University of Cape Town

STA4028Z

Assignment 2: Portfolio Theory

Backtesting a Black-Litterman Portfolio

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PART I : Report

Deriving the Black-Litterman approach

Q1: The Mutual Fund Separation Theorem

Proof below based on (Merton, 1972) and (Elton and Gruber, 1995).

1.1.1 Use Kuhn-Tucker methods to find the Lagrange multiplier and eliminate it from the constraint equations to find the mutual fund separation theorem:

Assumptions

- $\boldsymbol{\omega} \in \mathbb{R}^n$: portfolio weights vector.
- $\boldsymbol{\mu} \in \mathbb{R}^n$: vector of expected returns.
- $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$: covariance matrix .
- $\mathbf{1} \in \mathbb{R}^n$: vector of ones.
- $\gamma > 0$: risk-aversion (variance penalty) parameter.
- Budget constraint: $\boldsymbol{\omega}^\top \mathbf{1} = 1$.
- For validity, $\boldsymbol{\Sigma}$ must be positive definite (hence invertible) and $\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} \neq 0$.

Lagrangian and first-order condition

We start with the Lagrangian (given in the assignment brief):

$$\mathcal{L}(\boldsymbol{\omega}, \lambda) = \boldsymbol{\omega}^\top \boldsymbol{\mu} - \frac{\gamma}{2} \boldsymbol{\omega}^\top \boldsymbol{\Sigma} \boldsymbol{\omega} - \lambda\Big(\boldsymbol{\omega}^\top \mathbf{1} - 1\Big),$$

where λ is the Lagrange multiplier for the budget constraint.

Next, we take the gradient of $\mathcal{L}$ with respect to $\boldsymbol{\omega}$ and set to zero:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\omega}} = \boldsymbol{\mu} - \gamma \boldsymbol{\Sigma} \boldsymbol{\omega} - \lambda \mathbf{1} = \mathbf{0}. \tag{1}$$

Rearranging (1):

$$\gamma\, \boldsymbol{\Sigma} \boldsymbol{\omega} = \boldsymbol{\mu} - \lambda \mathbf{1} \quad \Longrightarrow \quad \boldsymbol{\omega} = \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \frac{\lambda}{\gamma} \boldsymbol{\Sigma}^{-1} \mathbf{1}. \tag{2}$$

To impose the budget (full–investment) constraint, we multiply both sides of $\boldsymbol{\omega}^\top \mathbf{1} = 1$ by $\mathbf{1}^\top$ and set the result equal to one:

$$\mathbf{1}^\top \boldsymbol{\omega} = 1.$$

Substituting the expression for $\boldsymbol{\omega}$ gives

$$\begin{aligned} 1 &= \mathbf{1}^\top \left(\frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \frac{\lambda}{\gamma} \boldsymbol{\Sigma}^{-1} \mathbf{1} \right) \\ &= \frac{1}{\gamma} \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \frac{\lambda}{\gamma} \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}. \end{aligned}$$

Multiply through by γ:

$$\gamma = \boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} - \lambda \, \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}.$$

Solve for λ:

$$\lambda = \frac{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} - \gamma}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}}. \tag{3}$$

Substitute (3) into (2):

$$\begin{aligned} \boldsymbol{\omega}^* &= \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} - \frac{1}{\gamma} \cdot \frac{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1} - \gamma}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} \boldsymbol{\Sigma}^{-1} \mathbf{1} \\ &= \frac{\boldsymbol{\Sigma}^{-1} \mathbf{1}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} + \frac{1}{\gamma} \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\mu} - \frac{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} \mathbf{1} \right). \end{aligned} \tag{4}$$

Hence, (4) is the Mutual fund separation theorem
Let:

$$\begin{aligned} \boldsymbol{\omega}_B &:= \frac{\boldsymbol{\Sigma}^{-1} \mathbf{1}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} && \text{(fully-invested minimum-variance portfolio)}, \\ \boldsymbol{\omega}_A &:= \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\mu} - \frac{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} \mathbf{1} \right) && \text{(zero-cost portfolio)}. \end{aligned}$$

Then the optimal portfolio is,

$$\boldsymbol{\omega}^* = \boldsymbol{\omega}_B + \frac{1}{\gamma}\boldsymbol{\omega}_A \tag{5}$$

Hence, (5) is the alternative form of Mutual fund separation theorem defined in (4).

Verification of properties

- Fully invested portfolio: $\mathbf{1}^\top \boldsymbol{\omega}_B = \dfrac{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}} = 1.$
- Zero-cost active portfolio:

$$\mathbf{1}^\top \boldsymbol{\omega}_A = \mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}}\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1} = 0.$$

- Hence, $\mathbf{1}^\top \boldsymbol{\omega}^* = \mathbf{1}^\top \boldsymbol{\omega}_B + \frac{1}{\gamma}\mathbf{1}^\top \boldsymbol{\omega}_A = 1 + 0 = 1.$

1.1.2 Discuss the constraint equation:

In this optimisation problem, the only constraint that appears inside the Lagrangian is the budget constraint:

$$\boldsymbol{\omega}^\top \mathbf{1} = 1,$$

which ensures that all wealth is invested across the assets i.e. the portfolio weights must sum to one. No other constraint, such as a fixed target return or non–negativity restrictions, was specified in the setup. Therefore, it is the only condition that needs to be enforced when forming the Lagrangian.

The Kuhn–Tucker framework is a method for handling both equality and inequality constraints. In this case, we have only an equality constraint, so the KKT framework leads to the standard Lagrange first–order conditions that we already solved. If inequality constraints (e.g. no short-selling $\boldsymbol{\omega}_i \geq 0$) are added, the two-fund separation may no longer hold as corner solutions arise.

Q2: The Mutual Fund Separation Theorem in terms of an equilibrium view

Proof below based on (Black and Litterman, 1992; He and Litterman, 2002; Idzorek, 2007).

We start from the Mutual fund separation theorem:

$$\boldsymbol{\omega}^* = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}} + \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \frac{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}\mathbf{1}\right). \tag{1}$$

Next, we introduce the equilibrium (policy) view $\boldsymbol{\Pi}$. and substitute the identity,

$$\boldsymbol{\mu} = \boldsymbol{\Pi} + (\boldsymbol{\mu} - \boldsymbol{\Pi})$$

into the second term of the (1) above.

Thus,

$$\begin{aligned}\boldsymbol{\omega}^* = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}} &+ \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\Pi} - \frac{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}\mathbf{1}\right) \\ &+ \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left((\boldsymbol{\mu} - \boldsymbol{\Pi}) - \frac{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\Pi})}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}\mathbf{1}\right).\end{aligned} \tag{2}$$

Then we group terms to define the funds named in the assignment brief:

- Benchmark fund (fully invested minimum-variance portfolio):

$$\boldsymbol{\omega}_B = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}. \tag{3}$$

  This is the first term in $\boldsymbol{\omega}^*$.

- Strategic fund (comes from equilibrium view $\boldsymbol{\Pi}$): define the strategic active vector

$$\boldsymbol{a}_S = \boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\Pi} - \frac{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}\mathbf{1}\right).$$

  This vector has zero sum i.e. $\mathbf{1}^\top\boldsymbol{a}_S = 0$. The strategic portfolio is then the scaled version

$$\boldsymbol{\omega}_S = \frac{1}{\gamma}\boldsymbol{a}_S. \tag{4}$$

- Tactical fund: define

$$\boldsymbol{a}_T \;=\; \boldsymbol{\Sigma}^{-1}\left((\boldsymbol{\mu}-\boldsymbol{\Pi}) - \frac{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\boldsymbol{\Pi})}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}\,\mathbf{1}\right),$$

  and set

$$\boldsymbol{\omega}_T \;=\; \frac{1}{\gamma}\,\boldsymbol{a}_T. \tag{5}$$

  It is important to note $\mathbf{1}^\top\boldsymbol{a}_T = 0$ so $\boldsymbol{\omega}_T$ is the zero-cost active portfolio.

Putting it all together,

$$\boldsymbol{\omega}^* = \boldsymbol{\omega}_B + \boldsymbol{\omega}_S + \boldsymbol{\omega}_T = \underbrace{\Big(\boldsymbol{\omega}_B + \boldsymbol{\omega}_S\Big)}_{\boldsymbol{\omega}_P} + \boldsymbol{\omega}_T,$$

which is equation (3) in the assignment brief: $\boldsymbol{\omega}^* = \boldsymbol{\omega}_P + \boldsymbol{\omega}_T$.

To express $\boldsymbol{\omega}_S, \boldsymbol{\omega}_T$ in the matrix forms required (equation (5) and (6) in assignment brief), we can rewrite each active vector in the equivalent form:

$$\boldsymbol{a}_S = \boldsymbol{\Sigma}^{-1}\Big(\boldsymbol{\Pi}\mathbf{1}^\top - \mathbf{1}\boldsymbol{\Pi}^\top\Big)\,\frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}},$$
$$\boldsymbol{a}_T = \boldsymbol{\Sigma}^{-1}\Big((\boldsymbol{\mu}-\boldsymbol{\Pi})\mathbf{1}^\top - \mathbf{1}(\boldsymbol{\mu}-\boldsymbol{\Pi})^\top\Big)\,\frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}.$$

Multiplying each by $1/\gamma$:

$$\boldsymbol{\omega}_S = \frac{1}{\gamma}\,\boldsymbol{a}_S, \qquad \boldsymbol{\omega}_T = \frac{1}{\gamma}\,\boldsymbol{a}_T,$$

Important to note:

- Both $\boldsymbol{\omega}_S$ and $\boldsymbol{\omega}_T$ are zero-cost active portfolios i.e. their weights sum to zero, so adding them to $\boldsymbol{\omega}_B$ preserves the full-investment condition.
- $\boldsymbol{\Sigma}$ invertible and $\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1} \neq 0$.

Q3: The Black-Litterman mean and covariance

Proofs below based on Meucci (2010a), Walters (2014), and based in the Bayesian linear estimation theorem of Theil (1971).

3.1 Certain Views

Assumptions

We consider an equilibrium model of returns $\boldsymbol{R} \in \mathbb{R}^{n\times 1}$ with

$$\mathbb{E}[\boldsymbol{R}] \sim \mathcal{N}\Big(\boldsymbol{\Pi},\, \tau\boldsymbol{\Sigma}\Big), \qquad \boldsymbol{\Sigma} \in \mathbb{R}^{n\times n} \text{ positive definite,} \tag{1}$$

where:

- $\boldsymbol{\Pi}$ is the $n\times 1$ vector of equilibrium expected returns implied by the market portfolio,
- $\boldsymbol{\Sigma}$ is the $n \times n$ equilibrium covariance matrix of returns,
- $\tau > 0$ is a scalar reflecting the uncertainty in the prior mean .

Investors express k linear views about combinations of asset returns:

$$\boldsymbol{y} = \boldsymbol{P}\,\mathbb{E}[\boldsymbol{R}] = \boldsymbol{V} + \boldsymbol{e}, \tag{2}$$

where:

- $\boldsymbol{P} \in \mathbb{R}^{k\times n}$ encodes each of the k views as a linear combination of the n assets,
- $\boldsymbol{V} \in \mathbb{R}^{k\times 1}$ is the vector of the view expectations,
- $\boldsymbol{e} \in \mathbb{R}^{k\times 1}$ represents the random view errors, with $\mathbb{E}[\boldsymbol{e}] = \boldsymbol{0}$ and $\mathrm{Cov}(\boldsymbol{e}) = \boldsymbol{\Omega}$.

In the "certain-views" case, the views are assumed to be exact:

$$\boldsymbol{e} = \boldsymbol{0}, \qquad \text{so that} \qquad \boldsymbol{P}\,\mathbb{E}[\boldsymbol{R}] = \boldsymbol{V}. \tag{3}$$

Optimization Problem

When the views are certain, the posterior expected returns, denoted $\mathbb{E}[\boldsymbol{R}]^*$, are obtained by minimizing the Mahalanobis distance between $\mathbb{E}[\boldsymbol{R}]$ and the prior mean $\boldsymbol{\Pi}$, subject to the linear view constraints:

$$\mathbb{E}[\boldsymbol{R}]^* = \arg\min_{\mathbb{E}[\boldsymbol{R}]\in\mathbb{R}^{n\times 1}} (\mathbb{E}[\boldsymbol{R}] - \boldsymbol{\Pi})^\top (\tau\boldsymbol{\Sigma})^{-1} (\mathbb{E}[\boldsymbol{R}] - \boldsymbol{\Pi}) \quad \text{s.t.} \quad \boldsymbol{P}\mathbb{E}[\boldsymbol{R}] = \boldsymbol{V}. \tag{4}$$

Lagrangian Formulation

Then, we introduce a $k \times 1$ vector of Lagrange multipliers $\boldsymbol{\lambda}$ for the equality constraint. The Lagrangian function is

$$\mathcal{L}(\mathbb{E}[\boldsymbol{R}], \boldsymbol{\lambda}) = (\mathbb{E}[\boldsymbol{R}] - \boldsymbol{\Pi})^\top (\tau\boldsymbol{\Sigma})^{-1} (\mathbb{E}[\boldsymbol{R}] - \boldsymbol{\Pi}) - 2\,\boldsymbol{\lambda}^\top (\boldsymbol{P}\mathbb{E}[\boldsymbol{R}] - \boldsymbol{V}). \tag{5}$$

Differentiating $\mathcal{L}$ with respect to $\mathbb{E}[\boldsymbol{R}]$ and set the derivative to zero:

$$\begin{aligned} & \frac{\partial \mathcal{L}}{\partial \mathbb{E}[\boldsymbol{R}]} = 2(\tau\boldsymbol{\Sigma})^{-1}(\mathbb{E}[\boldsymbol{R}] - \boldsymbol{\Pi}) - 2\boldsymbol{P}^\top \boldsymbol{\lambda} = \boldsymbol{0}, \\ \Rightarrow \quad & (\tau\boldsymbol{\Sigma})^{-1}(\mathbb{E}[\boldsymbol{R}] - \boldsymbol{\Pi}) = \boldsymbol{P}^\top \boldsymbol{\lambda}. \end{aligned} \tag{6}$$

Solving for $\boldsymbol{\lambda}$

Rearranging (6) gives

$$\mathbb{E}[\boldsymbol{R}] - \boldsymbol{\Pi} = \tau\boldsymbol{\Sigma}\boldsymbol{P}^\top \boldsymbol{\lambda}. \tag{7}$$

Applying the constraint $\boldsymbol{P}\mathbb{E}[\boldsymbol{R}] = \boldsymbol{V}$:

$$\begin{aligned} & \boldsymbol{P}\Big(\boldsymbol{\Pi} + \tau\boldsymbol{\Sigma}\boldsymbol{P}^\top \boldsymbol{\lambda}\Big) = \boldsymbol{V}, \\ \Rightarrow \quad & (\boldsymbol{P}\,\tau\boldsymbol{\Sigma}\,\boldsymbol{P}^\top)\boldsymbol{\lambda} = \boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}. \end{aligned} \tag{8}$$

Assuming the $k \times k$ matrix $\boldsymbol{P}\,\tau\boldsymbol{\Sigma}\,\boldsymbol{P}^\top$ is non-singular, we can solve for the Lagrange multipliers:

$$\boldsymbol{\lambda} = \Big(\boldsymbol{P}\,\tau\boldsymbol{\Sigma}\,\boldsymbol{P}^\top\Big)^{-1}(\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}). \tag{9}$$

Posterior Mean

Substituting (9) into (7) to obtain

$$\mathbb{E}[\boldsymbol{R}]^* = \boldsymbol{\Pi} + \tau\boldsymbol{\Sigma}\boldsymbol{P}^\top \Big(\boldsymbol{P}\,\tau\boldsymbol{\Sigma}\,\boldsymbol{P}^\top\Big)^{-1}(\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}). \tag{10}$$

Interpretation:

- The first term, $\boldsymbol{\Pi}$, represents the equilibrium (prior) expected returns.
- The second term adjusts these returns based on the exact views. The direction and magnitude of the adjustment depend on the view matrix $\boldsymbol{P}$, the view vector $\boldsymbol{V}$,

and the scaled covariance $\tau\boldsymbol{\Sigma}$.

Posterior Covariance. Because the views are assumed exact ($\boldsymbol{e} = \mathbf{0}$), the posterior covariance remains the equilibrium covariance:

$$\boldsymbol{\Sigma}^* = \boldsymbol{\Sigma}. \tag{11}$$

3.2.1 Uncertain views: Bayesian formulation

Assumptions

- $\mathbb{E}[\boldsymbol{R}] \in \mathbb{R}^{n\times 1}$ is the unknown vector of expected returns.
- $\boldsymbol{\Pi} \in \mathbb{R}^{n\times 1}$ is the equilibrium mean (prior) and $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$ is the equilibrium covariance matrix (positive definite).
- $\tau > 0$ is a scalar scaling parameter for prior uncertainty.
- $\boldsymbol{P} \in \mathbb{R}^{k\times n}$ incorporates k linear views on the n assets, with corresponding view vector $\boldsymbol{V} \in \mathbb{R}^{k\times 1}$.
- The view errors $\boldsymbol{e} \in \mathbb{R}^{k\times 1}$ satisfy $\mathbb{E}[\boldsymbol{e}] = \mathbf{0}$ and $\mathrm{Cov}(\boldsymbol{e}) = \boldsymbol{\Omega} \in \mathbb{R}^{k\times k}$ (symmetric and positive definite).

Prior distribution

From the assignment brief,

$$\mathbb{E}[\boldsymbol{R}] \sim \mathcal{N}(\boldsymbol{\Pi},\, \tau\boldsymbol{\Sigma}). \tag{4}$$

Hence, the Gaussian prior density up to normalising constant k_1 is,

$$p(\mathbb{E}[\boldsymbol{R}]) = k_1 \, \exp\Big[-\tfrac{1}{2}(\boldsymbol{\Pi} - \mathbb{E}[\boldsymbol{R}])^\top (\tau\boldsymbol{\Sigma})^{-1}(\boldsymbol{\Pi} - \mathbb{E}[\boldsymbol{R}])\Big]. \tag{5}$$

Likelihood based on the views

The k views are modelled as

$$\boldsymbol{y} = \boldsymbol{P}\mathbb{E}[\boldsymbol{R}] = \boldsymbol{V} + \boldsymbol{e}, \qquad \boldsymbol{e} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Omega}). \tag{6}$$

Conditional on $\mathbb{E}[\boldsymbol{R}]$, the likelihood of the observed views $\boldsymbol{V}$ is therefore

$$p(\boldsymbol{V} \mid \mathbb{E}[\boldsymbol{R}]) = k_2 \, \exp\Big[-\tfrac{1}{2}(\boldsymbol{V} - \boldsymbol{P}\mathbb{E}[\boldsymbol{R}])^\top \boldsymbol{\Omega}^{-1}(\boldsymbol{V} - \boldsymbol{P}\mathbb{E}[\boldsymbol{R}])\Big]. \tag{7}$$

Posterior distribution using Bayes' theorem

By Bayes' theorem,

$$p(\mathbb{E}[\boldsymbol{R}] \mid \boldsymbol{V}) = \frac{p(\boldsymbol{V} \mid \mathbb{E}[\boldsymbol{R}])\, p(\mathbb{E}[\boldsymbol{R}])}{p(\boldsymbol{V})}.$$

Writing the prior and likelihood with their normalising constants k_1, k_2 gives

$$p(\mathbb{E}[\boldsymbol{R}] \mid \boldsymbol{V}) = \frac{k_1 k_2}{p(\boldsymbol{V})}\, \exp\Big[-\tfrac{1}{2}(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{\Pi})^\top(\tau\boldsymbol{\Sigma})^{-1}(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{\Pi})\Big]\, \exp\Big[-\tfrac{1}{2}(\boldsymbol{V}-\boldsymbol{P}\mathbb{E}[\boldsymbol{R}])^\top\boldsymbol{\Omega}^{-1}(\boldsymbol{V}-\boldsymbol{P}\mathbb{E}[\boldsymbol{R}])\Big].$$

Since $p(\boldsymbol{V})$ is a scalar that is independent of $\mathbb{E}[\boldsymbol{R}]$, set

$$k := \frac{k_1 k_2}{p(\boldsymbol{V})},$$

and hence

$$p(\mathbb{E}[\boldsymbol{R}] \mid \boldsymbol{V}) = k\, \exp\Big[-\tfrac{1}{2}(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{\Pi})^\top(\tau\boldsymbol{\Sigma})^{-1}(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{\Pi})\Big]\, \exp\Big[-\tfrac{1}{2}(\boldsymbol{V}-\boldsymbol{P}\mathbb{E}[\boldsymbol{R}])^\top\boldsymbol{\Omega}^{-1}(\boldsymbol{V}-\boldsymbol{P}\mathbb{E}[\boldsymbol{R}])\Big]. \tag{8}$$

Equation (8) expresses the posterior of $\mathbb{E}[\boldsymbol{R}]$ as the product of two Gaussian exponentials, the prior (based on the equilibrium distribution) and the likelihood (based on the uncertain views).

3.2.2 Uncertain views: deriving the mean of the conditional distribution

We start from the posterior density defined in section 3.2.1,

$$p(\mathbb{E}[\boldsymbol{R}] \mid \boldsymbol{V}) \propto \exp\Big[-\tfrac{1}{2}(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{\Pi})^\top(\tau\boldsymbol{\Sigma})^{-1}(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{\Pi})\Big]\exp\Big[-\tfrac{1}{2}(\boldsymbol{P}\mathbb{E}[\boldsymbol{R}]-\boldsymbol{V})^\top\boldsymbol{\Omega}^{-1}(\boldsymbol{P}\mathbb{E}[\boldsymbol{R}]-\boldsymbol{V})\Big],$$

where:

- $\mathbb{E}[\boldsymbol{R}]$ is the vector of unknown expected returns.
- $\boldsymbol{\Pi}$ and $\tau\boldsymbol{\Sigma}$ denote the prior mean and prior covariance.
- $\boldsymbol{P}$, $\boldsymbol{V}$, and $\boldsymbol{\Omega}$ capture the views, view values, and view uncertainty respectively.

Expanding both quadratic forms and removing additive constants that do not depend on $\mathbb{E}[\boldsymbol{R}]$ gives:

$$\begin{aligned}&(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{\Pi})^\top(\tau\boldsymbol{\Sigma})^{-1}(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{\Pi})+(\boldsymbol{P}\mathbb{E}[\boldsymbol{R}]-\boldsymbol{V})^\top\boldsymbol{\Omega}^{-1}(\boldsymbol{P}\mathbb{E}[\boldsymbol{R}]-\boldsymbol{V})\\ &=\mathbb{E}[\boldsymbol{R}]^\top\Big[(\tau\boldsymbol{\Sigma})^{-1}+\boldsymbol{P}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{P}\Big]\mathbb{E}[\boldsymbol{R}]-2\,\mathbb{E}[\boldsymbol{R}]^\top\Big[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi}+\boldsymbol{P}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{V}\Big].\end{aligned} \tag{8}$$

The expression in (8) is a quadratic form in $\mathbb{E}[\boldsymbol{R}]$ which matches the canonical exponent of a multivariate Gaussian density.

Comparing (8) to the canonical log-density form:

$$-\tfrac{1}{2}\mathbb{E}[\boldsymbol{R}]^\top(\boldsymbol{\Sigma}^*)^{-1}\mathbb{E}[\boldsymbol{R}]+\mathbb{E}[\boldsymbol{R}]^\top(\boldsymbol{\Sigma}^*)^{-1}\mathbb{E}[\boldsymbol{R}]^*,$$

we identify:

$$(\boldsymbol{\Sigma}^*)^{-1}=(\tau\boldsymbol{\Sigma})^{-1}+\boldsymbol{P}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{P}, \tag{9}$$

$$(\boldsymbol{\Sigma}^*)^{-1}\,\mathbb{E}[\boldsymbol{R}]^*=(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi}+\boldsymbol{P}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{V}. \tag{10}$$

These two equations determine the posterior conditional mean and covariance, provided $(\boldsymbol{\Sigma}^*)^{-1}$ is non-singular which holds if $\tau\boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}$ are positive definite and $\boldsymbol{P}$ has full row rank.

Inverting (9) yields:

$$\boldsymbol{\Sigma}^*=\Big[(\tau\boldsymbol{\Sigma})^{-1}+\boldsymbol{P}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{P}\Big]^{-1}.$$

Pre-multiplying (10) by $\boldsymbol{\Sigma}^*$ gives the posterior mean:

$$\mathbb{E}[\boldsymbol{R}]^* = \Big[(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{P}\Big]^{-1}\Big[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi} + \boldsymbol{P}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{V}\Big]. \tag{11}$$

Equation (11) is precisely equation (13) from the assignment and represents the mean of the conditional distribution of $\mathbb{E}[\boldsymbol{R}]$ given the views.

Equivalent Black–Litterman form

Using the Woodbury matrix inversion identity:

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1},$$

with $A = (\tau\boldsymbol{\Sigma})^{-1}$, $U = \boldsymbol{P}^\top$, $C = \boldsymbol{\Omega}^{-1}$, and $V = \boldsymbol{P}$, we can re-express $\mathbb{E}[\boldsymbol{R}]^*$ in the familiar Black–Litterman form:

$$\mathbb{E}[\boldsymbol{R}]^* = \boldsymbol{\Pi} + \tau\boldsymbol{\Sigma}\boldsymbol{P}^\top\Big(\boldsymbol{P}\tau\boldsymbol{\Sigma}\boldsymbol{P}^\top + \boldsymbol{\Omega}\Big)^{-1}(\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}). \tag{12}$$

3.2.3 Variance of the posterior distribution

Proof below based on (Satchell and Scowcroft, 2000) (see Appendix: Proof of Theorem 1)

Using Bayes' theorem and the model assumptions, the posterior density for $\mathbb{E}[\boldsymbol{R}] = \mathbb{E}[\boldsymbol{R}]$, dropping normalising constants, can be written as

$$p(\mathbb{E}[\boldsymbol{R}] \mid \boldsymbol{V}) \propto \exp\Big\{-\tfrac{1}{2}\Big[(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{\Pi})^\top(\tau\boldsymbol{\Sigma})^{-1}(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{\Pi})+(\boldsymbol{P}\mathbb{E}[\boldsymbol{R}]-\boldsymbol{V})^\top\boldsymbol{\Omega}^{-1}(\boldsymbol{P}\mathbb{E}[\boldsymbol{R}]-\boldsymbol{V})\Big]\Big\}. \tag{1}$$

Next, we expand the two quadratic forms and collect terms involving $\mathbb{E}[\boldsymbol{R}]$. After expansion and re-ordering we obtain an expression of the form

$$\begin{aligned} &(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{\Pi})^\top(\tau\boldsymbol{\Sigma})^{-1}(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{\Pi})+(\boldsymbol{P}\mathbb{E}[\boldsymbol{R}]-\boldsymbol{V})^\top\boldsymbol{\Omega}^{-1}(\boldsymbol{P}\mathbb{E}[\boldsymbol{R}]-\boldsymbol{V}) \\ &= \mathbb{E}[\boldsymbol{R}]^\top\Big[(\tau\boldsymbol{\Sigma})^{-1}+\boldsymbol{P}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{P}\Big]\mathbb{E}[\boldsymbol{R}] - 2\,\mathbb{E}[\boldsymbol{R}]^\top\Big[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi}+\boldsymbol{P}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{V}\Big] + \Big(\boldsymbol{V}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{V}+\boldsymbol{\Pi}^\top(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi}\Big). \end{aligned} \tag{2}$$

Defining H, C, A (Based on hint)

$$\boldsymbol{H} := (\tau\boldsymbol{\Sigma})^{-1}+\boldsymbol{P}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{P}, \qquad \boldsymbol{C} := (\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi}+\boldsymbol{P}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{V}, \qquad A := \boldsymbol{V}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{V}+\boldsymbol{\Pi}^\top(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\Pi}. \tag{3}$$

Note that $\boldsymbol{H}$ is symmetric and under the stated assumptions positive definite.
With these definitions the quadratic expression (2) becomes up to constants

$$\mathbb{E}[\boldsymbol{R}]^\top\boldsymbol{H}\mathbb{E}[\boldsymbol{R}] - 2\,\mathbb{E}[\boldsymbol{R}]^\top\boldsymbol{C} + A. \tag{4}$$

Completing the square with respect to $\mathbb{E}[\boldsymbol{R}]$:

$$\begin{aligned} \mathbb{E}[\boldsymbol{R}]^\top\boldsymbol{H}\mathbb{E}[\boldsymbol{R}] - 2\,\mathbb{E}[\boldsymbol{R}]^\top\boldsymbol{C} + A &= (\mathbb{E}[\boldsymbol{R}]-\boldsymbol{H}^{-1}\boldsymbol{C})^\top\boldsymbol{H}(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{H}^{-1}\boldsymbol{C}) - \boldsymbol{C}^\top\boldsymbol{H}^{-1}\boldsymbol{C} + A \\ &= (\mathbb{E}[\boldsymbol{R}]-\boldsymbol{H}^{-1}\boldsymbol{C})^\top\boldsymbol{H}(\mathbb{E}[\boldsymbol{R}]-\boldsymbol{H}^{-1}\boldsymbol{C}) + \Big(A-\boldsymbol{C}^\top\boldsymbol{H}^{-1}\boldsymbol{C}\Big). \end{aligned} \tag{5}$$

The final scalar $\Big(A-\boldsymbol{C}^\top\boldsymbol{H}^{-1}\boldsymbol{C}\Big)$ does not depend on $\mathbb{E}[\boldsymbol{R}]$ and therefore is absorbed into the normalising constant of the density.

After absorbing the constant, the posterior density takes the canonical Gaussian form:

$$p(\mathbb{E}[\boldsymbol{R}] \mid \boldsymbol{V}) \propto \exp\Big\{ -\tfrac{1}{2}(\mathbb{E}[\boldsymbol{R}] - \boldsymbol{H}^{-1}\boldsymbol{C})^\top \boldsymbol{H}(\mathbb{E}[\boldsymbol{R}] - \boldsymbol{H}^{-1}\boldsymbol{C})\Big\}. \tag{6}$$

Hence, the conditional posterior mean and covariance are:

$$\mathbb{E}[\boldsymbol{R}]^* = \boldsymbol{H}^{-1}\boldsymbol{C}, \qquad \boldsymbol{\Sigma}^* = \boldsymbol{H}^{-1}. \tag{7}$$

Substituting the definition of $\boldsymbol{H}$, yields the required variance formula

$$\boldsymbol{\Sigma}^* = \Big[(\tau\boldsymbol{\Sigma})^{-1} + \boldsymbol{P}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{P}\Big]^{-1} \tag{8}$$

Q4: The Black–Litterman portfolio weights

Proof below based on (Meucci, 2010b).

Recall the mutual-fund separation theorem for a generic mean $\boldsymbol{\mu}$:

$$\boldsymbol{\omega}^* = \boldsymbol{\omega}_B + \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \frac{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}\mathbf{1}\right), \tag{1}$$

with

$$\boldsymbol{\omega}_B := \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}. \tag{2}$$

In the Black–Litterman model replace $\boldsymbol{\mu}$ by the posterior mean $\mathbb{E}[\boldsymbol{R}]^*$ (derived in section 3.2.2: equation (12)). Therefore, direct substitution of $\mathbb{E}[\boldsymbol{R}]^*$ into (1) yields,

$$\boldsymbol{\omega}^*_{BL} = \boldsymbol{\omega}_B + \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left(\mathbb{E}[\boldsymbol{R}]^* - \frac{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbb{E}[\boldsymbol{R}]^*}{\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbf{1}}\mathbf{1}\right), \tag{3}$$

We substitute the posterior mean derived in section 3.2.2,

$$\mathbb{E}[\boldsymbol{R}]^* = \boldsymbol{\Pi} + \tau\boldsymbol{\Sigma}\boldsymbol{P}^\top\left(\boldsymbol{P}\tau\boldsymbol{\Sigma}\boldsymbol{P}^\top + \boldsymbol{\Omega}\right)^{-1}(\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}) \tag{4}$$

into the mutual-fund separation expression (3). For convenience let

$$\boldsymbol{M} := \boldsymbol{P}\tau\boldsymbol{\Sigma}\boldsymbol{P}^\top + \boldsymbol{\Omega}. \tag{5}$$

First, we derive $\boldsymbol{\Sigma}^{-1}\mathbb{E}[\boldsymbol{R}]^*$. Using $\boldsymbol{\Sigma}^{-1}(\tau\boldsymbol{\Sigma}) = \tau\boldsymbol{I}$ we get

$$\begin{aligned}\boldsymbol{\Sigma}^{-1}\mathbb{E}[\boldsymbol{R}]^* &= \boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi} + \boldsymbol{\Sigma}^{-1}\left(\tau\boldsymbol{\Sigma}\boldsymbol{P}^\top\right)\boldsymbol{M}^{-1}(\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}) \\ &= \boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi} + \tau\,\boldsymbol{P}^\top\boldsymbol{M}^{-1}(\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}).\end{aligned} \tag{6}$$

Next, form the scalar appearing in the projection onto $\mathbf{1}$:

$$\begin{aligned}s &:= \mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\mathbb{E}[\boldsymbol{R}]^* \\ &= \mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi} + \mathbf{1}^\top\left(\tau\boldsymbol{P}^\top\boldsymbol{M}^{-1}(\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi})\right) \\ &= \mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi} + \tau\left(\boldsymbol{P}\mathbf{1}\right)^\top\boldsymbol{M}^{-1}(\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}),\end{aligned} \tag{7}$$

where we used $\mathbf{1}^\top\boldsymbol{P}^\top = (\boldsymbol{P}\mathbf{1})^\top$.

Let $d := \mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}$ and $\boldsymbol{\omega}_B = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{d}$. Now we substitute (6) and (7) into equation (16) of the assignment brief to obtain the Black–Litterman weights:

$$\begin{aligned}
\boldsymbol{\omega}^*_{BL} &= \boldsymbol{\omega}_B + \frac{1}{\gamma}\,\boldsymbol{\Sigma}^{-1}\Big(\mathbb{E}[\boldsymbol{R}]^* - \frac{s}{d}\,\mathbf{1}\Big) \\
&= \boldsymbol{\omega}_B + \frac{1}{\gamma}\left[\boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi} + \tau\,\boldsymbol{P}^\top \boldsymbol{M}^{-1}(\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}) - \frac{s}{d}\,\boldsymbol{\Sigma}^{-1}\mathbf{1}\right] \\
&= \boldsymbol{\omega}_B + \frac{1}{\gamma}\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi} + \frac{\tau}{\gamma}\,\boldsymbol{P}^\top \boldsymbol{M}^{-1}(\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}) - \frac{s}{d\gamma}\,\boldsymbol{\Sigma}^{-1}\mathbf{1}.
\end{aligned} \tag{8}$$

Finally , we use $\boldsymbol{\Sigma}^{-1}\mathbf{1} = d\,\boldsymbol{\omega}_B$ to collect the $\boldsymbol{\omega}_B$ terms :

$$\boldsymbol{\omega}^*_{BL} = \boldsymbol{\omega}_B\Big(1 - \frac{s}{\gamma}\Big) + \frac{1}{\gamma}\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\Pi} + \frac{\tau}{\gamma}\,\boldsymbol{P}^\top \boldsymbol{M}^{-1}(\boldsymbol{V} - \boldsymbol{P}\boldsymbol{\Pi}). \tag{9}$$

PART II : Implement and Historically Simulate the Black-Litterman Model

Problem Specification

This assignment focused on implementing and backtesting the Black-Litterman (BL) portfolio optimisation model using the same historical dataset from Assignment 1. The assignment required developing a fully documented R function "blacklitterman" that calculates the posterior expected returns and optimal portfolio weights by combining market equilibrium with investor views. Using the rolling-window methodology from Assignment 1, a historical backtest simulation is conducted under full investment and no-short-selling constraints. The analysis included computing and comparing performance metrics (Sharpe ratio, Jensen's alpha, portfolio beta, and turnover) relative to a chosen strategic benchmark portfolio. Appropriate choices for the long-term equilibrium portfolio and risk aversion parameter must be justified, targeting tracking errors between 3-10% relative to the benchmark

Data Specification

The Tactical Asset Allocation data is loaded from "PT-DATA-ALBI-JIBAR-JSEIND-Daily-1994-2017.xlsx" with the following asset information:
Key: (Number of rows, number of columns)

1. Sheet 1: (8439,2) ALBI (All Bond Index (ALBI) Total Return Index (TRI) Data
2. Sheet 2: (8405,4) Money Market Data: JIBAR and STEFI TRI
3. Sheet 3: (8439,28) JSE ICB Industrial Level Indices
4. Sheet 4: (8439,20) JSE Various Indices: JSE Growth, JSE Value, JSE ALSI, JSE SRI

Configuration control

Coding for Part II was completed using RStudio 2024.09.0+375 ("Cranberry Hibiscus" Release) and was based on R and MATLAB code provided by Professor Tim Gebbie(STA4028Z).
Version control: managed with Git and GitHub.
To view repository, click on link: https://github.com/NesanNaidoo/PT-A2-Black-Litterman-Portfolio

Methodology

Rolling Window Framework

This experiment applied a rolling window approach, adapted from Assignment 1. A fixed-length training window of 60 months was incremented forward by one month at a time. At each test interval, returns are generated by applying the weights optimised at the start of that interval, rather than rebalancing within the window. This ensured that the returns for each period directly correspond to the initial portfolio allocation .This avoids artificially improving performance metrics such as Sharpe ratio, alpha, or volatility by preventing intra-window adjustments that would not occur in a realistic out-of-sample setting.

For each training window, the Black-Litterman (BL) portfolio weights were estimated using the function `blacklitterman`. These weights were then applied out-of-sample to the following 12 months of test data to compute portfolio returns. This procedure produced a time series of overlapping in-sample (IS) and out-of-sample (OOS) statistics, Sharpe ratios, Jensen's alpha, beta, and turnover for each window.

Black-Litterman Portfolio Estimation

For each training window, arithmetic monthly returns were used to compute the sample mean vector μ and covariance matrix Σ. Market equilibrium implied returns (Π) were derived from the strategic benchmark portfolio, while investor views were incorporated via matrices P and Q with associated uncertainty Ω. The posterior expected returns and covariance matrix were computed according to the Black-Litterman framework.
The optimal portfolio weights w^* were obtained through mean-variance optimization of the posterior estimates, under no-short-selling and fully invested constraints:

$$\sum_{i=1}^{N} w_i = 1, \quad w_i \geq 0 \quad \forall i$$

Small negative weights were set to zero and the remaining weights re-normalized. The risk-free rate for excess return calculations was proxied by the STEFI asset.

Strategic Portfolio Equilibrium Choice

The long-term strategic portfolio equilibrium was set to the market-implied equilibrium returns (Π) which was derived from historical in-sample averages. This choice is justified because it represents the market consensus. Therefore, providing a neutral benchmark for measuring the effect of investor views. Furthermore, using the market equilibrium as the reference ensures that the Black-Litterman adjustments do not deviate too much from a realistic long-term allocation.

Risk-Aversion Parameter (γ)

A risk-aversion parameter of $\gamma = 3$ was chosen. This value was selected to maintain full investment and avoid short-selling, while allowing moderate exposure to higher-risk assets. A higher γ would overweight safer assets, reducing the potential impact of investor views, and a lower γ could force short positions to achieve higher expected returns. Thus, $\gamma = 3$ represents a balance between risk control and responsiveness to views.

Tau Parameter (τ)

The scalar $\tau = 0.05$ was chosen to show moderate uncertainty in the prior equilibrium returns. A small τ keeps the portfolio close to the strategic benchmark, minimizing deviations due to investor views. Conversely, a larger τ would overweight the views, increasing volatility and tracking error.

Tracking Error

The portfolio is designed to maintain a tracking error between 3% and 10% relative to the ALSI benchmark. This ensures that the BL portfolio achieves some differentiation from the strategic equilibrium without deviating excessively.Thus, keeping risk and exposure aligned with long-term objectives.

Results and Analysis

See Appendix A for training and test periods used that correspond to the different rolling windows.

Portfolio Turnover

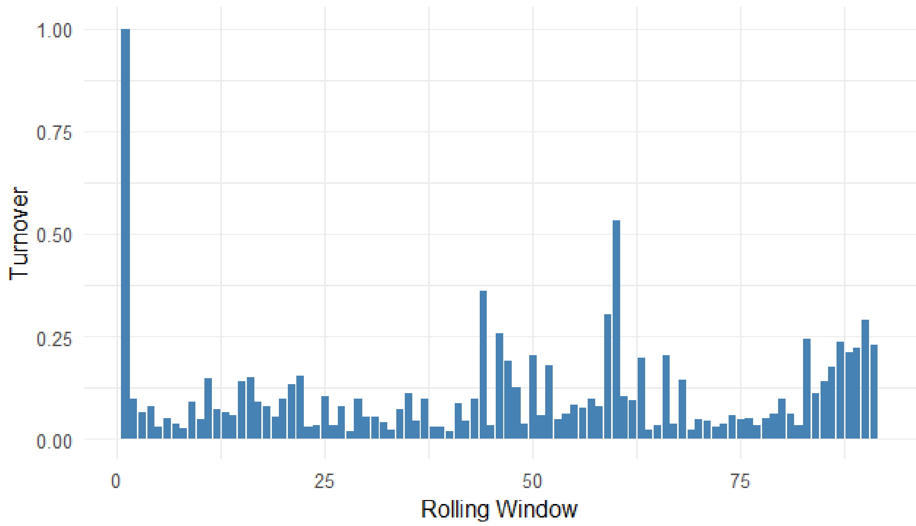

Figure 1: Portfolio turnover along testing period 2009-2017.

Portfolio turnover measures the proportion of portfolio holdings that change between consecutive rebalancing periods. As seen in Figure 1, window 1 shows complete turnover (1.00) as the portfolio is constructed from scratch. Turnover then stabilises within the 0.02–0.15 range through Windows 2–20. This corresponds to the post-2008 recovery period where the model adjusts gradually to market normalisation. Notable spikes occur in Windows 44–46 (0.26–0.36) and Windows 50–60 (peaking at 0.53). This suggests that these periods were where the optimiser made significant reallocations. These spikes likely reflect the model responding to unstable covariance estimates or shifting market dynamics. After Window 60, turnover returns to lower levels (0.03–0.18). This indicates that the portfolio settles into more stable allocations.

Portfolio mean returns

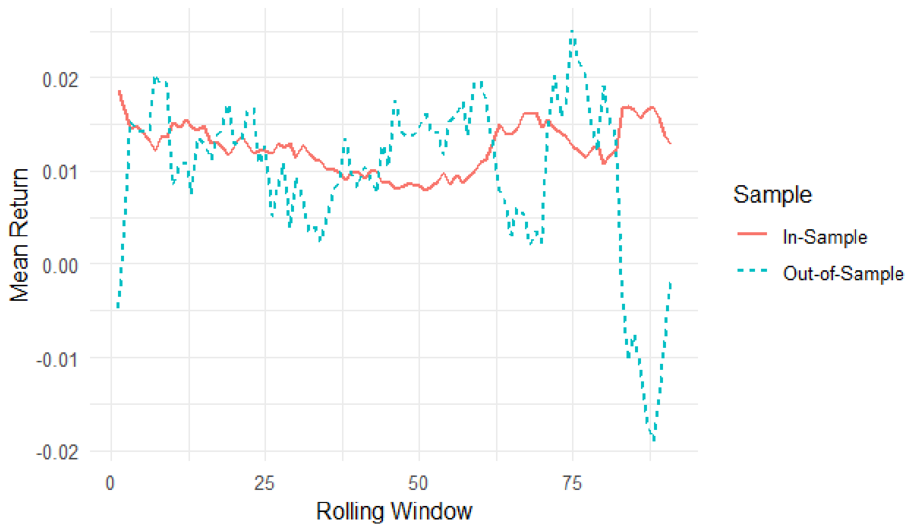

Figure 2: Mean portfolio returns (IS vs OOS) along testing period 2009-2017.

Figure 2 shows the in-sample portfolio returns across windows 1–40, range approximately from 0.012 to 0.025. The corresponding out-of-sample returns are generally lower, between 0.003 and 0.022. During periods of market stress (windows 10–20), the out-of-sample returns initially drop. This reflects adverse market conditions at the time. However, the OOS returns recover in later windows (32-56) and exceed the in-sample returns. This suggests the model is adaptive to changing markets. However, Figure 2 also highlights that between window 58 and 72 and after window 80, the in-sample returns consistently exceed out-of-sample returns. This indicates potential over-fitting to historical data and reduced predictive ability for future periods.

Variance of portfolio returns

Figure 3: Variance of portfolio returns (IS vs OOS) along testing period 2009-2017.

Figure 3 illustrates that the in-sample variance, remains relatively stable around 0.0015–0.0020. The out-of-sample variance increases during market stress (windows 10–20) to approximately 0.003–0.005 before declining in subsequent windows. Higher out-of-sample variance during stressful periods (between windows 1 to 8 and between windows 70 and 80) indicates increased portfolio risk when applied to new data, whereas the stability in between windows 9 to 73 suggests risk mitigation as markets normalise.

Sharpe ratios

Figure 4: Sharpe ratios of portfolios (IS vs. OOS) along testing period 2009-2017.

Figure 4 shows that the in-sample Sharpe ratios, are consistently positive across all windows (approximately 0.10–0.70). This indicates strong risk-adjusted performance during training. The Out-of-sample Sharpe ratios, drop during stress periods (Windows 10–20) and sometimes fall below zero. This signals poor risk-adjusted performance in adverse market conditions. From Windows 41–70, the out-of-sample Sharpe ratios improve and occasionally surpass in-sample values. This suggests periods of effective generalisation. However, in later windows (57 to 91), out-of-sample Sharpe ratios overall declines significantly, further pointing to over-fitting.

Jensen's alpha and beta

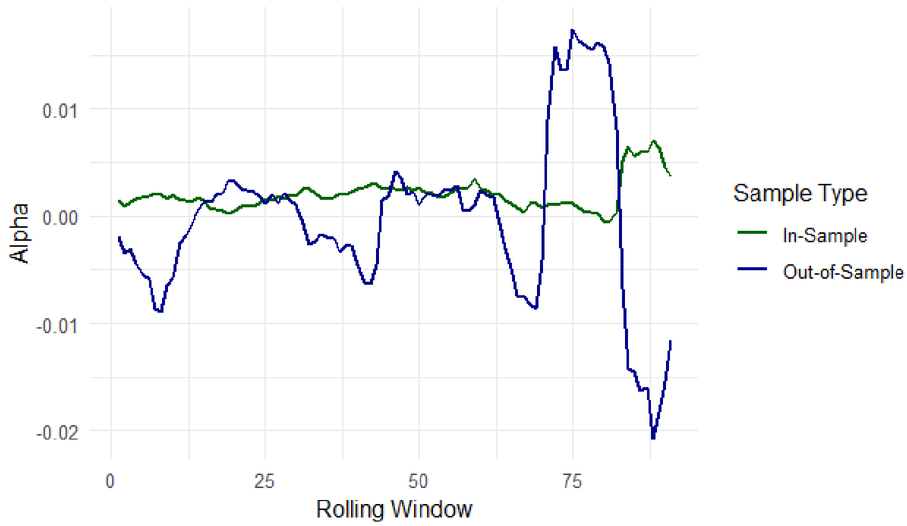

Figure 5: Jensen's alpha (IS vs. OOS) along testing period 2009-2017.

Jensen's alpha measures excess return relative to the benchmark after accounting for systematic risk (beta). Figure 5 shows that the in-sample Jensen's Alpha across windows, generally range from 0.005 to 0.02. This indicates positive excess returns relative to the benchmark during training. The out-of-sample alpha fluctuates more widely and occasionally drops below zero, particularly during market stress periods (Windows 10–20). This suggests that while the model can outperform the benchmark on historical data, its ability to generate excess returns in new market conditions is less consistent. Windows 71–80 show positive out-of-sample alpha (0.008–0.018). This indicates the portfolio outperformed its benchmark on a risk-adjusted basis. Negative out-of-sample alpha in Windows 64–69 and 83–90 suggests under performance relative to the benchmark during these periods.

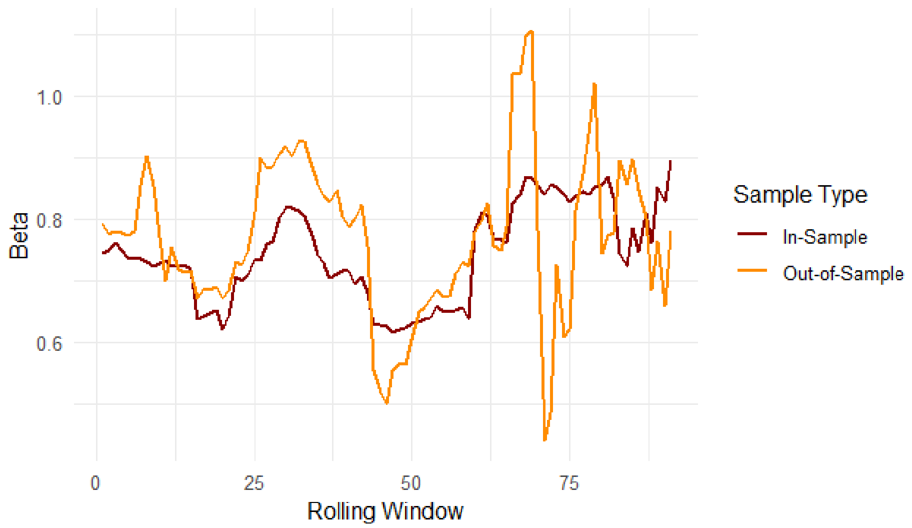

Figure 6: Jensen's beta (IS vs. OOS) along testing period 2009-2017.

Beta measures the portfolio's sensitivity to market movements. Figure 6 illustrates that the in-sample Beta is mostly between 0.8 and 1.2, suggesting the portfolio has similar systematic risk exposure to the benchmark. The Out-of-sample beta, varies during (windows 10–20), sometimes exceeding 1.3, indicating higher sensitivity to market movements than intended. In later periods, beta stabilises near in-sample levels. Lower beta values (0.50–0.70) in Windows 44–49 and 71–74 suggest more defensive positioning during these periods. The generally sub-1.0 beta reflects the inclusion of bonds (ALBI) as a core holding index, which reduces overall market sensitivity.

Tracking error

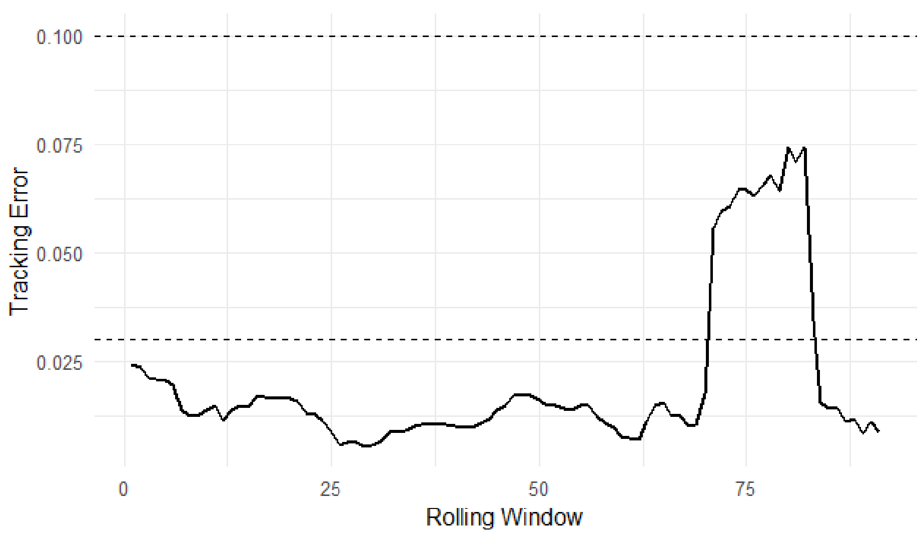

Figure 7: Tracking error relative to the ALSI index along testing period 2009-2017.

Tracking error measures the standard deviation of the difference between portfolio returns and benchmark returns. Figure 7 shows in-sample tracking error generally remains low (approx. 0.01–0.03). This indicates the portfolio closely follows the benchmark during training. The out-of-sample tracking error increases during volatile periods (Windows 10–20), reflecting deviations from benchmark performance. A rising tracking error signals reduced effectiveness of the strategy in replicating benchmark behaviour and potentially higher active risk when applied to new data. Higher tracking error appears in Windows 70–82 (0.055–0.074), indicating the portfolio deviated more substantially from the benchmark. This deviation could reflect active bets taken by the Black-Litterman framework based on investor views diverging from market equilibrium.

Portfolio Weights Evolution

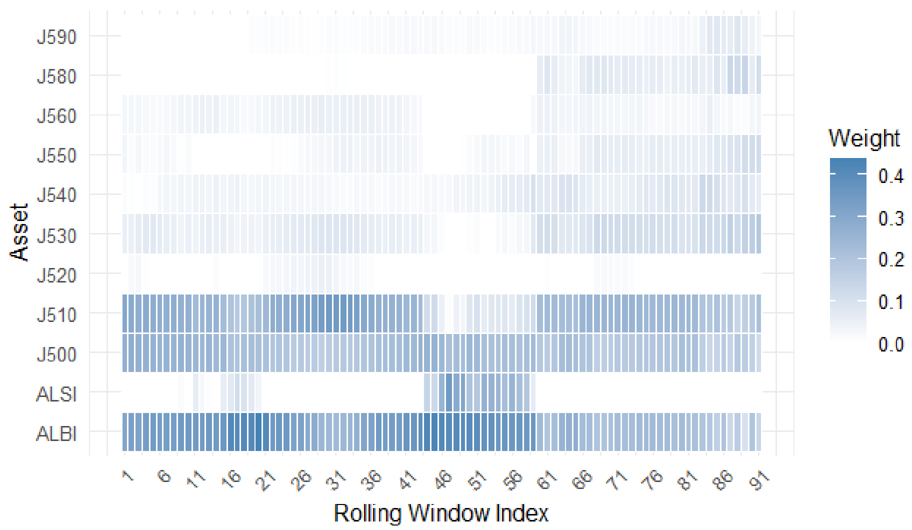

Figure 8: Evolution of Black-Litterman Portfolio Weights along testing period 2009-2017.

The weight evolution across the 91 rolling windows reveals a portfolio anchored by two core holdings: J500 (large-cap equities) and ALBI (bonds). J500 consistently receives allocations around 0.3–0.4 throughout most windows. ALBI maintains notable weight, particularly in the early to middle windows (1–50), serving as a stabilising component. Mid-cap equities (J510–J530) show intermittent allocations with periods of near-zero exposure. Smaller-cap equities (J550–J590) generally receive minimal weight until later windows (60–91), where allocations increase slightly. This suggests the model gradually shifts exposure towards mid- and small-cap equities over time. This is likely in response to evolving market conditions. Early periods favour core equities and bonds, while later windows diversify more broadly as the model adapts to new data. ALBI allocations remain fairly consistent despite slight reductions in Windows 40–50. Overall, the strategy balances market upside with risk control. Increased equity exposure in later windows indicates more risk-taking or confidence in market trends.

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Appendix A:

Rolling window : training vs. test periods.

Table 1: Training and testing windows.

Window	Training Period	Testing Period
1	2003-09-30 / 2008-08-31	2008-09-30 / 2009-08-31
2	2003-10-31 / 2008-09-30	2008-10-31 / 2009-09-30
3	2003-11-30 / 2008-10-31	2008-11-30 / 2009-10-31
4	2003-12-31 / 2008-11-30	2008-12-31 / 2009-11-30
5	2004-01-31 / 2008-12-31	2009-01-31 / 2009-12-31
6	2004-02-29 / 2009-01-31	2009-02-28 / 2010-01-31
7	2004-03-31 / 2009-02-28	2009-03-31 / 2010-02-28
8	2004-04-30 / 2009-03-31	2009-04-30 / 2010-03-31
9	2004-05-31 / 2009-04-30	2009-05-31 / 2010-04-30
10	2004-06-30 / 2009-05-31	2009-06-30 / 2010-05-31
11	2004-07-31 / 2009-06-30	2009-07-31 / 2010-06-30
12	2004-08-31 / 2009-07-31	2009-08-31 / 2010-07-31
13	2004-09-30 / 2009-08-31	2009-09-30 / 2010-08-31
14	2004-10-31 / 2009-09-30	2009-10-31 / 2010-09-30
15	2004-11-30 / 2009-10-31	2009-11-30 / 2010-10-31
16	2004-12-31 / 2009-11-30	2009-12-31 / 2010-11-30
17	2005-01-31 / 2009-12-31	2010-01-31 / 2010-12-31
18	2005-02-28 / 2010-01-31	2010-02-28 / 2011-01-31
19	2005-03-31 / 2010-02-28	2010-03-31 / 2011-02-28
20	2005-04-30 / 2010-03-31	2010-04-30 / 2011-03-31
21	2005-05-31 / 2010-04-30	2010-05-31 / 2011-04-30
22	2005-06-30 / 2010-05-31	2010-06-30 / 2011-05-31
23	2005-07-31 / 2010-06-30	2010-07-31 / 2011-06-30
24	2005-08-31 / 2010-07-31	2010-08-31 / 2011-07-31
25	2005-09-30 / 2010-08-31	2010-09-30 / 2011-08-31
26	2005-10-31 / 2010-09-30	2010-10-31 / 2011-09-30
27	2005-11-30 / 2010-10-31	2010-11-30 / 2011-10-31
28	2005-12-31 / 2010-11-30	2010-12-31 / 2011-11-30
29	2006-01-31 / 2010-12-31	2011-01-31 / 2011-12-31
30	2006-02-28 / 2011-01-31	2011-02-28 / 2012-01-31
31	2006-03-31 / 2011-02-28	2011-03-31 / 2012-02-29
32	2006-04-30 / 2011-03-31	2011-04-30 / 2012-03-31

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Window	Training Period	Testing Period
33	2006-05-31 / 2011-04-30	2011-05-31 / 2012-04-30
34	2006-06-30 / 2011-05-31	2011-06-30 / 2012-05-31
35	2006-07-31 / 2011-06-30	2011-07-31 / 2012-06-30
36	2006-08-31 / 2011-07-31	2011-08-31 / 2012-07-31
37	2006-09-30 / 2011-08-31	2011-09-30 / 2012-08-31
38	2006-10-31 / 2011-09-30	2011-10-31 / 2012-09-30
39	2006-11-30 / 2011-10-31	2011-11-30 / 2012-10-31
40	2006-12-31 / 2011-11-30	2011-12-31 / 2012-11-30
41	2007-01-31 / 2011-12-31	2012-01-31 / 2012-12-31
42	2007-02-28 / 2012-01-31	2012-02-29 / 2013-01-31
43	2007-03-31 / 2012-02-29	2012-03-31 / 2013-02-28
44	2007-04-30 / 2012-03-31	2012-04-30 / 2013-03-31
45	2007-05-31 / 2012-04-30	2012-05-31 / 2013-04-30
46	2007-06-30 / 2012-05-31	2012-06-30 / 2013-05-31
47	2007-07-31 / 2012-06-30	2012-07-31 / 2013-06-30
48	2007-08-31 / 2012-07-31	2012-08-31 / 2013-07-31
49	2007-09-30 / 2012-08-31	2012-09-30 / 2013-08-31
50	2007-10-31 / 2012-09-30	2012-10-31 / 2013-09-30
51	2007-11-30 / 2012-10-31	2012-11-30 / 2013-10-31
52	2007-12-31 / 2012-11-30	2012-12-31 / 2013-11-30
53	2008-01-31 / 2012-12-31	2013-01-31 / 2013-12-31
54	2008-02-29 / 2013-01-31	2013-02-28 / 2014-01-31
55	2008-03-31 / 2013-02-28	2013-03-31 / 2014-02-28
56	2008-04-30 / 2013-03-31	2013-04-30 / 2014-03-31
57	2008-05-31 / 2013-04-30	2013-05-31 / 2014-04-30
58	2008-06-30 / 2013-05-31	2013-06-30 / 2014-05-31
59	2008-07-31 / 2013-06-30	2013-07-31 / 2014-06-30
60	2008-08-31 / 2013-07-31	2013-08-31 / 2014-07-31
61	2008-09-30 / 2013-08-31	2013-09-30 / 2014-08-31
62	2008-10-31 / 2013-09-30	2013-10-31 / 2014-09-30
63	2008-11-30 / 2013-10-31	2013-11-30 / 2014-10-31
64	2008-12-31 / 2013-11-30	2013-12-31 / 2014-11-30
65	2009-01-31 / 2013-12-31	2014-01-31 / 2014-12-31
66	2009-02-28 / 2014-01-31	2014-02-28 / 2015-01-31
67	2009-03-31 / 2014-02-28	2014-03-31 / 2015-02-28
68	2009-04-30 / 2014-03-31	2014-04-30 / 2015-03-31

Continued on next page

Table 1 – continued from previous page

Window	Training Period	Testing Period
69	2009-05-31 / 2014-04-30	2014-05-31 / 2015-04-30
70	2009-06-30 / 2014-05-31	2014-06-30 / 2015-05-31
71	2009-07-31 / 2014-06-30	2014-07-31 / 2015-06-30
72	2009-08-31 / 2014-07-31	2014-08-31 / 2015-07-31
73	2009-09-30 / 2014-08-31	2014-09-30 / 2015-08-31
74	2009-10-31 / 2014-09-30	2014-10-31 / 2015-09-30
75	2009-11-30 / 2014-10-31	2014-11-30 / 2015-10-31
76	2009-12-31 / 2014-11-30	2014-12-31 / 2015-11-30
77	2010-01-31 / 2014-12-31	2015-01-31 / 2015-12-31
78	2010-02-28 / 2015-01-31	2015-02-28 / 2016-01-31
79	2010-03-31 / 2015-02-28	2015-03-31 / 2016-02-29
80	2010-04-30 / 2015-03-31	2015-04-30 / 2016-03-31
81	2010-05-31 / 2015-04-30	2015-05-31 / 2016-04-30
82	2010-06-30 / 2015-05-31	2015-06-30 / 2016-05-31
83	2010-07-31 / 2015-06-30	2015-07-31 / 2016-06-30
84	2010-08-31 / 2015-07-31	2015-08-31 / 2016-07-31
85	2010-09-30 / 2015-08-31	2015-09-30 / 2016-08-31
86	2010-10-31 / 2015-09-30	2015-10-31 / 2016-09-30