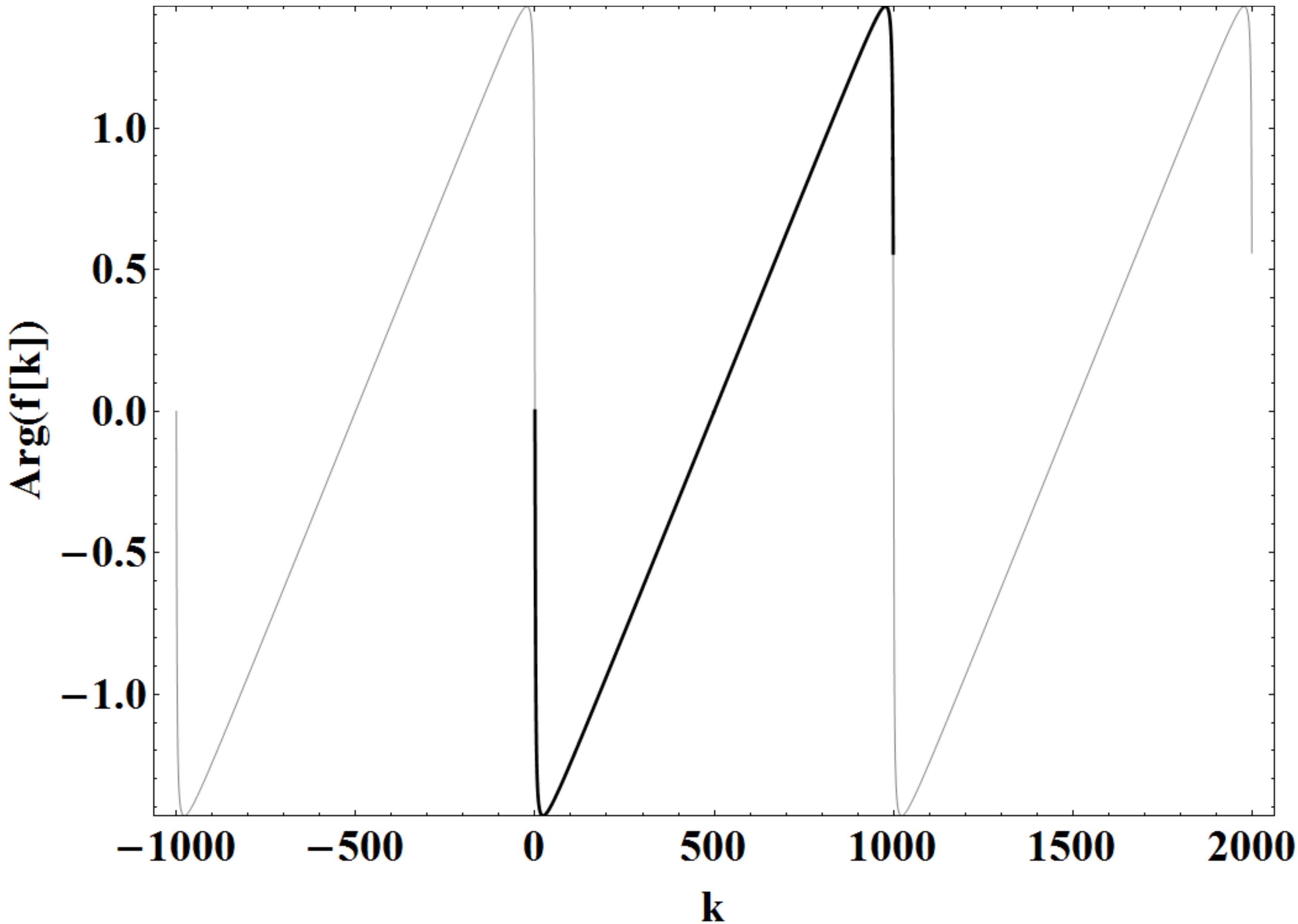
Numerical Examples

- Discrete Fourier Transforms can be calculated via a very memory-speed-efficient algorithm called the Fast Fourier Transform
- The algorithm is available in any computational environment: MatLab™, Mathematica™ , C++

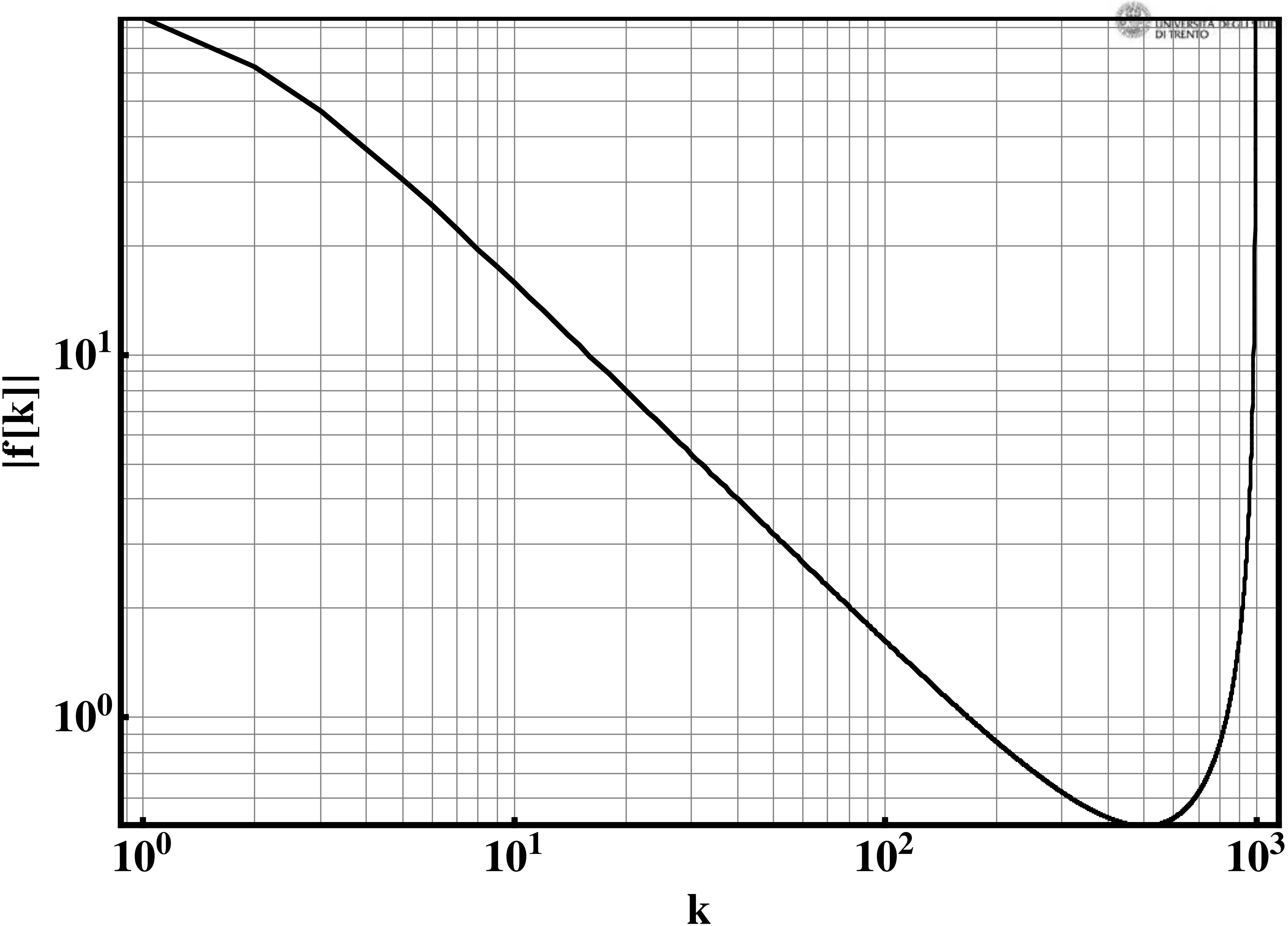



1.0
0.8
0.6
0.4
0.2
0.0
f[n]
$e^{-\frac{n}{100}}$
0
200
400
600
800
1000
n


100
80
60
40
20
|f[k]|
−1000
−500
0
500
1000
1500
2000
k



Arg(f[k])
1.0
0.5
0.0
−0.5
−1.0
−1000
−500
0
500
1000
1500
2000
k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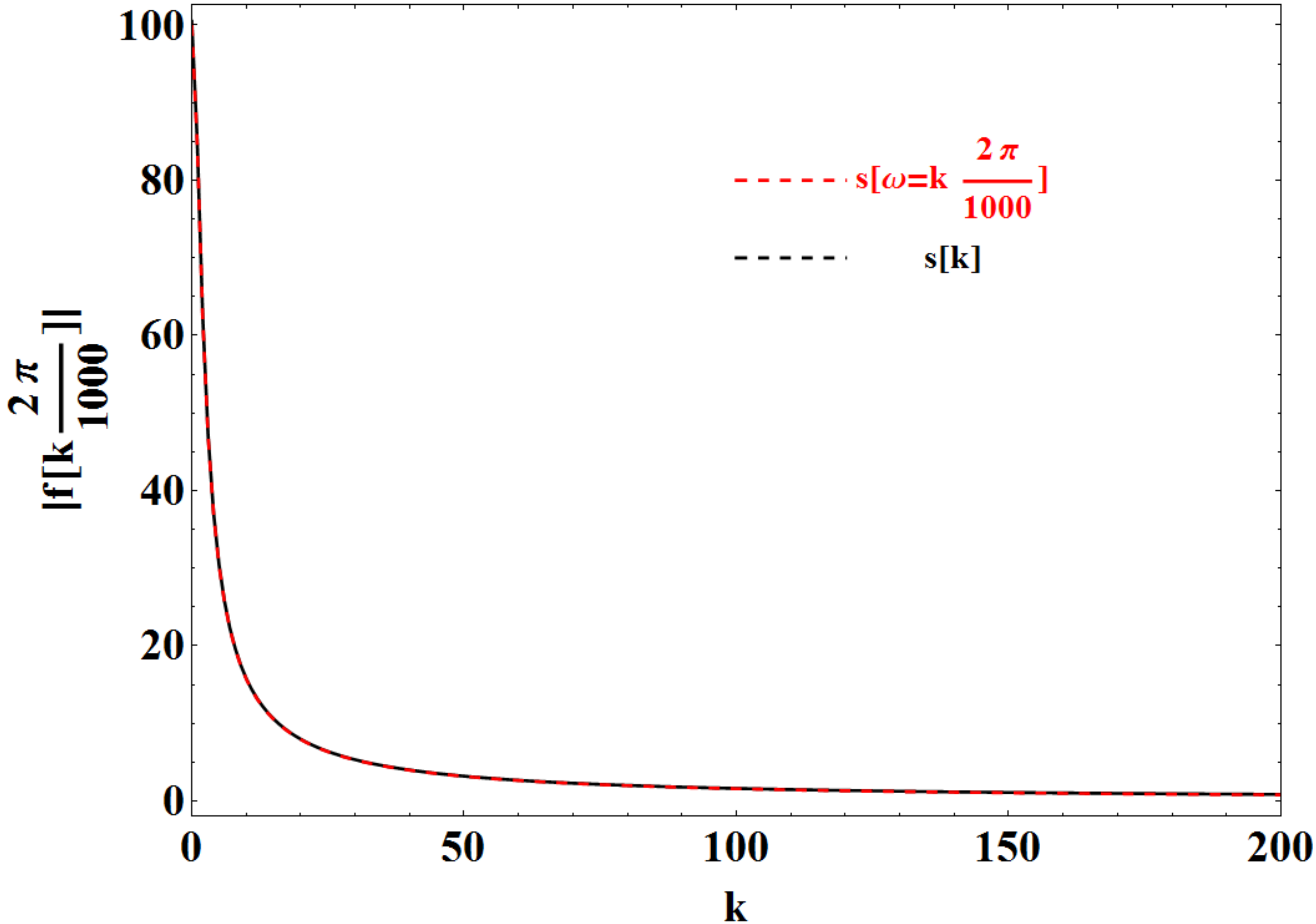
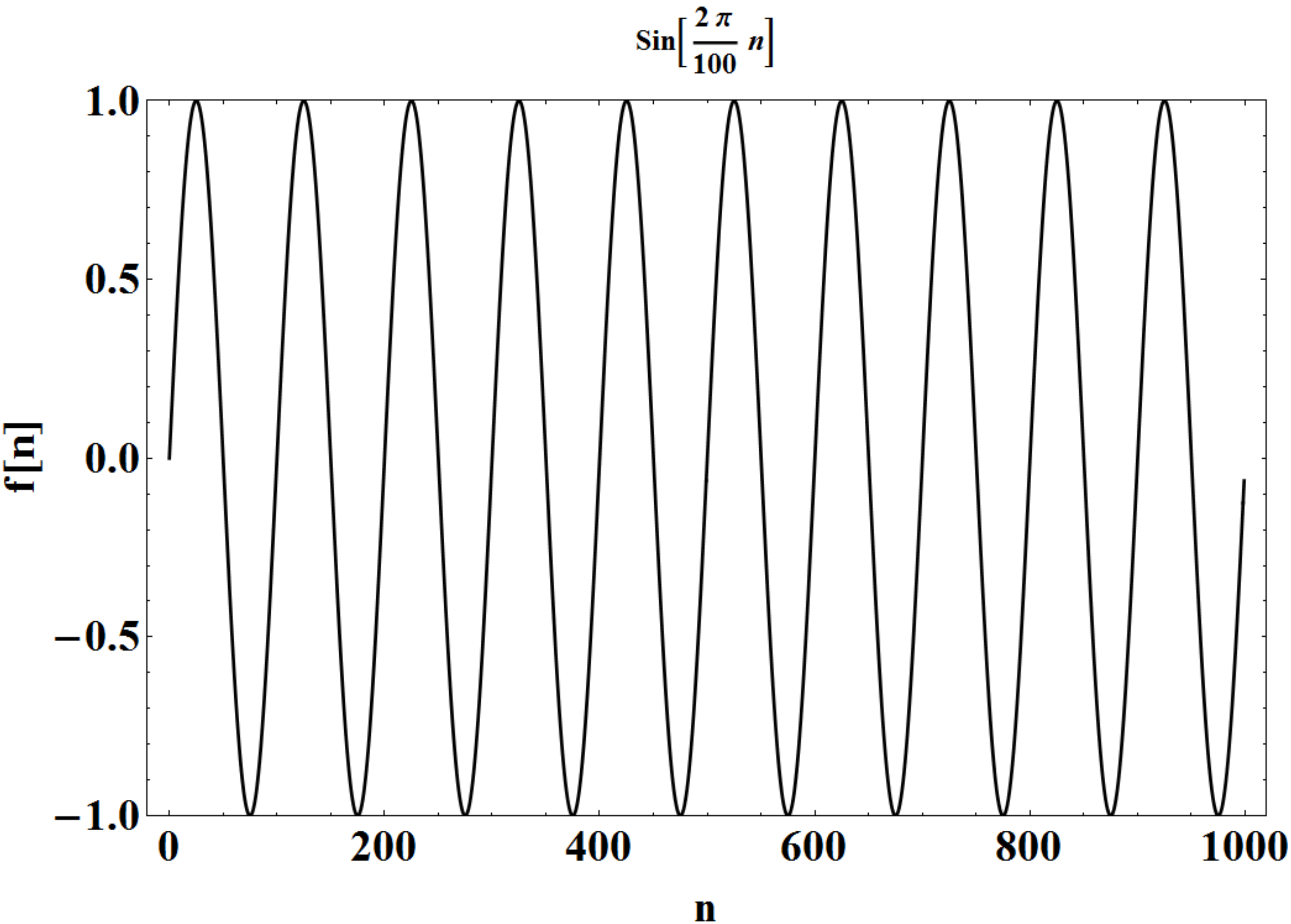
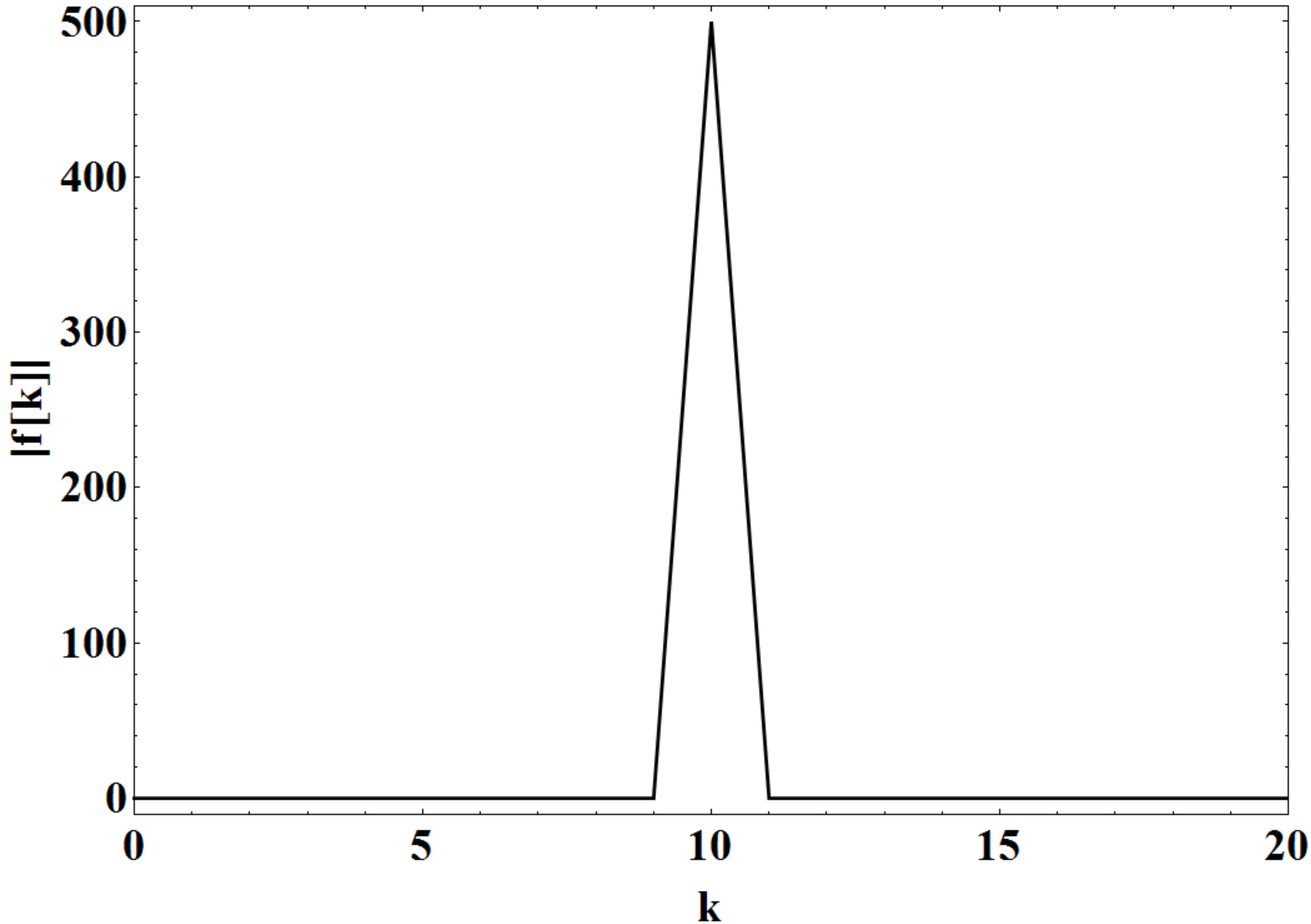
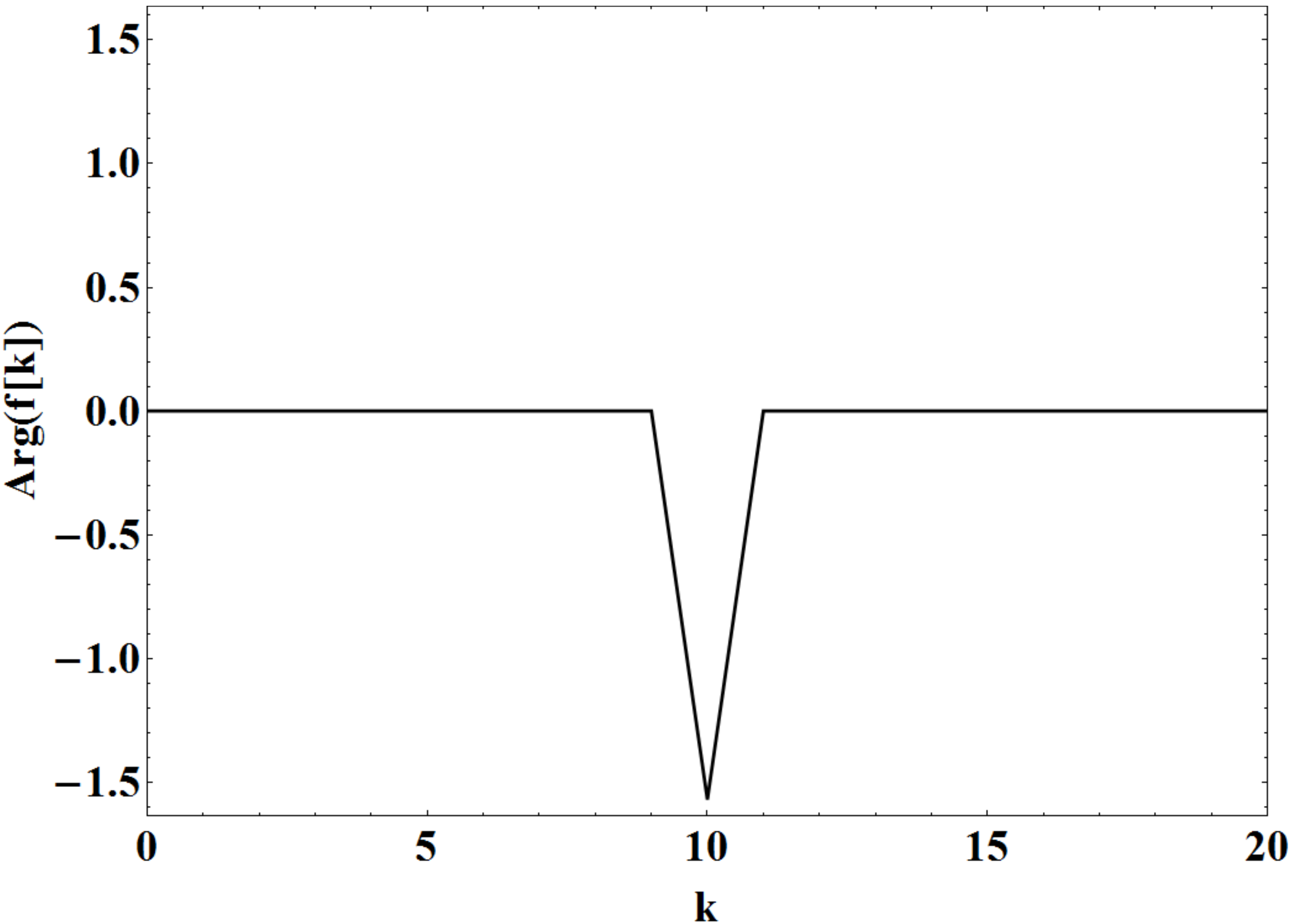
10^1
10^0
|f[k]|
10^0
10^1
10^2
10^3
k


100
80
60
40
20
0
0
50
100
150
200
k
|f[k 2π/1000]|
s[ω=k 2π/1000]
s[k]


$\mathrm{Sin}\left[\frac{2\pi}{100}n\right]$
1.0
0.5
0.0
−0.5
−1.0
f[n]
0
200
400
600
800
1000
n


500
400
300
200
100
0
|f[k]|
0
5
10
15
20
k


Arg(f[k])
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
5
10
15
20
k

FFT: a very fast algorithm