Assignment 2 solution

Netanel Albert

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Problem 1:

Let G be a graph and let M be a matching in G. Let M' be a maximum matching in G, that amongst all such matchings maximizes $|M' \cap M|$ as well. Lets assume that $\exists x\ s.t.\ x \in V(M) \wedge x \notin V(M')$. Let y be the neighbor of x in M. $y \in V(M')$, otherwise M' isn't a maximum matching (we can add the edge xy). Define $M'' := \{M' \setminus \{yz\}\} \cup \{xy\}$. $|M'| = |M''|$ therefore M'' is a maximum matching, but $|M \cap M'| < |M \cap M''|$ in contradiction to $|M \cap M'|$ maximum. Therefore $V(M) \subseteq V(M')$ $\blacksquare$

Problem 2:

First direction:
Let G be a graph s.t. any two of its edges lie on a common cycle, in particular any two of its vertices lie on a common cycle. Therefore by Whitneys theorem the graph is 2-connected.

Second direction:
Let G be a 2-connected graph. Assuming not any two of its edges lie on a common cycle, let $ab, xy \in E(G)$ such. $\varrho(\{x,y\},\{a,b\}) \leq 1$ so by Mengers theorem $\kappa(\{x,y\},\{a,b\}) \leq 1$ in contradiction to that G is 2-connected $\blacksquare$

Problem 3:

$2-connector(G)$:

1. : Let $T :=$ the blocks tree of G [1].
2. : Let $l := \lceil \frac{l(T)}{2} \rceil$ where $l(T)$ is the number of the leaves in T.
3. : Let $d := \Delta(T) - 1$.
4. : Return $max\{l, d\}$.

Correctness prove: let $n := max\{\lceil \frac{l(T)}{2} \rceil, \Delta(T) - 1\}$, and let $m :=$ the least number of (non-)edges that upon adding those to the graph a 2-connected graph is attained. $m \leq n$ is trivial, because if we add less than $\lceil \frac{l(T)}{2} \rceil$ edges we still have leaves, so the leaf neighbor is a disconnector. and if we add less than $\Delta(T) - 1$ edges so the vertex that had $\Delta(T)$ neighbors is a disconnector.
To show that $n \leq m$ we'll prove in induction on n that a 2-connected graph can be attain by adding n edges.
Base: $n = 1$ so we have at most 2 leaves or the max degree is 2, therefore T is a single point which is already 2-connected, or it's a line which we close to cycle by 1 edge.
Step:
Case 1: $\Delta(T) - 1 < \lceil \frac{l(T)}{2} \rceil$. in this case we know that T is not a star graph, so we will connect 2 leaves with distance > 2. That way $l(T)$ become smaller in 2 so m also become smaller.
Case 2: $\lceil \frac{l(T)}{2} \rceil \leq \Delta(T) - 1 \Rightarrow l(T) \leq 2(\Delta(T) - 1)$. in this case there are no more than 2 vertices with degree $\Delta(T)$. If T is a star graph we'll connect 2 leaves of it. otherwise we'll connect 2 leaves that came from the 2 highest degree vertices. any way m become smaller.
In any case we can 2-connect the new tree we made with $m - 1$ edges by the inductive hypothesis $\blacksquare$

Running time:

1. : $O(V(G))$ - It's preformed by one DFS on G.
2. : $O(V(T)) \leq O(V(G))$
3. : $O(V(T)) \leq O(V(G))$
4. : $O(1)$

In conclusion - the running time is linear - $O(V(G))$

Problem 4:

The wish list of problems given B, a 2-connected graph (v, u are vertices):
$longest(B)$ return the longest path in B.
$longest(B, v)$ return the longest path in B start with v.
$longest(B, v, u)$ return the longest path in B start with v and end with u.

Lets define a helper algorithm given a block B and a vertex v:
$visit(B, v)$:

1. : Let $M := 0$
2. : For each unvisited $C \in N_T(B)$
3. : Let x be the common vertex of B and C in G.
4. : $M = max\{M, visit(C, x) + longest(B, v, x)\}$.
5. : return $max\{M, longest(B, v)\}$.

Now lets define the main algorithm given a connected graph G:
$longestPath(G)$:

1. : Let $T :=$ the blocks tree of G [1]
2. : Let $M := 0$
3. : For each vertex $B \in V(T)$
4. : For each $C \in N_T(B)$
5. : Let x be the common vertex of B and C in G
6. : $M = max\{M, visit(C, x) + longest(B, x)\}$.
7. : $M = max\{M, longest(B)\}$.
8. : return M.

Bibliography

[1] See lecture slide 4.A.