Portfolio Theory: Assignment 1

The Statistics of Strategy Back-Testing

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Problem Specification

This assignment examined the statistics of strategy backtesting within the context of portfolio theory. Part I focused on proving that the estimated annualized Sharpe ratio(SR) converges asymptotically to a normal distribution. Furthermore, part I motivated that for a sufficiently large number of samples, the mean of the sample maximum of standard normally distributed random variables can be approximated. Lastly, part I focused on the derivation and discussion of the minimum backtest length.

Part II focused on mean–variance backtesting of the tangency portfolio under full investment and no-short-selling constraints. Sharpe ratio–maximising portfolios are computed from rolling windows of the historical data. The historical dataset is divided into in-sample (IS) and out-of-sample (OOS) datasets. Two experiments are conducted: 1) compared IS and OOS Sharpe ratios and 2) evaluated OOS backtest performance using a rolling window approach.

Data Specification

The Tactical Asset Allocation data is from PT-DATA-ALBI-JIBAR-JSEIND-Daily-1994-2017.xlsx

1. ICB Industrial Level Indices
2. ALBI (All Bond Index (ALBI) Total Return Index (TRI) Data)
3. Money Market Data: JIBAR and STEFI TRI

4. Various Indices: JSE Growth, JSE Value, JSE ALSI, JSE SRI

Configuration control

Version control: managed with Git and GitHub.

Packages: managed with `renv` for reproducibility.

Project structure:

Repository: github.com/NesanNaidoo/Portfolio-Theory-Assignment-1-Backtesting

PART I : Introduction to Strategy Backtesting

Question 1 : Sample Error when Estimating the Sharpe Ratio

Proof below based on asymptotic distributions of Sharpe Ratio estimators (Lo 2002) (See Appendix A. IID Returns)

Q1 — Sample error when estimating the Sharpe Ratio

Assumptions

- We assume IID excess returns $r_1, \dots, r_n$ with $r_t \sim \mathcal{N}(\mu, \sigma^2)$.
- Let q be returns per year, y the number of years, and $n = qy$.
- The true annualised Sharpe is $SR = \sqrt{q}\,\frac{\mu}{\sigma}$.
- Estimators used in proof:

$$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} r_t, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{t=1}^{n}(r_t - \hat{\mu})^2, \quad \widehat{SR} = \sqrt{q}\,\frac{\hat{\mu}}{\hat{\sigma}}.$$

Distribution of the sample mean

Since $r_t \sim \mathcal{N}(\mu, \sigma^2)$ and using the fact the the sum of independent Normal variables are Normal. Therefore,

$$\sum_{t=1}^{n} r_t \sim N(n\mu,\ n\sigma^2). \tag{1}$$

Therefore, the sample mean $\hat{\mu} = \dfrac{1}{n} \sum_{t=1}^{n} r_t$ follows a Normal distribution

$$\hat{\mu} \sim N\Big(\mu,\ \frac{\sigma^2}{n}\Big). \tag{2}$$

Hence, by centering and scaling by $\sqrt{n}$, this leads to

$$\sqrt{n}\,(\hat{\mu} - \mu) \sim \mathcal{N}(0, \sigma^2). \tag{3}$$

Distribution of the sample variance

Given $\hat{\sigma}^2 = \frac{1}{n} \sum_{t=1}^{n} (r_t - \hat{\mu})^2$, then the result based on Normal theory is $\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}$.

If $U \sim \chi^2_k$ then $\mathbb{E}[U] = k$ and $\mathrm{Var}(U) = 2k$.

Hence, to standardise U we subtract its mean k and divide by its standard deviation $\sqrt{2k}$. Let $k = n-1$ and $U = \dfrac{n\hat{\sigma}^2}{\sigma^2}$. Using the fact that χ^2_{n-1}, is the sum of $n-1$ independent Z_i^2 terms, where each Z_i^2 has mean 1 and variance 2 and by the Central Limit Theorem , the centered and scaled sum converges to $\mathcal{N}(0,1)$ as $n \to \infty$. Therefore,

$$\frac{\dfrac{n\hat{\sigma}^2}{\sigma^2} - (n-1)}{\sqrt{2(n-1)}} \xrightarrow{d} \mathcal{N}(0,1), \qquad (n \to \infty). \tag{4}$$

Next the numerator and denominator is multiplied by σ^2:

$$\frac{n\hat{\sigma}^2 - (n-1)\sigma^2}{\sigma^2\sqrt{2(n-1)}} \xrightarrow{d} \mathcal{N}(0,1). \tag{5}$$

Then the numerator can be rewritten as $\mathcal{N}(\hat{\sigma}^2 - \sigma^2) + \sigma^2$ and fraction can be split into 2 terms:

$$\frac{\mathcal{N}(\hat{\sigma}^2 - \sigma^2)}{\sigma^2\sqrt{2(n-1)}} + \frac{1}{\sqrt{2(n-1)}} \xrightarrow{d} \mathcal{N}(0,1). \tag{6}$$

Since $\dfrac{1}{\sqrt{2(n-1)}} \to 0$ as $n \to \infty$, therefore

$$\frac{\mathcal{N}(\hat{\sigma}^2 - \sigma^2)}{\sigma^2\sqrt{2(n-1)}} \xrightarrow{d} \mathcal{N}(0,1). \tag{7}$$

Let $A_n = \sqrt{n}(\hat\sigma^2 - \sigma^2)$. Therefore,

$$\frac{\sqrt{n}\,A_n}{\sigma^2\sqrt{2(n-1)}} \xrightarrow{d} \mathcal{N}(0,1). \tag{8}$$

Hence $A_n \xrightarrow{d} N\Big(0, \frac{\sigma^4 2(n-1)}{n}\Big)$, and letting $n \to \infty$ leads to

$$\sqrt{n}(\hat\sigma^2 - \sigma^2) \xrightarrow{d} \mathcal{N}(0, 2\sigma^4). \tag{9}$$

Using Delta method to obtain $\sqrt{n}(\hat\sigma - \sigma)$

Let $h(x) = \sqrt{x}$, so $h'(\sigma^2) = 1/(2\sigma)$. Then by Taylor expansion,

$$\hat\sigma - \sigma \approx h'(\sigma^2), (\hat\sigma^2 - \sigma^2) = 1/(2\sigma)(\hat\sigma^2 - \sigma^2).$$

By multiplying by $\sqrt{n}$:

$$\sqrt{n}(\hat\sigma - \sigma) \approx \frac{\sqrt{n}}{2\sigma}(\hat\sigma^2 - \sigma^2)$$

Using (9):

$$\sqrt{n}(\hat\sigma - \sigma) \xrightarrow{d} N\Big(0, \Big(\frac{1}{2\sigma}\Big)^2 \cdot 2\sigma^4\Big) = N\Big(0, \frac{\sigma^2}{2}\Big).$$

Combining results so far

$$\sqrt{n}\begin{pmatrix}\hat\mu - \mu \\ \hat\sigma - \sigma\end{pmatrix} \xrightarrow{d} N\Big(0,\ \Sigma\Big) \qquad \text{where} \qquad \Sigma = \begin{pmatrix}\sigma^2 & 0 \\ 0 & \sigma^2/2\end{pmatrix}.$$

Using multivariate delta method

Let $g(\mu, \sigma) = \sqrt{q}\,\mu/\sigma = SR$. Then the Taylor expansion leads to

$$\sqrt{n}(g(\hat\mu, \hat\sigma) - g(\mu, \sigma)) = \nabla g(\mu, \sigma)^\top \sqrt{n}\begin{pmatrix}\hat\mu - \mu \\ \hat\sigma - \sigma\end{pmatrix} + o_p(1).$$

Thus by the Central Limit Theorem,

$$\sqrt{n}(\widehat{SR} - SR) \xrightarrow{d} N\Big(0, \nabla g^\top \Sigma \nabla g\Big).$$

Calculating the gradient and variance

Partial derivatives:

$$\frac{\partial g}{\partial \mu} = \frac{\sqrt{q}}{\sigma} \qquad \text{and} \qquad \frac{\partial g}{\partial \sigma} = -\frac{SR}{\sigma},$$

so $\nabla g = (\sqrt{q}/\sigma,\ -SR/\sigma)^\top$. Then

$$V = \nabla g^\top \Sigma \nabla g = \left(\frac{\sqrt{q}}{\sigma}\right)^2 \sigma^2 + \left(\frac{SR}{\sigma}\right)^2 \frac{\sigma^2}{2} = q + \frac{SR^2}{2}.$$

Hence,

$$\sqrt{n}(\widehat{SR} - SR) \xrightarrow{d} N\left(0,\ q + \frac{SR^2}{2}\right).$$

Converting to variance per year

Therefore,

$$\mathrm{Var}(\widehat{SR}) \approx \frac{q + \frac{SR^2}{2}}{n}.$$

By dividing numerator and denominator by q:

$$\frac{q + \frac{SR^2}{2}}{n} = \frac{1 + \frac{SR^2}{2q}}{n/q} = \frac{1 + \frac{SR^2}{2q}}{y}.$$

Since $n/q = y$. Thus the final result is

$$\widehat{SR} \xrightarrow{d} N\left(SR,\ \frac{1 + \frac{SR^2}{2q}}{y}\right).$$

Question 2 : The Maximum of the Sample

Proof below based on (D. H. Bailey and López de Prado 2014) (See Appendix: Proof of proposition 1) ,(Embrechts, Klüppelberg, and Mikosch 1997) (pp.138–147) and (Resnick 2008) (see Proposition 2.1(iii))

Question 3 : Minimum Backtest Length

PART II : Backtest Performance of the Tangency Portfolio

Experiment 1 : In-Sample and Out-Of-Sample Sharpe Ratios

Experiment 2 : Out-Of-Sample Backtesting using a Rolling Window

References

Bailey, David H., and Marcos López de Prado. 2014. "The Deflated Sharpe Ratio:correcting for Selection Bias, BacktestOverfitting, and Non-Normality." *The Journal of Portfolio Management* 40 (September): 94–107. https://doi.org/10.3905/jpm.2014.40.5.094.

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Resnick, Sidney I. 2008. *Extreme Values, Regular Variation and Point Processes*. Springer, Cop.

Appendix A : Code

Appendix B : Session Information