# Experimental Methods
# Lecture 6

October 1st, 2020

# Discrete Fourier Transform

- N samples of $s(t)$, sampling time T, signal duration ~NT
- The transform and its inverse

$$s_k = \sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}nk} \qquad s_n = \frac{1}{N}\sum_{k=0}^{N-1} s_k e^{+i\frac{2\pi}{N}nk}$$

- Symmetry

$$s_{N-k} = s_{-k} = s_k^*$$

- $\simeq \frac{\text{N}}{2}$ complex coefficient, N real coefficients

# Discrete Fourier Transform

- Relation to continuous transform

$$s_k = \sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}nk} = \sum_{n=-\infty}^{\infty} (\Theta(n) - \Theta(n-N+1))s_n e^{-i\frac{2\pi}{N}nk}$$

$$s_k = s'_d\left(\phi = k\frac{2\pi}{N}\right)$$

- $s'_d$ discrete-time Fourier transform of

$$s'_n = (\Theta(n) - \Theta(n-N+1))s_n$$

- But

$$s'_d(\phi) = \frac{1}{\mathrm{T}}\sum_{k=-\infty}^{\infty} s'_c\left(\frac{\phi}{T} + n\frac{2\pi}{T}\right)$$

- $s'_c$ continuous Fourier Transform of

$$s'(t) = s(t)(\Theta(t) - \Theta(t-NT))$$

- Thus

$$s_k = \frac{1}{\mathrm{T}}\sum_{k=-\infty}^{\infty} s'_c\left(k\frac{2\pi}{NT} + n\frac{2\pi}{T}\right)$$

# Discrete Fourier Transform

$$s_k = \frac{1}{\mathrm{T}} \sum_{k=-\infty}^{\infty} s_c' \left( k \frac{2\pi}{NT} + n \frac{2\pi}{T} \right)$$

- Notice from

$$s'(t) = s(t)(\Theta(t) - \Theta(t - NT))$$

- Using convolution

$$s'(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} s(\omega') NT e^{-\frac{i(\omega-\omega')NT}{2}} Sinc\left( \frac{(\omega - \omega')NT}{2} \right) d\omega'$$

- If $s(t)$ is long duration, $\Delta T > NT$ , it is narrow band, $\Delta\omega \simeq 1/\Delta T$.
- Extrapolating:

$$s'(\omega) \simeq \frac{1}{2\pi} s(0) \Delta\omega NT e^{-\frac{i\omega NT}{2}} Sinc\left( \frac{\omega T}{2} \right)$$

- Which is not band limited
- Aliasing may come from truncation. Signals should be truncated after they have decayed to zero

# Convolution

- Convolution of infinite length series.

$$z_n = \sum_{k=-\infty}^{\infty} y_{n-k} x_k$$

- Reverse y, shift it by n and do the "scalar product" with x

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=-3 |
| $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | $y_{-5}$ | $y_{-6}$ | $y_{-7}$ | $y_{-8}$ | |
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=-2 |
| $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | $y_{-5}$ | $y_{-6}$ | $y_{-7}$ | |
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=-1 |
| $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | $y_{-5}$ | $y_{-6}$ | |
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=0 |
| $y_5$ | $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | $y_{-5}$ | |
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=1 |
| $y_6$ | $y_5$ | $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | |
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=2 |
| $y_7$ | $y_6$ | $y_5$ | $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | |
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=3 |
| $y_8$ | $y_7$ | $y_6$ | $y_5$ | $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | |

# Convolution

- Convolution of two series of finite and different length
- No element is defined

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ? | ? | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | ? | ? | $n=-3$ |
| $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | ? | ? | ? | ? | |
| ? | ? | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | ? | ? | $n=-2$ |
| $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | ? | ? | ? | |
| ? | ? | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | ? | ? | $n=-1$ |
| $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | ? | ? | |
| ? | ? | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | ? | ? | $n=0$ |
| ? | $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | ? | |
| ? | ? | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | ? | ? | $n=1$ |
| ? | ? | $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | |
| ? | ? | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | ? | ? | $n=2$ |
| ? | ? | ? | $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | |
| ? | ? | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | ? | ? | $n=3$ |
| ? | ? | ? | ? | $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | |

# Convolution

- Convolution of two series of finite and equal length
- The case for n=0 is defined

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=-3 |
| $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | $y_{-5}$ | ? | ? | ? | |
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=-2 |
| $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | $y_{-5}$ | ? | ? | |
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=-1 |
| $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | $y_{-5}$ | ? | |
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=0 |
| $y_5$ | $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | $y_{-5}$ | |
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=1 |
| ? | $y_5$ | $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | $y_{-4}$ | |
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=2 |
| ? | ? | $y_5$ | $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | $y_{-3}$ | |
| $x_{-5}$ | $x_{-4}$ | $x_{-3}$ | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | n=3 |
| ? | ? | ? | $y_5$ | $y_4$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | |

# Linear convolution

- To convolute finite length series you need to pad them and make them infinite.
- Case 1: padding with zeros
- For series of equal length $N$, resulting series has $2N - 1$ meaningful elements.
- Other zeros can be ignored

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | n=-5 |
| $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | n=-4 |
| 0 | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | n=-3 |
| 0 | 0 | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | n=-2 |
| 0 | 0 | 0 | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | n=-1 |
| 0 | 0 | 0 | 0 | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | 0 | 0 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | n=0 |
| 0 | 0 | 0 | 0 | 0 | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | 0 | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | n=1 |
| 0 | 0 | 0 | 0 | 0 | 0 | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | 0 | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | n=2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | 0 | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | n=3 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | 0 | 0 | |
| 0 | 0 | 0 | 0 | 0 | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | n=4 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | 0 | |
| 0 | 0 | 0 | 0 | 0 | $x_{-2}$ | $x_{-1}$ | $x_0$ | $x_1$ | $x_2$ | 0 | 0 | 0 | 0 | 0 | n=5 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $y_2$ | $y_1$ | $y_0$ | $y_{-1}$ | $y_{-2}$ | |

# Circular convolution of equal length series

- Case 2: repeat series periodically
- Invert one of the two, shift it by n , and truncate to $N$
- Result is periodic with same period $N$

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | n=-3 |
| $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | |
| $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | n=-2 |
| $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | |
| $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | n=-1 |
| $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | |
| $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | n=0 |
| $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | |
| $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | n=1 |
| $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | |
| $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | n=2 |
| $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | |
| $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_0$ | $x_1$ | $x_2$ | n=3 |
| $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | $y_0$ | $y_3$ | $y_2$ | $y_1$ | |

# Fourier transform of circular convolution

- A natural way of continuing y periodically: extend the Fourier anti-transform to any integer:

$$y_{N+m}=\frac{1}{N}\sum_{k=0}^{N-1} y_k e^{ik\frac{2\pi}{N}(N+m)} = \frac{1}{N}\sum_{k=0}^{N-1} y_k e^{ik\frac{2\pi}{N}m} e^{ik2\pi} = y_m$$

- Then

$$z_n=\sum_{j=0}^{N-1}\left(\frac{1}{N^2}\sum_{k=0}^{N-1} y_k e^{ik\frac{2\pi}{N}(n-j)}\sum_{l=0}^{N-1} x_l e^{il\frac{2\pi}{N}j}\right)$$

- That is

$$z_n=\frac{1}{N^2}\sum_{k,l=0}^{N-1} y_k x_l\, e^{ik\frac{2\pi}{N}n}\sum_{j=0}^{N-1} e^{ij\frac{2\pi}{N}(l-k)}$$

- But

$$\sum_{j=0}^{N-1} e^{ij\frac{2\pi}{N}(l-k)} = N\delta_{kl}$$

# Fourier transform of circular convolution

- From

$$z_n = \frac{1}{N^2} \sum_{k,j=0}^{N-1} y_k x_l \, e^{ik\frac{2\pi}{N}n} \sum_{j=0}^{N-1} e^{ij\frac{2\pi}{N}(l-k)}$$

- And

$$\sum_{j=0}^{N-1} e^{ij\frac{2\pi}{N}(l-k)} = N\delta_{kl}$$

- It follows

$$z_n = \frac{1}{N} \sum_{k=0}^{N-1} y_k x_k \, e^{ik\frac{2\pi}{N}n}$$

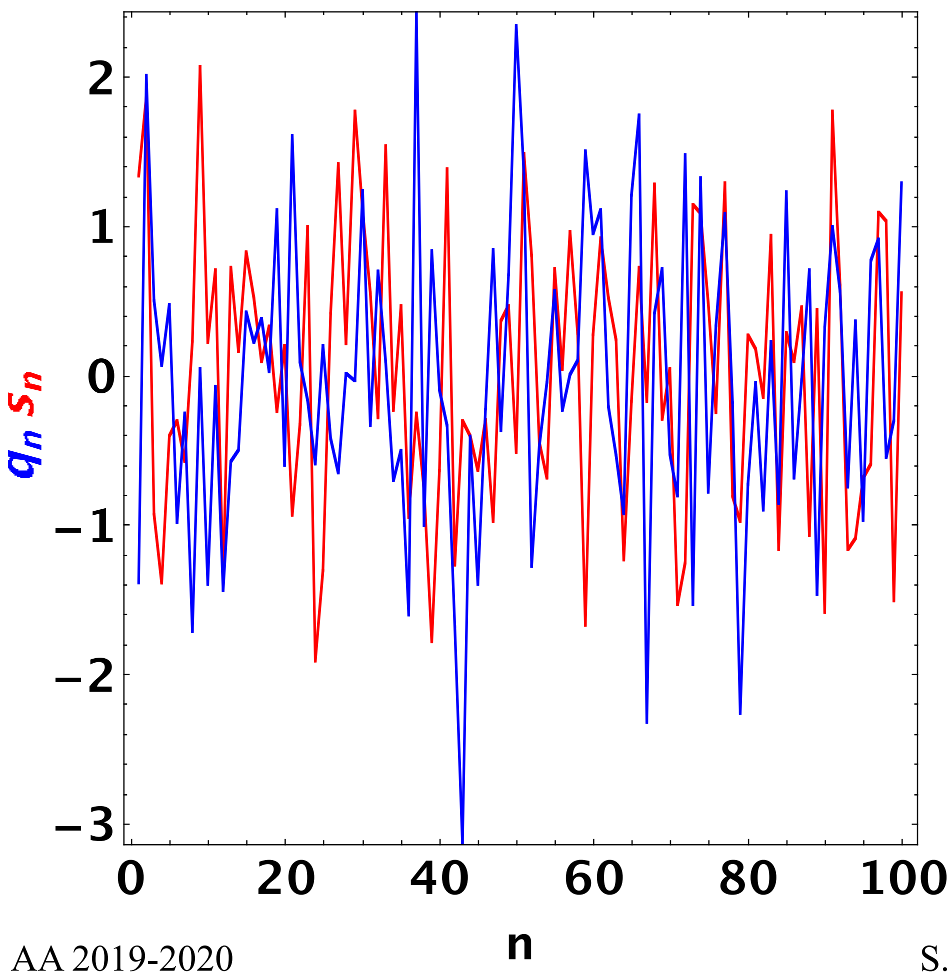
- Going back to the definition of discrete Fourier transform we get

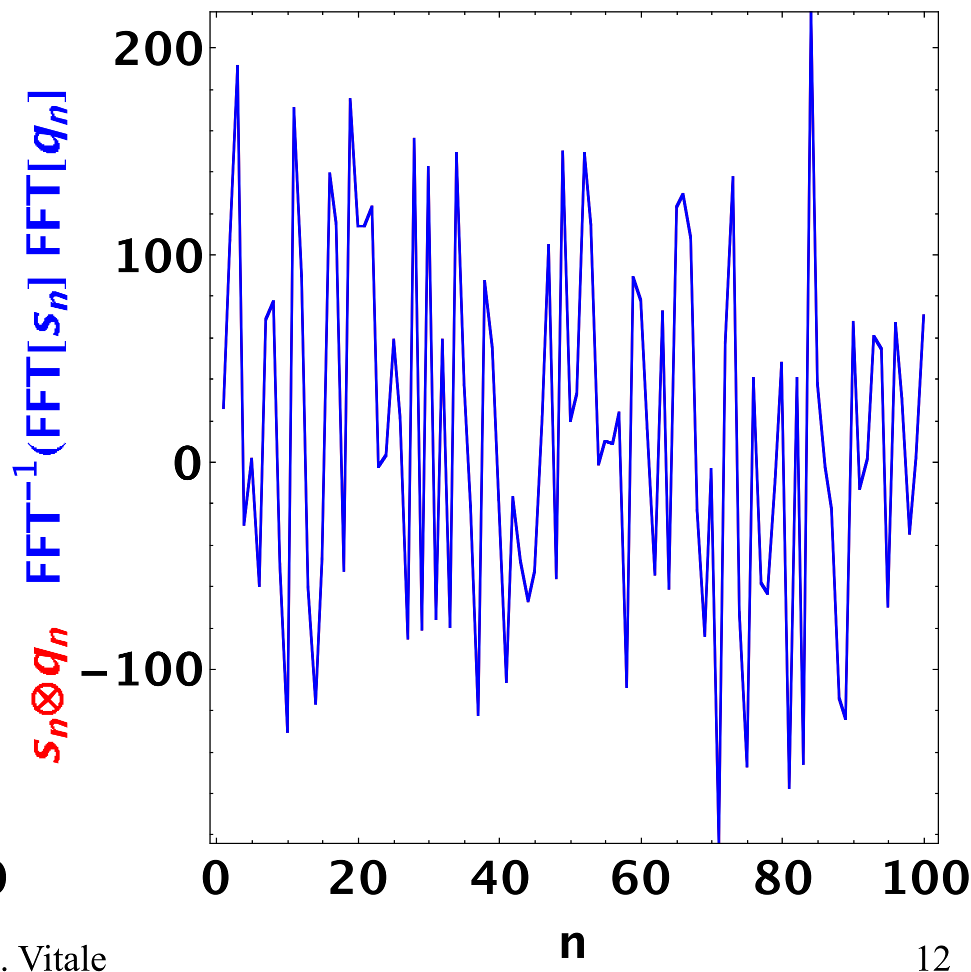
$$z_k = y_k x_k$$

- Which is the convolution theorem for discrete Fourier transforms

# Examples with random numbers

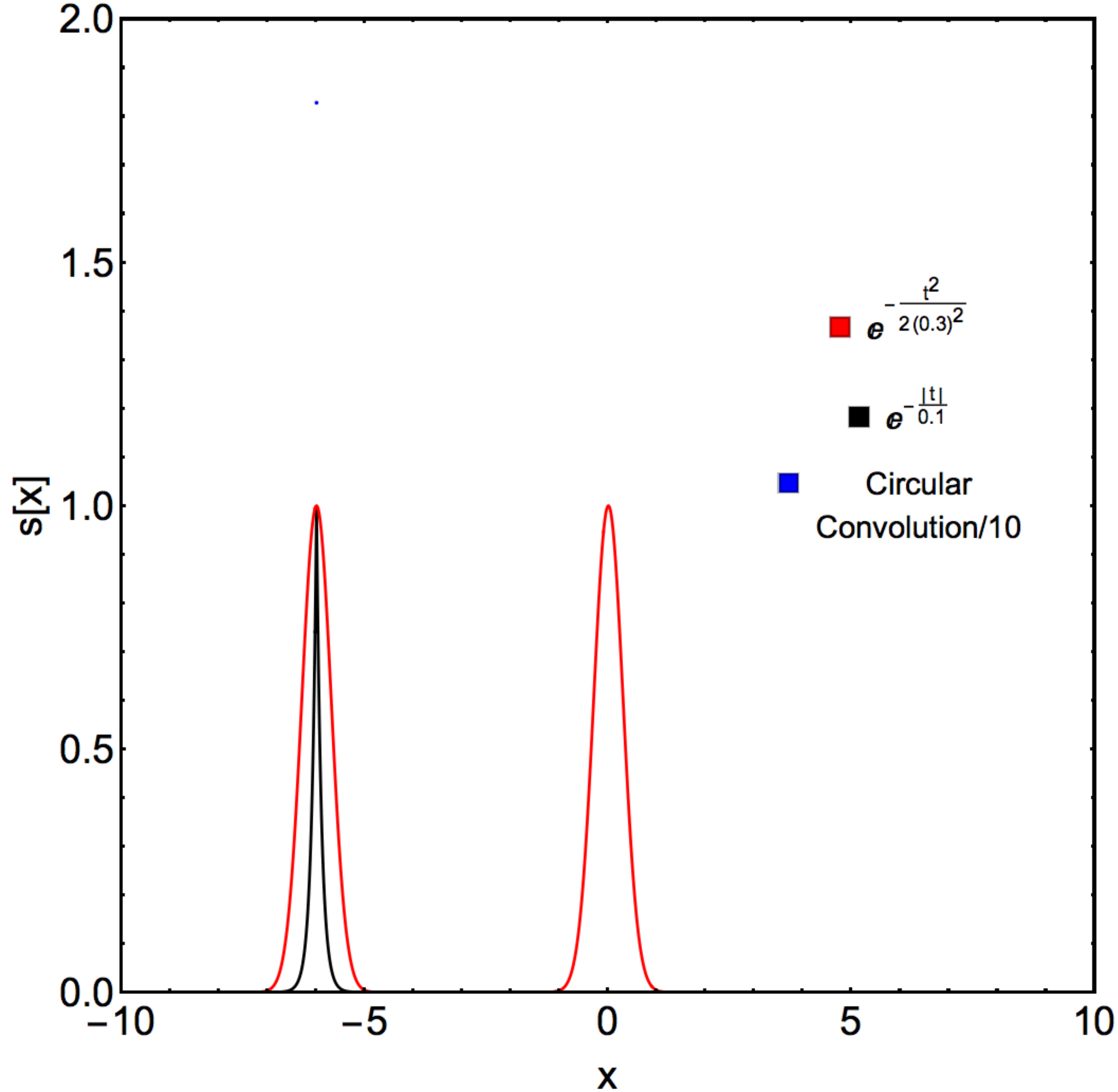
- Two series of $10^4$ random Gaussian numbers with zero mean and unit variance



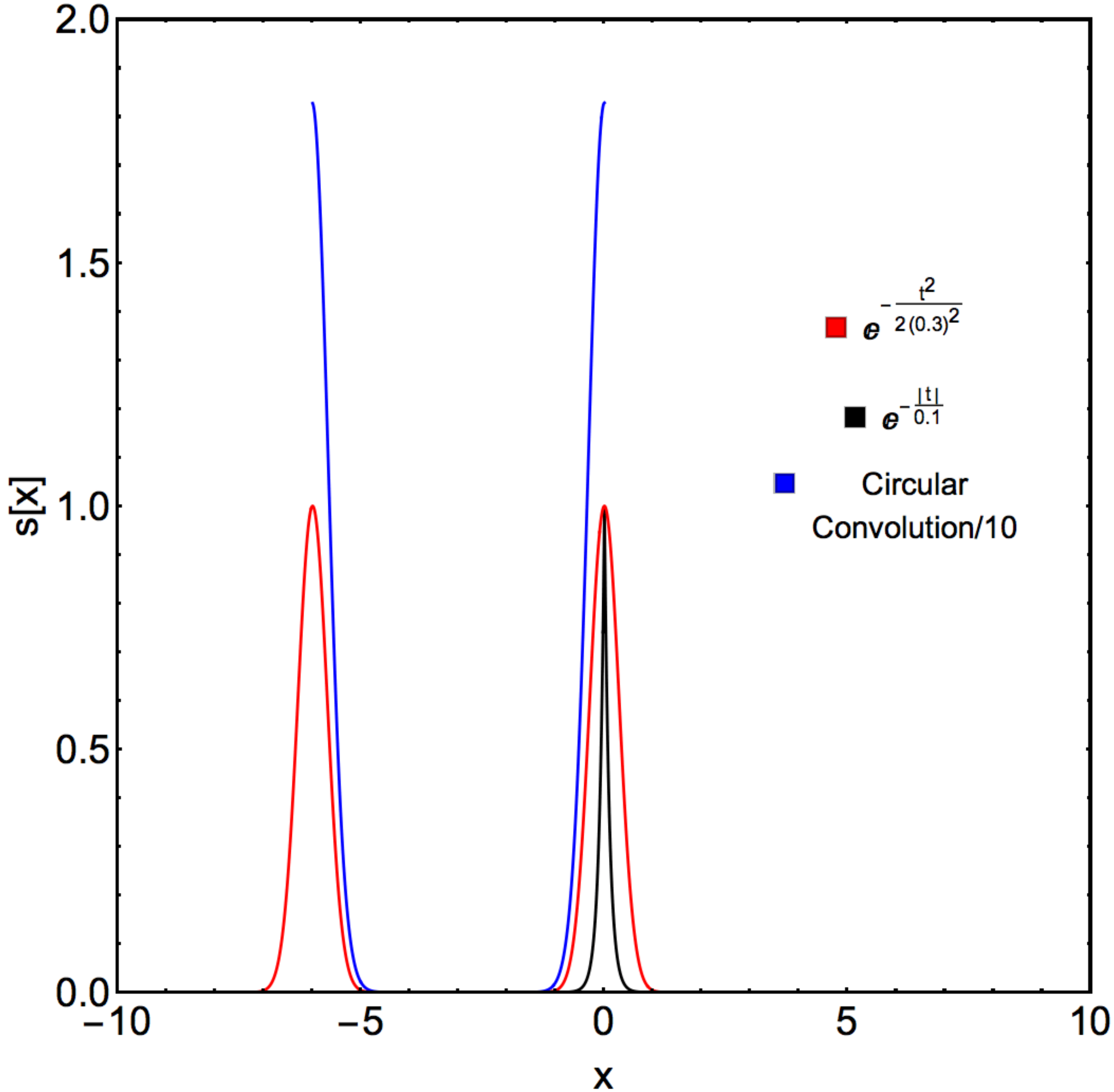
# Can we use DFT to calculate true convolutions?

- For finite duration signal yes, provided that they go to zero fast enough at both ends



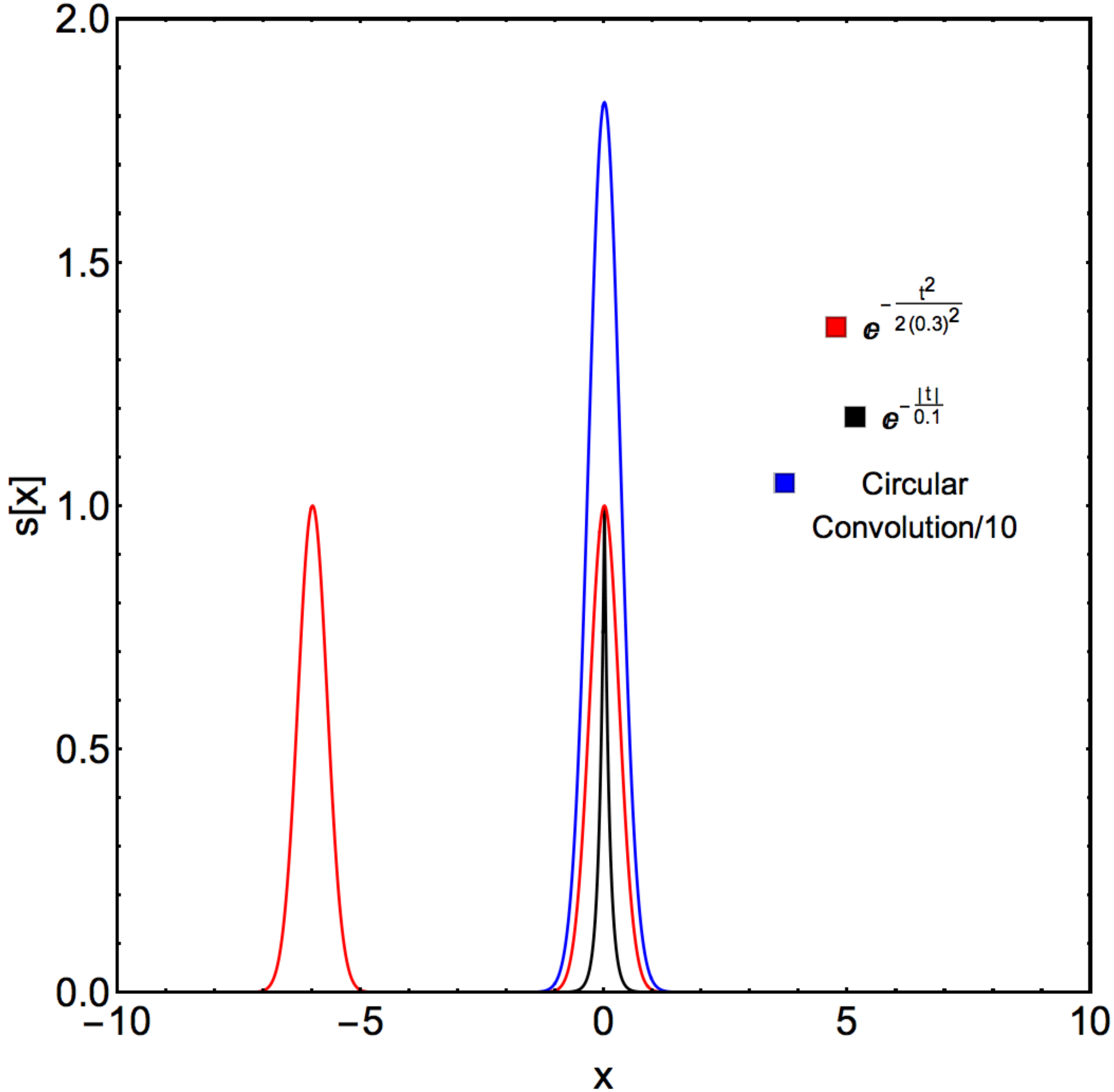
# Can we use DFT to calculate true convolutions?

- For finite duration signal yes, provided that they go to zero fast enough at both ends



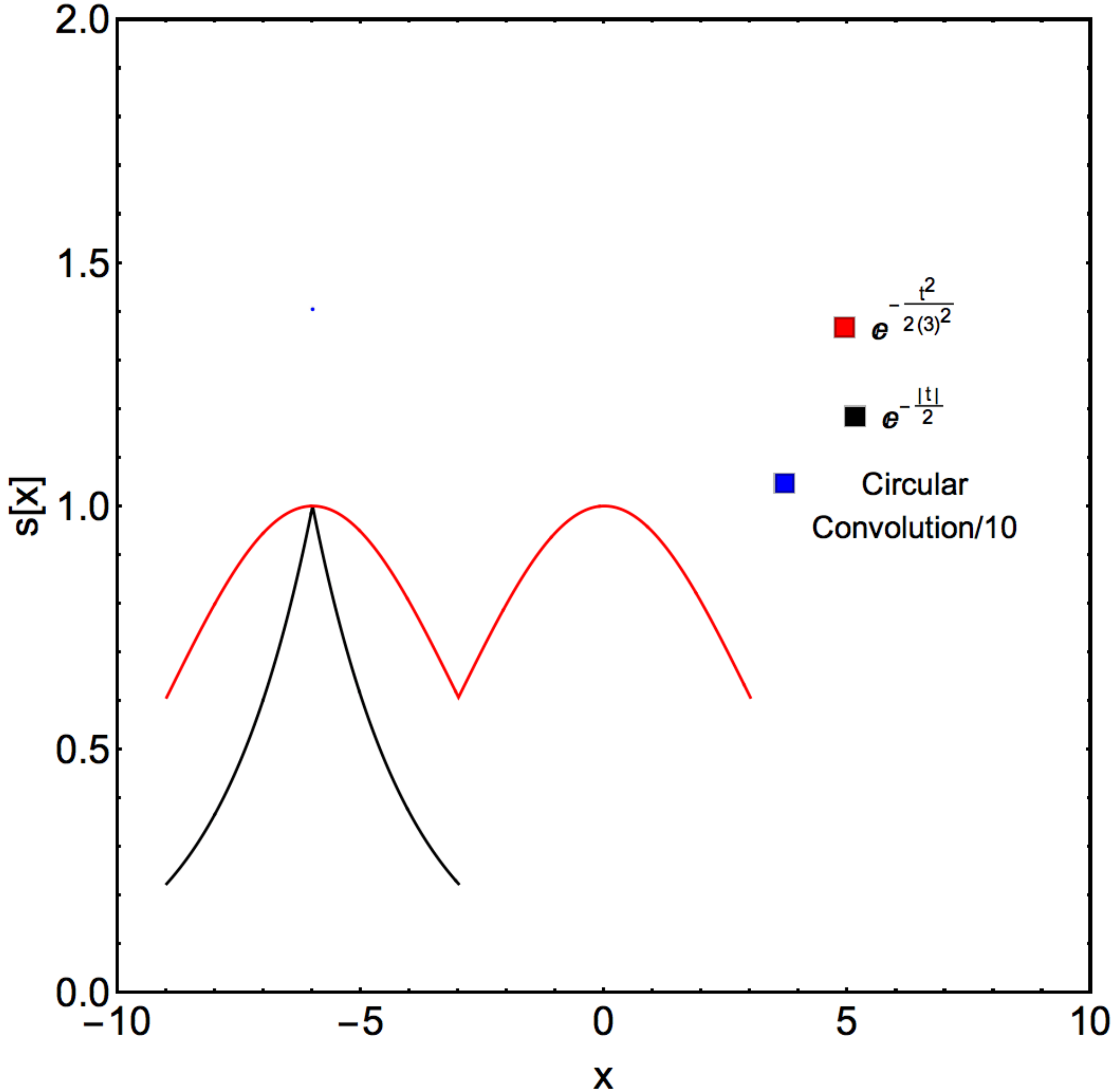
# Can we use DFT to calculate true convolutions?

- As I used signals with $-\frac{N}{2} \leq \text{n} \leq \frac{N}{2}$ results must be appropriately shifted to give true circular convolution



# Can we use DFT to calculate true convolutions?

- A counterexample



# Circular convolution: frequency domain

$$\hat{s}_k = \sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}kn}$$

- Is already periodic. The padding takes place the same way

$$\hat{s}_{k+N} = \sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}(k+N)n} = \sum_{n=0}^{N-1} s_n e^{-i\frac{2\pi}{N}kn} e^{-i\frac{2\pi}{N}Nn} = \hat{s}_k$$

- If

$$\hat{q}_k = \frac{1}{N}\sum_{j=0}^{N-1} \hat{p}_j \hat{s}_{k-j} \quad 0 \le j \le N-1$$

- Then

$$q_n = p_n s_n$$

- Notice the factor 1/N in the definition of the convolution

# Practical advice

- You may use FFT to numerically estimate Fourier transform of continuous signals but:
  - Signal must be sampled densely enough
  - Signal must have been truncated appropriately
  - If you want to use convolution, both signals (and/or Fourier transforms) must decay fast enough that they are naturally padded with zeros at both ends
  - In some applications signals are multiplied by "spectral windows" that taper their ends to get the previous condition. To be discussed later

# What should you know about sampling

- Sampling theorem
- The Nyquist rule
- Discrete Fourier Transform
- How to use a FFT on a computer
- Circular convolution and effect of truncation

# Exercise

1. Calculate the Continuous Fourier Transform of

$$e^{-\frac{t^2}{2}}\mathrm{Sin}(8\pi t)$$ All units in SI

2. Calculate the FFT of the curve above sampled at the sampling frequency of 20 Hz, and truncated at $|t|\leq 10$ s
3. Calculate the FFT of the two following functions

$$e^{-\frac{t^2}{2}} \qquad \mathrm{Sin}(8\pi t)$$

sampled at the sampling frequency of 20 Hz, and truncated at $|t|\leq 10$ s

4. Calculate their circular convolution and compare with 2. above

# A physical instrument as a “system”

- A system transforms an input signal into an output one

$$i(t) \xrightarrow{\text{Input}} \boxed{\text{System}} \xrightarrow{\text{Output}} o(t)$$

- The output at any time may depend on the input at all times
- A system is then an operator or a functional in the vector space of signals

$$o(t) = \Im\left[i(t)\right]$$

- A discrete system (in data analysis or calculations) converts an input data series into an output one. The output series is in principle a function of all samples of the input series

$$o_n = f\left[......i_{-2}, i_{-1}, i_0, i_1, i_2.....\right]$$

# Mathematical examples

- System regulated by a linear differential equation: Newton law

$$m\ddot{x} = F_x$$

  - Input: force $F_x(t)$
  - Output: particle coordinate $x(t)$
  - Input-output relation (particle at rest in the origin at infinite past)

$$x(t) = \int_0^{\infty} t' F(t - t')dt'$$

- Non linear system: Ginzburg Landau Equation for superconducting current **j**

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m}\left(-i\hbar\nabla - 2e\mathbf{A}\right)^2\psi = 0$$

$$\mathbf{j} = \frac{2e}{m}\mathrm{Re}\left\{\psi^*\left(-i\hbar\nabla - 2e\mathbf{A}\right)\psi\right\}$$

  - Input: vector potential **A** (better: its applied part)
  - Output: current density **j**

# Special properties

## Causality

- For physical systems the *principle of causality* holds:

$$o(t)=\Im\left[i(t')\right] \quad t\geq t'$$

- For arbitrary "mathematical systems", encountered in calculations or in data analysis, causality is not required

## Linearity

- A system is linear if:

$$\Im\left[a_1 i_1(t)+a_2 i_2(t)\right]=a_1\Im\left[i_1(t)\right]+a_2\Im\left[i_2(t)\right]$$

i.e. if it obeys the principle of superposition

- For discrete systems, linearity implies:

$$o_n=\sum_{k=-\infty}^{\infty} h_{n,k} i_k$$

- with $h_{n,k}$ a numerical coefficient

# Linearity and the input-output relation

- For a band-limited signal: $$i(t)=\sum_{k=-\infty}^{\infty} i(kT)\,\mathrm{Sinc}\left[\frac{\pi}{T}(t-kT)\right]$$
- Linearity implies:

$$o(t)=\Im\left[i(t)\right] \;=\sum_{k=-\infty}^{\infty} i(kT)\,\Im\left[\mathrm{Sinc}\left[\frac{\pi}{T}(t-kT)\right]\right]$$

- Now remember that

$$\operatorname*{Lim}_{a\to 0}\left(\frac{1}{a}\mathrm{Sinc}\left[\pi x/a\right]\right)=\delta(x)$$

- Then taking the limit of infinite fast sampling

$$\operatorname*{Lim}_{T\to 0}\left[o(t)\right]=\sum_{k=-\infty}^{\infty} i(kT)\operatorname*{Lim}_{T\to 0}\left(\frac{1}{T}\Im\left[\mathrm{Sinc}\left[\frac{\pi}{T}(t-kT)\right]\right]\right)T=\int_{-\infty}^{\infty} i(t')h(t,t')dt'$$

- where $kT\to t'$ and $\qquad h(t,t')=\Im\left[\delta(t-t')\right]$
- $h(t,t')$ is called the impulse response of the system, as it is the output at time t for a $\delta$ arriving at time t'.

# A central formula for linear systems

- In general $$o(t)=\int_{-\infty}^{\infty} i(t')h(t,t')\,dt'$$
- For causal systems $$o(t)=\int_{-\infty}^{t} i(t')h(t,t')\,dt'$$

# System may have a free evolution

- Free evolution: output for no input

$$o_n=o_n^o+\sum_{k=-\infty}^{\infty} h_{n,k}i_k \qquad o(t)=o^o(t)+\int_{-\infty}^{\infty} i(t')h(t,t')\,dt'$$

- $o^o$ depends on internal states of the systems (initial conditions)
- Principle of superposition holds for $o_n-o_n^o$ and $o(t)-o^o(t)$

# An example: the calorimeter

- A body of heat capacitance C, temperature T, receiving a heat input with power W, and with a loss path to the thermostat (at $T_o$) of conductance κ.
- Input: heat flow. Output: temperature
- First principle of thermodynamics

$$C\left(dT/dt\right) = W - \kappa\left(T - T_o\right)$$

- The derivative of $T_o$ is zero

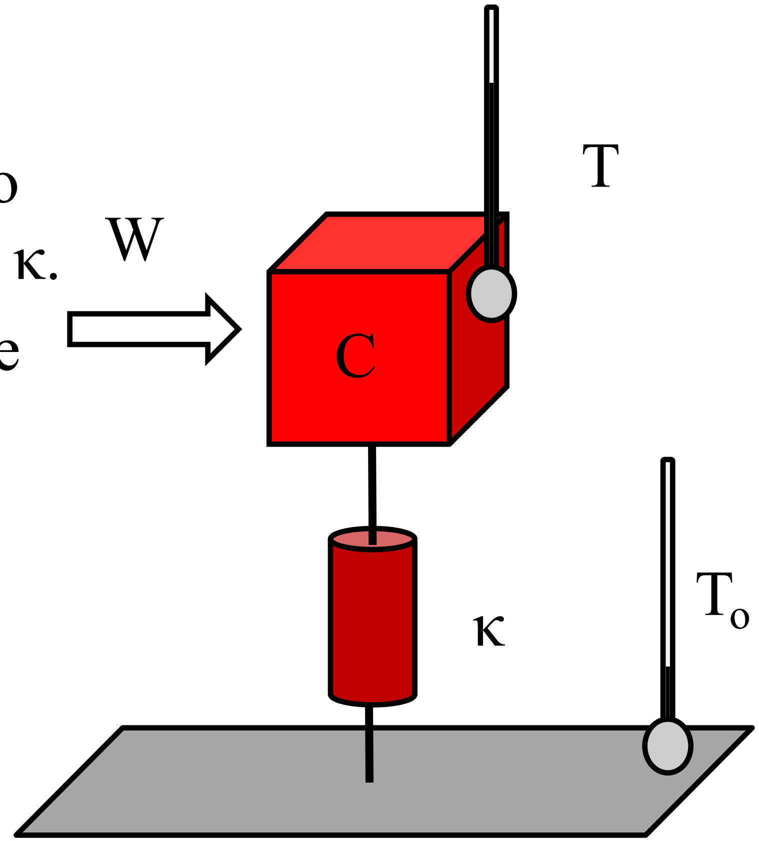
$$C\left[d\left(T - T_o\right)/dt\right] = W - \kappa\left(T - T_o\right)$$

- A change of variables and names

$$\Delta T = T - T_o \quad \tau^{-1} = \frac{\kappa}{C}$$

- Finally

$$d\Delta T/dt + \Delta T/\tau = W/C$$



# Disclaimer

- To make things dimensionally more transparent, a better way to write the equation would have been
$$\frac{d\Delta T}{dt}+\frac{\Delta T}{\tau}=\frac{W}{\kappa}\frac{\kappa}{C}=\frac{1}{\tau}\frac{W}{\kappa}\equiv\frac{\Delta T_o}{\tau}$$
- So that input and output have the same dimensions
- The following slides suffer a bit because of the choice of using $\frac{W}{C}$ as the input, instead of $\Delta T_o = W/\kappa$

# A linear system

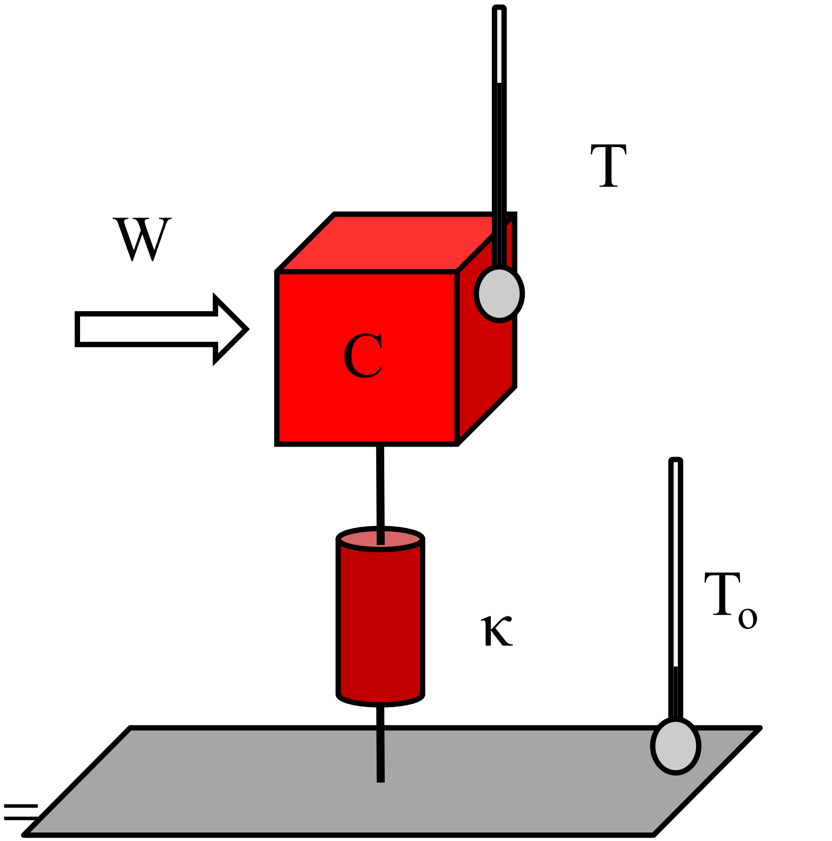
- If C and $\tau$ are independent of $\Delta T$ the system is linear.
- Indeed, suppose that

$$d\Delta T_1/dt + \Delta T_1/\tau = W_1/C$$

- and that

$$d\Delta T_2/dt + \Delta T_2/\tau = W_2/C$$

- Then




$$d\left(c_1\Delta T_1 + c_2\Delta T_2\right)/dt + \left(c_1\Delta T_1 + c_2\Delta T_2\right)/\tau =$$

$$= d\left(c_1\Delta T_1\right)/dt + \left(c_1\Delta T_1\right)/\tau + d\left(c_2\Delta T_2\right)/dt + \left(c_2\Delta T_2\right)/\tau =$$

$$= c_1 W_1/C + c_2 W_2/C$$

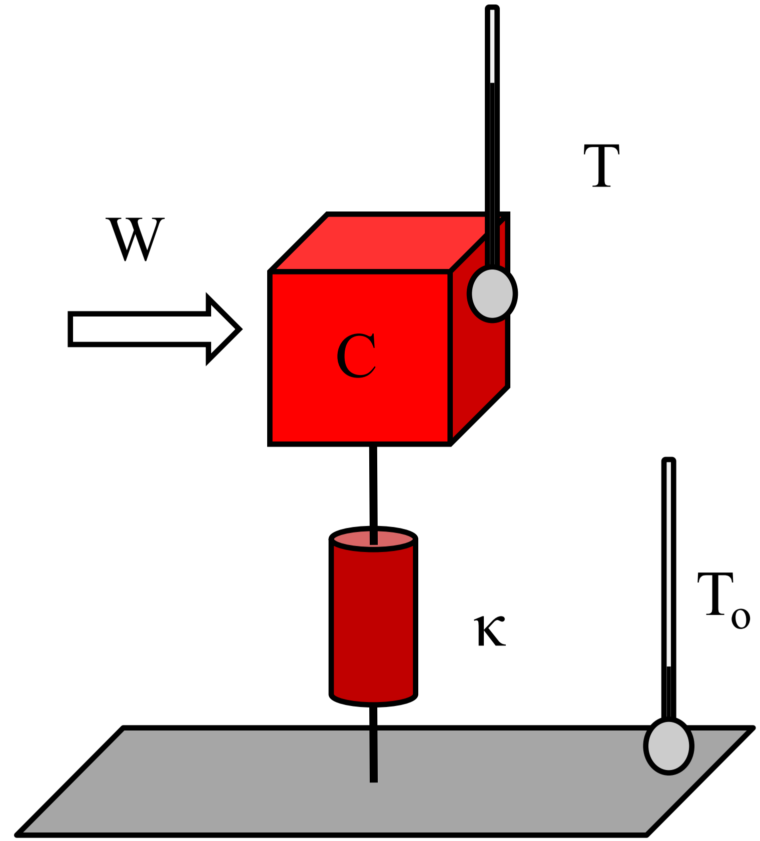
# Input-output relations

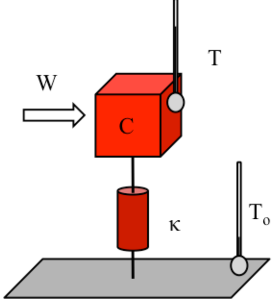
- Let's find the input-output relation from the equation

$$d\Delta T/dt + \Delta T/\tau = W/C$$

- From calculus, in order to find the general solution for $0 \leq t \leq \infty$ you need to:
  1. Find the general solution of associated homogeneous equation
  2. Add a special solution of complete equation

(Assume C and $\tau$ constant)







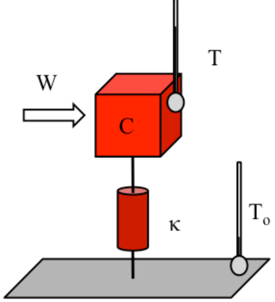

# Input-output relations

- Homogeneous equation $\quad d\Delta T/dt + \Delta T/\tau = 0$
- General solution $\quad \Delta T_h(t) = \Delta T_o e^{-\frac{t}{\tau}}\Theta(t)$
- $\Delta T_o$ to be found later from initial conditions
- A well known algorithm to find a special solution of a differential equation of the n$^{th}$ degree:
  - Take the homogeneous solution for all initial conditions equal to 0, except that of order n-1 that must be set to 1

$$\Delta T_h(0) = 1 \rightarrow \Delta T_h(t) = e^{-\frac{t}{\tau}}\Theta(t)$$

  - The solution is

$$\Delta T_{nh}(t) = \int_0^{\infty} \frac{W(t')}{C} e^{-\frac{t-t'}{\tau}}\Theta(t-t')dt' = \int_0^{t} \frac{W(t')}{C} e^{-\frac{t-t'}{\tau}}dt' = \int_0^{t} \frac{W(t-t'')}{C} e^{-\frac{t''}{\tau}}dt''$$



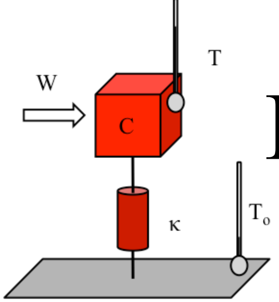

# Input-output relations

- Let's check that $\Delta T_{nh}(t)=\int_0^t \left(W(t')/C\right)e^{-\frac{t-t'}{\tau}}dt' = e^{-\frac{t}{\tau}}\int_0^t \left(W(t')/C\right)e^{\frac{t'}{\tau}}dt'$
- Is a solution to $d\Delta T/dt+\Delta T/\tau = W/C$
- Take the derivative

$$d\Delta T_{nh}/dt=-\frac{1}{\tau}e^{-\frac{t}{\tau}}\int_0^t \left(W(t')/C\right)e^{\frac{t'}{\tau}}dt'+e^{-\frac{t}{\tau}}\left(W(t)/C\right)e^{\frac{t}{\tau}}$$
$$=-\frac{1}{\tau}\Delta T_{nh}(t)+\left(W(t)/C\right)$$

- Then

$$d\Delta T_{nh}/dt+\Delta T_{nh}/\tau \quad =-\frac{1}{\tau}\Delta T_{nh}+\left(W/C\right)+\frac{1}{\tau}\Delta T_{nh} \quad = W/C$$




# Impulse response and free evolution

- In summary the input-output relation

$$\Delta T(t) = \Delta T_o e^{-\frac{t}{\tau}} + \int_0^{\infty} \frac{W(t')}{C} e^{-\frac{t-t'}{\tau}} \Theta(t-t') dt' =$$

$$= \Delta T_o e^{-\frac{t}{\tau}} + \int_0^{t} \frac{W(t-t'')}{C} e^{-\frac{t''}{\tau}} \Theta(t'') dt''$$

- In the language of the previous slides the impulse response of the system is

$$h(t,t') = e^{-\frac{t-t'}{\tau}} \Theta(t-t')$$

- While its free evolution is

$$\Delta T_o e^{-\frac{t}{\tau}}$$

- Notice that at t=0, only the free evolution term is different from zero