Foreword

The final product of the exam must be a short report containing the answers to the various questions (Q) listed in the following section, together with a brief justification for those answers. The report must be in the form of a .pdf file and should be uploaded, at the end of the exam, onto the same website from which you have downloaded this pdf document. Please put you full name and today's date in the title. You are allowed, during the test, to use notes, books, and lecture slides. You are also allowed to consult Wikipedia and other databases. The use of computing tools like Mathematica, Matlab, or any other tool of your choice is warmly encouraged. You are allowed to use whatever method you find fit to answer the questions: analytical calculations, numerical calculations, simulations etc. You are allowed to approximate and neglect terms at your will, provided such approximations are physically sound and justified. You are not required to answer all the questions to obtain full credit. However, you are supposed to grasp the essence of the exercise. It is absolutely forbidden to exchange any information with third parties, either fellow students or external experts. Good luck!

Test

Please note: all needed numerical values are listed in table I on page 4.

The experiment

Inspired by a recent Nature paper [1] where a small mechanical oscillator, based on a piezo crystal, is used to convert a voltage signal out of a q-bit into an electromagnetic wave, I would like you to analyse the following combination of the exercises on the piezo transducer and on the Mach-Zender interferometer respectively.

The scheme of the experiment is shown in fig 1. A piezo crystal is attached to a small mass, of mass m, to form a harmonic oscillator, which is kept at thermodynamic equilibrium

[1] https://doi.org/10.1038/s41586-020-3038-6

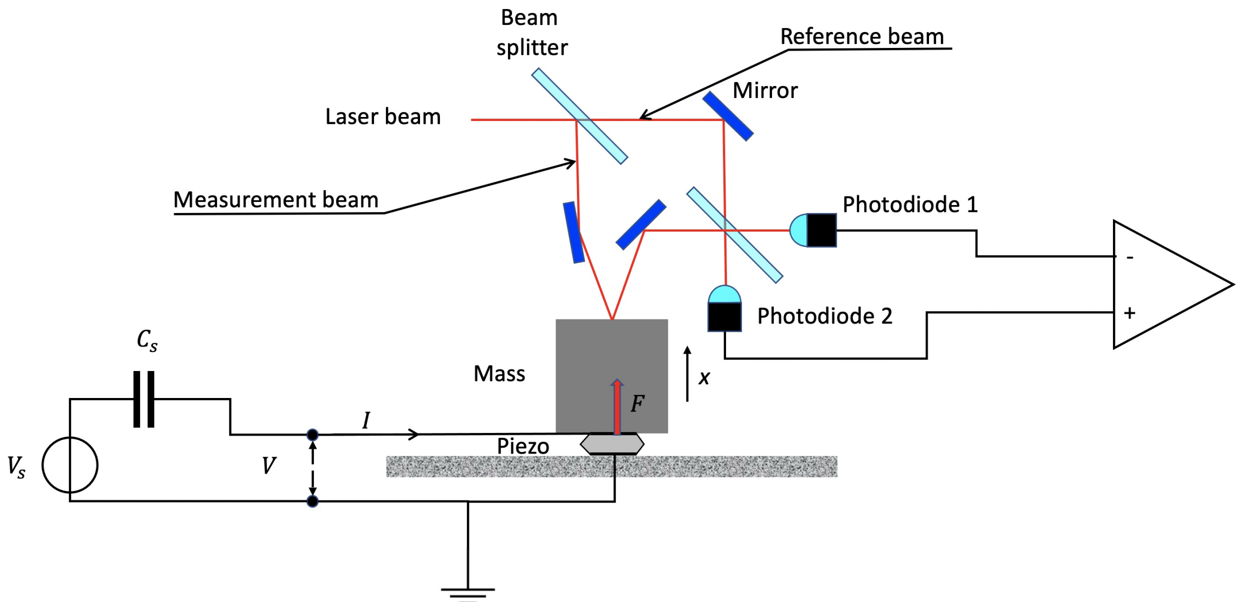

FIG. 1.

at temperature T. The piezo crystal is the spring of the oscillator, and its electrical port is also the input to the device. A voltage source, generating the voltage V_s, and with capacitive impedance of capacitance C_s, is attached to such an electrical input port as shown in the figure.

For your convenience, I am copying here, from the exercise on the piezo, the small signal, linearised constitutive equations the describe the behaviour of a piezoelectric crystal:

$$\begin{aligned} I &= s\left(\mathcal{C}_p\, V - d\, F\right) \\ x &= d\, V - \frac{F}{k} \end{aligned} \tag{1}$$

V, x and F are defined in fig. 1 and k, d and C_s are numerical coefficients.

In addition, we describe the mechanical losses connected to the deformation of the piezo, with an equivalent viscous damping acting on the oscillator mass, exerting the force $F_v =$

$-\beta\dot{x}$. In table I we give the values for d and C_s, but those for k and β are given indirectly via the 'bare' resonant frequency $\nu = \sqrt{k/m}/(2\pi)$, and the q-factor $Q = (m/\beta)\sqrt{k/m}$

The motion of the mass is sensed by a Mach-Zender interferometer: a laser beam is split into a reference beam and a measurement beam (see fig 1), with intensities I_{ref} and I_m respectively, and with $I_{ref} \gg I_m$. The measurement beam is reflected almost normally out of the surface of the oscillator mass. Thus a displacement δx, along the x axis of the figure, produces a change of the optical path length of the reference beam, given by $\delta l \simeq -2\delta x$.

As explained in the abovementioned homework, in a Mach-Zender interferometer, the two photodiodes of the figure collect two different intensities I_1 and I_2, given by:

$$\begin{aligned} I_1 &= \frac{1}{2}\left(I_{ref} + I_m - \sqrt{I_m I_{ref}}Cos(\Delta\phi)\right) \\ I_2 &= \frac{1}{2}\left(I_{ref} + I_m + \sqrt{I_m I_{ref}}Cos(\Delta\phi)\right) \end{aligned} \tag{2}$$

where $\Delta\phi$ is the total phase difference between the two beams. Here we have assumed that all beam splitters have 50/50 splitting ratio. We also assume that the voltage signals out of both photodiodes are proportional to their relative collected intensities, with same values of the proportionality constant.

With this assumptions, the voltage difference $\Delta V = V_2 - V_1$ is proportional to $I_2 - I_1$, and thus gives a measurement of $Cos(\Delta\phi)$. As usual in interferometric systems, we assume that the static phase shift between the two beams can be adjusted at will, within 2π, to pick the proper working point and linearise the response of the interferometer to $\Delta\phi$, if needed.

We also assume that any non intrinsic amplitude instability of the laser is negligible (even if it would cancel out in a perfect Mach-Zender scheme), and that the two light paths have the same length, within a high enough precision to suppress the effect of the laser frequency noise.

Finally we consider the noise of the photodiode readout electronics as negligible, and that such electronics is fast enough to follow the GHz range signals of the exercise.

Questions

Q1 Find the linearised small signal transfer function between V_s and x

Q2 Find the power spectral density (PSD) of x in the absence of any applied signal.

Q3 Find the PSD of the readout noise due to the intrinsic fluctuation of the beams' intensities. Express it as an equivalent PSD for x

Q4 Suppose one wants to resolve the PSD of the true motion of x at resonance against the readout background of Q3. Assume he wants a frequency resolution sufficient to resolve the narrow line, and a relative error ρ on the peak value. What would be the minimum needed measurement time to perform such a measurement?

Q5 Suppose that V_s is a very short pulse approximated as $V_s(t) = \phi_o \delta(t)$, what is the minimum uncertainty one can reach on the measurement of ϕ_o?

Parameter	Value	Units	Parameter	Value	Units
ν	1.	GHz	Q	10000	
m	0.5	pg	$\mathcal{C}_p$	60.	fF
$\mathcal{C}_s$	5.	fF	d	5.	pm/V
$\mathcal{I}_m$	100.	μW	$\mathcal{I}_{\text{ref}}$	100.	mW
λ	1.	μm	T	4.	K
ρ	0.1				

TABLE I. List of parameter values.