EE6302 Control System Design

Frequency Response Technique

Introduction

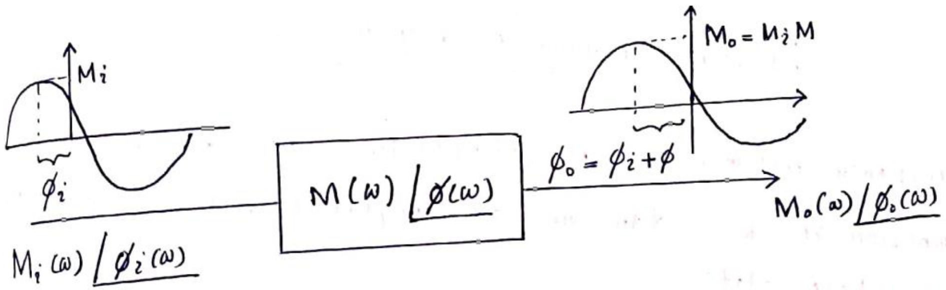
- Frequency response methods were developed by Nyquist and Bode in the 1930s.
- Frequency response yields a new vantage point from which to view feedback control systems.
  - When modeling transfer functions from physical data
  - When designing lead compensators to meet a steady-state error requirement and a transient response requirement
  - When finding the stability of nonlinear system
  - In settling ambiguities when sketching a root locus
  - …….

The concept of frequency response

- Sinusoids can be represented as complex numbers called phasors.

$$M_1 \cos(\omega t + \phi_1) = M_1 \angle \phi_1$$

- In steady state, sinusoidal input to a linear system generates sinusoidal response of the same frequency.
- Even though the inputs and outputs are of the same frequency, they may differ in in amplitude and phase angle from input



$$M_o(\omega) \angle \phi_o(\omega) = M_i(\omega) M(\omega) \angle \phi_i(\omega) + \phi(\omega)$$

$$M(\omega) = \frac{M_o(\omega)}{M_i(\omega)}$$

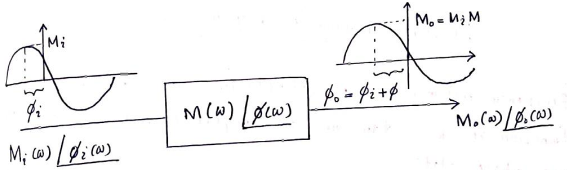
$$\phi(\omega) = \phi_o(\omega) - \phi_i(\omega).$$

Magnitude frequency response

Phase frequency response

FREQUENCY RESPONSE

The concept of frequency response



$$M_o(\omega)\angle\phi_o(\omega) = M_i(\omega)\,M(\omega)\angle\phi_i(\omega)+\phi(\omega)$$

<u>Magnitude frequency response</u>

- Ratio of the output sinusoid's magnitude to input sinusoid's magnitude

$$M(\omega) = \frac{M_o(\omega)}{M_i(\omega)}$$

<u>Phase frequency response</u>

- Difference in phase angle between the output and the input sinusoid.

$$\phi(\omega) = \phi_o(\omega) - \phi_i(\omega).$$

Both responses are a function of frequency and apply only to the steady state sinusoidal response of the system

The frequency response of a system whose transfer function is G(s) is;

$$\boxed{G(j\omega) = G(s)|_{s\rightarrow j\omega}}$$

Plotting frequency response

- Frequency response can be plotted in several ways.
- One way is to
  - Plot the magnitude curve in decibels(dB) vs log ω
  - Plot the phase curve as phase angle vs log ω
- The log-magnitude and phase frequency response curves as functions of log ω are called Bode plots or Bode diagrams.

Asymptotic Approximations: Bode Plots

- Consider the following transfer function.

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^m(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- The magnitude frequency response

$$|G(j\omega)| = \frac{K\,|j\omega+z_1|\,|j\omega+z_2|\ldots|j\omega+z_k|}{|(j\omega)^m|\,|j\omega+p_1|\,|j\omega+p_2|\ldots|j\omega+p_n|}$$

Magnitude response in dB

$$|G(j\omega)|\,dB = 20\log|G(j\omega)|$$

$$|G(j\omega)|\,dB = 20\log K + 20\log|j\omega+z_1| + 20\log|j\omega+z_2| + \ldots + 20\log|j\omega+z_k| - \left[20\log(j\omega)^m + 20\log|j\omega+p_1| + \ldots 20\log|j\omega+p_n|\right]$$

- Phase frequency response is the sum of the phase frequency response curves of the zero terms minus the sum of the phase frequency response curves of the pole terms.

Creating the bode plots

- **Bode plot for G(s) = s+a**

<u>Magnitude response</u>

$G(s) = s + a$

$G(j\omega) = j\omega + a$

<u>Magnitude Response</u>

At low frequencies, $\omega << a$

$G(j\omega) \approx a$

$|G(j\omega)|\ dB = 20 \log a$

At high frequencies, $\omega >> a$

$G(j\omega) \approx j\omega$

$G(j\omega) = \omega \angle 90^\circ$

$|G(j\omega)|\ dB = 20 \log \omega$

$$|G(j\omega)|\ dB \begin{cases} 20 \log a & ; \ \omega << a \\ 20 \log \omega & ; \ \omega >> a \end{cases}$$

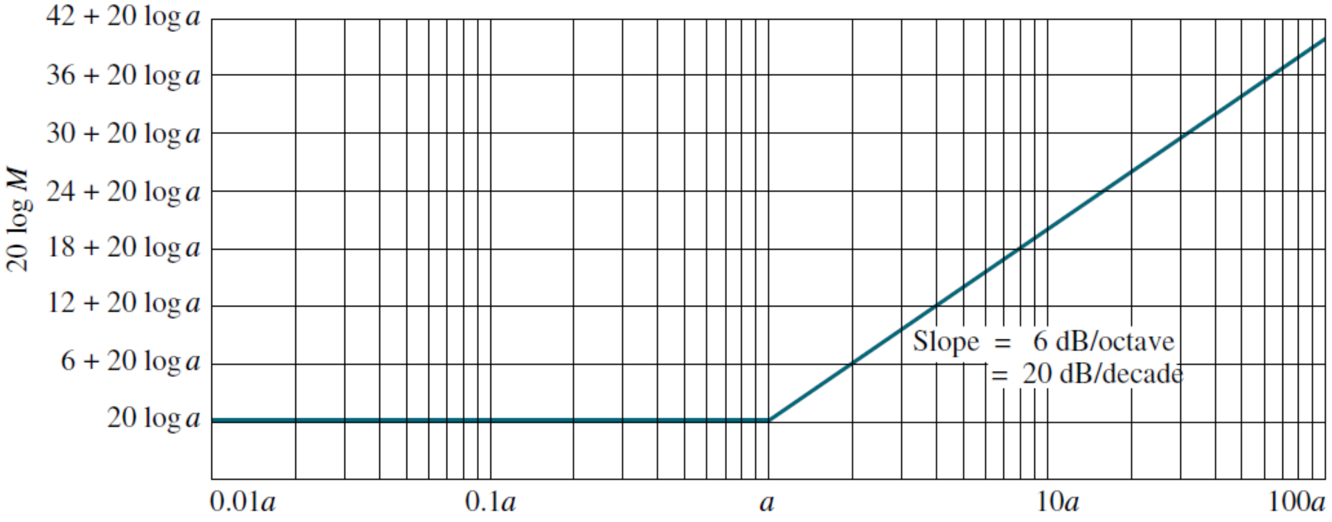
at $\omega = a$.

$G(j\omega) = a(j+1)$

$G(j\omega) = \sqrt{2}a \angle 45^\circ$

$|G(j\omega)|\ dB = 20 \log \sqrt{2}a$

High freq. asymptote



Low freq. asymptote

Break freq.

Creating the bode plots

- **Bode plot for G(s) = s+a**

Phase response

$G(j\omega) = j\omega + a$

At low frequencies, $\omega \ll a$

$G(j\omega) = a$

$\angle G(j\omega) = 0$

At the break frequency, $\omega = a$

$G(j\omega) = a(j+1) = \sqrt{2}a\angle 45^\circ$

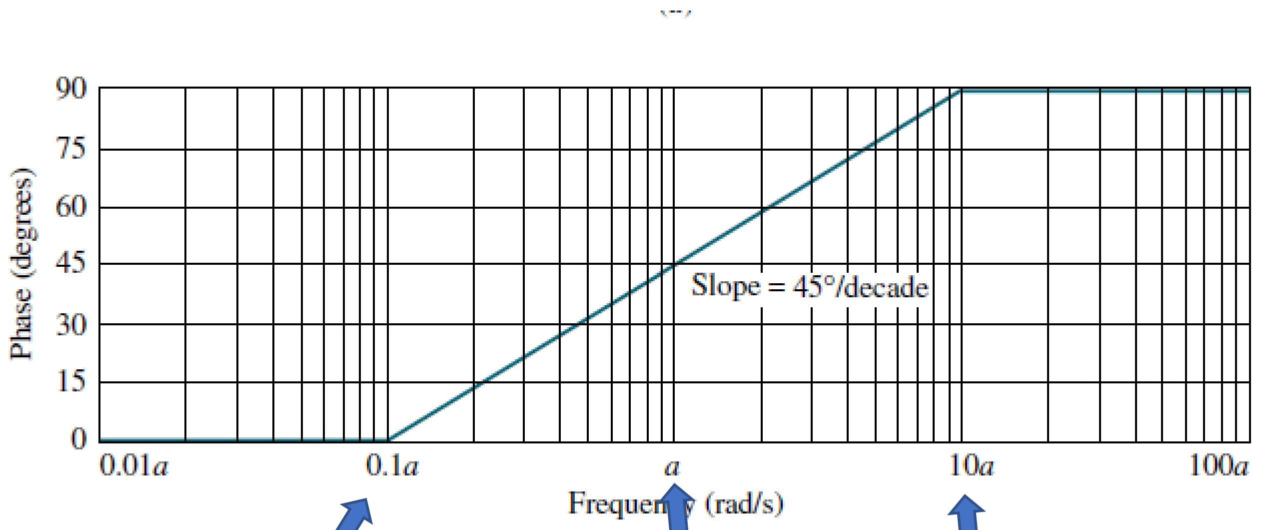
$\angle G(j\omega) = 45^\circ$

At high frequencies, $\omega \gg a$

$G(j\omega) = j\omega = \omega\angle 90^\circ$

$\angle G(j\omega) = 90^\circ$

Slope = 45deg/decade



Start 1 decade below the break frequency

Break frequency

1 decade above the break frequency

Normalized Magnitude and Phase plots

- It is often convenient to normalize the magnitude and scale the frequency so that log magnitude plot will be 0 dB at a break frequency of unity.
- Normalization helps to create the frequency responses of the functions in the form of;

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^m(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- We will create normalized Bode plots for following functions

1. G(s) = s+a
2. G(s) = 1/(s+a)
3. G(s) = s
4. G(s) = 1/s

Normalized Magnitude and Phase plots

1. Normalized frequency response for G(s) = s+a

1) Normalized bode plot for $G(s) = s+a$

$$G(s) = a\left(\frac{s}{a}+1\right)$$

- Define new frequency variable $s' = s/a$
- Magnitude is devided by the quantity a to yield 0 dB at the break frequency.

$$\frac{G(s)}{a} = s'+1$$

Normalized magnitude $= \frac{|G(s)|}{a}$

Normalized magnitude response $= |s'+1|$

Normalized Frequency response $\Rightarrow G(j\omega) = j\omega+1$

$$|G(j\omega)| \begin{cases} 1 & ; \omega \ll 1 \\ j\omega & ; \omega \gg 1 \end{cases}$$

$$|G(j\omega)|\,dB \begin{cases} 20\log 1 = 0\,dB & ; \omega \ll 1 \\ 20\log\sqrt{2} \approx 0 & ; \omega = 1 \\ 20\log\omega & ; \omega \gg 1 \end{cases}$$

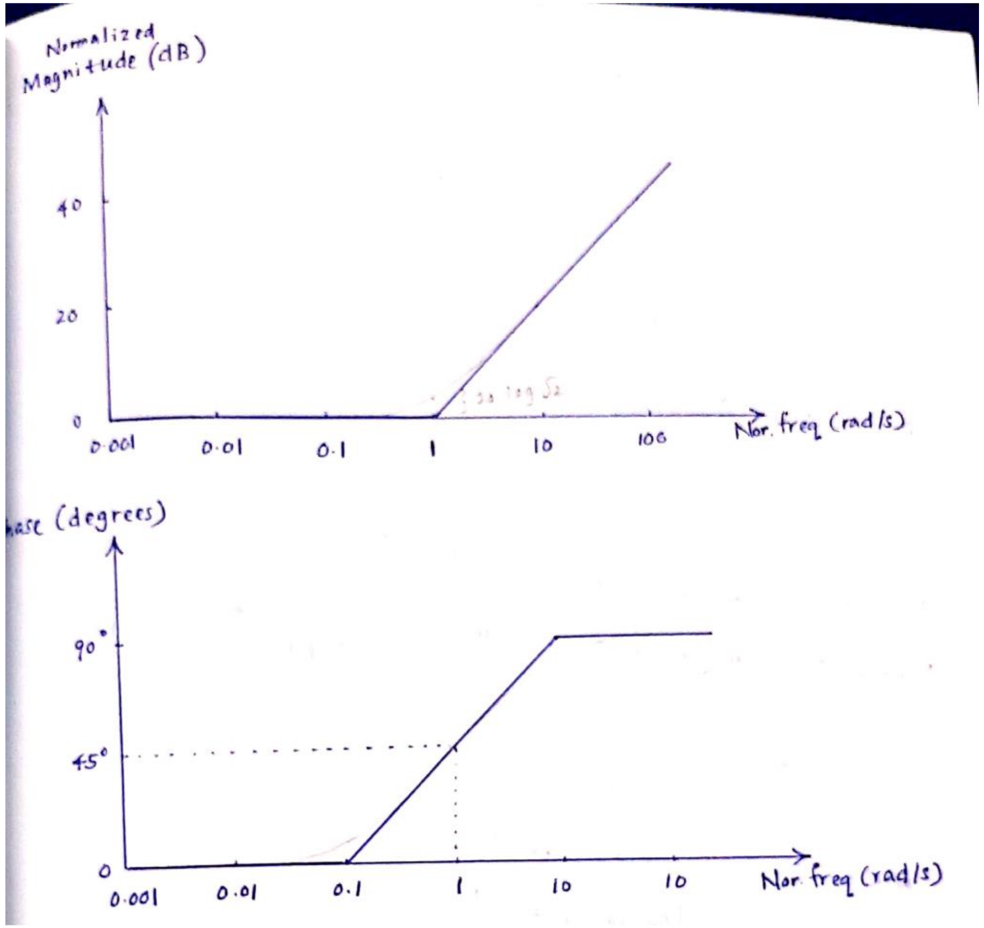
Normalized Magnitude and Phase plots

1. Normalized frequency response for G(s) = s+a

Normalized Frequency response $\Rightarrow$ $G(j\omega) = j\omega + 1$

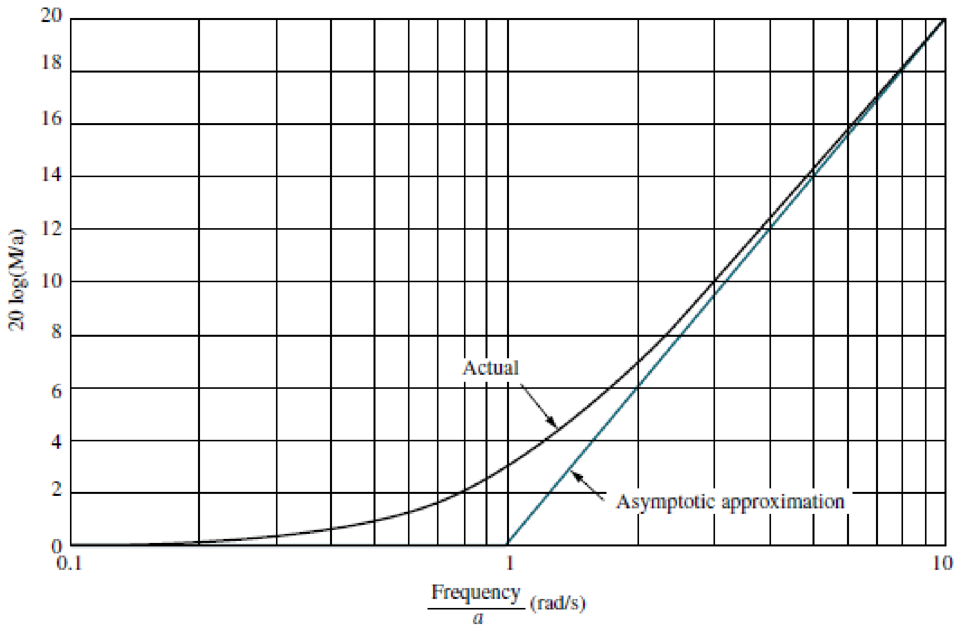
$$|G(j\omega)| \begin{cases} 1 & ; \ \omega \ll 1 \\ j\omega & ; \ \omega \gg 1 \end{cases}$$

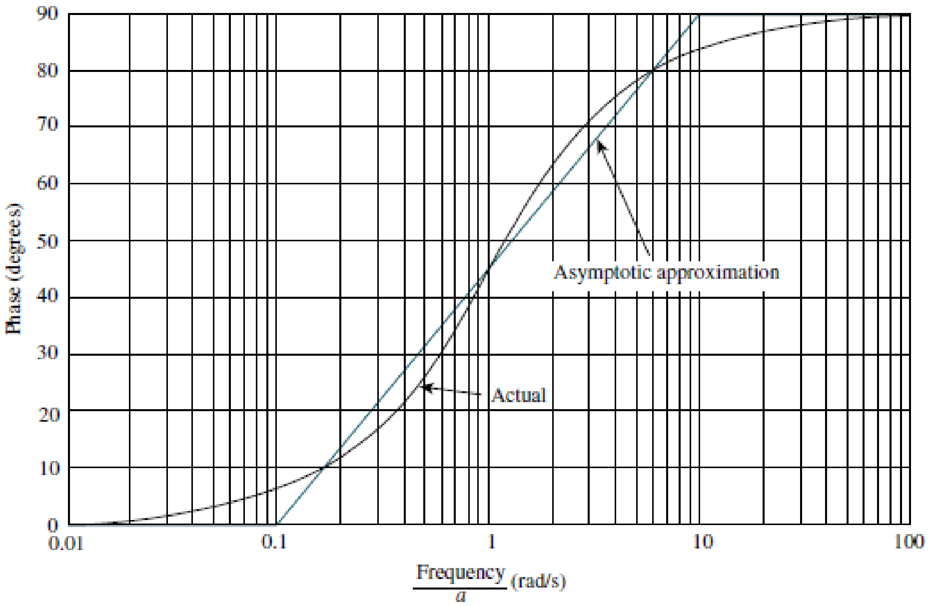
$$|G(j\omega)|\,dB \begin{cases} 20 \log 1 = 0 \text{ dB} & ; \ \omega \ll 1 \\ 20 \log \sqrt{2} \approx 0 & ; \ \omega = 1 \\ 20 \log \omega & ; \ \omega \gg 1 \end{cases}$$



Normalized Magnitude and Phase plots

1. Normalized frequency response for G(s) = s+a

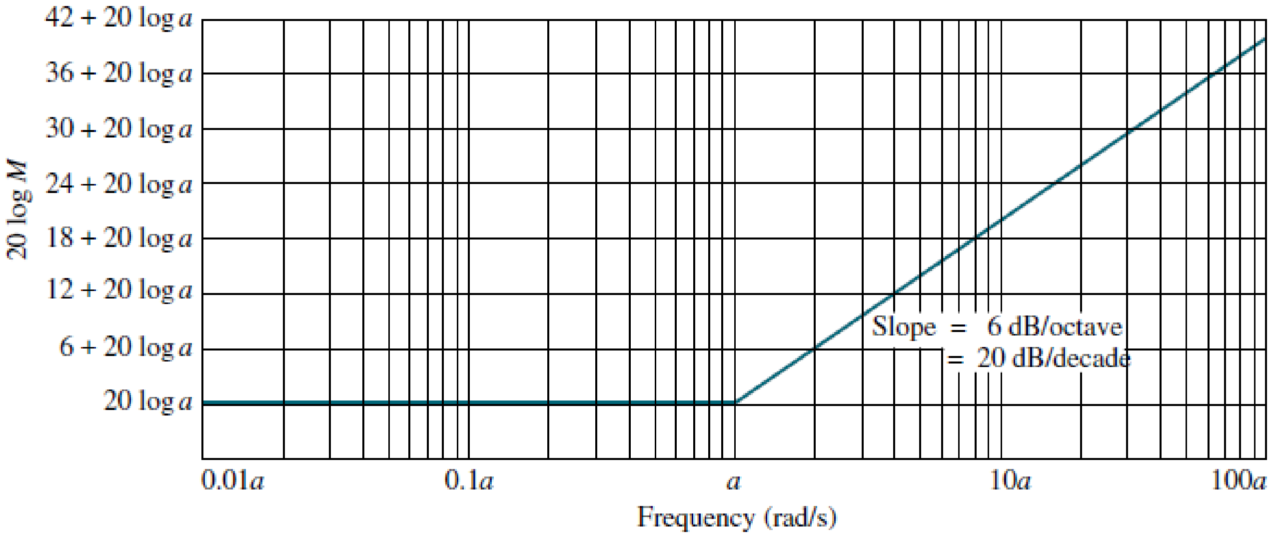


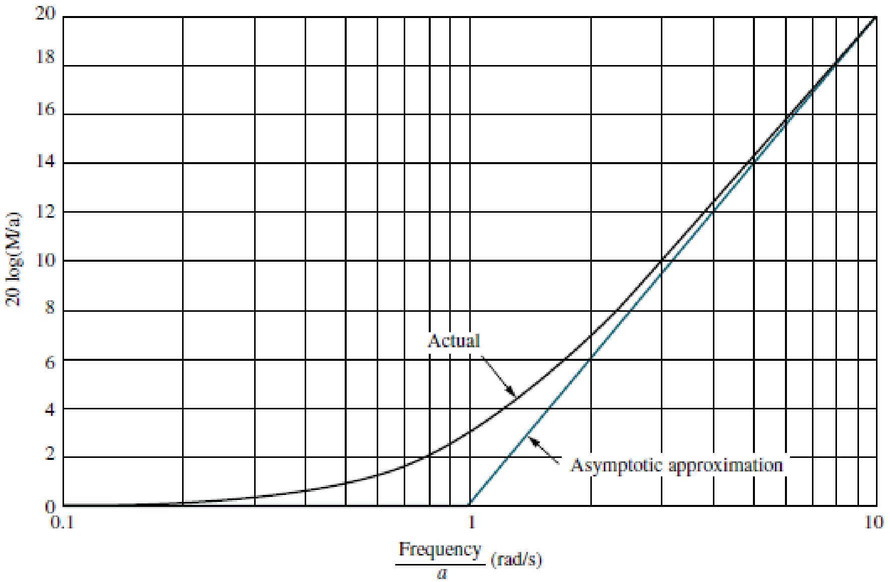


- Note the actual Bode plots superimposed on their respective asymptotic approximations
- Observe that our asymptotic approximations provide close approximations to the actual Bode plots

Comparison

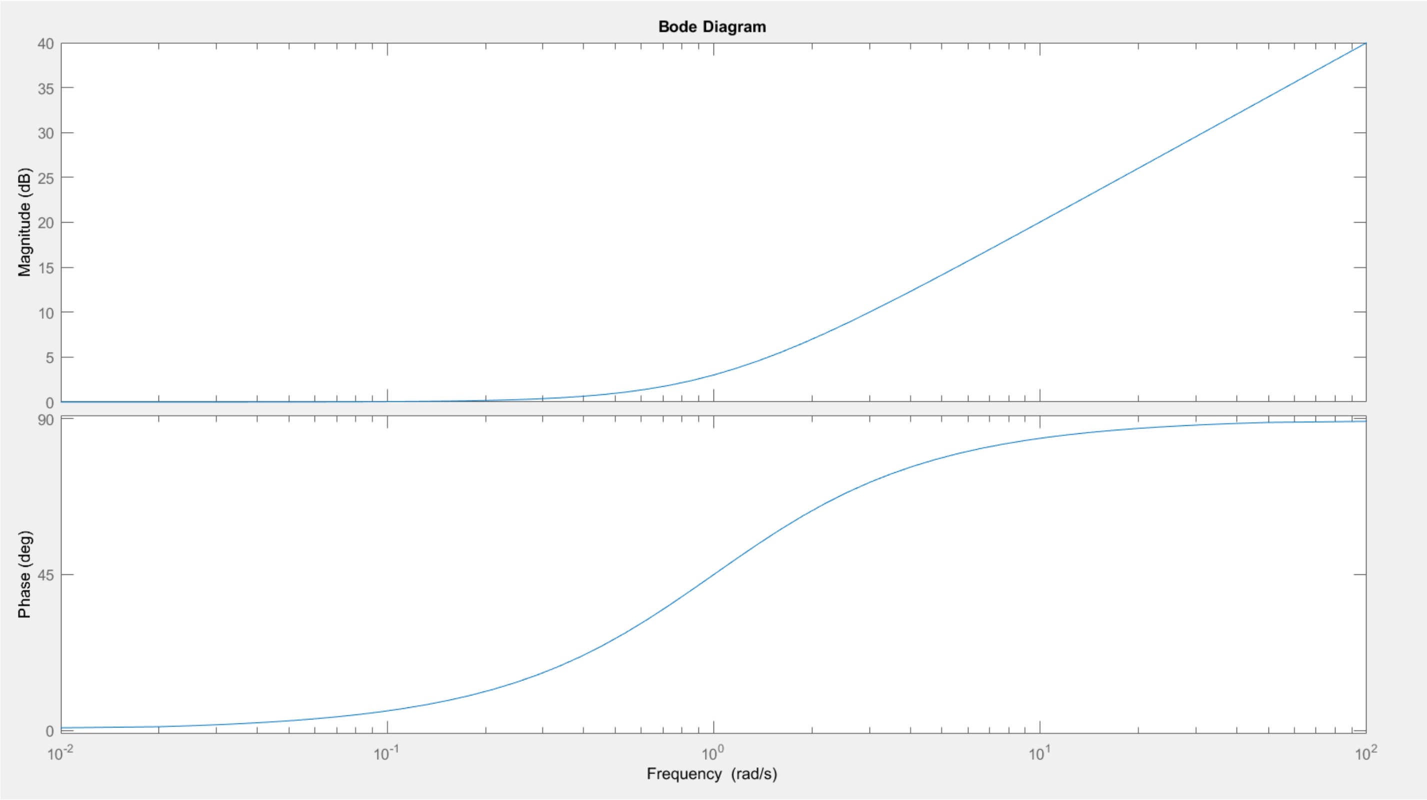
- **Bode plot for G(s) = s+a**
- Compare the two plots





- To obtain the original frequency response, the magnitude and frequency has to be multiplied by the quantity *a*

Bode plot for G(s) = s+1 – Using MATLAB




```
s = tf('s');
sys = s+1

bode(sys)
```

Normalized Magnitude and Phase plots

2. Normalized frequency response for G(s) = 1/(s+a)

$$G(s) = \frac{1}{a(s/a + 1)}$$

$$a\,G(s) = \frac{1}{s/a + 1}$$

Normalized frequency variable $s' = s/a$

$$a\,G(s) = \frac{1}{s'+1}.$$

Normalized magnitude $= a|G(s)|$

Normalized magnitude response $= \frac{1}{s'+1} = \frac{1}{j\omega+1}$

$$G(j\omega) = \frac{1}{j\omega+1}$$

$\omega << 1$; $G(j\omega) \approx 1$

$$|G(j\omega)| = 1$$

$$|G(j\omega)|\,dB = 0$$

$$\angle G(j\omega) = 0.$$

$\omega = 1$; $G(j\omega) = \frac{1}{j+1} = \frac{1}{\sqrt{2}} \angle -45^\circ$

$|G(j\omega)|\,dB = 20 \log \frac{1}{\sqrt{2}} \approx 0$ (asymptotic approx-mation

$$\angle G(j\omega) = -45^\circ$$

$\omega >> 1$; $G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega} \angle -90^\circ$

$$|G(j\omega)|\,dB = -20 \log \omega$$

$$\angle G(j\omega) = -90^\circ$$

Normalized Magnitude and Phase plots

2. Normalized frequency response for G(s) = 1/(s+a)

$$G(j\omega) = \frac{1}{j\omega + 1}$$

$\omega << 1$; $G(j\omega) \approx 1$

$|G(j\omega)| = 1$

$|G(j\omega)|_{dB} = 0$

$\angle G(j\omega) = 0$

$\omega = 1$; $G(j\omega) = \frac{1}{j+1} = \frac{1}{\sqrt{2}} \angle -45^\circ$

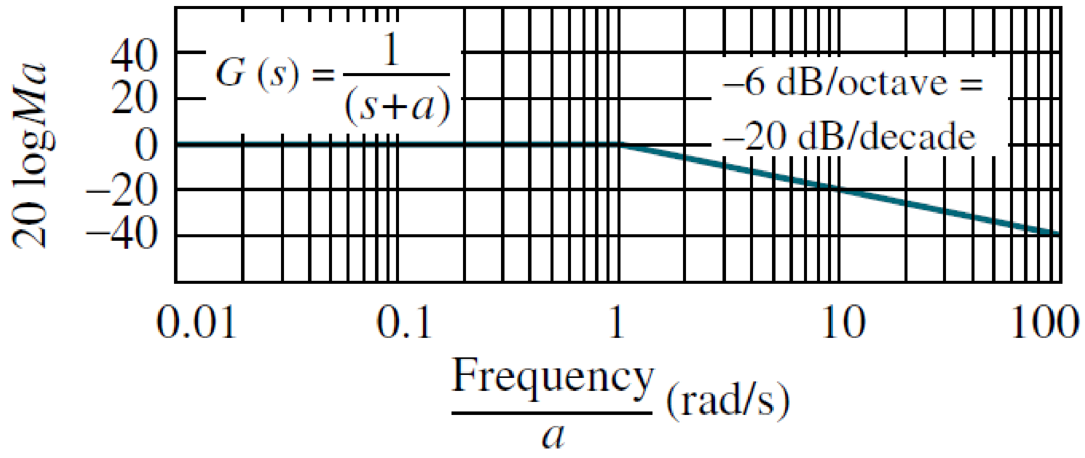
$|G(j\omega)|_{dB} = 20 \log \frac{1}{\sqrt{2}} \approx 0$ (asymptotic approx-mation)

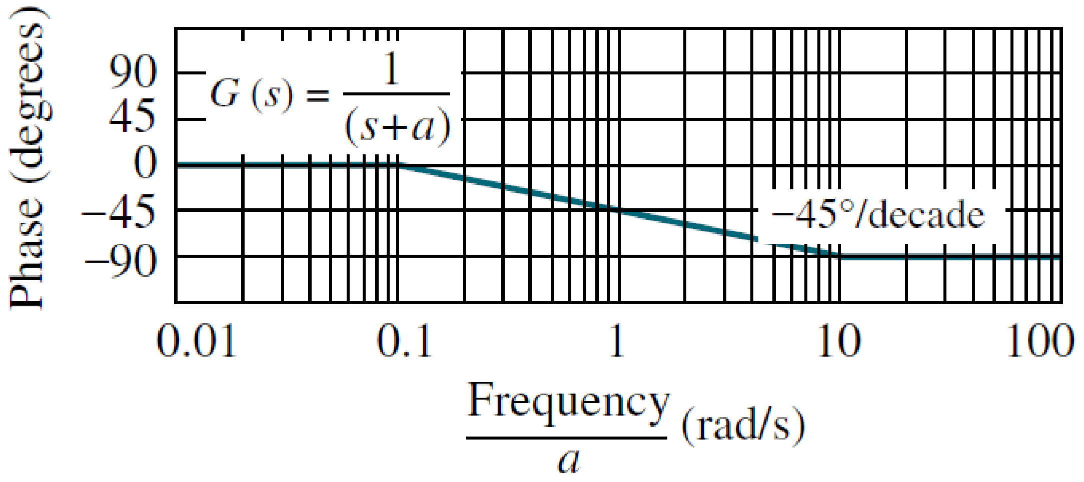
$\angle G(j\omega) = -45^\circ$

$\omega >> 1$; $G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega} \angle -90^\circ$

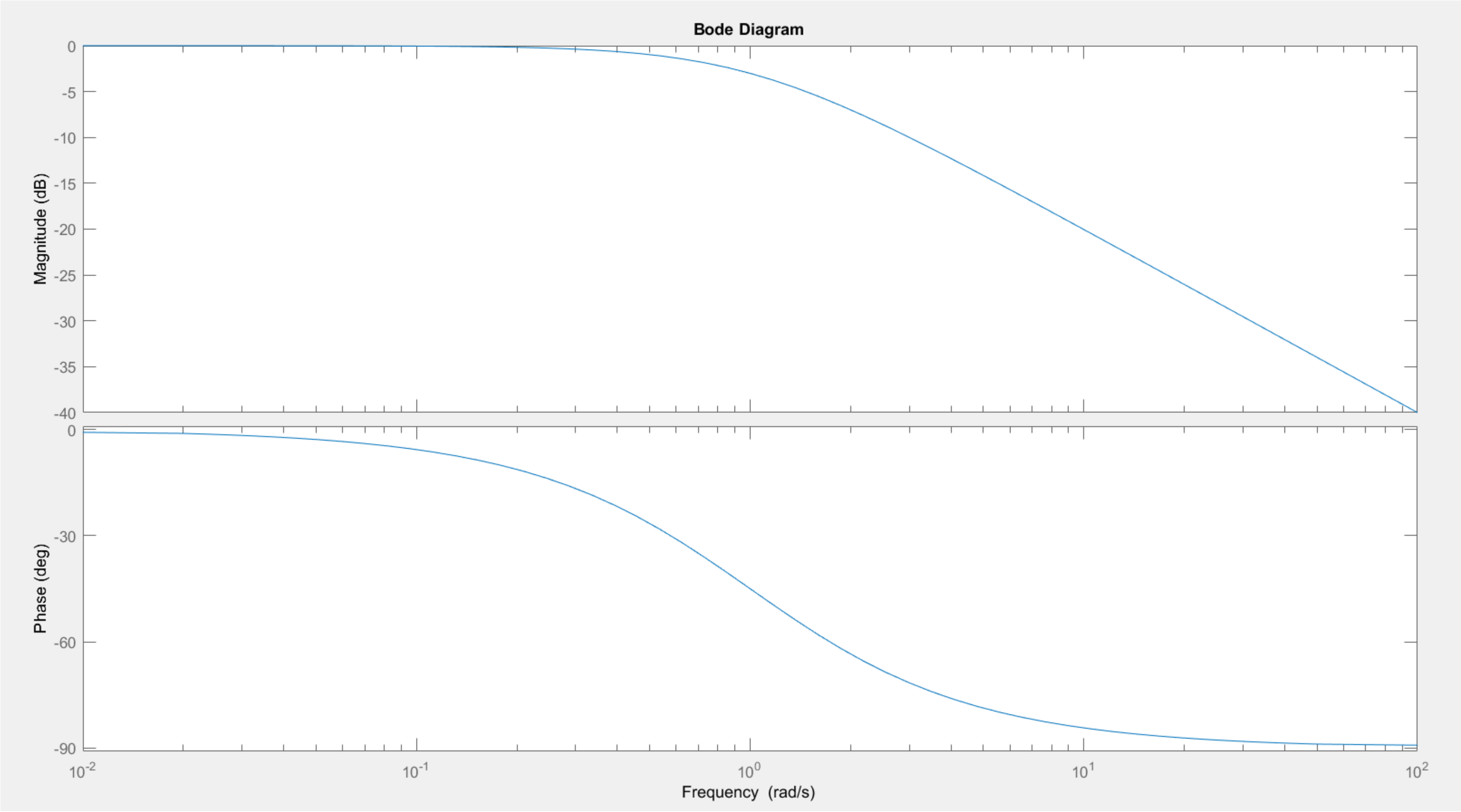
$|G(j\omega)|_{dB} = -20 \log \omega$

$\angle G(j\omega) = -90^\circ$





Bode plots for G(s) = 1/(s+a) – Using MATLAB



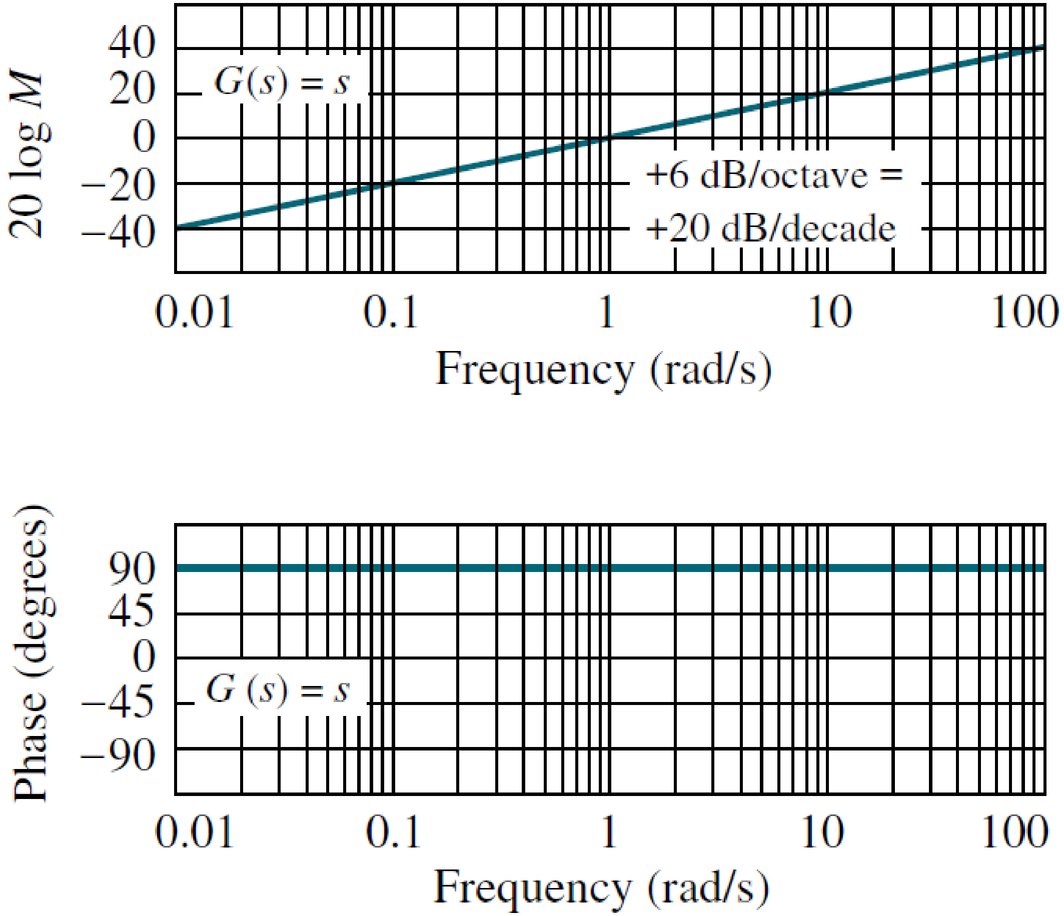

```
s = tf('s');
sys = 1/(s+1)

bode(sys)
```

Normalized Magnitude and Phase plots

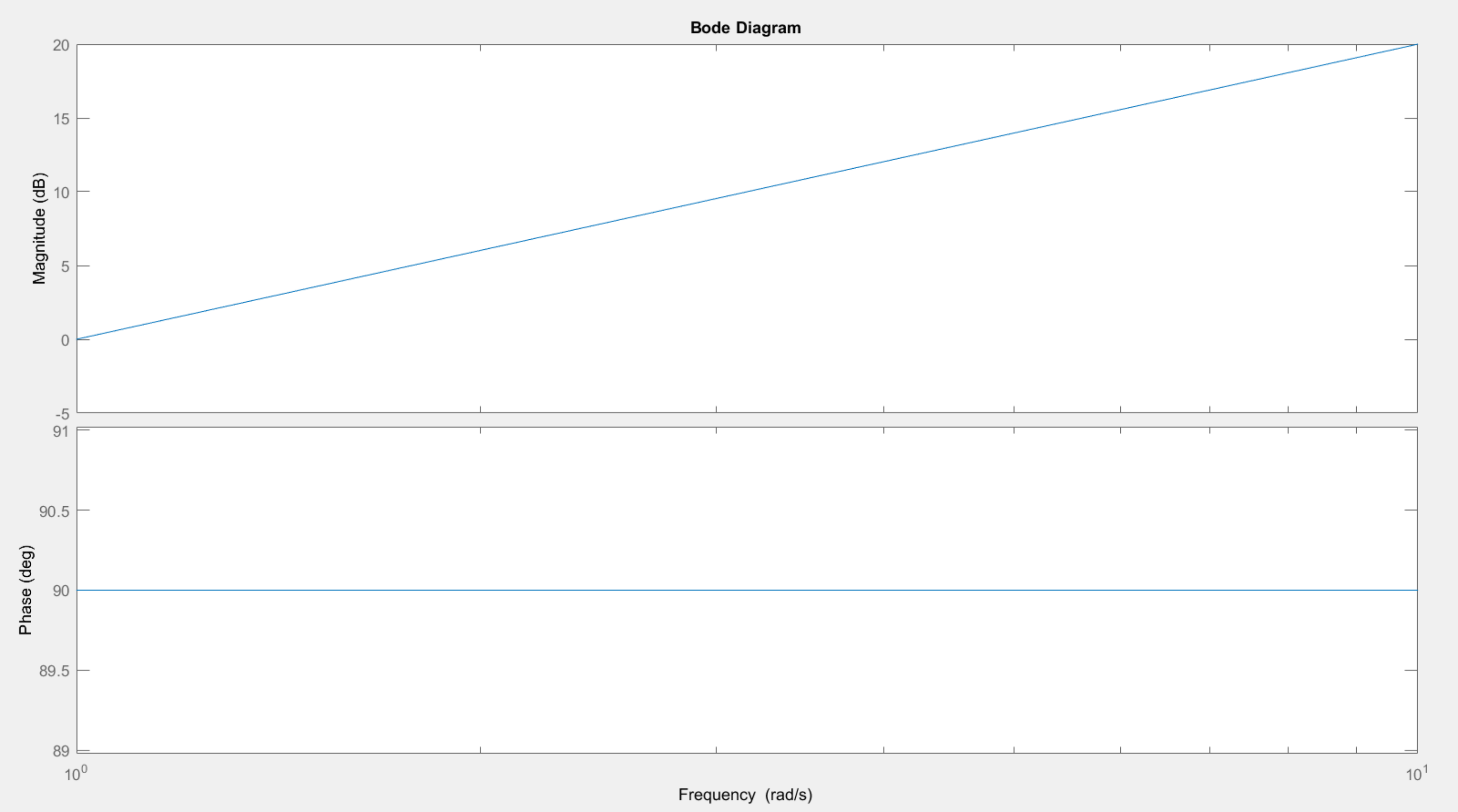
3. Normalized frequency response for G(s) = s

$$G(j\omega) = j\omega$$
$$= \omega \angle 90^\circ$$

$$|G(j\omega)|\, dB = 20 \log \omega$$
$$\angle G(j\omega) = 90^\circ$$



Bode plot for G(s) = s – Using MATLAB




```
s = tf('s');
sys = s

bode(sys)
```

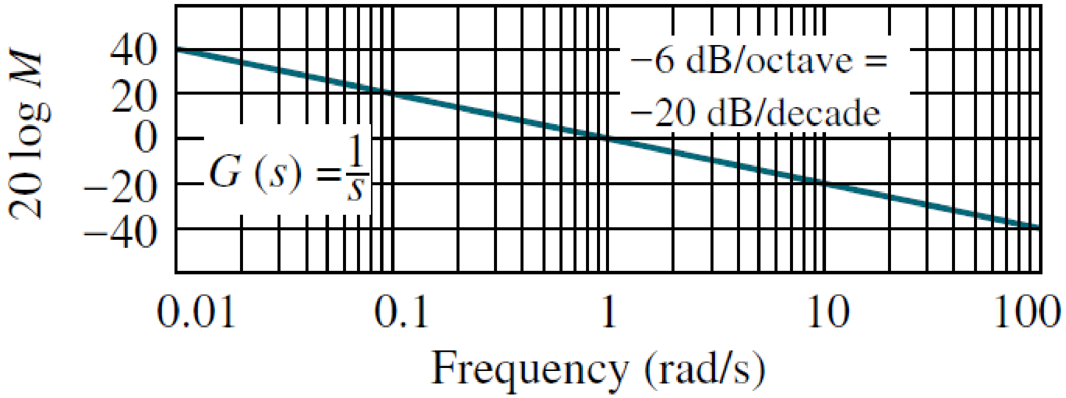
Normalized Magnitude and Phase plots

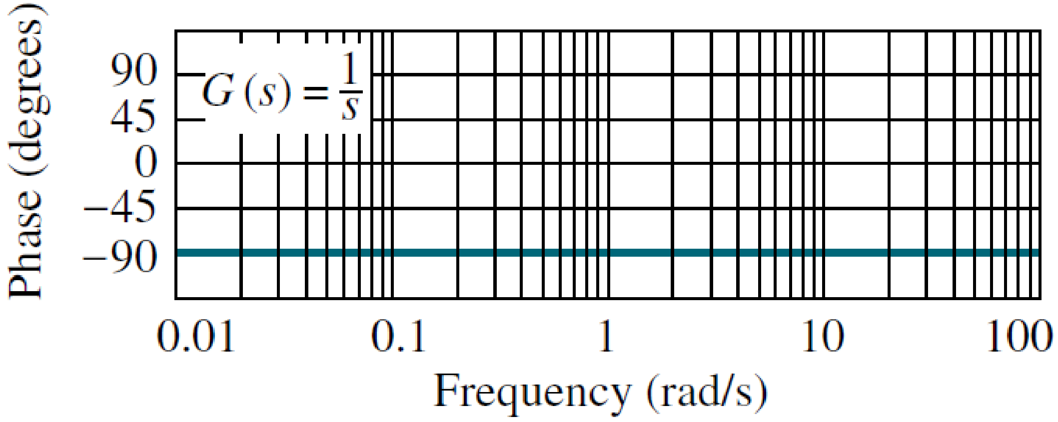
4. Normalized frequency response for G(s) = 1/s

$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega}\angle -90^\circ$$

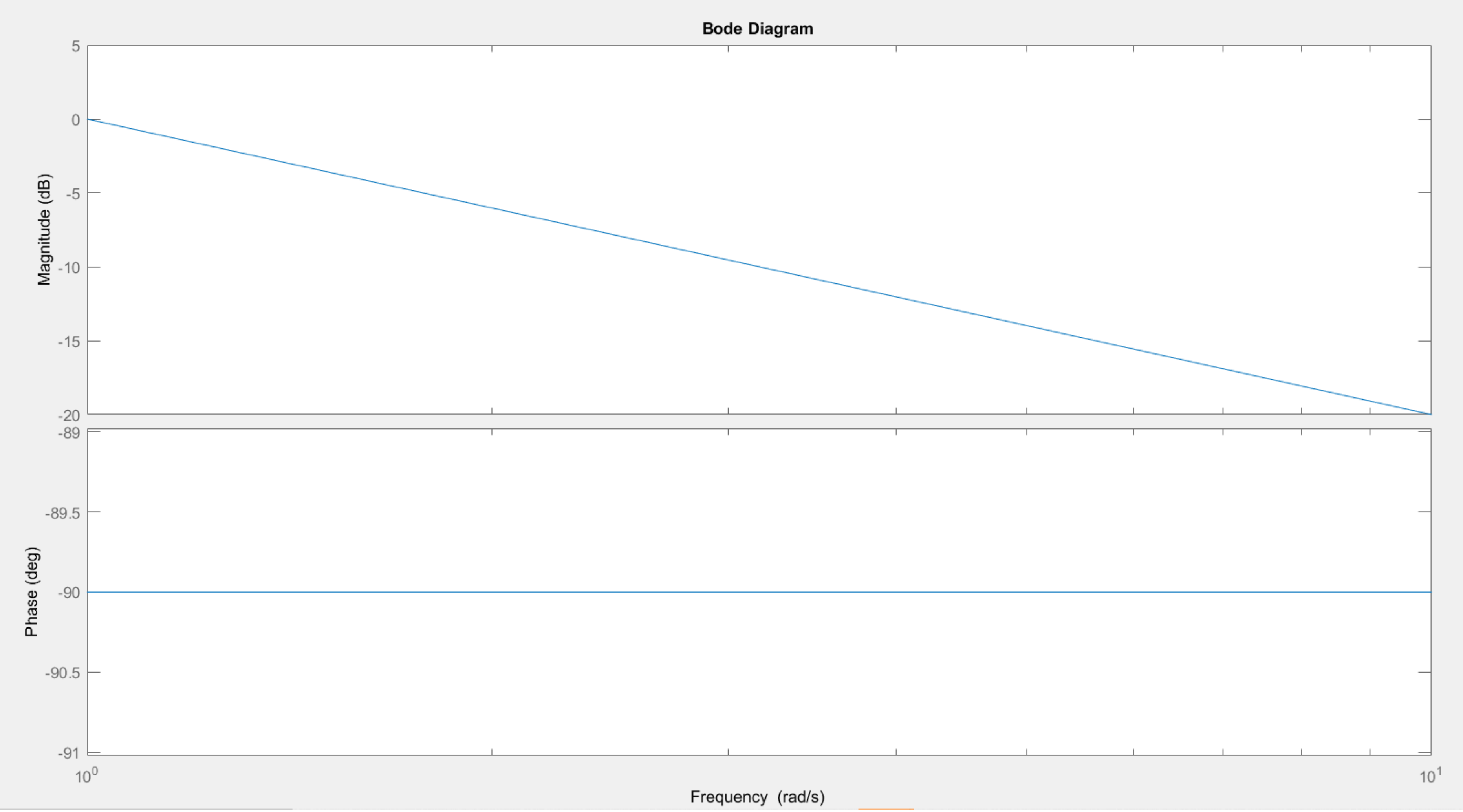
$$|G(j\omega)|\,dB = 20\log\omega^{-1} = -20\log\omega$$

$$\angle G(j\omega) = -90^\circ$$





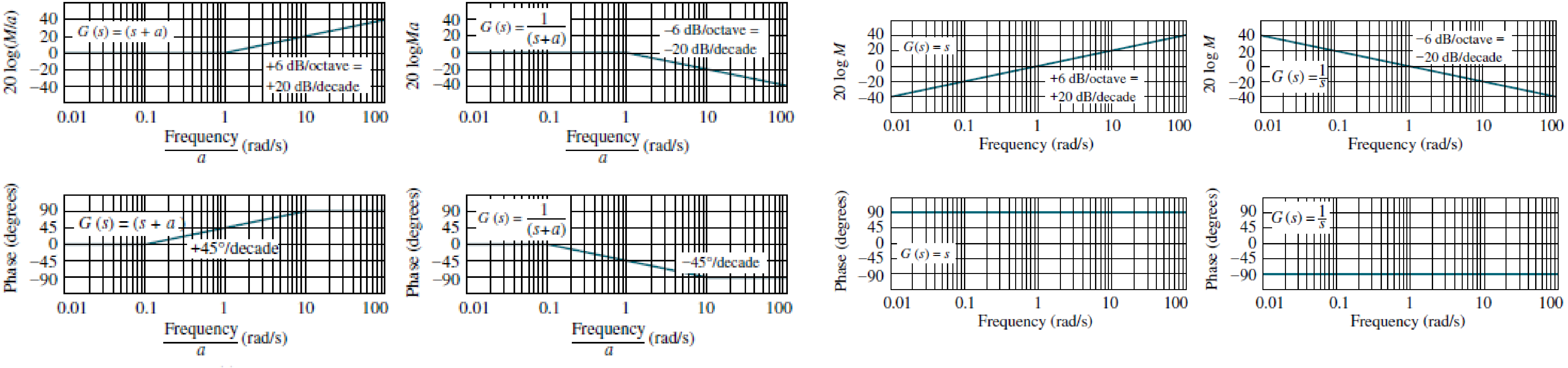
Bode plot for G(s) = 1/s – Using MATLAB




```
s = tf('s');
sys = 1/s

bode(sys)
```

Exercise

- Compare the four Bode plots and identify any patterns that exist.



$G(s) = (s + a)$ $\quad$ $G(s) = 1/(s + a)$ $\quad$ $G(s) = s$ $\quad$ $G(s) = 1/s$

- Make sure that you can logically derive above Bode plots
- Understanding on Bode plots for these basic functions are helpful to create Bode plots for systems in the form of ;

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^m(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Creating bode plots for a given system

- We will now study how to create Bode plots of a function of the following form;

$$G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_k)}{s^m(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- The best way to understand the procedure is through an example.

Example

Draw the Bode plot of the following system

$$G(s) = \frac{s+3}{s(s+1)(s+2)}$$

- Normalize and scale the freq.

$$G(s) = \frac{3\left(\frac{s}{3}+1\right)}{2s\,(s+1)\left(\frac{s}{2}+1\right)}$$

$$G(j\omega) = \frac{3\left(\frac{j\omega}{3}+1\right)}{2j\omega\,(j\omega+1)\left(\frac{1}{2}j\omega+1\right)}$$

Magnitude plot

$$|G(j\omega)|\,dB = 20\log\frac{3}{2} + 20\log\left|\frac{\omega}{3}j+1\right| + 20\log\left|\frac{1}{j\omega}\right| + 20\log\left|\frac{1}{j\omega+1}\right| + 20\log\left|\frac{1}{\frac{\omega}{2}j+1}\right|$$

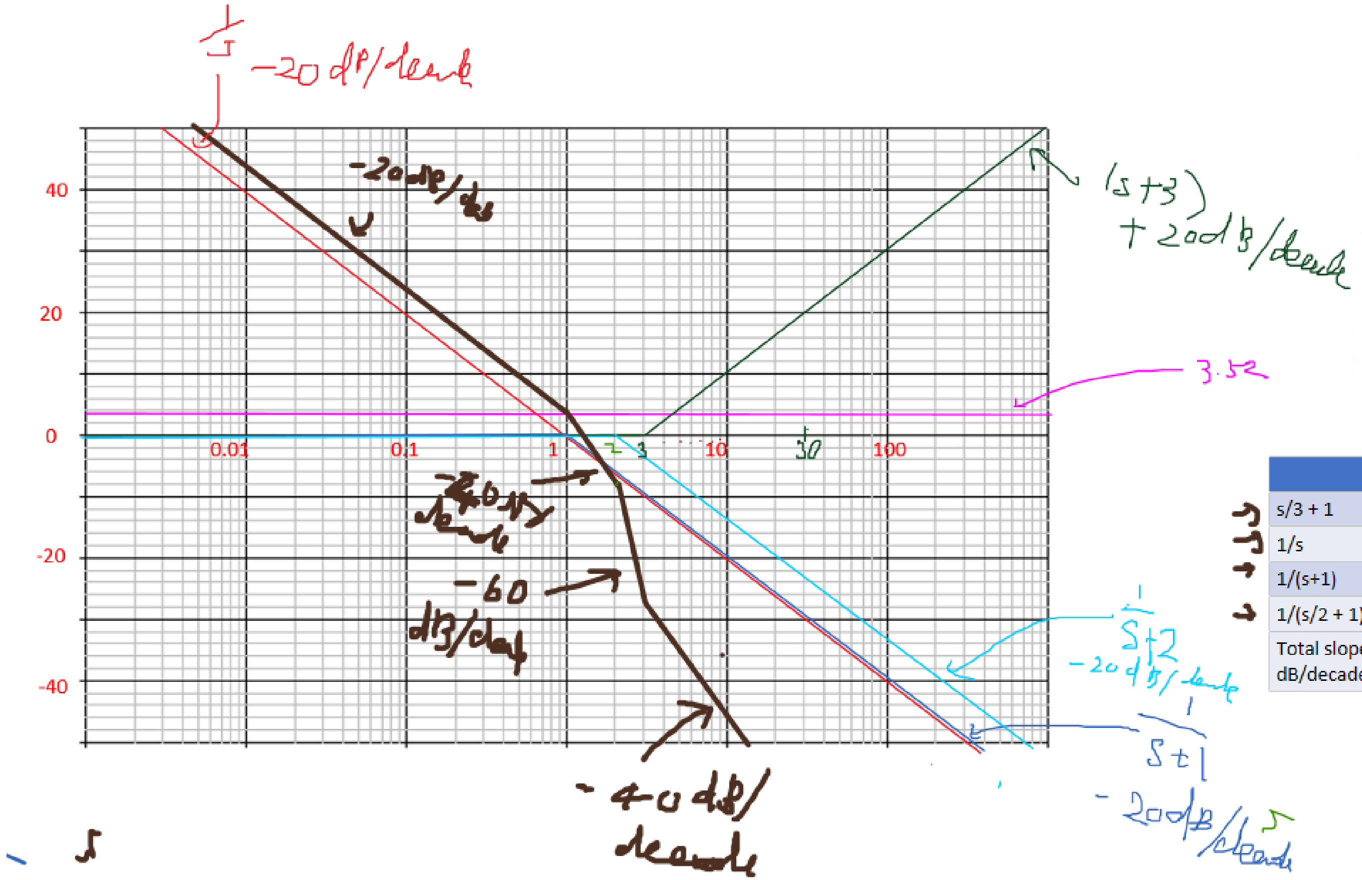
$$= 3.522 + 20\log\left|\frac{\omega}{3}j+1\right| + 20\log\left|\frac{1}{j\omega}\right| + 20\log\frac{1}{|j\omega+1|} + 20\log\frac{1}{\left|\frac{\omega}{2}j+1\right|}$$

Example - Solution

Draw the Bode plot of the following system

$$G(s) = \frac{s+3}{s(s+1)(s+2)}$$

Magnitude frequency response



$$G(s) = \frac{s+3}{s(s+1)(s+2)}$$

Magnitude plot

$$|G(j\omega)|\ dB = 20\log\frac{3}{2} + 20\log\left|\frac{\omega}{3}j+1\right| + 20\log\left|\frac{1}{j\omega}\right| + 20\log\left|\frac{1}{j\omega+1}\right| + 20\log\left|\frac{1}{\frac{\omega}{2}j+1}\right|$$

$$3.522 + 20\log\left|\frac{\omega}{3}j+1\right| + 20\log\left|\frac{1}{j\omega}\right| + 20\log\frac{1}{|j\omega+1|} + 20\log\frac{1}{|\frac{\omega}{2}j+1|}$$

	>0.01	>1	>2	>3
s/3 + 1	0	0	0	20
1/s	-20	-20	-20	-20
1/(s+1)	0	-20	-20	-20
1/(s/2 + 1)	0	0	-20	-20
Total slope dB/decade	-20	-40	-60	-40

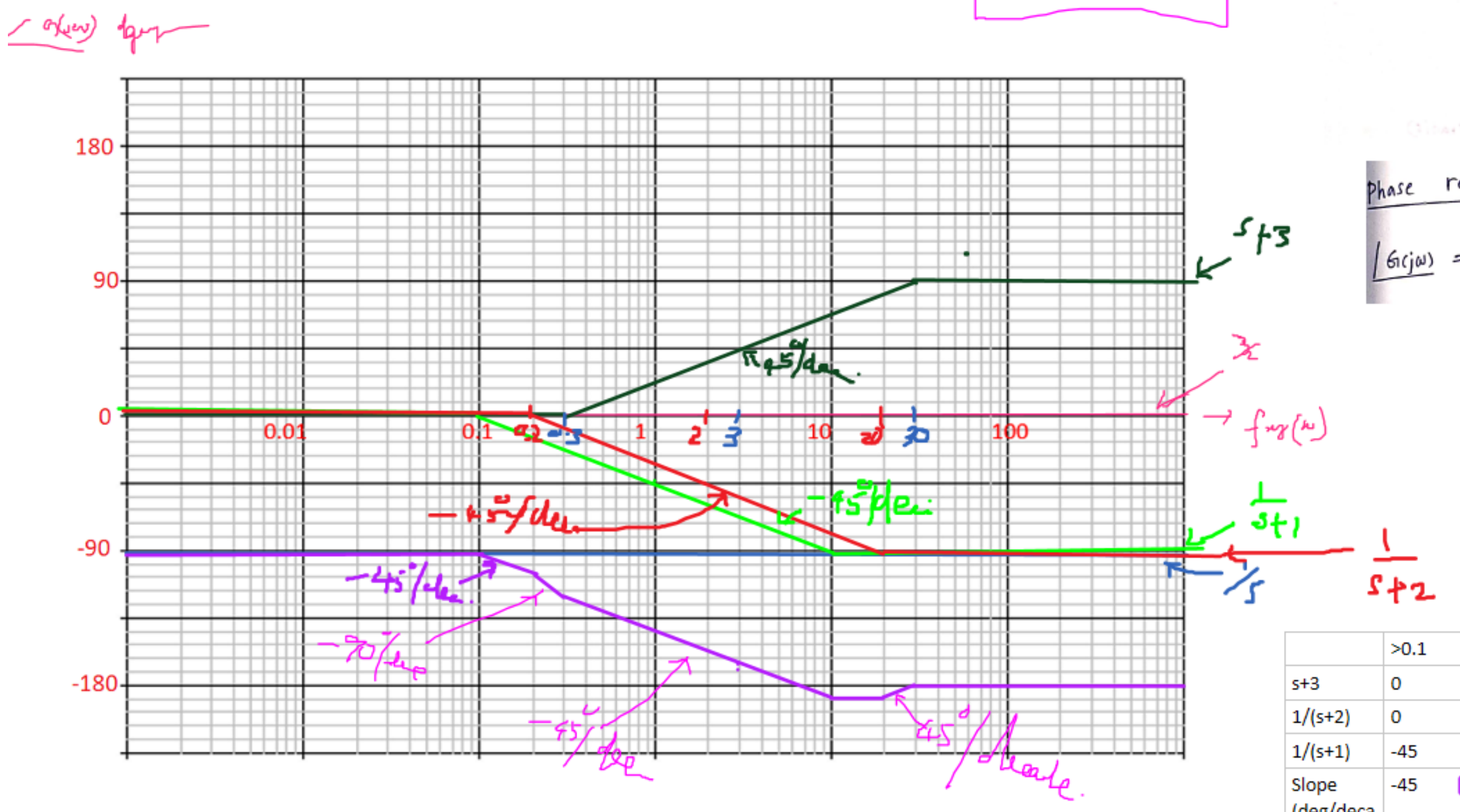
Example - Solution

Draw the Bode plot of the following system

$$G(s) = \frac{s+3}{s(s+1)(s+2)}$$

Phase frequency response

$$\boxed{G(s) = \frac{s+3}{s(s+1)(s+2)}}$$



Magnitude plot

$$|G(j\omega)|\,dB = 20\log\frac{3}{2} + 20\log\left|\frac{\omega}{3}j+1\right| + 20\log\left|\frac{1}{j\omega}\right| + 20\log\left|\frac{1}{j\omega+1}\right| + 20\log\left|\frac{1}{\frac{\omega}{2}j+1}\right|$$

$$= 3.522 + 20\log\left|\frac{\omega}{3}j+1\right| + 20\log\left|\frac{1}{j\omega}\right| + 20\log\frac{1}{|j\omega+1|} + 20\log\frac{1}{|\frac{\omega}{2}j+1|}$$

Phase response

$$\angle G(j\omega) = \angle\frac{1}{j\omega} + \angle\frac{j\omega}{3}+1 + \angle\frac{1}{j\omega+1} + \angle\frac{1}{j\omega/2+1} + \angle\frac{3}{2}$$

	>0.1	>0.2	>0.3	>10	>20	>30	
s+3	0	0	+45	+45	+45	0	
1/(s+2)	0	-45	-45	-45	0	0	
1/(s+1)	-45	-45	-45	0	0	0	
Slope (deg/decade)	-45	-90	-45	0	45	0	

Exercise

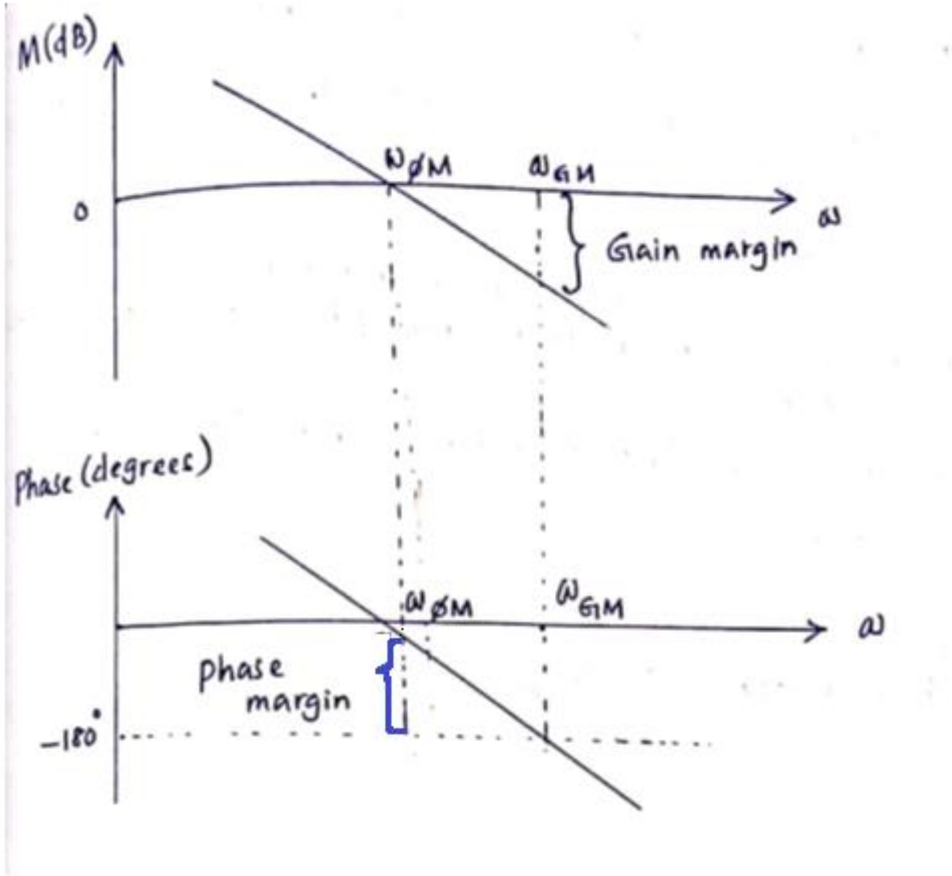
1. Create the Bode plots for the following system transfer functions (in the form of asymptotic approximations)

   1. $G(s) = \frac{3(s+3)}{s}$
   2. $G(s) = \frac{s}{(s+2)(s+3)}$
   3. $G(s) = \frac{1}{s(s+1)(s+2)}$

2. Use MATLAB to plot the Bode plots for each of the systems above and compare them with your asymptotic approximations.

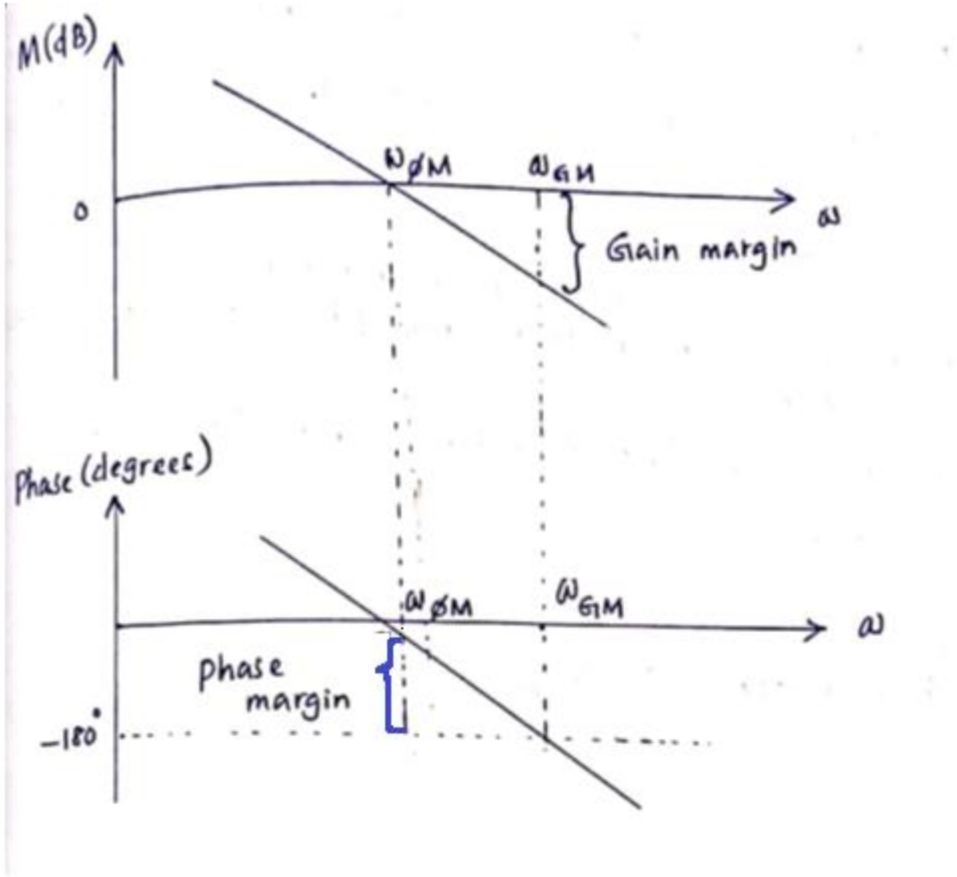
Stability assessment via Bode Plots

An open loop stable system is stable in closed-loop, if the open loop magnitude frequency response has a gain of less than 0 dB at the frequency where the phase frequency response is -180 degrees.



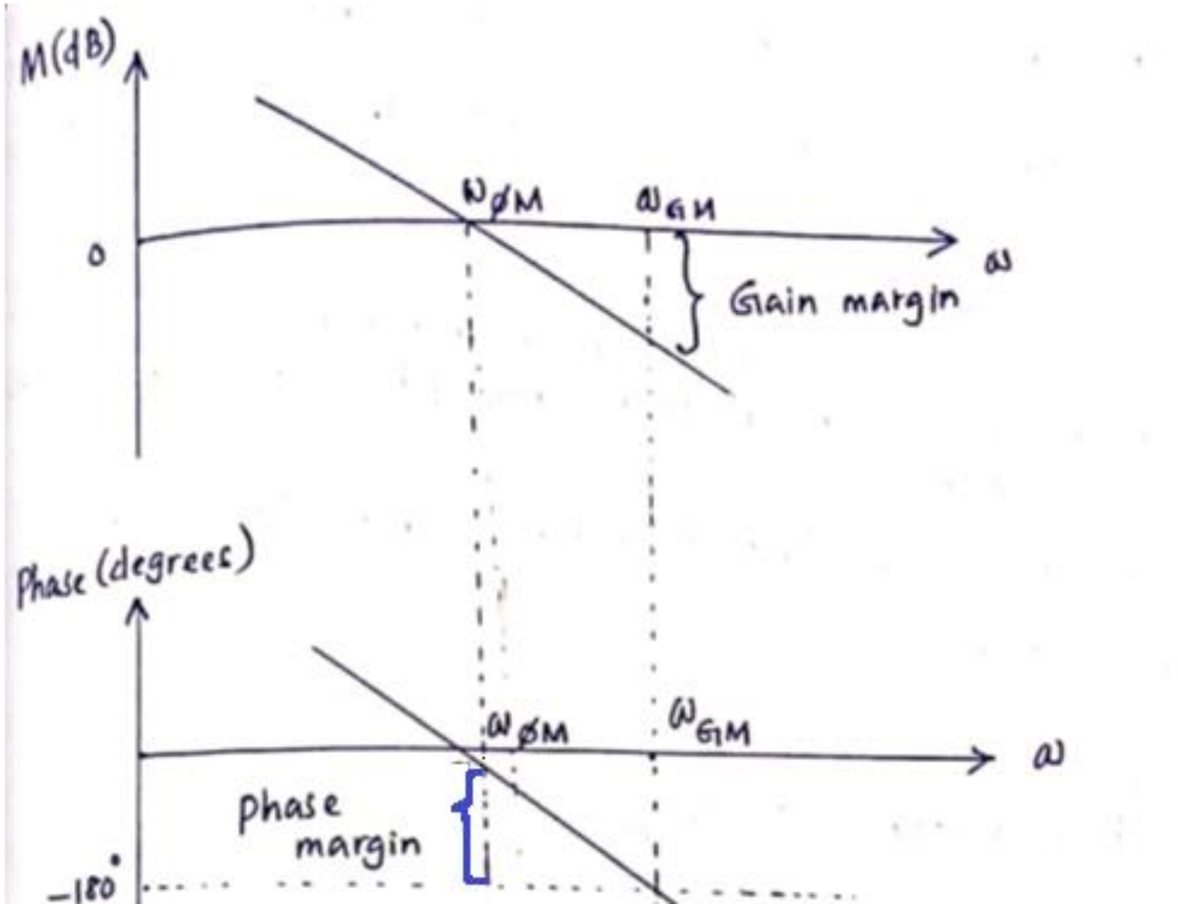
Gain margin G_M

Gain margin is defined as the change in open-loop gain, expressed in dB, required at -180^0 of phase shift to make the closed-loop system unstable



Phase margin Φ_M

The phase margin is defined as the change in open loop phase shift required at unity gain (0 dB) to make the closed loop system unstable.



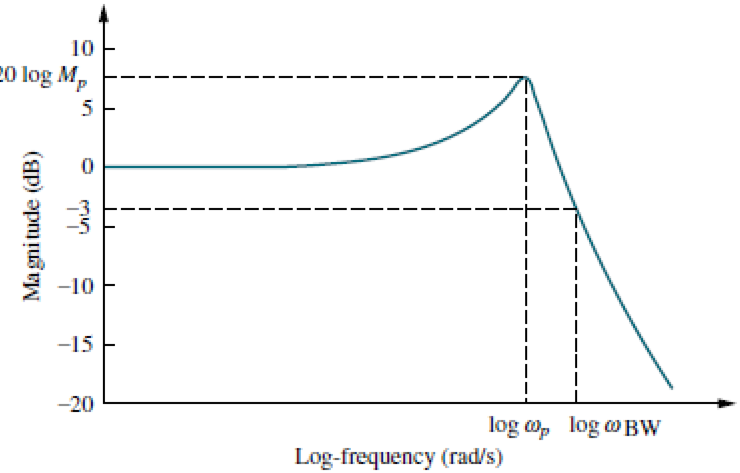
- Gain margin and phase margin of the open-loop system tell how stable the system would be in closed –loop.
- Systems with greater gain and phase margins can withstand greater changes in system parameters before becoming unstable.
- This can be qualitatively related to the fact that a system with poles farther away from the imaginary axis has a greater degree of stability.

Relation Between Closed-Loop Transient and Closed-Loop Frequency Responses

Assumption: Closed- loop system can be approximated by a second order system.

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Magnitude freq. response of a second order system is given below



$$\omega_{BW} = \omega_n \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$\omega_{BW} = \frac{4}{T_s\zeta} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Bandwidth of the closed-loop frequency response → the frequency, ω_{BW}, at which the magnitude response curve is 3 dB down from its value at zero frequency

- Note that the settling time Ts and peak time Tp of the closed-loop step response are inversely proportional to the bandwidth of the closed loop frequency response.
- Speed of the closed-loop response is increased by increasing the bandwidth of the closed-loop frequency response.

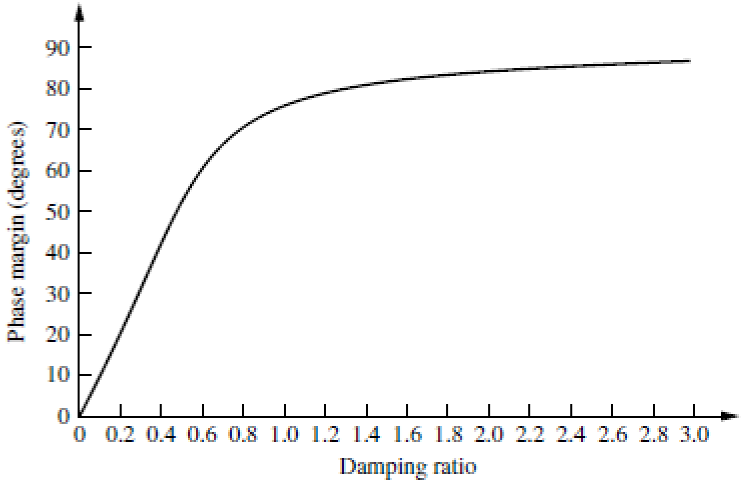
Relation Between Closed-Loop Transient and Open-Loop Frequency Responses

Assumption: Closed- loop system can be approximated by a second order system.

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Phase margin:

$$\Phi_M = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}$$



We know that the % overshoot of the closed loop system step response is given by,

$$M_p\,\% = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100$$

Percent overshoot of the closed loop step response can be reduced by increasing the phase margin of the open-loop frequency response.

Design via frequency response

1. Transient response design via gain adjustment
2. Design of compensators
   A. Lag compensation
   B. Lead compensation

Transient response design via gain adjustment

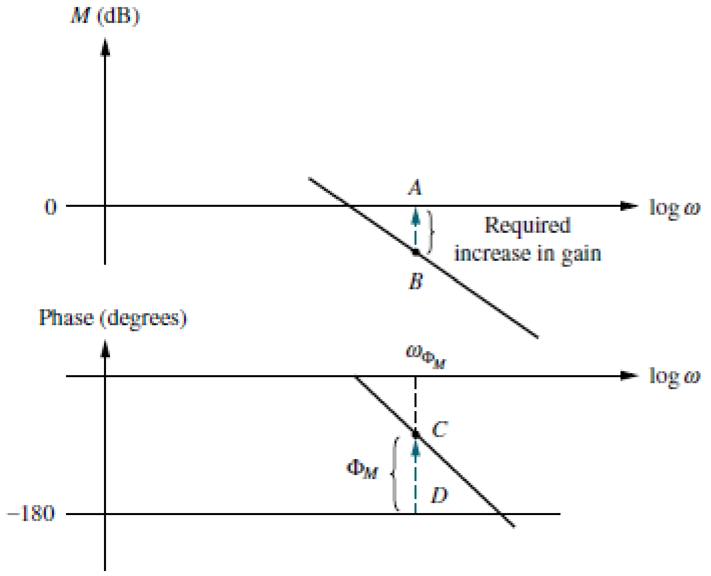
Design procedure to determine the gain to meet a percent overshoot requirement using open-loop frequency response is outlined below assuming dominant second-order closed-loop poles.

1. Draw the Bode magnitude and phase plots for a convenient value of gain.
2. Determine the required phase margin using the following equations.

$$M_p\% = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}}$$

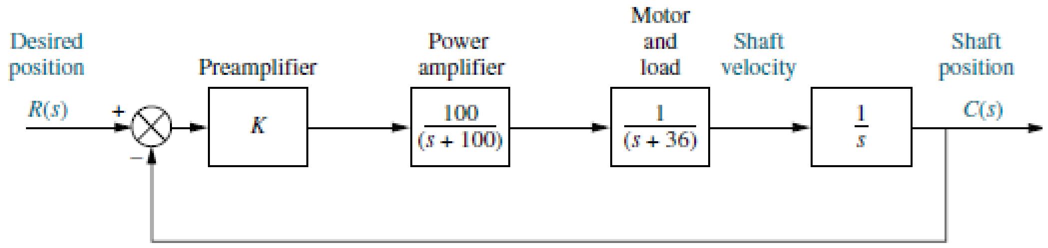
$$\phi_M = \tan^{-1}\frac{2\xi}{\sqrt{-2\xi^2+\sqrt{1+4\xi^4}}}$$

3. Find the frequency, $\omega_{\Phi M}$, on the Bode phase diagram that yields the desired phase margin Φ_M = CD
4. Change the gain by an amount AB to force the magnitude curve to go through 0 dB at $\omega_{\Phi M}$. The amount of gain adjustment is the additional gain needed to produce the required phase margin.



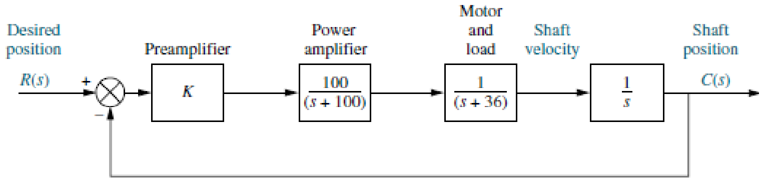
Example

For the position control system shown in the following figure, find the value of the gain K to yield a 9.5% overshoot in the transient response for a step input



Example - solution

For the position control system shown in the following figure, find the value of the gain K to yield a 9.5% overshoot in the transient response for a step input



- Calculate the damping ratio and the required phase margin to achieve a % overshoot of 9.5%

$$G_{OL}(s) = \frac{100K}{(s+100)(s+36)s}$$

$$M_p = 9.5\%$$

$$M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100\% = 9.5\%$$

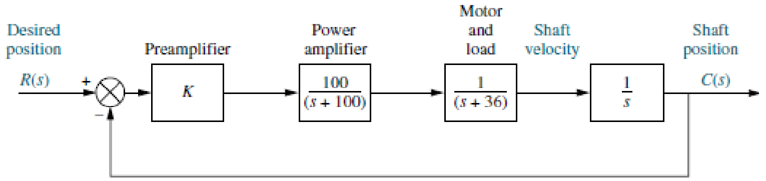
$$\xi = 0.6$$

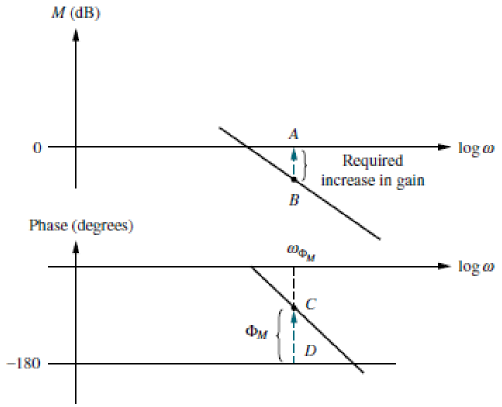
$$\phi_M = \tan^{-1}\frac{2\xi}{\sqrt{-2\xi^2+\sqrt{1+4\xi^4}}}$$

$$\phi_M = 59.2^\circ$$

Example - solution

For the position control system shown in the following figure, find the value of the gain K to yield a 9.5% overshoot in the transient response for a step input





$$G(j\omega) = \frac{100}{j\omega(j\omega+100)(j\omega+36)}$$
$$= \frac{100}{j\omega(-\omega^2+136j\omega+3600)}$$
$$= \frac{100}{-j\omega^3-136\omega^2+3600j\omega}$$
$$= \frac{100}{-136\omega^2+j\omega(3600-\omega^2)}$$
$$= \frac{100\left[-136\omega^2-j\omega(3600-\omega^2)\right]}{(-136\omega^2)^2+\omega^2(3600-\omega^2)^2}$$

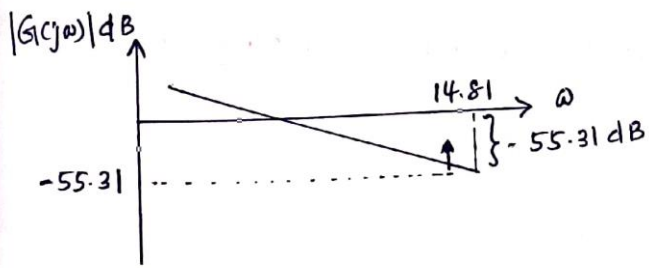
- Find the frequency, $\omega_{\Phi M}$, on the Bode phase diagram that yields the desired phase margin Φ_M

frequency at which $\phi_M = 59.2°$

$$\tan^{-1}\left[\frac{\omega(3600-\omega^2)}{136\omega^2}\right] = -180° + 59.2°$$
$$\frac{\omega(3600-\omega^2)}{136\omega^2} = \tan(-120.8°)$$
$$\omega(3600-\omega^2) = 1.678 \times 136\,\omega^2$$
$$3600 - \omega^2 - 228.208\,\omega = 0$$
$$\omega^2 + 228.208\,\omega - 3600 = 0$$
$$\omega = 14.81 \text{ rad/s}, \; -243.02 \text{ rad/s}$$
$$\omega = 14.81 \text{ rad/s} \quad (\omega > 0)$$

Magnitude at $\omega = 14.81$ rad/s

$$G(j\omega) = \frac{100\left[-136\times14.81^2 - j\,14.81(3600-14.81^2)\right]}{(-136\times14.81^2)^2 + 14.81^2(3600-14.81^2)^2}$$
$$G(j\omega) = -0.0009 - 0.0015j$$
$$G(j\omega) = 0.0017\angle{-120.8°}$$
$$|G(j\omega)| = 0.0017$$
$$|G(j\omega)|\,dB = 20\log 0.0017 = -55.31 \text{ dB}$$



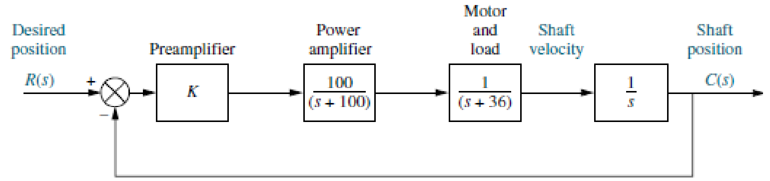
- Change the gain by an amount AB to force the magnitude curve to go through 0 dB at $\omega_{\Phi M}$

Requird gain adjustment = 55.31 dB

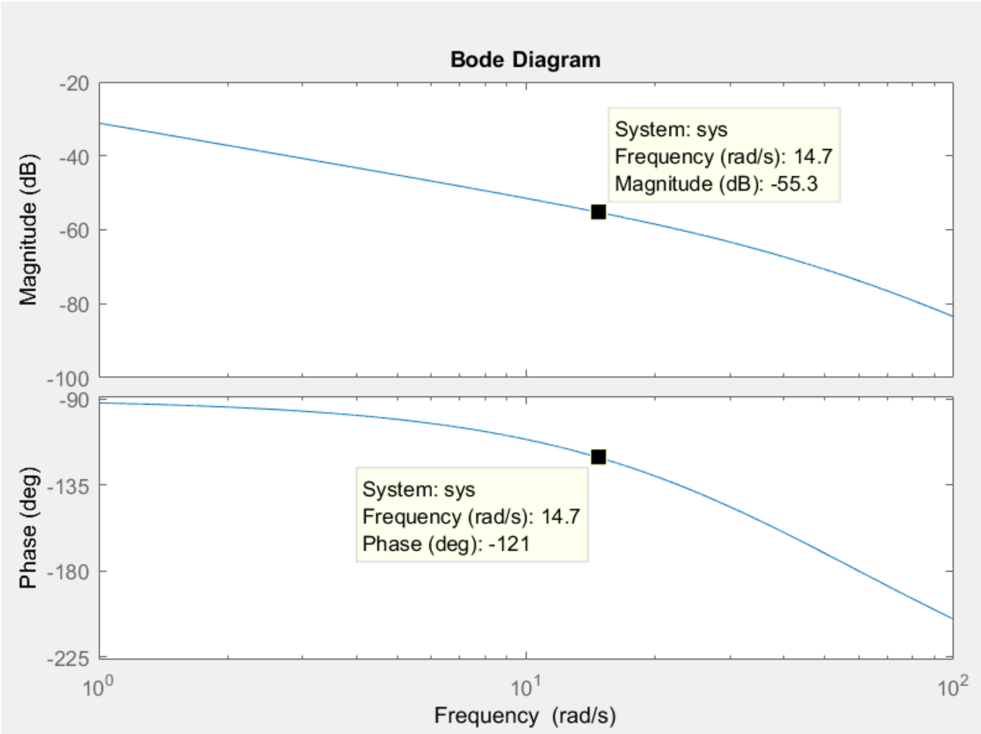
$$55.31 = 20\log K$$
$$\log K = 2.7655$$
$$K = 10^{2.7655}$$
$$K = 582.8$$

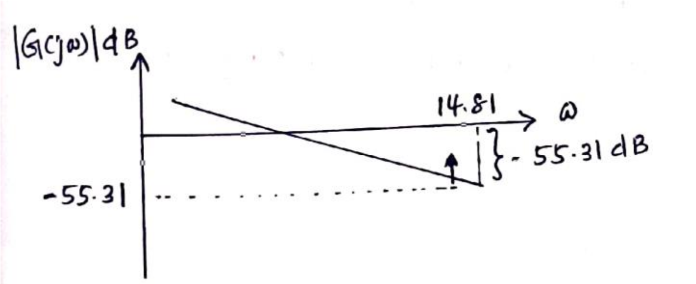
Example – solution – Using MATLAB

For the position control system shown in the following figure, find the value of the gain K to yield a 9.5% overshoot in the transient response for a step input




```
s = tf('s');

sys = 100/(s*(s+100)*(s+36))

w = logspace(0,2,10000);

bode(sys,w)
```





Requird gain adjustment = 55.31 dB

$55.31 = 20 \log K$

$\log K = 2.7655$

$K = 10^{2.7655}$

$K = 582.8$

Lag compensation

- Recall the concepts we learnt about designing a lag compensator using the root-locus

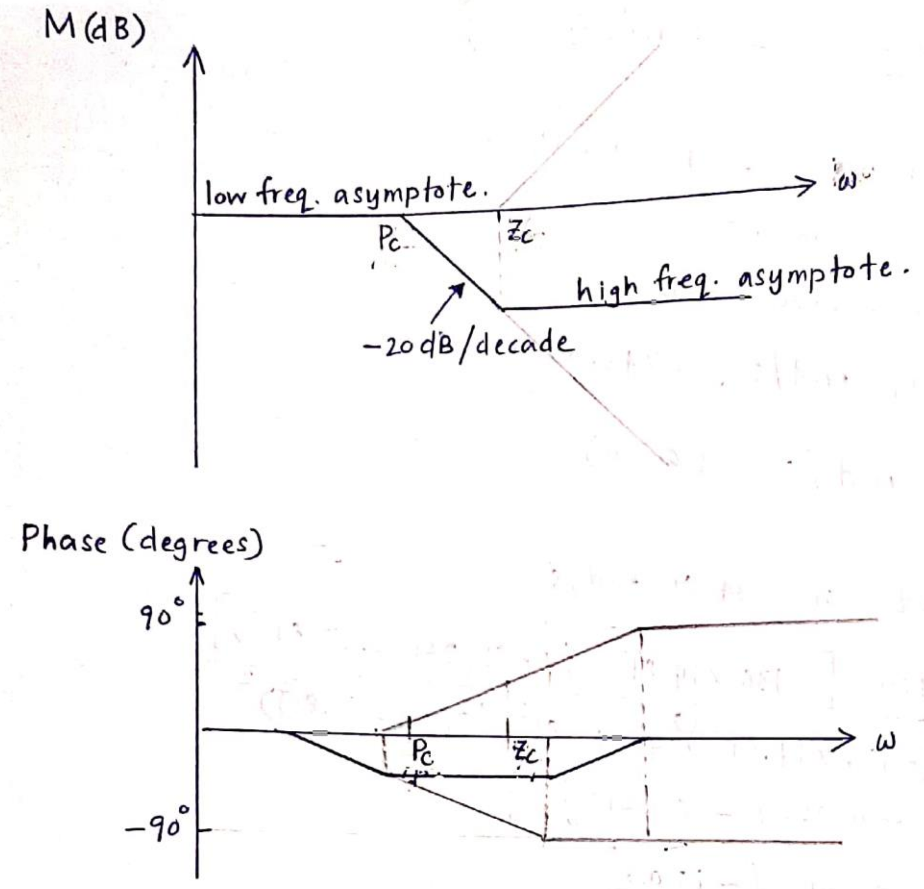
$$G_c(s) = \gamma \frac{s+z_c}{s+p_c} \quad ; \; z_c > p_c$$

- Lag compensator improves the steady state error of the step response.
- In this section, we will learn how to use frequency response technique to design a lag compensator.

- Lag compensation consists of
  - First setting the gain to meet the steady-state error requirement and
  - Then reducing the high-frequency gain to create stability and meet the phase-margin requirement for the transient response.

Lag compensation

$$G_c(s) = \gamma \cdot \frac{s+z_c}{s+p_c} \quad ; \; z_c > p_c$$



Lag compensation – Design procedure

$$G_c(s) = \gamma \cdot \frac{s+Z_c}{s+P_c} \quad ; \; Z_c > P_c$$

1. Set the gain, K, to the value that satisfies the steady-state error specification and plot the Bode magnitude and phase diagrams for this value of gain.
2. Find the frequency where the phase margin is 5^0 to 12^0 greater than the phase margin that yields the desired transient response.

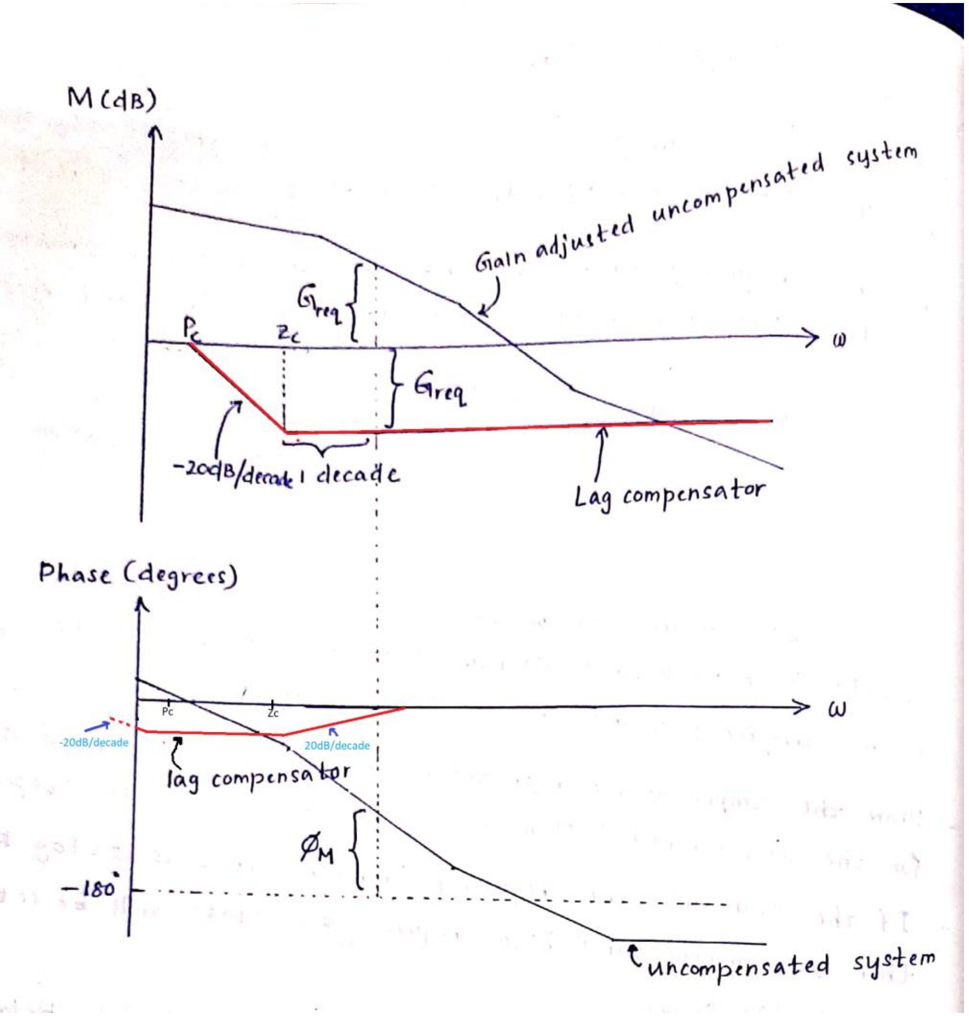
$$\phi_M = \phi_{M,\text{required}} + \underbrace{5^\circ \text{ to } 12^\circ}_{\text{to compensate the phase contribution from the lag compensator.}}$$

3. Select a lag compensator whose magnitude response yields a composite Bode magnitude diagram that goes through 0 dB at the frequency found in Step 2 as follows:
   - Draw the compensator's high-frequency asymptote to yield 0 dB for the compensated system at the frequency found in Step 2. (If the gain at the frequency found in Step 2 is 20 log K_{PM}, then the compensator's high-frequency asymptote will be set at -20 log K_{PM})
   - Select the upper break frequency (Zc) to be 1 decade below the frequency found in Step 2
   - Select the low frequency asymptote to be at 0 dB
   - Connect the compensator's high and low frequency asymptotes with a -20dB/decade line to locate the lower break frequency

4. Set the dc gain of the lag compensator to 1

$$G_{Lag}(s)\big|_{s=0} = 1 \;\Rightarrow\; 1 = \gamma \cdot \frac{Z_c}{P_c} \;\Rightarrow\; \gamma = \frac{P_c}{Z_c}.$$

Lag compensation – Design procedure

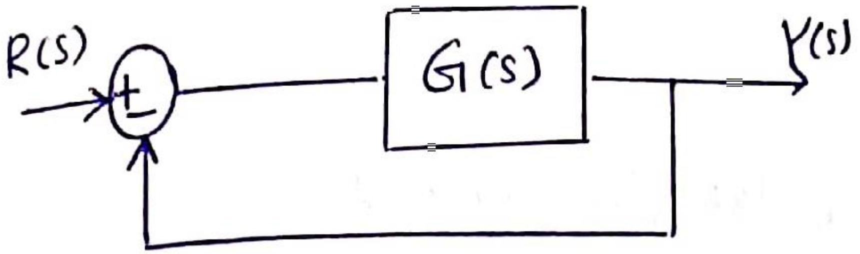


$$G_c(s) = \gamma \frac{s+Z_c}{s+P_c} \quad ; Z_c > P_c$$

Example

The unity feed back system shown in the following figure is operating with a 15% overshoot.

Design a suitable lag compensator to yield a five-fold improvement in steady state error without appreciably changing the transient response.

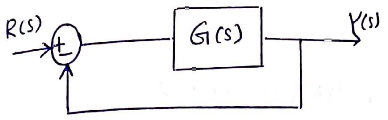


$$G(s) = \frac{K}{(s+2)(s+5)(s+7)}$$

Example

The unity feed back system shown in the following figure is operating with a 15% overshoot.

Design a suitable lag compensator to yield a five-fold improvement in steady state error without appreciably changing the transient response.



$$G(s) = \frac{K}{(s+2)(s+5)(s+7)}$$

- First find the required phase margin to have a 15% overshoot

$$M_p = 15\%$$

$$M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100\% = 9.5$$

$$\xi = 0.517$$

$$\phi_M = \tan^{-1}\frac{2\xi}{\sqrt{-2\xi^2+\sqrt{1+4\xi^4}}} = 53.17^\circ$$

- Calculate the required gain K to achieve a 15% overshoot. (steps are similar to "transient response design via gain adjustment")

$$G(s) = \frac{1}{(s+2)(s+5)(s+7)} = \frac{1}{(s+2)(s^2+12s+35)}$$

$$= \frac{1}{s^3+14s^2+59s+70}$$

Frequency response

$$G(j\omega) = \frac{1}{(j\omega)^3+14(j\omega)^2+59j\omega+70}$$

$$= \frac{1}{-j\omega^3-14\omega^2+59j\omega+70}$$

$$= \frac{1}{(70-14\omega^2)+j\omega(59-\omega^2)}$$

$$= \frac{70-14\omega^2 - j\omega(59-\omega^2)}{(70-14\omega^2)^2-\omega^2(59-\omega^2)^2}$$

$$\angle G(j\omega) = \tan^{-1}\left[\frac{-(59-\omega^2)\omega}{70-14\omega^2}\right]$$

Frequency at which $\phi_M = 53.17^\circ$

$$\tan^{-1}\left[\frac{-(59-\omega^2)\omega}{70-14\omega^2}\right] = 53.17^\circ - 180^\circ = -126.83^\circ$$

$$\frac{-(59-\omega^2)\omega}{70-14\omega^2} = \tan(53.17-180^\circ)$$

$$\omega(\omega^2-59) = 1.335(70-14\omega^2)$$

$$\omega^3+18.69\omega^2-59\omega-93.45 = 0$$

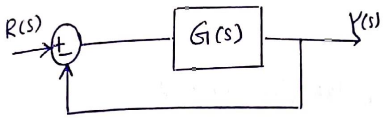
$$\omega = 3.74, -21.25, -1.17$$

$\omega > 0$, $\therefore \omega = 3.74$ rad/s

Example

The unity feed back system shown in the following figure is operating with a 15% overshoot.

Design a suitable lag compensator to yield a five-fold improvement in steady state error without appreciably changing the transient response.



$$G(s) = \frac{K}{(s+2)(s+5)(s+7)}$$

Magnitude at $\omega = 3.74$ rad/s.

$$G(j\omega) = \frac{1}{-j\,3.74^3 - 14 \times 3.74^2 + 59 \times 3.74 j + 70}$$

$$= 4.8 \times 10^{-3} \angle -126.78^\circ$$

$$|G(j\omega)|\ dB = 20 \log 4.76 \times 10^{-3} = -46.45\ dB$$

Required gain adjustment $= 46.45$ dB.

$$46.45 = 20 \log K$$

$$K = \log^{-1}\left(\frac{46.45}{20}\right)$$

$$\boxed{K = 210.14}$$

System will operate at a 15% overshoot at this value of the gain.
Our task is to reduce its steady state error while maintaining the % overshoot at 15%

- Find the steady state error of the uncompensated system

steady state error of the uncompensated system $= e_{ss,uncomp}$

$$e_{ss,uncomp} = \lim_{s \to 0} \frac{s\,R(s)}{1 + G(s)}$$

$$R(s) = 1/s$$

$$e_{ss,uncomp} = \lim_{s \to 0} \frac{1}{1 + G(s)} = \frac{1}{1 + G(0)}$$

$$G(0) = \frac{210.14}{2 \times 5 \times 7} = 3.002$$

$$e_{ss,uncomp} = \frac{1}{1 + 3.002} = 0.25$$

- We can calculate the expected steady state error of the compensated system

steady state error of the compensated system,

$$e_{ss,comp} = \frac{0.25}{5} = 0.05$$

- Find the new gain K required to reduce the steady state error

New gain to yield $e_{ss,comp} = 0.05$

$$e_{ss,comp} = \frac{1}{1 + G(0)}$$

$$1 + G(0) = \frac{1}{e_{ss,comp}}$$

$$\frac{K}{2 \times 5 \times 7} = \frac{1}{e_{ss,comp}} - 1$$

$$K = 70\left(\frac{1}{0.05} - 1\right)$$

$$K = 1330$$

Gain adjusted system

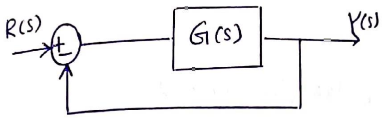
$$G'(s) = \frac{1330}{(s+2)(s+5)(s+7)}$$

At this new gain K = 1330, the steady state error will be at the required level.
But now, the %overshoot will NOT be 15%.
We can use a lag compensator to restore the phase margin at the value required to achieve a 15% overshoot.

Example

The unity feed back system shown in the following figure is operating with a 15% overshoot.

Design a suitable lag compensator to yield a five-fold improvement in steady state error without appreciably changing the transient response.



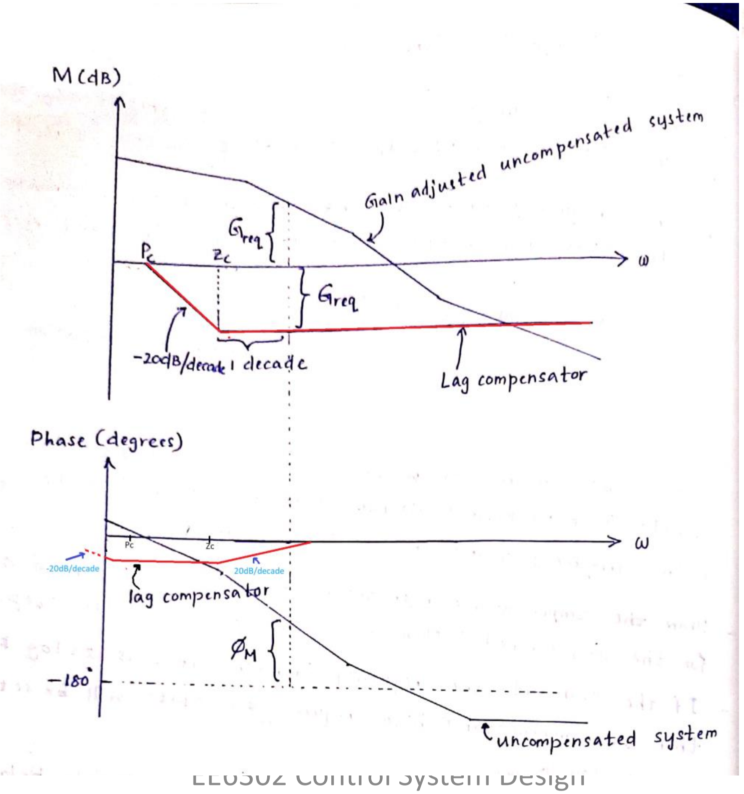
$$G(s) = \frac{K}{(s+2)(s+5)(s+7)}$$

$$G_c(s) = \gamma \frac{s+z_c}{s+p_c} \quad ; \; z_c > p_c$$

Required phase margin, $\phi_{M,req} = 53.17^\circ$

$$\phi_M = \phi_{M,req} + 10^\circ$$
$$= 53.17^\circ + 10^\circ$$
$$= 63.17^\circ$$

We consider a phase margin 5 to 12 degrees higher.
Why ? Go back to the phase frequency plot of the lag compensator. It will contribute anywhere from -5 to -12 degrees of phase at the phase-margin frequency.



Find the frequency where phase margin is 63.17°

Frequency at which $\phi_M = 63.17^\circ$,

$$\tan^{-1}\left[\frac{-(59-\omega^2)\omega}{70-14\omega^2}\right] = -180^\circ + 63.17^\circ$$

$$\frac{\omega(\omega^2-59)}{70-14\omega^2} = \tan(-116.83^\circ)$$

$$\omega^3 - 59\omega = 1.98(70-14\omega^2)$$

$$\omega^3 + 27.72\omega^2 - 59\omega - 138.6 = 0$$

$$\omega = 3.27, \; -29.56, \; -1.43$$

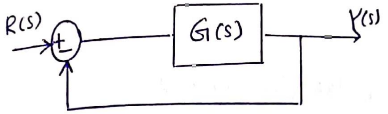
$\omega > 0$, $\omega = 3.27$ rad/s

phase margin frequency $\phi_{WM} = 3.27$ rad/s

Example

The unity feed back system shown in the following figure is operating with a 15% overshoot.

Design a suitable lag compensator to yield a five-fold improvement in steady state error without appreciably changing the transient response.



$$G(s) = \frac{K}{(s+2)(s+5)(s+7)}$$

Connect the compensator's high and low frequency asymptotes with a -20dB/decade line to locate the lower break frequency

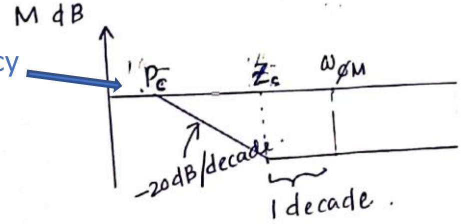
Gain adjusted system

$$G'(s) = \frac{1330}{(s+2)(s+5)(s+7)}$$

- At $\omega = 3.27$ rad/s, the compensated system's magnitude must go to 0 dB. The magnitude at $\omega = 3.27$ rad/s

$$|G'(j\omega)| = 7.517$$

$$|G'(j\omega)|\,dB = 17.52 \text{ dB}.$$

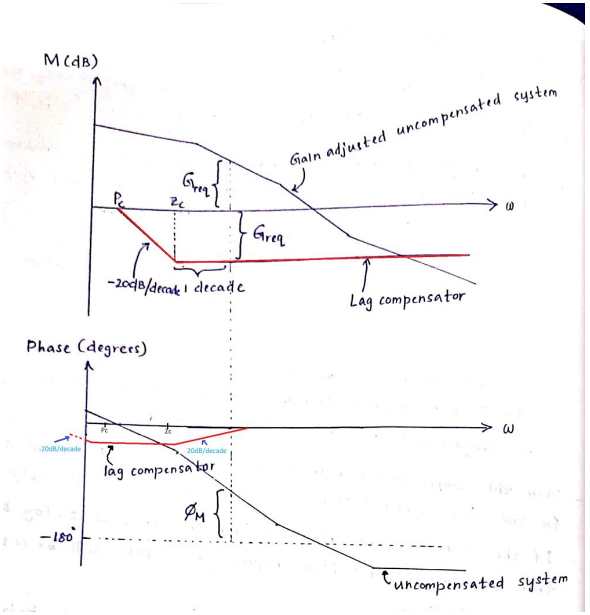


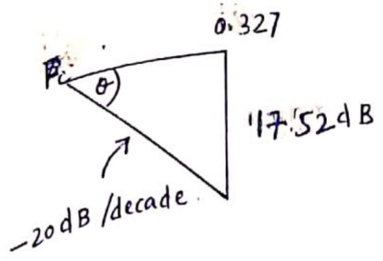
Low frequency asymptote

High-frequency asymptote

$$Z_c = \frac{\omega_{\phi M}}{10} = \frac{3.27}{10} = 0.327 \text{ rad/s}$$

Calculate Zc such that it is one decade below phase margin freq





$$\tan\theta = 20 \text{ dB/decade}.$$

$$\frac{17.52}{\log 0.327 - \log P_c} = 20$$

$$\log P_c = \log 0.327 - \frac{17.52}{20}$$

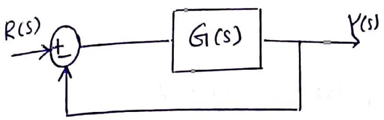
$$\log P_c = -1.361$$

$$P_c = 0.044$$

Example

The unity feed back system shown in the following figure is operating with a 15% overshoot.

Design a suitable lag compensator to yield a five-fold improvement in steady state error without appreciably changing the transient response.



$$G(s) = \frac{K}{(s+2)(s+5)(s+7)}$$

The lag compensator should have a dc gain of unity to retain the value of $e_{ss,comp}$.

$$G_{Lag}(s) = \gamma \frac{s+0.327}{s+0.044}$$

$$G_{Lag}(s)\big|_{s=0} = 1$$

$$\frac{0.327\,\gamma}{0.044} = 1$$

$$\gamma = 0.135$$

$$G_{Lag}(s) = \frac{0.135\,(s+0.327)}{(s+0.044)}$$

$$G_{OL,comp}(s) = \frac{0.135\,(s+0.327)}{s+0.044} \cdot \frac{1330}{(s+2)(s+5)(s+7)}$$

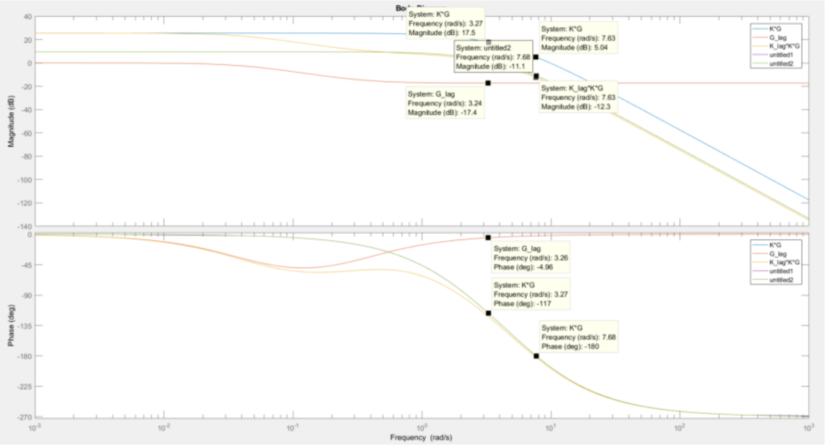
Exercise

Plot the bode plots for the different transfer functions as explained in the lecture (FRT_8) and understand the effect of designed lag compensator.

- What can you say about the closed loop stability of the original system without any adjustments (i.e 210 G(s) open loop tf)?
- What can you say about the closed loop stability of the gain adjusted system to improve the steady state error.

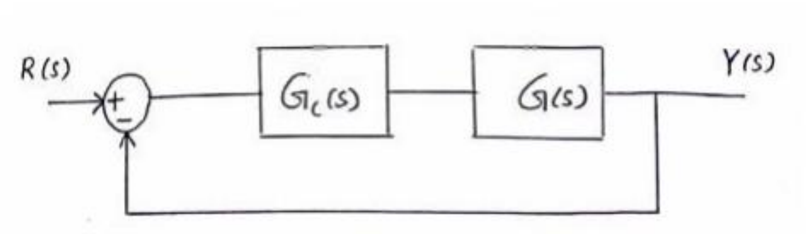
Check in the plots;
- The phase margin of the compensated system and see whether it corresponds to the required phase margin to achieve a 15% overshoot
- Whether the compensated system will be stable in closed loop

Logically explain using the plot, why we considered a correction of +10 degrees in our calculations for the phase margin.




```
s = tf('s');
w = logspace(-3,3,1000000);

G = 1/((s+2)*(s+5)*(s+7)) % G(s)

G_lag = 0.135*(s+0.327)/(s+0.044) %designed lag compensator

bode(210.14*G,1330*G,G_lag, G_lag*1330*G,w)
legend('210.14*G','1330*G','G_lag', '1330*G_lag*G')
```

Lead compensation

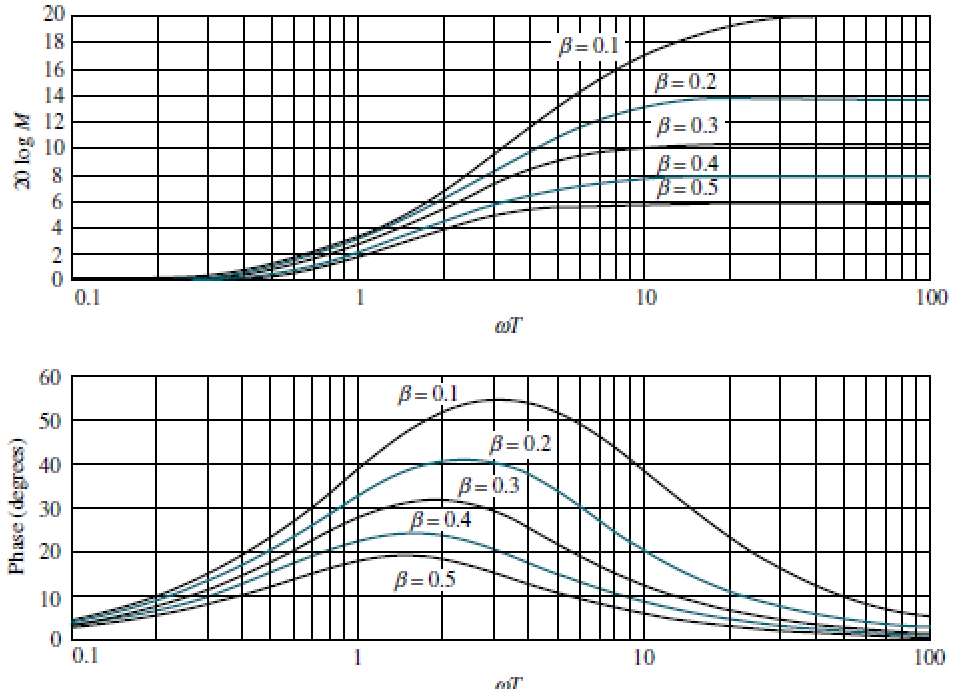


$$G_c(s) = K \cdot \frac{s + z_c}{s + P_c} \quad ; \quad P_c > z_c$$

We can write the transfer function or a lead compensator in the following alternative format

$$G_c(s) = \frac{1}{\beta} \frac{s + \dfrac{1}{T}}{s + \dfrac{1}{\beta T}}$$

where β < 1.

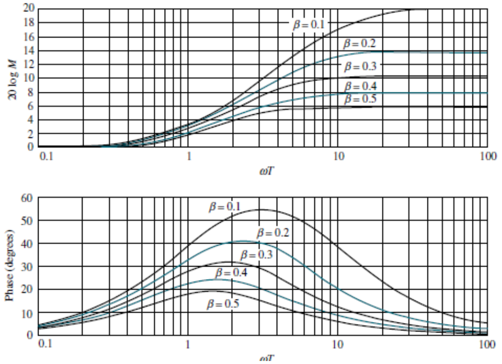


<u>Exercise</u>: Draw the bode plots for the lead compensator in terms of asymptotic approximations

Lead compensation

$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

where β < 1.



$$G_c(j\omega) = \frac{1}{\beta}\,\frac{j\omega + 1/T}{j\omega + 1/\beta T} = \frac{j\omega T + 1}{j\omega\beta T + 1}$$

phase angle of the lead compensato

$$\phi_c = \tan^{-1}\omega T - \tan^{-1}\omega\beta T$$

differentiating with respect to ω,

$$\frac{d\phi_c}{d\omega} = \frac{T}{1+(\omega T)^2} - \frac{\beta T}{1+(\omega\beta T)^2}$$

at $\phi_c = \phi_{c,max}$, $\frac{d\phi_c}{d\omega} = 0$

$$\frac{T}{1+(\omega_{max}T)^2} - \frac{\beta T}{1+(\omega_{max}\beta T)^2} = 0$$

$$\frac{1}{1+(\omega_{max}T)^2} = \frac{\beta}{1+(\omega_{max}\beta T)^2}$$

$$1+(\omega_{max}\beta T)^2 = \beta + \beta(\omega_{max}T)^2$$

$$(\omega_{max}\beta T)^2 - \beta(\omega_{max}T)^2 = \beta - 1$$

$$\omega_{max}^2 T^2 \beta(\beta - 1) = \beta - 1$$

$$\omega_{max} = \frac{1}{T\sqrt{\beta}}$$

Freq when $\phi_c = \phi_{c,max}$

$$\phi_{c,max} = \tan^{-1}(\omega_{max}T) - \tan^{-1}(\omega_{max}\beta T)$$

$$= \underbrace{\tan^{-1}\left(\frac{1}{\sqrt{\beta}}\right)}_{\phi_1} - \underbrace{\tan^{-1}\sqrt{\beta}}_{\phi_2}$$

$$\tan\phi_{c,max} = \tan(\phi_1 - \phi_2)$$

$$\tan\phi_{c,max} = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\cdot\tan\phi_2}$$

$$\tan\phi_{c,max} = \frac{\frac{1}{\sqrt{\beta}} - \sqrt{\beta}}{1+\frac{1}{\sqrt{\beta}}\cdot\sqrt{\beta}}$$

$$\tan\phi_{c,max} = \frac{1-\beta}{2\sqrt{\beta}}$$

Maximum phase angle

$$\phi_{c,max} = \tan^{-1}\left(\frac{1-\beta}{2\sqrt{\beta}}\right) = \sin^{-1}\left(\frac{1-\beta}{1+\beta}\right)$$

The lead compensator's magnitude at $\omega = \omega_{max}$

$$G_c(j\omega_{max}) = \frac{j\omega_{max}T+1}{j\omega_{max}\beta T+1} = \frac{j\frac{1}{\sqrt{\beta}}+1}{j\sqrt{\beta}+1} = \frac{j+\sqrt{\beta}}{j\beta+\sqrt{\beta}}$$

$$|G_c(j\omega_{max})| = \frac{\sqrt{1+\beta}}{\sqrt{\beta^2+\beta}} = \frac{1}{\sqrt{\beta}}$$

$$|G_c(j\omega_{max})| = \frac{1}{\sqrt{\beta}}$$

Lead compensation

Lead compensation increases the bandwidth by increasing the gain crossover frequency.

At the same time, phase margin is raised at higher frequencies

The result of this is a larger phase margin and a higher phase margin frequency.

In the time domain, this corresponds to a lower percent overshoot with smaller peak time and settling time.

Lead compensation – Design Procedure

$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

1. Lead compensator has negligible effect at low frequencies. Therefore, set the gain K of the uncompensated system to the value that satisfies the steady state error requirement.
2. Plot the Bode plots at this value of the gain and determine the uncompensated system's phase margin
3. Find the phase margin required to meet the damping ratio or % overshoot requirement. Then evaluate the additional phase contribution required from the compensator.

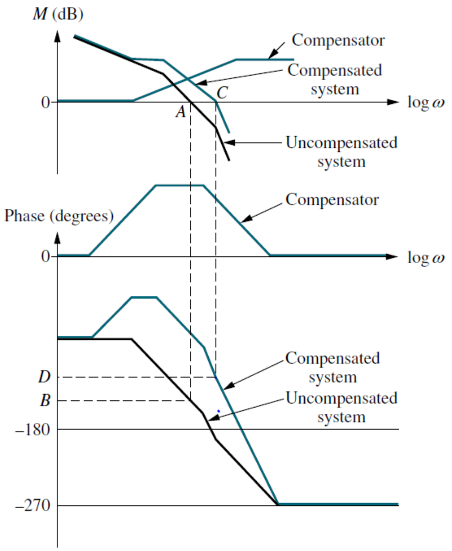
$$\phi_{M,req} = \phi_{PM,org} + \phi_{Lead} - \underbrace{\phi_{corr}}_{\text{correction factor}}$$

ϕ_{corr} = Correction factor to compensate for the lower frequency uncompensated system's phase angle at the higher phase margin frequency.

$\phi_{corr} \approx 10^\circ$

Phase contribution required from ~~the~~ lead compensator

$$\phi_{Lead} = \phi_{M,req} - \phi_{PM,org} + \phi_{corr}$$



4. Determine the value of β from the lead compensator's required phase margin

$$\phi_{Lead} = \phi_{Lead,max} = \tan^{-1}\left(\frac{1-\beta}{2\sqrt{\beta}}\right) = \sin^{-1}\left(\frac{1-\beta}{1+\beta}\right)$$

5. Determine the compensators magnitude at the peak of the phase curve.

$$|G_c(j\omega_{max})| = \frac{1}{\sqrt{\beta}}$$

6. Determine the new phase margin frequency $\omega_{\phi M}$ by finding where the uncompensated system's magnitude curve is the negative of the lead compensator's magnitude at the peak of the phase curve.
7. Determine the value of T

$$\omega_{\phi M} = \omega_{max}. \qquad \omega_{\phi M} = \frac{1}{T\sqrt{\beta}}$$

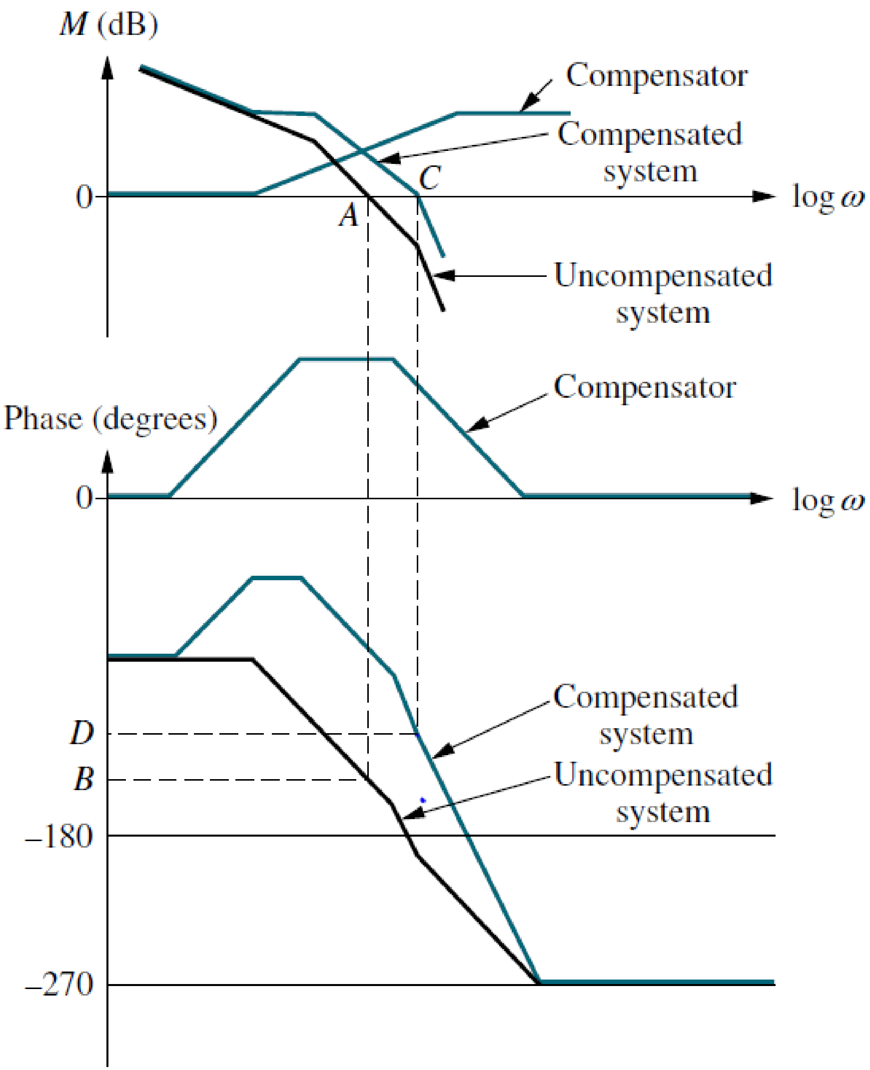
$$T = \frac{1}{\omega_{\phi M}\sqrt{\beta}}$$

8. Check if the closed loop bandwidth requirement to meet the settling time, peak time specification etc. is met in your design

$$\omega_{BW} = \frac{4}{T_s\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Lead compensation – Design Procedure



$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

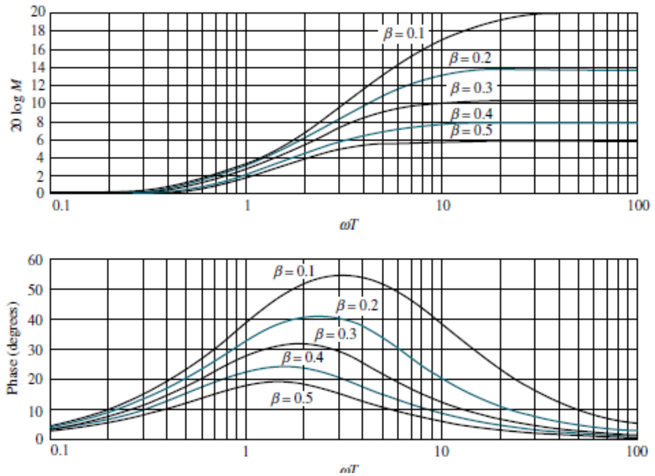
$$\phi_{PM,req} = \phi_{PM,org} + \phi_{Lead} - \underbrace{\phi_{corr}}_{\text{correction factor}}.$$

ϕ_{corr} = Correction factor to compensate for the lower frequency uncompensated system's phase angle at the higher phase margin frequency.

$\phi_{corr} \approx 10^\circ$

Phase contribution required from ~~the~~ lead compensator

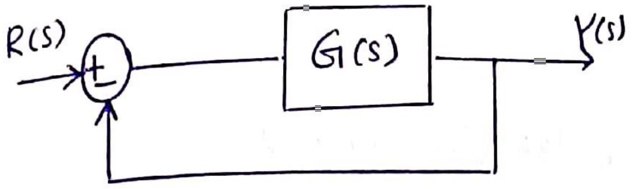
$$\phi_{Lead} = \phi_{PM,req} - \phi_{PM,org} + \phi_{corr}$$



Lead compensation – Example

The unity feedback system shown in the following figure should operate with a 5% steady state error and 15% percent overshoot for a unit step response.

Design a suitable lead compensator to improve the settling time.

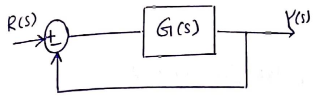


$$G(s) = \frac{K}{(s+2)(s+5)}$$

Lead compensation – Example - Solution

The unity feedback system shown in the following figure should operate with a 5% steady state error and 15% percent overshoot for a unit step response.

Design a suitable lead compensator to improve the settling time.



$$G(s) = \frac{K}{(s+2)(s+5)}$$

$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

$$e_{ss} = \frac{1}{1+K\,G(0)} = 0.05$$

$$\frac{1}{1+\frac{K}{2\times 5}} = 0.05$$

$$1+\frac{K}{10} = 20$$

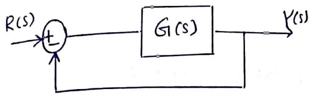
$$K = 190$$

Gain adjusted system, $G(s) = \dfrac{190}{(s+2)(s+5)}$

Lead compensation – Example - Solution

The unity feedback system shown in the following figure should operate with a 5% steady state error and 15% percent overshoot for a unit step response.

Design a suitable lead compensator to improve the settling time.



$$G(s) = \frac{K}{(s+2)(s+5)}$$

$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

Gain adjusted system, $G(s) = \frac{190}{(s+2)(s+5)}$

Frequency response, $G(j\omega) = \frac{190}{(j\omega+2)(j\omega+5)}$

$$G(j\omega) = \frac{190}{-\omega^2+7j\omega+10} = \frac{190}{(10-\omega^2)+7\omega j}$$

$$= \frac{190\,(10-\omega^2-7\omega j)}{(10-\omega^2)^2+49\omega^2}$$

$$|G(j\omega)| = 190\left\{\left[\frac{10-\omega^2}{(10-\omega^2)^2+49\omega^2}\right]^2+\left[\frac{-7\omega}{(10-\omega^2)^2+49\omega^2}\right]^2\right\}^{1/2}$$

Phase margin frequency of the gain adjusted system

$$|G(j\omega)| = 1$$

$$190\left\{\left[\frac{10-\omega^2}{(10-\omega^2)^2+49\omega^2}\right]^2+\left[\frac{-7\omega}{(10-\omega^2)^2+49\omega^2}\right]^2\right\}^{1/2} = 1$$

$$190^2\left[(10-\omega^2)^2+49\omega^2\right] = \left[(10-\omega^2)^2+49\omega^2\right]^2$$

$$190^2 = (10-\omega^2)^2+49\omega^2$$

$\omega^2 = x$, $\quad 190^2 = (10-x)^2+49x$

$$36100 = 100-20x+x^2+49x$$

$$x^2+29x-36000 = 0$$

$$x = 175.79, \ \cancel{-204.79}$$

$$\omega^2 = 175.79$$

$$\omega = \omega_{\phi M} = 13.258 \text{ rad/s}$$

Phase margin of the gain adjusted system.

$$G(j\omega)\Big|_{\omega=13.258} = \frac{190}{(13.258j+2)(13.258j+5)}$$

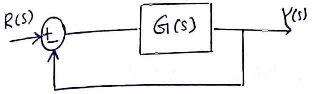
$$= 1\angle -150.76^\circ$$

$$\phi_{PM_{org}} = -150.76^\circ + 180^\circ = 29.26^\circ$$

Lead compensation – Example - Solution

The unity feedback system shown in the following figure should operate with a 5% steady state error and 15% percent overshoot for a unit step response.

Design a suitable lead compensator to improve the settling time.



$$G(s) = \frac{K}{(s+2)(s+5)}$$

$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

Phase margin required to maintain 15% overshoot,

$$M_p = 15\%$$

$$M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times \cancel{1}00 = 15$$

$$\xi = 0.517$$

$$\phi_{PM,req} = \tan^{-1}\frac{2\xi}{\sqrt{-2\xi^2+\sqrt{1+4\xi^4}}} = 53.17^\circ$$

Phase contribution required from the lead compensator

$$\phi_{Lead} = \phi_{M,req} - \phi_{PM,org} + \phi_{corr}$$

$$= 53.17^\circ - 29.26^\circ + 10^\circ$$

$$= 33.91^\circ$$

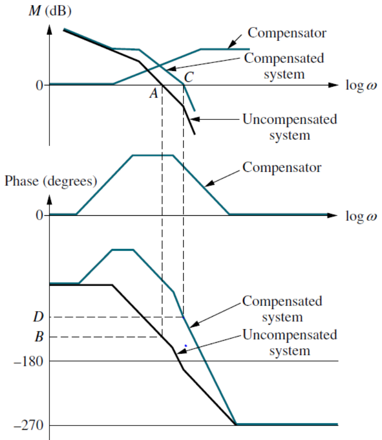
$$\phi_{Lead} = \phi_{Lead,max} = \sin^{-1}\left(\frac{1-\beta}{1+\beta}\right)$$

$$\sin^{-1}\left(\frac{1-\beta}{1+\beta}\right) = 33.91^\circ$$

$$(1-\beta) = (1+\beta)\ 0.5579$$

$$\beta = \frac{1-0.5579}{1+0.5579}$$

$$\beta = 0.284$$



Compensator's magnitude at the peak of the phase curve.

$$|G_{Lead}(j\omega_{max})| = \frac{1}{\sqrt{\beta}} = \frac{1}{\sqrt{0.284}} = 1.876 = 5.47\ dB.$$

Frequency at which the uncompensated system's magnitude ~~is~~ equals to the negative of the lead compensator's magnitude at the peak of the compensator's phase curve.

$$|G(j\omega)| = -5.47\ dB = 0.533.$$

$$190^2 = 0.533^2 \ldots \left[(10-\omega^2)^2 + 49\omega^2\right]$$

$$\omega^2 = x, \quad 190^2 = 0.533^2\left[(10-x)^2 + 49x\right]$$

$$127072.8 = 100 - 20x + x^2 + 49x.$$

$$x^2 + 29x - 126972.8 = 0$$

$$x = 342.1,\ \cancel{-371.1}$$

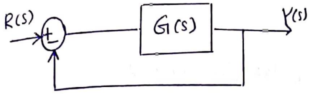
$$\omega^2 = 342.1$$

$$\omega = 18.49\ rad/s$$

Lead compensation – Example - Solution

The unity feedback system shown in the following figure should operate with a 5% steady state error and 15% percent overshoot for a unit step response.

Design a suitable lead compensator to improve the settling time.



$$G(s) = \frac{K}{(s+2)(s+5)}$$

$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

Compensator's magnitude at the peak of the phase curve.

$$|G_{Lead}(j\omega_{max})| = \frac{1}{\sqrt{\beta}} = \frac{1}{\sqrt{0.284}} = 1.876 = 5.47 \text{ dB}.$$

Frequency at which the uncompensated system's magnitude equals to the negative of the lead compensator's magnitude at the peak of the compensator's phase curve.

$$|G(j\omega)| = -5.47 \text{ dB} = 0.533$$

$$190^2 = 0.533^2 \cdot [(10-\omega^2)^2 + 49\omega^2]$$

$$\omega^2 = x, \quad 190^2 = 0.533^2[(10-x)^2 + 49x]$$

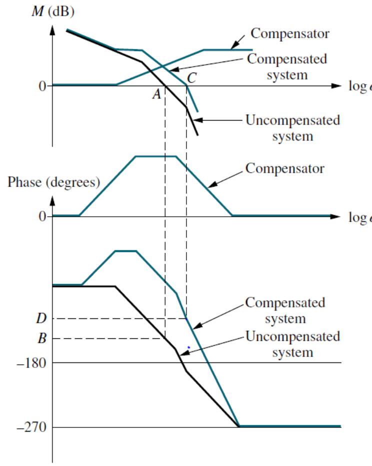
$$127072.8 = 100 - 20x + x^2 + 49x$$

$$x^2 + 29x - 126972.8 = 0$$

$$x = 342.1, \; \cancel{-371.1}$$

$$\omega^2 = 342.1$$

$$\omega = 18.49 \text{ rad/s}$$



Phase margin frequency of the compensated system = 18.49 ra

$$\omega_{PM,comp} = 18.49 \text{ rad/s}$$

Lead compensator's $\omega_{max} = \omega_{PM,comp} = 18.49$ rad/s

$$\omega_{max} = \frac{1}{T\sqrt{\beta}}$$

$$T = \frac{1}{\omega_{max}\sqrt{\beta}} = \frac{1}{18.49\sqrt{0.284}} = 0.101$$

$$G_{lead}(s) = \frac{1}{\beta} \cdot \frac{s + 1/T}{s + 1/\beta T}$$

$$= \frac{1}{0.284} \frac{s + 1/0.101}{s + \frac{1}{0.284 \times 0.101}}$$

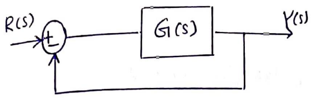
$$= 3.521 \left(\frac{s+9.9}{s+34.86}\right)$$

$$\text{Compensated system} = 3.521\left(\frac{s+9.9}{s+34.86}\right)\frac{190}{(s+2)(s+5)}$$

Exercise

The unity feedback system shown in the following figure should operate with a 5% steady state error and 15% percent overshoot for a unit step response.

Design a suitable lead compensator to improve the settling time.



$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

$$G_{lead}(s) = \frac{1}{\beta} \cdot \frac{s+1/T}{s+1/\beta T}$$

$$= \frac{1}{0.284} \frac{s + 1/0.101}{s + \frac{1}{0.284 \times 0.101}}$$

$$= 3.521 \left(\frac{s + 9.9}{s + 34.86} \right)$$

Compensated system $= 3.521 \left(\frac{s+9.9}{s+34.86} \right) \frac{190}{(s+2)(s+5)}$

Plot the bode plots of the uncompensated system, lead compensator and the compensated system on using MATLAB
Understand the effect of compensation using your plot.
You may refer to the MATLAB script explained for the lag compensator design.