Experimental Methods
Lecture 28

November 25th, 2020

Digital Noise Estimation

The periodogram

- Estimate of PSD of zero-mean Gaussian process $x(t)$ from N of its samples $x[n]=x(n\Delta T) \quad 0\leq n\leq N$; Measurement duration: $T=N\Delta T$
- Periodogram

$$S_k=\frac{\Delta T}{N}\left|\sum_{n=0}^{N-1}x[n]e^{-ikn\frac{2\pi}{N}}\right|^2$$

- Statistics of periodogram

$$\frac{S_k}{\frac{\Delta \mathrm{T}}{2\pi}\int_{-\infty}^{\infty}\left|H\left(\omega-k\frac{2\pi}{T}\right)\right|^2\ S(\omega)d\omega}$$

- is a χ^2 with 2 degrees of freedom

$$|H(\omega)|^2=\frac{1}{N}\frac{Sin^2\left(\frac{\omega T}{2}\right)}{Sin^2\left(\frac{\omega\Delta T}{2}\right)}$$

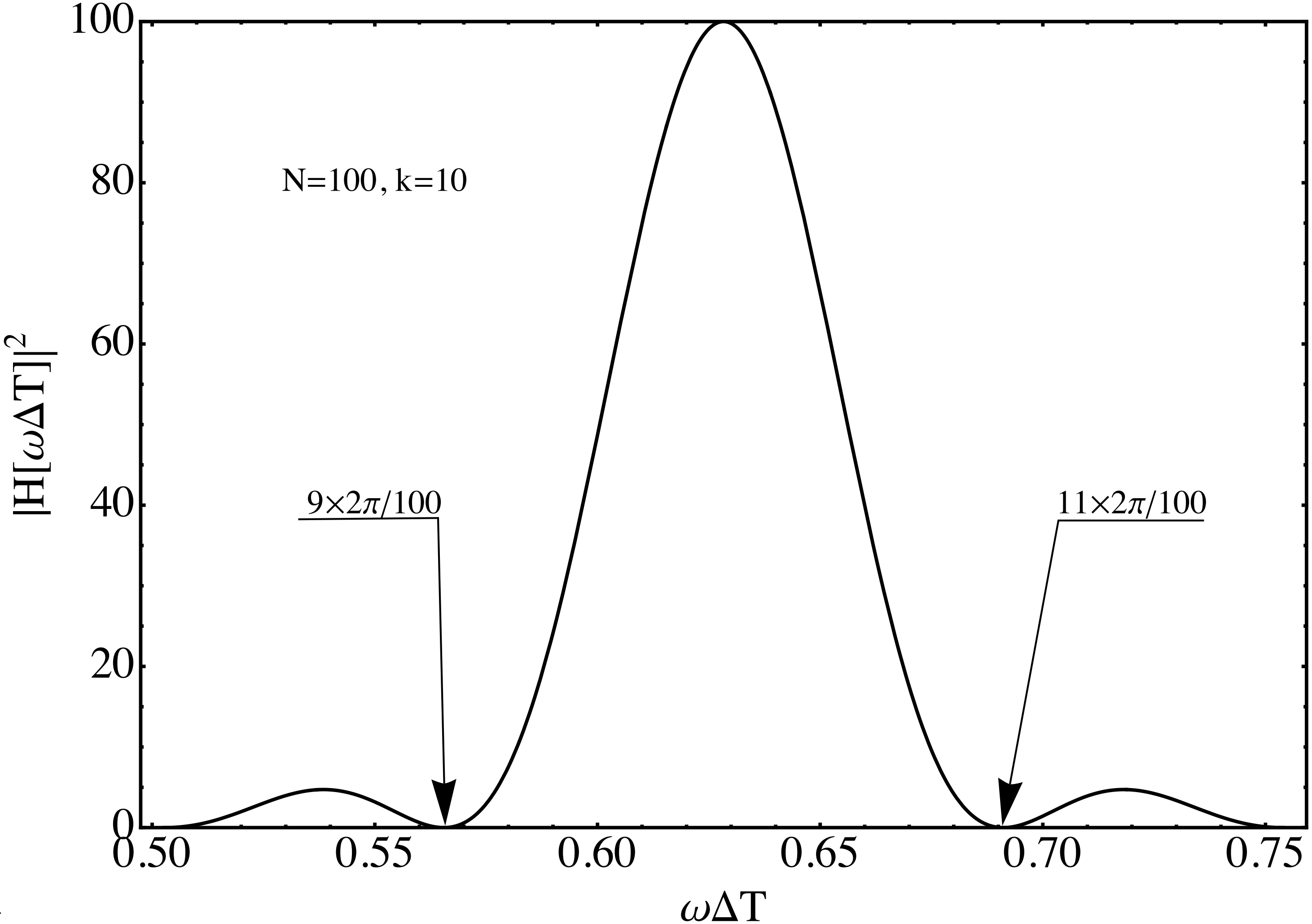
- If aliasing cured, $S(\omega)\simeq 0$ for $|\omega|>\frac{\pi}{\Delta T}$, then:

$$\frac{S_k}{\frac{\Delta \mathrm{T}}{2\pi}\int_{-\infty}^{\infty}\left|H\left(\omega-k\frac{2\pi}{T}\right)\right|^2\ S(\omega)d\omega}=\frac{S_k}{\frac{\Delta \mathrm{T}}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}}\left|H\left(\omega-k\frac{2\pi}{T}\right)\right|^2\ S(\omega)d\omega}$$

Digital estimate of PSD and Discrete Fourier Transform

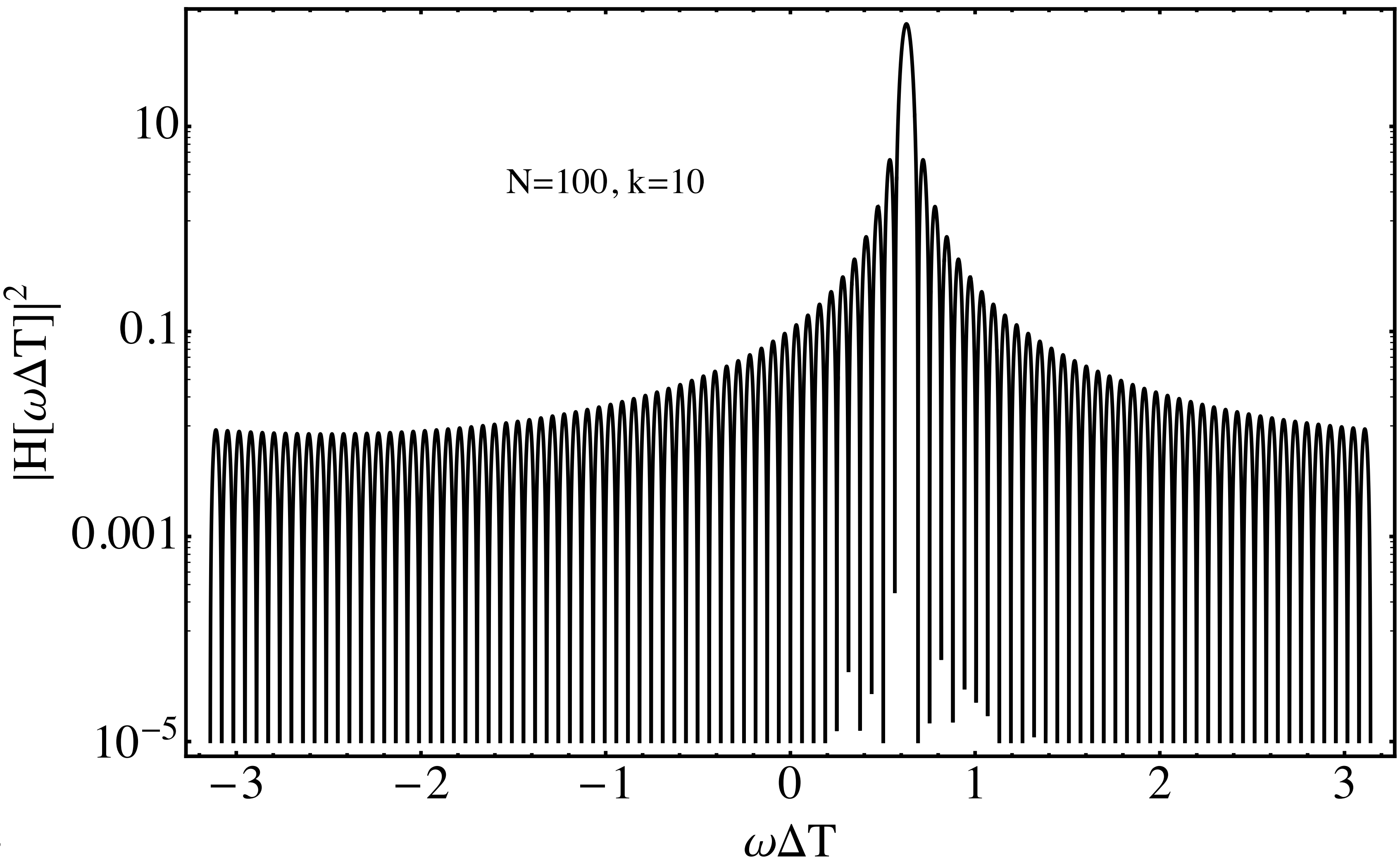
A blow-up, the first lobe is zero for: $\omega \in (k \pm 1)(2\pi/N\Delta T) = (k \pm 1)(2\pi/T)$

Spectral resolution is $\pm 2\pi/T$!



Digital estimate of PSD and Discrete Fourier Transform

A better representation of side lobes. Amplitude decays very slowly. Each coefficient of the spectral estimate is also partly contributed by the power within the side lobes.



Precision

- Recalling the properties of chi-square

$$0.7S_k \leq S\left(k\,\frac{2\pi}{T}\right) \leq 2.4\ \mathrm{S_k}$$

- This is a very imprecise estimate, not at all surprising as the spectral resolution is ±1/T and the duration of the measurement is T so that the radiometric formula would give a 100% error.

- In conclusion S_k is indeed an estimator for $S(\omega = \ k\frac{2\pi}{T})$ with two problems:

1. The relative precision is low, worse than 100%
2. The accuracy is poor due to the “leakage” from side lobes

Digital estimate of PSD and Discrete Fourier Transform

Thus $S_k = \left(\Delta T/N\right)\left|\sum_{j=0}^{N-1} x\left[n\right]e^{-ikn\left(2\pi/N\right)}\right|^2$

is an estimator for $S_{xx}(\omega= k\ 2\pi/T)$ with low relative precision, worse than 100%.

In order to improve relative precision you need to average over many independent estimates. There are two ways of doing that:

- Average the estimates for many data series $\bar{S}_k = \sum_{j=1}^{M} S_{k,j}\Big/M$
- Average many nearby coefficients $S_{\bar{k}} = \sum_{k=\bar{k}-M/2}^{\bar{k}+M/2} S_k\Big/M$

In both cases the error decreases like $1/\sqrt{M}$. In the first case the total length of the data series increases to $N_{tot}= N\ M$ so that the radiometric formula gives indeed

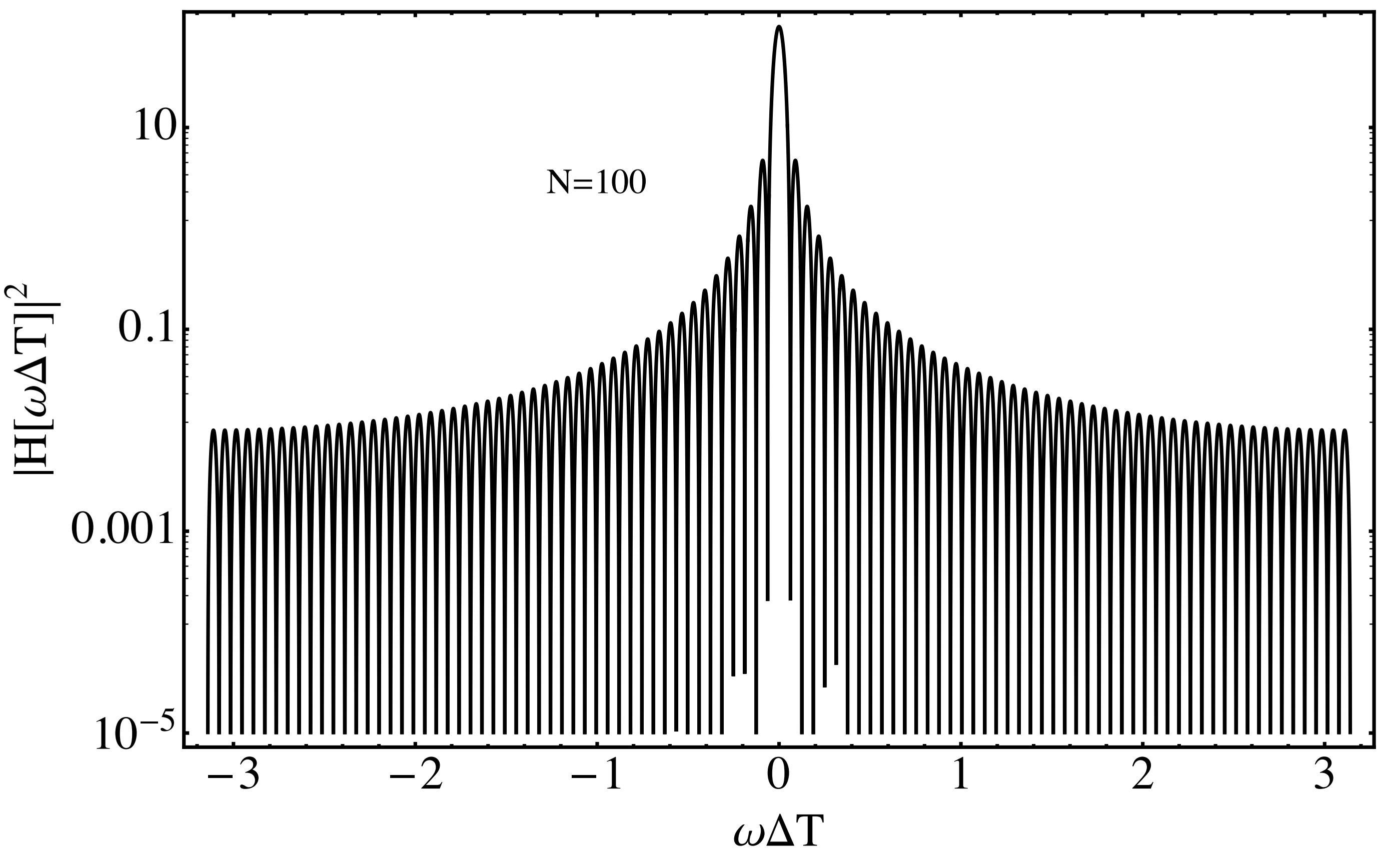
$$\Delta S_{\bar{k}}/S_{\bar{k}} \approx 1\Big/\sqrt{NM\Delta T \times \left(1/N\Delta T\right)} = 1\Big/\sqrt{M} \approx \Delta S_k/S_k\Big/\sqrt{M}$$

With the second trick the spectral resolution worsen to M 1/T thus

$$\Delta S_{\bar{k}}/S_{\bar{k}} \approx 1\Big/\sqrt{N\Delta T \times \left(M/N\Delta T\right)} = 1\Big/\sqrt{M} \approx \Delta S_k/S_k\Big/\sqrt{M}$$

Accuracy

Before examining in more details the effect of averaging, we will first consider the problem of the leakage of side lobes. The starting point is that
$$\langle S_k \rangle = \left(\Delta T/2\pi\right)\int_{-\infty}^{\infty} S_{xx}(\omega)\left|H\left(\omega - k\,2\pi/T\right)\right|^2 d\omega$$
With
$$H(\omega) = \left(1/\sqrt{N}\right)\sum_{n=0}^{N-1} e^{-i\,n\omega\Delta T}$$
a function whose square modulus decays very slowly with $\omega\to\infty$



Accuracy

Notice that by defining $\phi=\omega\Delta T$ the function H can be written as

$$w(\phi)\equiv H(\omega=\phi/\Delta T)=\left(1/\sqrt{N}\right)\sum_{n=0}^{N-1}e^{-i\,n\phi}$$

This is the discrete-time Fourier Transform of the window function

$$w[n]=\left(1/\sqrt{N}\right)\Pi\left(n/(N-1)-1/2\right)$$

which we (indirectly) calculated to be

$$w(\phi)=\left(1/\sqrt{N}\right)\sum_{k=0}^{N-1}e^{-i\phi k}=\left(1/\sqrt{N}\right)\left(1-e^{-iN\phi}\right)/\left(1-e^{-i\phi}\right)$$

Thus our spectral estimator can be written as

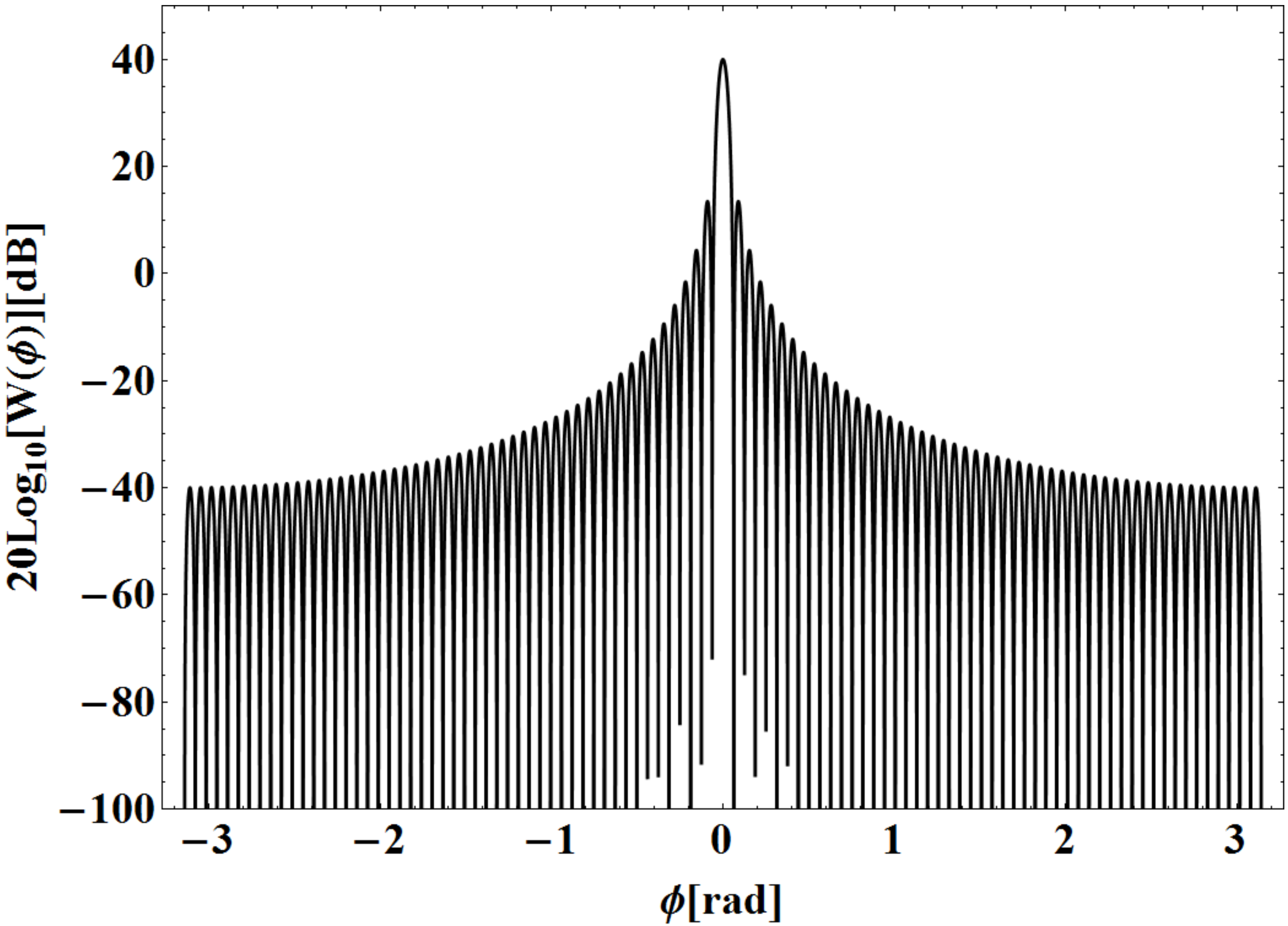
$$S_k=\Delta T\left|\sum_{n=-\infty}^{\infty}x[n]w[n]e^{-ikn(2\pi/N)}\right|^2$$

This suggest a way to decrease spectral leakage by selecting a window function w[n] whose discrete-time Fourier transform $w(\phi)$ has smaller side-lobes.

A few examples of such windows follow

The rectangular window

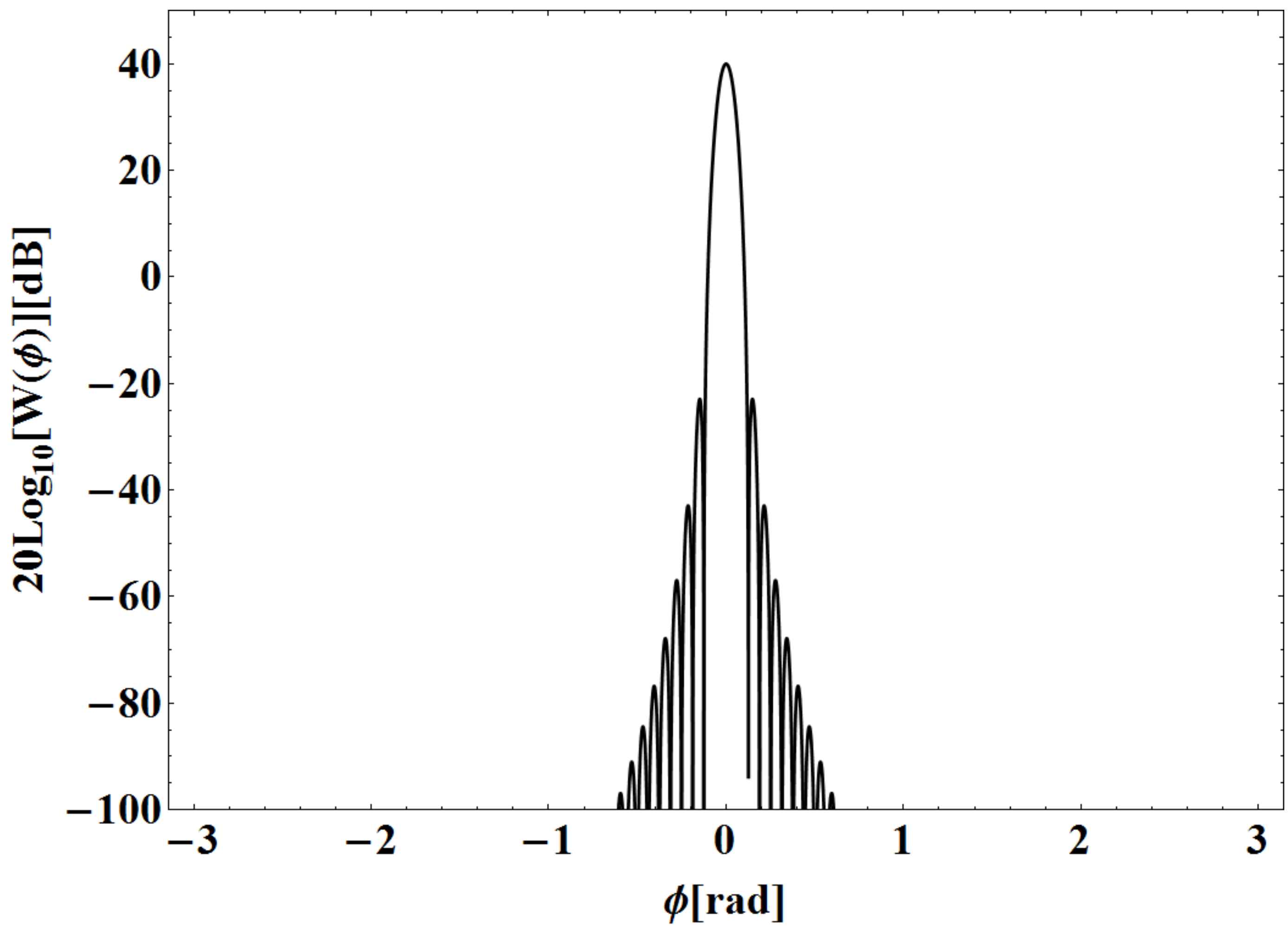
- The formula $w[n]=1/\sqrt{N}\ \ 0\le n\le N-1\ \ (N=100)$
- Width of the main lobe $\pm 2\pi/N$
- Suppression of side lobes $\approx 30\text{dB}$



The Hann window

$$w[n] = \sqrt{2/3N}\left[1 - \mathrm{Cos}\left[n(2\pi/N)\right]\right] \qquad 0 \le n \le N-1 \quad (N = 100)$$

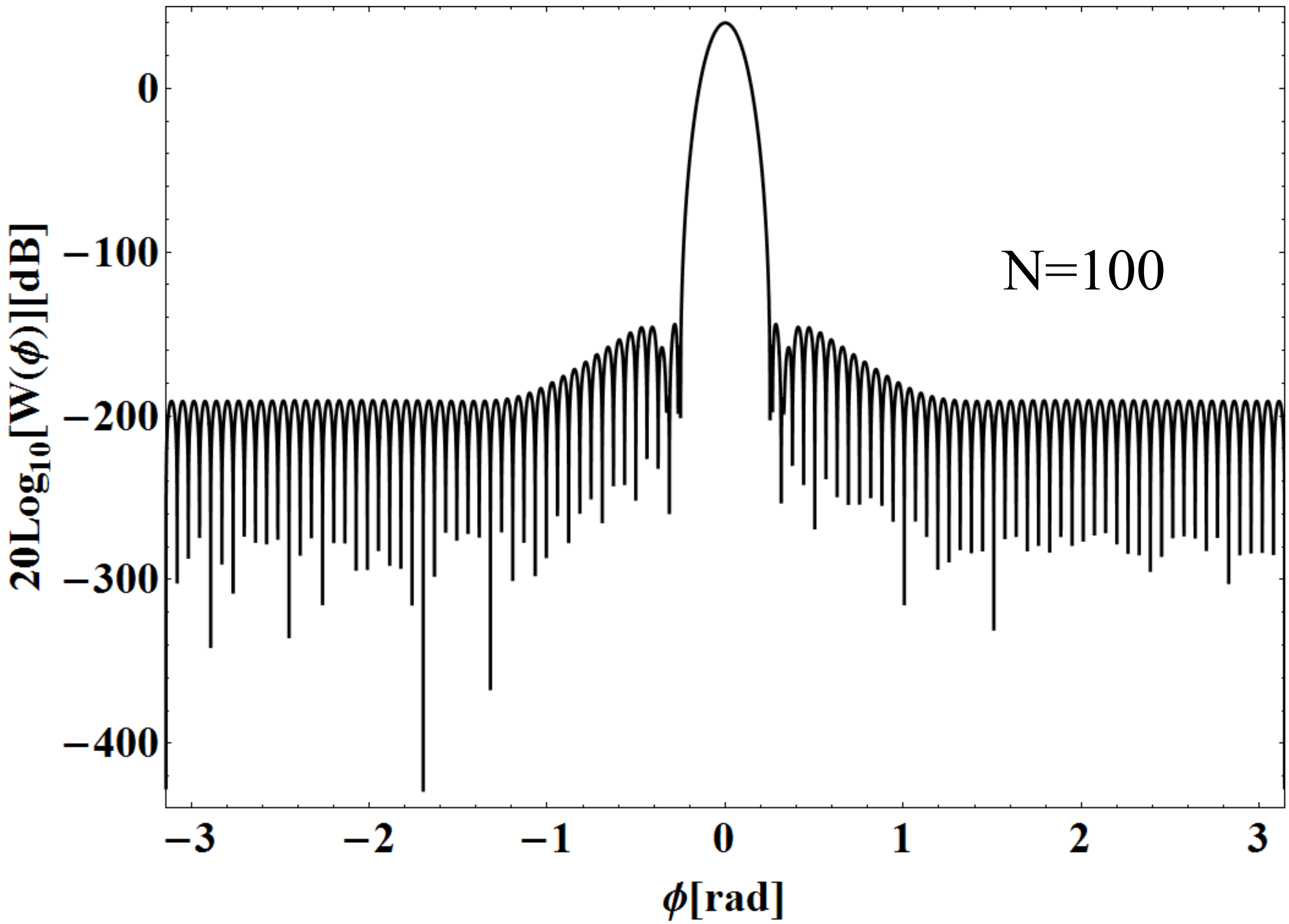
- Width of main lobe $\quad \pm 2(2\pi/N)$
- Suppression of side lobes >60 dB



Blackman-Harris

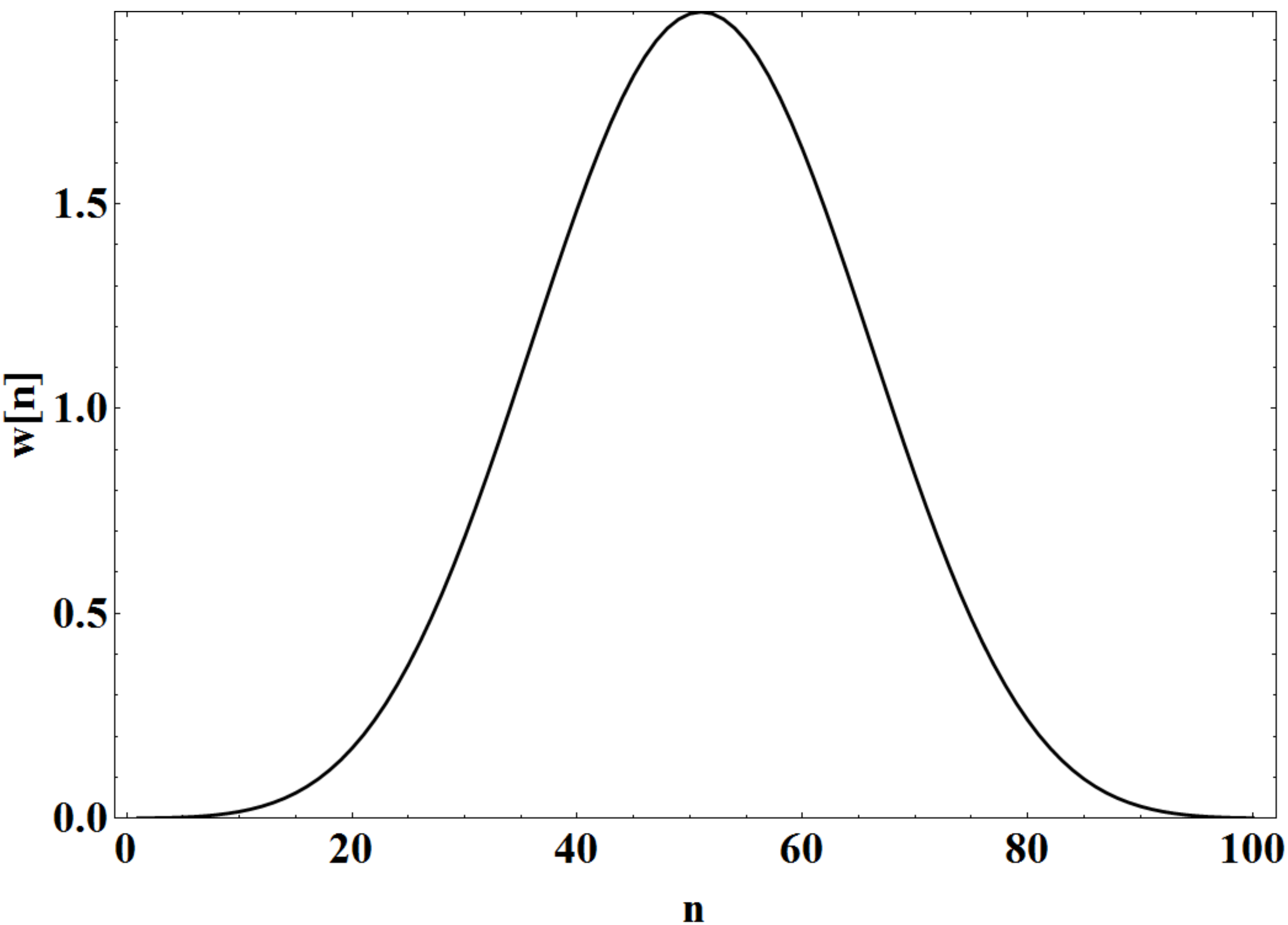
$$0.70634 - 0.96139 \times \mathrm{Cos}\left[\frac{2\pi}{N} n\right] + 0.27816 \times \mathrm{Cos}\left[2\frac{2\pi}{N} n\right] - 0.02300 \times \mathrm{Cos}\left[3\frac{2\pi}{N} n\right] \quad 0 \le n \le N-1$$

- Width of main lobe $\pm 3\left(2\pi/N\right)$
- Suppression of side lobes >140 dB



In the time domain

$$0.70634 - 0.96139 \times \mathrm{Cos}\left[\frac{2\pi}{N} n\right] + 0.27816 \times \mathrm{Cos}\left[2\frac{2\pi}{N} n\right] - 0.02300 \times \mathrm{Cos}\left[3\frac{2\pi}{N} n\right] \quad 0 \le n \le N-1$$



The effect of the window

- The window avoids sharp truncation of data that would give Sinc(f) tails in the frequency domain
- The higher the order of the window, the higher the order of the lowest derivative that gets truncated
- However the higher the order of the window, the larger the width of first lobe
- Notice that this implies that, for small k, S_k is also contributed by power at low frequency down to dc.
- Thus for Hannings even the third coefficient (or the second at non-zero frequency) picks up the power from frequencies down to dc, and for Blackmann-Harris even the fourth coefficient (or the third at non-zero frequency) does so.

De-trending

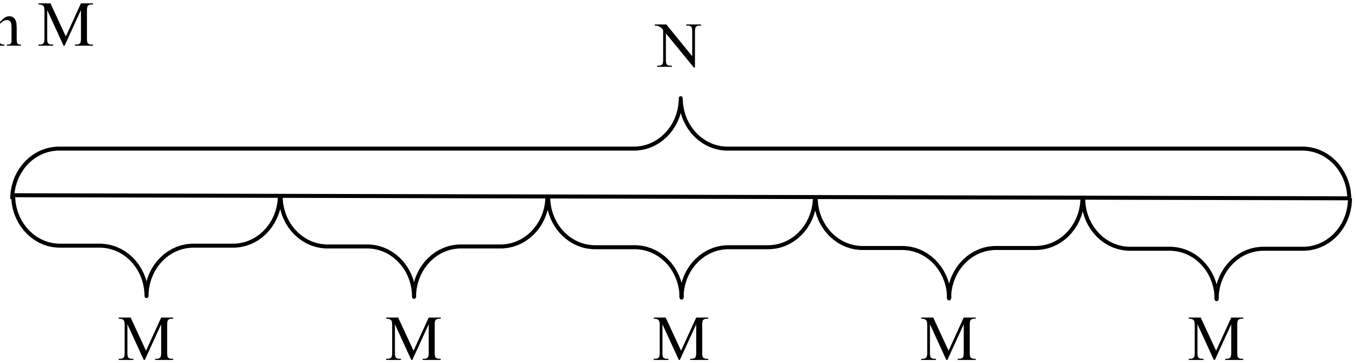
- To mitigate the effect of noise at very low frequency leaking into the spectral estimates at high frequency, we may use two approaches:
  - Either drop the coefficients that are affected by the spectral leakage: e.g. <4 for Blackmann Harris Window
  - Or filter data with a high pass filter with a lower roll-off at 1/T
    - A popular high pass filter, called de-trending, consists of fitting a line to the data with least squares, and subtract then the best fit line from the same data to generate the de-trended data.
    - As noise at frequencies lower than 1/T appears substantially as a drift in the data, the method is rather effective.
    - It must be stressed that the method is quite empirical, and statistical properties are not straightforward.

Combining averaging and windowing

Thus spectral leakage is suppressed by windowing (and de-trending).

Fluctuations are reduced by averaging. The most common approach for averaging consists of:

1. dividing the available data series of length N in stretches of equal length M

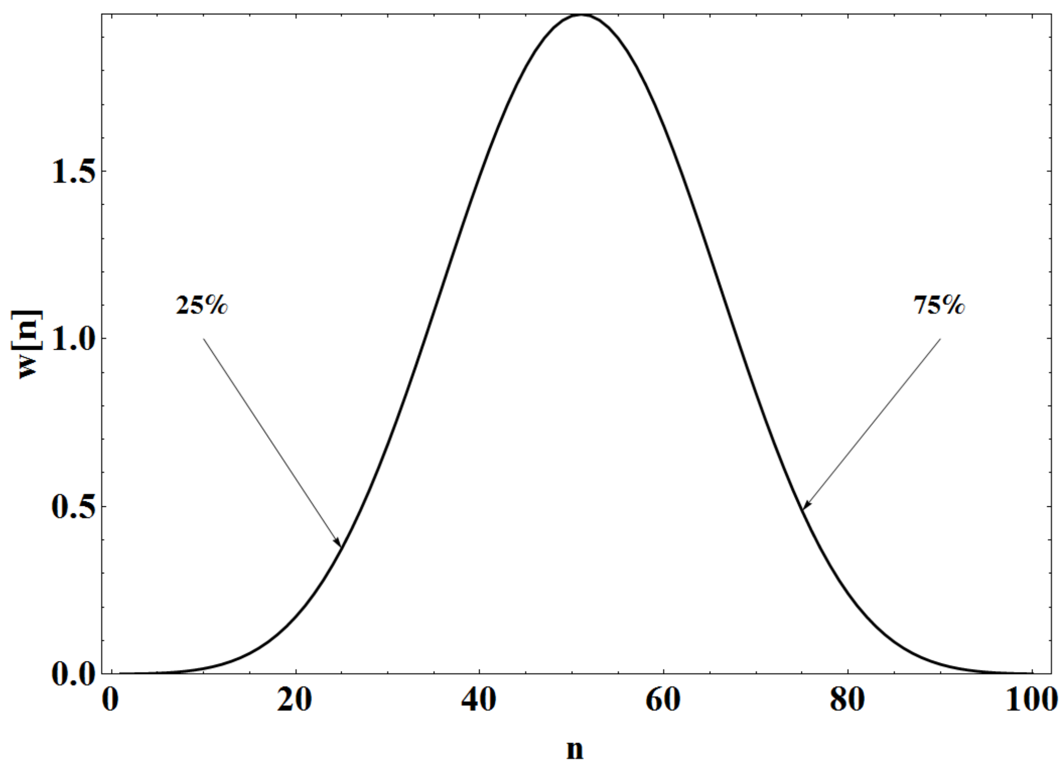


2. detrending each stretch and windowing it with the appropriate window .
3. evaluating $S_{k,j}$ for each stretch j
4. For each value of k, evaluate $\overline{S}_k = \sum_{j=1}^{N/M} S_{k,j} \Big/ \left(N/M\right)$
5. There is however a more efficient way to divide the data. See the following slide.

Overlapping stretches

The spectral window suppresses the data in the tails.

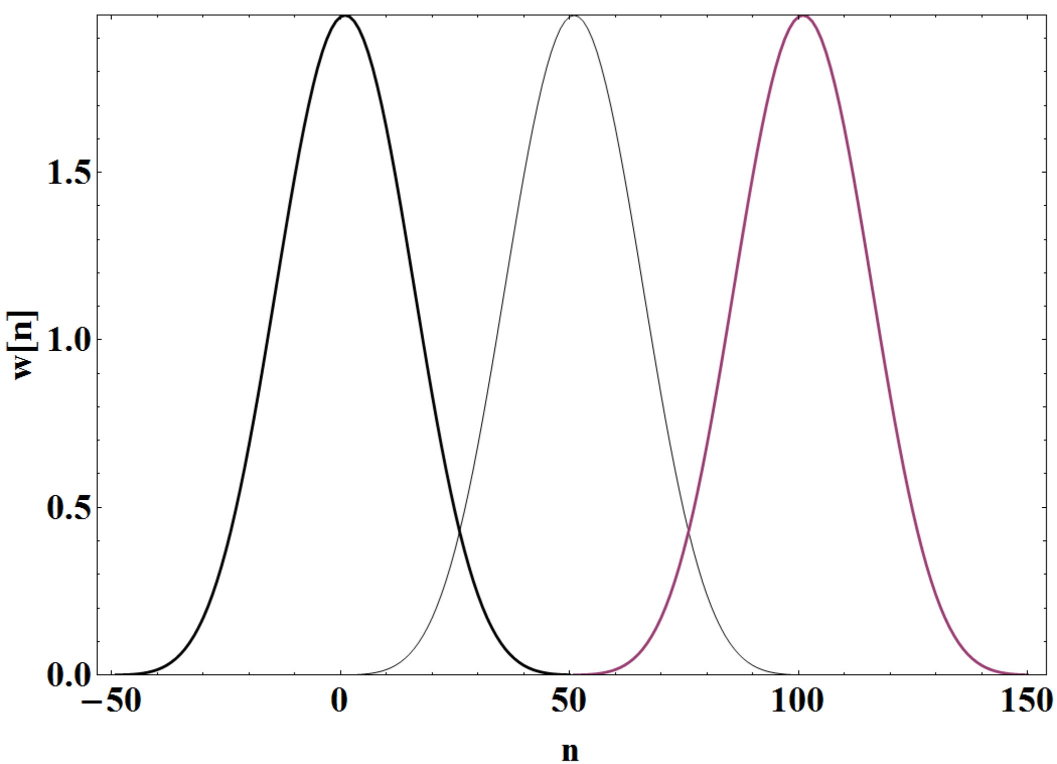
The picture is for the Blackman-Harris window. Arrows show the two 25% tails. In these tails data are weighted by less than 1/4



Overlapping stretches

Thus adjoining stretches can be somewhat overlapped, never really weighting the data within the overlapping tails more than the remaining ones

For the Blackman-Harris window one can show that 50% overlap or slightly more produce best results



Combining averaging and windowing

Thus best recipe consists of

1. dividing the available data series of length N in overlapping stretches of equal length M
2. Windowing (and de-trending) each stretch with the appropriate window, and finally evaluating $S_{k,j}$ for that stretch (that we label with j)
3. averaging the spectral estimates from different stretches

$$\overline{S}_k = \sum_{j=1}^{N_s} S_{k,j} \Big/ N_s$$

The spectral resolution depend on the width of the central lobe of the window. This is usually n times the basic frequency $1/M\Delta T$

The relative error on the spectral estimate depends on the number of stretches N_s entering in the average
$$\Delta\overline{S}_k / \overline{S}_k \approx 1 \Big/ \sqrt{\left(n/M\Delta T\right) M\Delta T}\sqrt{N_s}$$

However a more accurate calculation shows that the formula holds with n=1

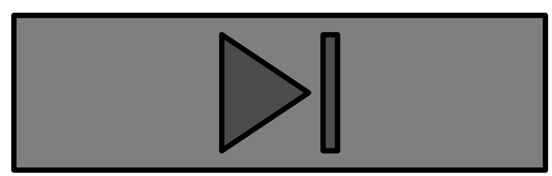
For 50% overlap $(N_s+1)(M/2)=N$:

$$\Delta\overline{S}_k / \overline{S}_k \approx 1 \Big/ \sqrt{\frac{2N}{M} - 1}$$

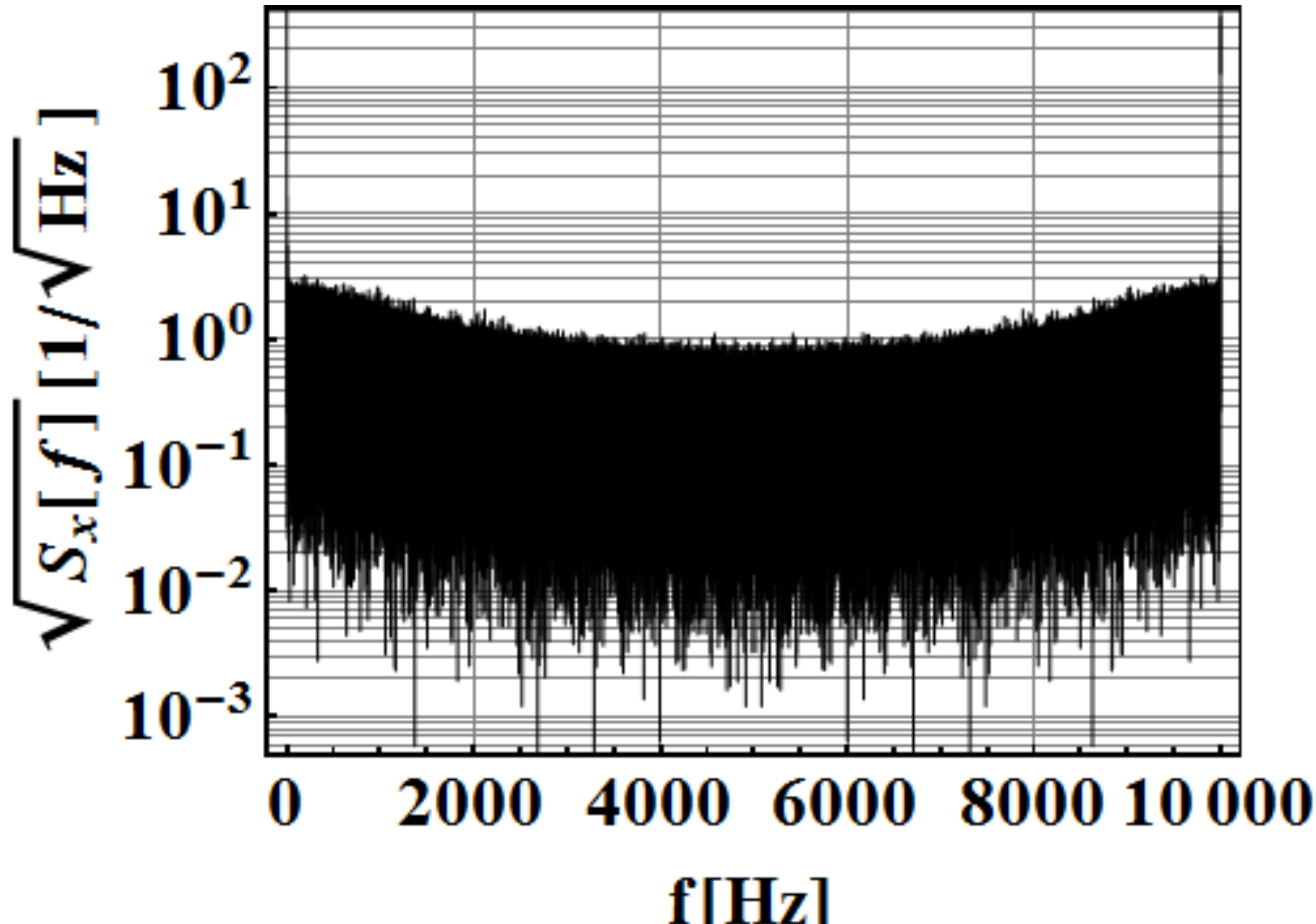
Spectral estimate by FFT: summary

1. Divide the data in stretches
2. Multiply by window
3. Make FFT
4. Take Square Modulus
5. Multiply by ΔT (or 2 ΔT if you need the one-sided PSD)
6. Average different spectra
7. Take error from rms of different spectra

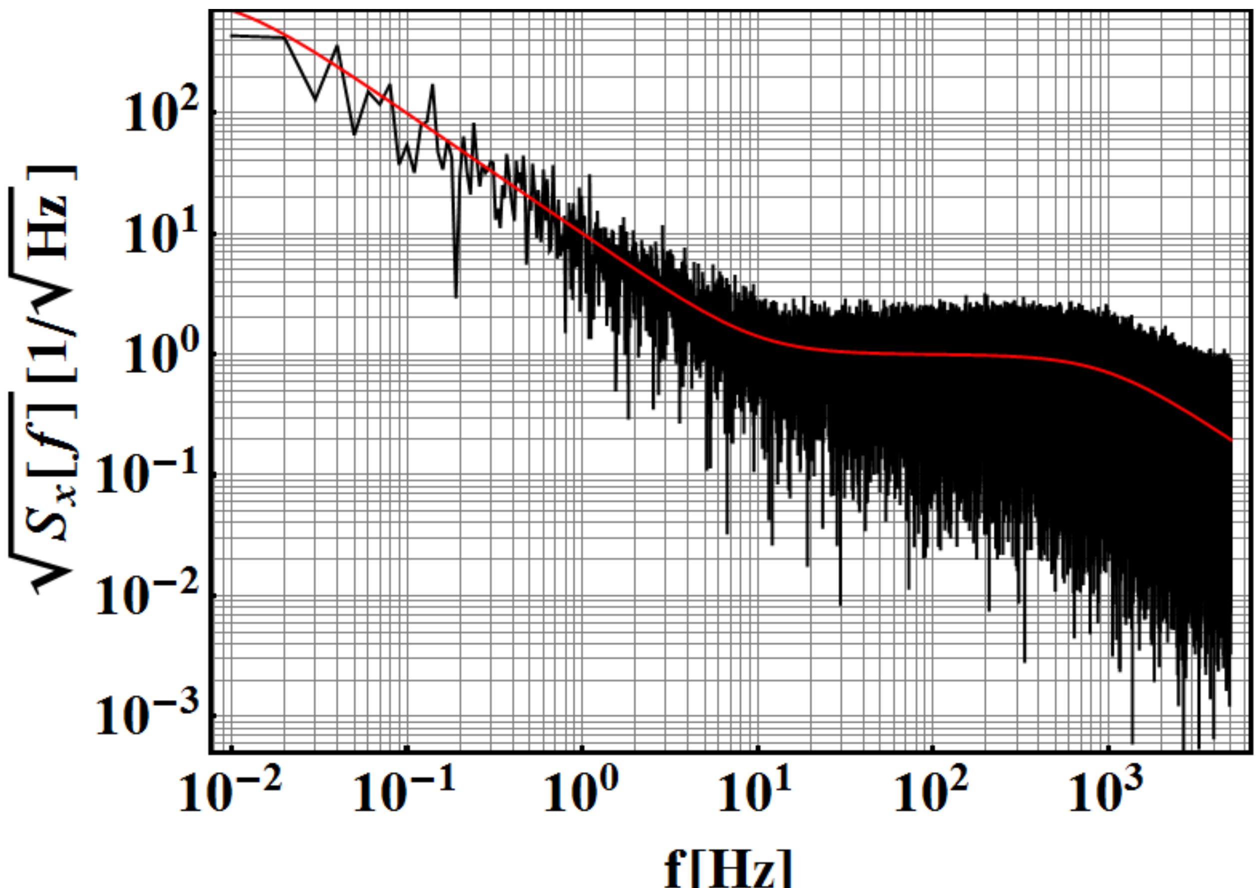
Continuing the previous example



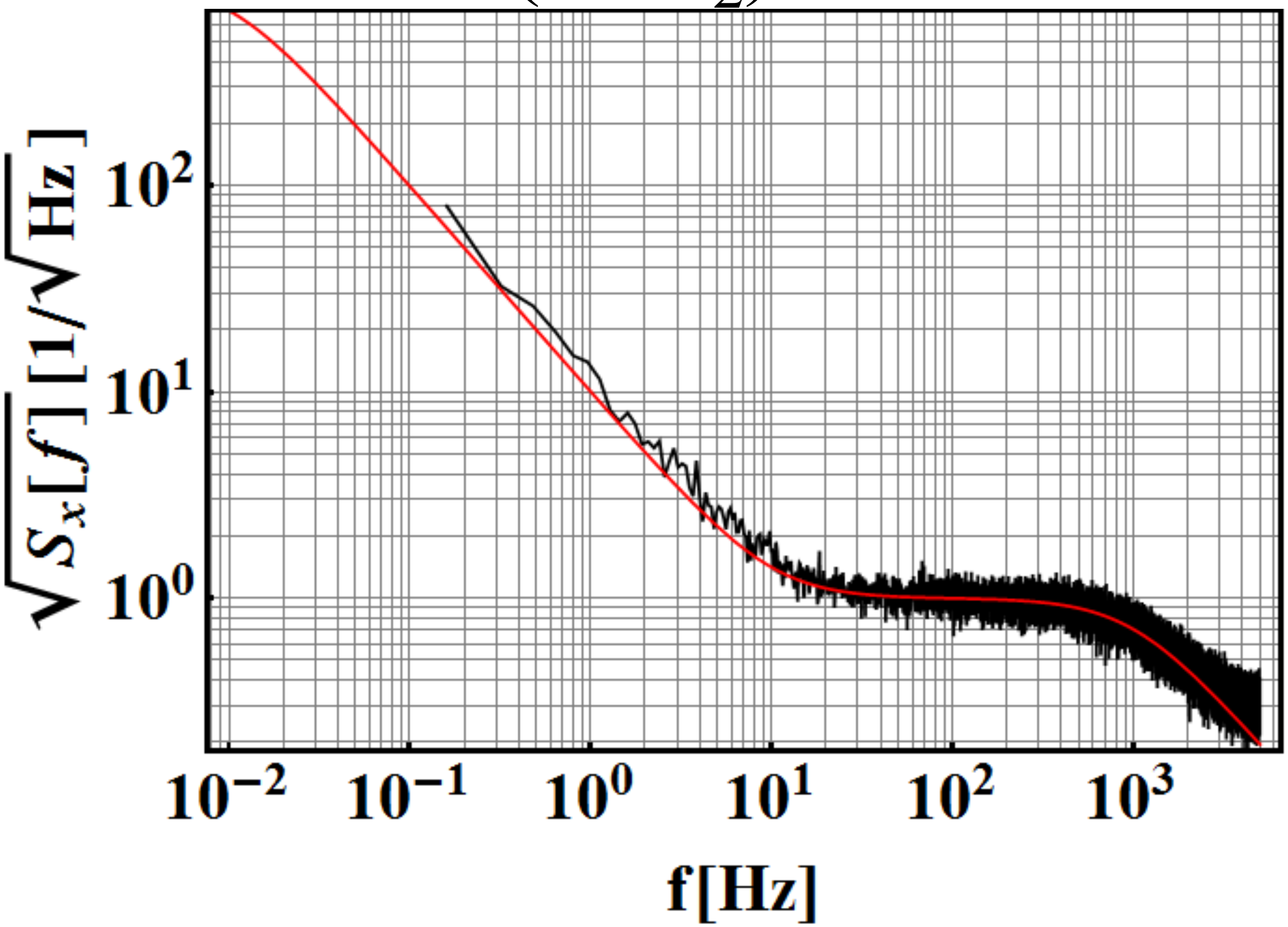
All numbers, no average, log-lin scale (PSD_1 first plot)



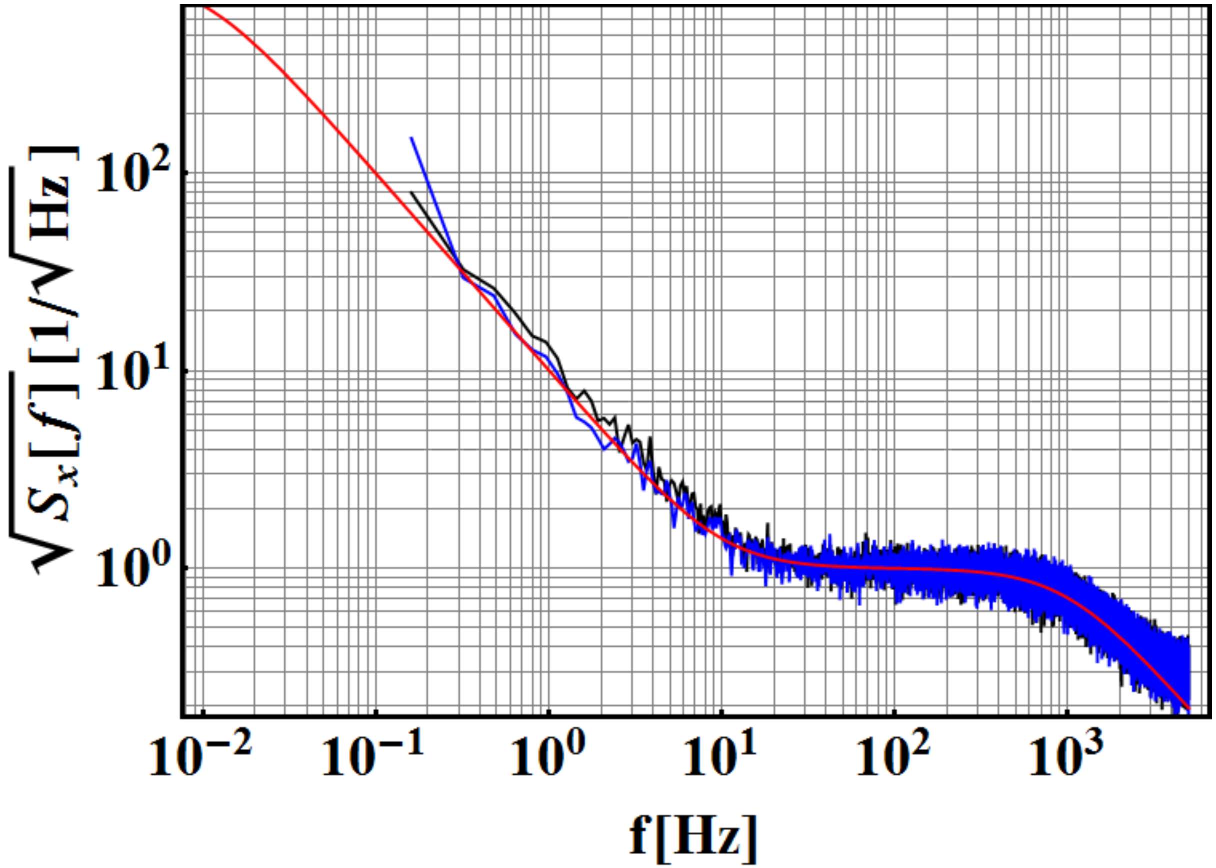
Log-log scale only N/2 coefficients (PSD_1 second plot)



Average of 16 non overlapping stretches (PSD_2)



Hanning window



Blackmann-Harris window