Experimental Methods
Lecture 24

November 16th, 2020

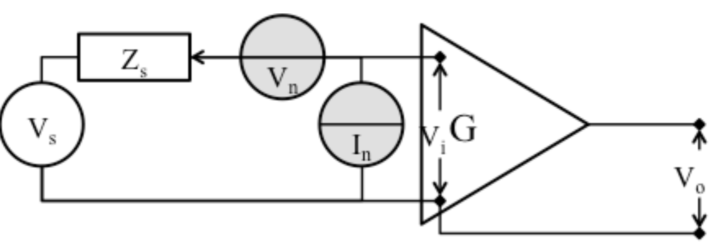
Critical noise parameters for a 2-port device

- Noise at input $V_n^{in} = V_n + I_n Z_s$
- Before proceeding further let's give a few important definitions:

1. Noise energy or temperature $E_n(\omega) = k_B T_n(\omega) \equiv \sqrt{S_{V_nV_n}(\omega) S_{I_nI_n}(\omega)}$
2. Noise Resistance $R_n(\omega) \equiv \sqrt{S_{V_nV_n}(\omega) / S_{I_nI_n}(\omega)}$
3. Noise cross-coherence $\rho_n(\omega) = S_{V_nI_n}(\omega) \Big/ \sqrt{S_{V_nV_n}(\omega) S_{I_nI_n}(\omega)}$

- Inverting these definitions

1. Voltage PSD: $S_{V_nV_n}(\omega) = k_B T_n(\omega) R_n(\omega)$
2. Current PSD $S_{I_nI_n}(\omega) = k_B T_n(\omega) / R_n(\omega)$
3. Voltage-current cross spectrum $S_{V_nI_n}(\omega) = k_B T_n(\omega) \rho_n(\omega)$



Minimizing noise by impedance matching

- Total noise PSD at input, at a given frequency

$$S_{V_n^{in}V_n^{in}}(\omega)=k_B T_n(\omega)\left[R_n(\omega)+\left|Z_s(\omega)\right|^2\Big/R_n(\omega)+2\,\mathrm{Re}\left\{Z_s(\omega)\rho_n(\omega)\right\}\right]$$

- Can be minimized by adjusting R_n

$$\partial S_{V_n^{in}V_n^{in}}(\omega)\Big/\partial R_n(\omega)=k_B T_n(\omega)\left[1-\left|Z_s(\omega)\right|^2\Big/R_n^2(\omega)\right]=0$$

- That gives $R_n(\omega)=\left|Z_s(\omega)\right|$
- When this condition is fulfilled the contribution of voltage noise and of current noise become equal and the PSD becomes

$$S_{V_n^{in}V_n^{in}}(\omega)=2k_B T_n(\omega)\left[\left|Z_s(\omega)\right|+\mathrm{Re}\left\{Z_s(\omega)\rho_n(\omega)\right\}\right]$$

- Or, for uncorrelated generators

$$S_{V_n^{in}V_n^{in}}(\omega)=2k_B T_n(\omega)\left|Z_s(\omega)\right|$$

- Watch out! This is not Nyquist law, and T_n is not T

Noise figure and matched source

- From the PSD

$$S_{V_n^{in}V_n^{in}} = k_B T_n \left[R_n + \left|Z_s\right|^2 \big/ R_n + 2\,\mathrm{Re}\left\{Z_s \rho_n\right\} \right] + 2k_B T\,\mathrm{Re}\left\{Z_s\right\}$$

- We define the noise figure as the ratio, *measured in decibels,* between the total noise PSD and that of the thermal noise alone

$$F = 20\mathrm{Log}_{10}\left\{ \frac{k_B T_n \left[R_n + \left|Z_s\right|^2 \big/ R_n + 2\,\mathrm{Re}\left\{Z_s \rho_n\right\} \right] + 2k_B T\,\mathrm{Re}\left\{Z_s\right\}}{2k_B T\,\mathrm{Re}\left\{Z_s\right\}} \right\}$$

- That is $F = 20\mathrm{Log}_{10}\left\{ 1 + \frac{T_n}{T}\left[\frac{R_n}{2\,\mathrm{Re}\left\{Z_s\right\}} + \frac{\left|Z_s\right|^2}{2R_n\,\mathrm{Re}\left\{Z_s\right\}} + \frac{\mathrm{Re}\left\{Z_s \rho_n\right\}}{\mathrm{Re}\left\{Z_s\right\}} \right] \right\}$

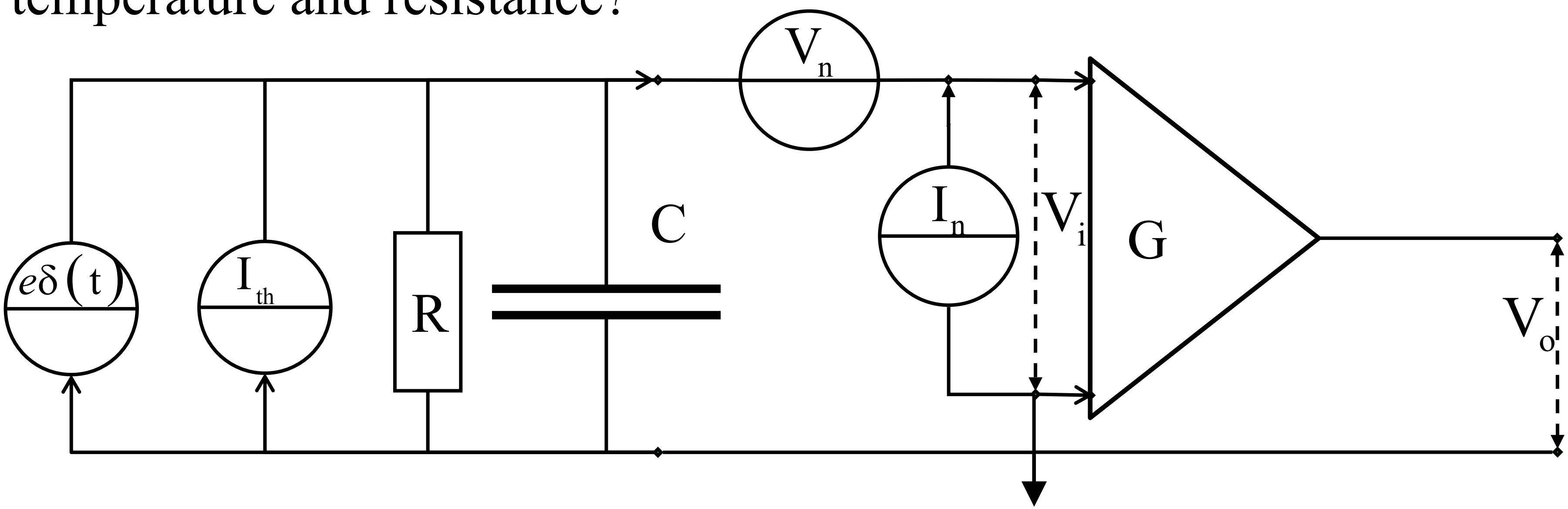
- For uncorrelated noise sources this figure has a minimum when

$$Z_s = R_n$$

- Whose value is $F_{min} = 20\mathrm{Log}_{10}\left\{ 1 + \frac{T_n}{T} \right\}$

The meaning of noise energy. One example: charge detector

- A charge detector consisting of a lossy capacitor read out by a low noise amplifier. I_{th} is the Nyquist current noise generator associated with R. We will assume V_n and I_n to be uncorrelated.
- The signal at input consists of a single charge *e*. That is, the signal consists of a current impulse $I(t)=e\delta(t)$
- What is the minimum measurable value of *e*, given the amplifier noise temperature and resistance?



Charge uncertainty

- Thus,

$$\sigma_e = \sqrt{2S_{Io}\tau}$$

- And

$$S_{I_{n,e}I_{n,e}}(\omega) = \left(S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2}\right)\left(1 + \frac{S_{V_nV_n}C^2}{S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2}}\omega^2\right)$$

- with

$$S_{I_{n,e}I_{n,e}}(\omega) = S_{Io}(1 + \tau^2\omega^2)$$

- Then

$$\sigma_e = \sqrt{2\left(S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2}\right)\sqrt{\frac{S_{V_nV_n}C^2}{S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2}}}} = \sqrt{2C}\left(S_{V_nV_n}\left(S_{I'_nI'_n} + \frac{S_{V_nV_n}}{R^2}\right)\right)^{\frac{1}{4}}$$

- Now take the limit for $R \to \infty$ (low dissipations)
- Then

$$\sigma_e \to \sqrt{2C\sqrt{S_{V_nV_n}S_{I'_nI'_n}}} = \sqrt{2\,Ck_BT'_n}$$

The noise energy

- In conclusion

$$\sigma_e = \sqrt{2\,Ck_B T_n'}$$

- Now suppose that a charge σ_e is deposited on the capacitor. This will get an energy

$$E = \frac{1}{2} C\sigma_e^2 = k_B T_n'$$

- This is the crucial result: the minimum detectable energy is the noise energy of the amplifier
- Notice that if $T \to 0$ $T_n' \to T_n$ as the thermal noise becomes negligible

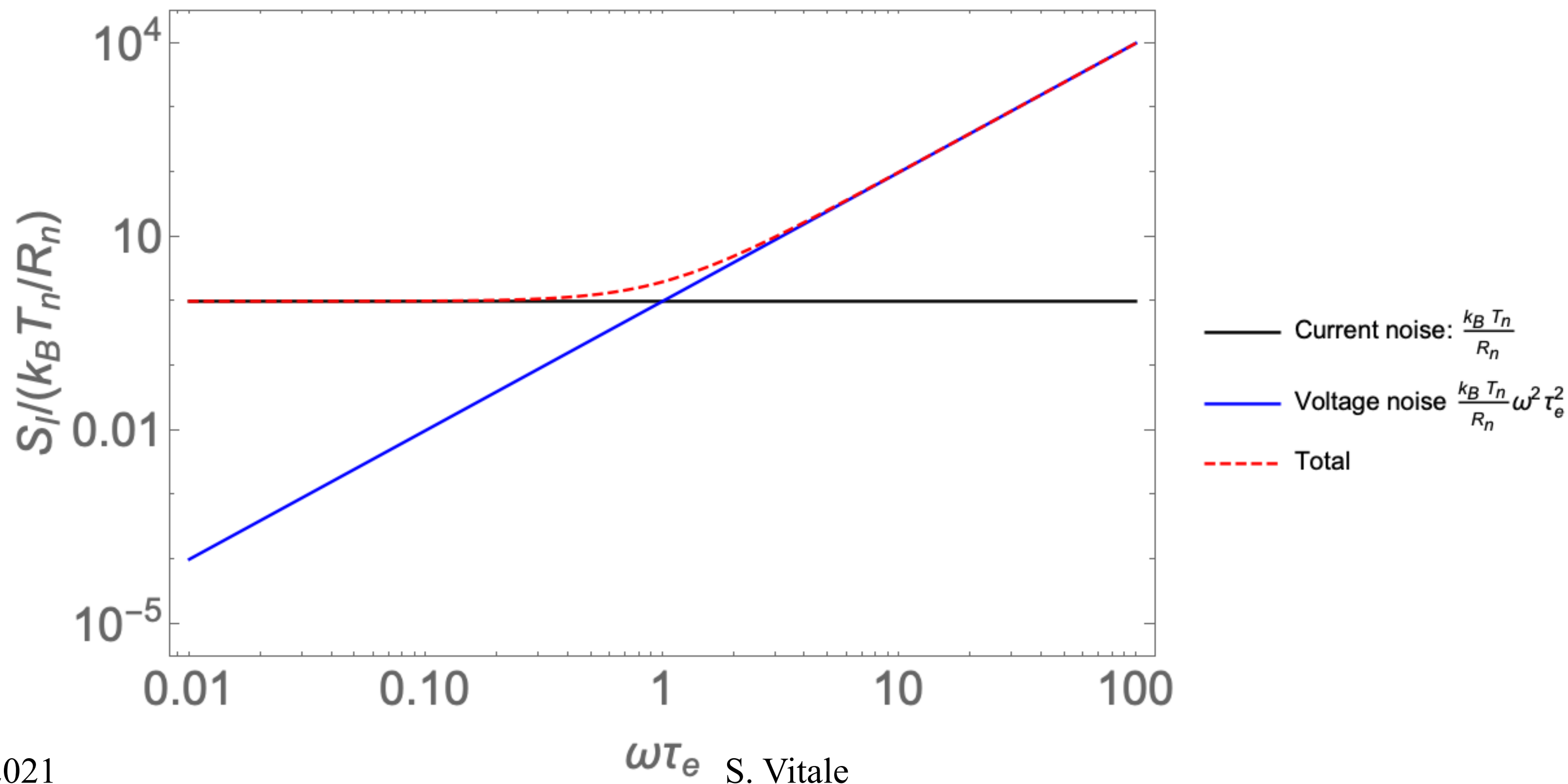
Total noise at input

- Total equivalent current noise through Z

$$S_{I_n I_n} = S_{Io}(1 + \omega^2\tau^2)$$

$$S_{Io} = \frac{2k_B T}{R} + \frac{k_B T_n}{R_n} + \frac{k_B T_n R_n}{R^2} \quad \tau^2 = C^2 R_n^2 \frac{T_n/R_n}{\frac{2T}{R} + \frac{T_n}{R_n} + \frac{T_n R_n}{R^2}}$$

- For $R \to \infty$ $S_{Io} \to \frac{k_B T_n}{R_n}$ $\tau \to C R_n$



charge detector

- The template.
  - The (unit-amplitude) signal

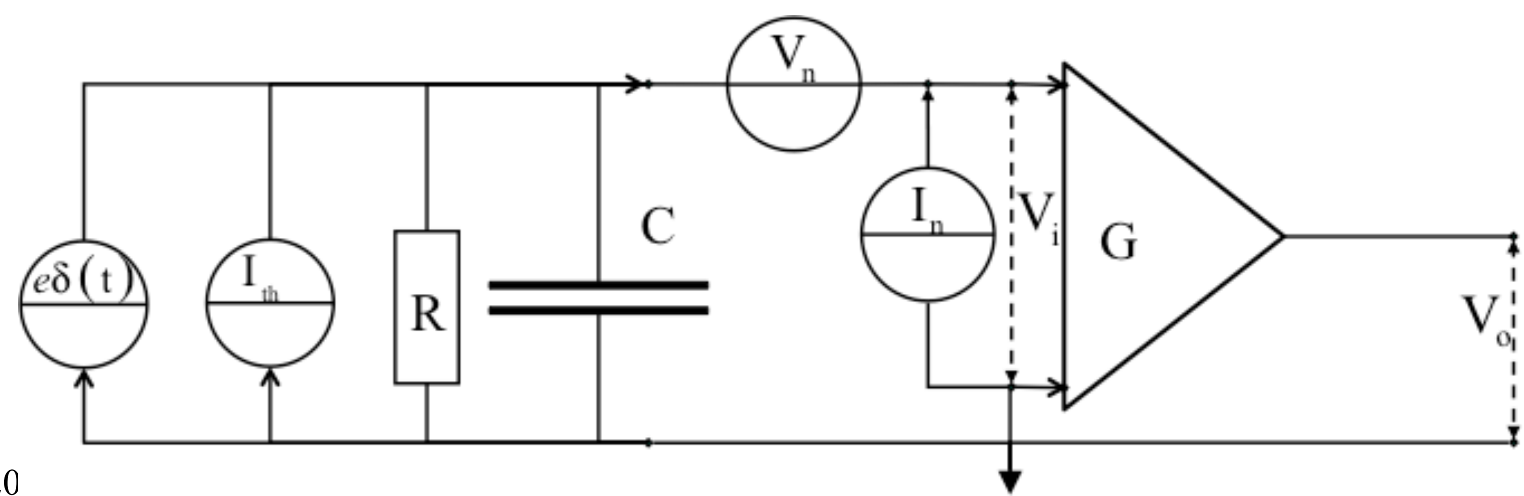
$$f(\omega) = Z(\omega)$$

  - Noise

$$S_{V_iV_i}(\omega) = S_{I_n'I_n'}|Z(\omega)|^2 + S_{V_n'V_n'}$$

  - Template

$$h(\omega) = \frac{\sigma_e^2 Z(\omega)}{S_{I_n'I_n'}|Z(\omega)|^2 + S_{V_n'V_n'}} = \frac{\sigma_e^2}{S_{Io}(1+\omega^2\tau^2)} \frac{Z(\omega)}{|Z(\omega)|^2}$$



Example: charge detector

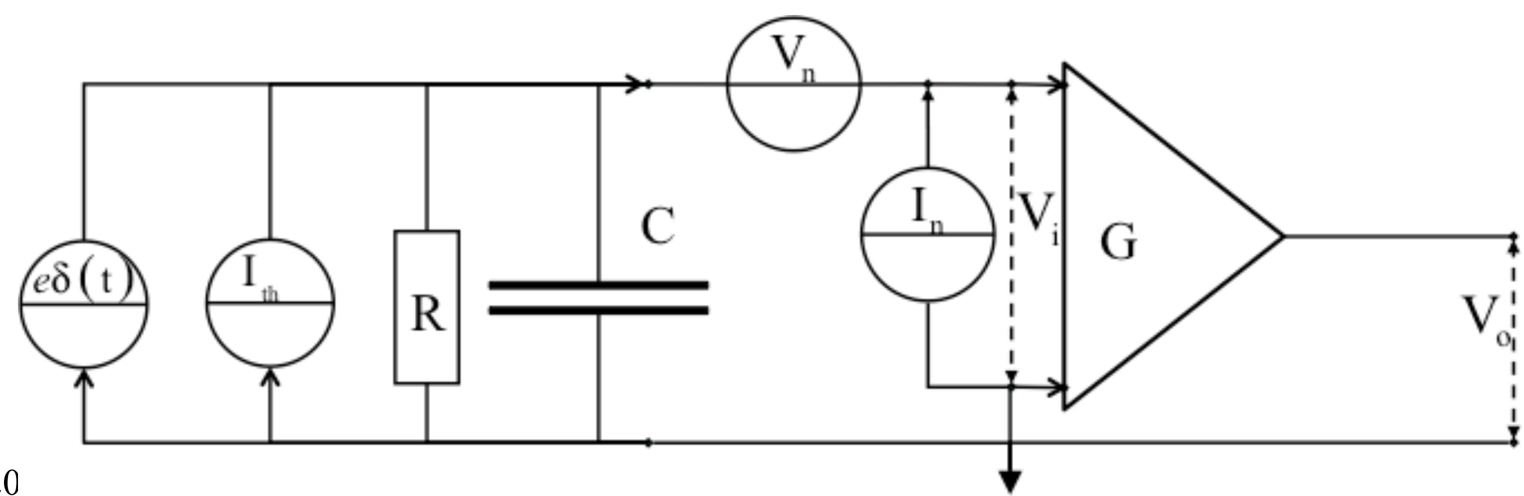
- Continuing

$$h(\omega) = \frac{\sigma_e^2 Z(\omega)}{S_{I_n' I_n'}|Z(\omega)|^2 + S_{V_n' V_n'}} = \frac{\sigma_e^2}{S_{Io}(1+\omega^2\tau^2)} \frac{Z(\omega)}{|Z(\omega)|^2}$$

$$= \frac{\sigma_e^2}{S_{Io}(1+\omega^2\tau^2)} \frac{1}{Z(\omega)^*} = \frac{\sigma_e^2}{S_{Io}R} \frac{1-i\omega RC}{1+\omega^2\tau^2}$$

- Remember $\sigma_e^2 = 2S_{Io}\tau$ then

$$h(\omega) = \frac{2\tau}{R} \frac{1-i\omega RC}{1+\omega^2\tau^2}$$



The optimum template

- The filter template:

$$h(\omega) = \frac{2\tau}{R}\frac{1 - i\omega RC}{(1 + \omega^2\tau^2)}$$

- Fourier transform:

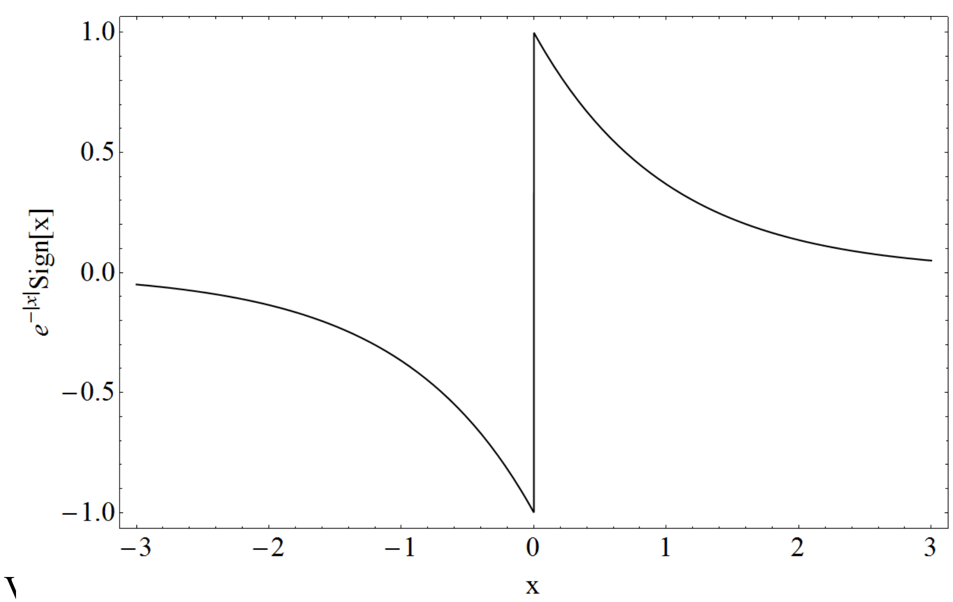
$$\frac{2\,\tau}{(1 + \omega^2\tau^2)} \rightarrow e^{-\frac{|t|}{\tau}}$$

$\frac{2\,\tau\, i\omega RC}{(1+\omega^2\tau^2)}$ is the Fourier transform of $RC\,\frac{d}{dt}\left(e^{-\frac{|t|}{\tau}}\right) \rightarrow -\frac{RC}{\tau}\,e^{-\frac{|t|}{\tau}}Sign[t]$

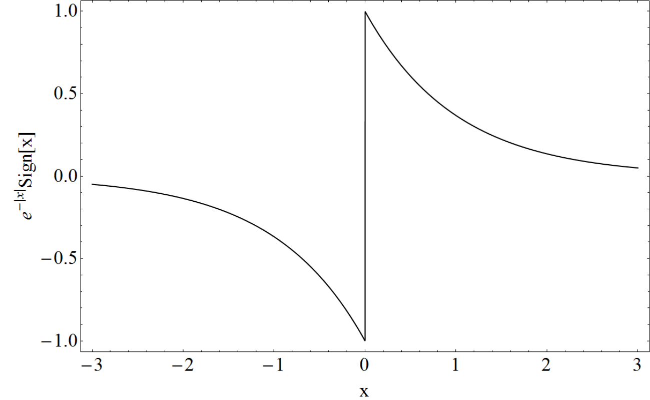
- Thus

$$h(t) = \frac{1}{R}e^{-\frac{|t|}{\tau_e}}\left(1 + \frac{RC}{\tau}Sign[t]\right) \rightarrow \lim_{R\rightarrow\infty} h(t) = \frac{1}{R_n}e^{-\frac{|t|}{\tau_e}}Sign[t]$$

A numerical example is shown here:



The optimum template



- The filter template $h(t) \simeq \frac{1}{R_n} e^{-\frac{|t|}{\tau}} Sign[t]$.
- Multiplication by the data and integration delivers the time average of the data over a time $\simeq \tau$ *after* the charge arrival time (supposed to be known) *minus* the average, again over a time $\simeq \tau$, *before* the arrival of the charge.
- As the arrival of the charge causes a step (an exponential decaying over a time $RC \gg \tau$), the above procedure will estimate the height of the step.
- Integrating longer than τ will decrease the effect of voltage noise, that ramps up at high frequency, but will enhance that of current noise that can simulate a step if given the right amount of time. τ is the optimal integration time

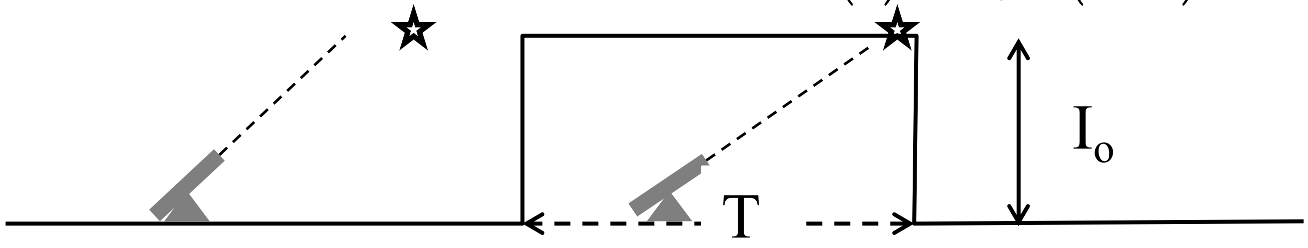
Frequency conversion and narrow-band signals

Introduction: frequency conversion

- PSD of most disturbances is not white, thus frequency ranges where the noise is lower may allow high precision measurements otherwise made impossible by low SNR.
- Thus in many experiments, physicists need changing the frequency of a signal and moving it to frequencies where PSD of disturbances is lower. This is called frequency conversion.

Example 1: infrared detection

- There is a common problem in application of visible-infrared optics shared by astronomy, laboratory spectroscopy etc. We will use infrared astronomy as an example, but all considerations applies to the remaining applications too.
- In infrared astronomy the source signal, typically a star image from a telescope, is basically at 0 frequency. For instance the telescope may be pointed to the star and the light intensity on one pixel of the photo-detector be recorded for some measurement time T. By subtracting the intensity signal measured when the telescope was pointing to the empty sky, one can measure the intensity due to the source. The signal in time looks then like a "box" function $I(t) = I_o \Pi(t/T)$



Detecting a box signal

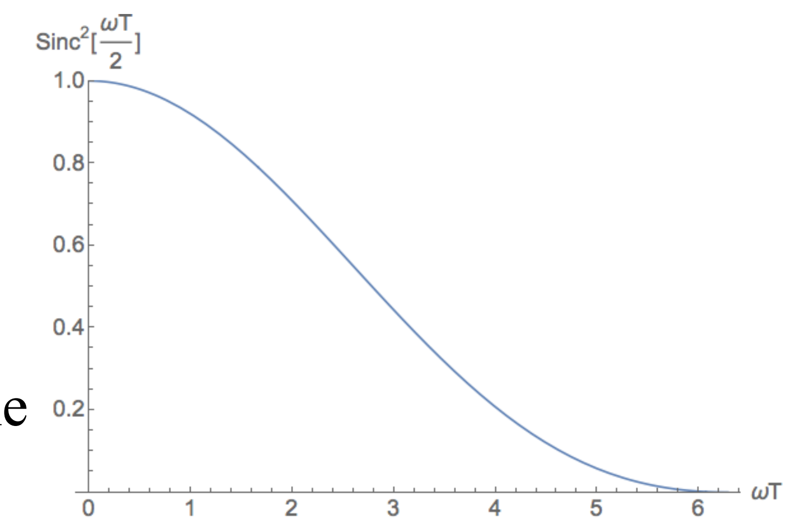
- The Fourier transform of a box is notoriously

$$I(\omega)=I_o T \mathrm{Sinc}(\omega T/2)$$

- If the PSD of total intensity noise in the photo-detector is $S_{II}(\omega)$, the minimum error on I_o is

$$\sigma_{I_o}=\left\{(1/2\pi)\int_{-\infty}^{\infty}d\omega\left(T^2\left[\mathrm{Sinc}(\omega T/2)\right]^2 \Big/ S_{II}(\omega)\right)\right\}^{-1/2}$$

- Most of the SNR is concentrated in the frequency range where the signal has the highest energy density $|\omega|\leq 2\pi/T$.
- On the other hand the error decreases as $1/T^{1/2}$ forcing then to use long enough observation times, and then concentrating the signal at lower frequency
- Low frequencies are noisy! See next page.



Noise can be very "red", that is very intense at low frequency. Here below is shown just the contribution of the sky background . Add low-frequency electronic noise, mechanical drift etc.

The processing of infrared sky noise by chopping, nodding and filtering

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Summary. The available observations of sky fluctuations have been used to define their spectrum and space correlation properties. Using these, it was possible to follow the noise content throughout the different stages of IR signal processing, and thus determine the detection sensitivity and its depence on the system parameters. It was found that the ultimate sensitivity is set by the photon noise component, a limit which, with the current nodding technique, can only be reached under very stringent mechanical, optical and meteorological conditions. It is shown that, when these conditions are not perfectly matched, the actual sensitivity can be improved if the nodding rate is increased or if an array of several detectors is used. The gain is most significant in the mid-infrared and can be traded against a much larger gain in observation time.

Key words: sky noise – chopping – nodding – infrared

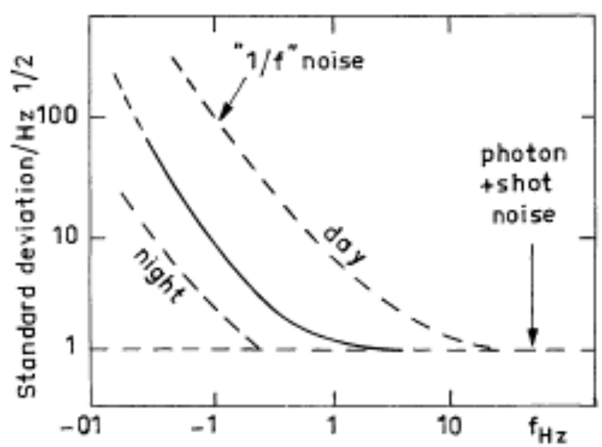
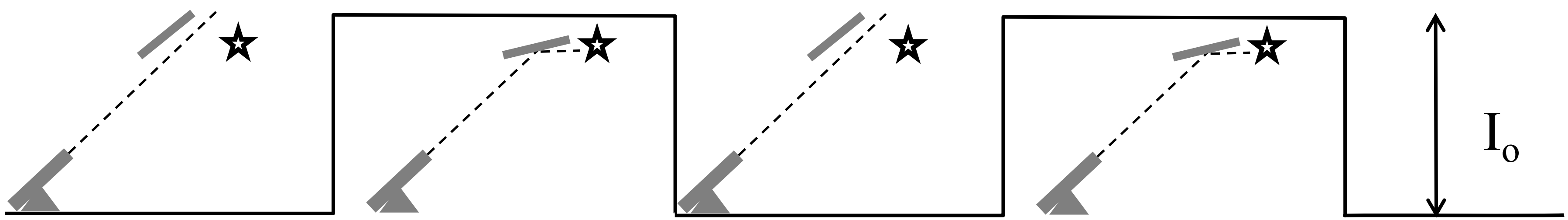


Fig. 1. Typical spectral power densities of background fluctuations

Shifting the frequency of the signal

- A classical way out of this problem is the chopper. Light is collected through a rotatable mirror that can be tilted back and forth, by a large enough angle to switch from the star to empty sky. Tilting is repeated periodically so that the intensity signal becomes a square wave at the frequency ω_o of the tilt. I_o becomes the amplitude of the wave.



- For such a signal, if the duration is T, we the know that the error on the estimate of I_o is approximately

$$\sigma_{I_o} = \sqrt{2S_{II}(\omega_o)/T}$$

- By picking the proper frequency, usually the highest compatible with the mechanics of the mirror, the error can be minimized

$$\texttt{FourierSinSeries[Sign[t], t, 5]} = \frac{4\,\mathrm{Sin}[t]}{\pi} + \frac{4\,\mathrm{Sin}[3\,t]}{3\,\pi} + \frac{4\,\mathrm{Sin}[5\,t]}{5\,\pi} \ldots$$

SOFIA Science Center
Stratospheric Observatory for Infrared Astronomy

Information for Researchers

Overview of SOFIA

The Stratospheric Observatory for Infrared Astronomy (SOFIA) is a 2.7 meter telescope (with an effective aperture of 2.5 m) carried aboard a Boeing 747-SP aircraft. It is the successor to the smaller Kuiper Airborne Observatory (KAO), which was operated by NASA

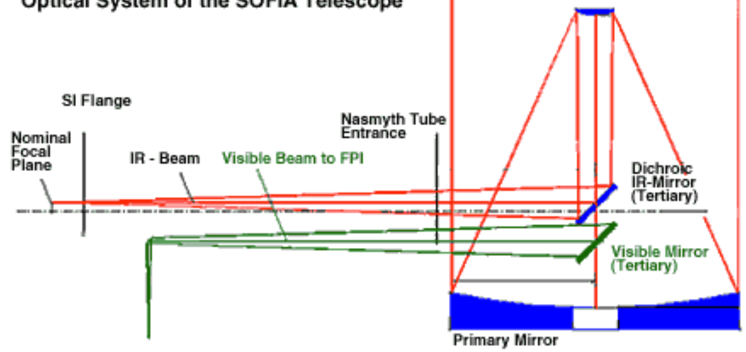
SOFIA is a joint project of NASA and the Deutsches Zentrum für Luft- und Raumfahrt (DLR, the German Space Agency). Flight

SOFIA Telescope

Under an agreement with NASA, **DLR** (the German Aerospace Center) supplied the telescope for the SOFIA observatory, as well as operation support, in exchange for observation time aboard the SOFIA.

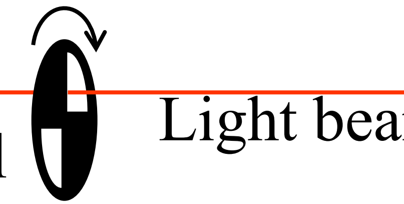
The SOFIA telescope consists of a parabolic 2.7m primary mirror and a hyperbolic secondary mirror in a bent Cassegrain configuration with two Nasmyth foci, the nominal IR focus and an additional visible light focus for guiding. The secondary mirror is attached to a chopping mechanism providing chop amplitudes of up to ± 5 arcmin at chop frequencies between 0 and 20 Hz, programmable by either a user supplied analogue or TTL curve or by the telescope control electronics. A flat tertiary mirror reflects the IR beam into the infrared Nasmyth focus, 300mm behind the instrument flange. If the fully reflecting tertiary is replaced with a dichroic mirror, the transmitted optical light is reflected by a second tertiary 289.2mm behind the dichroic and sent to the visible Nasmyth focus. There it is fed into the Focal Plane Imager (FPI), an optical focal plane guiding camera system. Independent of the FPI there are two other imaging and guiding cameras available: the Wide Field Imager (WFI) and the Fine Field Imager (FFI). Both of these cameras are attached to the front ring of the telescope.

Optical System of the SOFIA Telescope

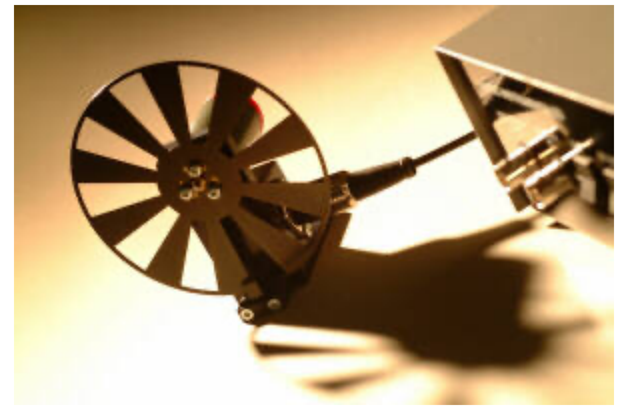


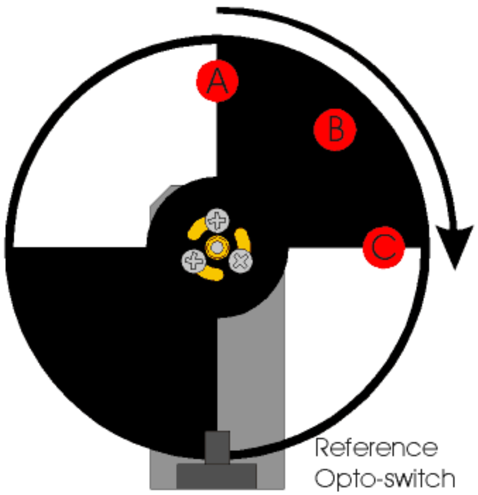
An example form laboratory spectroscopy: the chopper

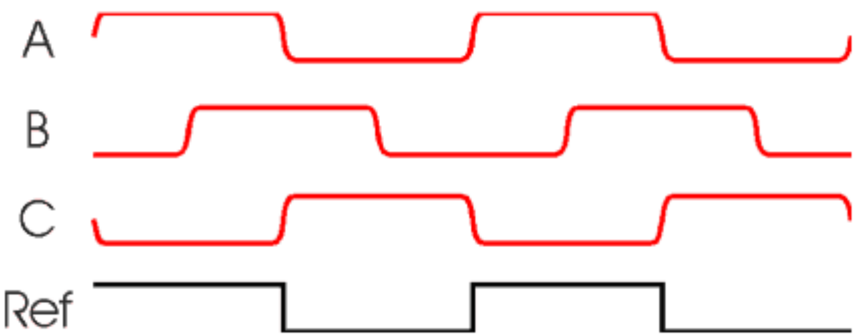
Rotating wheel



Light beam







In-phase

In-quadrature

Reference

Light intensity signal modulates the amplitude of a square wave carrier

A second example: measuring thermal forces on a free body

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- The body is LISA Pathfinder test mass. Thermal gradient induce forces because of:
  - Larger momentum transferred by hottest molecules in the residual very low gas pressure (radiometric effect)
  - Infrared (black-body) radiation pressure becomes asymmetric
  - Gas molecules trapped on hot surfaces get off more easily than from colder surfaces

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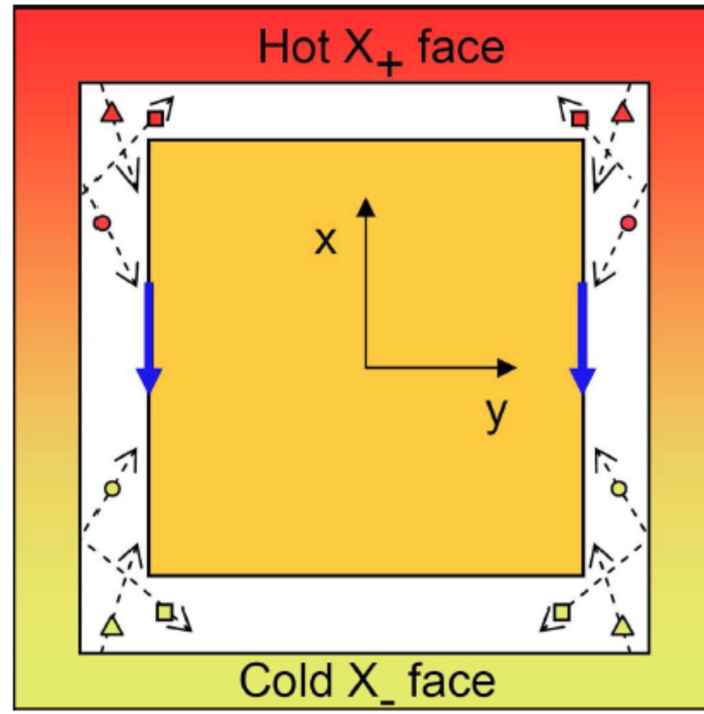

Thermal gradient forces

- Forces (torques) are measured with a torsion pendulum, by looking at the pendulum rotation
- Gradients may be applied via a set of heaters

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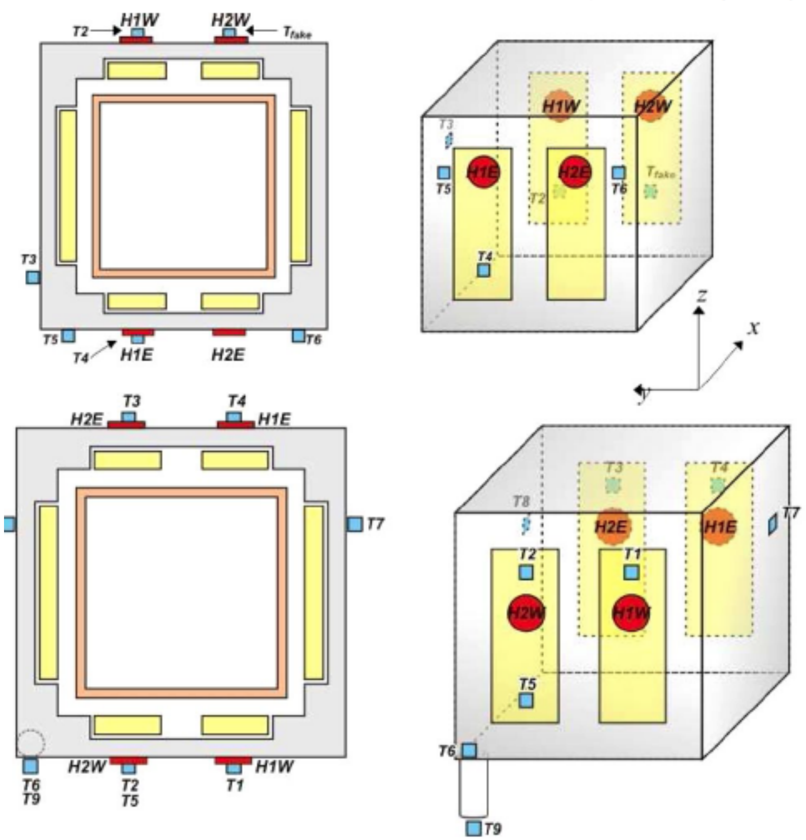

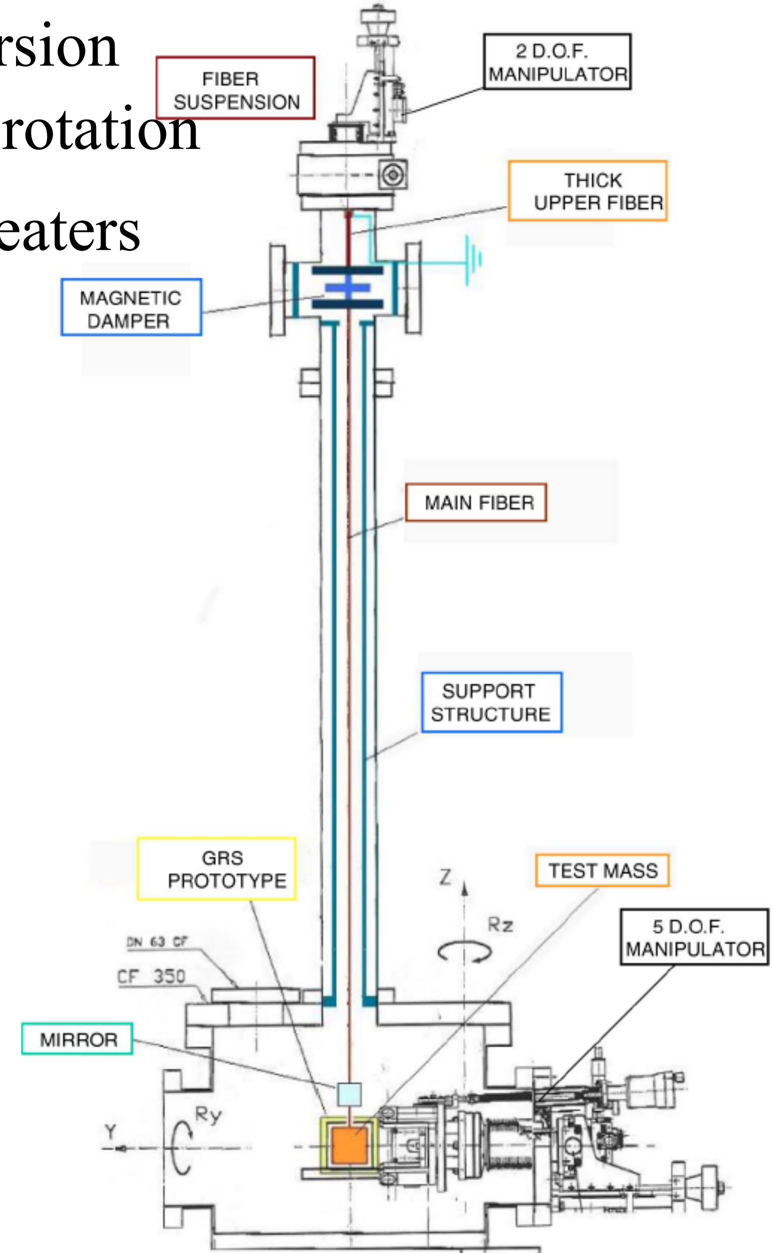

Thermal gradient forces

- Applying a steady gradient and looking at the steady rotation of the pendulum suffers from the usual problem of low frequency noise

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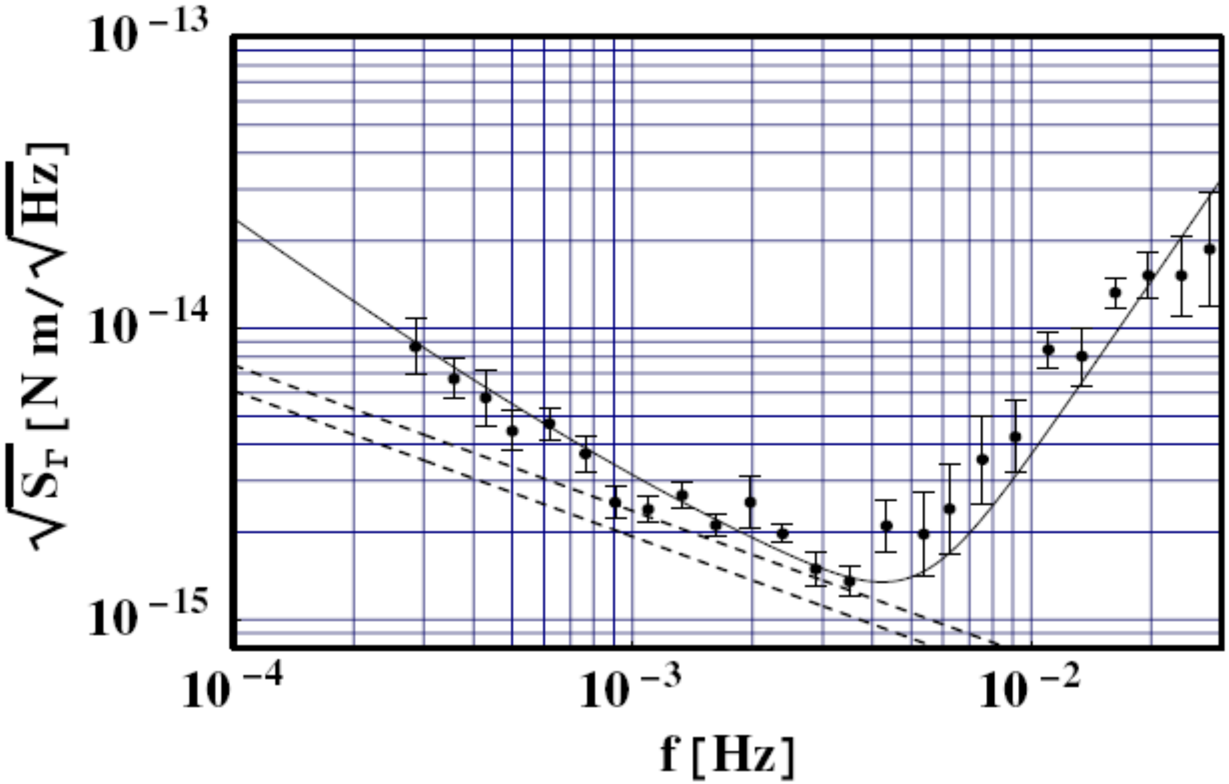

- Pendulum equivalent torque noise: a sweet spot around 2 mHz+ a $1/f^2$ branch at low frequency going worse below 0.1 mHz

Thermal gradient forces

- The trick: make the gradient oscillating by switching heaters on and off
- Ideal would be 2 mHz (noise minimum). Achievable, limited by thermal relaxation time, 0.5 mHz. Torque noise ≃6 fN m/√Hz
- Resolution over 1 hour integration ≃(6 fN m/√Hz) /√3600s =0.1 fN m

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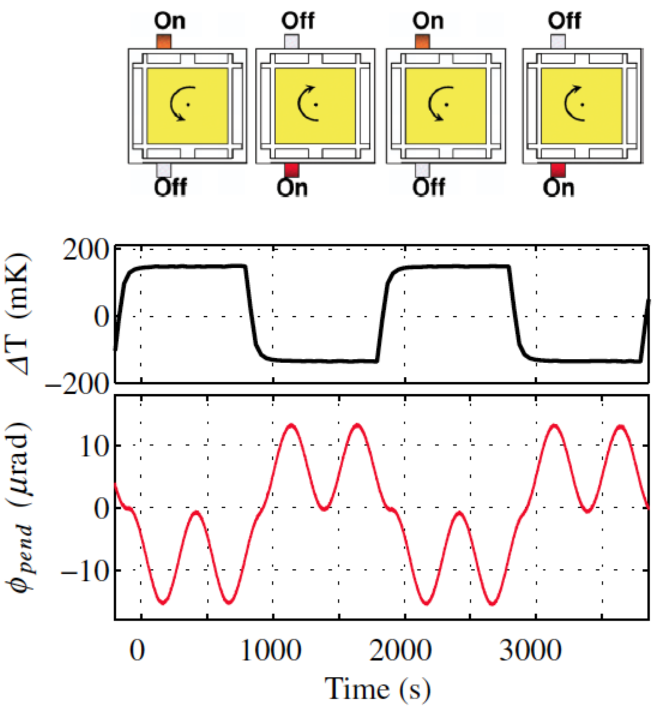
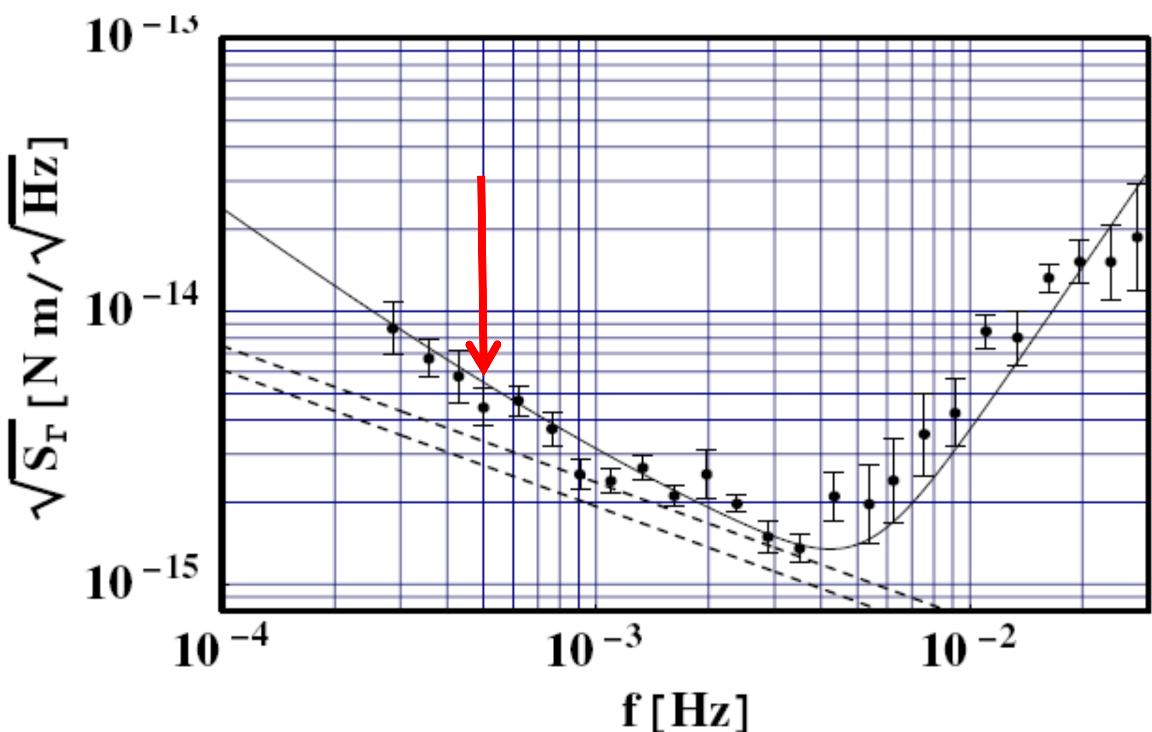
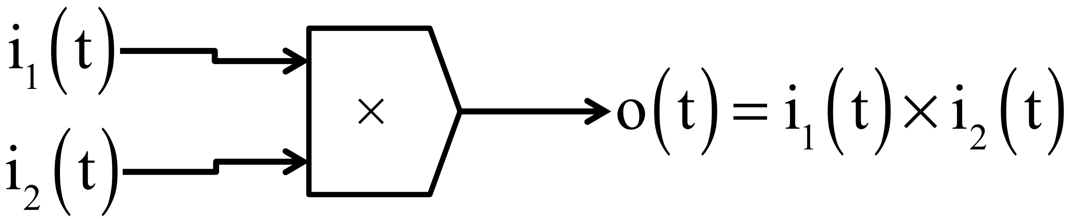


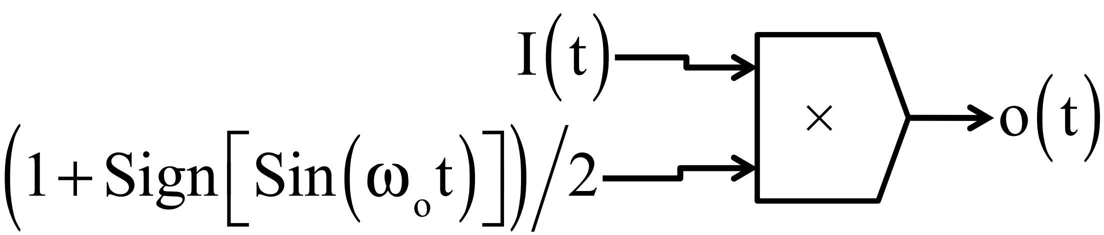


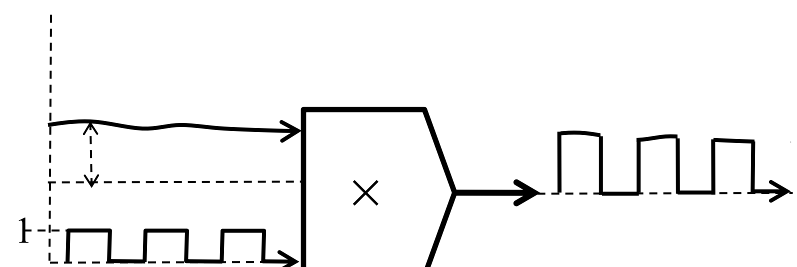
Frequency conversion and the mixer

- Thus in many experiments, one need to change (increase) the frequency of a signal.
- Linear systems *cannot* change signal frequency. To achieve frequency conversion you need some non-linear element. The most important example of such an element is the multiplier or *mixer*.
- The mixer is an ideal memory-less element that performs the product of two input signals like in the following scheme.



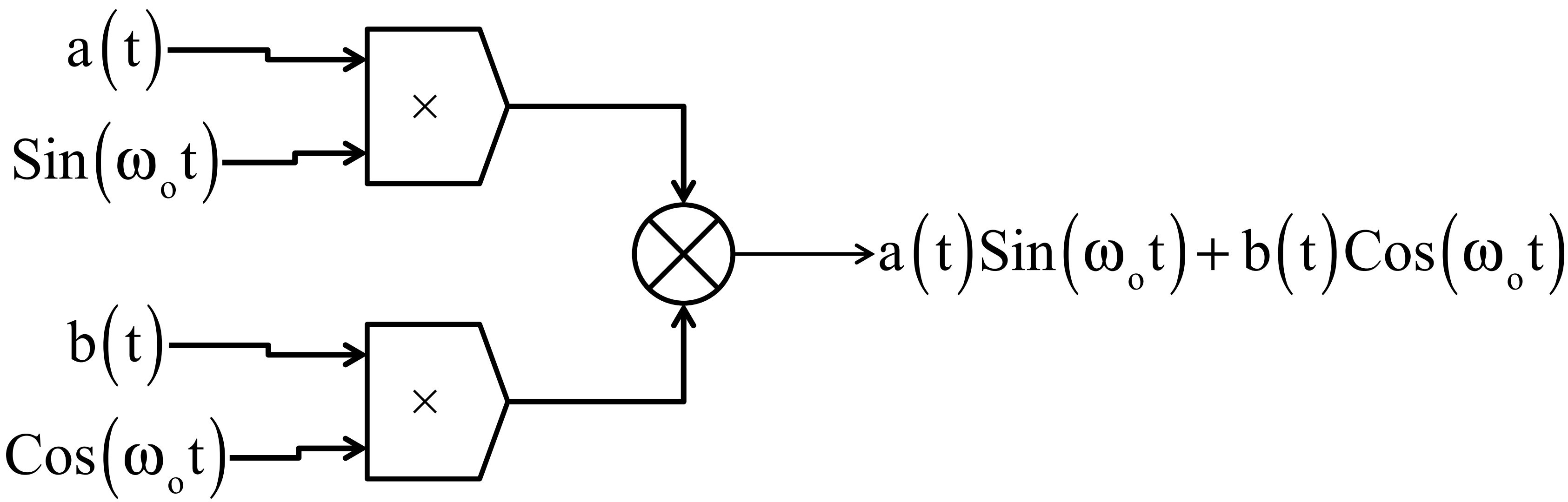
- The chopper is a mixer!





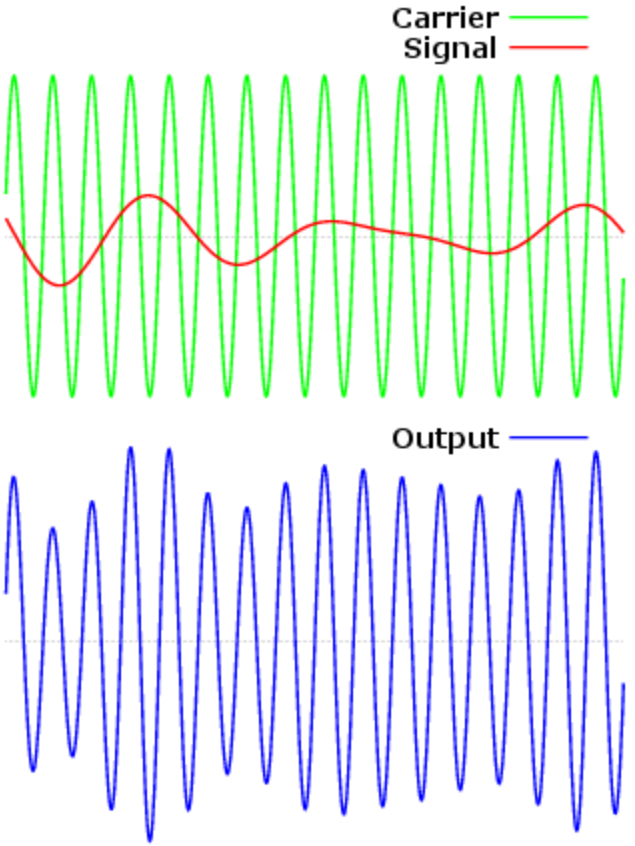
The more general narrow band signal

- Suppose you have two (slowly varying signals) a(t) and b(t). You may use two mixers to multiply them by the two phase of a "carrier" at frequency ω_o, according to the following scheme:



- And generate a "narrow-band signal" at the "carrier frequency" ω_o
- a(t) is called the "in-phase" component of the signal and b(t) the "quadrature" component.

AM radio



- The sound signal from a microphone modulates the amplitude of an electromagnetic carrier at frequency ω_o
- The electromagnetic signal s(t) is

$$s(t) = s_o\left[1 + m(t)\right]\mathrm{Sin}\left(\omega_o t + \phi\right)$$

- s_o and ϕ are constants. Max(|m(t)|) is called the modulation depth
- Notice that the signal can be re-written as

$$s(t) = s_o\left[1 + m(t)\right]\mathrm{Cos}(\phi)\mathrm{Sin}(\omega_o t) + s_o\left[1 + m(t)\right]\mathrm{Sin}(\phi)\mathrm{Cos}(\omega_o t)$$

- Thus the in phase and quadrature components are

$$a(t) = s_o\left[1 + m(t)\right]\mathrm{Cos}(\phi)$$

$$b(t) = s_o\left[1 + m(t)\right]\mathrm{Sin}(\phi)$$

FM Radio

- The sound signal modulates the phase of an electromagnetic carrier

$$s(t) = s_o \mathrm{Sin}\left(\omega_o t + \phi(t)\right) \equiv s_o \mathrm{Sin}\left(\varphi(t)\right)$$

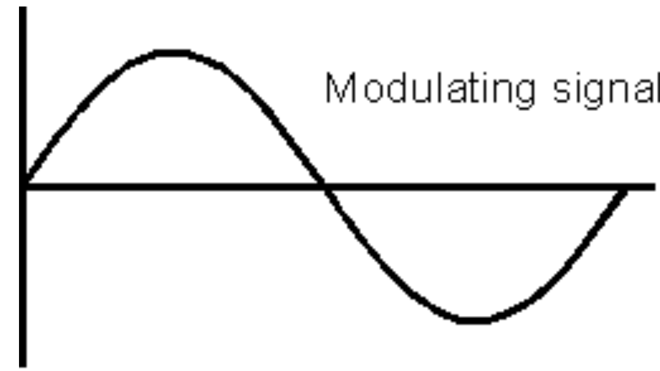
- The instantaneous frequency is

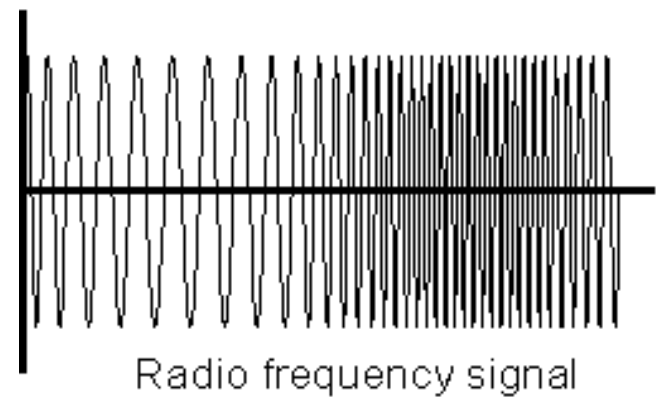
$$\omega(t) = \dot{\varphi}(t) = \omega_o + \dot{\phi}(t)$$

- And the modulation depth is

$$\mathrm{Max}\left(\left|\dot{\phi}(t)\right|\right) / \omega_o$$

- Also this signal can be written in the general form





$$s(t) = s_o \mathrm{Cos}\left[\phi(t)\right] \mathrm{Sin}\left(\omega_o t\right) + s_o \mathrm{Sin}\left[\phi(t)\right] \mathrm{Cos}\left(\omega_o t\right)$$

- So that

$$a(t) = s_o \mathrm{Cos}\left[\phi(t)\right]$$

$$b(t) = s_o \mathrm{Sin}\left[\phi(t)\right]$$