Portfolio Theory: Assignment 1

The Statistics of Strategy Back-Testing

Nesan Naidoo : NDXNES005

2025-09-13

Problem Specification

This assignment examined the statistics of strategy backtesting within the context of portfolio theory. Part I focused on proving that the estimated annualized Sharpe ratio(SR) converges asymptotically to a normal distribution. Furthermore, part I motivated that for a sufficiently large number of samples, the mean of the sample maximum of standard normally distributed random variables can be approximated. Lastly, part I focused on the derivation and discussion of the minimum Backtest length.

Part II focused on mean–variance backtesting of the tangency portfolio under full investment and no-short-selling constraints. Sharpe ratio–maximising portfolios are computed from rolling windows of the historical data. The historical dataset is divided into in-sample (IS) and out-of-sample (OOS) datasets. Two experiments are conducted: 1) compared IS and OOS Sharpe ratios and 2) evaluated OOS backtest performance using a rolling window approach.

Data Specification

The Tactical Asset Allocation data is from PT-DATA-ALBI-JIBAR-JSEIND-Daily-1994-2017.xlsx

1. ICB Industrial Level Indices
2. ALBI (All Bond Index (ALBI) Total Return Index (TRI) Data)
3. Money Market Data: JIBAR and STEFI TRI

4. Various Indices: JSE Growth, JSE Value, JSE ALSI, JSE SRI

Configuration control

Version control: managed with Git and GitHub.

Packages: managed with `renv` for reproducibility.

Project structure:

Repository: Nesan Naidoo Github

PART I : Introduction to Strategy Backtesting

Question 1 : Sample Error when Estimating the Sharpe Ratio

Proof based on ()

Assumptions

We assume excess returns r_t are IID normal, $r_t \sim N(\mu, \sigma^2)$. Let q be the number of return periods per year. Let y the number of years.
The total number of observations is $n = qy$.

Sample mean

By the Central Limit Theorem,

$$\sqrt{n}\,(\hat{\mu} - \mu) \ \xrightarrow{d}\ N(0, \sigma^2).$$

Therefore, $\mathrm{Var}(\hat{\mu}) \approx \sigma^2/n$.

Sample standard deviation

For a normal sample,

$$\sqrt{n}\,(\hat{\sigma}^2 - \sigma^2) \ \xrightarrow{d}\ N(0, 2\sigma^4).$$

By applying the delta method with $h(x) = \sqrt{x}$ at $x = \sigma^2$:

$$h'(x) = \tfrac{1}{2\sqrt{x}}, \quad h'(\sigma^2) = \tfrac{1}{2\sigma}.$$

Therefore,

$$\sqrt{n}\,(\hat{\sigma}-\sigma) \;\xrightarrow{d}\; N\Big(0, \tfrac{\sigma^2}{2}\Big).$$

Hence $\mathrm{Var}(\hat{\sigma}) \approx \sigma^2/(2n)$.

Furthermore, for normal data $\hat{\mu}$ and $\hat{\sigma}$ are asymptotically independent.

Sharpe ratio mapping

The annualised Sharpe ratio is $SR = \sqrt{q}\,\frac{\mu}{\sigma}$ The estimated annualised Sharpe Ratio is $\hat{SR} = \sqrt{q}\,\frac{\hat{\mu}}{\hat{\sigma}}$. Let $g(\mu,\sigma) = \sqrt{q}\,\mu/\sigma$ with gradient,

$$\nabla g(\mu,\sigma) = \begin{pmatrix} \frac{\sqrt{q}}{\sigma} \\ -\frac{SR}{\sigma} \end{pmatrix}.$$

The asymptotic covariance of $(\hat{\mu}, \hat{\sigma})$ is

$$\Sigma = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2/2 \end{pmatrix}.$$

By the delta method, $\sqrt{n}\,(\hat{SR} - SR) \;\xrightarrow{d}\; N(0, V)$ where $V = \nabla g^\top \Sigma \nabla g$.

Next,

$$V = \Big(\tfrac{\sqrt{q}}{\sigma}\Big)^2 \sigma^2 + \big(\tfrac{SR}{\sigma}\big)^2 \tfrac{\sigma^2}{2} = q + \tfrac{SR^2}{2}.$$

Asymptotic variance of $\hat{SR}$

Therefore,

$$\sqrt{n}\,(\hat{SR} - SR) \;\xrightarrow{d}\; N(0,\; q + \tfrac{SR^2}{2}).$$

Hence,

$$\mathrm{Var}(\hat{SR}) \;\approx\; \frac{q + \frac{SR^2}{2}}{n}.$$

Substituting $n = qy$:

$$\mathrm{Var}(\hat{SR}) \;\approx\; \frac{q + \frac{SR^2}{2}}{q\,y} \;=\; \frac{1 + \frac{SR^2}{2q}}{y}.$$

Therefore the asymptotic distribution is

$$\hat{SR} \sim N\Big(SR, \frac{1 + \frac{SR^2}{2q}}{y}\Big).$$

Question 2 : The Maximum of the Sample

Question 3 : Minimum Backtest Length

PART II : Backtest Performance of the Tangency Portfolio

Experiment 1 : In-Sample and Out-Of-Sample Sharpe Ratios

Experiment 2 : Out-Of-Sample Backtesting using a Rolling Window

References

Bailey, David H., Jonathan M. Borwein, Marcos López de Prado, and Qiji Jim Zhu. 2014. "Pseudo-Mathematics and Financial Charlatanism: The Effects of Backtest Overfitting on Out-of-Sample Performance." *Notices of the American Mathematical Society* 61 (May): 458. https://doi.org/10.1090/noti1105.

Bailey, David H., and Marcos López de Prado. 2014. "The Deflated Sharpe Ratio:correcting for Selection Bias, BacktestOverfitting, and Non-Normality." *The Journal of Portfolio Management* 40 (September): 94–107. https://doi.org/10.3905/jpm.2014.40.5.094.

Bailey, David, Jonathan Borwein, Marcos López de Prado, and Qiji Jim Zhu. 2016. "The Probability of Backtest Overfitting." *The Journal of Computational Finance*, September. https://doi.org/10.21314/jcf.2016.322.

Embrechts, Paul, Claudia Klüppelberg, and Thomas Mikosch. 1997. *Modelling Extremal Events for Insurance and Finance*. Springer.

Lee, Wai. 2000. *Theory and Methodology of Tactical Asset Allocation*. John Wiley & Sons.

Liu, Ying, Marie Rekkas, and Augustine Wong. 2012. "Inference for the Sharpe Ratio Using a Likelihood-Based Approach." *Journal of Probability and Statistics* 2012: 1–24. https://doi.org/10.1155/2012/878561.

Lo, Andrew W. 2002. "The Statistics of Sharpe Ratios." *Financial Analysts Journal* 58 (July): 36–52. https://doi.org/10.2469/faj.v58.n4.2453.

Resnick, Sidney I. 2008. *Extreme Values, Regular Variation and Point Processes*. Springer, Cop.

Appendix A : Code

Appendix B : Session Information