

Hydrogen Spectrum

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1 Introduction

In an atom, as electrons jump from higher energy to lower energy states they emit photons with specific wavelengths, which produce an emission spectrum. The energy associated with the photons correlates with the difference in energy of those states. Since the energy of each level n is fixed, a specific electron transition always releases a photon with the same wavelength (and energy) defined by Rydberg's formula. This model roots in Planck's effort to describe why objects glow as they are heated, introducing the concept of quantization.

2 Setup

We apply an electric discharge to a Hydrogen-Deuterium mixture which excites the electrons in the gas. The excitation states last for a very short period of time before the electrons transition back to their respective energy state. During this transition electrons release photons varying in energy and wavelength (ultraviolet, visible, infrared, etc). Using a diffraction grating we separate the light emitted into its component wavelengths (spectral lines), creating the emission spectrum of Hydrogen.



Figure 1: Hydrogen spectrum setup equipment.

As light leaves the Hydrogen lamp, it passes through a single slit and then through a collimating lens that aligns the light parallel to the surface of the diffraction grating. Light then reaches the diffraction grating where the different wavelengths reflect off the grating surface at different angles which we will observe through the eye piece of the telescoping lens as we rotate it (Figure 2).

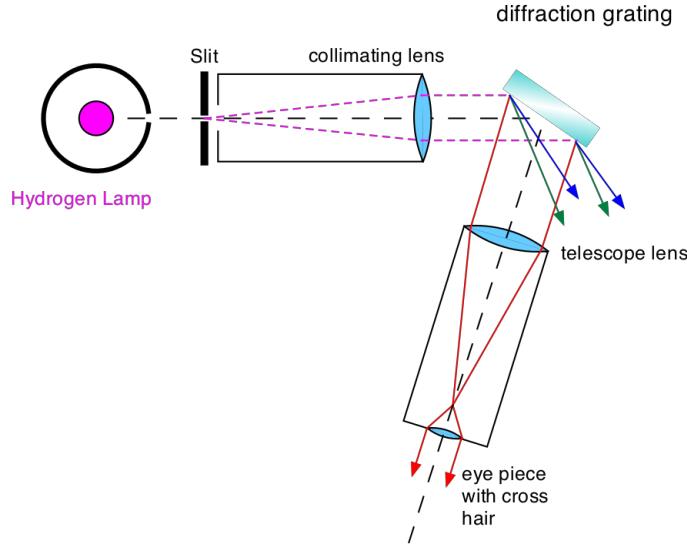


Figure 2: Diffraction grating spectroscope.

This diffraction grating, also known as a blazed grating, consists of a numerous collection of small, parallel, triangular ridges on its surface separated by a fixed length D (Figure 3). All light is incident at the same angle θ_{in} and different wavelengths diffract at an angle θ_{out} with respect to the grating surface.

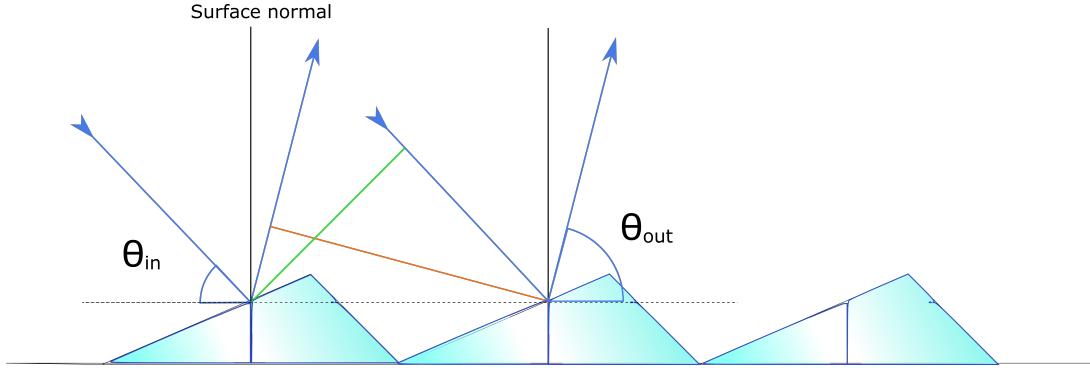


Figure 3: Grating Surface.

The path length difference between the incident light rays Δ_1 and between the diffracted light rays Δ_2 (Figure 4) are

$$\Delta_1 = D \cos(\theta_{in}) \quad \text{and} \quad \Delta_2 = D \cos(\theta_{out})$$

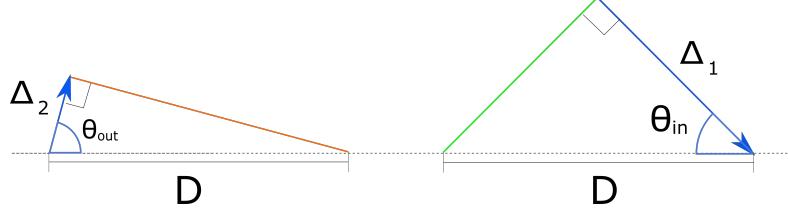


Figure 4: Path lengths of incident and diffracted light.

Therefore the path length difference between incident and diffracted light rays is $\Delta_1 - \Delta_2$. By the grating equation, we have

$$\Delta_1 - \Delta_2 = \Delta_{tot} = m_d\lambda \quad (1)$$

where λ corresponds to the wavelength of diffracted light, and m_d is an integer describing the diffraction order. If $\theta_{in} = \theta_{out}$, $m_d = 0$. If $\theta_{in} > \theta_{out}$, $m_d < 0$. And if $\theta_{in} < \theta_{out}$, $m_d > 0$.

We take our measurements relative to an arbitrary angle θ_a corresponding to the face-to-face configuration of the lenses, which define the optical axis (horizontal dashed line in Figure 5). We next place the diffraction grating at an angle of about 40 degrees with respect to the optical axis and rotate the telescoping lenses until we find the reflection of the incident light; that is, when $\theta_{in} = \theta_{out}$.

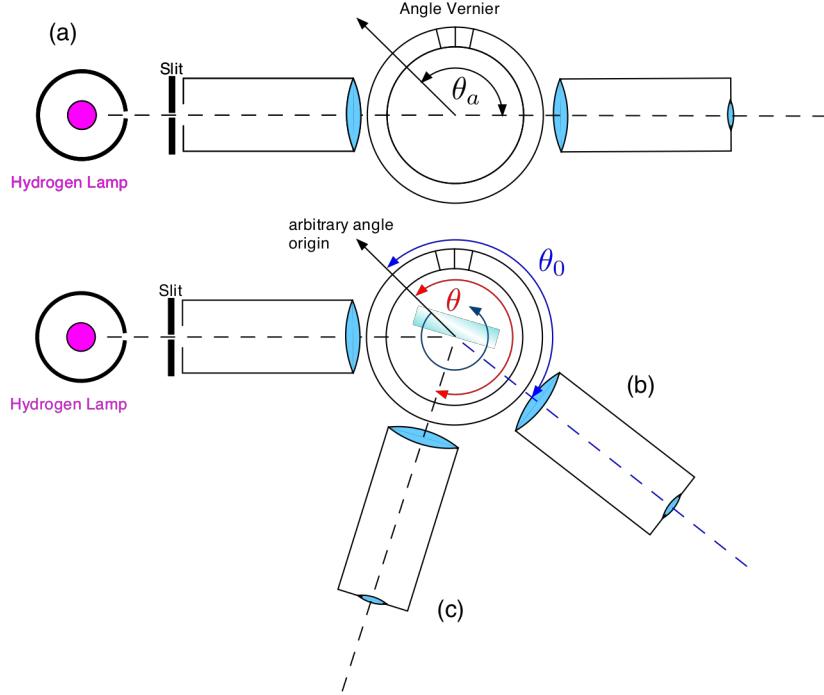


Figure 5: (a) Spectroscope alignment. (b) 0th order angle, θ_0 . (c) Spectral angle.

We define this angle as θ_0 measured relative to θ_a . Now we are ready to further rotate the telescoping lens to observe the spectral lines each coinciding with an angle θ .

From this arrangement, the relationships among θ_{in} , θ_{out} , θ_a , and θ_0 are

$$\theta_{in} = \frac{\theta_0 - \theta_a}{2} \quad (2)$$

$$\theta_{out} = \theta - (\theta_a + \theta_{in}) = \theta - \frac{\theta_a + \theta_0}{2} \quad (3)$$

Theoretical Derivation

Consider the atomic model of Hydrogen. We apply the next assumptions postulated by Bohr¹:

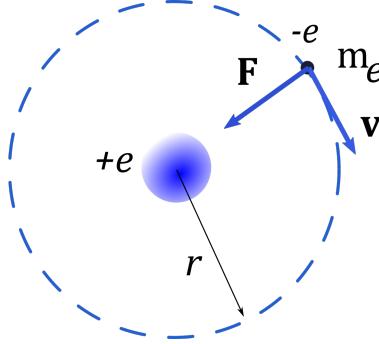


Figure 6: Bohr's atomic model for Hydrogen.

1. The electron moves in circular orbits around the proton, subjected by a Coulomb attractive force.
2. Only certain number of these orbits are stable; that is, the energy of such orbits is fixed in time.
3. If a electron is to transition from one orbit to another, it must absorb or release quantized amounts of energy, each corresponding to specific transition levels. In the case of an electron jumping from a higher energy level E_1 to a lower energy level E_2 , it must release a photon of wavelength

$$\frac{hc}{\lambda} = E_1 - E_2 = \Delta E_\lambda \quad (4)$$

where h is Planck's constant.

4. The radii r of the orbits is constrained by the quantization of the electron's angular momentum. That is, the angular momentum of the electron must be an integer multiple of $\hbar = h/2\pi$:

$$|\mathbf{r} \times m_e \mathbf{v}| = m_e v r = n\hbar \quad (5)$$

where m_e and v are the mass and velocity of the electron.

The potential energy of the electron is

$$U = k \frac{(+e)(-e)}{r} = -k \frac{e^2}{r}$$

where k is Coulomb's constant. Since the electron moves in a circular orbit, we have

$$F = k \frac{|(+e)(-e)|}{r^2} = \frac{m_e v^2}{r} \quad (6)$$

$$\frac{ke^2}{2r} = \frac{m_e v^2}{2} \quad (7)$$

$$\frac{ke^2}{2r} = K \quad (8)$$

The total energy of this system is therefore

$$\begin{aligned} E &= K + U \\ &= \frac{ke^2}{2r} - \frac{ke^2}{r} \\ &= -\frac{ke^2}{2r} \end{aligned} \quad (9)$$

We can combine Equations 5 and 6 to get an expression for the radius corresponding to the n th energy level:

$$\begin{aligned} \frac{ke^2}{2r} &= \frac{m_e v^2}{2} \\ &= \frac{m_e}{2} \left(\frac{n\hbar}{m_e r} \right)^2 \\ r_n &= \frac{n^2 \hbar^2}{m_e k e^2}, \quad n = 1, 2, 3, \dots \end{aligned}$$

and we define the ground state radius ($n = 1$), also known as the Bohr radius a_B , as

$$a_B = \frac{\hbar^2}{m_e k e^2} = 0.0529 \text{ nm}$$

Therefore, the energy of the n th energy level of the Hydrogen electron is

$$E_n = -\frac{ke^2}{2a_B} \frac{1}{n^2}. \quad (10)$$

By using $\Delta E_\lambda = hc/\lambda$, we can establish a relation between two energy levels n_1 and n_2 ($n_2 > n_1$), and the wavelength of light emitted:

$$\begin{aligned} \frac{hc}{\lambda} &= \frac{ke^2}{2a_B} \left[\frac{1}{n_1^2} - \frac{1}{n_2^2} \right] \\ \frac{1}{\lambda} &= R_H \left[\frac{1}{n_1^2} - \frac{1}{n_2^2} \right] \end{aligned} \quad (11)$$

where $R_H = ke^2/2a_B hc$ is known as Rydberg's constant.

3 Data

We are able to measure seven different diffraction angles, four in first order and three in second order. We tabulate the data below:

1 st Order Diffraction		
Color	θ (degrees)	σ_θ (degrees)
Violet	89.2	0.5
Blue-violet	87.5	0.5
Blue-green	84.0	0.5
Red	71.9	0.5
2 nd Order Diffraction		
Violet	60.3	0.5
Blue-violet	57.5	0.5
Blue-green	50.0	0.5

Table 1: Spectral Angles.

Other Measured Quantities

Name	Variable	Value
Origin angle	θ_a	132.7°
0th order angle	θ_0	173.9°
Uncertainty in origin angle	σ_{θ_a}	1.0°
Uncertainty in 0th order angle	σ_{θ_0}	1.0°
Number of ridges	N	30,480
Ridge separation	D	1/1,200 mm

4 Analysis

4.1 Error, and Uncertainty

To calculate the uncertainty in the wavelength λ , we first need the uncertainties in Δ_1 and Δ_2 :

$$\sigma_{\Delta_1} = -D \sin(\theta_{in}) \sigma_{\theta_{in}} \quad \text{and} \quad \sigma_{\Delta_2} = -D \sin(\theta_{out}) \sigma_{\theta_{out}}$$

Using Equations 2 and 3, the uncertainties in θ_{in} and θ_{out} are

$$\begin{aligned} \sigma_{\theta_{in}} &= \sqrt{\left(\sigma_{\theta_a} \frac{\partial \theta_{in}}{\partial \theta_a}\right)^2 + \left(\sigma_{\theta_0} \frac{\partial \theta_{in}}{\partial \theta_0}\right)^2} \\ \sigma_{\theta_{out}} &= \sqrt{\left(\sigma_{\theta_a} \frac{\partial \theta_{out}}{\partial \theta_a}\right)^2 + \left(\sigma_{\theta_0} \frac{\partial \theta_{out}}{\partial \theta_0}\right)^2 + \left(\sigma_{\theta_0} \frac{\partial \theta_{out}}{\partial \theta}\right)^2} \end{aligned}$$

where

$$\frac{\partial \theta_{in}}{\partial \theta_a} = \frac{\partial \theta_{in}}{\partial \theta_0} = -\frac{\partial \theta_{out}}{\partial \theta_a} = -\frac{\partial \theta_{out}}{\partial \theta_0} = \frac{1}{2} \quad \text{and} \quad \frac{\partial \theta_{out}}{\partial \theta} = 1$$

Therefore,

$$\sigma_{\theta_{in}} = \frac{1}{2} \sqrt{\sigma_{\theta_a}^2 + \sigma_{\theta_0}^2}$$

$$\sigma_{\theta_{out}} = \sqrt{\sigma_\theta^2 + \frac{1}{4}(\sigma_{\theta_a}^2 + \sigma_{\theta_0}^2)}$$

Then the uncertainty in λ is

$$\sigma_\lambda = \frac{1}{m_d} \sqrt{\sigma_{\Delta_1}^2 + \sigma_{\Delta_2}^2} = \frac{D}{m_d} \sqrt{\sin^2(\theta_{in}) \sigma_{\theta_{in}}^2 + \sin^2(\theta_{out}) \sigma_{\theta_{out}}^2} \quad (12)$$

Using Equation 4, we calculate the uncertainty in the energy of photons of wavelength λ by

$$\sigma_{E_\lambda} = \frac{hc}{\lambda^2} \sigma_\lambda. \quad (13)$$

4.2 Results

Based on the previous discussion, we compute the values for the wavelengths λ and the difference in energy of two levels E_λ , namely the energy of a photon emitted from the transition from the third, fourth, fifth, and sixth levels back to the second level. We used Equations 1, 2, and 3 to calculate the results for λ and Equation 4 for E_λ .

1 st Order Diffraction			
λ (nm)	σ_λ (m)	E_λ (eV)	σ_{E_λ} (eV)
416	11.9	2.98	0.0718
438	12.0	2.83	0.0657
485	12.3	2.55	0.0544
655	13.0	1.89	0.0314
2 nd Order Diffraction			
412	6.54	3.01	0.0478
432	6.52	2.87	0.0433
486	6.39	2.55	0.0336

Table 2: Wavelength and Energy of Photons Emitted

These data promptly correlate to the Balmer series.

The published values for the wavelengths emission lines of hydrogen are:

Color	λ^* (nm)
Red	410.1740
Blue-green	434.0462
Blue-violet	486.1362
Violet	656.4538

Table 3: Accepted wavelength values for the emission spectrum of Hydrogen².

We use Equation 10 to calculate the theoretical energy of the emitted photons, E_λ^* , as they transition from the third, forth, fifth, and sixth levels back to the second level.

Level Transition	E_λ^* (eV)
$6 \rightarrow 2$	3.02499
$5 \rightarrow 2$	2.85862
$4 \rightarrow 2$	2.55234
$3 \rightarrow 2$	1.89062

Table 4: Theoretical energy of photons.

We compare the published and theoretical results in Figures 7 and 8.

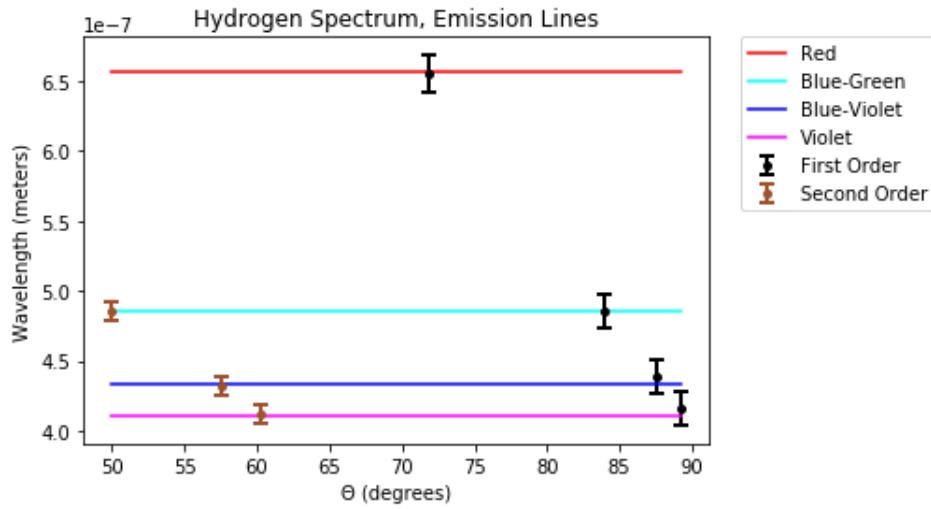


Figure 7: Emission lines of Hydrogen. Each color represents the published values λ^* .

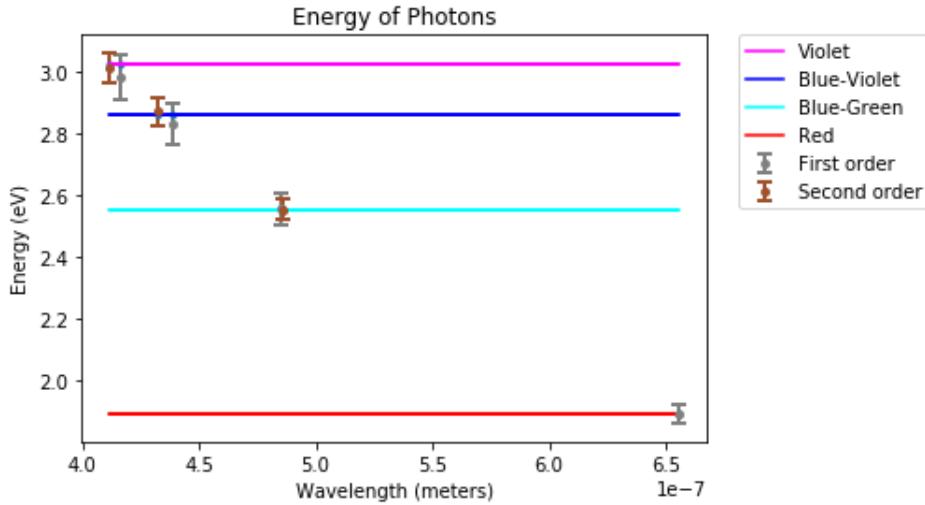


Figure 8: Energy of Photons. Each color represents the theoretical prediction of each energy E_λ .

Lastly, we utilize Equation 11 to plot a graph and determine the numerical value of Rydberg's constant using a linear fit. However, we also account for the set of points provided by setting the lower energy levels $n_1 = 1$ and $n_1 = 3$. We plot three sets for $n_1 = 1$, two sets for $n_1 = 2$, and two sets for $n_1 = 3$.

The published value of Rydberg's constant is

$$R_H^* = 10973731.6 \text{ m}^{-1}$$

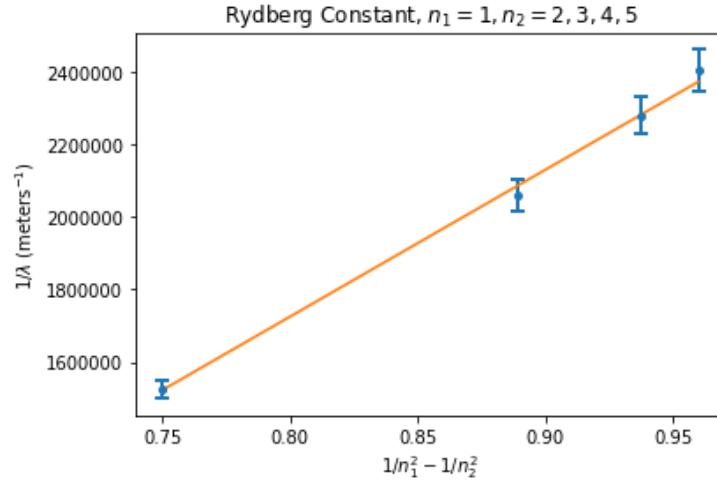


Figure 9: Transition to 1st level, first set. $R_H = 4048894.7 \pm 219121.3$

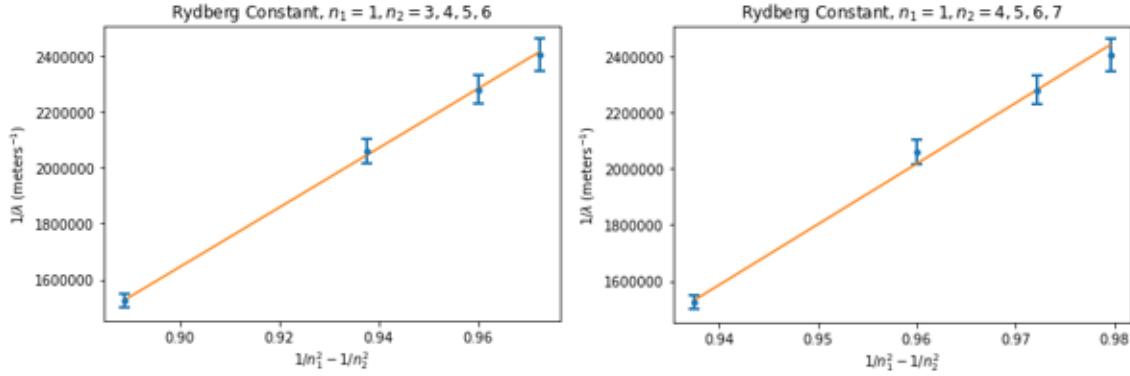


Figure 10: Transition to 1st level, second and third sets;

$$R_H = 10644536.8 \pm 575677.4, R_H = 21596897.8 \pm 1170112.9$$

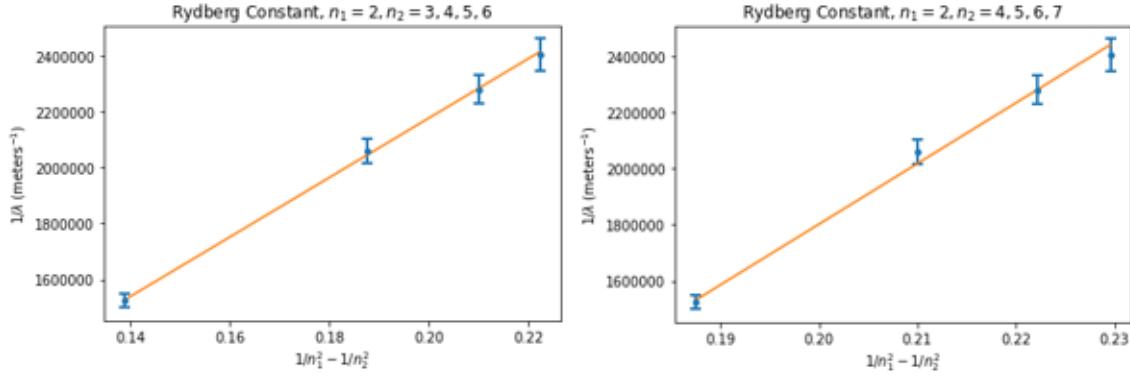


Figure 11: Transition to 2nd level, first and second sets;

$$R_H = 10644536.8 \pm 575677.4, R_H = 21596897.8 \pm 1170112.9$$

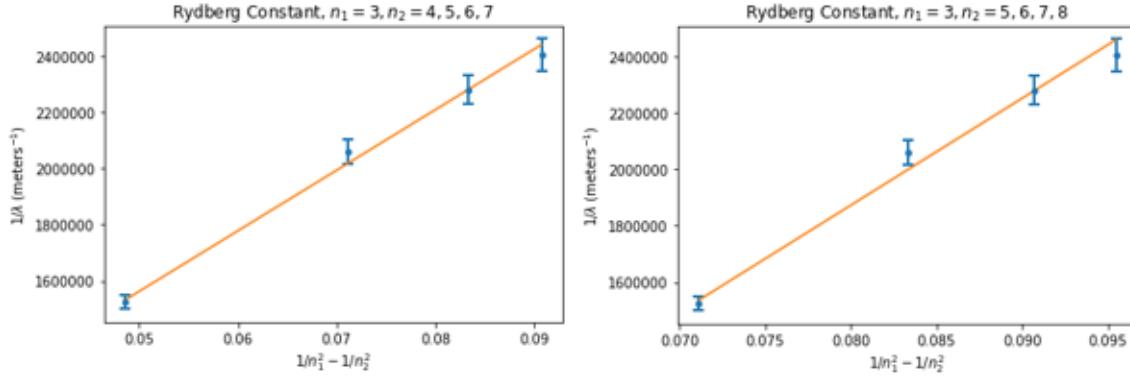


Figure 12: Transition to 3rd level, first and second sets;

$$R_H = 21596897.8 \pm 1170112.9, R_H = 37880586.8 \pm 2057272.5$$

5 Conclusion

According to Table 2 and our resulting plots, all our measurements lie within one standard deviation respectively. As a rough estimate, we can determine the average number of standard deviations in wavelength and energy as

$$\frac{1}{n} \sum^n \frac{|\lambda^* - \lambda|}{\sigma_\lambda} \quad \text{and} \quad \frac{1}{n} \sum^n \frac{|E_\lambda^* - E_\lambda|}{\sigma_{E_\lambda}}$$

where n is the number of data points. We obtain an average number of 0.220 standard deviations for λ and 0.245 standard deviations for E_λ .

As expected, we observe a linear relationship between λ and $1/n_2^2$. Figures 9, 10, 11, and 12 reveal that the best estimate for Rydberg's constant are the sets ($n_1 = 1, n_2 = 3, 4, 5, 6$) and ($n_1 = 2, n_2 = 3, 4, 5, 6$), with value $R_H = (106 \pm 5) \times 10^5 \text{ m}^{-1}$. Our result corresponds to 0.572 standard deviations away from the published value.

6 References

- [1] Serway, Raymond A., et al. Modern Physics. Thomson Brooks/Cole, 2004.
- [2] <https://physics.nist.gov/PhysRefData/Handbook/Tables/hydrogentable2.htm>