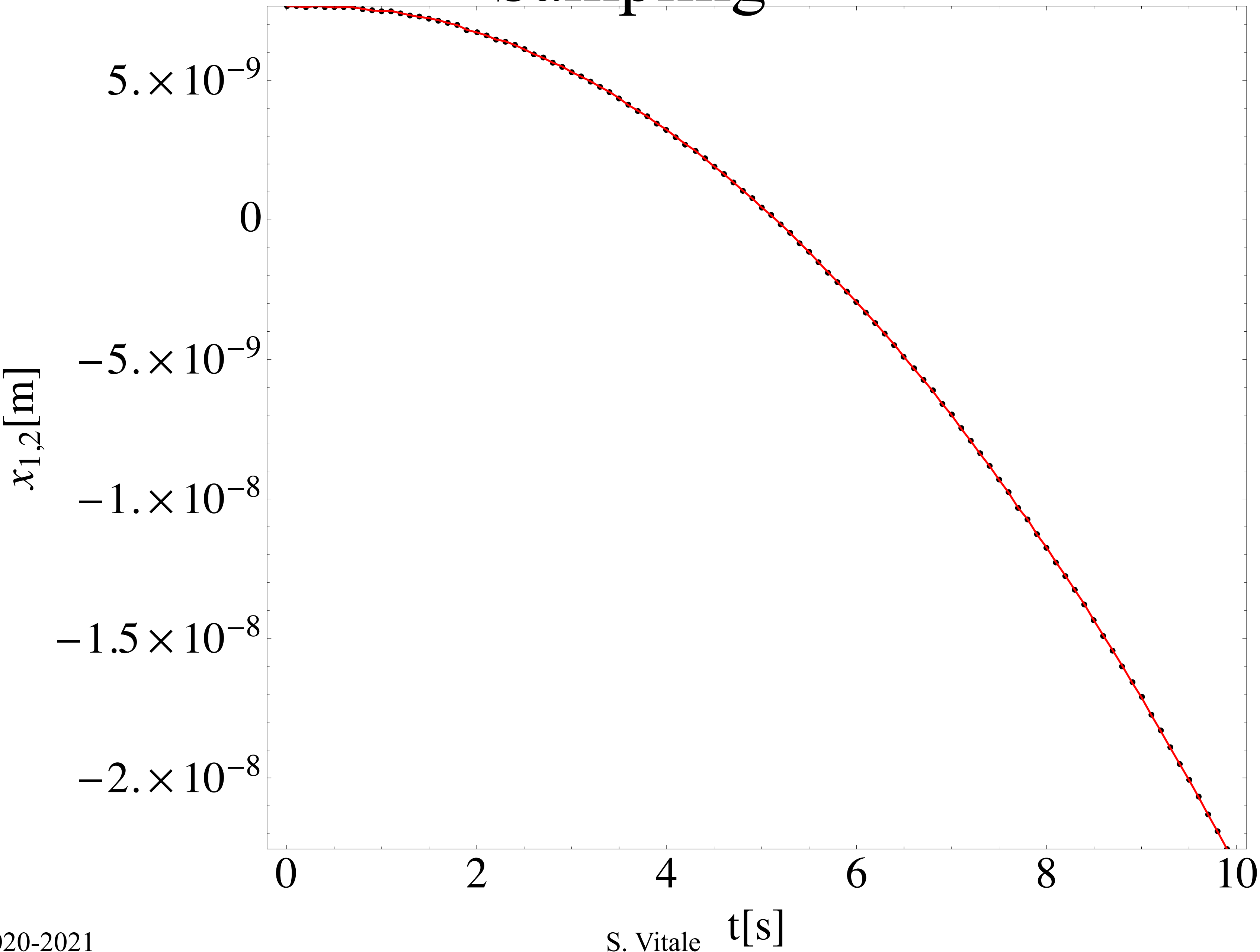
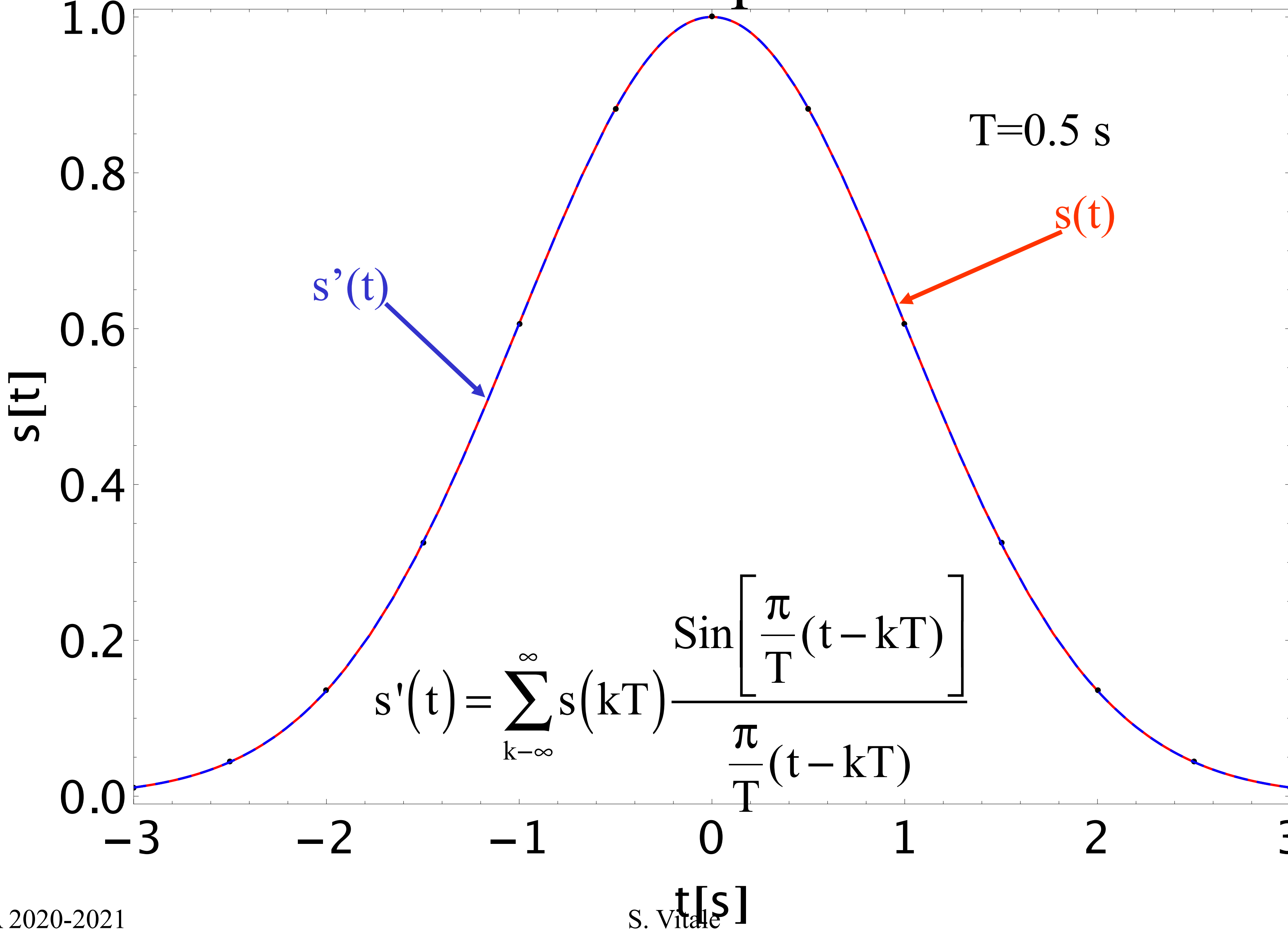
Experimental Methods
Lecture 4

September 28th, 2020

Sampling


$5. \times 10^{-9}$
0
$-5. \times 10^{-9}$
$-1. \times 10^{-8}$
-1.5×10^{-8}
$-2. \times 10^{-8}$
$x_{1,2}$[m]
0
2
4
6
8
10
t[s]

Shannon interpolation



$$s'(t) = \sum_{k-\infty}^{\infty} s(kT) \frac{\mathrm{Sin}\left[\frac{\pi}{T}(t-kT)\right]}{\frac{\pi}{T}(t-kT)}$$

Sampling theorem

Let's recap

$$s'[t] = \sum_{k=-\infty}^{\infty} s[k\,T]\,\frac{\mathrm{Sin}\left[\frac{\pi}{T}\,(t-k\,T)\right]}{\frac{\pi}{T}\,(t-k\,T)}$$

and

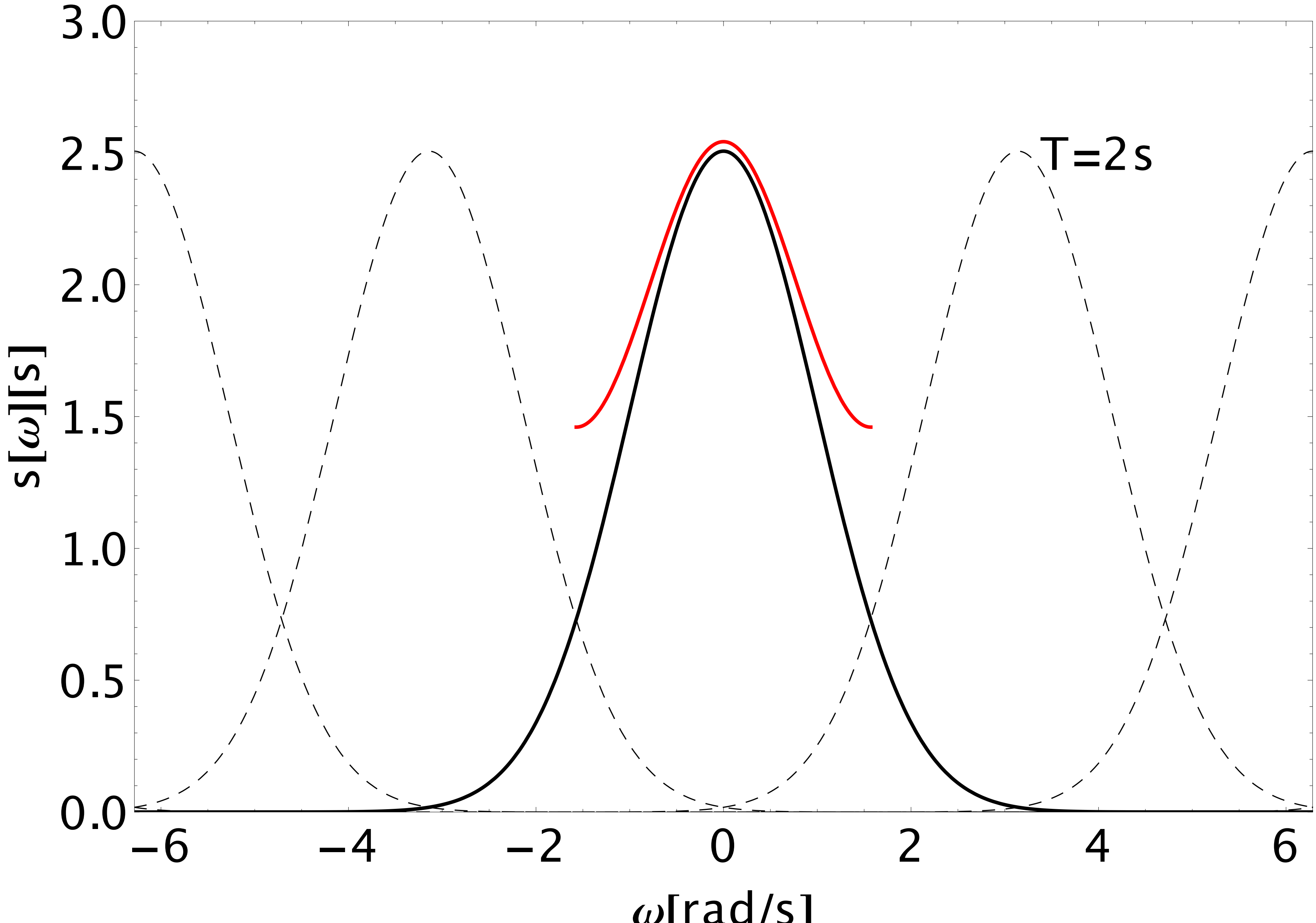
$$s'[\omega] = \left(\Theta\left[\omega+\frac{\pi}{T}\right]-\Theta\left[\omega-\frac{\pi}{T}\right]\right)\sum_{n=-\infty}^{\infty} s\left[\omega+n\,\frac{2\,\pi}{T}\right]$$

Notice that:

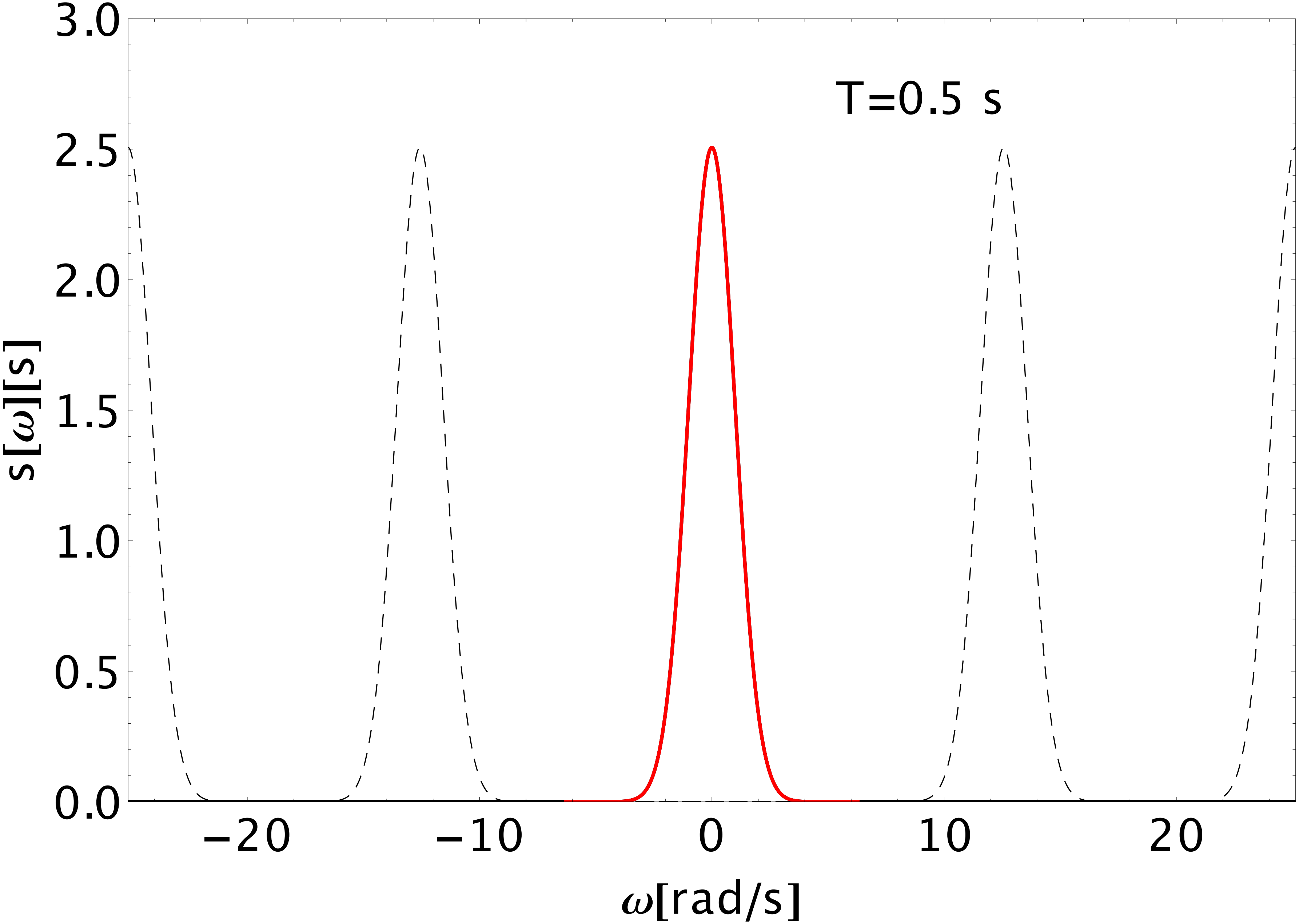
$$s'[\omega] \neq 0 \;\; \text{only for} \;\; -\frac{\pi}{T} \leq \omega \leq \frac{\pi}{T}$$

Within this range the "spectrum" of s' is the sum of infinite many "alias" of s[ω] shifted by integer multiples of the sampling angular frequency 2π/T

Fourier transform of s'



Fourier transform of s'



Sampling theorem

Notice that from

$$s'[\omega] = \left(\Theta\left[\omega + \frac{\pi}{T}\right] - \Theta\left[\omega - \frac{\pi}{T}\right]\right) \sum_{n=-\infty}^{\infty} s\left[\omega + n\frac{2\pi}{T}\right]$$

it follows that if also

$$\boxed{s[\omega] \neq 0 \ \text{ only for } |\omega| < \frac{\pi}{T}}$$

Then

$$\left(\Theta\left[\omega + \frac{\pi}{T}\right] - \Theta\left[\omega - \frac{\pi}{T}\right]\right) s\left[\omega + n\frac{2\pi}{T}\right] \neq 0$$

$$\text{only for } \left|\omega + n\frac{2\pi}{T}\right| < \frac{\pi}{T} \ \text{ and } |\omega| < \frac{\pi}{T}$$

That is only for n=0. In this case

$$\boxed{\mathrm{s}'[\omega] = \mathrm{s}[\omega] \ \rightarrow \ \mathrm{s}'[\mathrm{t}] = \mathrm{s}[\mathrm{t}]}$$

The sampling theorem

- If a signal is "band-limited" such that

$$s(\omega) = 0 \quad for \ |\omega| > f_n$$

with f_n the Nyquist frequency

- Then if the signal is sampled at a sampling frequency:

$$f_s > 2f_n$$

- The entire information is contained in its discrete samples from which the continuous signal can (in principle) be reconstructed through Shannon interpolation

Note on Shannon expansion

- Shannon interpolation of bandlimited functions is in fact an expansion on an orthogonal basis
- Indeed

$$\int_{-\infty}^{\infty} Sinc\left[\frac{\pi}{T}(t-k\,T)\right] Sinc\left[\frac{\pi}{T}(t-j\,T)\right] dt$$
$$= T\delta_{i,j}$$

- Then

$$\frac{1}{T}\int_{-\infty}^{\infty} s[t] Sinc\left[\frac{\pi}{T}(t-j\,T)\right] dt =$$
$$= \sum_{k=-\infty}^{\infty} \frac{s[kT]}{T} \int_{-\infty}^{\infty} Sinc\left[\frac{\pi}{T}(t-k\,T)\right] Sinc\left[\frac{\pi}{T}(t-j\,T)\right] dt=$$

$$= s[jT]$$

Energy conservation

- Notice that from

$$s[t] = \sum_{k=-\infty}^{\infty} s[kT] Sinc\left[\frac{\pi}{T}(t - k\,T)\right]$$

- It follows that

$$\int_{-\infty}^{\infty} |s[t]|^2 dt = \sum_{k,j=-\infty}^{\infty} s[kT]s[jT]\times$$

$$\times \int_{-\infty}^{\infty} Sinc\left[\frac{\pi}{T}(t - k\,T)\right] Sinc\left[\frac{\pi}{T}(t - j\,T)\right] dt$$

$$= T \sum_{k=-\infty}^{\infty} s[kT]^2$$

Band limited vs. time-limited

- A signal such that s[t]≠0 only for -ΔT/2 < t < ΔT/2 is called *time-limited* and its *duration* is ΔT
- Real physical signal are always time-limited
- Can signals be both time-limited and band-limited?
- No: here follows the proof
- Suppose $s[t]$ is band limited and its Nyquist frequency is ν_n. Then one can sample it at $\mathrm{T} = (2\nu_n)^{-1}$ and write

$$s[t] = \sum_{k=-\infty}^{\infty} s[k] Sinc\left(\frac{\pi}{T}(t - k\, T\right)$$

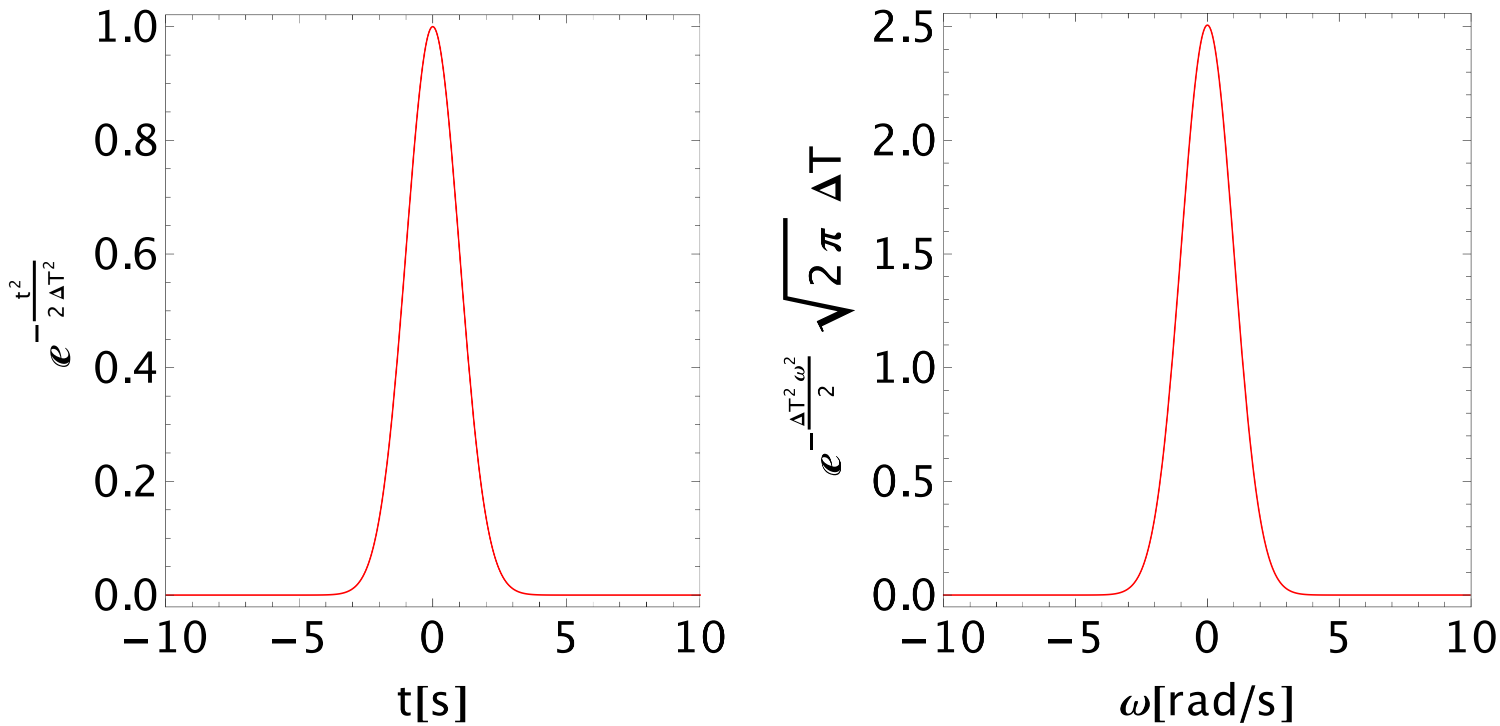
- But, if $s[t] = 0$ for $|t| \geq NT = \lceil \Delta T/(2T) \rceil \mathrm{T}$, then

$$s[t] = \sum_{k=-N}^{N} s[k] Sinc\left(\frac{\pi}{T}(t - k\, T\right)$$

- which has tails of infinite duration (well, is holomorphic and its zeros are isolated 🤓)

Band limited vs. time-limited

- However signals can be *approximately* both time-limited and band-limited



- Approximation may be arbitrarily accurate for many signals of interest

Truncated, band-limited signals

- Suppose we have a band-limited signal s[t] and we reconstruct it only from a limited number of samples

$$s'[t] = \sum_{k=-N}^{N} s[kT] Sinc\left[\frac{\pi}{T}(t - k\,T)\right]$$

- The square difference

$$|s[t] - s'[t]|^2 = \left| \sum_{|k|>N} s[kT] Sinc\left[\frac{\pi}{T}(t - k\,T)\right] \right|^2 \leq$$

$$\leq \sum_{|k|>N} \left|s[kT]\right|^2 \sum_{|k|>N} \left|Sinc\left[\frac{\pi}{T}(t - k\,T)\right]\right|^2 \leq$$

$$\leq \sum_{|k|>N} |s[kT]|^2 \sum_{k=-\infty}^{\infty} \left|Sinc\left[\frac{\pi}{T}(t - k\,T)\right]\right|^2$$

Truncated, band-limited signals

- But

```
In[•]:= Sum[Sinc[π/T (t - k T)]^2, {k, -∞, ∞}]

Out[•]= 1
```

$$\sum_{k=-\infty}^{\infty} \mathrm{Sinc}\left[\frac{\pi}{T}(t-k\,T)\right]^2$$

$$1$$

- Then

$|s[t]-s'[t]|^2 \leq$

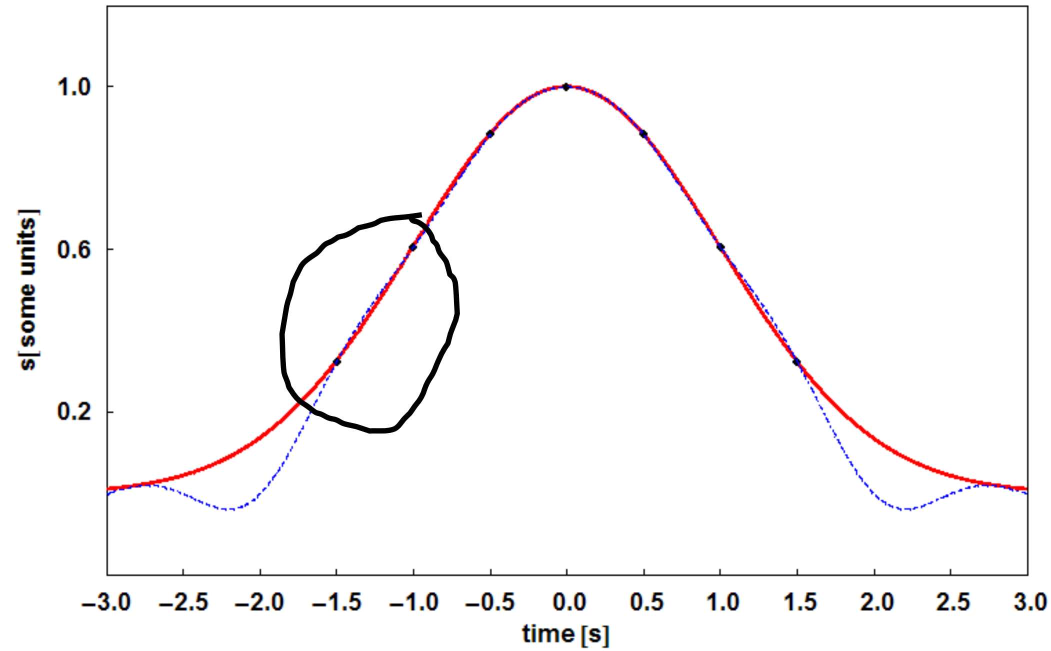
$$\leq \sum_{|k|>N} \left|s[kT]\right|^2 \sum_{k=-\infty}^{\infty} \left|Sinc\left[\frac{\pi}{T}(t-k\,T)\right]\right|^2 \leq$$

$$\leq \sum_{|k|>N} \left|s[kT]\right|^2 = \frac{1}{T}\int_{-\infty}^{\infty} |s[t]-s'[t]|^2 dt$$

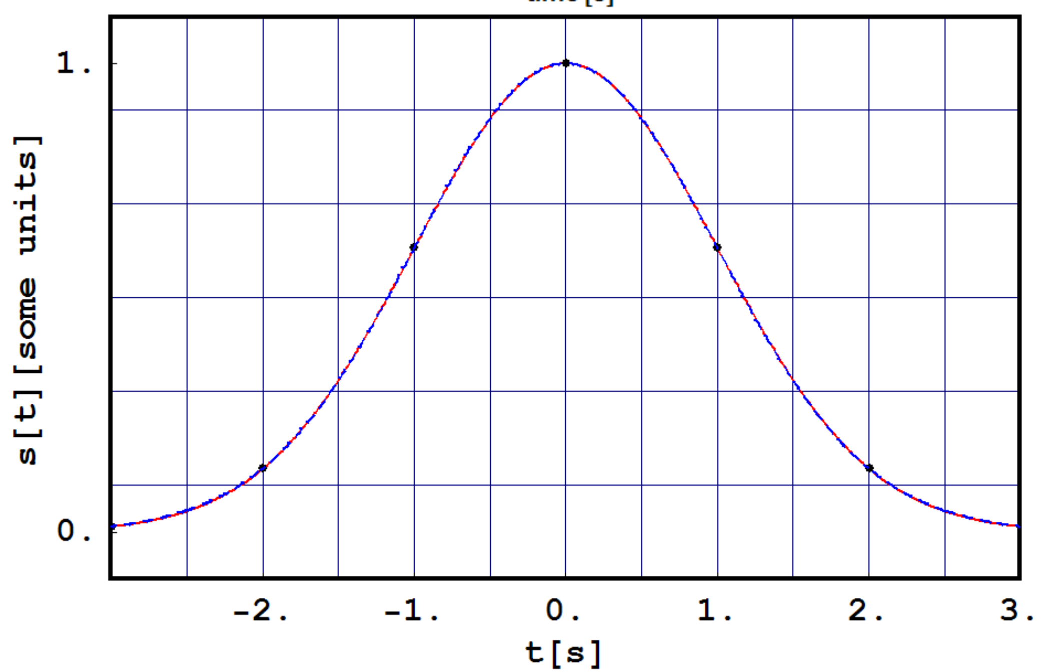
which converges to zero with the “total power” left out.

Effect of truncation

- Improperly truncated (7 samples)



- Properly truncated (30 samples)



Band-limited truncated samples

The entire information of a band-limited continuous truncated signal is contained in its

$$N_s = 2\,\Delta T\, f_n$$

samples, with accuracy increasing with ΔT

Analog to Digital conversion and measurement resolution

- Continuous signal are digitized and recorded with a finite number of bits N_b
  - Resolution: minimum representable variation of signal: $\propto$1 bit
  - Full scale: maximum representable number: $\propto 2^{Nb-1}-1$ (1 bit is used for sign representation)
  - Dynamic range: Full scale/Resolution=$2^{Nb-1}-1$

Dynamic range vs number of bits N_b

N_b	Dynamic Range	N_b	Dynamic Range	N_b	Dynamic Range
1.	0.	9.	2.6×10^{2}	17.	6.6×10^{4}
2.	1.	10.	5.1×10^{2}	18.	1.3×10^{5}
3.	3.	11.	$1.\times10^{3}$	19.	2.6×10^{5}
4.	7.	12.	$2.\times10^{3}$	20.	5.2×10^{5}
5.	1.5×10^{1}	13.	4.1×10^{3}	21.	$1.\times10^{6}$
6.	3.1×10^{1}	14.	8.2×10^{3}	22.	2.1×10^{6}
7.	6.3×10^{1}	15.	1.6×10^{4}	23.	4.2×10^{6}
8.	1.3×10^{2}	16.	3.3×10^{4}	24.	8.4×10^{6}

In summary

- The entire information of a signal of duration ΔT and Nyquist frequency f_n, sampled with N_b bits may be represented by $2Tf_nN_b$ bits

Exercise

Apply sampling principles to a "wave packet"

$$s(t) = e^{-\frac{t}{\Delta t}} Sin(2\,\pi \nu_o t)\Theta(t)$$

Take $\Delta t = 10\ s$ and $\nu_o = 10\ Hz$

Calculate continuous Fourier Transform

Sample and estimate alias for $\nu_s = 20, 21, 50, 100\ Hz$

Truncate at $t = [-1, +20]s$ and $[-1, +50]s$ and estimate error within the data range

Fourier Transforms of Discrete Data

- Two transforms:
  - Discrete-time Fourier Transform (infinite length data series)
  - Discrete Fourier Transform (finite length data series)
- Can they be used to estimate Fourier Transform of original continuous signals?

Fourier transform of time series

Sampled data form a series or a "discrete time sequence" s_k with $-\infty \leq k \leq \infty$. We can define the following function of the "angular frequency" ϕ

$$s[\phi] = \sum_{k=-\infty}^{\infty} s_k \, e^{-i \phi k}$$

which is called the Sequence Fourier Transform, or the Discrete-Time Fourier Transform of the discrete sequence.

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which is called the Sequence Fourier Transform, or the Discrete-Time Fourier Transform of the discrete sequence. It is indeed a transform, as there is an inversion formula

$$s_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} s[\phi] \, e^{i\,m\,\phi} \, d\phi$$

Indeed

$$s_m = \sum_{k=-\infty}^{\infty} s_k \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\,(m-k)\,\phi} \, d\phi \right) =$$

$$= \sum_{k=-\infty}^{\infty} s_k \frac{\mathrm{Sin}[(m-k)\,\pi]}{(m-k)\,\pi} = s_m$$

An obvious interpretation

The formula

$$s\left[\phi\right]=\sum_{k=-\infty}^{\infty} s_k e^{-ik\phi}$$

can be thought as the Fourier series expansion of s[ϕ], which is then a periodic function of ϕ with period 2 π.

Then the expansion coefficients s_k are given by the Fourier formula

$$s_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} s\left[\phi\right]e^{i\phi k}\,d\phi$$

Sequence transform vs continuous transform

- Assume $s_k = s(t = kT)$, that is the sequence results from the sampling of a continuous signal then both the following equations hold

$$s_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} s_d(\phi)e^{i\phi k}d\phi$$

$$s_k = s(t = kT) = \frac{1}{2\pi}\int_{-\infty}^{\infty} s_c(\omega)e^{i\omega kT}d\omega$$

- where $s_d(\phi)$ and $s_c(\omega)$ stand for the discrete time and continuous Fourier transforms respectively.
- One can split the infinite integral over an infinite sequence of intervals of width $\Delta\omega = 2\pi/T$ each, and rewrite the second equation as

$$s_k = s(t = kT) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{n\frac{2\pi}{T}-\frac{\pi}{T}}^{n\frac{2\pi}{T}+\frac{\pi}{T}} s_c(\omega)e^{i\omega kT}d\omega$$

- But

$$\int_{n\frac{2\pi}{T}-\frac{\pi}{T}}^{n\frac{2\pi}{T}+\frac{\pi}{T}} s_c(\omega)e^{i\omega kT}d\omega = \int_{-\frac{\pi}{T}}^{+\frac{\pi}{T}} s_c\left(\omega + n\frac{2\pi}{T}\right)e^{i\left(\omega+n\frac{2\pi}{T}\right)kT}d\omega = \int_{-\frac{\pi}{T}}^{+\frac{\pi}{T}} s_c\left(\omega + n\frac{2\pi}{T}\right)e^{i\omega kT}d\omega$$

- Then

$$s_k = \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}}\left(\sum_{n=-\infty}^{\infty} s_c\left(\omega + n\frac{2\pi}{T}\right)\right)e^{i\omega kT}d\omega \underset{\omega T\to\phi}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{T}\sum_{n=-\infty}^{\infty} s_c\left(\frac{\phi}{T} + n\frac{2\pi}{T}\right)\right)e^{ik\phi}d\phi$$

Sequence transform vs continuous transform

- Thus on one hand

$$s_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} s_d(\phi)e^{i\phi k}d\phi$$

- And on the other

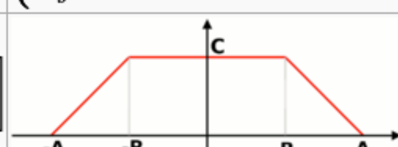
$$s_k = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{T}\sum_{n=-\infty}^{\infty} s_c\left(\frac{\phi}{T}+n\frac{2\pi}{T}\right)\right)e^{ik\phi}d\phi$$

- We conclude that

$$s_d(\phi) = \frac{1}{T}\sum_{n=-\infty}^{\infty} s_c\left(\frac{\phi}{T}+n\frac{2\pi}{T}\right) = \frac{1}{T}s_c'\left(\frac{\phi}{T}\right)$$

- where $s_c'(\omega)$ is the continuous Fourier transform of the Shannon interpolation of $s(t)$
- If data have been sampled fulfilling the sampling theorem, then

$$s_d(\phi) = \frac{1}{T}s_c\left(\frac{\phi}{T}\right) = \frac{1}{T}s_c'\left(\frac{\phi}{T}\right)$$

Time domain $x[n]$	**Frequency domain** $X(\omega)$	**Remarks**
$\delta[n]$	1	
$\delta[n-M]$	$e^{-i\omega M}$	integer *M*
$\sum_{m=-\infty}^{\infty} \delta[n-Mm]$	$\sum_{m=-\infty}^{\infty} e^{-i\omega Mm} = \frac{1}{M}\sum_{k=-\infty}^{\infty} \delta\left(\frac{\omega}{2\pi}-\frac{k}{M}\right)$	integer *M*
$u[n]$	$\frac{1}{1-e^{-i\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega-2\pi k)$	The $1/(1-e^{-i\omega})$ term must be interpreted as a distribution in the sense of a Cauchy principal value around its poles at $\omega = 2\pi k$.
$a^n u[n]$	$\frac{1}{1-ae^{-i\omega}}$	$\lvert a \rvert < 1$
e^{-ian}	$2\pi\delta(\omega+a)$	real number *a*
$\cos(an)$	$\pi\,[\delta(\omega-a)+\delta(\omega+a)]$	real number *a*
$\sin(an)$	$\frac{\pi}{i}\,[\delta(\omega-a)-\delta(\omega+a)]$	real number *a*
$\operatorname{rect}\left[\frac{(n-M/2)}{M}\right]$	$\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)}\, e^{-i\omega M/2}$	integer *M*
$\operatorname{sinc}[(a+n)]$	$e^{ia\omega}$	real number *a*
$W\cdot\operatorname{sinc}^2(Wn)$	$\operatorname{tri}\left(\frac{\omega}{2\pi W}\right)$	real number *W* $0 < W \le 0.5$
$W\cdot\operatorname{sinc}(Wn)$	$\operatorname{rect}\left(\frac{\omega}{2\pi W}\right)$	real numbers *W* $0 < W \le 1$
$\begin{cases} 0 & n=0 \\ \frac{(-1)^n}{n} & \text{elsewhere} \end{cases}$	$j\omega$	it works as a differentiator filter
$\frac{W}{(n+a)}\{\cos[\pi W(n+a)] - \operatorname{sinc}[W(n+a)]\}$	$j\omega\cdot\operatorname{rect}\left(\frac{\omega}{\pi W}\right) e^{ja\omega}$	real numbers *W*, *a* $0 < W \le 1$
$\frac{1}{\pi n^2}[(-1)^n-1]$	$\lvert\omega\rvert$	
$\begin{cases} 0; & n \text{ even} \\ \frac{2}{\pi n}; & n \text{ odd} \end{cases}$	$\begin{cases} j & \omega<0 \\ 0 & \omega=0 \\ -j & \omega>0 \end{cases}$	Hilbert transform
$\frac{C(A+B)}{2\pi}\cdot\operatorname{sinc}\left[\frac{A-B}{2\pi}n\right]\cdot\operatorname{sinc}\left[\frac{A+B}{2\pi}n\right]$		real numbers *A*, *B* complex *C*

- Exercise:
  - demonstrate this one
  - Compare with the Fourier Transform of $e^{-t/\tau}\Theta(t)$

Notice ω Wikipedia →φ Exp. Methods

One example

$$s_k = e^{-\frac{kT}{\tau}}\,\Theta[k]$$

$$s[\phi] = \sum_{k=0}^{\infty} e^{-\frac{kT}{\tau}}\, e^{-i\phi k}$$

Remember that the property of the geometric series

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad \text{if} \quad |r| < 1$$

Then, as indeed

$$\left| e^{-\frac{T}{\tau}}\, e^{-i\phi} \right| = e^{-\frac{T}{\tau}} < 1 \text{ for } \tau>0$$

we get

$$s[\phi] = \frac{1}{1 - e^{-\left(\frac{T}{\tau} + i\phi\right)}}$$

For $\frac{T}{\tau}$ and ϕ <<1

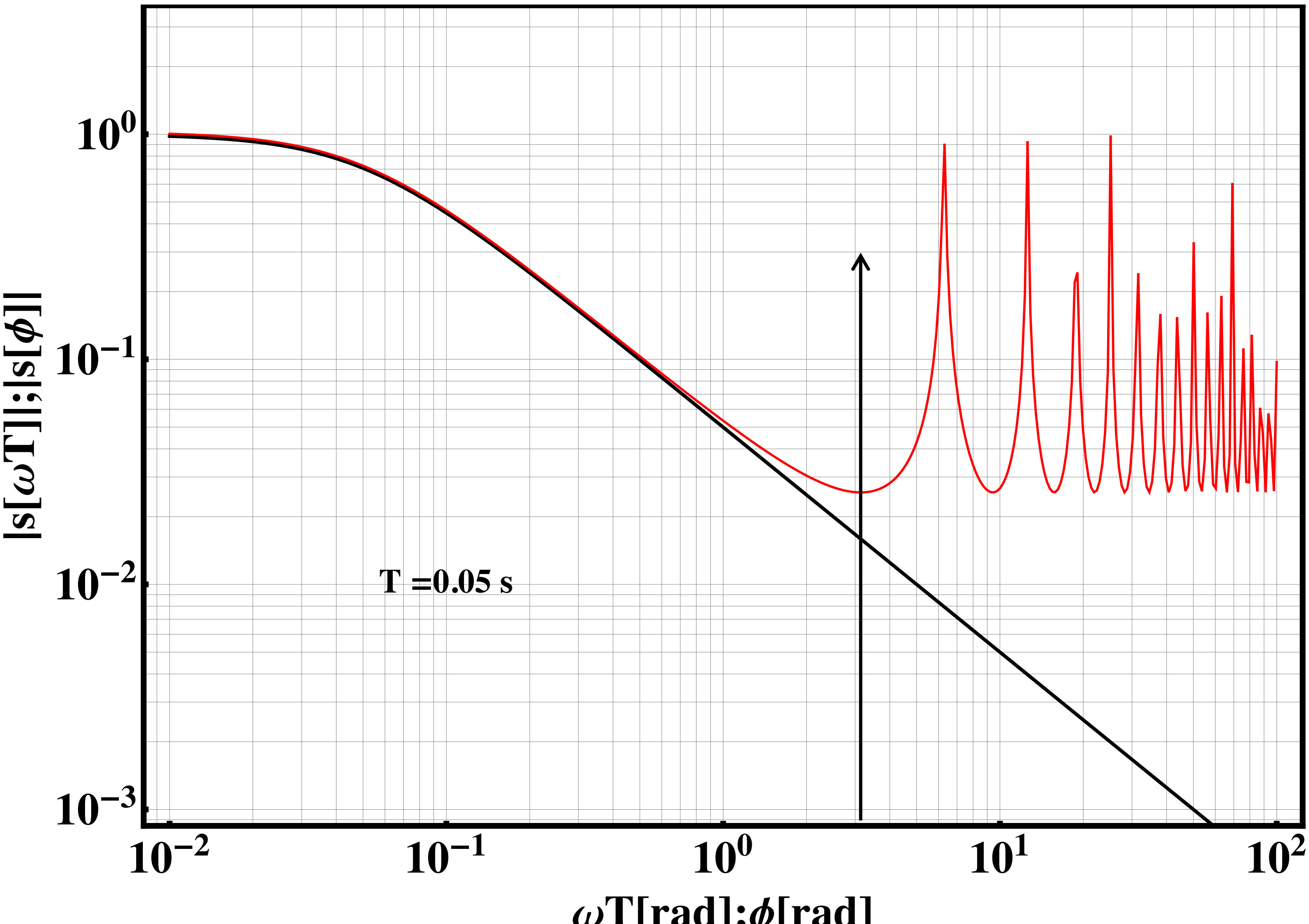
$$s[\phi] = \frac{1}{1 - e^{-\left(\frac{T}{\tau} + i\phi\right)}} \simeq \frac{1}{\frac{T}{\tau} + i\phi} =$$

$$= \frac{1}{T}\,\frac{\tau}{1 + i\phi\,\tau / T} = \frac{1}{T}\,\frac{\tau}{1 + i\omega\,\tau}$$

Where we have used $\phi=\omega T$. Thus the discrete time transform coincides with the continuous time transform, except for the multiplication for 1/T only if τ>>T and at frequencies ω<<1/T

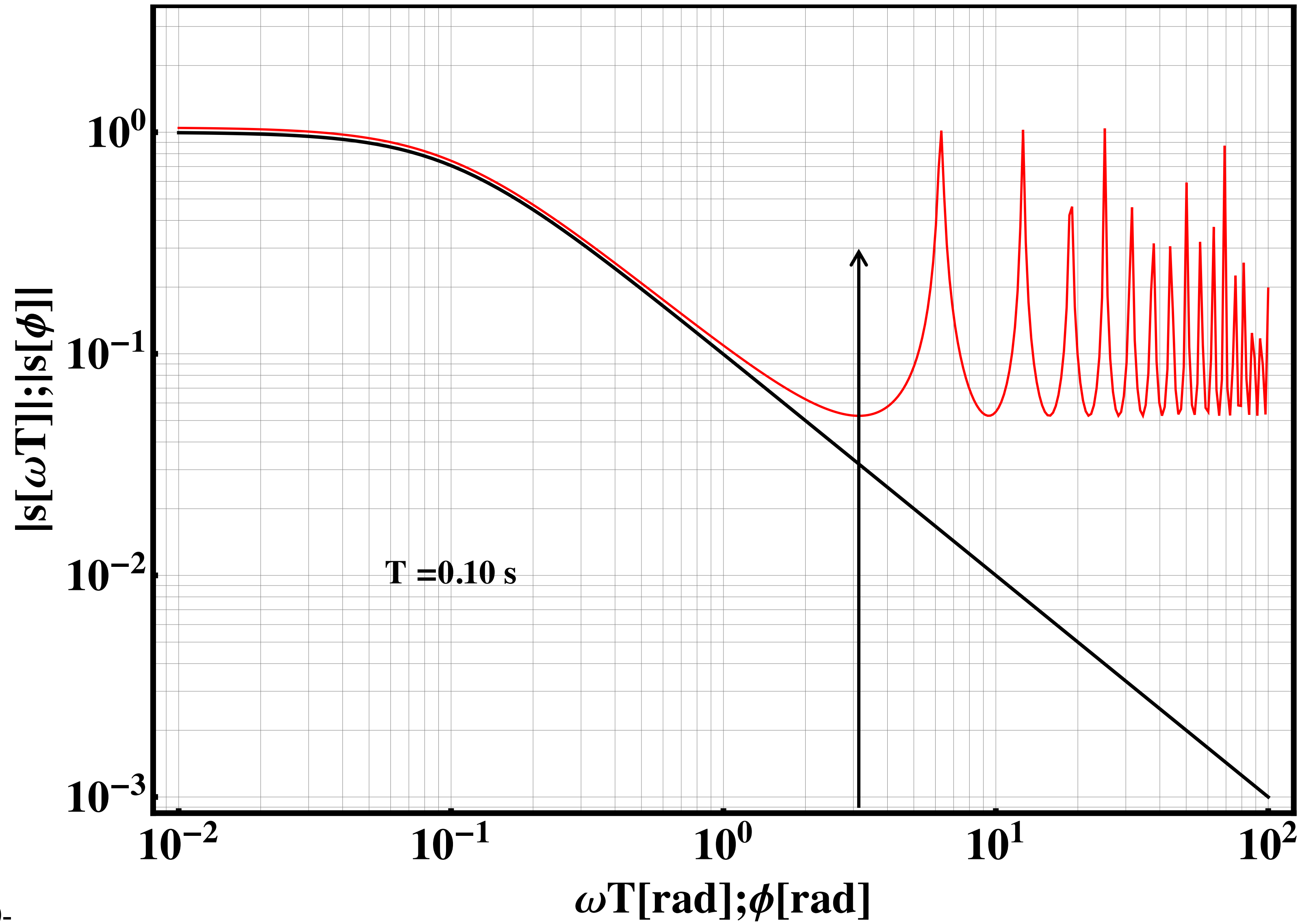
$e^{-t/\tau} \rightarrow 1/(1+i\omega\tau)$ $\quad e^{-nT/\tau} \rightarrow 1/(1-e^{-T/\tau - i\phi})$

$\tau = 1$ s



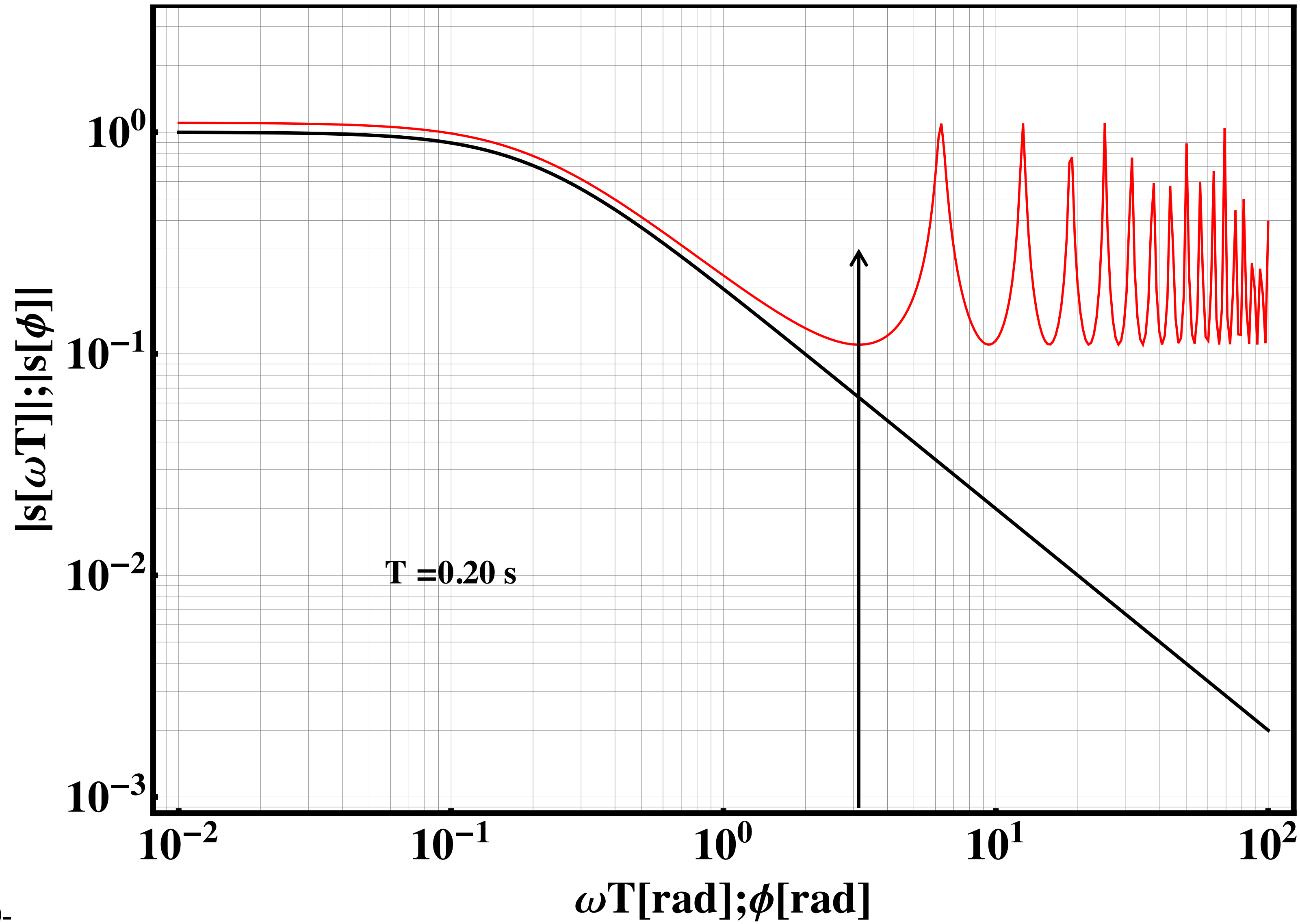
$e^{-t/\tau} \rightarrow 1/(1+i\omega\tau)$ $\quad$ $e^{-nT/\tau} \rightarrow 1/(1-e^{-T/\tau-i\phi})$

τ=1 s



$$e^{-t/\tau} \rightarrow 1/(1+i\omega\tau) \qquad e^{-nT/\tau} \rightarrow 1/(1-e^{-T/\tau-i\phi})$$

τ=1 s



$e^{-t/\tau} \rightarrow 1/(1+i\omega\tau)$ $e^{-nT/\tau} \rightarrow 1/(1-e^{-T/\tau-i\phi})$

$\tau = 1$ s