Experimental Methods.
Test of February 9, 2020

Data

Parameter	Value	Units	Parameter	Value	Units
ν	1.	GHz	Q	10 000	
m	0.5	pg	C_p	60.	fF
C_s	5.	fF	d	5.	pm/V
$\mathcal{I}_m$	100.	μW	$\mathcal{I}_{ref}$	100.	mW
λ	1.	μm	T	4.	K
ρ	0.1				

Q1

Starting form the linearised constitutive equations

$$I = s\, C_p\, V - s\, d\, F \tag{1}$$

and

$$F = -k\, x + k\, d\, V \tag{2}$$

you have to add Newton equation for the mass

$$\left(m\, s^2 + \beta\, s\right)\, x = F + F_b \tag{3}$$

with F_b the stochastic Brownian force to be used later, and the equation for the source

$$-\frac{I}{s\, C_s} + V_s = V \tag{4}$$

Combining we get

$$\left(m\, s^2 + \beta\, s + k\right)\, x - k\, d\, V == F_b \tag{5}$$

And

$$d\, k\, x + \left(C_p + C_s - d^2\, k\right)\, V == V_s\, C_s \tag{6}$$

And finally

$$\left(m\, s^2 + \beta\, s + k\left(1 + \frac{d^2\, k}{\left(C_p + C_s - d^2\, k\right)}\right)\right)\, x == k\, d\, \frac{C_s}{\left(C_p + C_s - d^2\, k\right)}\, V_s + F_b \tag{7}$$

then

$$x = \frac{k\, d\, \frac{C_s}{\left(C_p + C_s - d^2\, k\right)}}{\left(m\, s^2 + \beta\, s + k\left(1 + \frac{d^2\, k}{\left(C_p + C_s - d^2\, k\right)}\right)\right)}\, V_s \tag{8}$$

Thus the transfer function is

$$h[s] = (2\pi\nu)^2 d \frac{C_s}{C_p + C_s} \frac{\frac{1}{1-\frac{d^2 m (2\pi\nu)^2}{C_p+C_s}}}{s^2 + \frac{s\, 2\pi\nu}{Q} + (2\pi\nu)^2 \frac{1}{1-\frac{d^2 m (2\pi\nu)^2}{C_p+C_s}}} \tag{9}$$

Notice that

$$\frac{d^2 m (2\pi\nu)^2}{C_p + C_s} \simeq 7.6\times 10^{-6} \tag{10}$$

And can be neglected within the accuracy required here

Introducing

$$\tilde{s} = \frac{s}{2\pi\nu} \tag{11}$$

$$h[\tilde{s}] = d \frac{C_s}{C_p + C_s} \frac{1}{\tilde{s}^2 + \frac{\tilde{s}}{Q} + 1} = 0.38\ \mathrm{pm/V} \frac{1}{\tilde{s}^2 + \frac{\tilde{s}}{Q} + 1} \tag{12}$$

Q2

x fluctuates because of Brownian noise and because of the shot noise of the impinging photons of the measurement beam.

The esiest way to calculate the Browninan noise is to associate the proper force generator to viscous damping, with PSD

$$S_{F_B} = 2 k_B T \beta = 2 k_B T \frac{m\, 2\pi\nu}{Q} = 3.5\times 10^{-32}\ \mathrm{N^2/Hz} \tag{13}$$

The force due to the measurement beam is a shot noise with PSD

$$S_{F_{shot}} = \frac{4 h}{\lambda c} I_m = 8.8\times 10^{-40}\ \mathrm{N^2/Hz} \tag{14}$$

As the rate is $\frac{I_m}{h c/\lambda}$ and the exchange impulse per photon is $2\, h/\lambda$. This is completely negligible relative to Brownian force. This last one is transferred to x as

$$S_{x_B}[\omega] = \frac{2 k_B T\, 2\pi\nu}{m Q} \frac{1}{(\omega^2 - (2\pi\nu)^2)^2 + \frac{\omega^2 (2\pi\nu)^2}{Q^2}} \tag{15}$$

where again we have neglected $\frac{d^2 k}{C_p + C_s}$

By defining

$$y = \omega / (2\pi\nu) \tag{16}$$

this becomes

$$S_{x_B}[y] == \frac{2 k_B T}{m Q (2\pi\nu)^3} \frac{1}{(y^2-1)^2 + \frac{y^2}{Q^2}} = 8.9\times 10^{-41}\ \mathrm{m^2 s} \frac{1}{(y^2-1)^2 + \frac{y^2}{Q^2}} \tag{17}$$

Q3

As the voltages on the two photodiodes can be written as

$$V_{1\,(2)} = \frac{A}{2}\left(I_m + I_{ref} \pm \sqrt{I_m I_{ref}}\, \mathrm{Cos}[\Delta\phi]\right) \tag{18}$$

where A is some proportionality factor, $\Delta\phi$ is the fase difference between the measurement beam and reference beam, while I_m and I_{ref} are their intensities. Their difference is then

$$\Delta V = V_1 - V_2 = A\sqrt{I_m I_{ref}}\, \mathrm{Cos}[\Delta\phi] \tag{19}$$

If the angle of incidence of the light on the oscillator is near 90°, then the motion of the mass produces a phase shift ≃2x. By properly adjusting the optical path, and as we are talking of x<< λ

$$\Delta V \simeq A\sqrt{I_m I_{ref}}\; 2\pi \frac{2\,x}{\lambda} \tag{20}$$

As $I_m << I_{ref}$, the intensity on each photodiode is about $\frac{I_{ref}}{2}$ and thus the voltage out of each photodiode fluctuates due to the shot noise in the reference beam. The PSD of the difference is just the sum of the PSD out of the two photodiodes

$$S_{\Delta v} = 2\,\frac{A^2}{4}\, S_{I_{ref}} = \frac{A^2}{2}\, I_{ref}\, \frac{h\,c}{\lambda} \tag{21}$$

To convert into a displacement signal we need to use eq (14)

$$S_x = \frac{S_{\Delta v}}{\left(A\sqrt{I_m I_{ref}}\; 2\pi\,\frac{2}{\lambda}\right)^2} = \frac{1}{16\,\pi}\,\lambda^2\,\frac{\frac{\hbar\,c}{\lambda}}{I_m} = \frac{1}{16\,\pi}\,\frac{\hbar\,c\,\lambda}{I_m} = 6.3\times10^{-30}\ \mathrm{m^2/Hz} \tag{22}$$

Q4

As the width of the line is about $\Delta\nu=\pm\nu/2Q$, we require the measurement resolution to be about

$$\Delta\nu_{res} \simeq \frac{\nu}{3\,Q} \tag{23}$$

The peak value of the thermal noise density is

$$S_{x,max} \simeq S_{x_B}[1] = \frac{2\,k_B\,T\,Q}{m\,(2\,\pi\,\nu)^3} \tag{24}$$

If we want to resolve this peak value with a relative erroror of ρ=10%, we require that the measurement noise to be less or equal to that. The noise is dominated by the readout at all frequencies, which is dominated by the shot noise of the refence beam. thus one should do two measurements, one with the measurement beam blocked and one with with the measurement beam running and make the difference. The error is then $\sqrt{2}$ the value form the radiometric formula, using just the PSD of the readout noise:

$$\frac{1}{16\,\pi}\,\lambda^2\,\frac{\frac{\hbar\,c}{\lambda}}{I_m}\,\frac{\sqrt{2}}{\sqrt{2\,\pi\,\frac{\nu}{3\,Q}\,\Delta T}} == \rho\,\frac{2\,k_B\,T\,Q}{m\,(2\,\pi\,\nu)^3} \tag{25}$$

That is

$$\Delta T = \frac{3\,c^2\,m^2\,\pi^3\,\lambda^2\,\nu^5\,\hbar^2}{16\,Q\,T^2\,\rho^2\,k_B^2\,I_m^2} = 0.48\ \mathrm{ks} \tag{26}$$

Q5

One can use the wiener filter result. The impulse produces an x-signal

$$x[\omega] = \phi_o \frac{k\, d\, \frac{C_s}{C_p+C_s}}{(-m\,\omega^2 + i\,\omega\,\beta\, s + k)} = \phi_o \frac{(2\,\pi\,\nu)^2\, d\, \frac{C_s}{C_p+C_s}}{\left(-\omega^2 + \frac{i\,\omega\, 2\,\pi\,\nu}{Q} + (2\,\pi\,\nu)^2\right)} \tag{27}$$

Which is buried in a noise with PSD

$$S_{x,tot}[\omega] = \frac{2\, k_B\, T\, 2\,\pi\,\nu}{m\, Q} \frac{1}{(\omega^2 - (2\,\pi\,\nu)^2)^2 + \frac{\omega^2\,(2\,\pi\,\nu)^2}{Q^2}} + \frac{1}{16\,\pi}\,\lambda^2 \frac{\frac{\hbar\, c}{\lambda}}{I_m} \tag{28}$$

Thus the minimum uncertainty is

$$\sigma_{\phi_o} = \left(\frac{1}{2\,\pi}\int_{-\infty}^{\infty} \frac{\frac{\left((2\,\pi\,\nu)^2\, d\, \frac{C_s}{C_p+C_s}\right)^2}{\left(\omega^2 - (2\,\pi\,\nu)^2\right)^2 + \frac{\omega^2\,(2\,\pi\,\nu)^2}{Q^2}}}{\frac{2\, k_B\, T\, 2\,\pi\,\nu}{m\, Q} \frac{1}{\left(\omega^2 - (2\,\pi\,\nu)^2\right)^2 + \frac{\omega^2\,(2\,\pi\,\nu)^2}{Q^2}} + \frac{1}{16\,\pi}\,\lambda^2 \frac{\frac{\hbar\, c}{\lambda}}{I_m}}\, d\omega\right)^{-1/2} \tag{29}$$

A patient reshuffling

$$\sigma_{\phi_o} = \frac{\sqrt{\frac{1}{16\,\pi}\,\frac{\lambda^2}{\nu}\,\frac{\frac{\hbar\, c}{\lambda}}{I_m}}}{d\, \frac{C_s}{C_p+C_s}} \left(\int_{-\infty}^{\infty} \frac{1}{\frac{\frac{2\, k_B\, T}{m\, Q\,(2\,\pi\,\nu)^3}}{\frac{1}{16\,\pi}\,\lambda^2\,\frac{\frac{\hbar\, c}{\lambda}}{I_m}} + (y^2 - 1)^2 + \frac{y^2}{Q^2}}\, dy\right)^{-1/2} \tag{30}$$

As the constant

$$\delta = \frac{\frac{2\, k_B\, T}{m\, Q\,(2\,\pi\,\nu)^3}}{\frac{1}{16\,\pi}\,\lambda^2\,\frac{\frac{\hbar\, c}{\lambda}}{I_m}} = 1.4\times 10^{-11} \tag{31}$$

And

$$\delta + \left(y^2 - 1\right)^2 + \frac{y^2}{Q^2} = \left(y^2 - \sqrt{(1+\delta)}\right)^2 + y^2\left(2\left(\sqrt{1+\delta} - 1\right) + \frac{1}{Q^2}\right) \simeq \left(y^2 - 1\right)^2 + \frac{y^2}{Q^2} \tag{32}$$

Thus

$$\sigma_{\phi_o} = \frac{\sqrt{\frac{1}{16\,\pi}\,\frac{\lambda^2}{\nu}\,\frac{\frac{\hbar\, c}{\lambda}}{I_m}}}{d\, \frac{C_s}{C_p+C_s}} \left(\int_{-\infty}^{\infty} \frac{1}{(y^2 - 1)^2 + \frac{y^2}{Q^2}}\, dy\right)^{-1/2} \simeq 1.2\times 10^{-9}\ s\, V \tag{33}$$