Experimental Methods
Lecture 8

October 7th, 2020

Systems

$$i(t) \xrightarrow{\text{Input}} \boxed{\text{System}} \xrightarrow{\text{Output}} o(t)$$

$$o(t) = \Im[i(t)]$$

- Causality

$$o(t) = \Im[i(t' \leq t)]$$

- Linearity

$$\Im[a_1 i_1(t) + a_2 i_2(t)] = a_1 \Im[i_1(t)] + a_2 \Im[i_2(t)]$$

- Linear systems obey

$$o(t) = \int_{-\infty}^{\infty} h(t, t') i(t') dt'$$

- Causal linear system obey

$$o(t) = \int_{-\infty}^{t} h(t, t') i(t') dt'$$

- Causal linear system with free evolution

$$o(t) = o_o(t) + \int_{-\infty}^{t} h(t, t') i(t') dt'$$

Stationary systems

- A system is called stationary if a shift of the time axis does not affects its response, that is, if

$$o(t)=\Im\left[i(t')\right]\rightarrow\ o(t+T)=\Im\left[i(t'+T)\right]$$

- A stationary linear system must then obey

$$o(t+T)=\int_{-\infty}^{\infty} i(t')h(t+T,t')dt'=\int_{-\infty}^{\infty} i(t'+T)h(t,t')dt'=\int_{-\infty}^{\infty} i(t')h(t,t'-T)dt'$$

- That is $$h(t+T,t')=h(t,t'-T)$$
- It follows that $h(t'',t')=h(t''-T,t'-T)=h(t''-t')$
- And that

$$o(t)=\int_{-\infty}^{\infty} i(t')h(t-t')dt'=\int_{-\infty}^{\infty} h(t')i(t-t')dt'$$

- If the system is also causal $h(t)=h(t)\Theta(t)$

$$o(t)=\int_{-\infty}^{t} i(t')h(t-t')dt' \ =\int_{0}^{\infty} h(t')i(t-t')dt'$$

- Compare with calorimeter response

$$\Delta T(t)=\int_{0}^{\infty} e^{-\frac{t'}{\tau}}\left[W(t-t')/C\right]dt'$$

How common is linearity?

- Most systems may be linearized in a "small signal limit". Consider first a discrete system for which

$$o_n = f_n\left[......i_{-10}, i_{-9}, i_{-8}, ...i_8, i_9, i_{10}, ...\right]$$

- Suppose now that the input is different from zero only for $-N \leq k \leq N$ and that, within this interval, it only slightly deviates from a constant value i°. One can then linearize f_n as

$$o_n \approx f_n\left[i^o, i^o, i^o, ...i^o\right] + \sum_{k=-N}^{N} \left.\frac{\partial f_n}{\partial i_k}\right|_{i_k = i^o} \left(i_k - i^o\right) \equiv o_{no} + \sum_{k=-N}^{N} h_{n,k} \delta i_k$$

- By redefining the output signal, and in the limit $N \rightarrow \infty$

$$\delta o_n = o_n - o_{no} = \sum_{k=-\infty}^{\infty} h_{n,k} \delta i_k$$

How common is linearity?

- For continuous systems and signals, with a little courage, one could take the the limit for infinitely dense sampling

$$\delta o_n = \sum_{k=-N}^{N} \frac{h_{n,k}}{T} \delta i_k T \qquad \xrightarrow[T\to 0]{} \delta o(t) = \int_{-\infty}^{\infty} h(t,t')\delta i(t')$$

- Where $nT \to t$ $kT \to t'$ and $h_{n,k}/T \to h(t,t')$
- Otherwise one can show that linearization is possible, under certain circumstances, by functional derivatives (Fréchet derivative) and that, for stationary systems, the so called Volterra's expansion holds

$$o(t) = \Im[i(t)] = o_o(t) + \int_0^\infty h_1(\tau) i(t-\tau) d\tau +$$

$$+ \int_0^\infty \int_0^\infty h_2(\tau_1, \tau_2) i(t-\tau_1) i(t-\tau_2) d\tau_1 d\tau_2 + \cdots \ldots$$

A practical way to treat non-linearity

- Non-linearity cannot be always taken away by linearization
- Systems that are fast enough that delays can be neglected, can be treated as instantaneous-response, or without memory. In this case

$$o(t)=\mathcal{J}\left[i(t')\right]\rightarrow o(t)=f\left[i(t)\right]$$

- Non linear systems with memory may often be reasonably approximated as a non linear systems without memory followed by a linear system that describes the memory effects:

$$i(t)\longrightarrow\boxed{f\left[i(t)\right]}\longrightarrow\boxed{=\int_{-\infty}^{\infty}f\left[i(t')\right]h(t,t')dt'}\longrightarrow o(t)$$

Examples of linearization

- Germanium calorimeter
- C depends on temperature C(T)

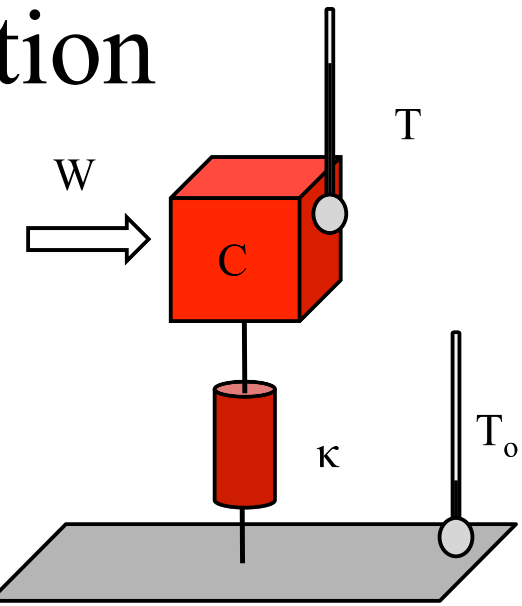
$$C(T)\left(d(T-T_o)/dt\right)=W-\kappa(T-T_o)$$

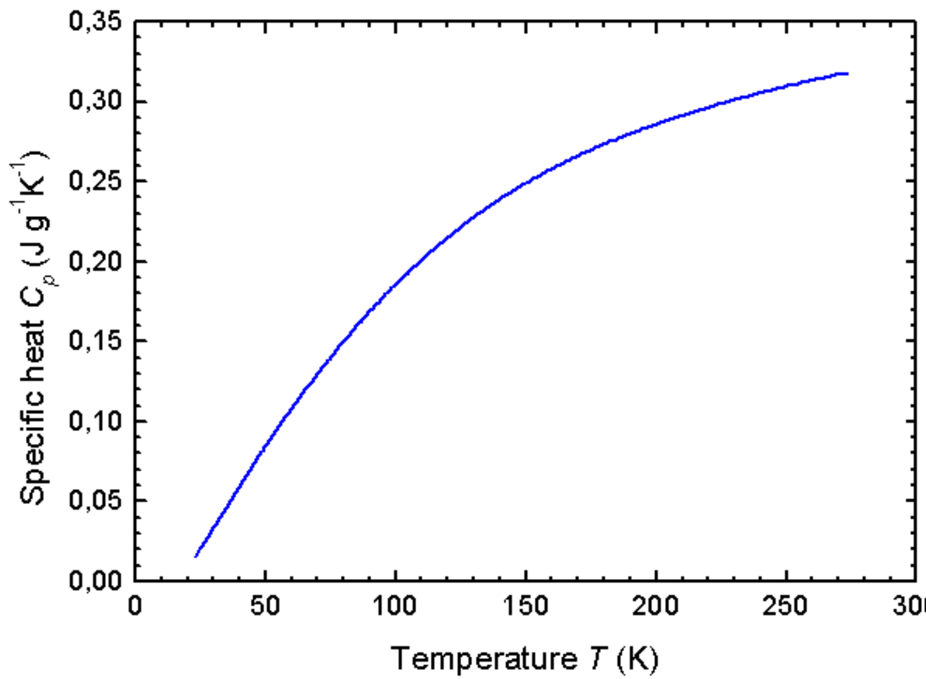
- Expanding around T_o

$$\left(C(T_o)+\left.\frac{dC}{dT}\right|_{T_o}(T-T_o)+\ldots\right)\times$$
$$\times\left(d(T-T_o)/dt\right)=W-\kappa(T-T_o)$$

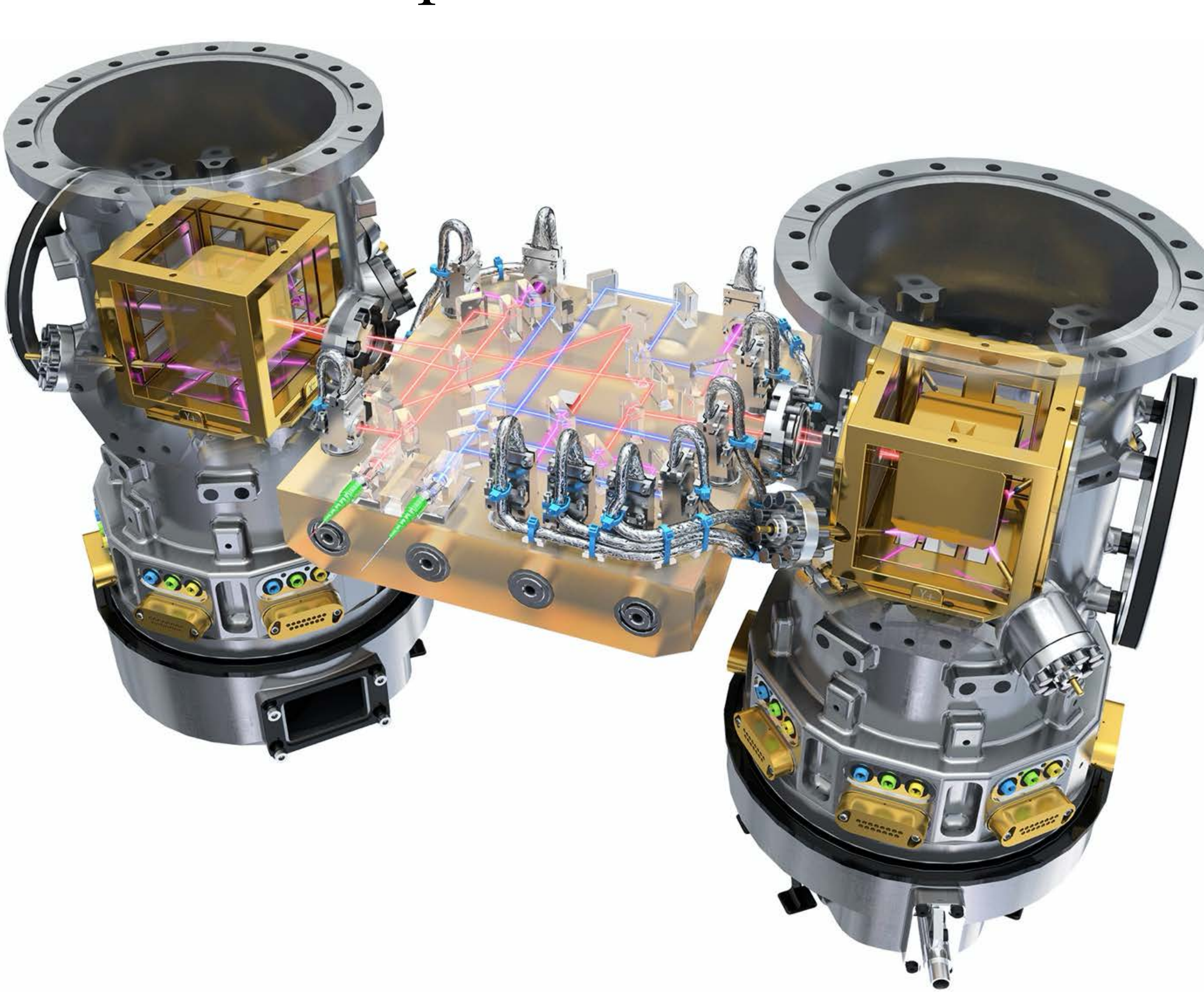
- To linear order in small signal $\Delta T=T-T_o$

$$C(T_o)\left(d\Delta T/dt\right)=W-\kappa\Delta T$$



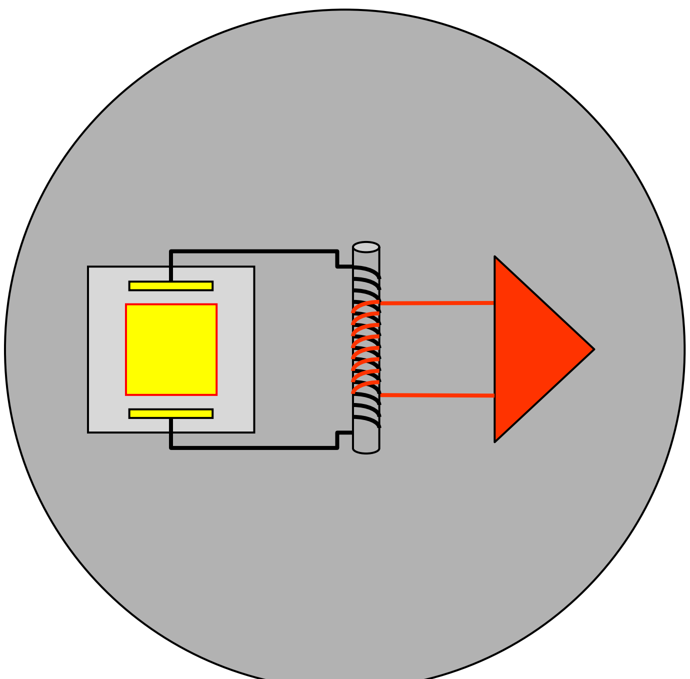


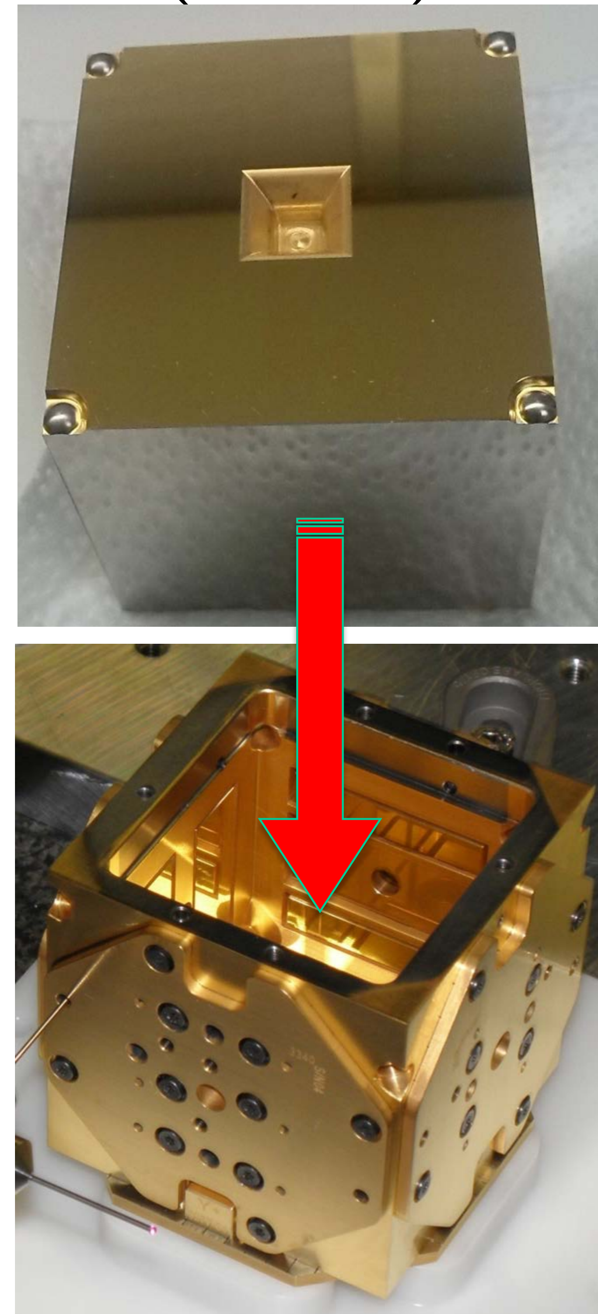
Examples of linearization



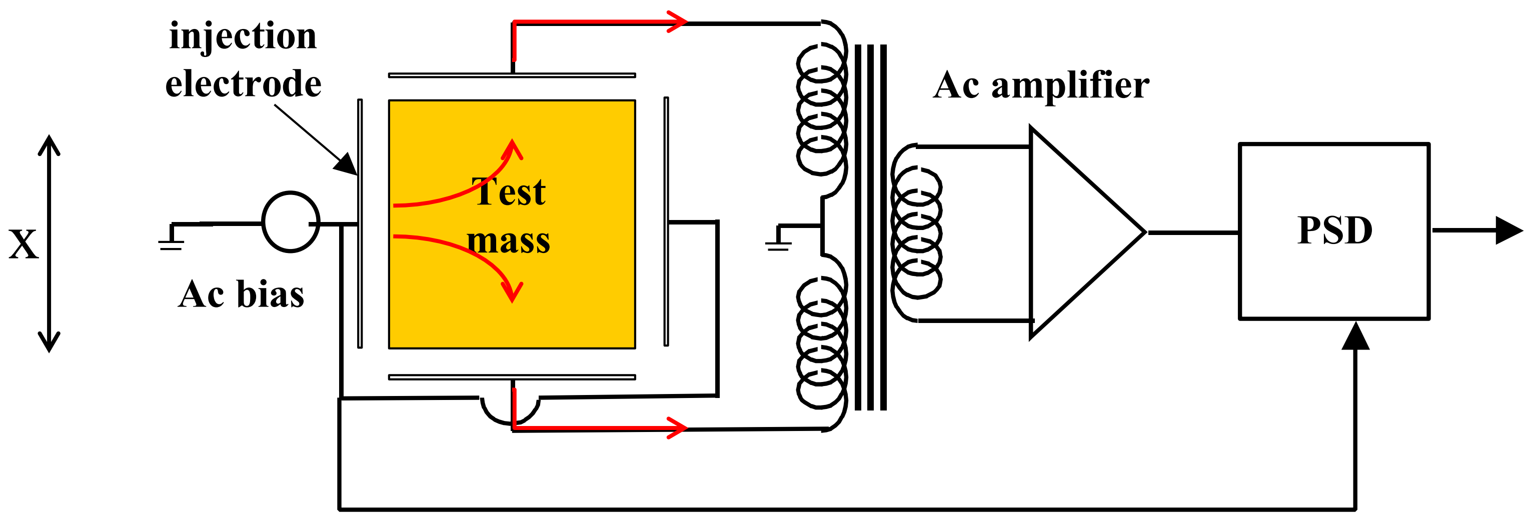
The Gravity reference Sensor (GRS)

- Drag-free along sensitive direction
- Other test-mass degrees of freedom controlled via electrostatic forces
- 3-4 mm clearance between test-mass and electrodes



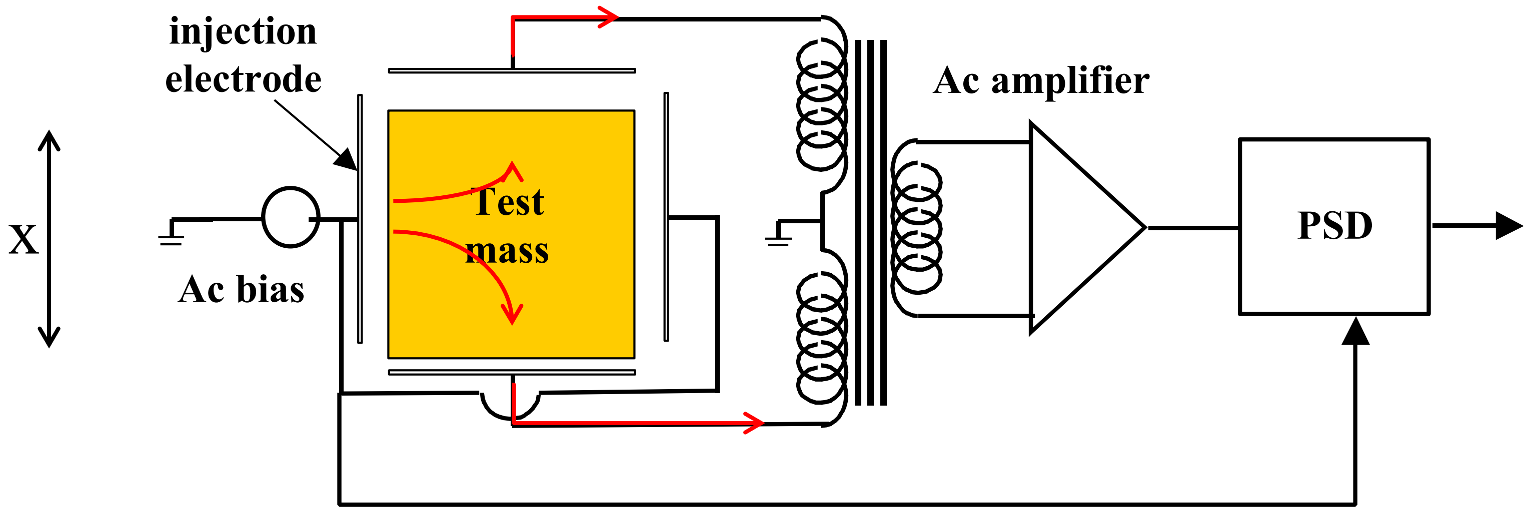


The displacement sensor



- Ac generator causes ac currents (red) to flow through capacitances at test-mass opposite faces normal to x
- Capacitances, nominally equal when test-mass is in the center, change with test-mass motion along x.

The displacement sensor



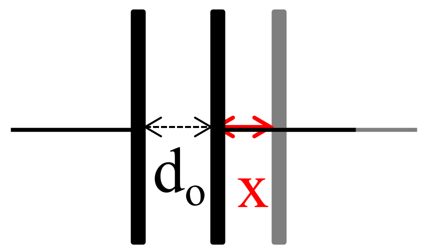
- Ac generator causes ac current (red) to flow through capacitances at opposite faces of the test-mass normal to x
- Capacitances, nominally equal when test-mass is in the center, change with test-mass motion along x.
- Ac currents become unbalanced. The *differential* transformers gets a current in the secondary.

Capacitors are non-linear motion sensors

- Capacitance of a standard parallel plate capacitor of area A and gap d_o

$$C_o = \frac{\varepsilon_o A}{d_o}$$

- If plate moves by x



- Capacitance becomes a non linear function of x

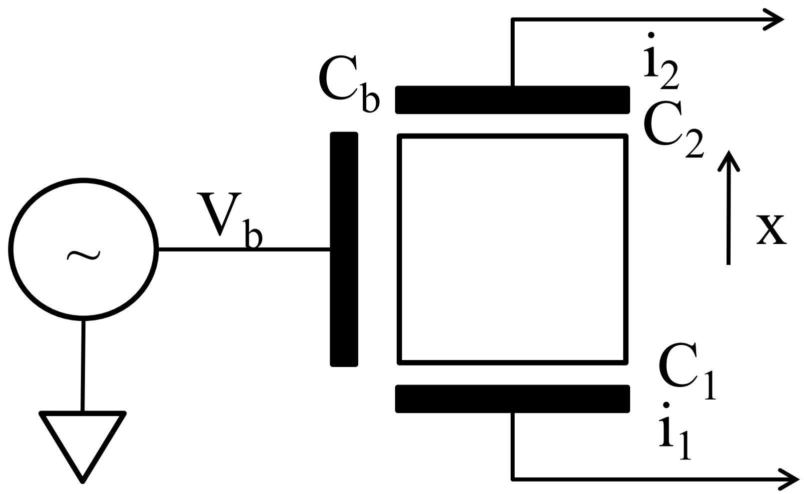
$$C(x) = \varepsilon_o A / (d_o + x)$$

- Can be expanded in Taylor series

$$C(x) = \frac{\varepsilon_o A}{d_o + x} = \frac{\varepsilon_o A}{d_o} - \frac{\varepsilon_o A}{d_o^2} x + \frac{\varepsilon_o A}{d_o^3} x^2 + O\left[x^3\right] = C_o\left(1 - \frac{x}{d_o} + \frac{x^2}{d_o^2}\right) + O\left[x^3\right]$$

Linearization of the displacement sensor

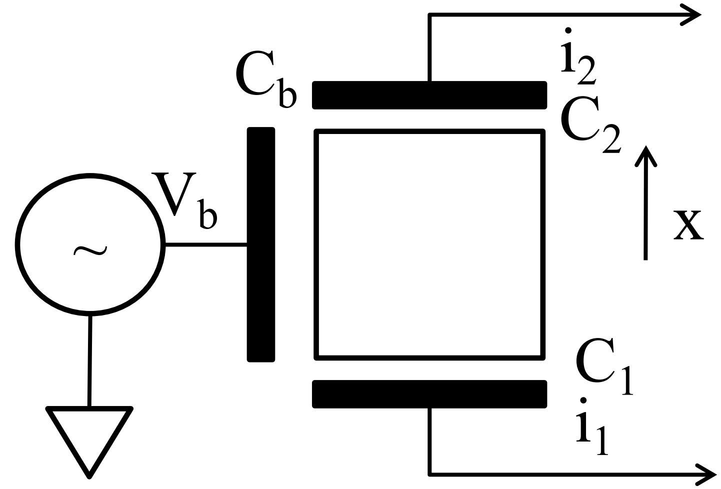
- Position dependent Capacitance $C(x) \approx C_o\left[1 - x/d_o + (x/d_o)^2\right]$
- Circuit



- Opposite capacitances have different dependences on x

$$C_1(x) \approx C_o\left[1 - x/d_o + (x/d_o)^2\right] \quad C_2(x) \approx C_o\left[1 + x/d_o + (x/d_o)^2\right]$$

Linearization of the displacement sensor



- Equations (ω_b: ac voltage frequency)

$$i_1 + i_2 = V_b \left[1/i\omega_b C_b + 1/i\omega_b \left(C_1 + C_2 \right) \right]^{-1}$$

$$i_1/i\omega_b C_1 = i_2/i\omega_b C_2$$

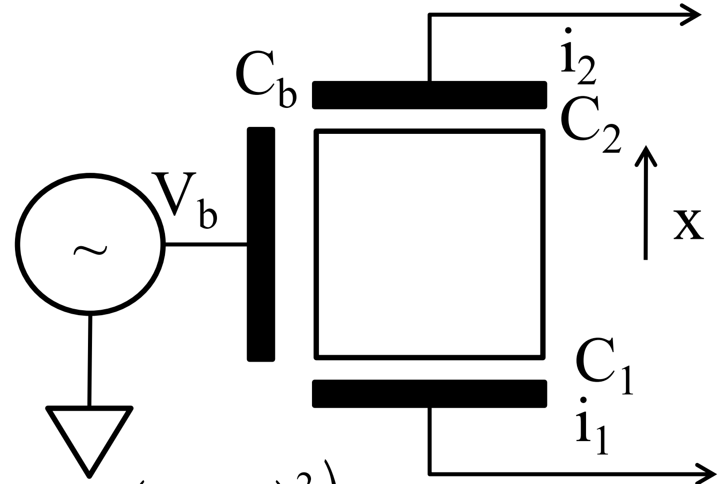
- Solving

$$i_1 = i\omega_b V_b \frac{C_1 C_b}{C_1 + C_2 + C_b},\ i_2 = i\omega_b V_b \frac{C_2 C_b}{C_1 + C_2 + C_b}$$

- Ac current difference

$$\Delta i \equiv i_2 - i_1 = i\omega_b V_b \frac{\left(C_2 - C_1 \right) C_b}{C_1 + C_2 + C_b}$$

Linearization of the displacement sensor



- Current difference

$$\Delta i = i\omega_b V_b \frac{\left(C_2 - C_1\right)C_b}{C_1 + C_2 + C_b}$$

- Using linearization of capacitances

$$\Delta i = i\omega_b V_b \frac{C_o\left(1 + x/d_o + \left(x/d_o\right)^2 - 1 + x/d_o - \left(x/d_o\right)^2\right)C_b}{C_o\left(1 + x/d_o + \left(x/d_o\right)^2 + 1 - x/d_o + \left(x/d_o\right)^2\right) + C_b}$$

- To first order in x

$$\Delta i = i\omega_b V_b \frac{2C_o C_b}{2C_o + C_b}\left(x/d_o\right)$$

- A linear, memory-less system

$$\Delta i\left(t\right) = \frac{i\omega_b V_b}{d_o}\frac{2C_o C_b}{2C_o + C_b}\int_0^\infty \delta\left(t'\right)x\left(t - t'\right)dt'$$

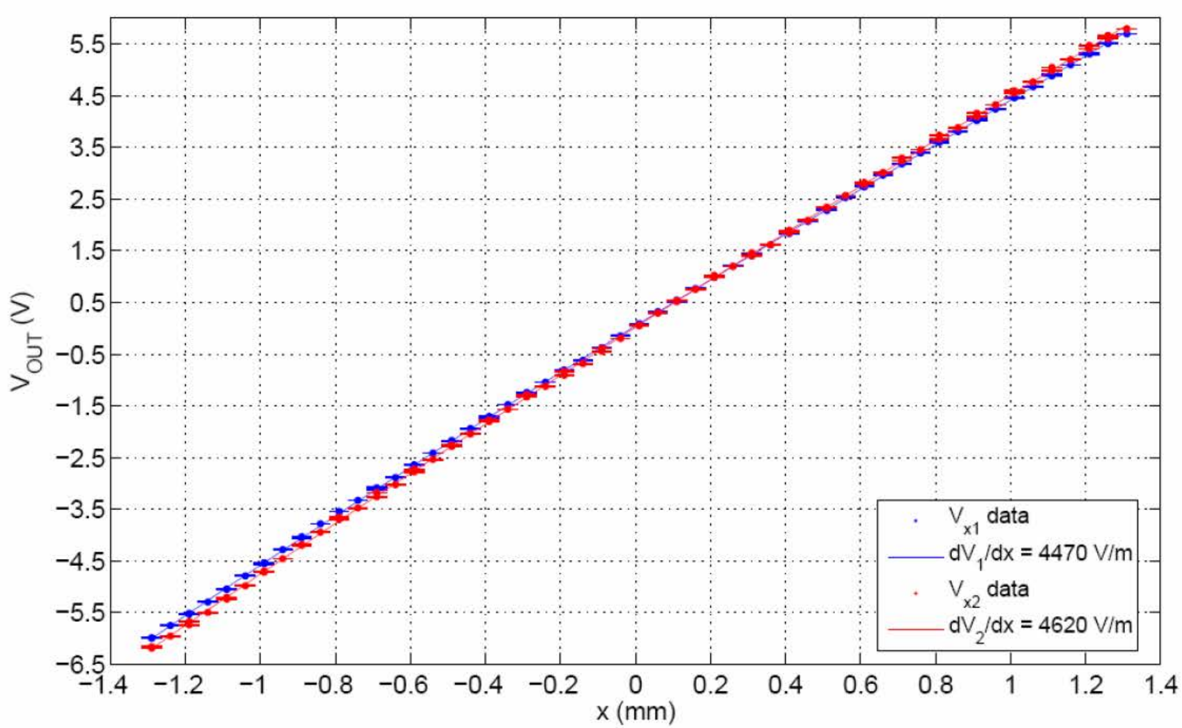
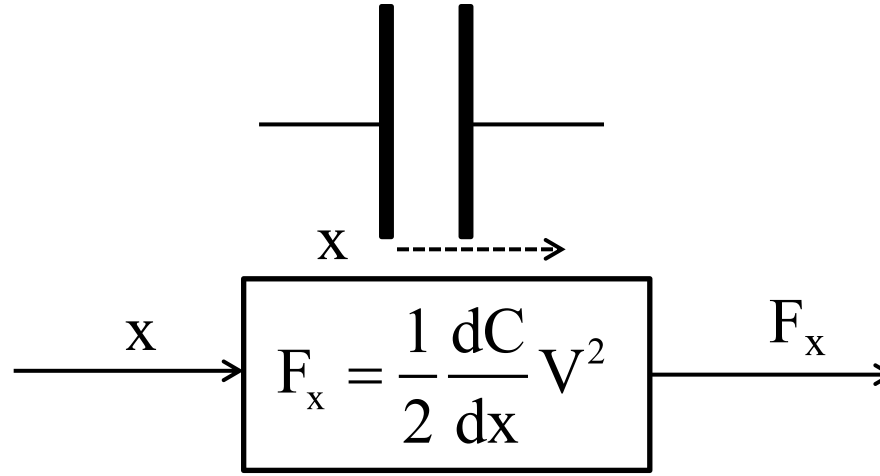


Figure 28: calibration of the x channels, obtained by comparison of the output voltages provided by the FEE when the EH is translated with respect of the TM by means of the linear micromanipulator. The experimental data are compared with a best-fit curve based on a 3rd order polynomial curve, accounting for the intrinsic nonlineary of the capacitive sensor and for the positioner residual "wobble"; the sensitivity shown in the caption corresponds to the linear term of the polynomial curve. In this configuration the TM voltage can be estimated from the sensor electrostatic model to be $V_M \approx 0.62$ V.

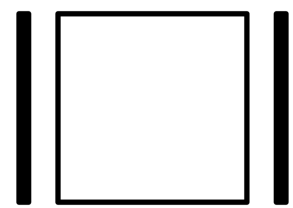
Notes

- The approximation hold until $(x/d_o)^2$ can be neglected. With 4 mm we see discrepancies already at 200 μm.
- The system is memory-less to our scopes, provided x varies slowly enough. Otherwise the change of capacitances are not "adiabatic" anymore for our circuit, and non-linearity would influence the time response.
- Memory effects are obviously introduced in all following stages (transformer, amplifiers) but these are reasonably linear.

Same system, different output: force



- The electric field exerts a force on capacitor plates that tends to squeeze the gap
- If one of the plates is the test-mass the force is exerted on the test mass. We have a system with displacement at input and force at the output
- Test-mass between plates. Assume: same test-mass at V and electrodes at zero



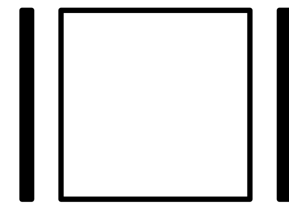
$$F_x = \frac{1}{2}V^2\left(\frac{dC_1}{dx} + \frac{dC_2}{dx}\right) = \frac{C_o V^2}{2}\frac{d}{dx}\left(1 + \frac{x}{d_o} + \left(\frac{x}{d_o}\right)^2 + 1 - \frac{x}{d_o} + \left(\frac{x}{d_o}\right)^2\right)$$

$$= \frac{C_o V^2}{2} 4\frac{x}{d_o^2} \quad = \frac{2C_o V^2}{d_o^2}x$$

Same system, different output: force

- System become memory-less and linear. Impulse response is

$$F_x\left(t\right)=\frac{2C_oV^2}{d_o^2}\int_0^{\infty}\delta\left(t'\right)x\left(t-t'\right)dt'$$



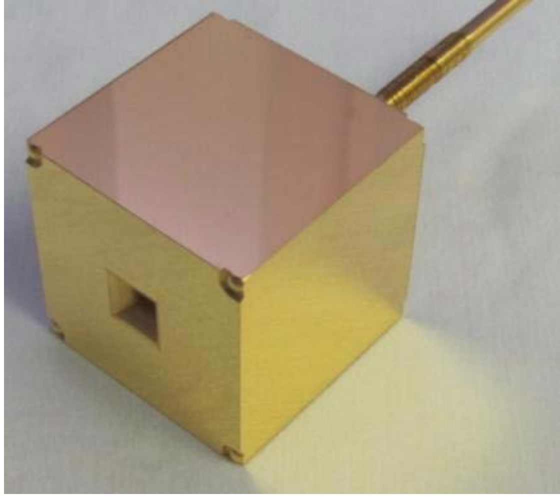
- Notice that the system behaves like a spring

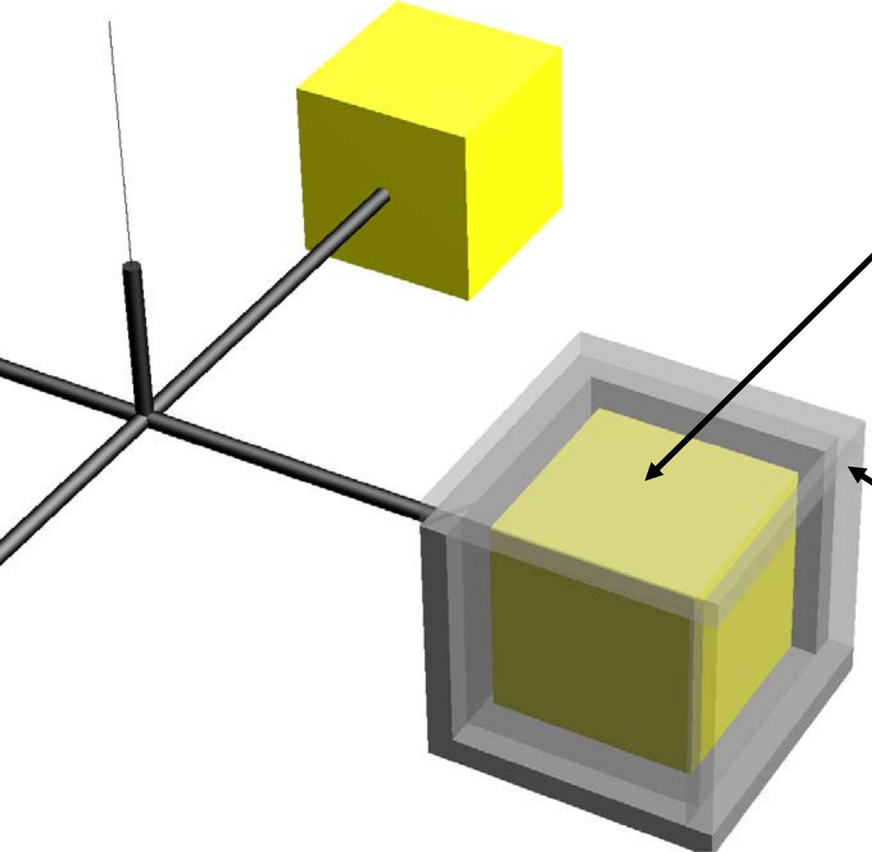
$$F_x=-\left(-\frac{2C_oV^2}{d_o^2}\right)x$$

- With negative stiffness

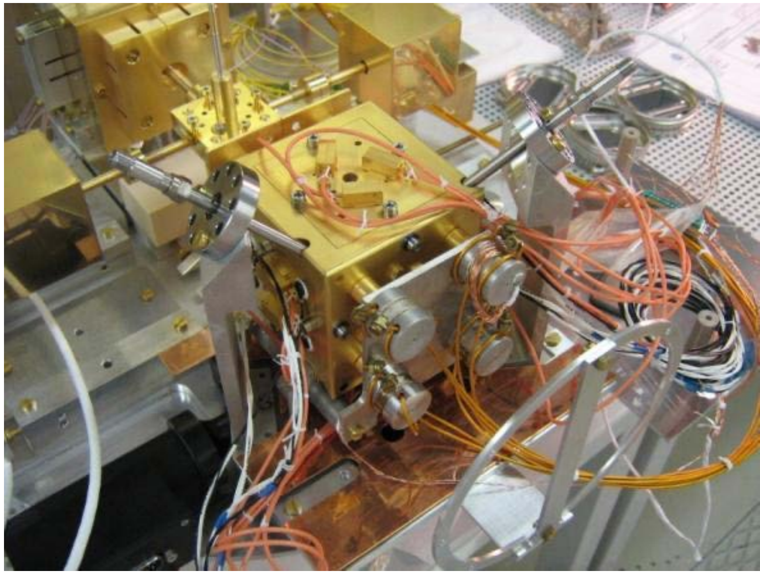
$$\kappa=-\frac{2C_oV^2}{d_o}<0$$

Testing on a torsion pendulum



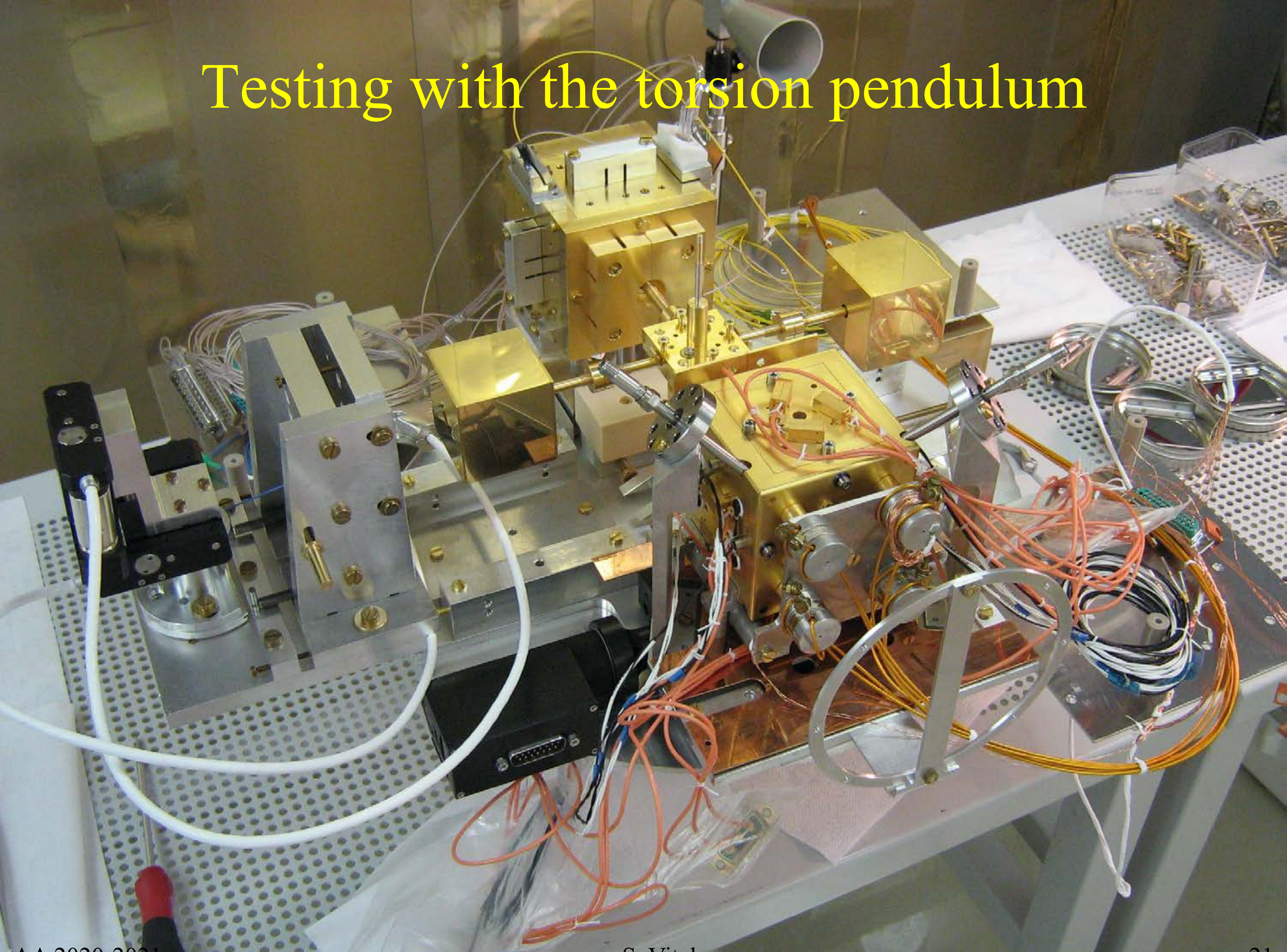


Test-mass (hollow)



Disturbing surroundings (GRS)

Testing with the torsion pendulum



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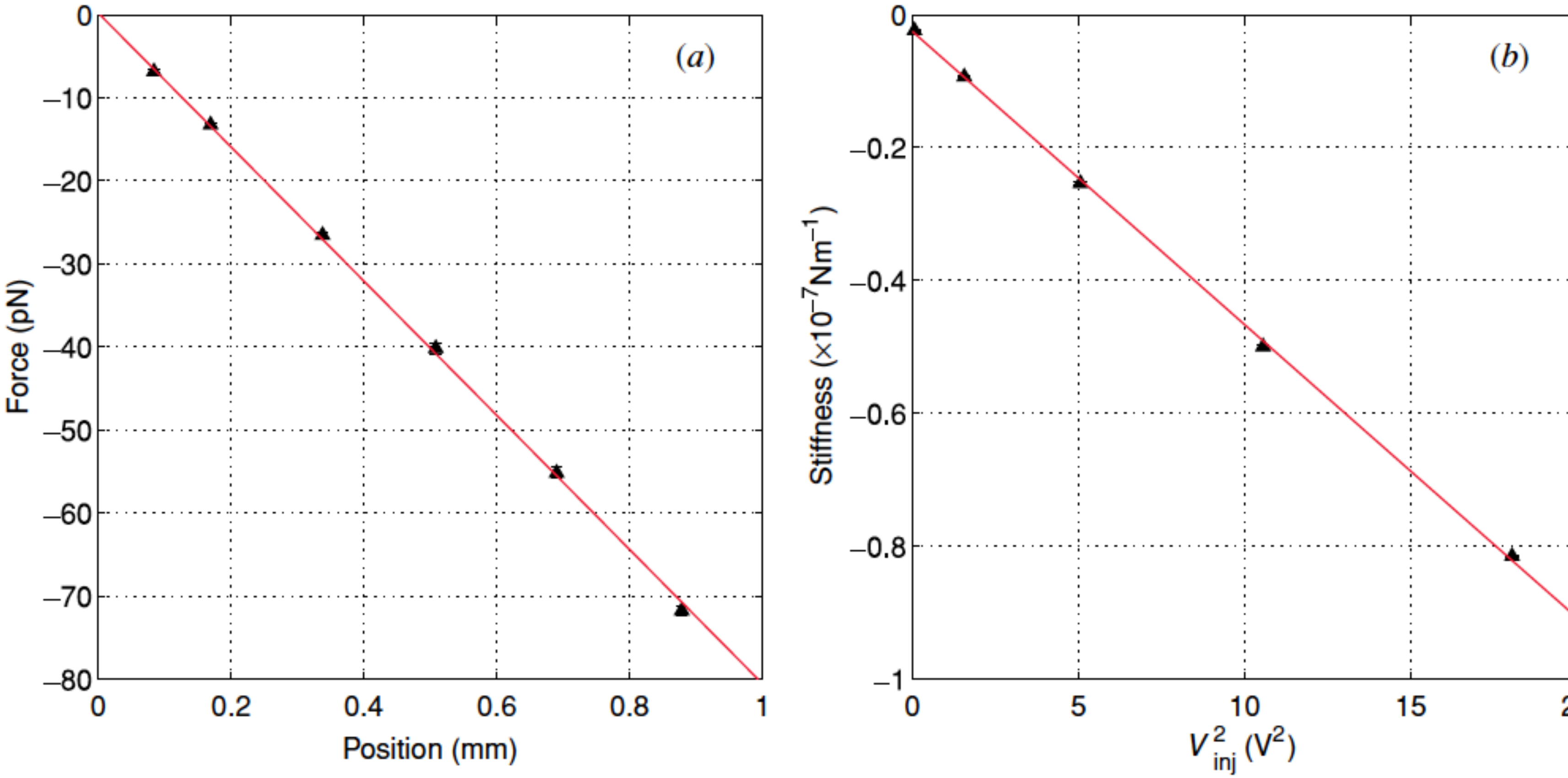


Figure 6. (*a*) A plot of the position-dependent force on the test mass. (*b*) The dependence of stiffness on injection voltage, the intercept is the remnant stiffness from all sources other than the AC injection voltages. Errorbars are smaller than the data markers.

Invertible operator, the role of input and output, and free evolution 1/3

- Some linear systems are described by operators that have an inverse:

$$o(t) = \Im\left[i(t)\right] \quad i(t) = \Im^{-1}\left[o(t)\right]$$

- That is, the equation

$$o(t) = \Im\left[i(t)\right]$$

  has just one solution
- For these systems the role of input and output can be interchanged
  - Example 1: multiplication by a constant

$$o(t) = c\,i(t) \quad i(t) = \frac{1}{c}o(t)$$

  - Example 2: Fourier transform

$$o(\omega) = \int_{-\infty}^{\infty} i(t)e^{-i\omega t}\,dt \qquad i(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} o(t)e^{i\omega t}\,dt$$

Invertible operator, the role of input and output, and free evolution 2/3

- Some linear systems (e.g. those described by differential equations) are better described by

$$D\left[o(t)\right]=i(t)$$

- For these system there may be n solutions:

$$D\left[o_j(t)\right]=0$$

- Because the operator is linear

$$D\left[o(t)+\sum_{j=1}^{n}c_j\,o_j(t)\right]=D\left[o(t)\right]+\sum_{j=1}^{n}c_j\,D\left[o_j(t)\right]=i(t)$$

- (c_j arbitrary constants)

Invertible operator, the role of input and output, and free evolution 3/3

$$D\left[o(t)+\sum_{j=1}^{n} c_j\, o_j(t)\right]=D\left[o(t)\right]+\sum_{j=1}^{n} c_j\, D\left[o_j(t)\right]=i(t)$$

- For all functions such that

$$\tilde{o}(t)\neq\sum_{j=1}^{n} c_j\, o_j(t)$$

- D[õ(t)]=i(t) has an inverse

$$\tilde{o}(t)=D^{-1}\left[i(t)\right]=\int_{-\infty}^{\infty} h(t,t')\,i(t')\,dt'$$

- Finally

$$o(t)=\underbrace{\sum_{j=1}^{n} c_j\, o_j(t)}_{\text{Free evolution: output without an input}}+\int_{-\infty}^{\infty} h(t,t')\,i(t')\,dt'$$

An example: the pendulum

- Equation of motion for the x coordinate

$$m\ddot{x} = -mg\,\mathrm{Sin}(\theta)\,\mathrm{Cos}(\theta) + F_{ex}$$

- Converting angles

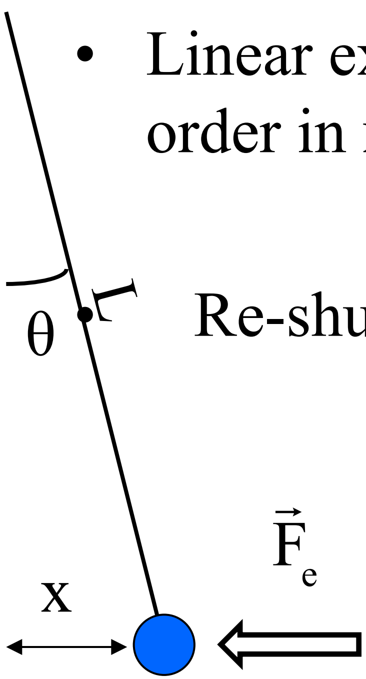
$$m\ddot{x} = -mg\frac{x}{L}\sqrt{1-\left(\frac{x}{L}\right)^2} + F_{ex}$$

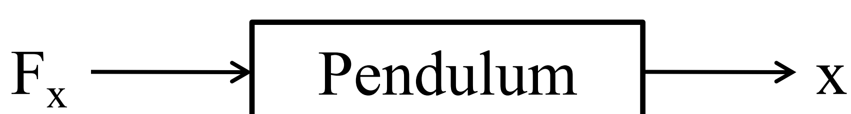
- Linear expansion to first order in x

$$m\ddot{x} = -mg\frac{x}{L} + F_{ex}$$

Re-shuffling

$$\ddot{x} + \frac{g}{L}x \equiv \ddot{x} + \omega_o^2 x = F_{ex}/m$$





A basic example: the pendulum

- Linearized equation for the x coordinate $\ddot{x} + \omega_o^2 x = F_{ex}/m$
- Special solution $\tilde{x}(t) = \frac{1}{m\,\omega_o}\int_0^\infty \mathrm{Sin}(\omega_o t')F_{ex}(t-t')dt'$
- Indeed $\dot{\tilde{x}}(t) = -\frac{1}{m\,\omega_o}\int_0^\infty \mathrm{Sin}(\omega_o t')\,\frac{dF_{ex}(t-t')}{dt'}dt' =$

$$= \frac{1}{m}\int_0^\infty \mathrm{Cos}(\omega_o t')F_{ex}(t-t')dt'$$

- and $\ddot{\tilde{x}}(t) = -\frac{1}{m}\int_0^\infty \mathrm{Cos}(\omega_o t')\,\frac{dF_{ex}(t-t')}{dt'}dt' =$

$$= -\frac{1}{m}\mathrm{Cos}(\omega_o t')F_{ex}(t-t')]_0^\infty - \frac{\omega_o}{m}\int_0^\infty \mathrm{Sin}(\omega_o t')F_{ex}(t-t')dt' =$$

$$= \frac{F_{ex}(t)]}{m} - \omega_o^2\tilde{x}(t)$$

- then:

$$\ddot{\tilde{x}}(t) + \omega_o^2\tilde{x}(t) = \frac{F_{ex}(t)]}{m} - \omega_o^2\tilde{x}(t) + \omega_o^2\tilde{x}(t) = \frac{F_{ex}(t)]}{m}$$