Assignment 6 solution

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Problem 1

Algorithm:

1. Solve the LP-relaxation of the weight vertex-cover problem and let $(x_v^*)_{v\in V(G)}$
2. Return $\mathcal{A} := \{v \in V(G) : x_v^* \geq 1/2\}$

Proof. We start by proving that the solution $\mathcal{A}$ returned by the algorithm is feasible. That is, that $\forall e = \{v, u\} \in E(G),\ v \in \mathcal{A} \vee u \in \mathcal{A}$. To that end, fix $e \in E(G)$. The LP-constraint for e in (P.SC) is that

$$\sum_{v\in e} x_v^* \geq 1$$

Each edge has exactly 2 vertices, hence, there exists a $v \in e$ such that $x_v^* \geq 1/2$ so that $e \in \mathcal{A}$.
Next, we address the approximation ratio of the algorithm. Let $(x_v^*)_{v\in V(G)}$ be an optimal solution for the LP-relaxation. For $v \in V(G)$, define

$$z_v := \begin{cases} 1, & v \in \mathcal{A} \\ x_v^*, & v \notin \mathcal{A} \end{cases}$$

Than, $z_v \leq 2 \cdot x_v^*$ holds for every $v \in V(G)$ by definition of $\mathcal{A}$. Observe that

$$c(\mathcal{A}) \leq \sum_{v\in V(G)} z_v \cdot c(v) \leq 2 \sum_{v\in V(G)} x_v^* \cdot c(v) = 2 \cdot OPT_f \leq 2 \cdot OPT$$

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Problem 2

part I

Let n_l denote the number of paths P satisfying $e(P) > l$. The paths are edge disjointed, so each one takes at least $l + 1$ edges, therefor

$$n_l \leq \frac{e(G)}{l+1} < \frac{e(G)}{l}$$

Moreover, Let k_l denote the number of paths P satisfying $e(P) \geq l$. The paths are edge disjointed, so each one takes at least l edges, therefor

$$k_l \leq \frac{e(G)}{l}$$

part II

If we add $E(P)$ back to G for some $P \in \mathcal{P}$, there is at most $e(P)$ new edge disjointed paths in G. Note that the paths in $\mathcal{P}$ can be divided to n paths of length r and $|\mathcal{P}| - n$ paths of length $\leq r - 1$, so

$$|OPT| \leq |\mathcal{P}| + \sum_{P\in\mathcal{P}} e(P) = |\mathcal{P}| + e(\mathcal{P}) \leq |\mathcal{P}| + n\cdot r + (r-1)(|\mathcal{P}| - n) = |\mathcal{P}| + n\cdot r + |\mathcal{P}|(r-1) - n\cdot r + n = |\mathcal{P}| + |\mathcal{P}|(r-1) + n = n + r|\mathcal{P}| \leq$$

part III

1. Set $\mathcal{P} := \phi$
2. **for** $i = 1$ to k **do**:
3. **if** s_i lies with t_i in the same connected component of G and $d_G(s_i, t_i) \leq \max\{\text{diam}(G), \sqrt{e(G)}\}$ **then**
4. Let P be a shortest $s_i t_i$-path in G.
5. Set $\mathcal{P} := \mathcal{P} \cup P$.
6. Set G := G E(P).
7. **end if**
8. **end for**
9. Return $\mathcal{P}$.

Let $m = max\{diam(G), \sqrt{e(G)}\}$, we know that $r \leq m$ and $e(G) \leq m^2$ (trivial), so:

$$|OPT| \leq \frac{e(G)}{r} + r|\mathcal{P}| \leq \frac{e(G)}{m} + m|\mathcal{P}| \leq \frac{m^2}{m} + m|\mathcal{P}| = m(|\mathcal{P}| + 1) \leq m(|\mathcal{P}| + |\mathcal{P}|) = 2m|\mathcal{P}|.$$

(Note that G is connected, and $d_G(s_1, t_1) \leq diam(G) \leq m$ so $|\mathcal{P}| \geq 1$) $\blacksquare$