ASSIGNMENT

INSTRUCTIONS

Deadline: June 12th, 2020, by 18:00.
Number of students per submission: At most 2 students.

- You are not to consult or discuss in any way shape or form anything related to this assignment with any person that is not your sole partner for this specific assignment.
- Any references you find and use in the literature or the web you are obligated to report in full following the style of the course notes.

PROBLEMS

PROBLEM 1. Let G be a graph and let $w := V(G) \to \mathbb{R}_{>0}$ be a weight function set on its vertices. The *minimum weight vertex-cover* problem calls for finding a vertex-cover C of G minimising $w(C) := \sum_{v \in C} w(v)$.

Adapt Algorithm 7.7.17 from the course notes, in order to devise a 2-approximation algorithm for the minimum weight vertex-cover problem. In your answer you are to include all the stages seen for Algorithm ? in the notes.

PROBLEM 2. Given a graph $G := (V, E)$ and a set of *terminal* pairs, namely $\mathcal{T} := \{(s_1, t_1), \ldots, (s_k, t_k)\}$, we seek to find the largest possible number of edge-disjoint paths connecting prescribed terminal pairs, i.e. terminal s_i must be connected to its counterpart t_i. We refer to such a system of edge-disjoint paths as a *linkage*. This problem is a special case of the so called *multi-commodity flow* problem. If k is fixed, then a quadratic time algorithm is known for this problem. If k is not fixed, then the problem is NP-complete. Algorithm 0.1 is a natural greedy algorithm for this problem.

Algorithm 0.1. A greedy algorithm

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Set 𝒫 := ∅
for i = 1 to k do
    if s_i lies with t_i in the same connected component of G then
        Let P be a shortest s_i t_i-path in G.
        Set 𝒫 := 𝒫 ∪ {P}.
        Set G := G − E(P).
    end if
end for
Return 𝒫.
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PART I. Prove that for any $\ell \in \mathbb{N}$, the number of paths P satisfying $e(P) > \ell$ in any linkage is at strictly less than $e(G)/\ell$.

Let $\mathcal{P}$ be denote the greedy linkage produced by Algorithm 0.1. Set

$$r := \max_{P \in \mathcal{P}} e(P)$$

to be the maximum length (in terms of edges) ever considered by the Algorithm 0.1 in producing $\mathcal{P}$. Let $\textit{OPT}$ denote an optimal linkage between the terminal pairs.

PART II. Prove that $|\textit{OPT}| \leq \frac{e(G)}{r} + r|\mathcal{P}|$.

The inequality seen in PART II is beneficial in the sense that it ties the greedy solution to an optimal solution. The first term in this inequality prompts us to redesign Algorithm 0.1 in such a way that a meaningful lower bound on r could be attained so that we could later extract the ratio of approximation of the algorithm.

For $u, v \in \binom{V(G)}{2}$, let $d_G(u, v)$ denote the length (in terms of edges) of a shortest uv-path in G; this being infinite of no uv-path exists in G. The *diameter* of G, denoted $\mathrm{diam}(G)$, is given by

$$\mathrm{diam}(G) := \max_{u,v \in \binom{V(G)}{2}} d_G(u, v).$$

PART III. Adapt Algorithm 0.1 as to attain an $O(\max\{\mathrm{diam}(G), \sqrt{e(G)}\})$-approximation for the problem with the latter restricted to connected graphs.