Assignment 5 solution

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Problem 1

Let G_x be the graph obtain by removing the edges $\{e \in E(G) : x_e = 0\}$ and the vertices left with degree 0 (we will see that there is no such vertices) .
The first constraint make sure that G_x has no cycles, therefore is a forest. Proof: if there is a cycle C so for $S = V_G(C)$ the condition falls.
The second constraint make sure that G_x is a **spanning tree** of G.
Proof: $e(G_x) = v(G) - 1$, so if there is more than 1 connectivity component, at least one of them must contain a cycle, but we saw that G_x has no cycles, so G_x is a **tree**. And because in a tree we know that $|E| \leq |V| - 1$ and $e(G_x) = v(G) - 1$ so $v(G_x) = v(G)$ and we get that G_x **spans** G.
The objective function make sure that $\sum_{e \in E(G_x)} c(e)$ is minimal. All together tell us that $c^T x$ captures the coast of minimum spanning tree of G. ■

Problem 2

The only different between the 2 relaxations is the first constraint of both, so we will prove that they are equivalent.
Let x_1 be a feasible solution for $(P1.MST)$. We need to show that

$$\sum_{e \in A} x_e \leq v(G) - \kappa(A),\ \forall A \subsetneq E(G)$$

Let $A_1 \subsetneq E(G)$ and let $P_1, P_2, ..., P_{\kappa(A)} \in V(G)$ be the connected components in $G[A]$.

$$\sum_{e \in A_1} x_e = \sum_{i=1}^{\kappa(A_1)} \sum_{e \in E_{G[A_1]}(P_i)} x_e \leq \sum_{i=1}^{\kappa(A_1)} (|P_i| - 1) = \sum_{i=1}^{\kappa(A_1)} |P_i| - \kappa(A_1) \leq v(G) - \kappa(A_1)$$

Let x_2 be a feasible solution for $(P2.MST)$. We need to show that $\sum_{e \in E_G(S)} x_e \leq |S| - 1,\ \forall S \subseteq V(G)$.
Let $S_2 \subseteq V(G),\ A_2 := E_G(S)$. Note that $\kappa(E_G(S)) \geq v(G) - |S| + 1, \forall S \subseteq V(G)$.
Case 1: $A_2 \neq E(G)$ so-

$$\sum_{e \in E_G(S_2)} x_e = \sum_{e \in A_2} x_e \leq v(G) - \kappa(A_2) \leq v(G) - (v(G) - |S_2| + 1) = |S_2| - 1$$

Case 2: $A_2 = E(G)$ so $S_2 = V(G)$ and-

$$\sum_{e \in E_G(S_2)} x_e = \sum_{e \in E(G)} x_e = v(G) - 1 = |S_2| - 1$$

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Problem 3

We will follow the recipe for moving from max to min and see if we'll get the same dual problem.
For each constraint $\sum_{e\in A} x_e \le v(G) - \kappa(A), A \subsetneq E(G)$ we define variable y_A such that $y_A \ge 0$. For the last constraint $\sum_{e\in E(G)} x_e = v(G) - 1$ we define variable y_E so that $y_E \in \mathbb{R}$. G is connected so $v(G) - 1 = v(G) - \kappa(E(G))$
And we looking for the min problem so we want to find the min of $\sum_{A\subseteq E(G)}(v(G) - \kappa(A)) \cdot y_A$ which is the old constraint times the new variables.
Now for every variable x_e $s.t.$ $x_e \ge 0$ we define constraint $\sum_{\substack{A:e\in A\\ A\subseteq E(G)}} y_A \ge -c_e$ which is $A_e^\intercal y \ge -c_e$ for the matrix A of the original problem.

Problem 4

1. Verify that y^* is feasible for (D.MST): We need to show that

$$\sum_{\substack{A:e\in A\\ A\subseteq E(G)}} y_A^* \ge -c(e),\ \forall e \in E(G)$$

Let $i \in [e(G)]$.

$$\sum_{\substack{A:e_i\in A\\ A\subseteq E(G)}} y_A^* = \sum_{j=i}^{e(G)} y_{R_j}^* = \sum_{j=i}^{e(G)-1} (c(e_{j+1}) - c(e_j)) - c(e_{e(G)}) = -c(e_i)$$

2. Verify complementary slackness conditions:
(0.1) We showed in 1. that $\sum_{\substack{A:e\in A\\ A\subseteq E(G)}} y_A^* = -c(e)$ whether or not $x_e^* > 0$.
(0.2) Let $A \in E(G)$. If $y{*}_A > 0$ so by y^* definition, $A = R_i = e_1, ..., e_i$ for some $i \in [v(G)]$ which is the a forest by Kruskal's algorithm, there for

$$\sum_{e\in A} x_e^* = |A| = v(G) - \kappa(A)$$

For the last equation we'll prove that for any forest G, $\kappa(E(G)) = v(G) - e(G)$. by induction on e(G) Proof: for $e(G) = 0$, $\kappa(E(G)) = v(G) = v(G) - e(G)$. For $e(G) > 0$, we choose a random edge e and set $G' = G \setminus \{e\}$. By the induction assumption we know that $\kappa(E(G')) = v(G') - e(G')$ and because there is no cycles, $\kappa(E(G)) = \kappa(E(G')) - 1$ so

$$\kappa(E(G)) = \kappa(E(G')) - 1 = v(G') - e(G') - 1 = v(G) - (e(G) - 1) - 1 = v(G) - e(G)$$

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