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Constructing Einstein Equation along with some Cosmological applications

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1 INTRODUCTION

When Einstein formulated his special theory of relativity in 1905, he based it on these two following postulates:

- The laws of physics are same in all inertial frames.
- The speed of light in vacuum is constant and same as observed from all inertial frames.

But what makes SR special, and GR general? Special relativity focuses on how the universe looks to two different observers that are moving relative to each other with constant velocity in a flat space and in the absence of gravitational; on the other hand, what makes GR general is because it allows us to describe the physics of the universe from the point of view of both accelerated and inertial observers and the GR theory extends SR to describe gravitation itself; which makes it more general form of relativity.

2 SPECIAL RELATIVITY[2]

Working Background:

1. flat spacetime, aka using only orthonormal (Cartesian-like) coordinates.

2. 4-dimensional spacetime: three of space, one of time: Minkowski space
3. The constructed coordinate system is an inertial frame.
4. A single moment in space and time, characterized uniquely by (t, x, y, z) and is defined as 'Event'.

We introduce spacetime interval between two events:

$$s^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

This quantity is invariant under changes of coordinates:

$$s^2 = -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2$$

2.1 Notation

Coordinates on spacetime will be denoted by letters with Greek superscript indices running from 0 to 3, with 0 generally denoting the time coordinate.

$$x^\mu : x^0 = ct, x^1 = x, x^2 = y, x^3 = z$$

in another frame:

$$x^{\mu'} : x^{0'} = ct, x^{1'} = x, x^{2'} = y, x^{3'} = z$$

2.2 Metric

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.1}$$

We introduce summation convention, in which indices which appear both as superscripts and subscripts are summed over:

$$s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$$

2.3 Coordinate transformations in spacetime

To transform it linear transformation is to multiply x μ by a (spacetime-independent) matrix:

$$x^{\mu'} = \Lambda^{\mu'}_{\nu} x^\nu$$

What kind of matrices will leave the interval invariant? The matrices which satisfy this condition are known as the Lorentz transformations.

2.4 FOUR-VECTORS

Four-vector is an object with four components, which transform in a specific way under Lorentz transformation. We see that any abstract vector A can be written as a linear combination of basis vectors:

$$\Delta z^{\alpha} = c\Delta x^{\alpha} + b\Delta y^{\alpha}$$

Example: four-velocity:

$$U^{\alpha} = \frac{dx^{\alpha}}{d\tau}$$

3 TENSORS

I used tensor algebra as a given for the following sections. For derivations, refer to [2] Chapter 3.

4 EINSTEIN'S EQUATION[1]

4.1 PARALLEL TRANSPORT AND GEODESICS

The definition of parallel transport is a tensor that doesn't change along a curve, such that is $\vec{U}_{\beta}$ be the tangent to a curve. A tensor field V is parallelly transported along the curve if $\nabla_{\vec{U}}\vec{V} = 0$.
geodesic definition : a geodesic generalizes the notion of a "straight line" to curved spacetime. If we choose $\vec{V} = \vec{U}$, we are parallel-transporting the tangent vector to the curve:

$$\nabla_{\vec{U}}\vec{U} = 0$$

In component notation:

$$U^{\beta}U^{\alpha}_{;\beta} = U^{\beta}U^{\alpha}_{,\beta} + U^{\beta}\Gamma^{\alpha}_{\nu\beta}U^{\nu} = \frac{d^2x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\nu\beta}\frac{dx^{\beta}}{d\tau}\frac{dx^{\nu}}{d\tau} = 0$$

4.2 RIEMANN TENSOR[4]

The Riemann tensor expresses how the components of the vector V_{α} vary when it is parallel-transported along a closed circuit. Deriving The Riemann tensor from this concept, we find that:

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} - \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\sigma}_{\beta\mu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\nu\beta}$$

4.3 BIANCHI IDENTITIES

First, we multiply Reimann tensor with $g_{\alpha\lambda}$ to get covariant Riemann tensor

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(g_{\alpha\nu,\beta\mu} - g_{\beta\nu,\alpha\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu})$$

Then by differentiating the above equation with respect to x^λ, gives:

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} - g_{\beta\mu,\alpha\nu\lambda} + g_{\beta\nu,\alpha\beta\lambda})$$

Using the fact that $g_{\alpha\beta}$ is symmetric and that partial derivatives commute, we find:

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0$$

in a general reference frame, becomes:

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$$

Which is known as the Bianchi identities.

4.4 RICCHI TENSOR

Ricci is contraction of the Riemann tensor:

$$R_{\beta\alpha} = R_{\nu\beta\mu\alpha} g^{\mu\nu}$$

Ricci tensor is symmetric:

$$R_{\beta\alpha} = R_{\alpha\beta}$$

4.4.1 RICCHI SCALAR TENSOR

We get Ricchi Scalar Tensor from contracting the Ricchi Tensor:

$$g^{\alpha\beta} R_{\alpha\beta} = R$$

4.5 EINSTEIN TENSOR

Apply Ricchi contraction to the Bianchi identities by multiplying the Bianchi identities with the metric tensor:

$$g^{\alpha\mu}[R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu}] = 0$$

we find that:

$$R_{\beta\nu;\lambda} + (-R_{\beta\lambda;\nu}) + R_{\mu\beta\nu\lambda} = 0$$

Then by contracting again, we find that:

$$(-2R^\nu_\lambda + \delta^\nu_\lambda R)_{;\nu} = 0$$

Next, we define Eienstein tensor:

$$G^\alpha_\beta = R^\nu_\lambda - \frac{1}{2}\delta^\nu_\lambda R$$

or,

$$G^{\alpha\beta} = R^{\nu\lambda} - \frac{1}{2}\delta^{\nu\lambda} R$$

However, we need to equate Eienstein tensor, which represents the curvuture of space, to the energy–momentum tensor , which represents the mass-energy of matter:

$$G^{\alpha\beta} = 8\pi G T^{\alpha\beta}$$

We will determine this constant in the next section.

4.6 Determining the constant in Einstein's Equation

In the Newton's version:

$$F = -m\nabla\Phi(x) = ma$$

such that

$$a = -\nabla\Phi(x)$$

Poisson field equation for the potential:

$$\nabla^2\Phi = 4\pi G\rho$$

then

$$\Phi = \frac{-MG}{r}$$

In the Newtonian limit, the 00 component of the metric g_{00}, corresponds to the Newtonian potential Φ from Schwarzchild raduis metric,

$$ds^2 = (1 - \frac{2MG}{r})dt^2 - \frac{dr^2}{1 - \frac{2MG}{r}} - r^2 d\Omega^2$$

we found that:

$$g_{00} = 1 - \frac{2MG}{r} = 1 + 2\Phi$$

then,

$$\nabla^2 = 8\pi G\rho$$

For the energy-momentum tensor: The four-momentum p is defined as (E, p^m)
Where p^0 is the energy density E of the particle and The other components are its spatial momentum.
let the density of the energy $p^0 = T^{00}(x)$
as the continuity equation of energy:

$$\frac{DT^{0\mu}}{Dx^\mu} = 0$$

The flow of energy is along direction $x\prime$ where in $T^{0\mu}$, 0 is the energy, and μ is the spacial component along $x\prime$
we can do the same for $T^{\nu 0}$ so in general:

$$\frac{DT^{\nu\mu}}{D^\mu} = 0$$

then

$$\nabla^2 g_{00} = 8\pi G\rho = 8\pi G T^{00}$$

so the right hand side of Eisenstein's equation must be:

$$G^{\mu\nu} = 8\pi G T^{\mu\nu}$$

5 APPLICATIONS

5.1 DERIVING SCHWARZSCHILD METRIC [3]

Let's assume spherical symmetry, means that the line element of the metric can depend only on rotational invariant, so there's no special direction in a spatial direction.
For rotational invariants quantities:

$$t, dt, r^2 = x.x, rdr = xdx, dx^2 = dr$$

$$dx^2 = dr^2 + r^2(d\theta^2 + sin(\theta)d\theta^2)$$

Hence, the most general spherical metric:

$$ds^2 = -C(r,t)dt^2 + E(r,t)dr^2 - 2D(r,t)dtdr + F(r,t)r^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

We can simplify this general equation by choosing different coordinates t', r', θ, ϕ and getting rid of some of these function

1. removing the function F by defining $r' = \sqrt{F(r,t)}r$ and $t' = t$
2. removing the function D by setting $dt' = f(t,r)(Cdt + Ddr)$

After some manipulations, finally we get that:

$$ds^2 = \frac{-1}{f^2}\frac{dt'^2}{C} + \frac{D^2}{C}dr^2 + Edr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

After combining the terms and dropping the primes, we get:

$$ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

Then the Metric is:

$$g_{\mu\nu} = \begin{bmatrix} -A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2(\theta) \end{bmatrix} \tag{5.1}$$

Now we can calculate the connection symbol, from:

$$\Gamma^{\lambda}_{\beta\mu} = \frac{1}{2}g^{\alpha\lambda}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})$$

Then we use the connection symbols to calculate the Reimann tensor, then we contract it to get the Ricci tensor.
Finally we solve for the function A and B, we found that:

$$B = \frac{1}{1 + \frac{k}{r}}$$

$$A = 1 + \frac{k}{r}$$

Therefore

$$ds^2 = -(1+\frac{k}{r})dt^2 + \frac{1}{1+\frac{k}{r}}dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

To solve for the constant k: at large r -> $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$
such that $h_{\mu\nu} \simeq \frac{k}{r}$
recall that $h_{00} = \frac{-2\phi}{c^2}$ and $\phi = \frac{-GM}{r}$
Note that ϕ is the Newtonian potential outside a spherical mass M.
Then

$$h_{00} \simeq \frac{-k}{r} = \frac{2GM}{rc^2}$$

$$k = \frac{-2GM}{c^2}$$

Therefore

$$ds^2 = -(1-\frac{2GM}{rc^2})dt^2 + \frac{1}{1-\frac{2GM}{rc^2}}dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2)$$

5.2 Deriving the Friedmann equation [5]

In this approach, we are using Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -c^2dt^2 + a^2(t)[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + sin^2(\theta)d\phi^2)]$$

Then the Metric is:

$$g_{\mu\nu} = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \frac{a^2(t)}{1-Kr^2} & 0 & 0 \\ 0 & 0 & a(t)^2r^2 & 0 \\ 0 & 0 & 0 & a(t)^2r^2sin^2(\theta) \end{bmatrix} \tag{5.2}$$

Now we calculate the connection symbol, such that:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})$$

Now we can calculate the Riemann tensor, such that:

$$R^{\mu}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho}$$

Ricci curvature tensor:

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$$

$$R_{00} = -3\frac{\ddot{a}(t)}{a(t)}$$

$$R_{oi} = 0$$

$$R_{ij} = \frac{\ddot{a}(t)a(t) + 2\dot{a}(t)^2 + 2K}{a(t)^2} g_{ij}$$

Ricci Scalar:

$$R = g^{\mu\nu} R_{\mu\nu} = g^{ij} R_{ij} = \frac{6(\ddot{a}(t)a(t) + \dot{a}(t)^2 + K)}{a(t)^2}$$

For the energy momentum tensor $T_{\mu\nu}$:

$$T_{00} = \rho(t)$$

$$T_{0i} = 0$$

$$T_{ij} = \rho(t) g_{ij}$$

Evaluate Eisenstein's equation for (00) component:
Eisenstein's equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

00 component:

$$-3\frac{\ddot{a}(t)}{a(t)} + \frac{3(\ddot{a}(t)a(t) + \dot{a}(t)^2 + K)}{a(t)^2} - \Lambda = 8\pi G T_{\mu\nu}$$

simplify:

$$\frac{\dot{a}(t) + K}{a(t)^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3} \rho(t)$$

Hence, this is the first Friedmann Equation.

$$D(\lambda) = \frac{S_o}{4}(\lambda - \frac{\lambda_o^4}{\lambda^3}) = \frac{0.09 ps/nm^2.km}{4}(1550nm - \frac{1312^4}{1550^3}) = 16.97 ps/nm.km \tag{5.3}$$

References

[1] List of formulas in Riemannian geometry. List of formulas in riemannian geometry — Wikipedia, the free encyclopedia, 2018. [Online; accessed 30-July-2018].

[2] Bernard F. Schutz. *A first course in general relativity.* Cambridge University Press, 2015.

[3] Neil Turok. Perimeter institute, lecture video: Relativity - lecture 9, 2011. [Online; accessed 30-July-2018].

[4] Fridolin Weber. San diego state university, lecture notes: Riemann curvature tensor and einstein tensor, 2018. [Online; accessed 30-July-2018].

[5] Timm Wrase. Vienna university of technology, lecture notes: Deriving the friedmann equations from general relativity, 2018. [Online; accessed 30-July-2018].