Gromov–Hausdorff distance between clouds of special type

Calculating distance to cloud of bounded metric spaces

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Gromov–Hausdorff distance

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Clouds

The Gromov–Hausdorff distance is a generalized pseudometric on the class of all metric spaces.
Example: $d_{GH}(\Delta_1, \mathbb{R}) = \infty$.
Clouds: all metric spaces that are at a finite distance from each other. $[X]$ — cloud containing X.

Cloud of bounded metric spaces

$$|X, \Delta_1| = \frac{1}{2} \operatorname{diam} X$$

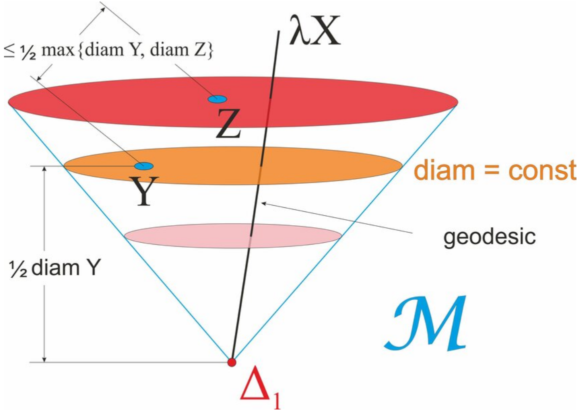
Cloud $[\Delta_1]$ — cloud of all <u>bounded</u> metric spaces.
Ultrametric inequality:

$$|X, Y| \leq \max \{|X, \Delta_1|, |Y, \Delta_1|\}$$

Geodesic:

$$|\lambda X, \mu X| = |\lambda - \mu| |X, \Delta_1|$$

Cloud of bounded metric spaces



Stabilizer group

Multiplying spaces by $\lambda > 0$ can produce interesting results. The distance $|X, \lambda X|$ can be infinite. For example, if X is a geometric progression.
Stabilizer group: all $\lambda > 0$ such that $[X] = [\lambda X]$.
Trivial: $\mathrm{St}\,([X]) = \{1\}$.
Center: metric space X, such that $X = \lambda X$
for all $\lambda \in \mathrm{St}([X])$.

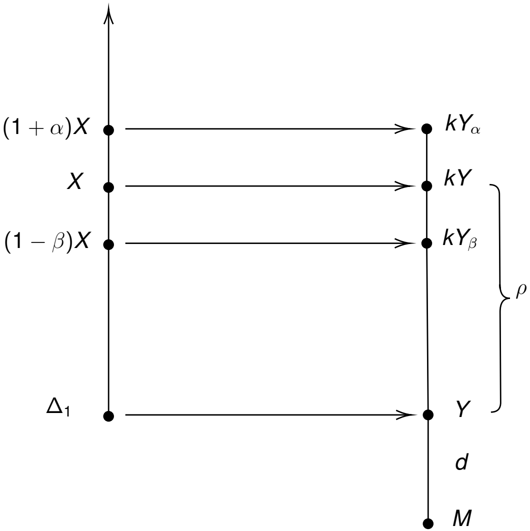
Gromov–Hausdorff distance between clouds

Metric spaces are sets by definition.
Clouds are not sets, they are proper classes. They contain sets of arbitrary cardinality.
Definition of Gromov–Hausdorff distance is modified.

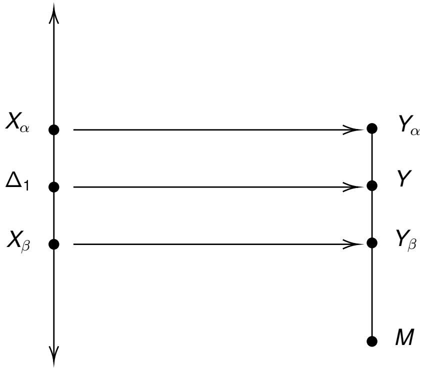
Center image Theorem

Clouds $[M], [\Delta_1]$, $[M]$ has a nontrivial stabilizer and M — center. R — correspondence between them, $\operatorname{dis} R = \epsilon < \infty$. Then $\left|R(\Delta_1), M\right| \leq 2\epsilon$.

Center image Theorem



Center image Theorem



Distance theorem

A cloud has a nontrivial stabilizer, and there are two spaces which break the ultrametric inequality. Then it's distance to $[\Delta_1]$ is infinite.
M is the center of $[M]$, Y_1 and Y_2 are such metric spaces that

$$|Y_1, M| \leq \rho,$$
$$|Y_1, M| \leq \rho,$$
$$|Y_1, Y_2| > \rho.$$

Distance Theorem

$$X_1 = R^{-1}(Y_1), X_2 = R^{-1}(Y_2)$$