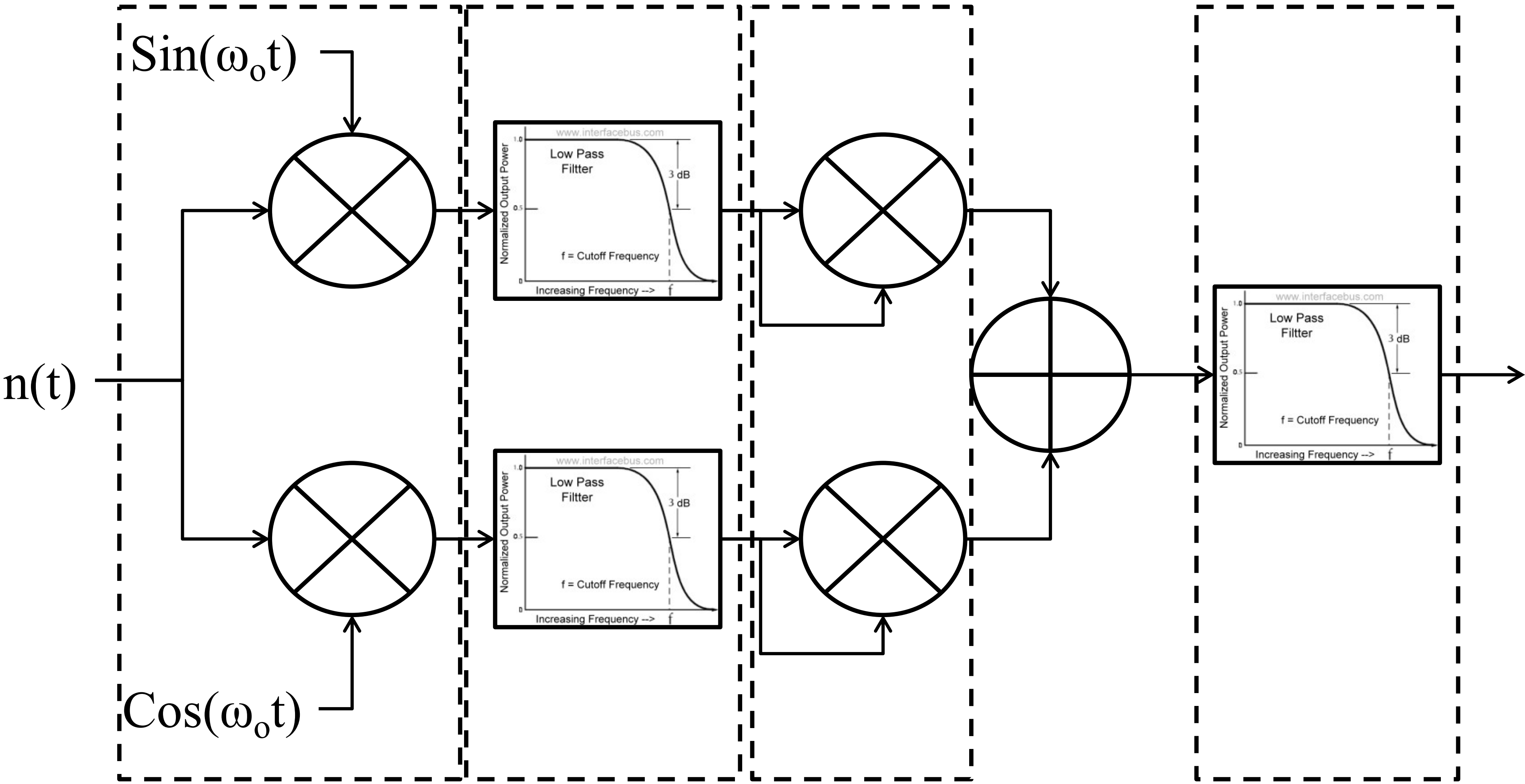
Experimental Methods
Lecture 27

November 23rd, 2020

Noise estimation

A general scheme for PSD estimation



sets frequency of estimation

sets frequency resolution

squares noise

sets precision

$$\frac{\Delta S}{S} \simeq \frac{1}{\sqrt{\Delta f \; T}}$$

Digital estimate of PSD and Discrete Fourier Transform

At low enough frequency, where data can be digitized and stored on a memory, the most efficient method to estimate a PSD of a stochastic process is to perform it numerically via a Fast Fourier Transform of the data.

In particular, if one acquires a time series made out of N samples of a zero-mean stochastic process x(t) sampled with sampling time ΔT

$$x\left[n\right]=x\left(n\Delta T\right) \qquad 0\leq n\leq N-1$$

one can calculate, preferably using the FFT algorithm, its Discrete Fourier Transform (DFT) $x\left[k\right]=\sum_{n=0}^{N-1}x\left[n\right]e^{-ikn\left(2\pi/N\right)}$

We will show that $S_k = \Delta T\,|x[k]|^2/N$, the so called *periodogram*, and some variations of this formula, give an estimate for $S_{xx}(\omega = k\,2\pi/T)$ where $T=N\Delta T$ is the time length of the data series, i.e. the duration of the measurement

Digital estimate of PSD and Discrete Fourier Transform

A good starting point is the Wiener-Kinchine theorem that states that the quantity $\tilde{x}(\omega)=\left(1/\sqrt{T}\right)\int_{-T/2}^{T/2} x(t)e^{-i\omega t}\,dt$

has the following property $\lim_{T\to\infty}\left\langle \tilde{x}(\omega)\tilde{x}^*(\omega)\right\rangle = S_{xx}(\omega)$

Consider now the DFT $x[k]=\sum_{n=0}^{N-1} x[n]e^{-ikn(2\pi/N)}$

It can be approximated as

$$x[k]=\left(1/\Delta T\right)\sum_{n=0}^{N-1} x(n\Delta T)e^{-ikn\Delta T(2\pi/N\Delta T)}\Delta T \approx \left(1/\Delta T\right)\int_0^T x(t)e^{-ik(2\pi/N\Delta T)t}\,dt$$

thus

$$x[k]\approx\sqrt{N/\Delta T}\;\tilde{x}(2\pi k/T)$$

and

$$\lim_{T\to\infty}\left\langle \left(\Delta T/N\right)x[k]x^*[k]\right\rangle = \lim_{T\to\infty}\left\langle S_k\right\rangle = S_{xx}(k2\pi/T)$$

Thus, at least for large T, S_k fluctuates around $S_{xx}(k\,2\pi/T)$

We will establish later the uncertainty of this estimate

Digital estimate of PSD and Discrete Fourier Transform

Let's now work out the result more precisely. The explicit definition of the periodogram $S_k = \Delta T \left| \sum_{n=0}^{N-1} x[n] e^{-ikn(2\pi/N)} / \sqrt{N} \right|^2$

$$= (\Delta T/N) \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} x[n] x[j] e^{-ik(n-j)(2\pi/N)}$$

Its mean value is: $\langle S_k \rangle = (\Delta T/N) \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} \langle x[n] x[j] \rangle e^{-ik(n-j)(2\pi/N)}$

Assuming that the original process is stationary

$$\langle S_k \rangle = (\Delta T/N) \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} R_{xx}\left((j-n)\Delta T\right) e^{-ik(n-j)(2\pi/N)}$$

$$= (\Delta T/2\pi) \int_{-\infty}^{\infty} d\omega S_{xx}(\omega) (1/N) \sum_{n=0}^{N-1} \sum_{j=0}^{N-1} e^{i\omega(n-j)\Delta T} e^{-ik(n-j)(2\pi/N)}$$

$$= (\Delta T/2\pi) \int_{-\infty}^{\infty} S_{xx}(\omega) \left| H(\omega - k 2\pi/T) \right|^2 d\omega$$

where $H(\omega) = \left(1/\sqrt{N}\right) \sum_{n=0}^{N-1} e^{i n\omega\Delta T}$

Digital estimate of PSD and Discrete Fourier Transform

From previous page $\langle S_k \rangle = (\Delta T/2\pi)\int_{-\infty}^{\infty} S_{xx}(\omega)\left|H(\omega - k\,2\pi/T)\right|^2 d\omega$

and $H(\omega) = \left(1/\sqrt{N}\right)\sum_{n=0}^{N-1} e^{i\,n\omega\Delta T}$

This is a well known function as $\sum_{k=0}^{N-1} e^{i\phi k} = \left(1-e^{iN\phi}\right)/\left(1-e^{i\phi}\right)$

so that
$$\left|H(\omega)\right|^2 = (1/N)\left|\left(1-e^{iN\omega\Delta T}\right)/\left(1-e^{i\omega\Delta T}\right)\right|^2 = \frac{1}{N}\left(\frac{\mathrm{Sin}(N\omega\Delta T/2)}{\mathrm{Sin}(\omega\Delta T/2)}\right)^2$$

Notice that $|H(\omega)|^2$ is periodic with period equal to the sampling frequency $2\pi/\Delta T$ so that using the usual old trick (see lecture on sampling theorem)

$$\langle S_k \rangle = (\Delta T/2\pi)\sum_{m=-\infty}^{\infty}\int_{-\pi/\Delta T}^{\pi/\Delta T} S_{xx}(\omega + m\,2\pi/\Delta T)\left|H(\omega - k\,2\pi/T)\right|^2 d\omega$$

$$\langle S_k \rangle = (\Delta T/2\pi)\int_{-\pi/\Delta T}^{\pi/\Delta T}\left|H(\omega - k\,2\pi/T)\right|^2\left[\sum_{m=-\infty}^{\infty} S_{xx}(\omega + m\,2\pi/\Delta T)\right]d\omega$$

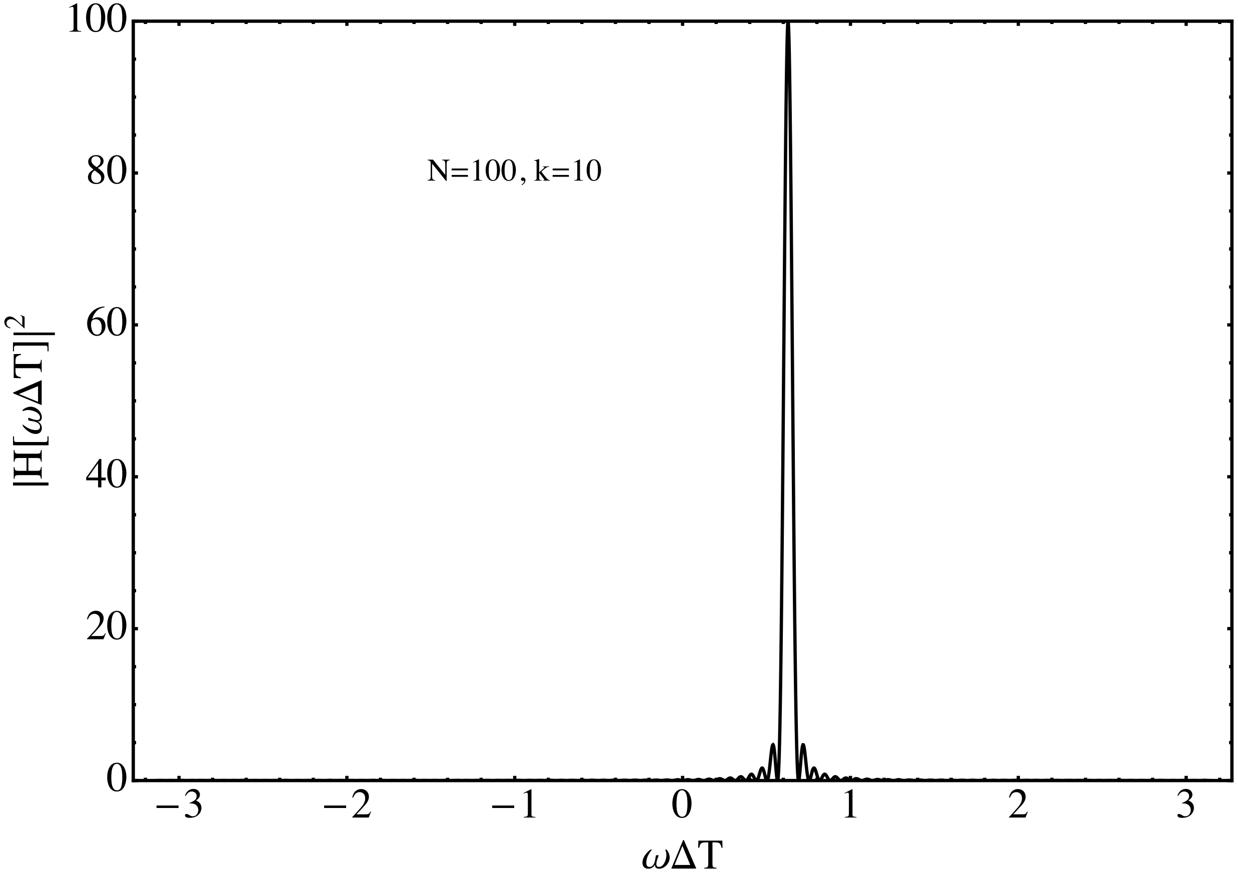
Thus, as rather immediate, DFT cannot help against aliasing, and if data have not been sampled at a sufficient high rate, S_k is contributed also by PSD aliases outside the frequency interval $\pm\pi/\Delta T$. In order for this not happening

$$S_{xx}(\omega) \simeq 0 \quad \text{for} \quad |\omega| \geq \pi/T$$

If sampling rate is adequate then

$$\left\langle S_k \right\rangle = \frac{\Delta T}{2\pi} \int_{-\frac{\pi}{\Delta T}}^{\frac{\pi}{\Delta T}} \frac{1}{N} \left(\frac{\mathrm{Sin}\left(\left(\omega - k\, 2\pi/T\right) N\Delta T/2\right)}{\mathrm{Sin}\left(\left(\omega - k\, 2\pi/T\right) \Delta T/2\right)} \right)^2 S_{xx}(\omega)\, d\omega$$

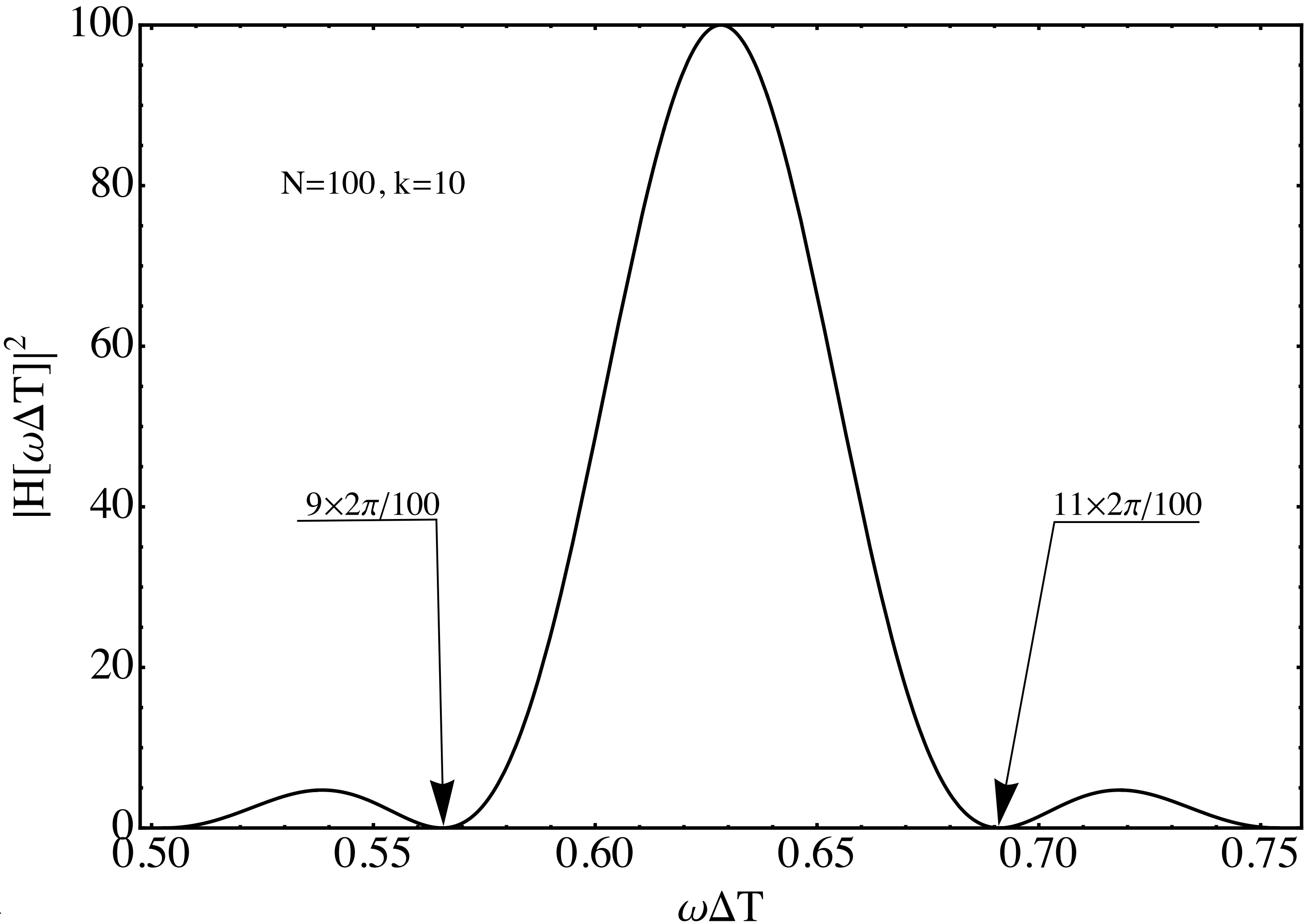
A plot of one example: a line at f=k/T with side lobes at all frequencies



Digital estimate of PSD and Discrete Fourier Transform

A blow-up, the first lobe is zero for: $\omega \in (k \pm 1)(2\pi/N\Delta T) = (k \pm 1)(2\pi/T)$

Spectral resolution is $\pm 2\pi/T$!



Digital estimate of PSD and Discrete Fourier Transform

You can check that, whatever are N and k<N

$$\frac{\Delta T}{2\pi}\int_{-\frac{\pi}{\Delta T}}^{\frac{\pi}{\Delta T}}\frac{1}{N}\left(\frac{\mathrm{Sin}\left(\left(\omega-k\,2\pi/T\right)N\Delta T/2\right)}{\mathrm{Sin}\left(\left(\omega-k\,2\pi/T\right)\Delta T/2\right)}\right)^2 d\omega=1$$

so that, if $S_{xx}(\omega)$ is approximately constant around $\omega=k2\pi/T$ then

$$\left\langle S_k\right\rangle=\frac{\Delta T}{2\pi}\int_{-\frac{\pi}{\Delta T}}^{\frac{\pi}{\Delta T}}\frac{1}{N}\left(\frac{\mathrm{Sin}\left(\left(\omega-k\,2\pi/T\right)N\Delta T/2\right)}{\mathrm{Sin}\left(\left(\omega-k\,2\pi/T\right)\Delta T/2\right)}\right)^2 S_{xx}\left(\omega\right)d\omega$$

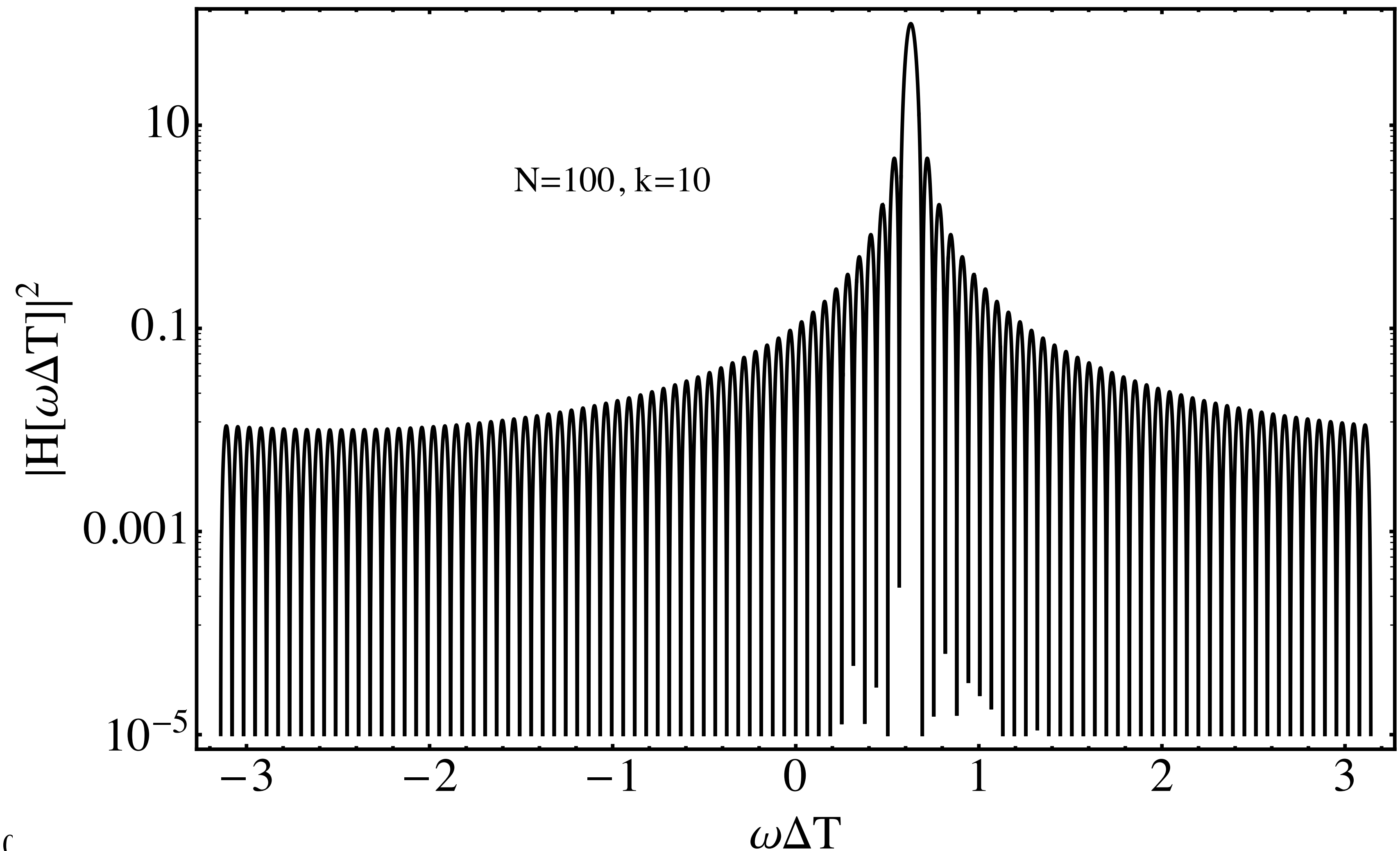
$$\simeq S_{xx}\left(k\,2\pi/T\right)\frac{\Delta T}{2\pi}\int_{-\frac{\pi}{\Delta T}}^{\frac{\pi}{\Delta T}}\frac{1}{N}\left(\frac{\mathrm{Sin}\left(\left(\omega-k\,2\pi/T\right)N\Delta T/2\right)}{\mathrm{Sin}\left(\left(\omega-k\,2\pi/T\right)\Delta T/2\right)}\right)^2 d\omega$$

$$=S_{xx}\left(k\,2\pi/T\right)$$

This a bit optimistic as it neglects the role of side lobes. See next page

Digital estimate of PSD and Discrete Fourier Transform

A better representation of side lobes. Amplitude decays very slowly. Each coefficient of the spectral estimate is also partly contributed by the power within the side lobes.



Digital estimate of PSD and Discrete Fourier Transform

Thus for correct sampling

$$\langle S_k \rangle = \frac{\Delta T}{2\pi} \int_{-\frac{\pi}{\Delta T}}^{\frac{\pi}{\Delta T}} \frac{1}{N} \left(\frac{\mathrm{Sin}\left(\left(\omega - k\,2\pi/T\right) N\Delta T/2\right)}{\mathrm{Sin}\left(\left(\omega - k\,2\pi/T\right)\Delta T/2\right)} \right)^2 S_{xx}(\omega)\, d\omega$$

is a spectral estimator of $S_{xx}(k2\pi/T)$.

This spectral estimator picks also the contribution of many side lobes.

We will come back to this later.

We want now to assess the uncertainty of this estimate. Let's define

$$\tilde{x}[k] = \left(\sqrt{\Delta T}/\sqrt{N}\right) \sum_{n=0}^{N-1} x[n] e^{-ikn(2\pi/N)}$$

so that $S_k = \left|\tilde{x}[k]\right|^2 = \mathrm{Re}^2\left\{\tilde{x}[k]\right\} + \mathrm{Im}^2\left\{\tilde{x}[k]\right\}$

Note that

$$\mathrm{Re}\left\{\tilde{x}[k]\right\} = \left(\sqrt{\Delta T}/\sqrt{N}\right) \sum_{n=0}^{N-1} x[n] \mathrm{Cos}\left(kn\left(2\pi/N\right)\right)$$

$$\mathrm{Im}\left\{\tilde{x}[k]\right\} = \left(\sqrt{\Delta T}/\sqrt{N}\right) \sum_{n=0}^{N-1} x[n] \mathrm{Sin}\left(kn\left(2\pi/N\right)\right)$$

Digital estimate of PSD and Discrete Fourier Transform

As
$$\mathrm{Re}\left\{\tilde{x}\left[k\right]\right\}=\left(\sqrt{\Delta T}/\sqrt{N}\right)\sum_{n=0}^{N-1}x\left[n\right]\mathrm{Cos}\left(kn\left(2\pi/N\right)\right)$$
$$\mathrm{Im}\left\{\tilde{x}\left[k\right]\right\}=\left(\sqrt{\Delta T}/\sqrt{N}\right)\sum_{n=0}^{N-1}x\left[n\right]\mathrm{Sin}\left(kn\left(2\pi/N\right)\right)$$
if x(t) is Gaussian then also the two variables above are Gaussian. If the samples x[n] are also uncorrelated, then the variables would be independent and
$$S_k/S_{xx}\left(k2\pi/T\right)=\left(\mathrm{Re}^2\left\{\tilde{x}\left[k\right]\right\}+\mathrm{Im}^2\left\{\tilde{x}\left[k\right]\right\}\right)/S_{xx}\left(k2\pi/T\right)$$
would be distributed as a reduced chi-square with 2 degrees of freedom. One can show that this is always the case, even if the x[n] are not independent. Thus, recalling the properties of chi-square

$$0.7S_k \le S_{xx}\left(k\,\frac{2\pi}{T}\right) \le 2.4\,\mathrm{S_k}$$

This is a very imprecise estimate, not at all surprising as the spectral resolution is ±1/T and the duration of the measurement is T so that the radiometric formula would give a 100% error.

Digital estimate of PSD and Discrete Fourier Transform

- As the DFT of x[n] is such that $x_k = x^*_{N-k}$
- It follows that $S_k = \left(\Delta T/N\right)\left|x_k\right|^2 = S_{N-k}$
- Thus only the first ≃N/2 coefficients have independent meanings, while the rest is just a specular copy of these.
- That is, the DFT spectral estimate ranges from zero frequency up to half the sampling frequency: $0 \leq f \leq 1/2\Delta T$ with a resolution of $\Delta f = 1/T$

In conclusion $S_k = \left(\Delta T/N\right)\sum_{n=0}^{N-1}\sum_{j=0}^{N-1} x\left[n\right]x\left[j\right]e^{-ik(n-j)(2\pi/N)}$

is indeed an estimator for $S_{xx}(\omega = k\,2\pi/T)$ with two problems:

1. The relative precision is low, worse than 100%
2. The accuracy is poor due to the “leakage” from side lobes

Here follows a numerical exercise