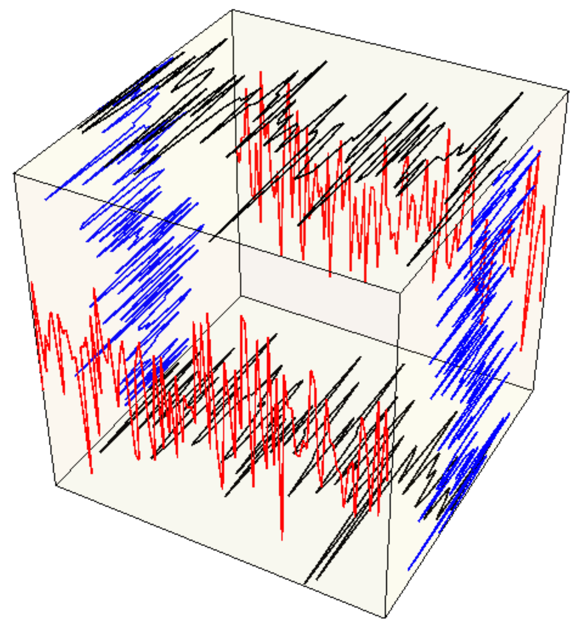
Experimental Methods
Lecture 14

October 21st, 2020

A stochastic process

- At each outcome of an experiment I get a function of time x(t)

dice



Stochastic dice

Moments of a stochastic process

- Key moments of a stochastic processes are
  - Mean value

$$\eta(t) =< x(t) > = \int_{-\infty}^{\infty} \chi f_{x(t)}(\chi) d\chi$$

  - Autocorrelation

$$R(t,t') =< x(t)x(t') > \int_{-\infty}^{\infty} \chi\psi f_{x(t)x(t')}(\chi\psi) d\chi d\psi$$

  - Auto-covariance

$$C(t,t') =< (x(t) - \eta(t))(x(t') - \eta(t')) \geq R(t,t') - \eta(t)\eta(t')$$

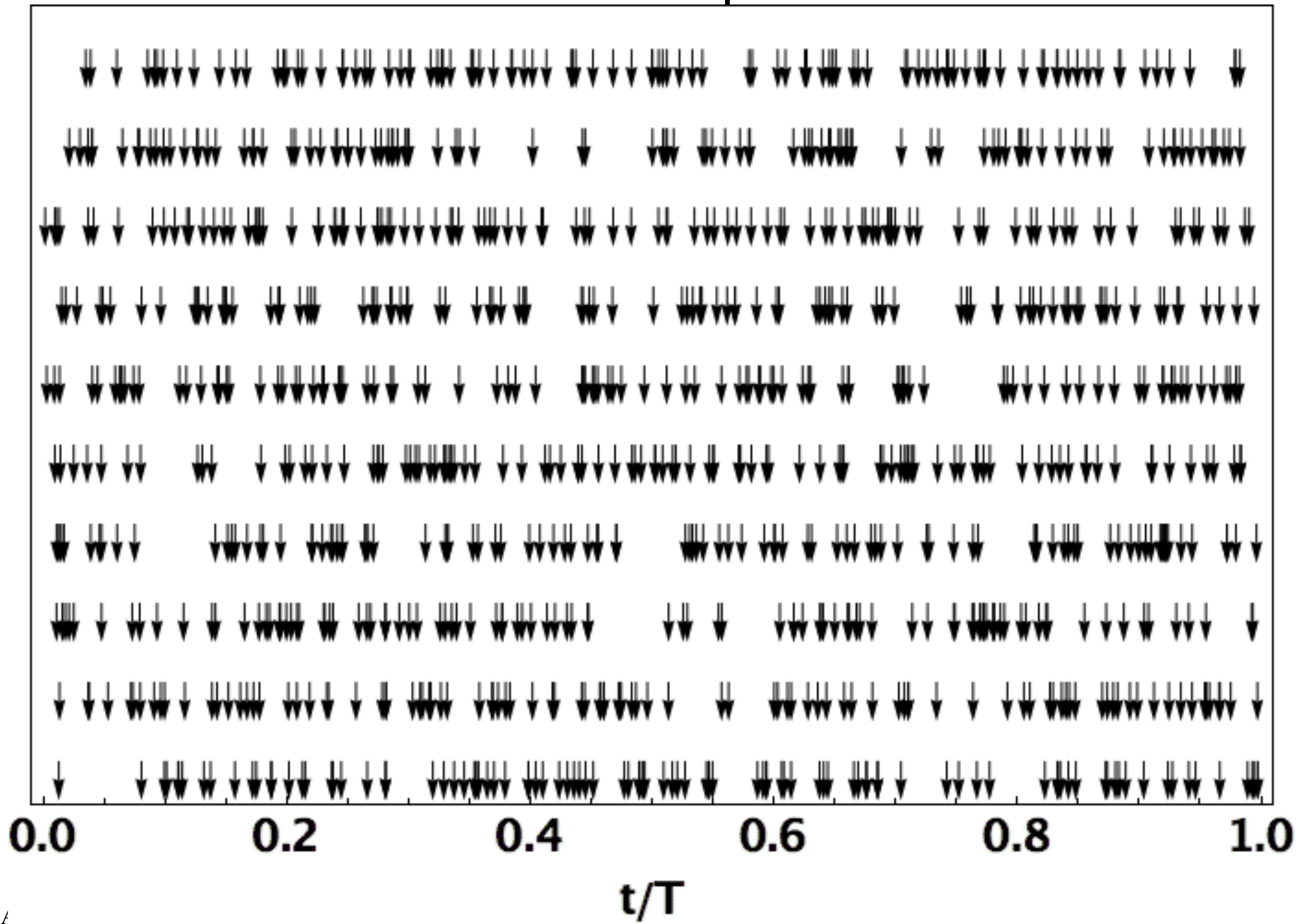
  - Which is the autocorrelation of the "noisy part" of $x(t)$

$$n(t) = x(t) - \eta(t)$$

Poisson process and shot noise

- A stochastic process based model, for all phenomena where carriers arrive at random at a collector/detector:
  - Photons in a laser beam;
  - Charge carriers across a junction;
  - Particles from a radioactive source;

Numerical example N=100



k: number of points between t_1 and t_2

- k has probability density function:

$$f_k(x) = e^{-\lambda(t_2-t_1)} \sum_{k=0}^{\infty} \frac{(\lambda(t_2-t_1))^k}{k!} \delta(x-k)$$

- With mean value

$$< k > = \lambda(t_2 - t_1)$$

- And variance

$$\sigma_k^2 = \lambda(t_2 - t_1)$$

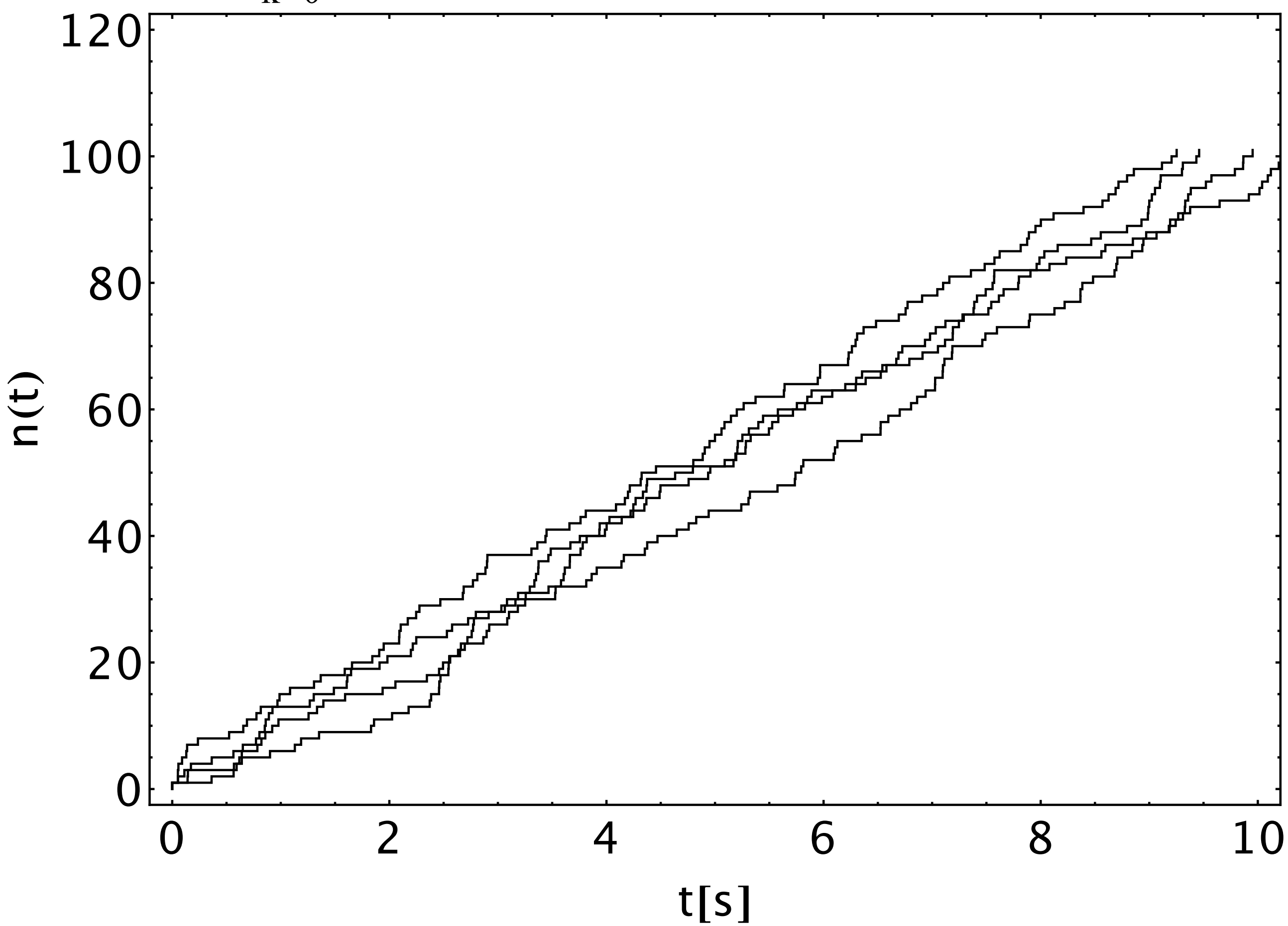
- If $t_1 \leq t_2 < t_3 \leq t_4$, k_1 is the number of points between t_1 and t_2, and k_2 is the number of points between t_3 and t_4, then **$\boldsymbol{k_1}$ and $\boldsymbol{k_2}$ are independent**

The Poisson stochastic process

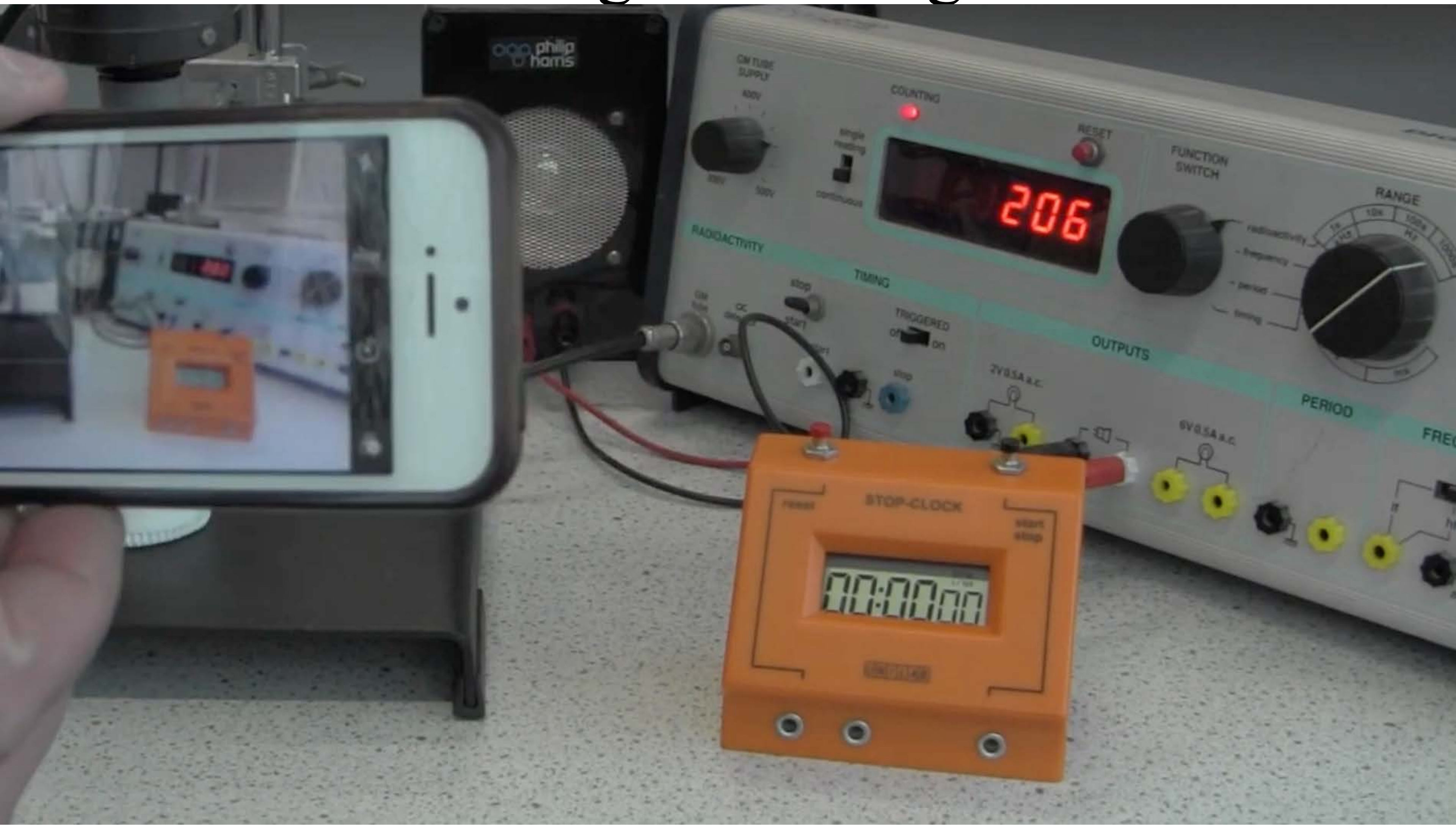
- Assume that the time instants above represent the time of arrival of some carrier at a given point. We define the process n(t) as the number of carriers that have already arrived at time t:

$$n(t) = \sum_{k=0}^{\infty} \Theta(t - t_k)$$

Example with $\lambda = 10\ s^{-1}$. Different lines corresponds to different outcomes of the experiment



Geiger counting



The Poisson stochastic process

- The process
$$n(t)=\sum_{k=0}^{\infty}\Theta(t-t_k)$$
- As the $t_k>0$, $n(0)=0$
- Furthermore, as
$$P\left(n, t\in\left[0,\tilde{t}\right]\right)=e^{-\lambda\tilde{t}}\frac{\left(\lambda\tilde{t}\right)^n}{n!}$$
- The probability density function of n(t) is
$$f_{n(t)}(x)=\sum_{k=0}^{\infty}e^{-\lambda t}\frac{(\lambda t)^k}{k!}\delta(x-k)$$
- It follows (see properties of Poisson distribution) that the mean value is
$$\langle n(t)\rangle=\lambda t$$
- Variance is
$$\sigma^2_{n(t)}=\lambda t$$
- And standard deviation
$$\sigma_{n(t)}=\sqrt{\lambda t}$$

The Poisson stochastic process

- From the Poisson process one can form the purely random process

$$\tilde{n}(t) = \sum_{k=0}^{\infty} \Theta(t - t_k) - \lambda t$$

- The mean value of which is

$$\langle \tilde{n}(t) \rangle = \langle n(t) - \lambda t \rangle = \langle n(t) \rangle - \lambda t = \lambda t - \lambda t = 0$$

- While the standard deviation is equal to that of n(t)

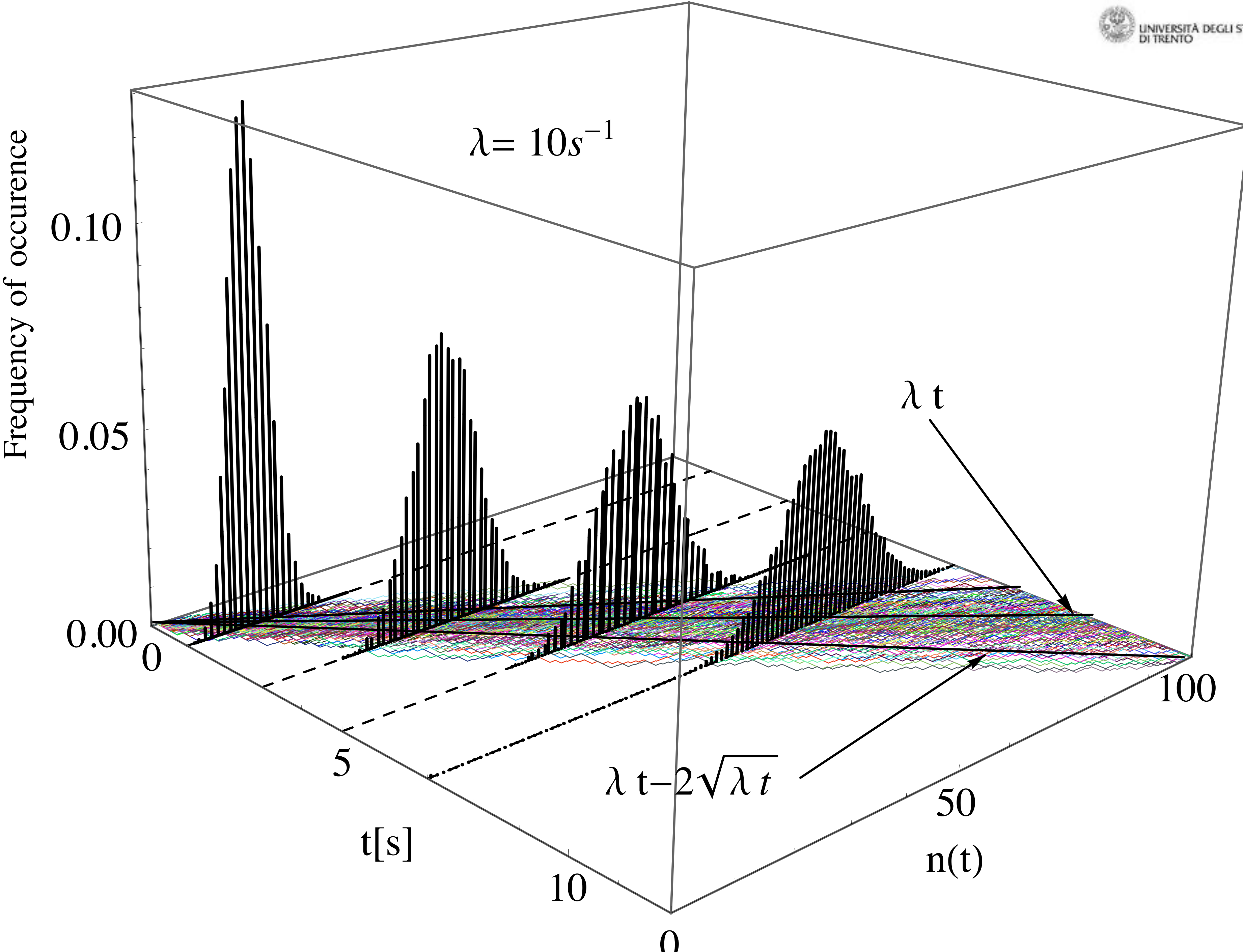
$$\sigma_{\tilde{n}(t)} = \sigma_{n(t)} = \sqrt{\lambda t}$$

- Thus the Poisson process can be written as the sum of the deterministic signal λt and the noisy part $\tilde{n}(t)$

$$n(t) = \tilde{n}(t) + \lambda t$$

Numerical example

- In the next slide a numerical example with $\lambda=10\ s^{-1}$
- In the $t \div n(t)$ plane you see samples of the process for $0 \leq t \leq 12$ s
- Also reported are the histograms of $n(t)$ as a function of t for t=1,3,5,7 s
- Histograms are plotted as lines whose height is proportional to frequency of occurrence, i.e. counts/n. of repetitions



Frequency of occurence
0.10
0.05
0.00
λ= 10s^−1
λ t
λ t−2√λ t
0
5
10
t[s]
0
50
100
n(t)

Simulating Poisson noise, a useful exercise

- Suppose one of the Poisson points has arrived at time t, what is the probability density function of the time Δt we have to wait till the next arrival?
- Remember that arrival times of different points are independent! Let's calculate the probability that $\Delta t \le x$ that is:

$$F(x) = 1 - P\{k = 0, t \in [t, t + x]\}$$

- From Poisson statistics

$$F(x) = \left(1 - e^{-\lambda x}\right)\Theta(x)$$

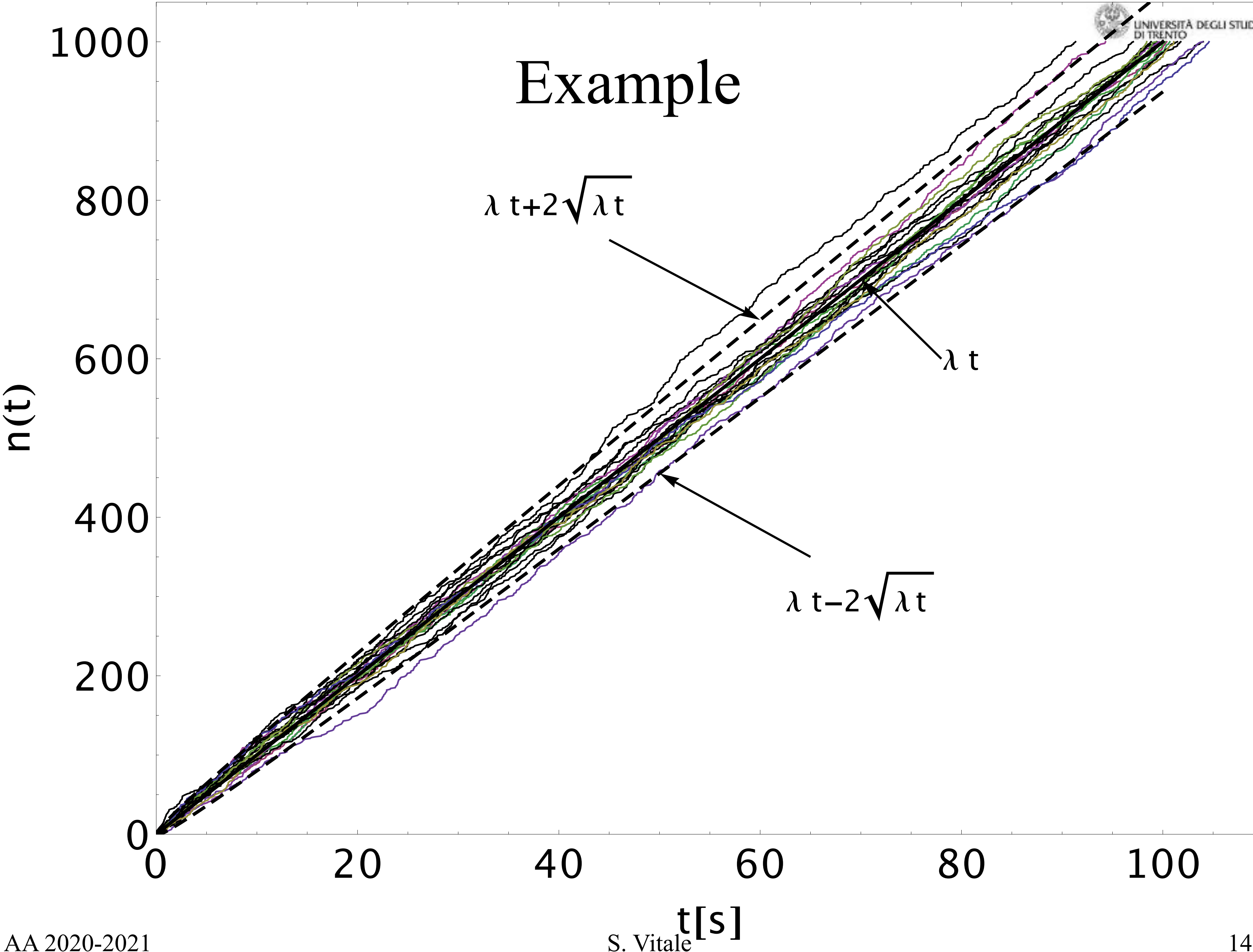
- But by definition this is the cumulative distribution function of Δt. Thus its probability density is

$$f_{\Delta t}(x) = dF(x)/dx = \lambda e^{-\lambda x}\Theta(x)$$

- If one can generate N random numbers with the above density Δt_1, Δt_2, Δt_3, Δt_4, …, then he gets Poisson events at times

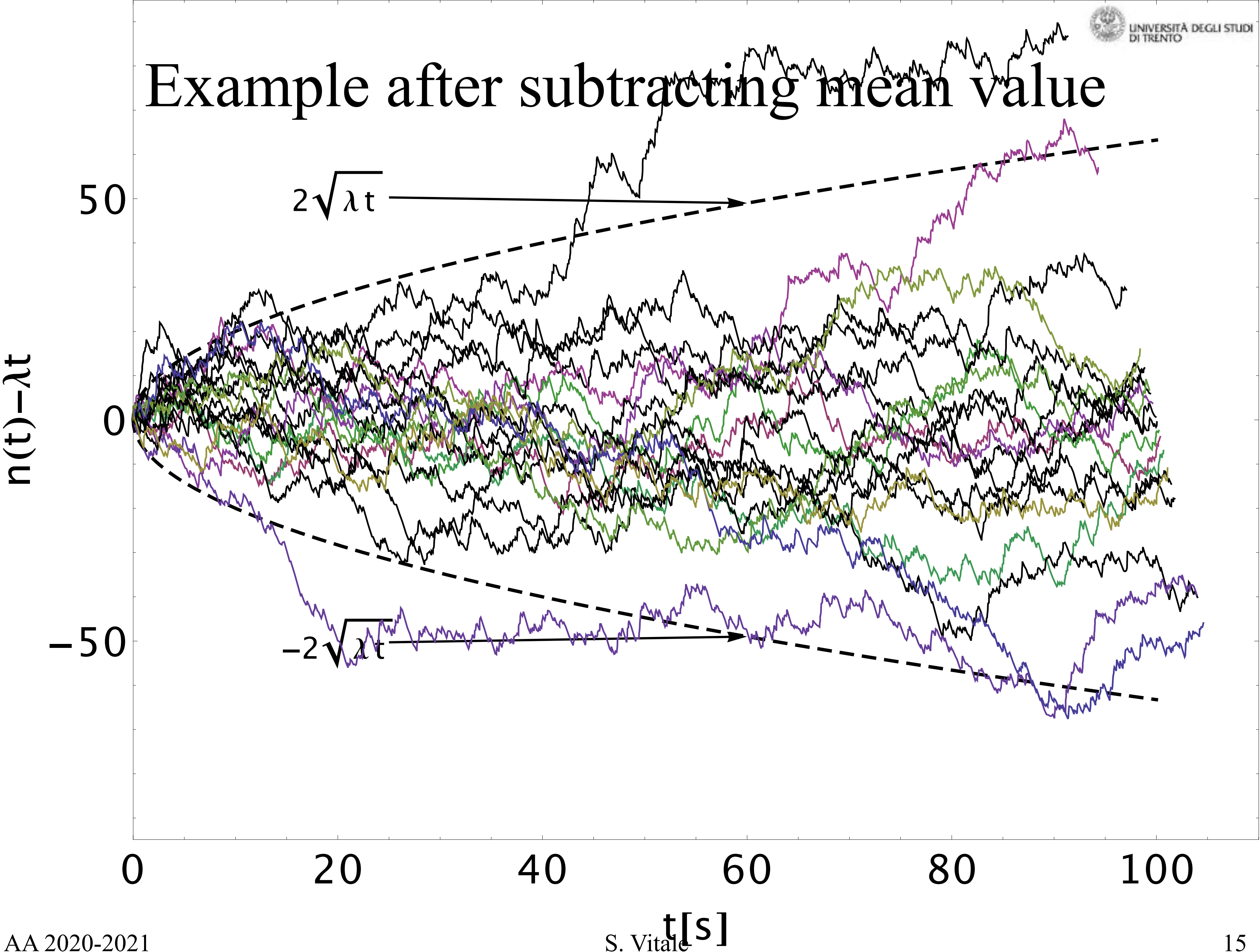
$$t_1 = \Delta t_1; \; t_2 = \Delta t_1 + \Delta t_2; \; t_3 = \Delta t_1 + \Delta t_2 + \Delta t_3$$

Example



1000
800
600
400
200
0
n(t)
0
20
40
60
80
100
t[s]
$\lambda\,t+2\sqrt{\lambda\,t}$
$\lambda\,t$
$\lambda\,t-2\sqrt{\lambda\,t}$

Example after subtracting mean value



Autocorrelation of Poisson process

- Let's calculate the autocorrelation of the Poisson process

$$R_{n,n}(t,t') = \langle n(t)n(t') \rangle = R(t',t)$$

- Consider the non-overlapping intervals [0,t] and [t,t'] with t'>t

$$\overbrace{\qquad\qquad\qquad\qquad\qquad\qquad}^{[0,t]}\overbrace{\qquad\qquad\qquad}^{[t,t']}$$

- The number of points in these intervals, n(t) and n(t)-n(t'), are independent random variables. Thus

$$\langle n(t)[n(t')-n(t)] \rangle = \langle n(t) \rangle \langle n(t')-n(t) \rangle$$

- By using Poisson formulas, and expanding the mean value:

$$\langle n(t)n(t') \rangle = \lambda t \lambda (t'-t) + \langle n(t)n(t) \rangle$$

- Again from Poisson formulas

$$R_{n,n}(t,t') = \langle n(t)n(t') \rangle = \lambda^2 t(t'-t) + \lambda t + \lambda^2 t^2 = \lambda^2 tt' + \lambda t$$

Autocorrelation of Poisson process

- Autocorrelation of the Poisson process (t<t')

$$R_{n,n}\left(t,t'\right)=\lambda^2 tt'+\lambda t$$

- As $$n\left(t\right)=\tilde{n}\left(t\right)+\lambda t$$

- The autocorrelation of the random part $\tilde{n}(t)$ is equal to the auto-covariance of n(t)

$$R_{\tilde{n},\tilde{n}}\left(t,t'\right)=C_{n,n}\left(t,t'\right)=R_{n,n}\left(t,t'\right)-\left\langle n\left(t\right)\right\rangle\left\langle n\left(t'\right)\right\rangle=\lambda^2 tt'+\lambda t-\lambda^2 tt'=\lambda t$$

- Here remember that t≤t' thus one can rewrite the above as

$$C_{n,n}\left(t,t'\right)==\lambda\left[t\,\Theta\left(t'-t\right)+t'\Theta\left(t-t'\right)\right]$$

- With $\Theta(0)=1/2$

- Notice: for any random noise $C(t,t)=\sigma^2(t)$

Examples of Physical Poisson processes

- Enegy carried by a laser beam E(t)=ħω n(t)
  - Mean value $\langle E(t)\rangle = \langle \hbar\omega\, n(t)\rangle = \hbar\omega \langle n(t)\rangle = \hbar\omega\lambda t$
  - Standard deviation $\sigma_{E(t)} = \hbar\omega\, \sigma_{n(t)} = \hbar\omega\sqrt{\lambda t}$
  - Autocorrelation of noisy part

$$R_{\tilde{E},\tilde{E}}(t,t') = \langle \hbar\omega\tilde{n}(t)\, \hbar\omega\tilde{n}(t')\rangle = (\hbar\omega)^2 R_{\tilde{n},\tilde{n}}(t,t') =$$

$$= (\hbar\omega)^2 \lambda\left[t\,\Theta(t'-t) + t'\Theta(t-t')\right]$$

- Charge on a junction Q(t)=*e*n(t)
  - Mean value $\langle Q(t)\rangle = e\lambda t$
  - Standard deviation $\sigma_{Q(t)} = e\sqrt{\lambda t}$
  - Autocorrelation of noisy part

$$R_{\tilde{Q},\tilde{Q}}(t,t') = e^2\lambda\left[t\,\Theta(t'-t) + t'\Theta(t-t')\right]$$

Types of stochastic processes: 1) stationary process

- A process is stationary if all its statistical properties are not affected by a translation of the time origin, that is, if for any N and T

$$f_{x(t_1+T)\,x(t_2+T)\ldots x(t_N+T)}\left(\chi_1,\chi_2\ldots\chi_N\right)=f_{x(t_1)\,x(t_2)\ldots x(t_N)}\left(\chi_1,\chi_2\ldots\chi_N\right)$$

- Immediate consequences: first order density is independent of time

$$f_{x(t+T)}\left(\chi\right)=f_{x(t)}\left(\chi\right)=f_x\left(\chi\right)$$

- Mean value

$$\eta\left(t\right)=\left\langle x\left(t\right)\right\rangle=\int_{-\infty}^{\infty}\chi f_x\left(\chi\right)d\chi=\text{Constant}\equiv\eta$$

- Same with standard deviation
- Two points density $f_{x(t+T)\,x(t+T+\Delta t)}\left(\chi_1,\chi_2\right)=f_{x(t)\,x(t+\Delta t)}\left(\chi_1,\chi_2\right)$

  may only depend on Δt
- Autocorrelation $R_{x,x}\left(t,t+\Delta t\right)=\left\langle x\left(t\right)x\left(t+\Delta t\right)\right\rangle=R_{x,x}\left(\Delta t\right)$
- Auto-covariance $C_{x,x}\left(t,t+\Delta t\right)=R_{x,x}\left(\Delta t\right)-\eta_o^2=C_{x,x}\left(\Delta t\right)$

stationary process

- For a stationary process

$$\left\langle x(t) \right\rangle = \eta \qquad\qquad \left\langle x(t)\, x(t+\Delta t) \right\rangle = R_{x,x}(\Delta t)$$

- The converse is not true. If the above holds the process is not necessarily stationary
- A process that obeys just the above is called “wide-sense” stationary

Types of stochastic processes: 2) normal process

- A process is called normal if for any N the joint probability densities of the samples of the process at any t_1, t_2, …..t_N is joint normal

$$f_{x(t_1),x(t_2),\ldots x(t_N)}\left(\chi_1,\chi_2,\ldots.\chi_N\right)=\frac{\sqrt{\|\mu\|}}{\left(2\pi\right)^{\frac{N}{2}}}e^{-\frac{1}{2}\sum_{i,j=1}^{N}\mu_{i,j}(\chi_i-\eta_i)(\chi_j-\eta_j)}$$

- with $\mu_{i,j}$ a positive definite matrix and η_i a real number. One can calculate that

$$\langle x(t_i)\rangle=\eta_i$$

- And that $\left(\mu^{-1}\right)_{i,j}=C\left(t_i,t_j\right)=\left\langle\left[x\left(t_i\right)-\eta_i\right]\left[x\left(t_j\right)-\eta_j\right]\right\rangle$

- Thus, for normal processes the entire information is contained within $\eta(t)$ and $C(t,t')$. All other moments may be derived from these functions

Types of stochastic processes: 2) normal process

- For a normal *and* stationary process the following holds:

$$\langle x(t)\rangle = \text{Constant} \equiv \eta \qquad R(t,t') = R(t'-t) \qquad C(t,t') = R(t'-t) - \eta^2$$

- Examples of distributions:
  - Single point (N=1)

$$\mu^{-1} = C(0) \equiv \sigma^2 \qquad \|\mu\| = \sigma^{-2} \qquad f_x(\chi) = \left(1/\sqrt{2\pi\sigma^2}\right) e^{-\frac{1}{2}\frac{(\chi-\eta)^2}{\sigma^2}}$$

  - Two points (N=2)

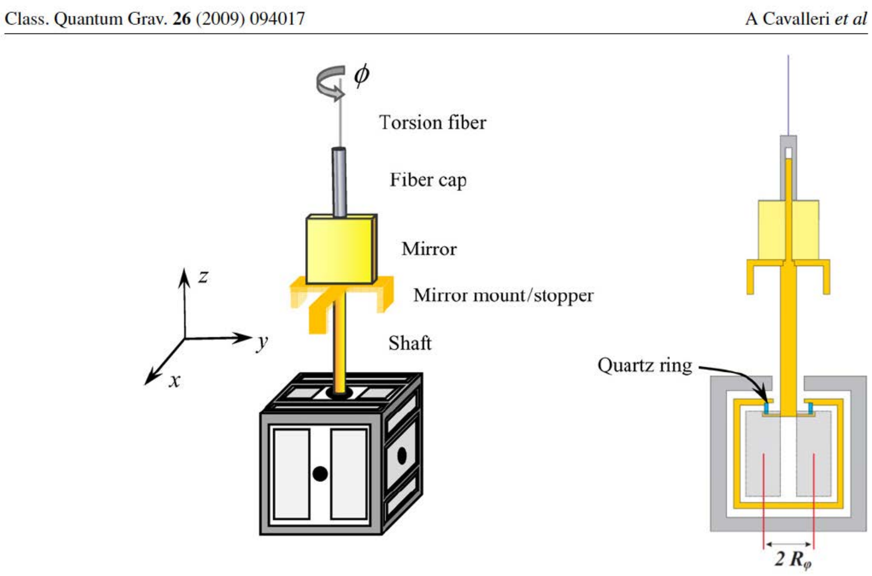
$$\mu^{-1} = \begin{Bmatrix} C(0) & C(\Delta t) \\ C(\Delta t) & C(0) \end{Bmatrix} \qquad \|\mu\| = \frac{1}{C(0)^2 - C(\Delta t)^2}$$

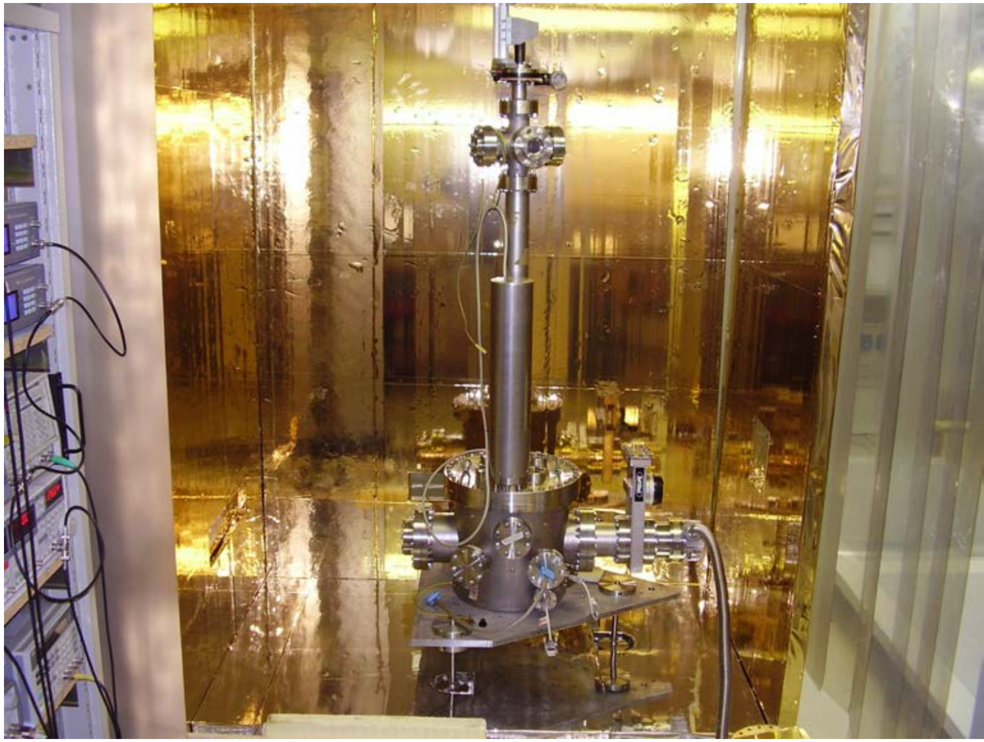
  - then

$$f_{x(t)x(t+\Delta t)}(\chi,\psi) = \frac{e^{-\frac{1}{2}\frac{1}{C(0)^2 - C(\Delta t)^2}\left[C(0)(\chi-\eta)^2 - 2C(\Delta t)(\chi-\eta)(\psi-\eta) + C(0)(\psi-\eta)^2\right]}}{2\pi\sqrt{C(0)^2 - C(\Delta t)^2}}$$

Normal processes

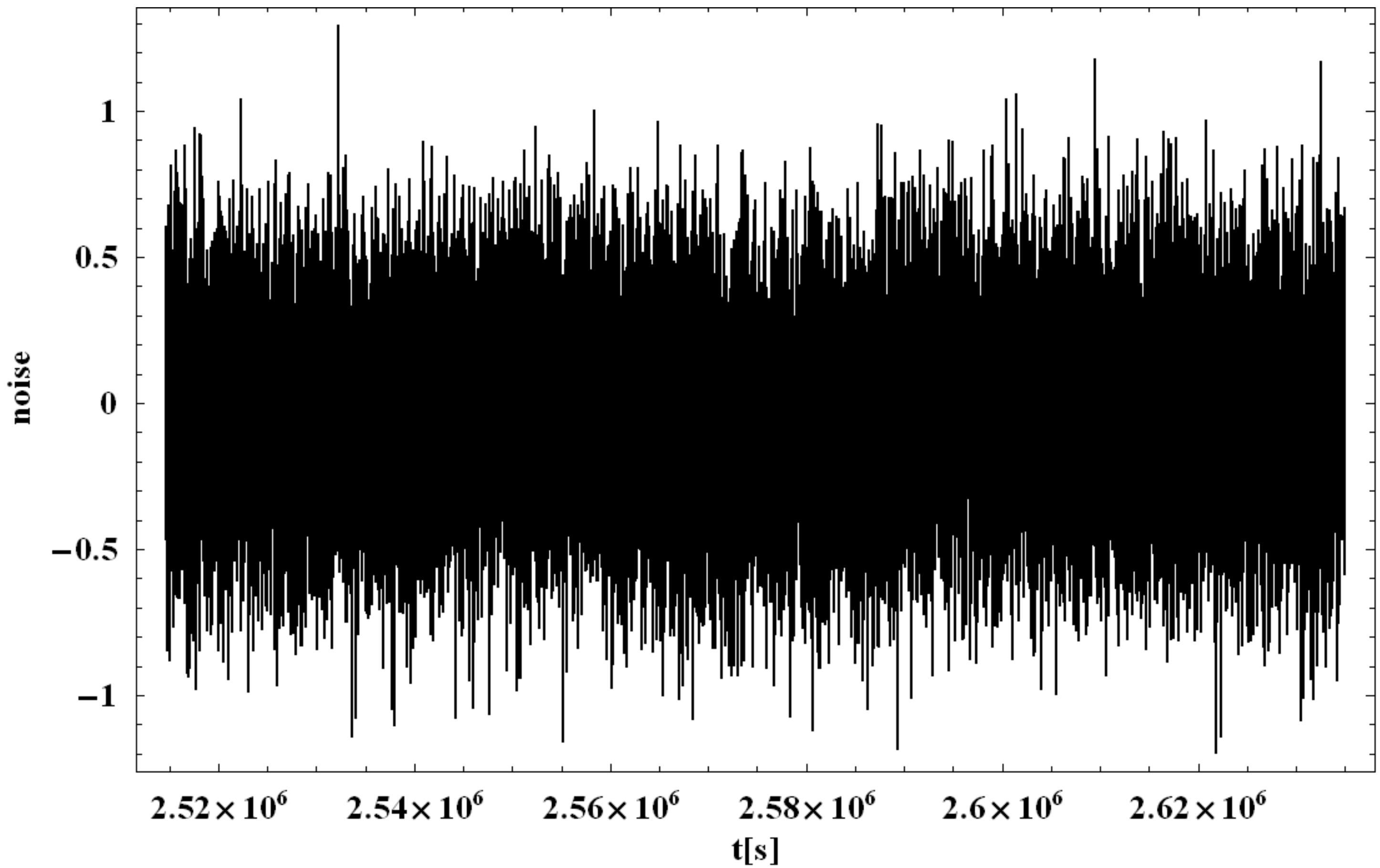
- Normal processes are expected in ordinary random noise due to central limit theorem.
- Large rare fluctuations may often appear beyond a few sigma
- Example 1: noise in torsion pendulum



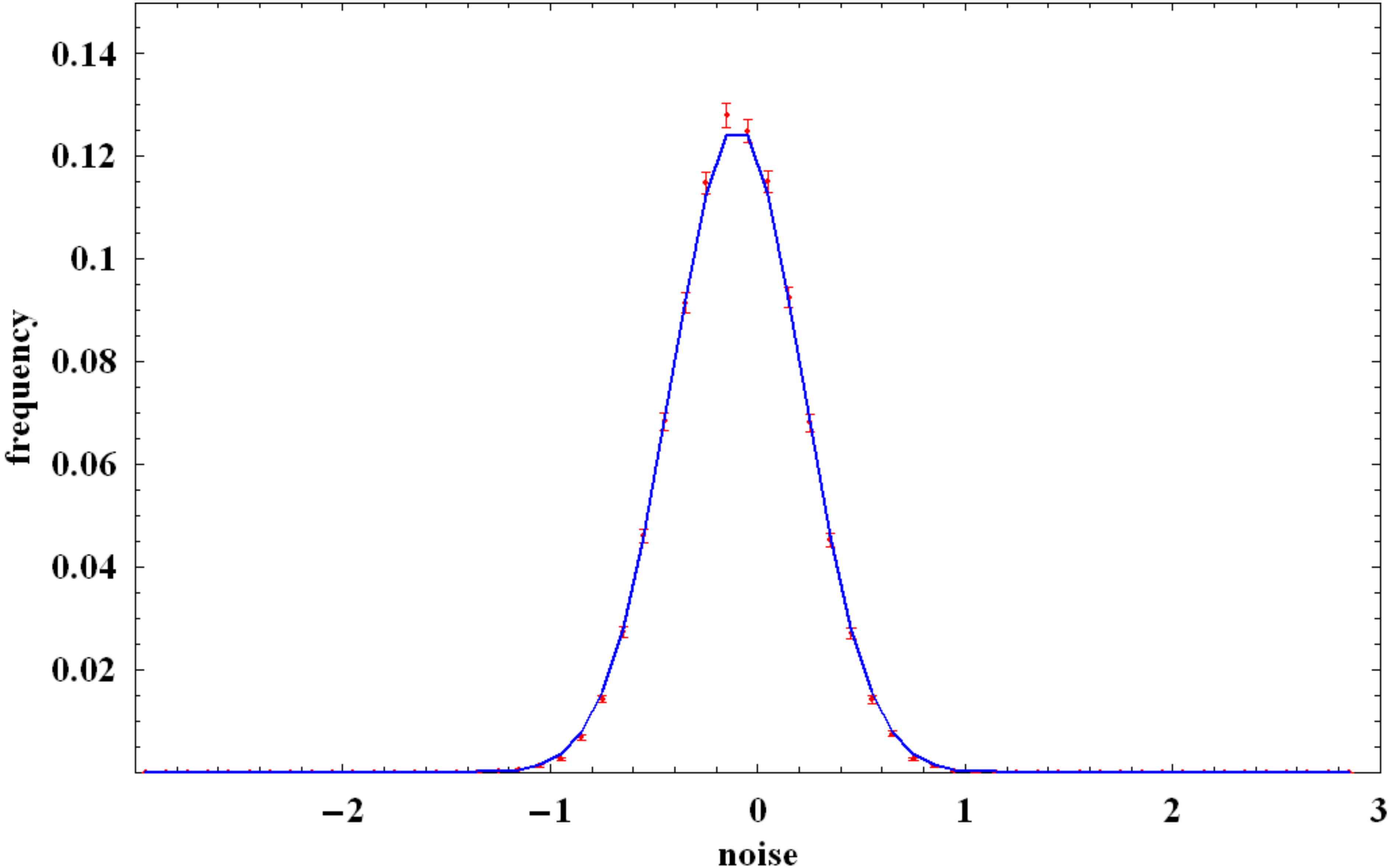


A Gaussian example

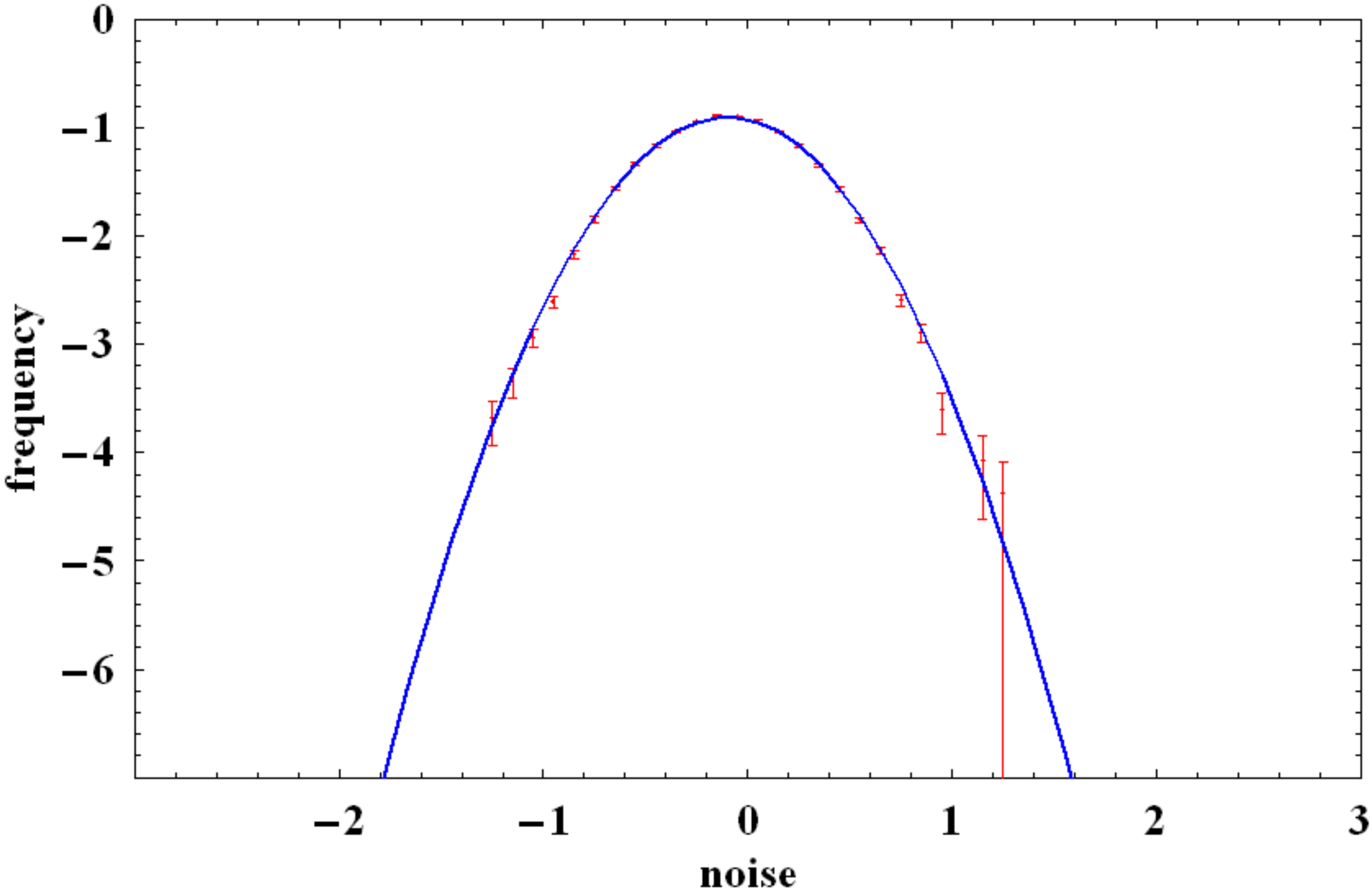
- Data are filtered to suppress autocorrelation (to be discussed later)
- Data are then independent random variables (if Gaussian)
- Data are normalized to have unit standard deviation



Histogram of data: an estimate of single point probability density.
Blue: theory from zero mean, unit standard deviation, normal distribution .

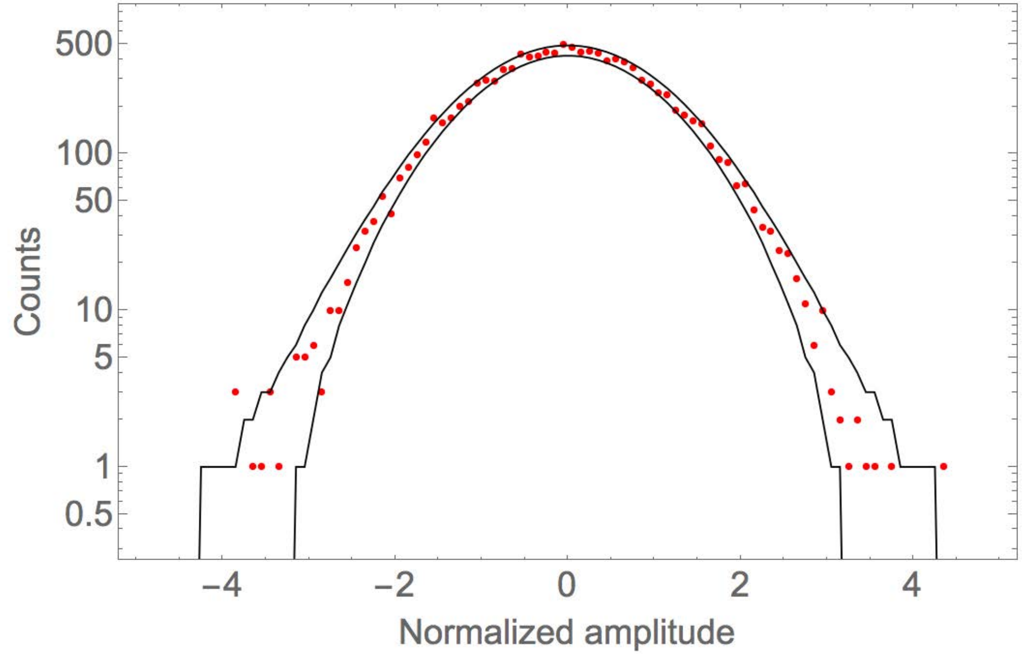


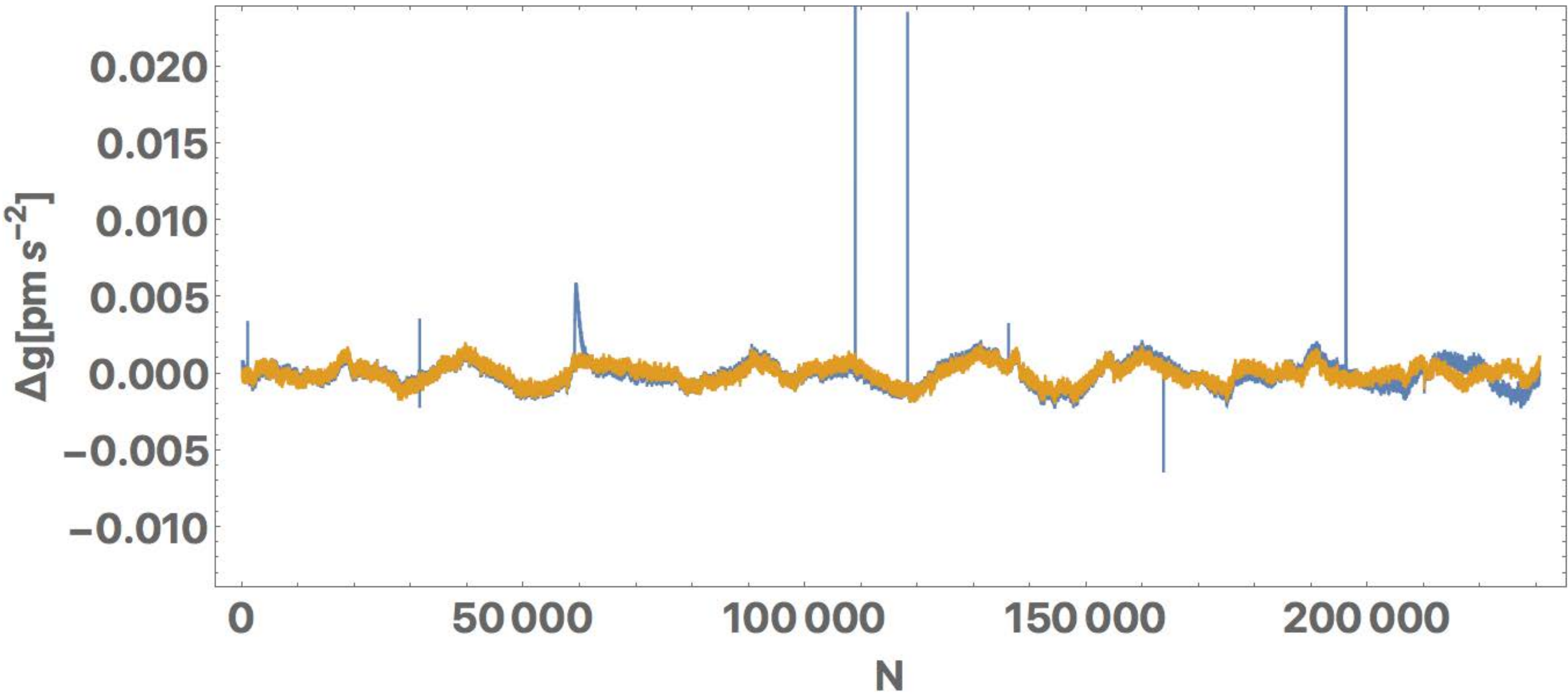
Same in log scale. Zero-count bins are omitted



LISA Pathfinder

- Histogram of data (after making them independent). Red: Histogram. Black lines: expected fluctuation of counting statistics
- But: spurious “events” plague the data. May be considered as signal



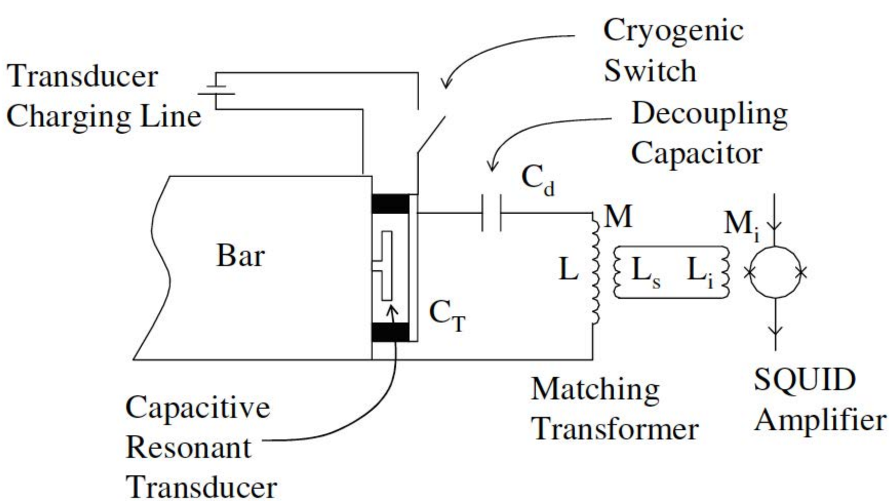


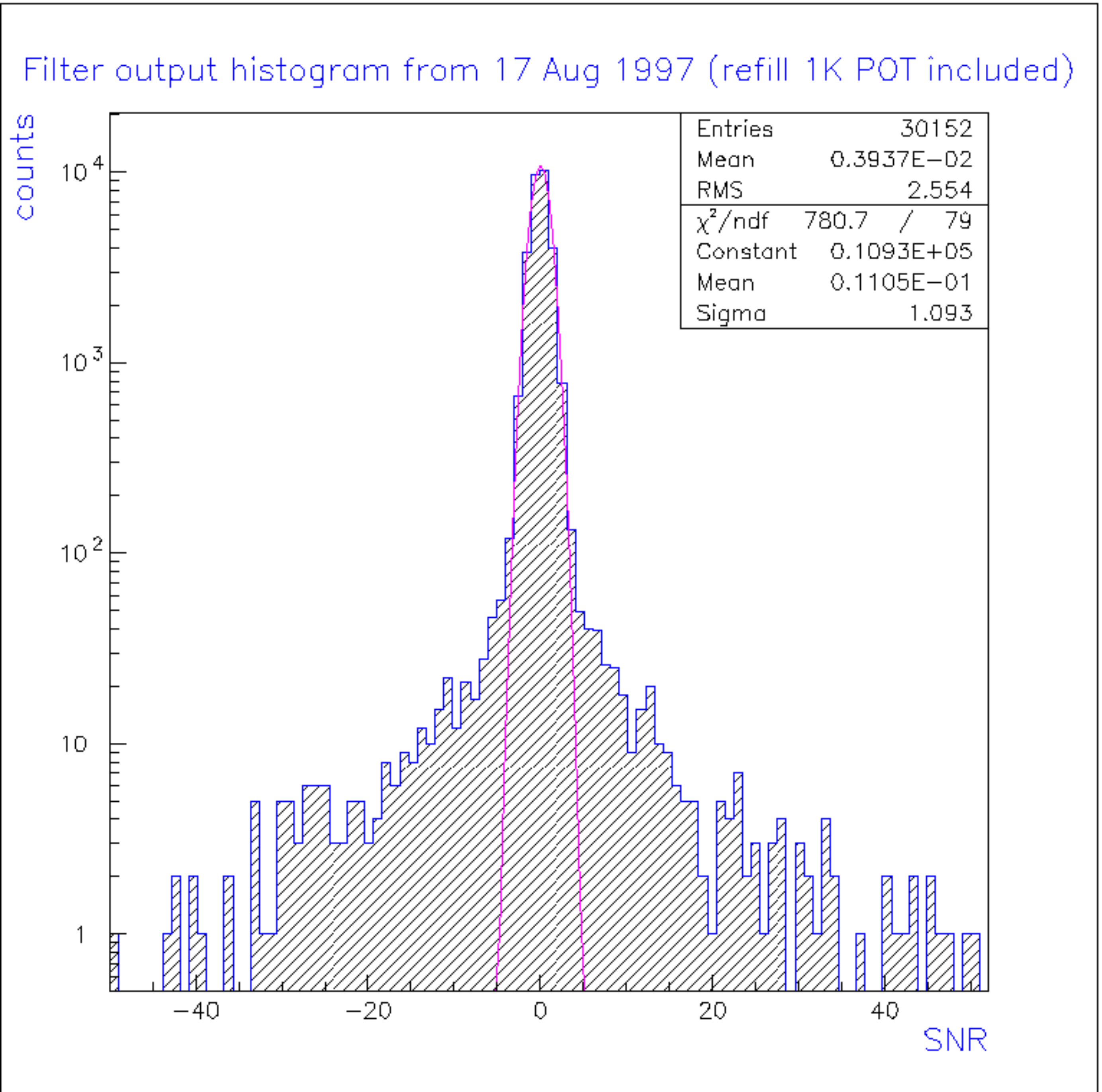
A non-Gaussian example

PRL **94,** 241101 (2005) PHYSICAL REVIEW LETTERS week ending 24 JUNE 2005

3-Mode Detection for Widening the Bandwidth of Resonant Gravitational Wave Detectors

L. Baggio,[1] M. Bignotto,[2] M. Bonaldi,[3] M. Cerdonio,[2] L. Conti,[2] P. Falferi,[3,*] N. Liguori,[2] A. Marin,[2] R. Mezzena,[1] A. Ortolan,[4] S. Poggi,[1] G. A. Prodi,[1] F. Salemi,[5] G. Soranzo,[6] L. Taffarello,[6] G. Vedovato,[6] A. Vinante,[1] S. Vitale,[1] and J. P. Zendri[6]





Histogram of output of filter for the search of short pulses