Assignment 3 solution

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Problem 1:

Let $n = max_{B \in \mathcal{B}(G)} \chi(B)$. We want to show that $n = \chi(G)$. Obviously $n \leq \chi(G)$, we want to show that $n \geq \chi(G)$ by coloring G using n colors, and we will do it like that:

1. : Color each block in n colors (separately).
2. : Choose a random Block B and look at his neighbours.
3. : For each block C that the common vertex of B and C, v doesn't have the same color in B and in C:
4. : define $vb =$ the color of v in B, $vc =$ the color of v in C.
5. : swap colors of all C vertexes between vb and vc.
6. : Go back to step 3 for C.

The blocks graph is a tree, so we have no cycles and eventually the process will end and we will get a legal coloring in n colors. ■

Problem 2:

Let $\varphi : V(G) \to [\chi(G)]$ be a coloring function of G. Define an order R using φ - $v \leq_R u \Leftrightarrow \varphi(v) \leq \varphi(u)$.
We want to show that using R, the greedy algorithm will color G leagally using $\chi(G)$ colors.
We'll show that the greedy algorithm will color each vertex v, in color $\leq \varphi(v)$ in induction on $\varphi(v)$.
$\varphi(v) = 1$: φ is a legal coloring so v has no neighbor u $s.t.$ $\varphi(u) = 1$, therefore the algorithm will color v in 1;
$\varphi(v) = k$: φ is a legal coloring so v has no neighbor u $s.t.$ $\varphi(u) = k$, and by the inductive hypothesis, no vertex colored in k in previous steps, therefore the algorithm will color v in k or less ■;

Problem 3:

Let G be a graph that admits an elegant ordering of its vertices, and let R be such an ordering, and let k be the colors amount that the greedy coloring algorithm applied to this ordering.
We want to show that $k = \chi(G)$. Let $v \in V(G)$ $s.t.$ *the greedy algorithm colored* v *in* k. v has at least $k-1$ neighbours that are before v in R, otherwise the algorithm would color v in less then k. R is an elegant ordering so those $k-1$ vertices are complete graph and they can't be paint in less then $k-1$ colors, therefore $\{v\} \cup N_G(v)$ can't be coloring in less then k colors ■

Problem 4:

Let G be a graph that admits an elegant ordering of its vertices, let R be such an ordering, and let C be a cycle in G of length at least 4. Let $v \in V(C)$ s.t. v is maximal in R. Let x, y be the neighbours of v in C. $x, y <_R v \wedge x, y \in N_G(v)$ so they are part of a complete graph, that mean the edge $xy \in E(G)$. $xy \notin E(C)$ because if so, C length $= 3$, but we now that it ≥ 4. ■