ASSIGNMENT

INSTRUCTIONS

Deadline: Original deadline is April 18, 2020, by 18:00. Due to Passover, this deadline is postponed up until April 20, 2020, by 18:00.
Number of students per submission: At most 2 students.

- You are not to consult or discuss in any way shape or form anything related to this assignment with any person that is not your sole partner for this specific assignment.
- Any references you find and use in the literature or the web you are obligated to report in full following the style of the course notes.

PROBLEMS

The following problem shows that the task of determining the chromatic number of a graph can be reduced to determining the chromatic number of its blocks.

PROBLEM 1. Let G be a graph. Prove that

$$\chi(G) = \max_{B \in \mathcal{B}(G)} \chi(B),$$

where $\mathcal{B}(G)$ denotes the set of blocks of G.

All upper bounds we have attained for $\chi(G)$ were through the greedy colouring algorithm. The proof of Brooks' theorem proceeds by cleverly feeding well-chosen orderings of the vertices of the graph in order to attain its result. The next problem asks you to prove that there are so called 'optimal' orderings.

PROBLEM 2. Prove that for every graph G there exists an ordering of its vertices such that if the greedy colouring algorithm is ran over the latter, then a legal vertex-colouring of G using $\chi(G)$ colours is attained.

A vertex ordering of a graph G, namely $v_1, \ldots, v_n$, is said to be *elegant* if for every $2 \leq i \in [n]$, it holds that $G[N_G(v_i) \cap \{v_1, \ldots, v_{i-1}\}]$ is complete. Not every graph has an elegant ordering of its vertices. Let us focus on the ones that do.

PROBLEM 3. Suppose a graph G admits an elegant ordering of its vertices, and suppose further that such an ordering is given. Prove that the greedy colouring algorithm applied to this ordering outputs a legal colouring of G using $\chi(G)$ colours.

Unfortunately, the class of graphs admitting an elegant ordering is quite restricted.

PROBLEM 4. Prove that any graph that has an elegant ordering of its vertices also has the property that it contains no induced cycles of length at least 4. That is, any cycle of such a graph of length 4 and above must admit an edge connecting two vertices on the cycle yet not being part of the cycle itself.