Experimental Methods Exam January 2021

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MAT. 223450

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Question 1

The sources of noise, with their relative PSD are:

- The thermal noise of the resistor R_p: $S_{V,V}(\omega) = 2k_bT_{th}R_p$
- The thermal noise of the resistor R_f: $S_{V,V}(\omega) = 2k_bT_{th}R_f$
- The amplifier's noise sources V_n and I_n

The capacitor isn't take in account as it's supposed to be ideal and with only an imaginary component. The resistances PSD are obtained through Nyquist formula.

The amplifier's noise sources can be evaluated from the quantity given in the following way:

$$V_n(\omega) = k_bT_n(\omega)R_n(\omega) = k_bT_n(1 + \frac{f_0}{f})R_n$$

$$I_n(\omega) = k_bT_n(\omega)/R_n(\omega) = k_b\frac{T_n}{R_n}(1 + \frac{f_0}{f})$$

All the PSD are reported as double sided PSD. One can transform them into one side PSD with the relation $S^{\text{exp phy}}(\omega) = 2S(\omega)$.

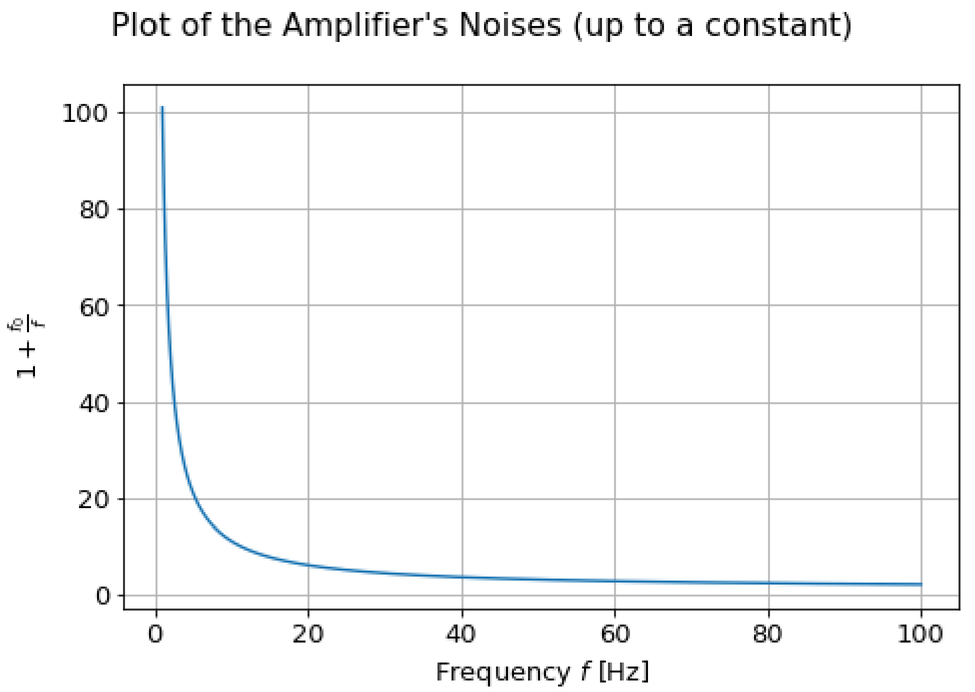
In [1]:
```
# packages used
import scipy.constants # https://docs.scipy.org/doc/scipy/reference/constants.html
from scipy.integrate import quad
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math


# Given constans
kb = scipy.constants.Boltzmann # J/K
Tnoise = 100 # K
Rnoise = 1.7e3 # Ohm
f0 = 100 # Hz

G = 10
Tth = 293 # K
C = 2.5e-12 #F
Rp = 10 # Ohm
Rf = 50e3 # Ohm
A = 0.7 # V^-1

Wt = 1.3e-6 # W
DeltaT = 60 # s

Fh = 500 # Hz

# Noise Temperature
Tn = lambda f: Tnoise*(1+f0/f)

# Show value
display(Math(r"k_b T_n R_n="+"{:.5} V".format(kb*Tnoise*Rnoise)))
display(Math(r"k_b \frac{T_n}{R_n}="+"{:.5}".format(kb*Tnoise/Rnoise )+r"\ \Omega^{-1}"))
```

$$k_b T_n R_n = 2.3471e-18V$$

$$k_b \frac{T_n}{R_n} = 8.1215e-25\ \Omega^{-1}$$

In [2]:
```
f = np.linspace(1, 100, int(1e6))
y = Tn(f)/Tnoise

plt.rcParams.update({'font.size': 13})

fig, axis = plt.subplots(1,1, figsize=(7, 5))
```

```
fig.suptitle("Plot of the Amplifier's Noises (up to a constant)")

axis.plot(f, y)
axis.set_xlabel("Frequency $f$ [Hz]")
axis.set_ylabel(r"$1+\frac{f_0}{f}$")
axis.grid(True, which="both")

plt.tight_layout()
plt.show()
```



Question 2

Regarding R_p:
It's voltage PSD can be propagated at input by noticing that the capacitor transforms the input current into an input voltage via $V(\omega) = i\omega C I(\omega)$.
Therefore R_p contributes with a term:

$$S_{R_p}^{in}(\omega) = 2k_b T_{th} R_p |\frac{1}{i\omega C}|^2$$

Regarding R_f:
Its noise current $S_{R_f}(\omega) = 2k_b\frac{T_{th}}{R_f}$ flows in the feedback branch of the feedback loop. The relation between the input current and the feedback current is

$$i_{fb} = -\frac{\beta G}{1+\beta G} i_e(\omega) \rightarrow \frac{i_e}{i_{fb}} = -\frac{1+\beta G}{\beta G}$$

with $\beta = R_f$ in our case. It's then possible to propagate the PSD and obtain:

$$S_{R_r}(\omega) = |\frac{1+R_f G}{R_f G}|^2\, 2k_b\frac{T_{th}}{R_f} = \frac{2k_b T_{th}}{R_f}\frac{(1+R_f G)^2}{R_f^2 G^2}$$

This value needs to be propagated at input by sending it through the resistor to get the Voltage PSD and then through the capacitor:

$$S_{R_r}^{in}(\omega) = \frac{2k_b T_{th}}{R_f}\frac{(1+R_f G)^2}{R_f^2 G^2}\,|\frac{R_p}{i\omega C}|^2$$

Regarding the Amplifier's noise sources:
V_n is already a voltage PSD while I_n needs to pass through R_f. After that both need to be propagated through the capacitor:

$$S_{I_n}^{in}(\omega) = k_b\frac{T_n}{R_n}(1+\frac{f_0}{f})|\frac{R_p}{i\omega C}|^2$$

$$S_{V_n}^{in}(\omega) = k_b T_n(1+\frac{f_0}{f})R_n|\frac{1}{i\omega C}|^2$$

To sum up, the total input PSD is

$$S_{R_p}^{in}(\omega) + S_{R_r}^{in}(\omega) + S_{I_n}^{in}(\omega) + S_{V_n}^{in}(\omega) =$$
$$2k_b\frac{T_{th}R_p}{\omega^2 C^2} + \frac{2k_b T_{th}}{R_f}\frac{(1+R_f G)^2}{R_f^2 G^2}\frac{R_p^2}{\omega^2 C^2} + \frac{k_b T_n}{R_n}(1+\frac{f_0}{f})\frac{R_p^2}{\omega^2 C^2} + (1+\frac{f_0}{f})\frac{k_b T_n R_n}{\omega^2 C^2} =$$
$$\left[R_p + \frac{(1+R_f G)^2}{R_f^2 G^2}\frac{R_p^2}{R_f} + \frac{(R_p^2+R_n^2)}{2R_n}(1+\frac{2\pi f_0}{\omega})\right]\frac{2k_b T_{th}}{\omega^2 C^2} = S_{tot}^{in}(\omega)$$

The PSD of an equivalent input optical power W can be evaluated by dividing everything by A^2

$$S_{P,P}(\omega) = \left[R_p + \frac{(1+R_fG)^2}{R_f^2G^2}\frac{R_p^2}{R_f} + \frac{(R_p^2+R_n^2)}{2R_n}(1+\frac{2\pi f_0}{\omega})\right]\frac{2k_bT_{th}}{\omega^2C^2A^2}$$

It's possible to notice that inside the square brackets there is the sum of somethings that have the units of Ω this implies that probably no math mistake was made.

The total PSD can be rewritten as

$$S_{P,P}(\omega) = \left[R_1 + R_2(1+\frac{2\pi f_0}{\omega})\right]\frac{R_3}{\omega^2}$$

with: $R_1 = R_p + \frac{(1+R_fG)^2}{R_f^2G^2}\frac{R_p^2}{R_f}$; $R_2 = \frac{(R_p^2+R_n^2)}{2R_n}$; $R_3 = \frac{2k_bT_{th}}{C^2A^2}$

Where the notation R_3 was chosen despite not being expressed in Ω as R_1 and R_2.

In [13]:
```
R1 = Rp + (Rp**2 / Rf)*((1+Rf*G)/(Rf*G))**2
R2 = (Rp**2 + Rnoise**2)/(2*Rnoise)
R3 = (2*kb*Tth)/(C**2 * A**2)
display(Math('R_1={:.3f} \Omega'.format(R1)) )
display(Math('R_2={:.3f} \Omega'.format(R2)) )
display(Math('R_3={:.3f}'.format(R3)+r"\ \frac{J V^2}{F^2}") )
```

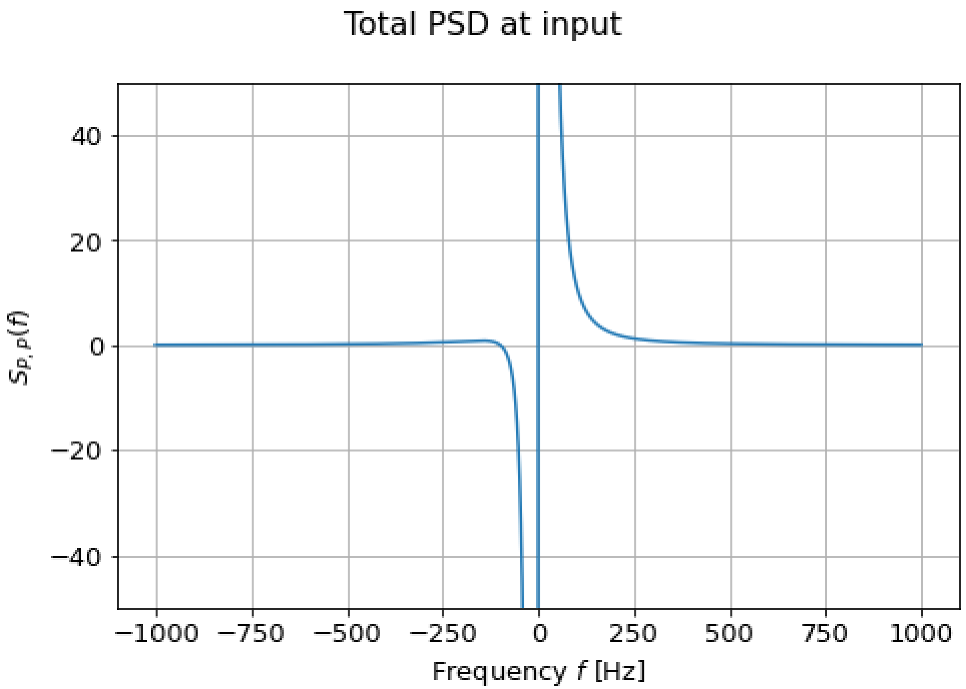
$$R_1 = 10.002\Omega$$

$$R_2 = 850.029\Omega$$

$$R_3 = 2641.830\ \frac{JV^2}{F^2}$$

In [4]:
```
def PSDtot(f):
    Reqiv =  R1+ R2* (1+f0/f)
    return Reqiv*R3/np.power(2*np.pi*f, 2)

f = np.linspace(-(1e3), 1e3, int(1e6))
PSD = PSDtot(f)

# Bode plot
```

```
plt.rcParams.update({'font.size': 13})

fig, axis = plt.subplots(1,1, figsize=(7, 5))
fig.suptitle("Total PSD at input")

axis.plot(f, PSD)
#axis.set_yscale('log')
axis.set_xlabel("Frequency $f$ [Hz]")
axis.set_ylabel(r"$S_{P,P}(f)$")
axis.grid(True, which="both")
axis.set_ylim([-50, 50])

plt.tight_layout()
plt.show()
```



Question 3

This is a Wiener's problem. The signal has the form $s(t) = Mf(t) = M\big(\Theta(t) - \Theta(t - \Delta T)\big)$ with $M = W_t\, 10^{-0.4m}$. Wiener's theory states that the minimum error is equal to:

$$\sigma_M^2 = \frac{1}{\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{|f(\omega)|^2}{S(\omega)}d\omega}$$

Where $f(\omega)$ is the Fourier transform of the signal's template:

$$f(\omega) = \mathcal{FT}\Big[\big(\Theta(t) - \Theta(t-\Delta T)\big)\Big](\omega) = \frac{i(e^{-i\Delta T\omega} - 1)}{\omega}$$

The $\mathcal{FT}$ was evaluated with WolframAlpha (normalization factor=1, oscillator factor=-1 to respect the convention used in the lectures). This is just an other form of the translated sinc.

One can write the signal to noise ratio as

$$SNR(\omega) = \frac{|f(\omega)|^2}{S(\omega)} = \frac{|i(e^{-i\Delta T\omega} - 1)|}{R_3}\frac{\omega}{(R_1+R_2)\omega + 2\pi f_0 R_2} = \frac{2 - 2cos(\Delta T\omega)}{R_3}\frac{\omega}{(R_1+R_2)\omega + 2\pi f_0 R_2}$$

The integral can be evaluated numerically:

In [5]:
```
def template(omega):
    return 1j*(np.exp(-1j*DeltaT*omega) -1)/omega

def integrand(omega):
    f = omega / (2*np.pi)
    return np.power( np.abs(template(omega)),2 )/ PSDtot(f)

I = quad(integrand, -np.inf, np.inf, limit=100)
display(Math(r'$\text{Value of the integral: }$'+'I={:.9f}'.format(I[0])+
             r'$\text{   Absolute error: }'+'{:.2}'.format(I[1]) ))

sigma2 = 2*np.pi/I[0]
display(Math('\sigma^2_M={:.3f}'.format(sigma2)) )
```

```
<ipython-input-5-fec884e921a5>:8: IntegrationWarning: The algorithm does not converge.  Roundoff error is detected
  in the extrapolation table.  It is assumed that the requested tolerance
  cannot be achieved, and that the returned result (if full_output = 1) is
  the best which can be obtained.
  I = quad(integrand, -np.inf, np.inf, limit=100)
```

Value of the integral: I=36.387198909 Absolute error: $7.1e+01$

$\sigma_M^2 = 0.173$

This integral does not converges due to the discontinuity at $\omega = -\frac{2\pi f_0 R_2}{R_1+R_2}$. A way to solve this problem may be to use the "Data whitening" and use the Wiener's theory for finite interval and white noise.

In [6]:
```
display(Math(r'-\frac{2\pi f_0 R_2}{R_1 + R_2}='+'{:.3f}'.format(-2*np.pi*f0*R2/(R1+R2)) ))
```

$$-\frac{2\pi f_0 R_2}{R_1 + R_2} = -621.011$$

Question 4

The chopper multiplies the signal by a square wave $\frac{1+Sign(sin(2\pi f_h t))}{2}$. As $\omega_0 = 2\pi f_h \simeq 3000Hz$ the signal can be expanded as a Taylor series so that:

$$1 + Sign(sin(|omega_0 t)) \simeq \frac{1}{2}\left[1 + \frac{4}{\pi} sin(\omega_0 t) + \ldots\right] \rightarrow \frac{1 + Sign(sin(\omega_0 t))}{2} \simeq \frac{1}{4} + \frac{sin(\omega_0 t)}{\pi}$$

The new signal template is therefore

$$f(t) = \frac{1}{4} + \frac{sin(\omega_0 t)}{\pi}$$

$$f(\omega) = \frac{\pi}{2}\delta(\omega) - i\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$$

By applying Wiener's theory:

$$\sigma_M^2 = \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|f(\omega)|^2}{S(\omega)} d\omega\right]^{-1} = \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|\frac{\pi}{2}\delta(\omega) - i\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]|^2}{S(\omega)} d\omega\right]^{-1}$$

Which can be separated in the sum of three contributes at the frequencies $0, \omega_0, -\omega_0$.

$$\sigma_M^2 = 2\pi\left[\frac{\pi^2}{4S(0)} + \frac{1}{S(\omega_0)} + \frac{1}{S(-\omega_0)}\right]^{-1}$$

The minimum detectable power (squared) is therefore:

In [7]:
```
# PSDtot(1e-100) instead of PSDtot(0) is used to avoid division by zero.
#For phyton that's an error, but in reality the two omega would cancel with on an other.
sigma2 = 2*np.pi/(np.pi**2/(4*PSDtot(1e-100)) + 1/PSDtot(Fh) + 1/(PSDtot(-Fh)) )
```

```
display(Math('\sigma^2_W={:.9f}'.format(sigma2)) )
```

$$\sigma^2_W = 0.694958263$$

To evaluate the maximum detectable magnitude is sufficient to invert the relation $W = W_t 10^{-0.4m} \rightarrow m = -2.5\ Log(\frac{\sigma_W}{W_t})$

In [8]:
```
display(Math('m={:.2f}'.format(-2.5*np.log(sigma2/Wt))) )
```

$$m = -32.97$$