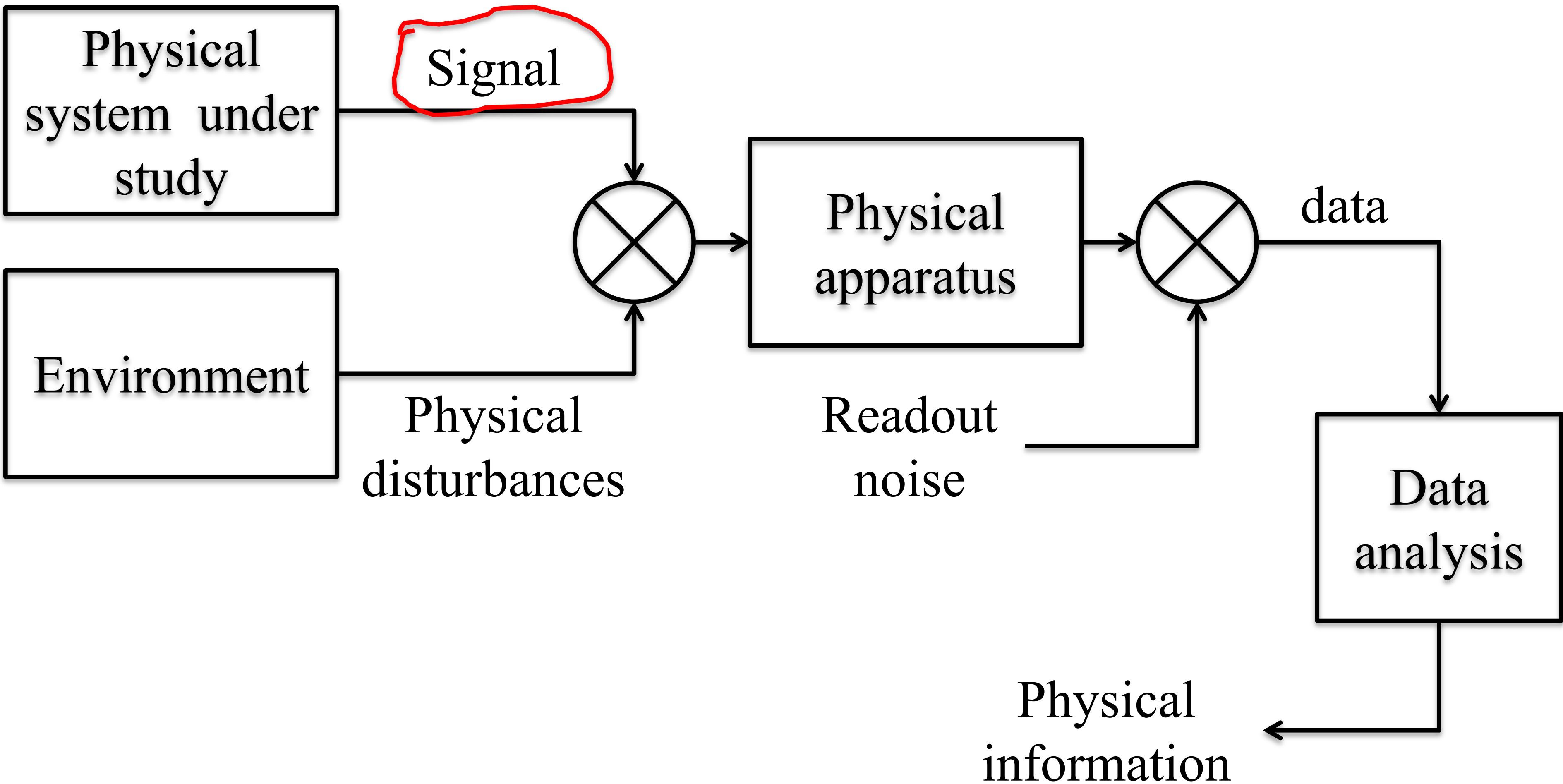
Experimental Methods
Lecture 2

September 23rd, 2020

My personal concept for a physical experiment



Signals

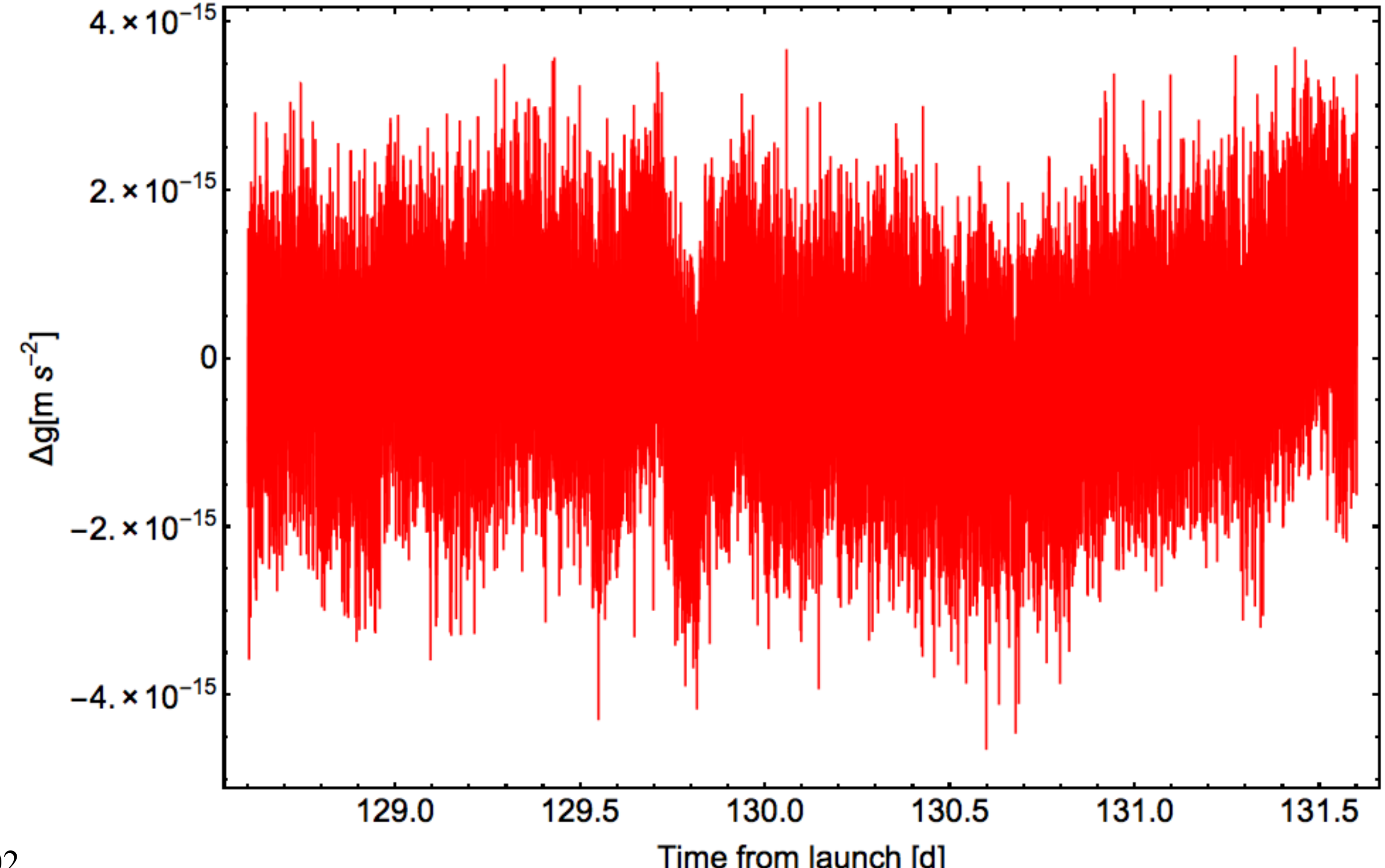
- A measurable quantity that depends on one or more measurable parameters

$$s\left(t,\theta_1,\theta_2.......\right)$$

Measured quantities

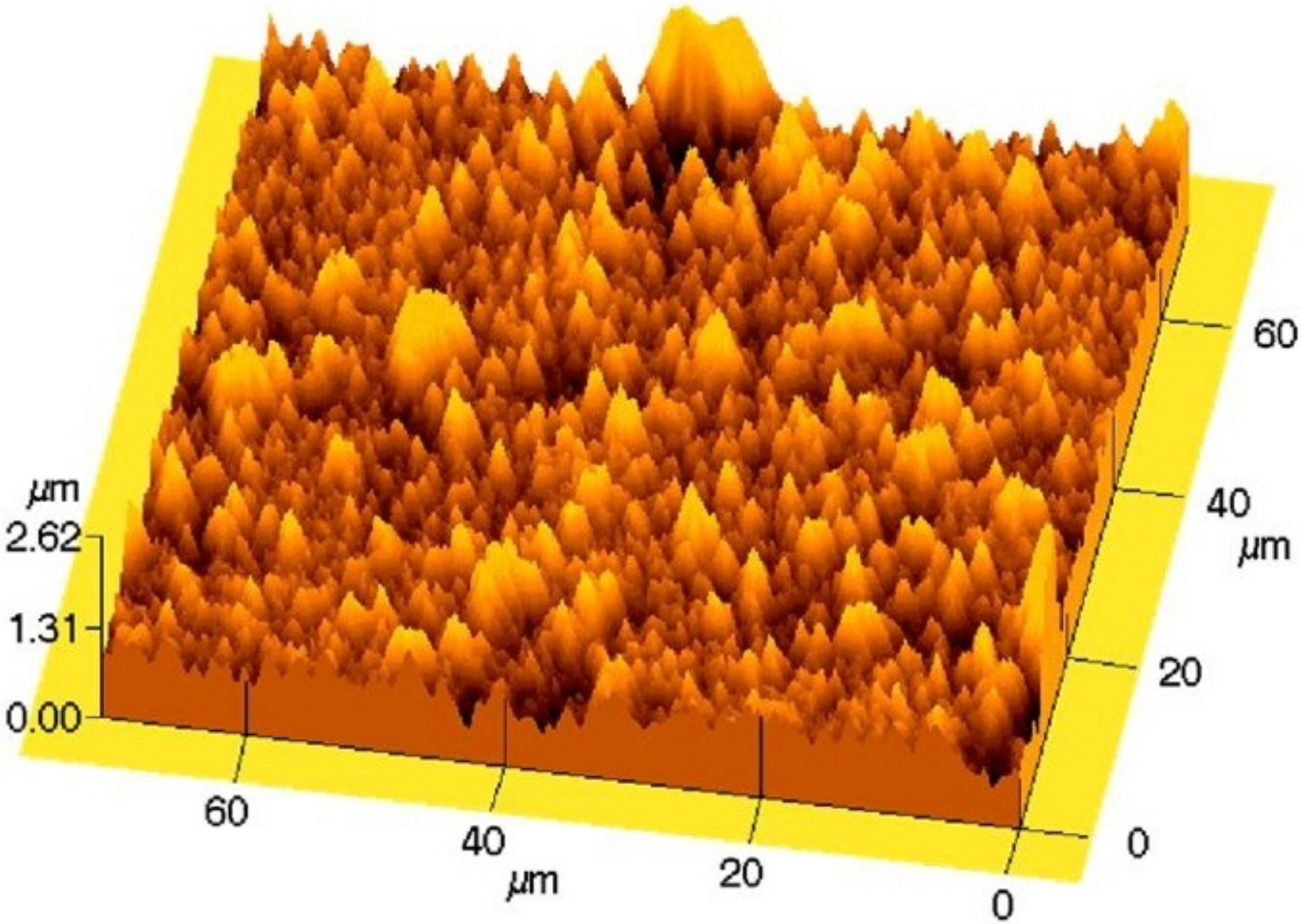
One example: LISA Pathfinder test-masses relative acceleration.

- Measurable quantity is acceleration
- Parameter is time



Atomic force microscopy

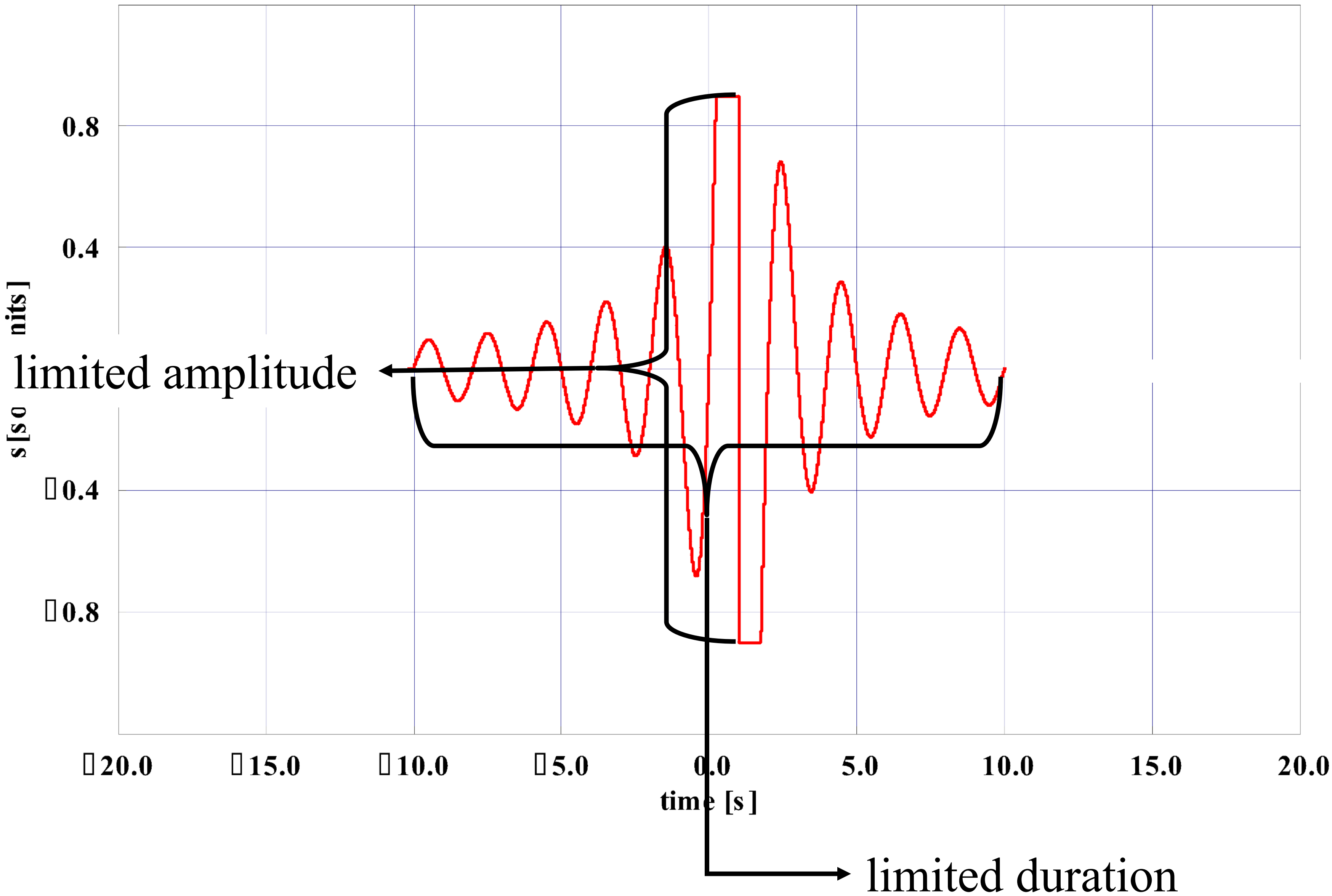
- Parameters are space coordinates. Measurable quantity is force (height)



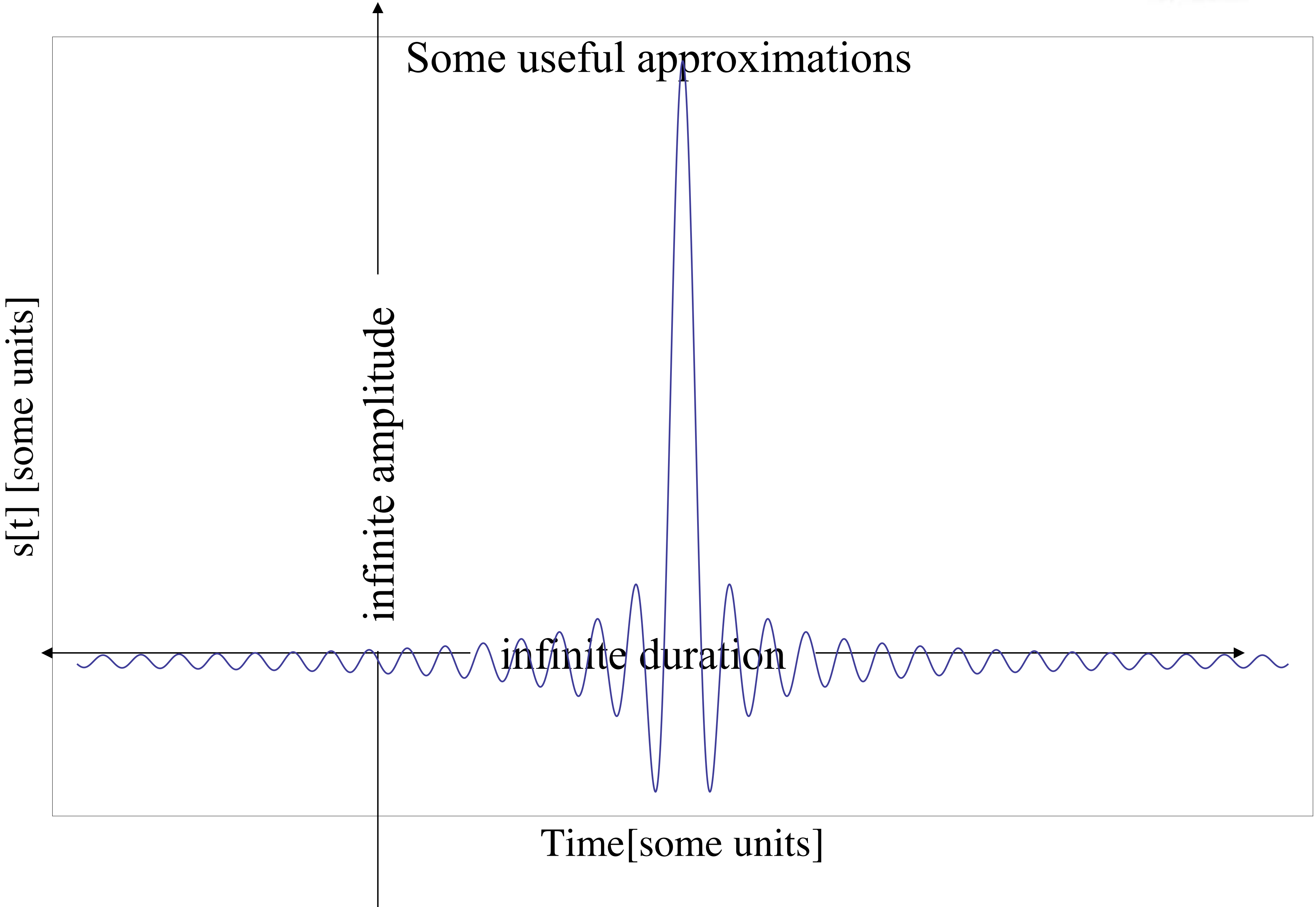
Real signals and mathematical signals

- Real signals are always limited in duration and amplitude
- Real signals cannot change instantaneously
- However in some calculations it is also useful to introduce “mathematical” signals of infinite duration and amplitudes, deltas and steps, etc.

Real Signals



Some useful approximations



Fourier transforms

- Here we start a quick refresh on Fourier transforms.
- Harmonic analysis is the most powerful tool to understand what information a signal carries on.
- The student is supposed to know the basics of harmonic analysis.
- The following pages are only meant to illustrate some implication of its most relevant results.

Fourier Transform (a quick primer)

- if

$$\int_{-\infty}^{\infty} |s(t)| dt < \infty$$

- (always true for real signals)
- (BTW: find some examples of “mathematical” signals for which this is not true)
- Then: $$s(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t} dt \quad \text{and} \quad s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s(\omega) e^{i\omega t} d\omega$$

- By the way, s(ω) is a signal!
- Very important transformation. Reasons will be clear later

One can Fourier-transform even some weird functions (distributions):

$$\int_{-\infty}^{\infty} \delta(t) e^{-i\omega t}\, dt = 1 \text{ and } \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{i\omega t}\, d\omega = \frac{1}{2\pi}$$

It follows, form inverse transformation, that:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} d\omega = \delta(t) \text{ and } \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t} dt = \delta(\omega)$$

Examples (with Mathematica™)

$f(t)$	$f(\omega)$
$e^{-\frac{t^2}{2\,\Delta T^2}}$	$e^{-\frac{1}{2}\Delta T^2\,\omega^2}\sqrt{2\,\pi}\;\Delta T$
$\delta[t]$	1
$\dfrac{-\theta[t-\frac{\Delta T}{2}]+\theta[t+\frac{\Delta T}{2}]}{\Delta T}$	$\dfrac{2\,\mathrm{Sin}[\frac{\Delta T\,\omega}{2}]}{\Delta T\,\omega}$
$\dfrac{e^{-\frac{\mathrm{Abs}[t]}{\Delta T}}}{\Delta T}$	$\dfrac{2}{1+\Delta T^2\,\omega^2}$
$\mathrm{Sin}[t\,\omega o]$	$-i\,\pi\,\mathrm{DiracDelta}[-\omega+\omega o]+i\,\pi\,\mathrm{DiracDelta}[\omega+\omega o]$
$\mathrm{Cos}[t\,\omega o]$	$\pi\,\mathrm{DiracDelta}[-\omega+\omega o]+\pi\,\mathrm{DiracDelta}[\omega+\omega o]$

Basic properties of Fourier transform

- Fourier transform is a linear operation

$$\int_{-\infty}^{\infty}\left[c_1 s_1(t)+c_2 s_2(t)\right]e^{-i\omega t}dt = c_1 s_1(\omega)+c_2 s_2(\omega)$$

- Fourier transform of a linear combination is the linear combination of the transforms

- Derivatives

$$\frac{ds(t)}{dt}=\frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}s(\omega)e^{i\omega t}d\omega\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}s(\omega)\frac{de^{i\omega t}}{dt}d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}i\omega s(\omega)e^{i\omega t}d\omega$$

- That is: the Fourier transform of the derivative is equal to the Fourier transform of the function multiplied by iω

$$\frac{ds(t)}{dt}\xrightarrow[\text{Fourier}]{}i\omega s(\omega)$$

A very important property: the convolution theorem

- In the following pages we'll show that

$$\int_{-\infty}^{\infty} s(t)q(t)e^{-i\omega t}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega')q(\omega-\omega')\,d\omega'$$

- That is the Fourier transform of a product of two functions

$$s(t)q(t)$$

- is equal to the convolution

$$s(\omega)*q(\omega) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega')q(\omega-\omega')\,d\omega'$$

of their transforms

- Symmetrically (watch out for the 2 π!)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(t')q(t-t')e^{-i\omega t}\,dt'\,dt = s(\omega)q(\omega)$$

Convolution theorem

$$\int_{-\infty}^{\infty} s(t)q(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega')e^{i\omega' t}d\omega'\frac{1}{2\pi}\int_{-\infty}^{\infty} q(\omega'')e^{i\omega'' t}d\omega''\right]e^{-i\omega t}dt$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega')d\omega'\frac{1}{2\pi}\int_{-\infty}^{\infty} q(\omega'')d\omega''\right]e^{-i(\omega-\omega'-\omega'')t}dt$$

but $\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega-\omega'-\omega'')t}dt = \delta(\omega-\omega'-\omega'') \quad$ so

$$\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega')d\omega'\frac{1}{2\pi}\int_{-\infty}^{\infty} q(\omega'')d\omega''\right]e^{-i(\omega-\omega'-\omega'')t}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega')d\omega'\int_{-\infty}^{\infty} q(\omega'')\delta(\omega-\omega'-\omega'')d\omega''$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega')q(\omega-\omega')d\omega'$$

A key example: narrow band functions

- Two (ever lasting) sinusoids $\mathrm{Sin}(\omega_o t) \quad \mathrm{Cos}(\omega_o t)$
- Their Fourier Transforms $\frac{\pi}{i}\left[\delta(\omega-\omega_o)-\delta(\omega+\omega_o)\right]$

$$\pi\left[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)\right]$$

- Product with two arbitrary functions $s(t)=a(t)\mathrm{Sin}(\omega_o t)+b(t)\mathrm{Cos}(\omega_o t)$
- Fourier transform using convolution theorem

$$s(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} a(\omega')\frac{\pi}{i}\left[\delta(\omega-\omega'-\omega_o)-\delta(\omega-\omega'+\omega_o)\right]d\omega'+$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} b(\omega')\pi\left[\delta(\omega-\omega'-\omega_o)+\delta(\omega-\omega'+\omega_o)\right]d\omega'$$

$$=\frac{a(\omega-\omega_o)-a(\omega+\omega_o)}{2i}+\frac{b(\omega-\omega_o)+b(\omega+\omega_o)}{2}$$

A key example: narrow band functions

- Fourier transform of a narrow band signal

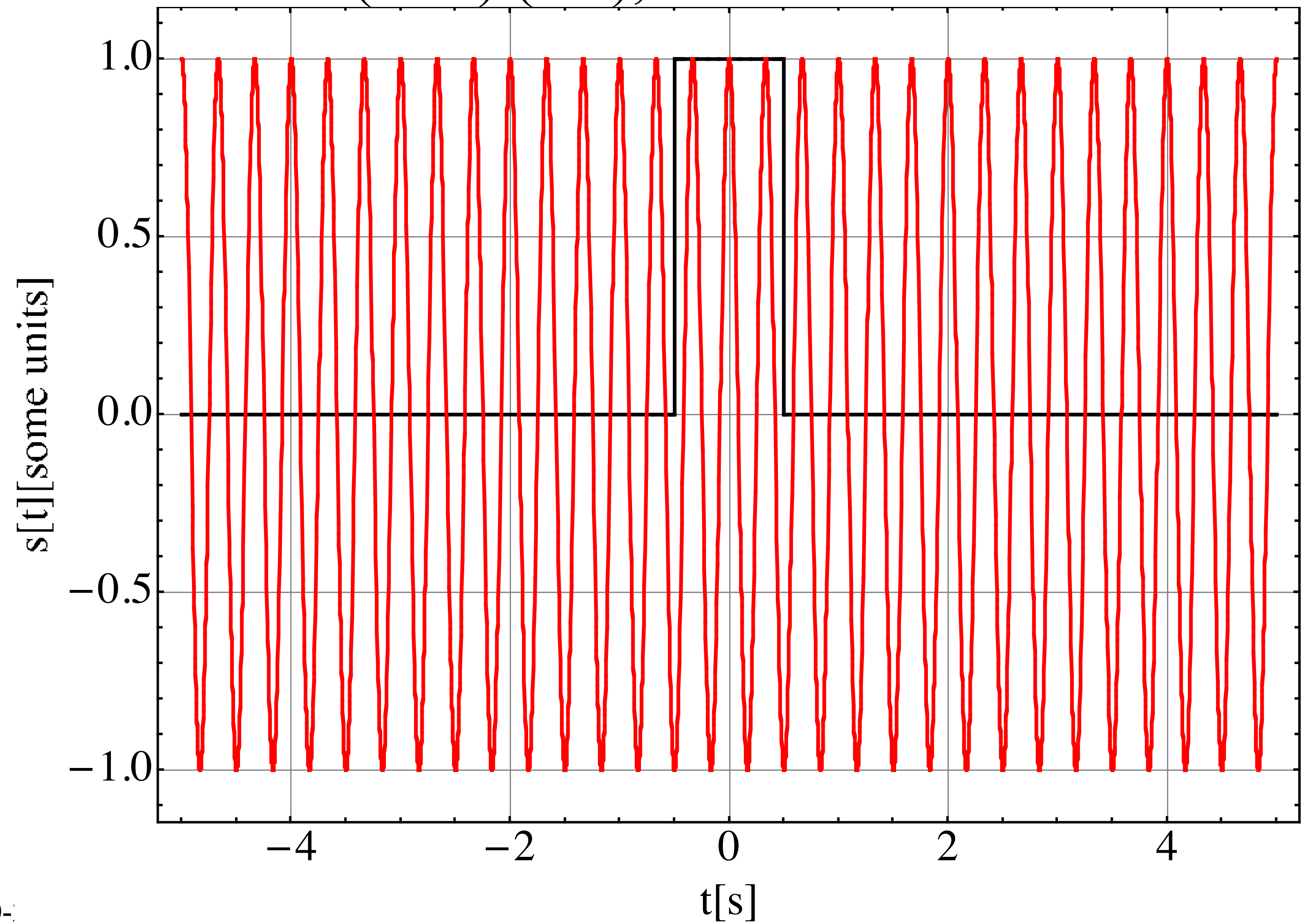
$$s(t)=a(t)\mathrm{Sin}(\omega_o t)+b(t)\mathrm{Cos}(\omega_o t)$$

$$s(\omega)=\frac{a(\omega-\omega_o)-a(\omega+\omega_o)}{2i}+\frac{b(\omega-\omega_o)+b(\omega+\omega_o)}{2}$$

- Two copies of the original spectrum centered at $\omega\pm\omega_o$ instead that at $\omega=0$

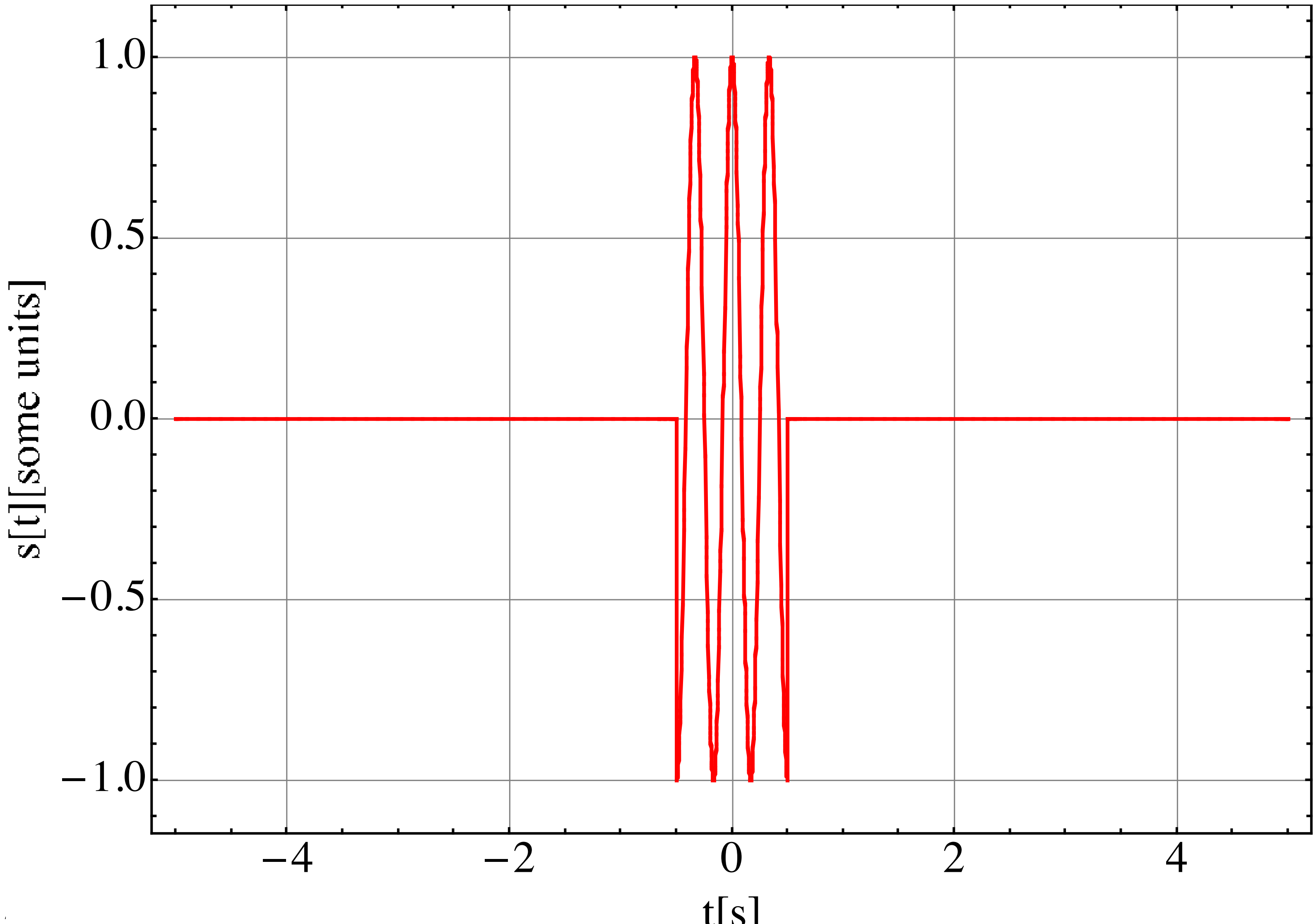
An example: the unit box

- A plot of the unit box Θ(t+ΔT/2)-Θ(t-ΔT/2) (black) and of the function Cos(2πνt) (red), with ΔT=1 s and ν=3 Hz



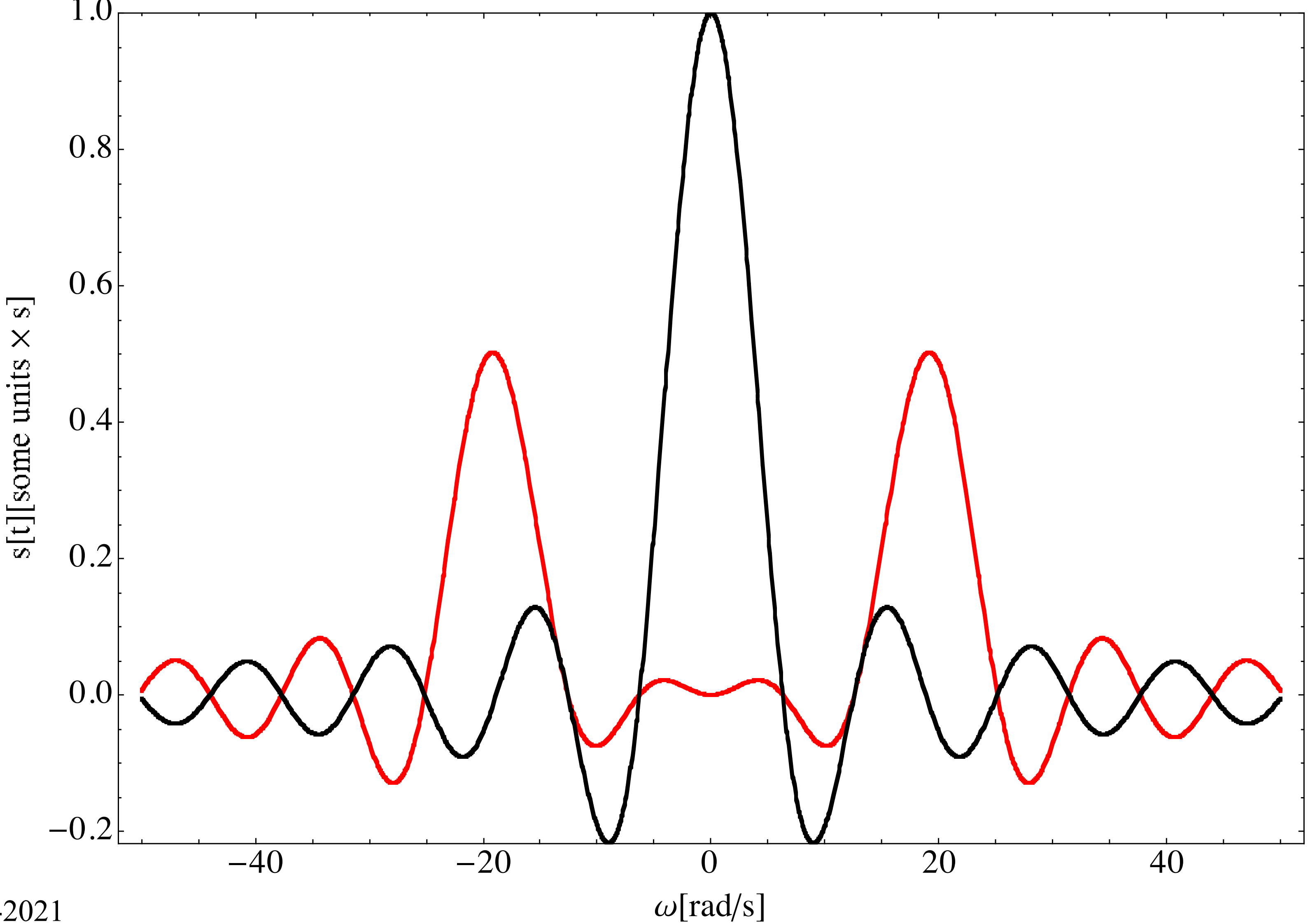
An example: the unit box

- A plot of their product $[\Theta(t+\Delta T/2)-\Theta(t-\Delta T/2)]\mathrm{Cos}(2\pi\nu t)$



An example: the unit box

- box's Fourier Transform, Sinc[ω/2](black) and its narrow band version (Sinc[ω/2-πν]+Sinc[ω/2+πν])/2 (red)



The symmetric formulation

- The convolution of two functions

$$s(t) * q(t) \equiv \int_{-\infty}^{\infty} s(t')q(t-t')dt'$$

- Its Fourier transform

$$F_{s(t)*q(t)}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega t} s(t')q(t-t')dt'dt$$

- Expanding

$$F_{s(t)*q(t)}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\omega t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega')e^{i\omega' t'}d\omega'\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} q(\omega'')e^{i\omega''(t-t')}d\omega''\right)dt'dt$$

- But

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega''-\omega)t}dt = \delta(\omega-\omega'') \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega'-\omega'')t'}dt' = \delta(\omega'-\omega'')$$

- then

$$F_{s(t)*q(t)}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(\omega')q(\omega'')\,\delta(\omega-\omega'')\delta(\omega'-\omega'')d\omega'd\omega'' = s(\omega)q(\omega)$$

Fourier transform conserve information

- Conjugate of transform

$$s^*(-\omega) = \left[\int_{-\infty}^{\infty} s(t)e^{i\omega t}dt\right]^* = \int_{-\infty}^{\infty} s^*(t)e^{-i\omega t}dt$$

  - * stands for conjugate.
- If s(t) is real, s(t)=s*(t), then $s^*(-\omega)=s(\omega)$
- That is

$$s'(\omega)=s'(-\omega) \text{ and } s''(\omega)=-s''(-\omega)$$

  - ´ stands for real part, ″ stands for imaginary part
- Fourier maps one real function s(t) on $-\infty< t < \infty$ into two real functions s' (ω) and s' ' (ω) on $0\leq \omega\leq \infty$
- There is no multiplication of information

Fourier transform conserve information

- If s(t) =s(-t)

$$s^*(-\omega) = s(\omega)$$

$$\int_{-\infty}^{\infty} s(t)e^{-i\omega t}\,dt = \int_{0}^{\infty} s(t)\left(e^{-i\omega t} + e^{+i\omega t}\right)dt = \int_{0}^{\infty} 2s(t)\,\mathrm{Cos}(\omega t)\,dt$$

- Then

$$s''(\omega) = 0$$

- If s(t)=- s(-t)

$$s'(\omega) = 0$$

Transforms conserve information

2) Signal “energy” $$E \underset{\text{def}}{\equiv} \int_{-\infty}^{\infty} \left|s(t)\right|^2 dt$$

$$E = \int_{-\infty}^{\infty} \left|s(t)\right|^2 dt \quad = \int_{-\infty}^{\infty} s(t)s(t)dt \; = \int_{-\infty}^{\infty} s(t)s(t)e^{-i0t}dt$$

Fourier transform of s(t)s (t) at ω=0

$$\int_{-\infty}^{\infty} s(t)q(t)e^{-i\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega')q(\omega-\omega')d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega')s(0-\omega')d\omega' \; = \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega')s(-\omega')d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega')s^{*}(\omega')d\omega' \; = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|s(\omega')\right|^2 d\omega'$$

$f(t)$	$f(\omega)$	$\int_{-\infty}^{\infty} \lvert f(t) \rvert^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert f(\omega) \rvert^2 d\omega$
$e^{-\frac{t^2}{2\,\Delta T^2}}$	$e^{-\frac{1}{2}\Delta T^2\omega^2}\sqrt{2\pi}\,\Delta T$	$\sqrt{\pi}\,\Delta T$	$\sqrt{\pi}\,\Delta T$
$\delta[t]$	1	$\int_{-\infty}^{\infty} \mathrm{Abs}[\delta[t]]^2\, dt$	∞
$\frac{-\theta\left[t-\frac{\Delta T}{2}\right]+\theta\left[t+\frac{\Delta T}{2}\right]}{\Delta T}$	$\frac{2\,\mathrm{Sin}\left[\frac{\Delta T\,\omega}{2}\right]}{\Delta T\,\omega}$	$\frac{1}{\Delta T}$	$\frac{1}{\Delta T}$
$\frac{e^{-\frac{\mathrm{Abs}[t]}{\Delta T}}}{\Delta T}$	$\frac{2}{1+\Delta T^2\,\omega^2}$	$\frac{1}{\Delta T}$	$\frac{1}{\Delta T}$
$\mathrm{Sin}[t\,\omega o]$	$-i\,\pi\,\delta[-\omega+\omega o]+i\,\pi\,\delta[\omega+\omega o]$	∞	$\frac{\int_{-\infty}^{\infty} \mathrm{Abs}[-i\,\pi\,\delta[-\omega+\omega o]+i\,\pi\,\delta[\omega+\omega o]]^2\, d\omega}{2\pi}$
$\mathrm{Cos}[t\,\omega o]$	$\pi\,\delta[-\omega+\omega o]+\pi\,\delta[\omega+\omega o]$	∞	$\frac{\int_{-\infty}^{\infty} \mathrm{Abs}[\pi\,\delta[-\omega+\omega o]+\pi\,\delta[\omega+\omega o]]^2\, d\omega}{2\pi}$

It's a special case of Parseval relation

- The "scalar product" of two functions

$$\int_{-\infty}^{\infty} f(t)g(t)dt$$

- Can always be read as

$$\int_{-\infty}^{\infty} f(t)g(t)e^{-i0t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega')g(0-\omega')d\omega' = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega')g^{*}(\omega')d\omega'$$

- Thus the scalar product is invariant for Fourier Transforms

$$\int_{-\infty}^{\infty} f(t)g(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega')g^{*}(\omega')d\omega'$$