UNIVERSITY OF CAPE TOWN
Department of Statistical Sciences Honours

Tim Gebbie : Portfolio Theory (STA)
Assignment 2 : Backtesting a Black-Litterman Portfolio

Submission deadline via vula:

PART I : Derive the Black-Litterman approach

It is strongly recommended that that MikTeX and the assocaited TeXworks editor is used to create the required assignment PDF for submission. This can be integrated into your R markdown file for PART II. Only a single PDF file will be accepted for the assignment submission. If multiple files are submitted only the first file (in alphabetic named ordering) will be graded.

Question 1 : The Mutual Fund Separation Theorem

Start with the mutual-fund Lagrangian [3] for some vector of mean returns $\boldsymbol{\mu}$ and its associated covariance matrix Σ in terms of the portfolio weight vector $\boldsymbol{\omega}$ (often termed portfolio controls)

$$L = \boldsymbol{\omega}^T \boldsymbol{\mu} - \frac{\gamma}{2} \boldsymbol{\omega}^T \Sigma \boldsymbol{\omega} - \lambda_\omega (\boldsymbol{\omega}^T \mathbf{1} - 1). \tag{1}$$

1.1 Use Kuhn-Tucker methods to find the lagrange multiplier and eliminate it from the contraint equations **to find** the mutual fund separation theorem:

$$\boldsymbol{\omega}^* = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} + \frac{1}{\gamma} \Sigma^{-1} \left(\boldsymbol{\mu} - \mathbf{1} \frac{\mathbf{1}^T \Sigma^{-1} \boldsymbol{\mu}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} \right). \tag{2}$$

The first term on the right is the lowest risk portfolio and the second term is the zero-cost portfolio that encapsulates the relative views of the assets.

1.2 Discuss the constraint equation arising from the lagrangian.

[10 marks]

Question 2 : The Mutual Fund Separation Theorem in terms of an equilibrium view

We can introduce long-term consensus equilibrium views $\boldsymbol{\Pi}$ by making the subsitution $\boldsymbol{\mu} = \boldsymbol{\mu} + \boldsymbol{\Pi} - \boldsymbol{\Pi}$ in equation (2). Here the optimal mutual fund controls can be then linearly separated into the component funds: (i) The policy fund, $\boldsymbol{\omega}_P = \boldsymbol{\omega}_B + \boldsymbol{\omega}_S$, which is a combination of a benchmark fund, $\boldsymbol{\omega}_B$, and a strategic fund, $\boldsymbol{\omega}_S$, and (ii) the tactical fund $\boldsymbol{\omega}_T$. The benchmark and policy funds are fully-invested, as is the complete mutual fund, the tactical fund is a zero-cost active portfolio whose leverage is determined by the risk-aversion parameter.

$$\boldsymbol{\omega}^* = \boldsymbol{\omega}_P + \boldsymbol{\omega}_T = (\boldsymbol{\omega}_B + \boldsymbol{\omega}_S) + \boldsymbol{\omega}_T. \tag{3}$$

2.1 Show that the respective funds are given by:

$$\boldsymbol{\omega}_B = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}}, \tag{4}$$

$$\boldsymbol{\omega}_S = \frac{\Sigma^{-1}}{\gamma} \left(\frac{\boldsymbol{\Pi}\mathbf{1}^T - \mathbf{1}\boldsymbol{\Pi}^T}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} \right) \Sigma^{-1}\mathbf{1}, \tag{5}$$

$$\boldsymbol{\omega}_T = \frac{\Sigma^{-1}}{\gamma} \left(\frac{(\boldsymbol{\mu} - \boldsymbol{\Pi})\mathbf{1}^T - \mathbf{1}(\boldsymbol{\mu} - \boldsymbol{\Pi})^T}{\mathbf{1}^T \Sigma^{-1} \mathbf{1}} \right) \Sigma^{-1}\mathbf{1}. \tag{6}$$

The strategic portfolio is determined by the relative value of assets in the long-term quasi-equilibrium where its views are often reverse engineered from policy constraints, for example, funding constraints or structural positions that are required for the long-term. This policy portfolio plays the role of the bogey that is being tracked by the tactical portfolio.

[10 marks]

Question 3 : The Black-Litterman mean and covariance

Consider an equilibrium distribution of returns $\boldsymbol{R}$ in terms of a equilibrium set of views $\boldsymbol{\Pi}$ and an equilbrium covariance matrix Σ and some constant τ [3]

$$\mathbb{E}[\boldsymbol{R}] \sim \mathrm{N}(\boldsymbol{\Pi}, \tau\Sigma) \tag{7}$$

We express the views as probability distribution of the form

$$\boldsymbol{y} = \boldsymbol{P}\mathbb{E}[\boldsymbol{R}] = \boldsymbol{V} + \boldsymbol{e} \tag{8}$$

where $\boldsymbol{P}$ is a $k \times n$ matrix for k views of the linear combinations of n assets, $\boldsymbol{V}$ is a $k \times 1$ vector and the error in the views is $\boldsymbol{e}$ a $k \times 1$ vector. The first row of $\boldsymbol{P}$ give the linear combination of expected returns that gives the first element of the views $\boldsymbol{y}$. We can assume that $\boldsymbol{y} \sim N(\boldsymbol{V}, \Omega)$.

3.1 **Certain views :** When there is certainty the error terms in the views are zero, $\boldsymbol{e} = 0$. We can then solve the following optimization

$$\mathbb{E}[\boldsymbol{R}]^* = \min_{\mathbb{E}[\boldsymbol{R}]} \left\{ (\mathbb{E}[\boldsymbol{R}] - \Pi)^T (\tau\Sigma)^{-1} (\mathbb{E}[\boldsymbol{R}] - \Pi) \right\} \text{ s.t. } \boldsymbol{P}\mathbb{E}[\boldsymbol{R}] = \boldsymbol{V}. \tag{9}$$

Solve the Lagrangian optimization to show:

$$\mathbb{E}[\boldsymbol{R}]^* = \Pi + \tau\Sigma\boldsymbol{P}^T (\boldsymbol{P}\tau\Sigma\boldsymbol{P}^T)^{-1} (\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}). \tag{10}$$

Here the associated covariance is $\Sigma^* = \Sigma$.

3.2 **Uncertain views:** Consider *Bayes Theorem* for this portfolio choice problem [2]

$$\mathbb{E}[\boldsymbol{R}]^* = \mathbb{P}(\mathbb{E}[\boldsymbol{R}]|\Pi) = \frac{\mathbb{P}(\Pi|\mathbb{E}[\boldsymbol{R}])\mathbb{P}(\mathbb{E}[\boldsymbol{R}])}{\mathbb{P}(\Pi)} \tag{11}$$

The *prior distribution* $\mathbb{P}(\mathbb{E}[\boldsymbol{R}])$ is as in equation . The marginal distribution of the equilibrium views is $\mathbb{P}(\Pi)$. The conditional distribution of equilibrium returns is $\mathbb{P}(\Pi|\mathbb{E}[\boldsymbol{R}])$. We now compute the *posterior distribution* for the expected returns $\mathbb{E}[\boldsymbol{R}]$ given the data $\boldsymbol{\Pi}$.

3.2.1 From Bayes Theorem and assuming that:

i. $\mathbb{P}(\mathbb{E}[\boldsymbol{R}]) = \mathbb{P}(\mathbb{E}[\boldsymbol{y}(\mathbb{E}[\boldsymbol{R}])]) \sim N(\boldsymbol{V}, \Omega)$, and

ii. $\mathbb{P}(\Pi|\mathbb{E}[\boldsymbol{R}]) \sim N(\mathbb{E}[\boldsymbol{R}], \tau\Sigma)$.

Show that:

$$\mathbb{P}(\mathbb{E}[\boldsymbol{R}]|\Pi) = \frac{k}{\mathbb{P}(\Pi)} e^{\left[-\frac{1}{2}(\Pi - \mathbb{E}[\boldsymbol{R}])^T (\tau\Sigma)^{-1} (\Pi - \mathbb{E}[\boldsymbol{R}])\right]} e^{\left[-\frac{1}{2}(\boldsymbol{P}\mathbb{E}[\boldsymbol{R}] - \boldsymbol{V})^T \Omega^{-1} (\boldsymbol{P}\mathbb{E}[\boldsymbol{R}] - \boldsymbol{V})\right]}. \tag{12}$$

Hint #1: The density function for a multivariate Gaussian is : $N(\boldsymbol{x}; \boldsymbol{\mu}, \Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T \Sigma^{-1} (\boldsymbol{x}-\boldsymbol{\mu})}$

3.2.2 Derive the mean of the conditional distribution using matrix algebra to re-arrange the quadratic term in the exponent of the distribution:

$$\mathbb{E}[\boldsymbol{R}]^* = \left[(\tau\Sigma)^{-1} + \boldsymbol{P}^T \Omega^{-1} \boldsymbol{P}\right]^{-1} \left[(\tau\Sigma)^{-1} \boldsymbol{\Pi} + \boldsymbol{P}^T \Omega^{-1} \boldsymbol{V}\right]. \tag{13}$$

This can be equivalently refactorized to take on the following form:

$$\mathbb{E}[\boldsymbol{R}]^* = \boldsymbol{\Pi} + (\tau\Sigma)\boldsymbol{P}^T \left[\boldsymbol{P}(\tau\Sigma)\boldsymbol{P}^T + \Omega\right]^{-1} (\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}). \tag{14}$$

Here we can set $\Omega = 0$ to recover equation (10).

3.2.3 Use the above calculation to show the variance of the posterior distribution is:

$$\Sigma^* = \left[(\tau\Sigma)^{-1} + \boldsymbol{P}^T\Omega^{-1}\boldsymbol{P}\right]^{-1} \tag{15}$$

Hint #2: Use variables:

i. $\boldsymbol{C} = (\tau\Sigma)^{-1}\boldsymbol{\Pi} + \boldsymbol{P}^T\Omega^{-1}\boldsymbol{V}$,

ii. $\boldsymbol{H} = (\tau\Sigma)^{-1} + \boldsymbol{P}^T\Omega^{-1}\boldsymbol{P}$ *(for symmetric* $\boldsymbol{H}$*), and*

iii. $\boldsymbol{A} = \boldsymbol{V}^T\Omega^{-1}\boldsymbol{V} + \boldsymbol{\Pi}^T(\tau\Sigma)^{-1}\boldsymbol{\Pi}$.

Then use that terms like $\boldsymbol{A} - \boldsymbol{C}^T\boldsymbol{H}\boldsymbol{C}$ *vanish into the constant of integration [2].*

[20 marks]

Question 4 : The Black-Litterman portfolio weights

Derive the Black-Litterman portfolio weights using the mutual fund separation theorem from equation (2) and substitute the Black-Litterman conditional views but retain the current equilibrium covariances to find:

$$\omega^*_{BL} = \omega_B + \frac{1}{\gamma}\Sigma^{-1}\left(\mathbb{E}[\boldsymbol{R}]^* - \mathbf{1}\frac{\mathbf{1}^T\Sigma^{-1}\mathbb{E}[\boldsymbol{R}]^*}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\right) \tag{16}$$

[5 marks]

PART II : Implement and Historically Simulate the Black-Litterman Model in a R script file

Using the same data as for Assignment I back-test your implementation of the BL model. For the test data compute the Black-Litterman portfolio weights from equation (). Include a scriptfile named: `T_A2_<Name>.R  that uses a function cal` file. The markdown file should include the code listing for the function in its appendix. The markdown file should include date and time information from when the file was excuted to prove that it works. Only a single PDF file will be accepted. The file should include the necessary graphics and equations to demonstrate the problem resolution. The objective is to present a historical simulation (a Backtest) of the approach.

1. **function header and help** Ensure that the function is correctly formated, commented and takes input arguments correctly the function header should follow pattern 1:

```r
a <- c(1,2,3)
b <- data.frame(this=a,that=c(3,4,5))
```

```r
blacklitterman <- function(varargin),
# BLACKLITTERMAN Black-Litterman mean and covariance are computed and the Black-Litterman portfolio weights are returned
```

2. **Variable input arguments**

   Variable input arguments should be parsed following the approach of pattern 2:

3. **Matrix algebra and code vectorization**

   The code should be vectorised in order to directly implement the matrix algebra. This may require using `t()` for tranpose, `inv()` for matrix inverse. Care should be taken using operators using different methods for matrix investion. Differentiate between the speudo inverse, and the inverse, if these are chosen in order, to better manage the matrix inversions. For example, the mutual fund separation theorem can be implemented as in code pattern 3 (here we use `solve()` but we could also use `inv()`, `cho2inv(chol())` for a Cholsky factorisation, or even `ginv()` for the Moore-Penrose inverse:

```r
# mean
f <- <- colMeans(x, na.rm=TRUE)
# covariance
H <- var(x, na.rm=TRUE)
#  diagonals of one
I <-  seq(1,1,length.out = length(H))
# invert the covariance matrix
invHI <- solve(H);
#  benchmark weights
wB <-  (t(I) %*%  invHI) \ (invHI);
# active weights
wA = (1/gamma)  %*%  (t(I)  %*%  invHI) \ ( ( H  \  (  f  %*%  t(I) - I %*% t(f))) / H ) %*% I;
```

4. **Test cases**

   Use the following inputs as test data for the code development:

```
Pi =
    0.2500
    0.1000
    0.0500
Sigma  =
    0.0900    0.0240   -0.0060
    0.0240    0.0100    0.0003
   -0.0060    0.0003    0.0025
V =
    0.2000
   -0.0500
P =
     1    -1     0
     0     1    -1
Omega =
         0.30    0
         0    0.55
```

5. **Backtest the BL model**

   Using the rolling-window method implemented in Assignment I backtest the Black-Litterman Method ensuring that the portfolio controls are fully invested and that no-short selling is permitted. Ensure that the Equity Curves are plotted, that you have at least one bar-graph representing an indicative selection of portfolio weights and comment on the amount of portfolio turn-over, the values of Jensen Alpha, the Sharpe Ratio and the portfolio Beta (using CAPM relative to the All-Share-Index) for you simulated portfolio. Make an appropriate choice for the long-term strategic portfolio equilibrium and motivate your choice. Make an appropriate choice of the risk-aversion and motivate your choice - tracking errors between 10% and 3% can be considered appropriate relative to the Strategic Benchmark. When choosing the risk-aversion ensure that no assets are sold short.

Additional credits will be assigned to those students who attempt to generate a variety of paths for different parameter choices; full marks are possilbe for those students who attempt to evaluate the probability of backtest overfitting and perhaps attempt to compute the deflated Sharpe ratio's.

Significant care should be taken in the presentation of how the weights and returns are used at each iteration in the simulation. Significant care should be taken that returns are correctly compounded and that the returns at the end of the time intervals are correctly related to the initial portoflio weights for each asset. Care should be taken to ensure that the loop structure used to generate the sequences of portfolio optimisations are compared like-for-like with the computation of the benchmark portfolio.

[20 marks]

Bibliography

[1] Black, F., Litterman, R., (1992), Global portfolio optimization, Financial Analyst Journal, Sep-Oct, pages 28-43

[2] Satchell, S., Scowcroft, A., (2000), A demystification of the Black-Litterman model: Managing quantitative and traditional portfolio construction, Journal of Asset Management, 1, 2, 138-150

[3] Lee, W. Theory and Methodology of Tactical Asset Allocation, Fabrozzi Associates, 2000