ASSIGNMENT

INSTRUCTIONS

Deadline: May 10, 2020, by 18:00.
Number of students per submission: At most 2 students.

- You are not to consult or discuss in any way shape or form anything related to this assignment with any person that is not your sole partner for this specific assignment.
- Any references you find and use in the literature or the web you are obligated to report in full following the style of the course notes.

PROBLEMS

PROBLEM 1. Prove that the following IP

$$\begin{aligned} \min \quad & \boldsymbol{c}^\intercal \boldsymbol{x} \\ \text{subject to} \quad & \\ & \sum_{e \in E_G(S)} \boldsymbol{x}_e \leq |S| - 1,\ \forall S \subseteq V(G) \\ & \sum_{e \in E(G)} \boldsymbol{x}_e = v(G) - 1 \\ & \boldsymbol{x}_e \in \{0, 1\}\ \forall e \in E(G). \end{aligned}$$

captures the minimum spanning tree problem for a connected graph G and a function $\boldsymbol{c} : E(G) \to \mathbb{R}_{\geq 0}$.

PROBLEM 2. Two LP relaxations are proposed for the IP seen in PROBLEM 1.

$$\begin{aligned} \min \quad & \boldsymbol{c}^\intercal \boldsymbol{x} \\ \text{subject to} \quad & \\ & \sum_{e \in E_G(S)} \boldsymbol{x}_e \leq |S| - 1,\ \forall S \subseteq V(G) \\ & \sum_{e \in E(G)} \boldsymbol{x}_e = v(G) - 1 \\ & \boldsymbol{x}_e \geq 0,\ \forall e \in E(G) \end{aligned} \tag{P1.MST}$$

and

$$\begin{aligned} \min \quad & \boldsymbol{c}^\intercal \boldsymbol{x} \\ \text{subject to} \quad & \\ & \sum_{e \in A} \boldsymbol{x}_e \leq v(G) - \kappa(A),\ \forall A \subsetneq E(G) \\ & \sum_{e \in E(G)} \boldsymbol{x}_e = v(G) - 1 \\ & \boldsymbol{x}_e \geq 0,\ \forall e \in E(G), \end{aligned} \tag{P2.MST}$$

where $\kappa(A)$ denotes the number of connected components in $G[A] := (V, A)$.

Prove that these two LPs are *equivalent* in the sense that every feasible solution of one is a feasible solution of the other.

PROBLEM 3. Upon replacing the objective function of the LP (P2.MST) with $\min -\boldsymbol{c}^{\intercal}\boldsymbol{x}$, use the recipe for taking duals in order to explain why the following LP

$$\begin{aligned} \min \quad & \sum_{A \subseteq E} (v(G) - \kappa(A))\boldsymbol{y}_A \\ \text{subject to} \quad & \\ & \sum_{\substack{A: e \in A \\ A \subseteq E(G)}} \boldsymbol{y}_A \geq -\boldsymbol{c}(e), \ \forall e \in E(G) \\ & \boldsymbol{y}_A \geq 0, \ \forall A \subsetneq E(G) \\ & \boldsymbol{y}_E \in \mathbb{R} \end{aligned} \tag{D.MST}$$

is the dual of the LP (P2.MST).

PROBLEM 4. Kruskal's algorithm is listed in Algorithm 0.1.

Algorithm 0.1. Kruskal's algorithm

INPUT: A connected graph $G = (V, E)$ and a non-negative cost function $\boldsymbol{c} : E(G) \to \mathbb{R}_{\geq 0}$.
OUTPUT: A minimum cost spanning tree.

```
Sort the members of E in non-decreasing order, i.e., e_1, e_2, ..., e_|E| such that c(e_1) ≤ c(e_2) ≤ ··· ≤ c(e_|E|).
Set F = ∅.
for i = 1 to |E| do
    if F ∪ {e_i} does not contain a cycle then
        Set F := F ∪ {e_i}.
    end if
end for
```

In this problem you are to use the LP (D.MST) in order to show that the solution $\boldsymbol{x}^*$ defined by Kruskal's algorithm (viewed as an integer vector and as a feasible solution for (P2.MST)) is optimal. We help you out along the way as follows. The aim is to use complementary slackness to do this verification. In particular, we shall momentarily define a prospective feasible dual solution, namely $\boldsymbol{y}^*$, for (D.MST) and then you will be asked to verify that $\boldsymbol{y}^*$ is feasible and that $\boldsymbol{y}^*$ and $\boldsymbol{x}^*$ satisfy complimentary slackness conditions. More specifically, after $\boldsymbol{y}^*$ is defined and you have shown it to be feasible you are to verify the following.

$$\text{if } \boldsymbol{x}^*_e > 0 \text{ for some } e \in E(G), \text{ then } \sum_{\substack{A: e \in A \\ A \subseteq E}} \boldsymbol{y}^*_A = -\boldsymbol{c}(e) \tag{0.1}$$

and

$$\text{if } \boldsymbol{y}^*_A > 0 \text{ for some } A \subseteq E(G), \text{ then } \sum_{e \in A} \boldsymbol{x}^*_e = v(G) - \kappa(A). \tag{0.2}$$

Moreover you also need to verify that whatever value $\boldsymbol{y}^*_E$, the unrestricted dual variable, receives, that its primal constraint is satisfied with $\boldsymbol{x}^*$.

As promised, we supply the definition of $\boldsymbol{y}^*$ to you. Let $e_1, \ldots, e_{e(G)}$ denote the ordering imposed by Kruskal's algorithm on the edges of G. For $i \in [e(G)]$, set $R_i := \{e_1, \ldots, e_i\}$. Define $\boldsymbol{y}^*$ as follows.

- Set $\boldsymbol{y}^*_{R_i} := \boldsymbol{c}(e_{i+1}) - \boldsymbol{c}(e_i)$ whenever $i \in [1, e(G) - 1]$.
- Set $\boldsymbol{y}^*_{R_{e(G)}} := -\boldsymbol{c}(e_{e(G)})$. Note that $R_{e(G)} = E(G)$.
- Set $\boldsymbol{y}^*_A := 0$ for all other subsets $A \subsetneq E(G)$ not considered above.

Complete the following.

1. Verify that $\boldsymbol{y}^*$ is feasible for (D.MST).
2. Verify complementary slackness conditions as explained above.