Experimental Methods
Lecture 16

October 26th, 2020

A final note: multiple stochastic processes

- Assume you have two different stochastic processes x(t) and y(t).
- Their statistics in known if for any N and any two sets of times $\{t_1, t_N\}$ and $\{t_1', t_M'\}$ we know the joint probability density

$$f_{x(t_1),y(t_1'),x(t_2),y(t_2')\dots x(t_N),y(t_M')}\left(\chi_1,\psi_1,\chi_2,\psi_2\dots\chi_N,\psi_M\right)$$

- One can define in addition to moments for each separate process, the joint moments: cross-correlation

$$R_{x,y}\left(t,t'\right)=\left\langle x\left(t\right)y\left(t'\right)\right\rangle = R_{y,x}\left(t',t\right)$$

- Cross covariance

$$C_{x,y}\left(t,t'\right)=R_{x,y}\left(t,t'\right)-\eta_x\left(t\right)\eta_y\left(t\right)$$

- The autocorrelation is just a special case when y(t)=x(t)

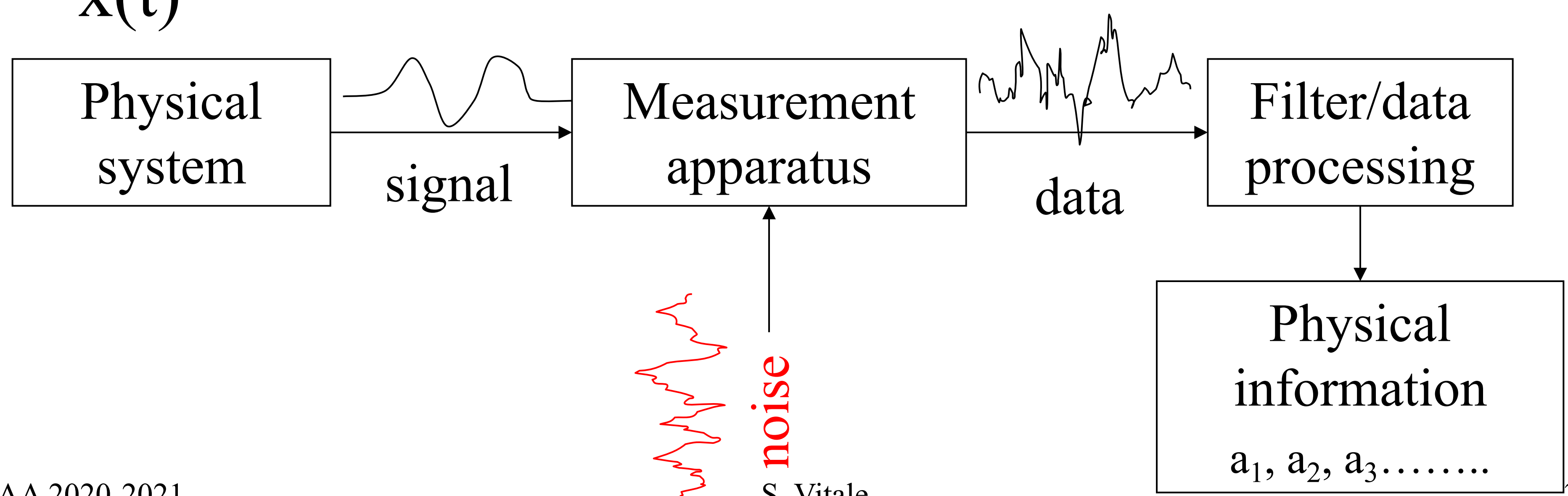
$$R_{x,x}\left(t,t'\right)=\left\langle x\left(t\right)x\left(t'\right)\right\rangle = R_{x,x}\left(t',t\right)$$

- Two processes are joint stationary if all their density functions are invariant under a time origin shift. They are normal if joint densities are normal etc.

Noise

Noise in physical experiments is described as a random signal x(t):

Independent (ensemble) repetitions of the same experiment produce different functions of time x(t)

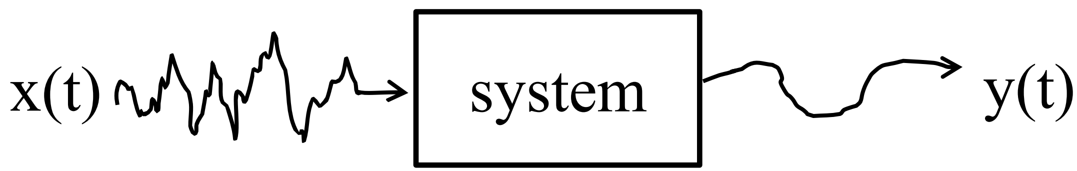


Noise and systems

- We now develop a model for the noise in physical apparatuses where noise is treated as consisting of stochastic signals.
- The signals are fed to systems and produce stochastic signals at output
- For linear systems, linear response theory allows to calculate the statistical properties of output given those at input.

Stochastic processes and systems

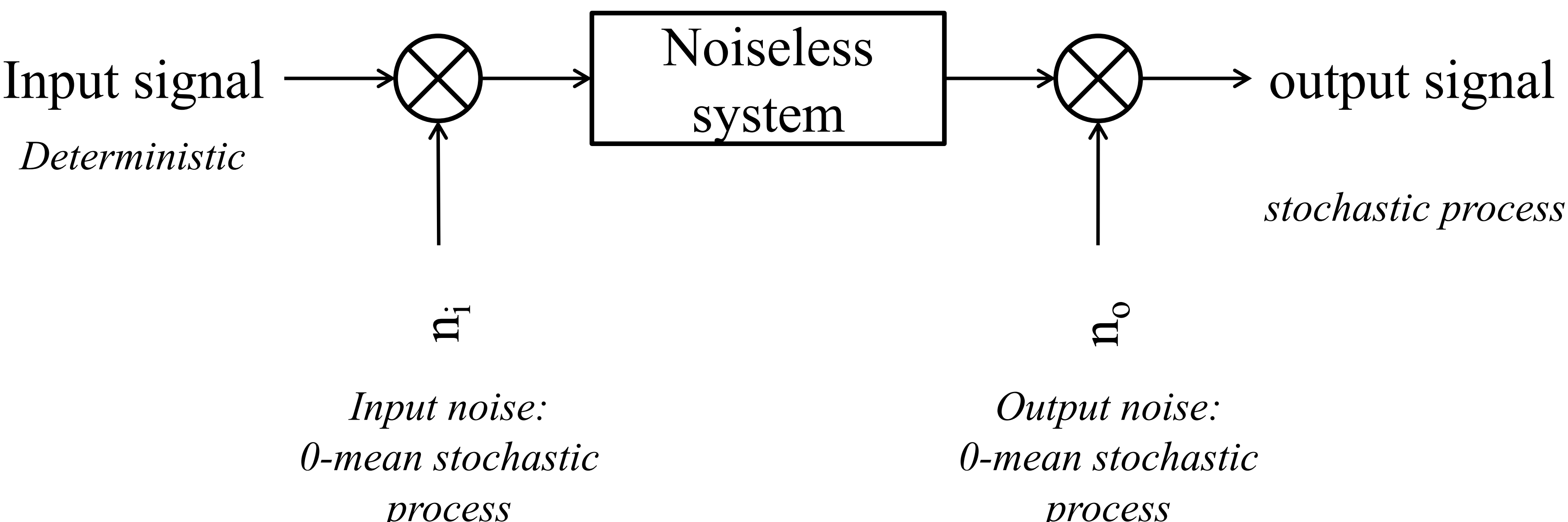
- For a single execution of an experiment, a stochastic process is just an ordinary signal



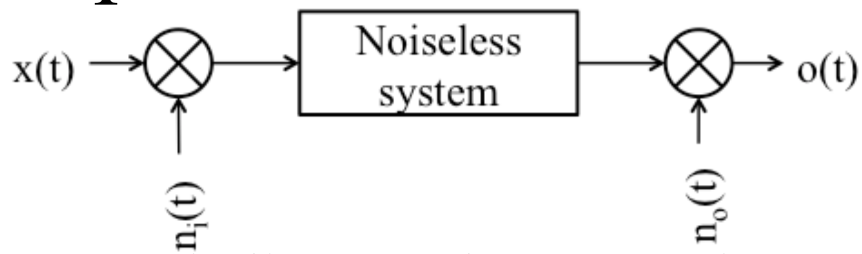
- The output is a signal
- The input is a random signal, then the output is also random
- The output is a stochastic process

The conceptual scheme for an apparatus in the presence of noise

- Disturbances in physical systems are successfully described as stochastic processes acting at input and at output of an intrinsically noiseless system:
- Notice: this implies that noise is independent of signal and signal levels.
- Not true for parametric noise



Stochastic processes and linear systems



- Suppose the system is linear. The output is

$$o(t) = n_o(t) + \int_{-\infty}^{\infty} h(t,t')\left[x(t') + n_i(t')\right]dt'$$

- As the system obeys the principle of superposition:

$$o(t) = n_o(t) + \int_{-\infty}^{\infty} h(t,t')x(t')dt' + \int_{-\infty}^{\infty} h(t,t')n_i(t')dt'$$

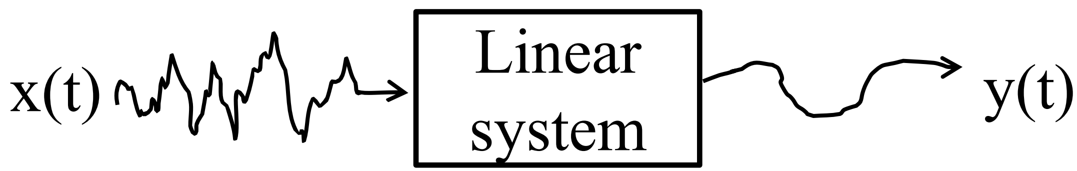
- Then the output can be written as o(t)= y(t)+ñ(t) where

$$\underbrace{y(t)}_{\text{Signal}} = \int_{-\infty}^{\infty} h(t,t')x(t')dt' \qquad \underbrace{\tilde{n}(t)}_{\text{Noise}} = n_o(t) + \int_{-\infty}^{\infty} h(t,t')n_i(t')dt'$$

- The response to noise is independent of the presence and the nature of the signal. Noise properties can be treated independently of signals. A reasonably accurate model for most systems.(not for parametric noise)

Stochastic processes and linear systems

- Suppose a stochastic process x(t) is fed to the input of a linear system whose impulse response is h(t,t')



- The output is

$$y(t) = \int_{-\infty}^{\infty} h(t,t')\,x(t')\,dt'$$

- One could in principle derive the joint probability densities of any order for y(t)
- An easier task is to derive moments

A parenthesis on mean value and linear operators

- Assume you have N random variables $x_1, x_2, \ldots x_N$ with a joint distribution $$f_{x_1,x_2....x_N}(\chi_1,\chi_2 \cdots \chi_N)$$
- Take now M functions of these random variables $y_1=g_1(x_1, x_2, \ldots x_N)$,$\ldots y_M=g_M(x_1, x_2, \ldots x_N)$, and calculate the mean value of some linear combination of them $\langle c_1 y_1 + c_2 y_2 + \ldots + c_N y_N \rangle =$

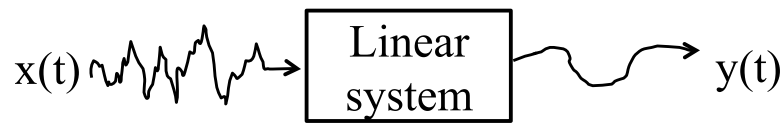
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}.....\int_{-\infty}^{\infty}\left[c_1 g_1(\chi_1,\chi_2..\chi_N) + c_2 g_1(\chi_1,\chi_2..\chi_N) + \ldots + c_N g_1(\chi_1,\chi_2..\chi_N)\right]\times$$
$$\times f_{x_1,x_2....x_N}(\chi_1,\chi_2 \cdots \chi_N)\, d\chi_1 d\chi_2 .. d\chi_N$$
$$= c_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}.....\int_{-\infty}^{\infty} g_1(\chi_1,\chi_2..\chi_N) f_{x_1,x_2....x_N}(\chi_1,\chi_2 \cdots \chi_N)\, d\chi_1 d\chi_2 .. d\chi_N + ..$$
$$+ c_N \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}.....\int_{-\infty}^{\infty} g_N(\chi_1,\chi_2..\chi_N) f_{x_1,x_2....x_N}(\chi_1,\chi_2 \cdots \chi_N)\, d\chi_1 d\chi_2 .. d\chi_N + ..$$

- Then
$$\langle c_1 y_1 + c_2 y_2 + \ldots + c_N y_N \rangle == c_1 \langle y_1 \rangle + c_2 \langle y_2 \rangle + .. + c_N \langle y_N \rangle$$
- In particular
$$\langle c_1 x_1 + c_2 x_2 + \ldots + c_N x_N \rangle == c_1 \langle x_1 \rangle + c_2 \langle x_2 \rangle + .. + c_N \langle x_N \rangle$$

Stochastic processes and linear systems

- Let's go back to our linear system

$$y(t)=\int_{-\infty}^{\infty}h(t,t')x(t')dt'$$



- And calculate the mean value of the output. Let's rewrite

$$y(t)=\lim_{\substack{\Delta t\to 0\\ N\to\infty}}\sum_{k=-N}^{N}h(t,k\Delta t)x(k\Delta t)\Delta t$$

- Then y(t) is a linear combination of the x(kΔt) and

$$\langle y(t)\rangle=\lim_{\substack{\Delta t\to 0\\ N\to\infty}}\sum_{k=-N}^{N}h(t,k\Delta t)\langle x(k\Delta t)\rangle\Delta t$$

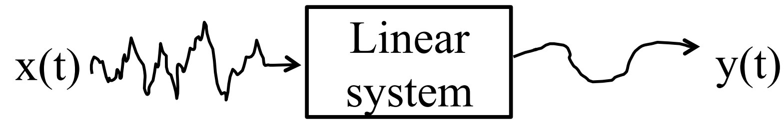
- Taking the limits

$$\langle y(t)\rangle=\int_{-\infty}^{\infty}h(t,t')\langle x(t')\rangle dt'$$

- Mean value and integration commute!

Stochastic processes and linear systems

$$y(t)=\int_{-\infty}^{\infty} h(t,t')x(t')dt'$$



- Autocorrelation. Use always

$$y(t)=\lim_{\substack{\Delta t\to 0\\ N\to\infty}}\sum_{k=-N}^{N} h(t,k\Delta t)x(k\Delta t)\Delta t$$

- Then

$$\langle y(t)y(t')\rangle=\lim_{\substack{\Delta t\to 0\\ N\to\infty}}\left\langle \sum_{k,j=-N}^{N} h(t,k\Delta t)h(t',j\Delta t)x(k\Delta t)x(j\Delta t)\Delta t^2 \right\rangle$$

- Using superposition

$$\langle y(t)y(t')\rangle=\lim_{\substack{\Delta t\to 0\\ N\to\infty}}\sum_{k,j=-N}^{N} h(t,k\Delta t)h(t',j\Delta t)\Delta t^2 \langle x(k\Delta t)x(j\Delta t)\rangle$$

- Taking the limits

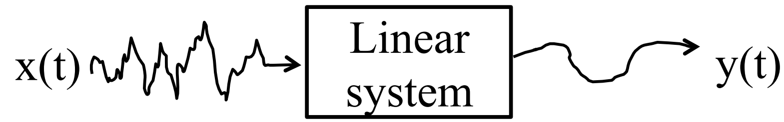
$$\langle y(t)y(t')\rangle=\int_{-\infty}^{\infty} h(t,t'')h(t',t''')\langle x(t'')x(t''')\rangle dt''dt'''$$

- Or

$$R_{y,y}(t,t')=\int_{-\infty}^{\infty} h(t,t'')h(t',t''')R_{x,x}(t'',t''')dt''dt'''$$

Stochastic processes and linear systems

$$y(t)=\int_{-\infty}^{\infty} h(t,t')\,x(t')\,dt'$$



- Holds for higher moments

$$\langle y(t)\,y(t')\,y(t'')\rangle =$$

$$=\int_{-\infty}^{\infty} h(t,t''')\,h(t',t'''')\,h(t'',t''''')\,\langle x(t''')\,x(t'''')\,x(t''''')\rangle\, dt'''\,dt''''\,dt'''''$$

- Etc.
- It’s a general rule for all linear operators (rather obvious if you allow for impulse response that are distributions)

$$\langle L_t[i(t)]\rangle = L_t\langle i(t)\rangle$$

$$\langle L_{1,t}[i(t)]\,L_{2,t'}[i(t')]\rangle = L_{1t}L_{2t'}\langle i(t)\,i(t')\rangle$$

$$\langle L_{1,t}[i(t)]\,L_{2,t'}[i(t')]\,L_{3,t''}[i(t'')]\rangle = L_{1t}L_{2t'}L_{3t''}\langle i(t)\,i(t')\,i(t'')\rangle$$

Stochastic processes and linear systems

- Additional examples: Derivatives

$$y(t)=\frac{dx(t)}{dt}=\lim_{dt\to 0}\frac{x(t+dt)-x(t)}{dt}$$

- The mean value $\lim_{dt\to 0}\left\langle \frac{x(t+dt)-x(t)}{dt}\right\rangle=\lim_{dt\to 0}\frac{\langle x(t+dt)\rangle-\langle x(t)\rangle}{dt}$

- Then $\left\langle \frac{dx}{dt}\right\rangle=\frac{d\langle x(t)\rangle}{dt}$

- Higher order

$$\left\langle \left.\frac{dx}{dt}\right|_t x(t')\right\rangle=\lim_{dt\to 0}\left\langle \frac{x(t+dt)-x(t)}{dt}x(t')\right\rangle$$

$$=\lim_{dt\to 0}\frac{\langle x(t+dt)x(t')\rangle-\langle x(t)x(t')\rangle}{dt}=\frac{\partial R_{x,x}(t,t')}{\partial t}$$

- Same way $\left\langle \left.\frac{dx}{dt}\right|_t \left.\frac{dx}{dt}\right|_{t'}\right\rangle=\frac{\partial^2 R_{x,x}(t,t')}{\partial t\,\partial t'}$

Example: shot noise

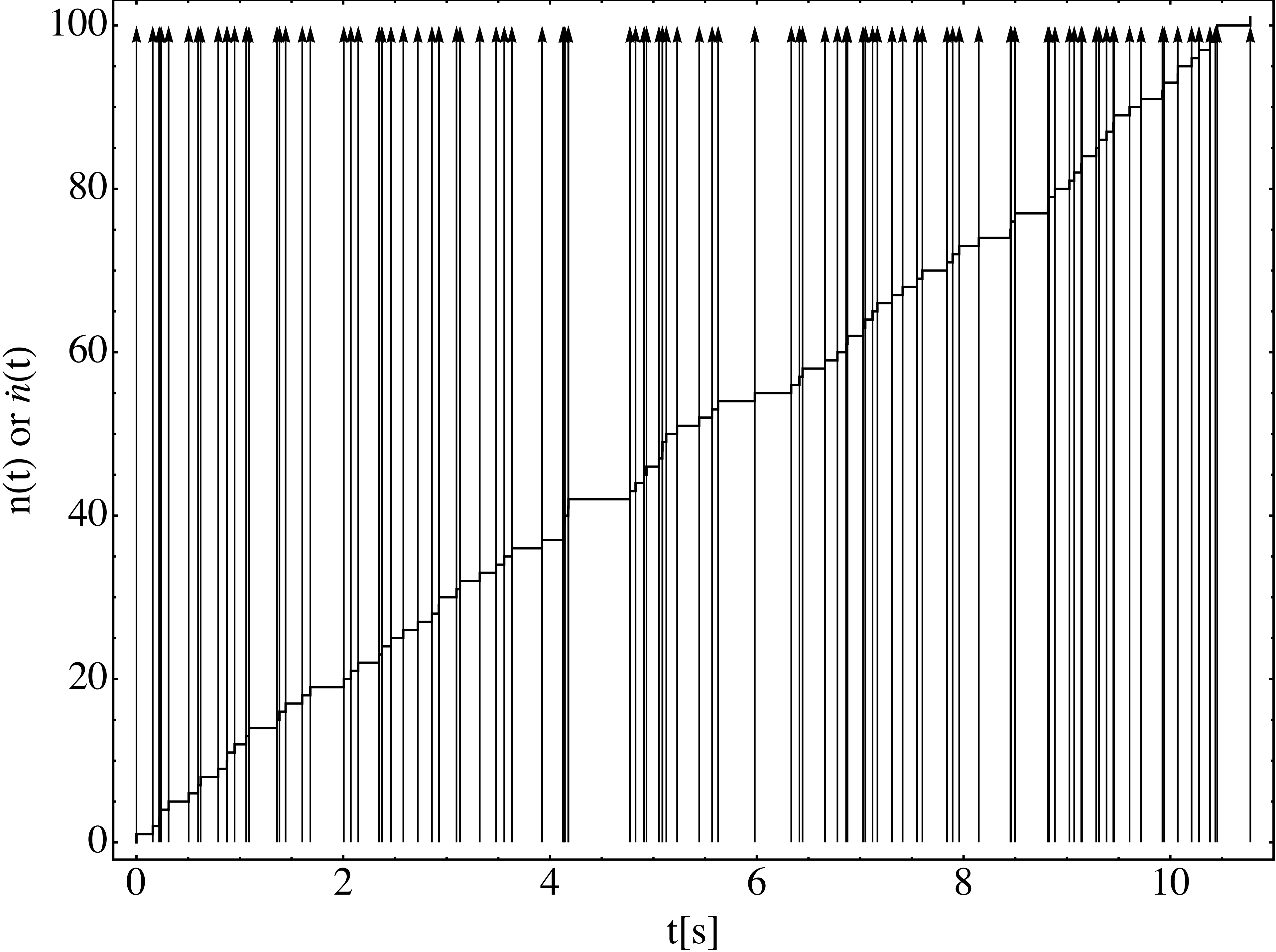
- Shot noise is the derivative of the Poisson process

$$\dot{n}(t) = \left.\frac{dn}{dt}\right|_t$$

- For photon beams, energy is proportional to a Poisson process: $E(t)=\hbar\omega n(t)$, power is proportional to shot noise $P(t) =\hbar\omega\, \dot{n}(t)$
- For random charge flow, $I(t)=e\, \dot{n}(t)$ is the current
- From the definition of shot noise

$$\dot{n}(t) = \frac{d}{dt}\left[\sum_{k=0}^{\infty}\Theta(t-t_k)\right] = \sum_{k=0}^{\infty}\delta(t-t_k)$$

- Thus shot noise consists of a series of impulses arriving at random times (see next page with $\lambda=10\ s^{-1}$)



100
80
60
40
20
0
n(t) or ṅ(t)
0
2
4
6
8
10
t[s]

Mean value and autocorrelation of shot noise

- Mean value of Poisson noise $\langle n(t) \rangle = \lambda t$
- Mean value of shot noise is equal to derivative of mean value:

$$\langle \dot{n}(t) \rangle = d\langle n(t) \rangle / dt = \lambda$$

- Autocorrelation of Poisson noise

$$\langle n(t) n(t') \rangle = \lambda^2 t t' + \lambda\left[t\Theta(t'-t) + t'\Theta(t-t') \right]$$

- Using derivative rules, the autocorrelation of shot noise is:

$$\langle \dot{n}(t) \dot{n}(t') \rangle = \frac{\partial^2 \langle n(t) n(t') \rangle}{\partial t \partial t'} = \frac{\partial}{\partial t'}\left\{ \lambda^2 t' + \lambda\left[\Theta(t'-t) - t\delta(t-t') + t'\delta(t-t') \right] \right\}$$

$$= \frac{\partial}{\partial t'}\left\{ \lambda^2 t' + \lambda\left[\Theta(t'-t) + (t'-t)\delta(t'-t) \right] \right\}$$

- As (t-t')δ(t-t')=0 → $= \dfrac{\partial}{\partial t'}\left\{ \lambda^2 t' + \lambda\Theta(t'-t) \right\} = \lambda^2 + \lambda\delta(t-t')$

- Finally the auto-covariance is $\langle \dot{n}(t) \dot{n}(t') \rangle - \langle \dot{n}(t) \rangle \langle \dot{n}(t') \rangle = \lambda\delta(t-t')$

Mean value and autocorrelation of shot noise

- In summary
- Mean value of shot noise is a constant: the "rate"

$$\eta_{\dot{n}}(t)=\langle\dot{n}(t)\rangle=\lambda$$

- Autocorrelation of Poisson noise only depends on delay t-t'

$$R_{\dot{n}(t)\dot{n}(t')}=\langle\dot{n}(t)\dot{n}(t')\rangle=\lambda^2+\lambda\delta(t-t')$$

- And so does the auto-covariance:

$$C_{\dot{n}(t)\dot{n}(t')}=\langle\dot{n}(t)\dot{n}(t')\rangle-\langle\dot{n}(t)\rangle\langle\dot{n}(t')\rangle=\lambda\delta(t-t')$$

- This only holds for t, t' ≥ 0. However the time origin can be moved $\rightarrow -\infty$, and then shot noise becomes at least wide-sense stationary.
- It is easy to convince yourself that it is also stationary

Stationary noise and linear stationary systems

- Let's consider the case where:
  - the input to a linear system x(t) is a stationary stochastic process
  - the linear system is time-invariant.
- The output signal y(t) is

$$y(t)=\int_{-\infty}^{\infty}h(t')x(t-t')dt'$$

- Then for any N

$$\langle y(t_1+T)y(t_2+T)....y(t_N+T)\rangle=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(t')h(t'')h(t'')\langle x(t_1+T-t')x(t_2+T-t'')x(t_3+T-t''')\rangle dt'dt''dt'''$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(t')h(t'')h(t'')\langle x(t_1-t')x(t_2-t'')x(t_3-t''')\rangle dt'dt''dt'''$$

$$=\langle y(t_1)y(t_2)....y(t_N)\rangle$$

- That is, y(t) is stationary.

Stationary noise and linear stationary systems

- A stationary noise at the input of a linear time-invariant system

1. Mean value

$$\langle y(t)\rangle = \int_{-\infty}^{\infty} h(t')\eta_x \, dt' = \eta_x \int_{-\infty}^{\infty} h(t')dt' = \text{Constant} \equiv \eta_y$$

2. Input-output cross correlation

$$R_{y,x}(t,t+\Delta t) = \langle y(t)x(t+\Delta t)\rangle = \int_{-\infty}^{\infty} h(t')\langle x(t-t')x(t+\Delta t)\rangle dt'$$

Using time-invariance of x

$$R_{y,x}(\Delta t) = \int_{-\infty}^{\infty} h(t')R_{x,x}(\Delta t + t')dt'$$

3. Output auto-correlation

$$R_{y,y}(t,t+\Delta t) = \langle y(t)y(t+\Delta t)\rangle =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt'dt''h(t')h(t'')\langle x(t-t')x(t+\Delta t - t'')\rangle$$

That is

$$R_{y,y}(\Delta t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt'dt''h(t')h(t'')R_{x,x}(\Delta t + t' - t'')$$

Examples

- Shot noise into a low-pass filter. A good model for charge detection.

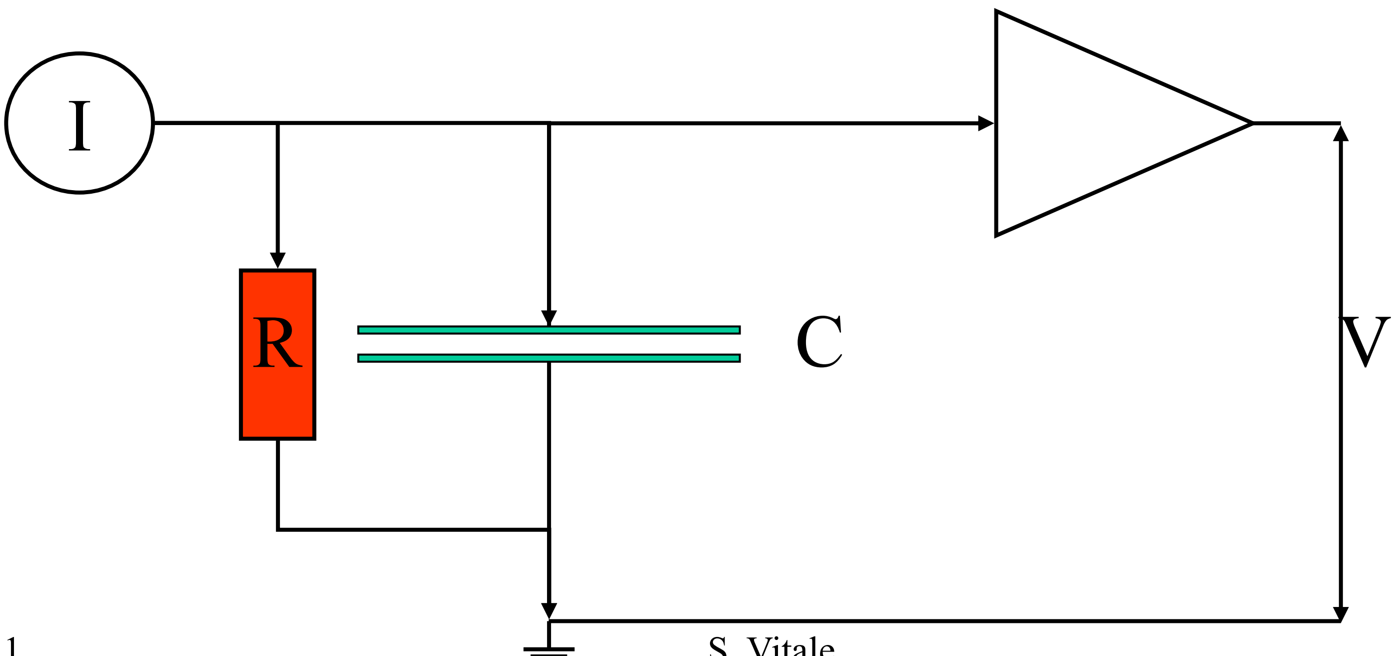
- White noise into a low-pass filter. The basic colored noise.

Random Charging

- Assume that a flow of particles, each of charge *e*, hits a detector represented by capacitor of capacitance C.
- Particles arrive as Poisson points with a rate λ. They are represented by a current generator generating a current

$$I(t) = \sum_{i=-\infty}^{\infty} e\delta(t - t_i)$$

- Losses in the capacitor are represented by a parallel resistor of Resistance R. Voltage V across C is read out by an ideal amplifier



Random Charging

- The circuit responds to the stochastic signal as to any other signal
- The impulse response of the circuit can be easily obtained from

$$C\left(dV(t)/dt\right) = -\left(V(t)/R\right) + I(t)$$

- Fourier transform gives the frequency response

$$V(\omega) = \left(I(\omega)/C\right)\left(i\omega + 1/RC\right)^{-1} \equiv h(\omega)I(\omega)$$

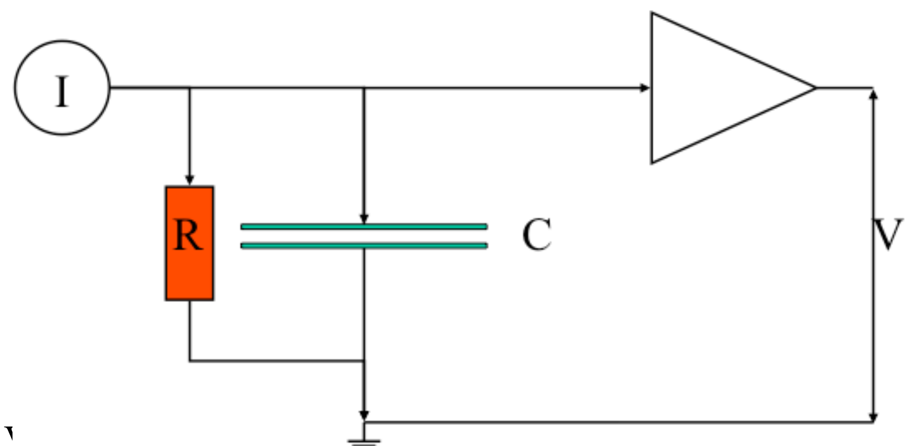
- The inverse Fourier transform of which is

$$h(t) = \Theta(t)e^{-t/RC}/C$$

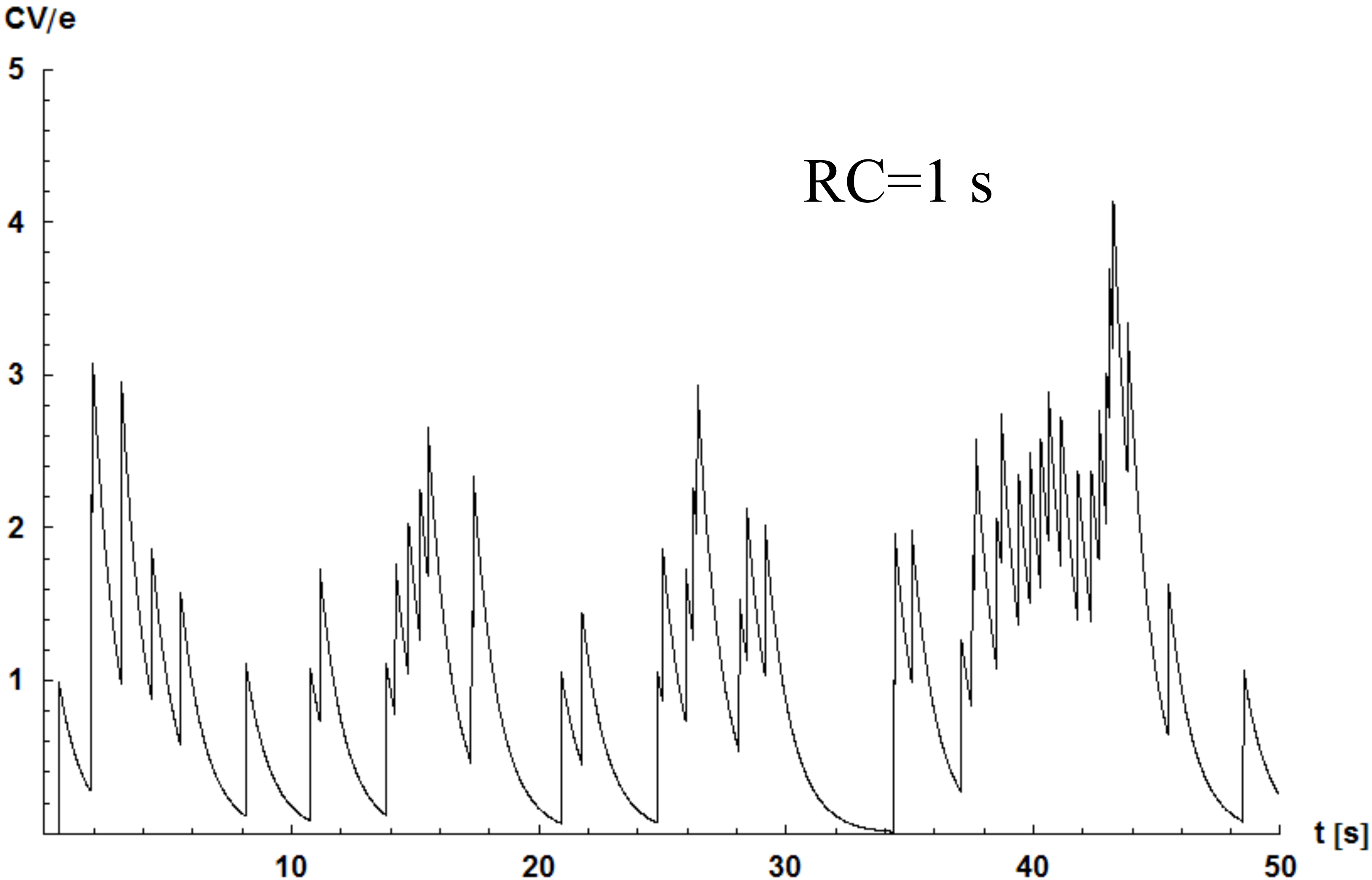
- As the current is just formed by impulses $\quad I(t) = \sum_{i=-\infty}^{\infty} e\delta(t - t_i)$
- and the system is linear, then

$$V(t) = \frac{e}{C}\sum_{i=-\infty}^{\infty}\Theta(t - t_i)e^{-(t-t_i)/RC}$$

- See next page for an example



Numerical example for charged particle detection



Random Charging

- Voltage mean value $\left\langle V(t)\right\rangle = \int_{-\infty}^{\infty}\Theta(t')\left(e^{-t'/RC}/C\right)\left\langle I(t-t')\right\rangle dt'$
- As $\left\langle I(t)\right\rangle = e\left\langle \dot{n}(t)\right\rangle = e\lambda$
- We obtain (rather obviously) $\left\langle V(t)\right\rangle = \dfrac{e\lambda}{C}\int_0^{\infty} e^{-t'/RC}dt' = Re\lambda$
- Autocorrelation
$$R_{VV}(\Delta t) = \int_0^{\infty}\int_0^{\infty}\left(e^{-t'/RC}/C\right)\left(e^{-t''/RC}/C\right)R_{II}\left(\Delta t + t' - t''\right)dt'dt''$$
- As $R_{II}\left(\Delta t + t' - t''\right) = e^2 R_{\dot{n}\dot{n}}\left(\Delta t + t' - t''\right) = e^2\lambda^2 + e^2\lambda\delta\left(\Delta t + t' - t''\right)$
- We get
$$R_{VV}(\Delta t) = e^2\lambda^2\left[\int_0^{\infty}\frac{e^{-t'/RC}}{C}dt'\right]^2 + \frac{e^2\lambda}{C^2}\int_0^{\infty}\int_0^{\infty}e^{-t'/RC}e^{-t''/RC}\delta\left(\Delta t + t' - t''\right)dt'dt''$$
- That is
$$R_{VV}(\Delta t) = (Re\lambda)^2 + \frac{e^2\lambda}{C^2}\int_{\mathrm{Max}[-\Delta t,0]}^{\infty}e^{-t'/RC}e^{-(\Delta t + t')/RC}dt'$$
$$= (Re\lambda)^2 + \frac{Re^2\lambda}{C}e^{-\Delta t/RC}\left[\Theta(\Delta t)\int_0^{\infty}e^{-2x}dx + \Theta(-\Delta t)\int_{-\Delta t/RC}^{\infty}e^{-2x}dx\right] =$$
$$= (Re\lambda)^2 + \frac{Re^2\lambda}{2C}e^{-|\Delta t|/RC}$$

Random Charging

- In summary
  - Mean value $\eta_V = Re\lambda$
  - Autocorrelation $R_{VV}(\Delta t) = (Re\lambda)^2 + \frac{Re^2\lambda}{2C} e^{-|\Delta t|/RC}$
  - Then auto-covariance is

$$C_{VV}(\Delta t) = R_{VV}(\Delta t) - \eta_V^2 = \frac{Re^2\lambda}{2C} e^{-|\Delta t|/RC}$$

- Thus voltage is the sum of a static dc voltage and of a noisy part $\tilde{V}(t)$

$$V(t) = Re\lambda + \tilde{V}(t)$$

- With autocorrelation $R_{\tilde{V}\tilde{V}}(\Delta t) = C_{VV}(\Delta t) = \frac{Re^2\lambda}{2C} e^{-|\Delta t|/RC}$
- Notice that variance of both $V(t)$ and $\tilde{V}(t)$ is:

$$\sigma_V^2 = \sigma_{\tilde{V}}^2 = R_{\tilde{V}\tilde{V}}(0) = C_{VV}(0) = Re^2\lambda/2C$$

- The then $\sigma_V^2/\eta_V = \frac{Re^2\lambda}{2CRe\lambda} = \frac{e}{2C}$
- A well known way of measuring electron charge independently of λ!