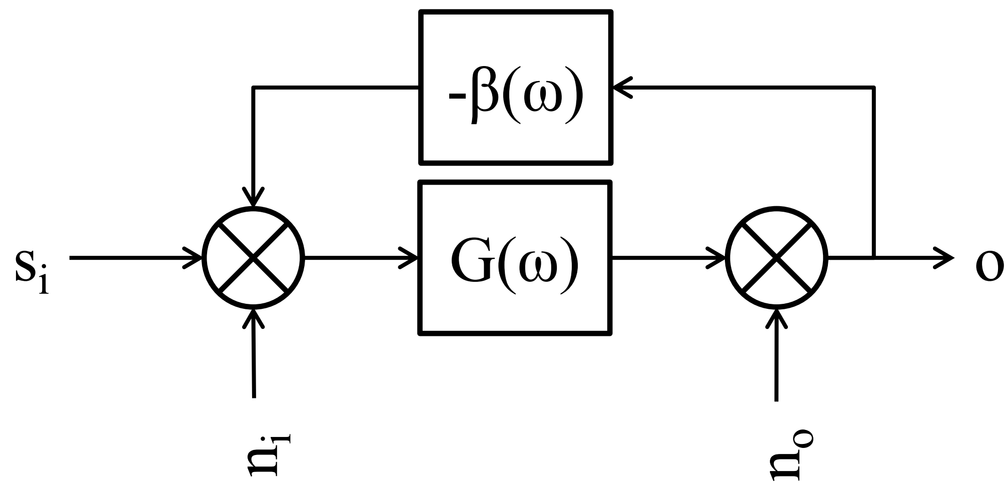
Experimental Methods
Lecture 22

November 9th, 2020

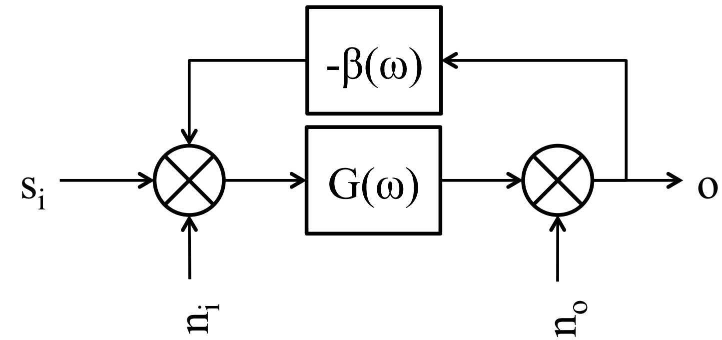
Feedback and noise

- Before proceeding further, let's consider the effect of a feedback loop, such as that used to stabilize and linearize a measurement instrument, on signal to noise ratio
- We will show that feedback cannot improve SNR
- Our model is the following:



- Here the feedback stage with transfer function $-\beta(\omega)$ is assumed to be noiseless.

Feedback and noise



- As before, we first treat noise as any other signal. With this we can calculate transfer functions to be used to propagate PSD
- The equation to be solved is

$$o(\omega)=n_o(\omega)+G(\omega)\big(s_i(\omega)+n_i(\omega)-\beta(\omega)o(\omega)\big)$$

- Then

$$o(\omega)=\frac{n_o(\omega)+G(\omega)\big(s_i(\omega)+n_i(\omega)\big)}{1+\beta(\omega)G(\omega)}$$

- As before we can introduce the equivalent noise at input by writing

$$o(\omega)=\frac{G(\omega)}{1+\beta(\omega)G(\omega)}\left(\frac{n_o(\omega)}{G(\omega)}+s_i(\omega)+n_i(\omega)\right)\equiv\frac{G(\omega)}{1+\beta(\omega)G(\omega)}\big(n_e(\omega)+s_i(\omega)\big)$$

- The formula for the equivalent noise at input is unchanged by feedback:

$$n_e(\omega)=\frac{n_o(\omega)}{G(\omega)}+n_i(\omega)$$

Feedback and noise

- Let's calculate the closed loop SNR:

1. SNR at output

$$\mathrm{SNR}_o^{cl}(\omega)=\frac{\left|G(\omega)/\left(1+\beta(\omega)G(\omega)\right)\right|^2\left|s_i(\omega)\right|^2}{\left|G(\omega)/\left(1+\beta(\omega)G(\omega)\right)\right|^2 S_{n_in_i}(\omega)+S_{n_on_o}(\omega)/\left|1+\beta(\omega)G(\omega)\right|^2}$$

By simplifying everywhere the denominator $\left|1+\beta(\omega)G(\omega)\right|^2$ we get

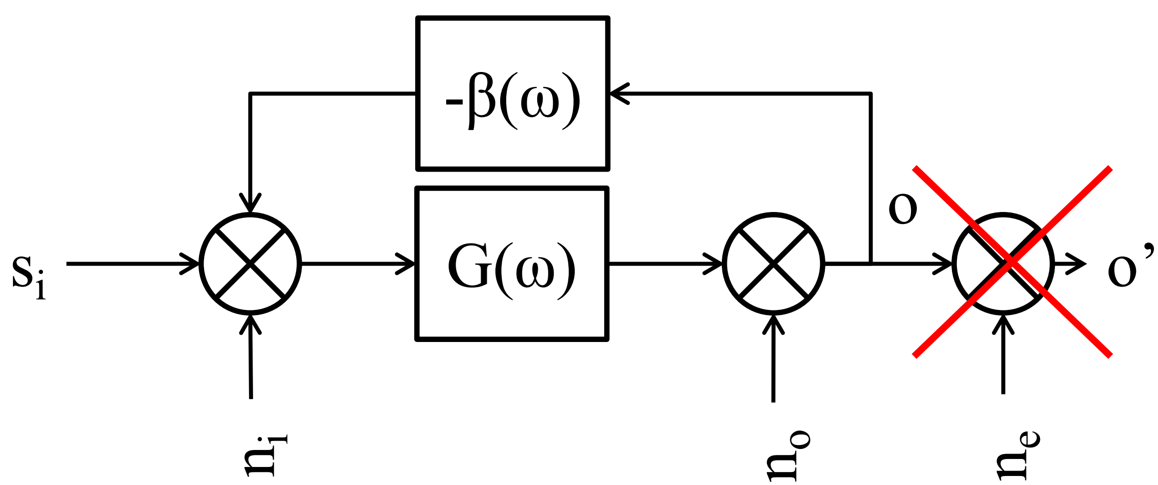
$$\mathrm{SNR}_o^{cl}=\left|G(\omega)\right|^2\left|s_i(\omega)\right|^2/\left(\left|G(\omega)\right|^2 S_{n_in_i}(\omega)+S_{n_on_o}(\omega)\right)=\mathrm{SNR}_o(\omega)$$

- As at input the equivalent noise and the signal are unaffected by the loop, the SNR stays unchanged too.
- *The feedback loop does not affect the signal to noise ratio*

Effect of additional noise

- Closed loop output $$o(\omega)=\frac{G(\omega)}{1+\beta(\omega)G(\omega)}\left(n_e(\omega)+s_i(\omega)\right)$$
- Open loop output $$o(\omega)=G(\omega)\left(n_e(\omega)+s_i(\omega)\right)$$
- If $|\beta G|>>1$ then $$\left|G(\omega)/\left[1+\beta(\omega)G(\omega)\right]\right|\approx 1/\left|\beta(\omega)\right|<<\left|G(\omega)\right|$$

- Thus in closed loop both signal and noise are suppressed by the same large factor. *In order to keep the signal to noise ratio constant, no significant noise must be added after the considered stage.*



Suppression of input variations due to noise

- Variations at the input of the apparatus *due to noise are*

$$i_{tot}(\omega)=n_i(\omega)-\frac{\beta(\omega)G(\omega)}{1+\beta(\omega)G(\omega)}\left(\frac{n_o(\omega)}{G(\omega)}+n_i(\omega)\right)$$

$$=\frac{n_i(\omega)}{1+\beta(\omega)G(\omega)}-\frac{\beta(\omega)}{1+\beta(\omega)G(\omega)}n_o(\omega)$$

- the PSD of which is

$$S_{n_{itot}n_{itot}}(\omega)=S_{n_in_i}(\omega)\Big/\left|1+\beta(\omega)G(\omega)\right|^2+S_{n_on_o}(\omega)\left|\beta(\omega)\right|^2\Big/\left|1+\beta(\omega)G(\omega)\right|^2$$

- Which goes to zero for $|G(\omega)|\rightarrow\infty$
- Thus a feedback loop suppress the noise dynamics at the instrument input as it does for any other signal.

Measurement of input noise

- Notice that while

$$i_{tot}(\omega) = \frac{n_i(\omega)}{1+\beta(\omega)G(\omega)} - \frac{n_o(\omega)\beta(\omega)}{1+\beta(\omega)G(\omega)} \to 0 \text{ for } G(\omega) \to \infty$$

- The feedback signal

$$i_{fb}(\omega) = -\beta(\omega)\ \frac{n_o(\omega)+G(\omega)n_i(\omega)}{1+\beta(\omega)G(\omega)} \to -n_i(\omega) \text{ for } G(\omega) \to \infty$$

- And is then a good measurement of $n_i(\omega)$

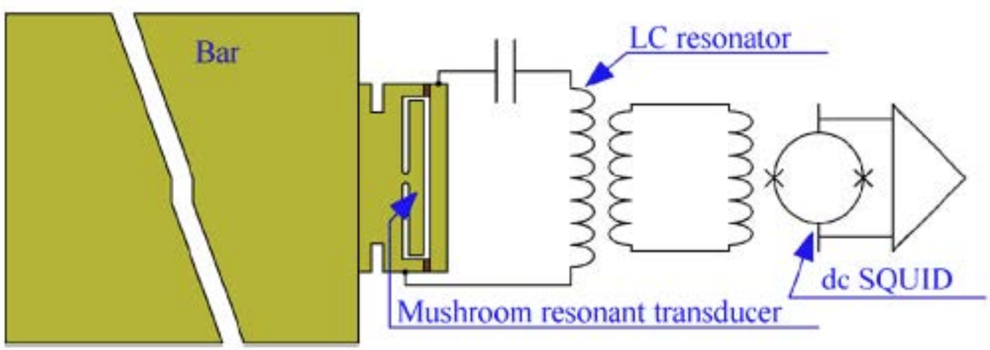
An example of noise reduction by feedback: cooling 2 Tons at 130μK

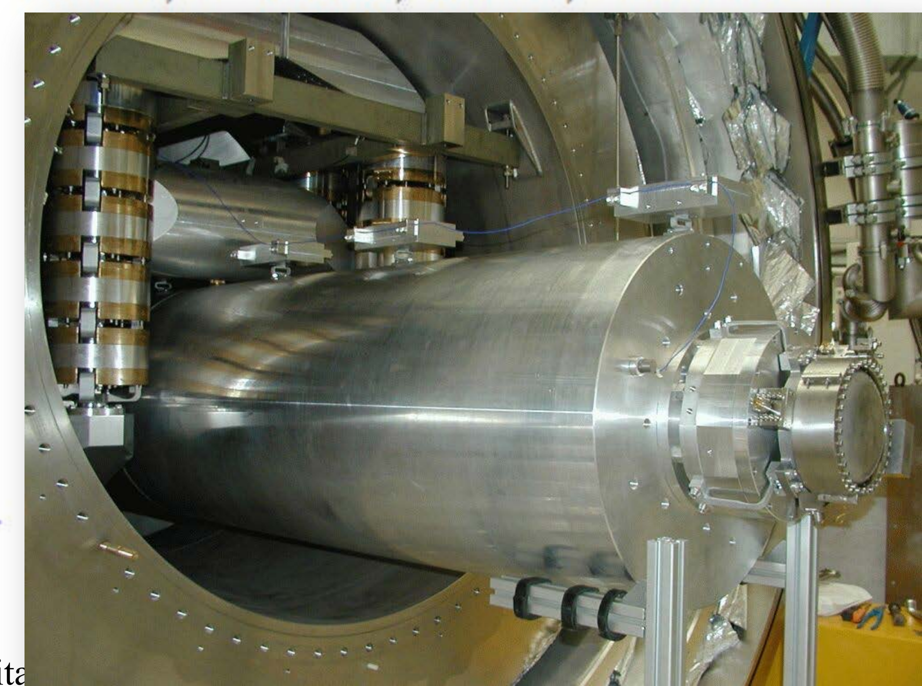
- The Auriga GW antenna at 4.2 K. Energy of oscillations of the fundamental mode = $k_B T/2$.

Selected for a Viewpoint in *Physics*
PRL **101,** 033601 (2008) PHYSICAL REVIEW LETTERS week ending 18 JULY 2008

Feedback Cooling of the Normal Modes of a Massive Electromechanical System to Submillikelvin Temperature

A. Vinante,[1,2,*] M. Bignotto,[3,4] M. Bonaldi,[1,2] M. Cerdonio,[3,4] L. Conti,[3,4] P. Falferi,[1,2] N. Liguori,[3,4] S. Longo,[5] R. Mezzena,[6,2] A. Ortolan,[5] G. A. Prodi,[6,2] F. Salemi,[6,2] L. Taffarello,[4] G. Vedovato,[4] S. Vitale,[6,2] and J.-P. Zendri[4]





The Feedback

- Measure the elongation of the bar. Feed back a force proportional to the elongation, in order to counteract the Brownian force.
- Oscillations decrease in amplitude (require a very sensitive motion detector and a very quiet actuator

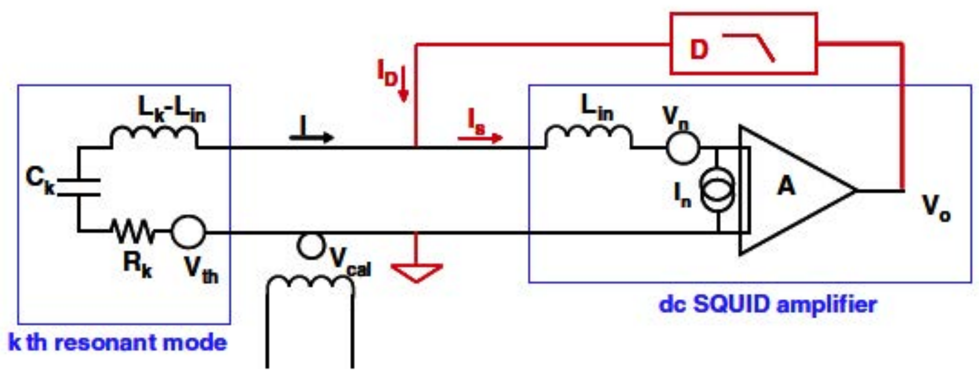


FIG. 2 (color online). The feedback cooling scheme. The kth normal mode is approximated, around its resonance frequency, by a series-RLC circuit. In this representation, different modes should be thought of as being in parallel with each other. The dc SQUID is represented as current amplifier. The electronic feedback cooling is obtained by sending back a current I_D phase shifted of $\pi/2$ with respect to I_s.

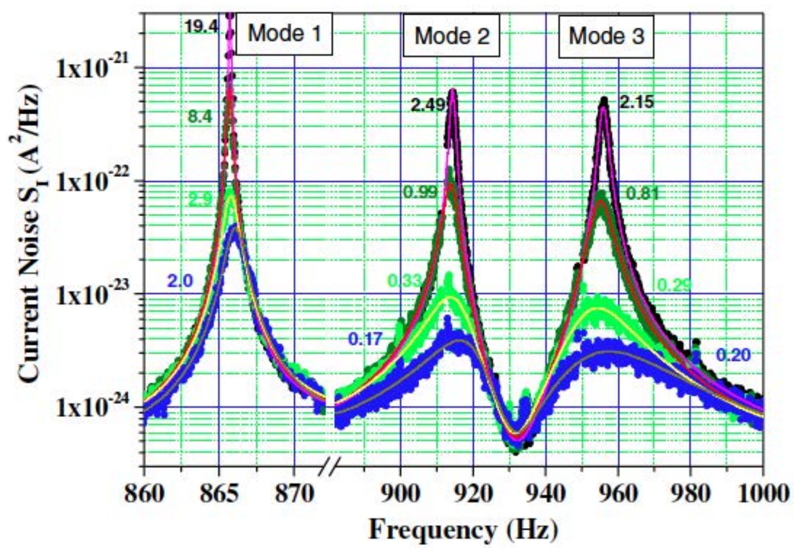


FIG. 3 (color online). Power spectrum of the current noise in the normal modes, as measured by the SQUID amplifier. The noise spectra are related to four different feedback settings. Each noise spectrum is well fitted by a proper combination of three Lorentzian curves. Each Lorentzian peak is labeled by the corresponding effective temperature, measured in mK.

Amplitude of oscillation decreases

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- Effective energy (effective temperature) in the mode decreases

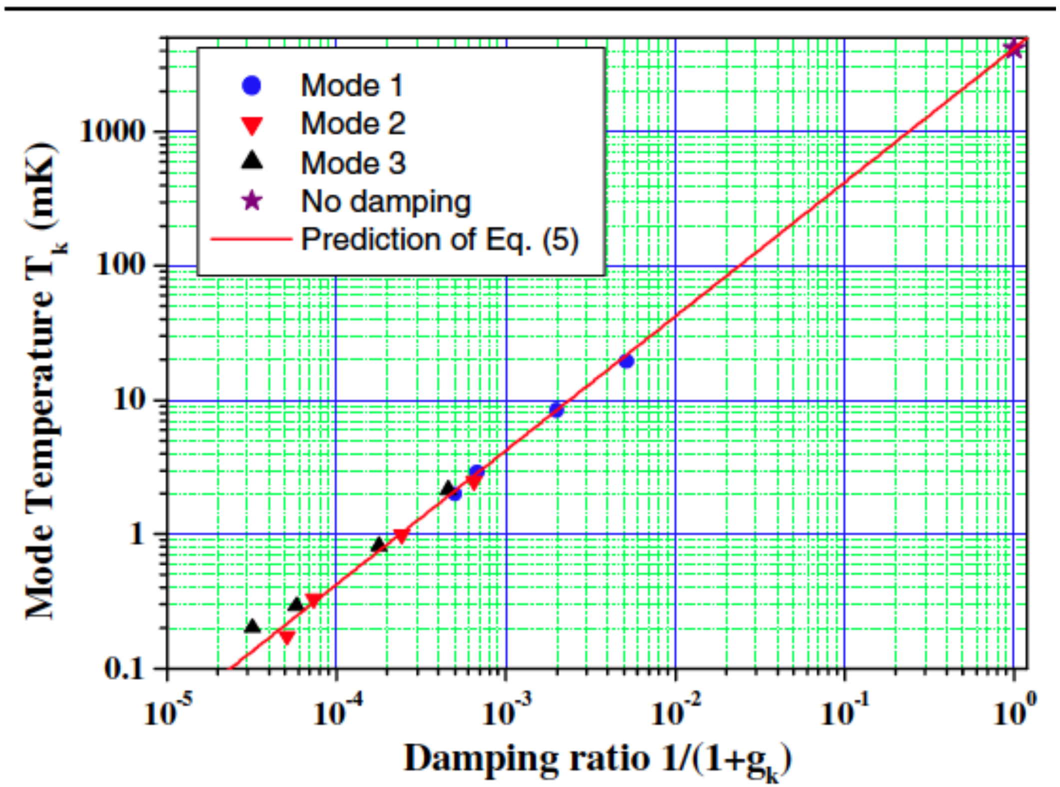
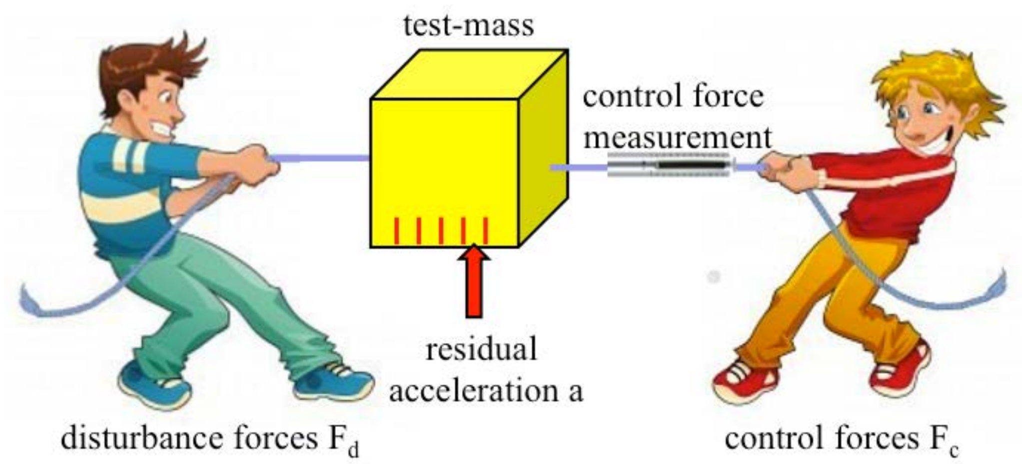
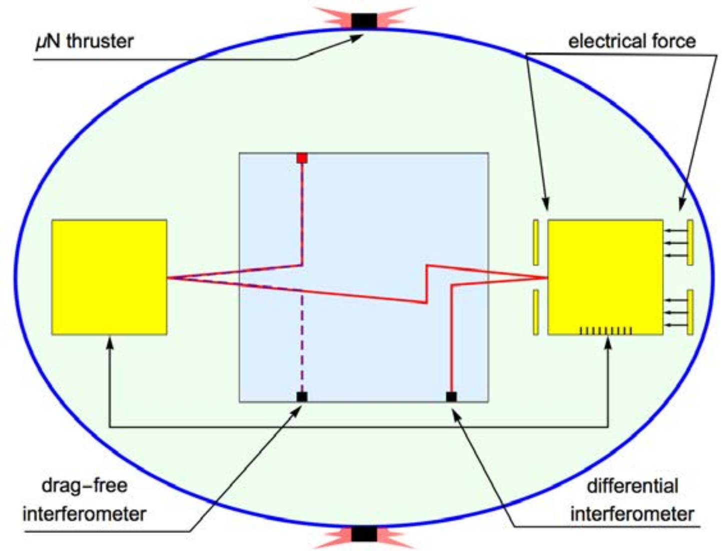


FIG. 4 (color online). Effective temperature of the modes as function of the damping ratio. The straight line is the mode temperature predicted by Eq. (5), with T_0 fixed to the value of the bath temperature $T_0 = 4.2$ K, and no free parameters. The theoretical limit at no feedback damping is also shown, to provide a graphical visualization of the achieved temperature reduction.

Extracting information from close loop measurement LISA Pathfinder

- Deriving out of the loop force



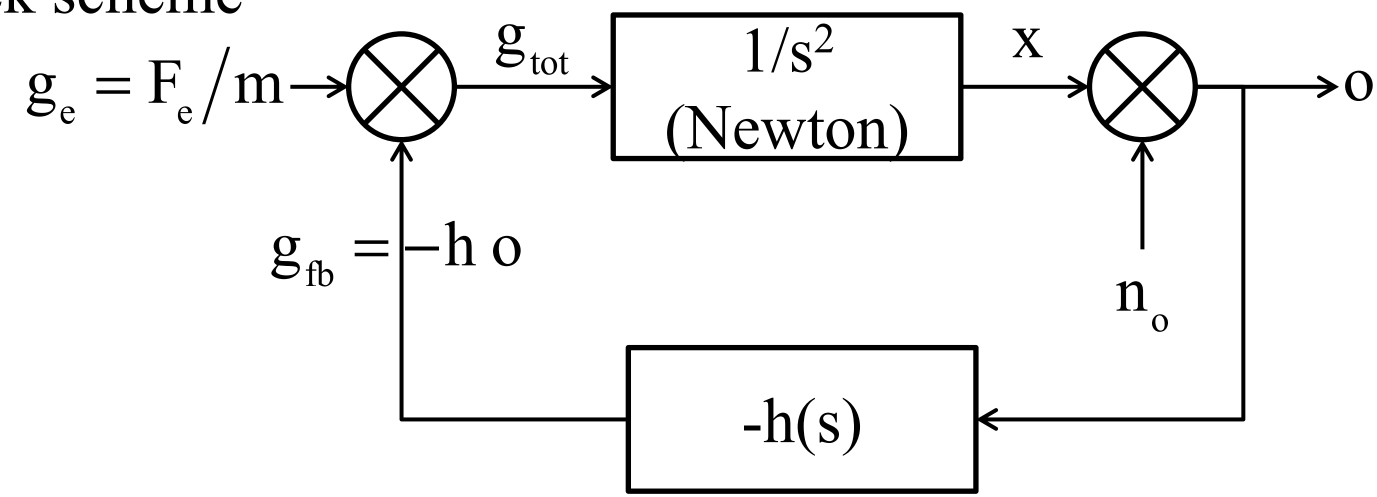


- Out of the loop forces are derived as

$$\frac{F_d(t)}{m} \equiv \Delta g(t) = \Delta a(t) - \frac{F_c(t)}{m}$$

In the language of systems

- A block scheme



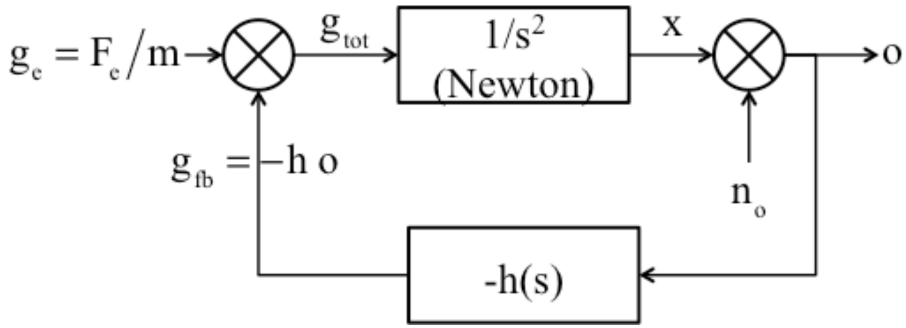
- The equation $$o = n_o + \frac{1}{s^2}\left(g_e - h\,o\right)$$
- The output $$o = \frac{n_o s^2 + g_e}{s^2 + h}$$
- Or its 2nd time derivative (the closed loop acceleration)

$$s^2 o = \frac{s^2}{s^2 + h}\left(n_o s^2 + g_e\right)$$

- Not a good measure of g_e. Underestimates force noise at low freq.

Estimating input force

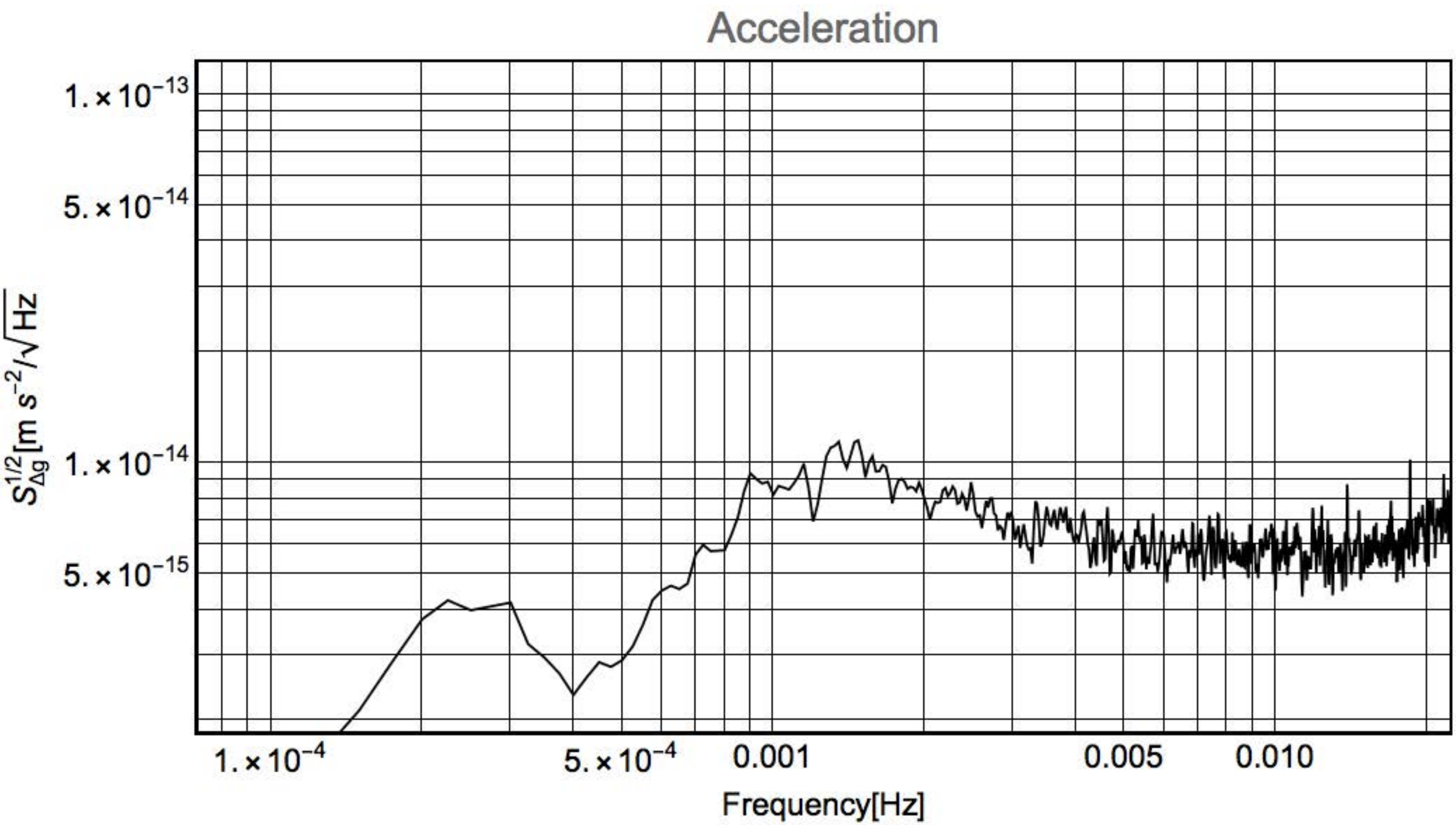
- A block scheme



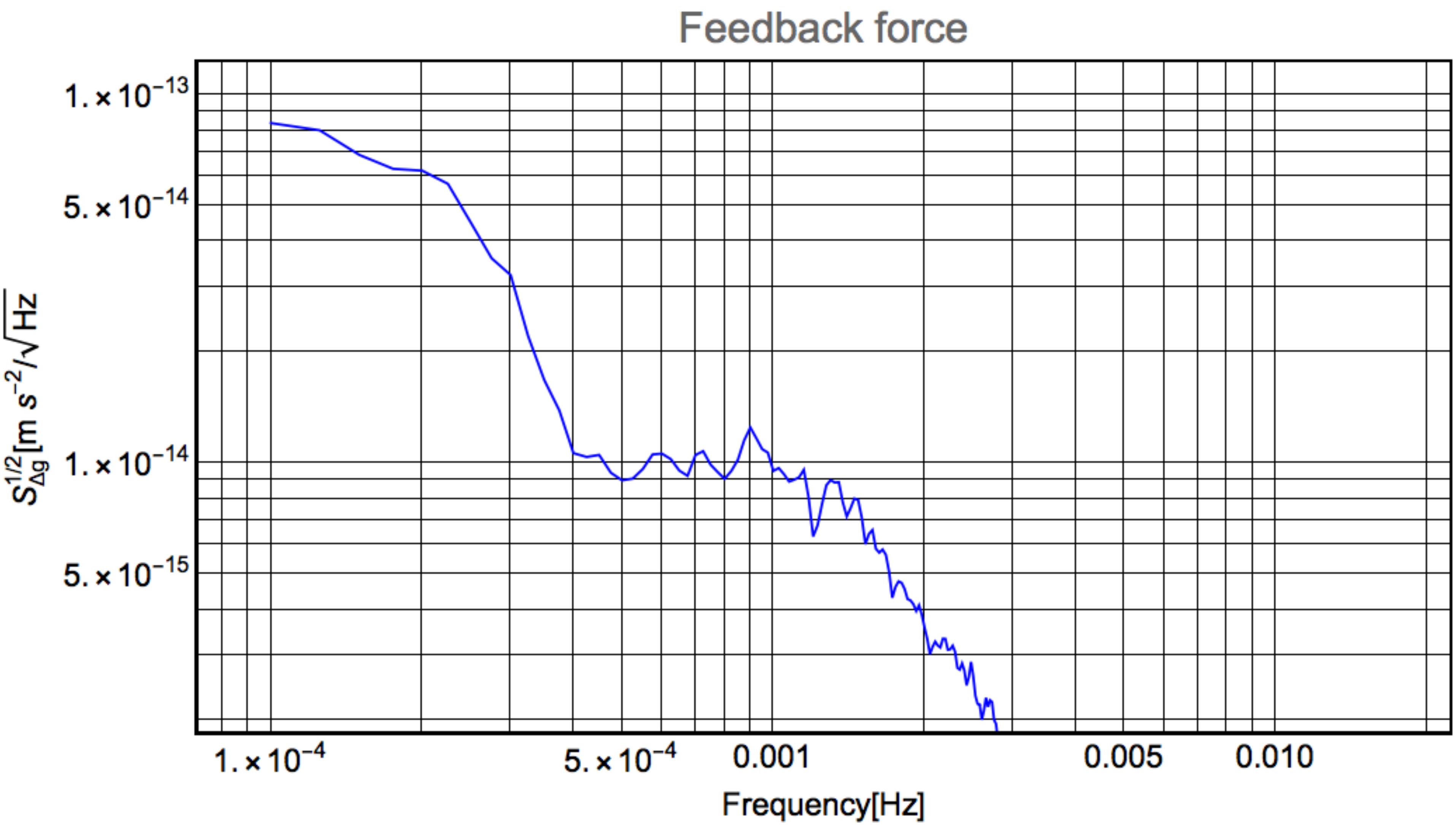
- The output $o = \dfrac{n_o s^2 + g_e}{s^2 + h}$
- Acceleration $s^2 o = \dfrac{s^2}{s^2 + h}\left(n_o s^2 + g_e\right)$

- Take instead $\ddot{o} - g_{fb}$ (we know g_{fb}!), that is $\left(s^2 + h\right)o = s^2 n_o + g_e$
- Which is obviously $\ddot{o} - g_{fb} = \ddot{n}_o + g_e$
- A good measure of g_e, just corrupted by the readout noise (unavoidable)

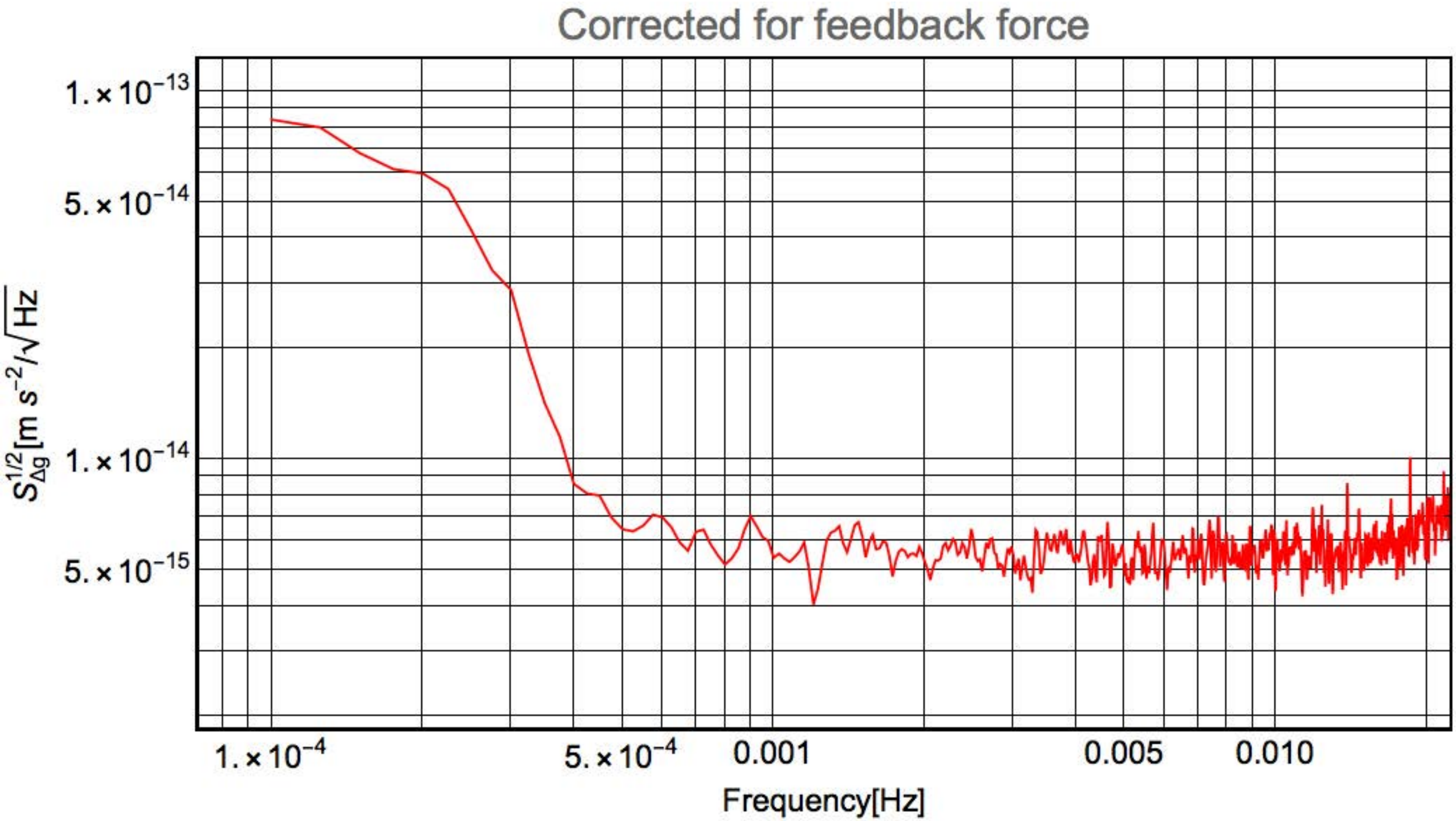
An example from real data



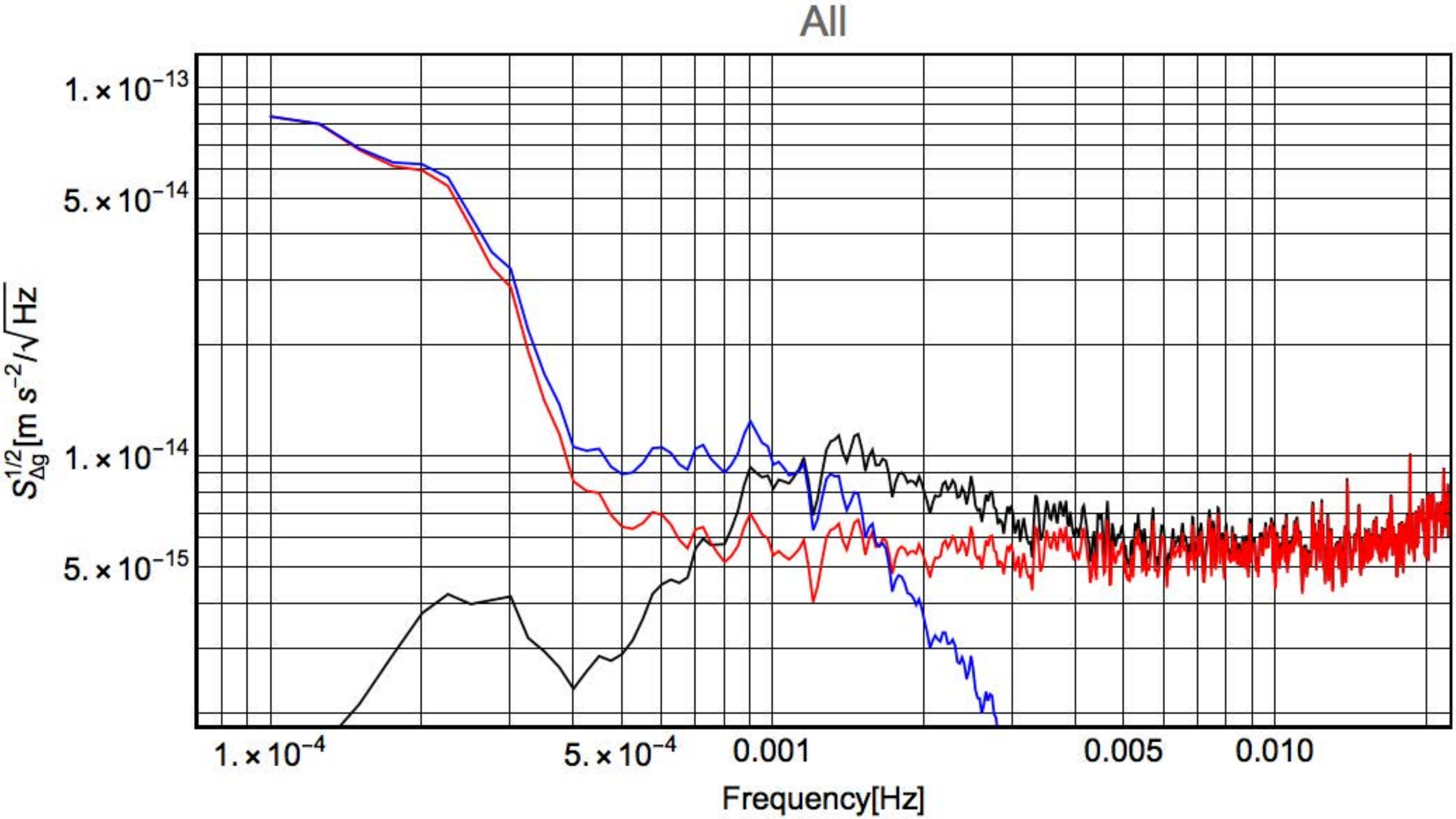
An example from real data



An example from real data



An example from real data



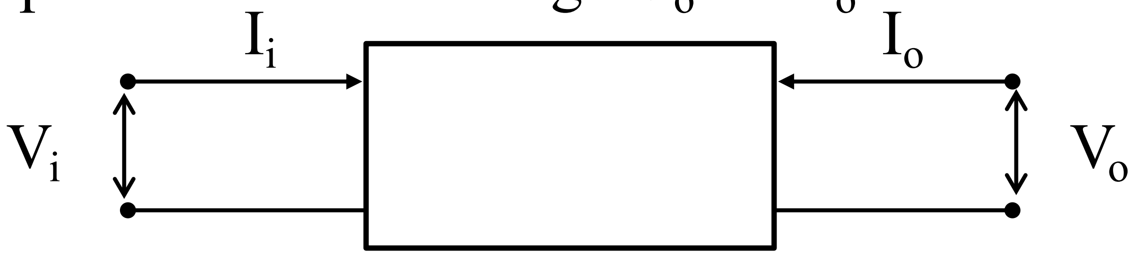
Two-port networks

- A linear two-port network is a system with multiple inputs and multiple outputs.

- This system is linear and time invariant. It constitutes a more accurate scheme for a measurement instrument, *as it also includes the perturbation that a measurement device exerts on the physical system at its input.*
- two-port networks need not to be passive, and may actively increase the physical energy of signals going through them. The most classical example are electrical active two-port devices, like voltage and/or current amplifiers.

Two-port electrical devices

- The device is constituted by two ports. In each port we find two signals:
  - The input current and voltage I_i and V_i
  - The output current and voltage V_o and I_o



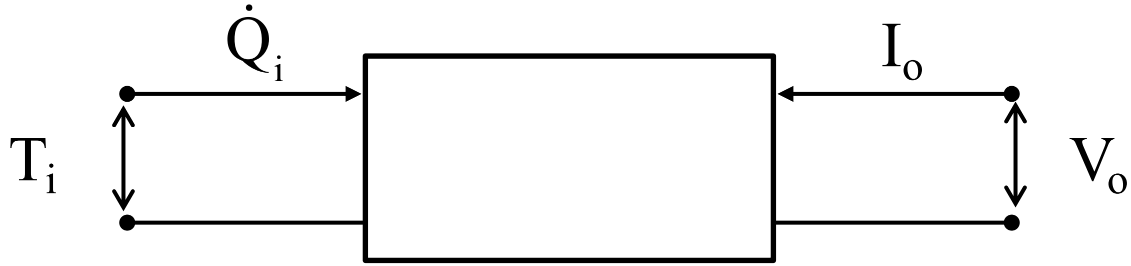
- In the frequency domain, as the system is linear, we can write (for instance)

$$\begin{pmatrix} V_i \\ V_o \end{pmatrix} = \begin{pmatrix} Z_i & Z_{io} \\ Z_{oi} & Z_o \end{pmatrix} \cdot \begin{pmatrix} I_i \\ I_o \end{pmatrix}$$

- Where the Zs are called impedances.

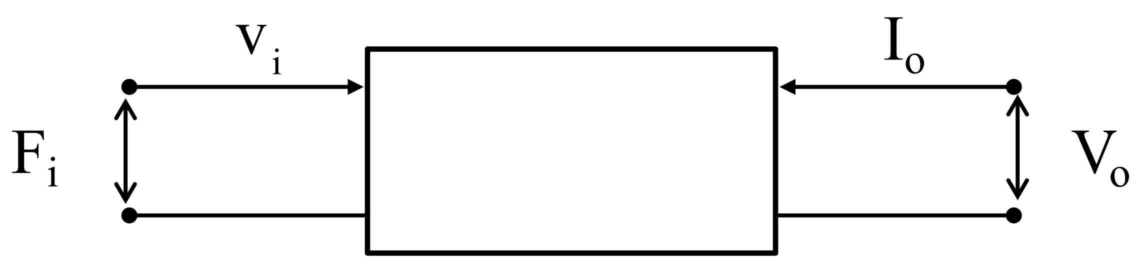
Other ports

- Ports don't need to be electrical. They may be
  - Thermal: quantities are temperature T (differences) heath flow $\dot{Q}$



Bolometers, calorimeters etc., Impedances are replaced by thermal conductance, thermal capacity…

  - Mechanical: quantities Force F , Velocity v



Motion transducers, piezoelectric. Impedances are replaced by springs, dashpot, inertia….

Other ports

- Optical

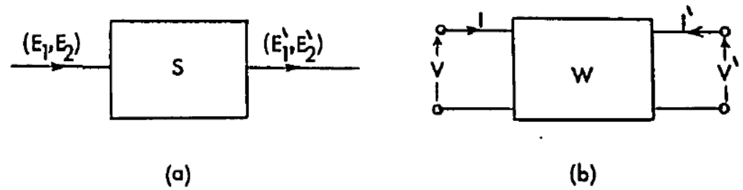


Fig. 1. (a) Linear optical system S; (b) linear two-port network W. For S and W the transformation of a pair of oscillating quantities between input and output is of interest.

The authors are with the Electrical Materials Laboratory, Department of Electrical Engineering, University of Nebraska, Lincoln, Nebraska 68508.

Received 13 March 1972.

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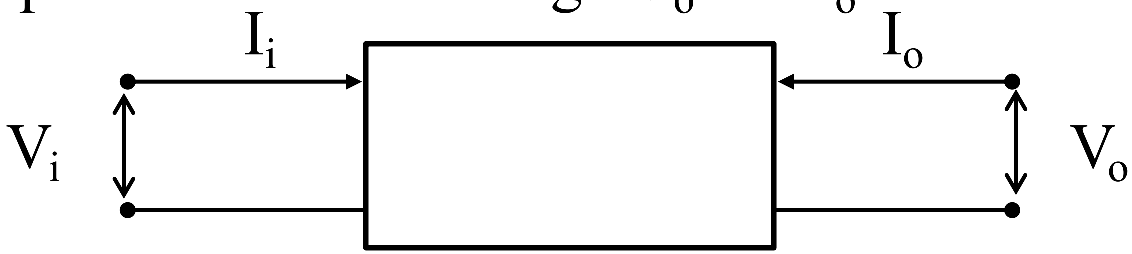
- We will analyze electrical ports as they are somewhat simpler, but we will also discuss exercises on other kinds of devices.
- The main facts we will establish for electrical two-ports will remain valid for generic two-port systems

The voltage amplifier

- The device is constituted by two ports. In each port we find two signals:
  - The input current and voltage I_i and V_i
  - The output current and voltage V_o and I_o



- In the frequency domain, as the system is linear, we can write, (for instance)

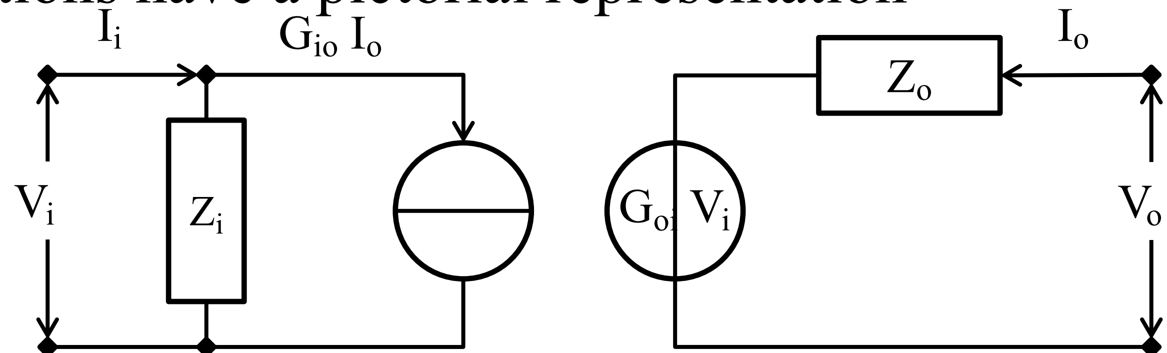
$$\begin{aligned} I_i &= V_i/Z_i + G_{io} I_o \\ V_o &= G_{oi} V_i + Z_o I_o \end{aligned} \rightarrow \begin{pmatrix} I_i \\ V_o \end{pmatrix} = \begin{pmatrix} 1/Z_i & G_{io} \\ G_{oi} & Z_o \end{pmatrix} \bullet \begin{pmatrix} V_i \\ I_o \end{pmatrix}$$

- Z_i is called input impedance, Z_o output impedance, and non diagonal coefficients are often called gains.

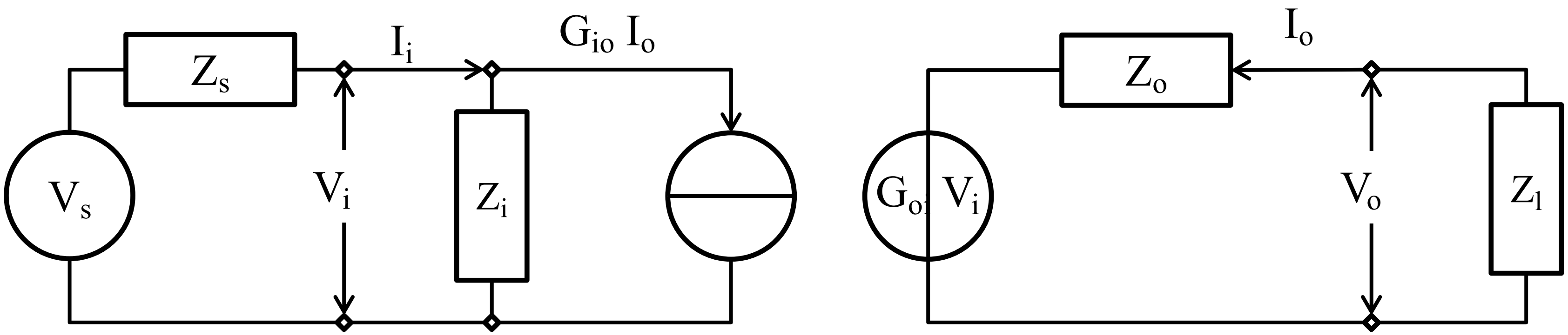
The voltage amplifier

$$\begin{pmatrix} I_i \\ V_o \end{pmatrix} = \begin{pmatrix} 1/Z_i & G_{io} \\ G_{oi} & Z_o \end{pmatrix} \cdot \begin{pmatrix} V_i \\ I_o \end{pmatrix}$$

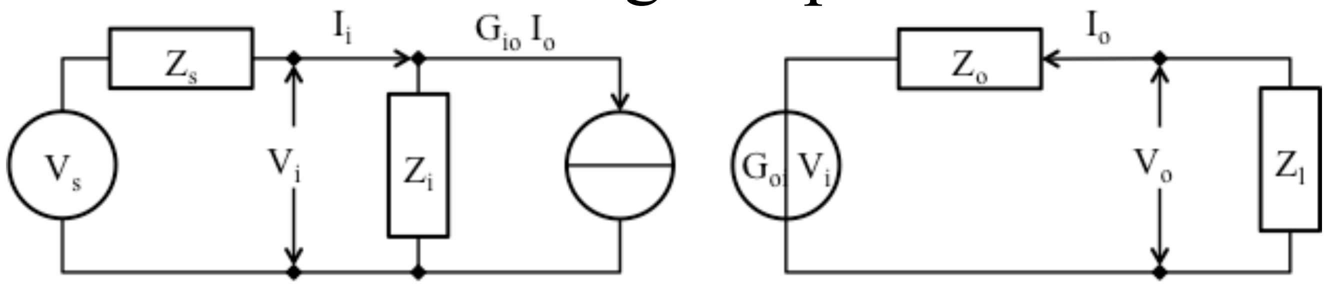
- These relations have a pictorial representation



- They can be entirely solved only if there are two additional relations among currents and voltages: the source (voltage generator V_s and impedance Z_s) and load (impedance Z_L) branches.



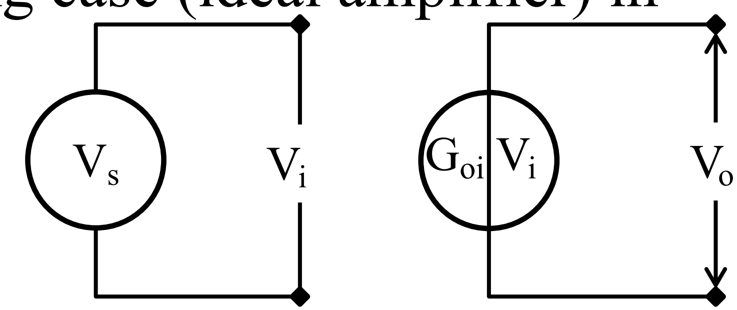
The voltage amplifier



- Adding the new branches, these are now the circuit equations:

$$I_i = \frac{V_i}{Z_i} + G_{io} I_o \quad V_o = G_{oi} V_i + Z_o I_o \quad I_i = \frac{V_s - V_i}{Z_s} \quad V_o = -Z_l I_o$$

- These can now be solved in the general case.
- We are however interested in the limiting case (ideal amplifier) in which, $Z_i \rightarrow \infty$ and $Z_L \rightarrow \infty$. Then

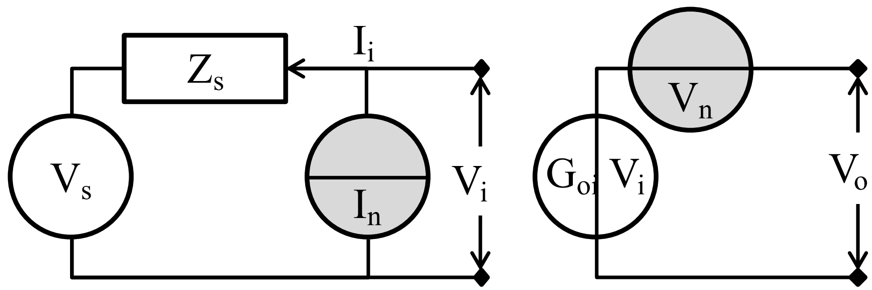


- $I_o = -\frac{V_o}{Z_l} \rightarrow 0 \quad I_i = \frac{V_i}{Z_i} - \frac{G_{io} V_o}{Z_l} \rightarrow 0$

- $V_i = V_s - Z_s I_i \rightarrow V_s \qquad V_o = G_{oi} V_i - \frac{Z_o}{Z_l} V_o \rightarrow G_{oi} V_i = G_{oi} V_s$

Noise in two port systems: voltage amplifier

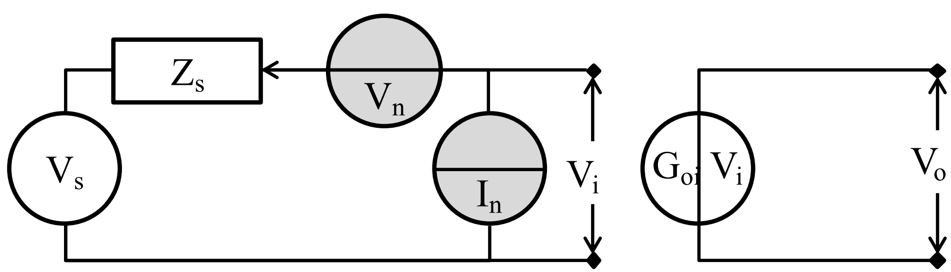
- If a port is noisy, but still linear, Norton and Thévenin theorem hold. Thus, if both the input and the output ports are noisy, we can represent these noise sources as two, possibly correlated generators V_n and I_n:



- The relevant quantities, I_i and V_o are now calculated to be

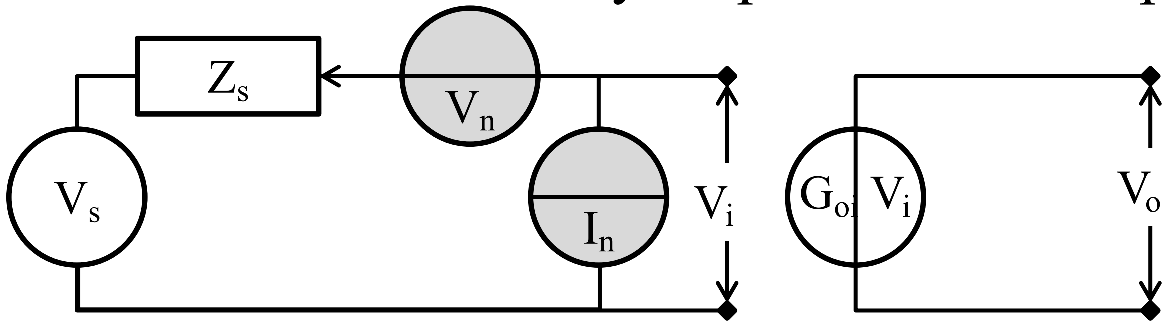
$$I_i = I_n \qquad V_i = V_s + I_n Z_s \rightarrow V_o = G_{oi}\left(V_s + I_n Z_s\right) + V_n$$

- Can take the voltage generator to input $V_n \rightarrow V_n / G_{oi}$

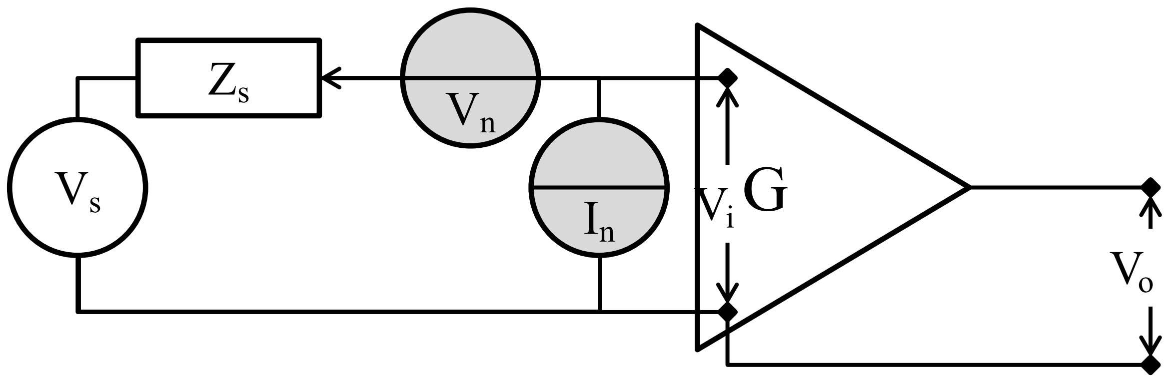


$$\rightarrow V_o = G_{oi}\left(V_s + V_n + I_n Z_s\right)$$

- Thus, in conclusion the ideal noisy amplifier can be represented as

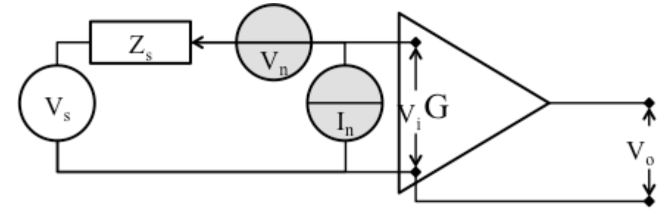


- Or, with a different graphics, and the simpler symbol G for the gain, as



- As the output voltage is $V_o = G\left(V_s + V_n + I_n Z_s\right)$ noise can be expressed at input as $V_n^{in} = V_n + I_n Z_s$
- Notice that I_n truly circulate within Z_s. Thus the amplifier perturbs the source. This perturbation is often called the amplifier *back-action*

Noise in two port systems: voltage amplifier



- In summary the noise at input of an ideal, noisy amplifier is $V_n^{in} = V_n + I_n Z_s$
- We need now to model the statistical properties of these noise generators.
  - In most experimental circumstances V_n and I_n are found to be stationary and Gaussian.
  - In some circumstances, they can be correlated. Remember that.

$$R_{V_n^{in}V_n^{in}}(\tau) = \langle V_n(t)V_n(t+\tau)\rangle + \int_0^\infty \int_0^\infty Z_s(t')Z_s(t'')\langle I_n(t-t')I_n(t+\tau-t'')\rangle dt'dt''$$

$$+\int_0^\infty Z_s(t')\langle I_n(t-t')V_n(t+\tau)\rangle dt' + \int_0^\infty Z_s(t')\langle I_n(t+\tau-t')V_n(t)\rangle dt'$$

$$= R_{V_nV_n}(\tau) + \int_0^\infty \int_0^\infty Z_s(t')Z_s(t'')R_{I_nI_n}(\tau - t'' + t')dt'dt''$$

$$+\int_0^\infty Z_s(t')R_{V_nI_n}(-\tau - t')dt' + \int_0^\infty Z_s(t')R_{V_nI_n}(\tau - t')dt'$$

  - Switching to Fourier transforms, the PSD is then

$$S_{V_n^{in}V_n^{in}}(\omega) = S_{V_nV_n}(\omega) + |Z_s(\omega)|^2 S_{I_nI_n}(\omega) + 2\,\mathrm{Re}\{Z_s(\omega)S_{V_nI_n}(\omega)\}$$