Coupled Pendula Experiment

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Coupled Pendula Experiment

***Abstract*—In this experiment, we examined the coupled pendulums to learn more about coupled oscillator. We investigated the coupled pendulum two normal modes, which are the odd and the even mode. We found that the frequencies for even modes are $0.75Hz$, and the frequencies for the odd modes are all lower than their theoretical value. Then we examined the beat frequency by choosing the appropriate spring constant and coupling length values, the beating frequencies are found to be $f = 0.097Hz$, $0.080Hz$, $0.106Hz$, $0.039Hz$, all of which are also below the theoretical values.**

I. Introduction

Many physical systems involve Coupled oscillators, for example, in solidstate physics, we can describe atoms oscillations around their equilibrium positions in terms of coupled oscillators. In this experiment, we are investigating coupled oscillators using coupled pendulums system.

in the coupled pendulums system, the two pendulums can exchange energy between them, such that, we can setup them to move in a such complex way, however, with certain initial conditions, we can examine certain unique frequencies to the system. The system consists of two pendulums with the same mass with a length L, the length is divided into four coupling lengths l. In the experiment, we are examining the system behaviour when combining different strings with string constant k, with different coupling lengths.

We are examining three initial conditions to the system which give rise to both oscillations in phase and opposite in phase and beating case. The first two oscillations are called normal modes, the mode in phase is called even mode, and the one opposite in phase is called odd mode. Also we are studying the beating case where one of the pendulum is at rest and the other is reaching it's maximum amplitude and so the energy is constantly transferred between them. In our experiment, we will be using Fourier analysis to determine the angular frequency ω from each collected dataset and then we will compare it to the theoretical predictions.

II. Theoretical Background [1]

A. In single pendulum case

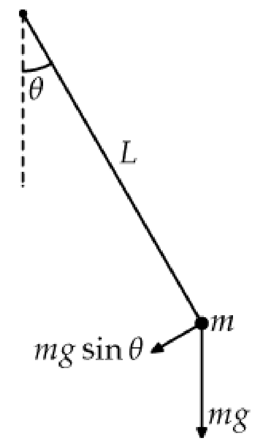

Fig. 1. Single simple pendulum diagram

Consider we have two pendulums in the system, they have the same mass and length and they are coupled by a spring of spring constant k, and they have been moved from their initial state of rest by initial conditions $\theta_1 = \theta_2$:

We apply Newton's second law for a rotational system which states the relation between net external torque and the angular acceleration of a body about a fixed axis.

$$\Gamma = I\alpha \tag{1}$$

such that,

Γ is the torque

α is the angular acceleration $\frac{d^2\theta}{dt^2}$ From the diagram we see that:

$$\Gamma = r \times mg = L\hat{r} \times mg(cos\theta\hat{r} - sin\theta\hat{\theta})$$
$$= -L\,mg\,sin\theta\,\hat{k}$$

the torque about the fixed point is directed in the k direction. Therefore,

$$\Gamma = I\alpha = -L\,mg\,sin\theta$$

Such that, The moment of inertia (I) of a point mass about the pivot point is $I = mL^2$. Then,

$$mL^2\,\alpha = -L\,mg\,sin\theta$$
$$\alpha = \frac{d^2\theta}{dt^2} = -\frac{g}{L}sin\theta$$

We can use the small angle approximation $sin\theta \approx \theta$ if the angle of oscillation is small.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

So we get an equation that reduces to the simple harmonic oscillator equation:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Therefore,

$$\omega^2 = \frac{g}{L}$$

Then, the angular frequency:

$$\omega_o = \sqrt{\frac{g}{L}} \quad (2)$$

B. Two coupled pendulums case

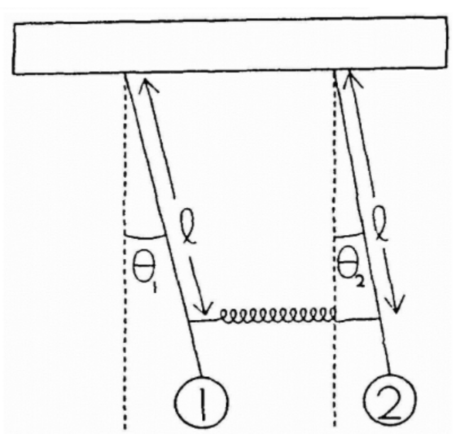

Fig. 2. Coupled Pendulums diagram [2]

Consider we have two pendulums in the system, they have the same mass and length and they are coupled by a spring of spring constant k, and they have been moved from their initial state of rest by initial conditions $\theta_1 = -\theta_2$:

$$\Gamma = r \times F \quad (3)$$

Such that F_s is the restoring force on either the left or the right pendulum due to the spring:

$$F_s = k\Delta x = k(x_2 - x_1) = kL(sin(\theta_2) - sin(\theta_1))$$

Now the driving force due to gravity on the system:

$$F_{1,2;g} = m_{1,2}gsin(\theta_{1,2})$$

Therefore the total force acting on the system:

$$F_{1,2;T} = m_{1,2}gsin(\theta_{1,2}) + kl(sin(\theta_2) - sin(\theta_1))$$

Such that for small oscillations:

$$F_{1,2;T} = m_{1,2}g\theta_{1,2} + kl(\theta_2 - \theta_1)$$

therefore,

$$\Gamma = I\frac{d^2\theta_1}{dt^2} = -mgL\theta_1 - kLl(\theta_1 - \theta_2)$$

$$\Gamma = I\frac{d^2\theta_2}{dt^2} = -mgL\theta_2 + kLl(\theta_1 - \theta_2)$$

We substitute the moment of inertia (I) of a point mass about the pivot point with $I = mL^2$:

$$mL^2\frac{d^2\theta_1}{dt^2} = -mgL\theta_1 - kLl(\theta_1 - \theta_2)$$

$$mL^2\frac{d^2\theta_2}{dt^2} = -mgL\theta_2 + kLl(\theta_1 - \theta_2)$$

Then we add and subtract the above two equations to obtain new simple harmonic differential equations,

$$I(\frac{d^2\theta_1}{dt^2} + \frac{d^2\theta_2}{dt^2}) = -mgL(\theta_1 + \theta_2)$$

$$I(\frac{d^2\theta_1}{dt^2} - \frac{d^2\theta_2}{dt^2}) = -(mgL + 2kl^2)(\theta_1 - \theta_2)$$

we have two solutions to these equations, and they are called normal modes, in which both pendulums in the system move with the same frequency:

$$\omega_{even}^2 = \frac{mgL}{I} = \omega_o^2$$

$$\omega_{even} = \omega_o = \sqrt{\frac{g}{L}} \quad (4)$$

where both pendulums' oscillations will be in phase and the angular frequency will just be ω_o.

$$\omega_{odd}^2 = \frac{mgL + 2kl^2}{I}$$

In this case, we have oscillations 180^o out of phase.]

$$\omega_{odd} = \omega_o + \frac{kl^2}{\omega_o mL^2} \quad (5)$$

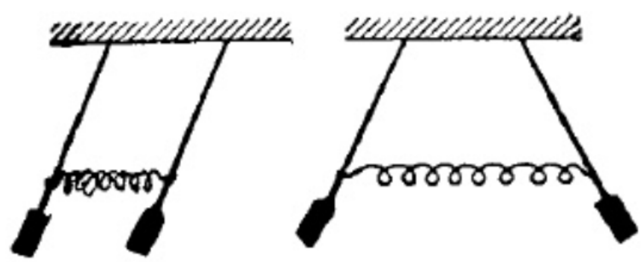

Fig. 3. Normal modes of coupled pendulums

C. Beat frequency

In the beating case, the two modes, odd and even, are interfering with each other and they are in superposition state. The two pendulums have been moved from their initial state of rest by initial conditions $\theta_1 = 0$ and $\theta_2 =$ some arbitrary angle. In the beating case, one of the pendulum is at rest and the other is reaching it's maximum amplitude and so the energy is constantly transferred between them. In this case, there is a phase difference of 90^o. Therefore, The Beat frequency of the system is given by :

$$\Delta\omega = \omega_{odd} - \omega_{even} = \frac{kl^2}{\omega_o m L^2} \tag{6}$$

III. EXPERIMENTAL DESIGN AND PROCEDURE

A. Description of the apparatus

A) Four springs
B) two pendulums with the same mass.
C) four mounting locations.
D) PC equipped with data acquisition software
E) Motion-signal converter device

B. Description of the experimental procedure

First, we selected 4 springs to determine the spring constants and then we averaged them, we measured the extension of the springs by hanging a mass from one spring and measure the associated displacement. We repeated this procedure for 6 masses for each spring.

In the setup of the system, we have four coupling different lengths, and we have four different strings. so for each possible combination of spring constant, k, and mounting location, l, we Measured the amplitude versus time signal using the provided software on the Lab's PC, for each combinations for both even and odd initial conditions.

Finally, to determine the beat frequency, we Measured the amplitude signal versus time for four beat trials, Such that, in each trial we choose the appropriate spring constant, k, and mounting location, l where it maximizes equation 6.

IV. ANALYSIS

A. Method of Analysis and Presentation of Results

The measurement of each of the spring's force constants consisted of plotting the mass against the displacement of the spring, and calculating the slope. Due to Hooke's Law, the slope multiplied by the gravitational constant g would give the force constant of the spring.

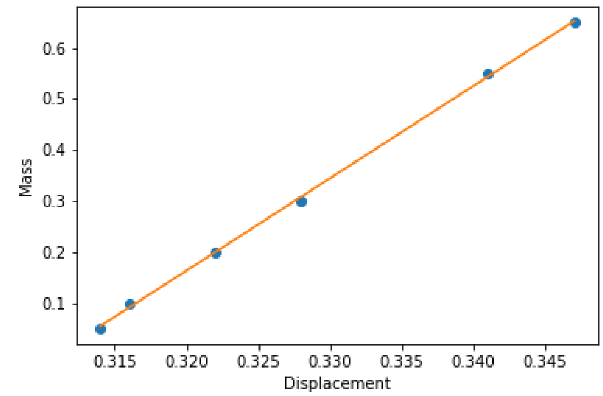

Fig. 4. Spring Constant for Spring 1

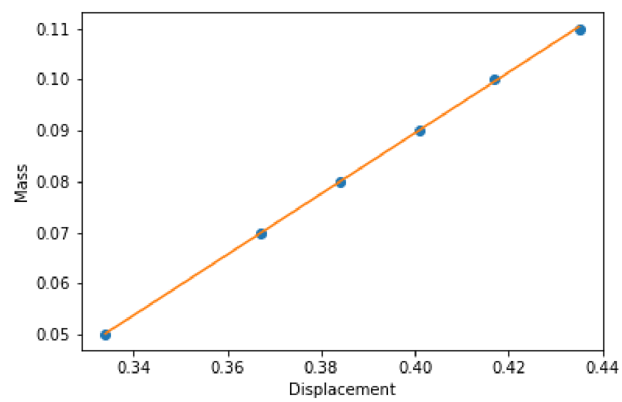

Fig. 5. Spring Constant for Spring 2

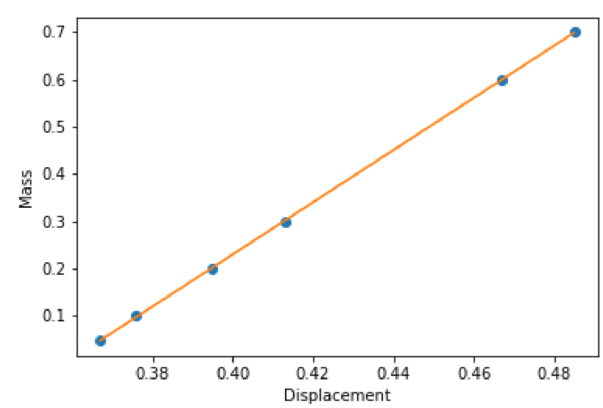

Fig. 6. Spring Constant for Spring 3

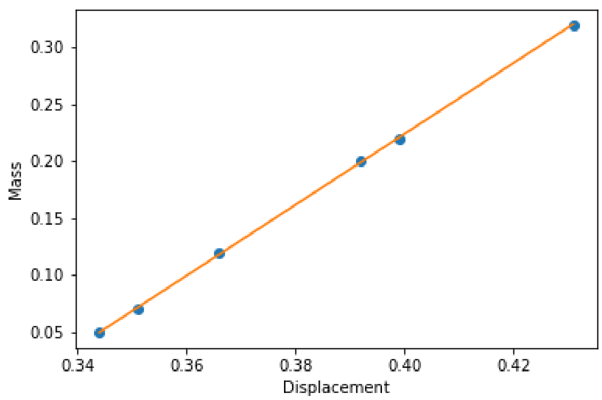

Fig. 7. Spring Constant for Spring 4

The force constant of the four springs used are given below:

- Spring 1: $k = 177.6Nm$
- Spring 2: $k = 5.8Nm$
- Spring 3: $k = 54.1Nm$
- Spring 4: $k = 30.5Nm$

The Even and Odd modes of each configuration of spring constant k and attachment distance l is given by Figures 12 to 43.

The even modes all have the same frequency, the fundamental frequency ω_0. This was found to be around $0.75Hz$. Note this is in Hertz, and using the method of analysis outlined in the theory section requires an extra factor of 2π, to make it the angular frequency.

By using Fourier analysis, the peak frequencies recorded for the odd modes of each spring for each level, is plotted against the spring force constant and the square of the spring level.

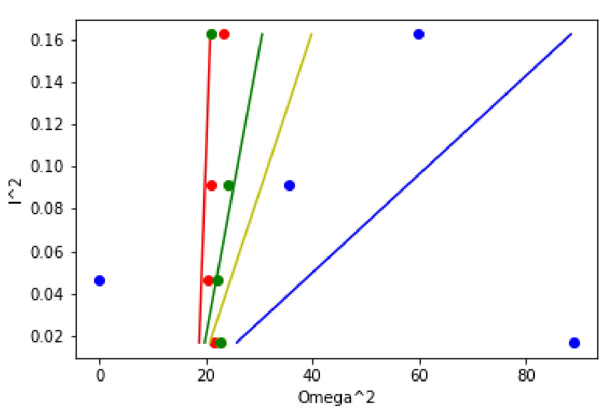

Fig. 8. Omega2̂ plotted against l2̂

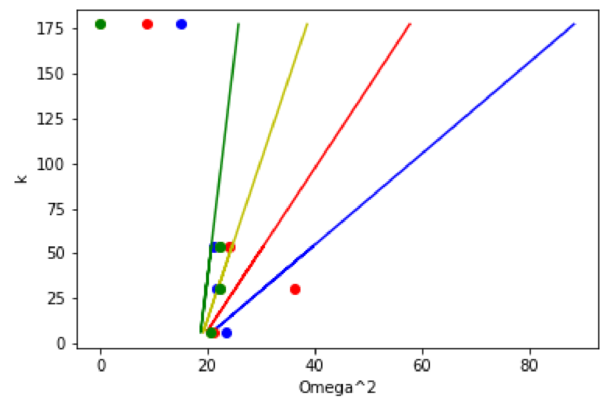

Fig. 9. Omega2̂ plotted against k

Since Fourier analysis turned out to be very inaccurate from the theoretical value, the frequency of the oscillation was also manually found using the original plots itself. This will be further discussed in the Discussion of Results section. The plot of frequencies obtained manually, without fourier analysis are given by Figures 10 and 11.

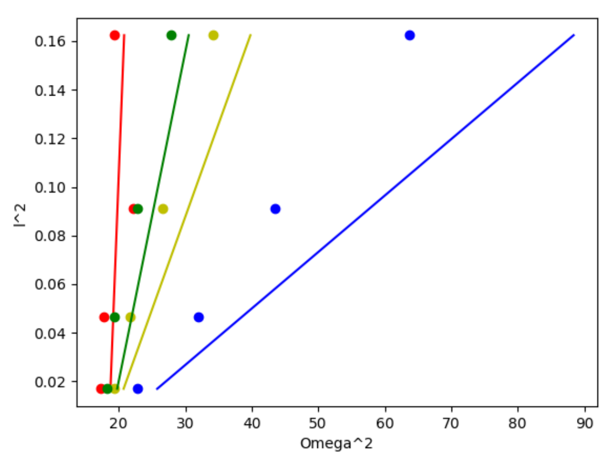

Fig. 10. Omega2̂ plotted against l2̂

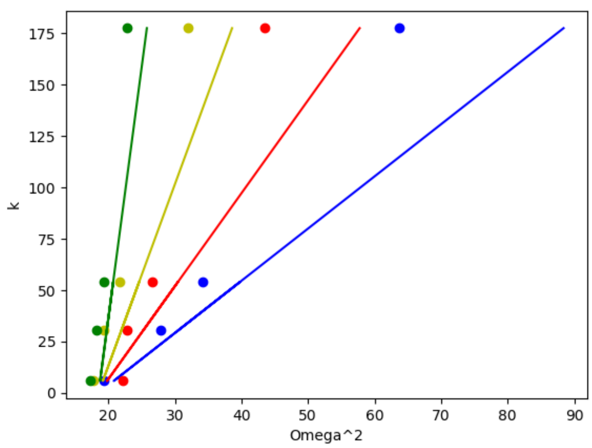

Fig. 11. Omega$\hat{2}$ plotted against k

Due to the fact that the lower frequencies tend to be lost in noise in our Fourier analysis, the beat frequency of the Part C was also calculated without Fourier analysis using the wave itself. The experimental beat frequencies were found to be the following:

- Beat 1: $f = 0.097Hz$
- Beat 2: $f = 0.080Hz$
- Beat 3: $f = 0.106Hz$
- Beat 4: $f = 0.039Hz$

Note this calculation was done in Hertz. A conversion of 2π must be factored to switch this to be ω the angular frequency. Using the spring constants above as well as the length of each level of the springs, the theoretical values for the beat frequencies are given as follows:

- Beat 1: $f = 0.135Hz$
- Beat 2: $f = 0.113Hz$
- Beat 3: $f = 0.125Hz$
- Beat 4: $f = 0.040Hz$

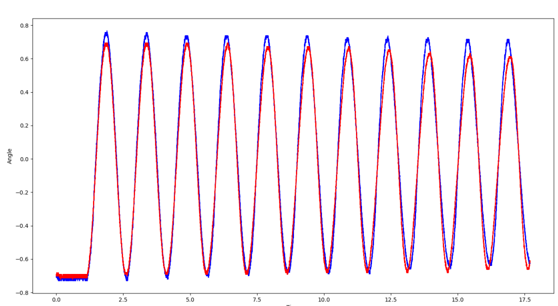

Fig. 12. Spring 1 Level 1 Even

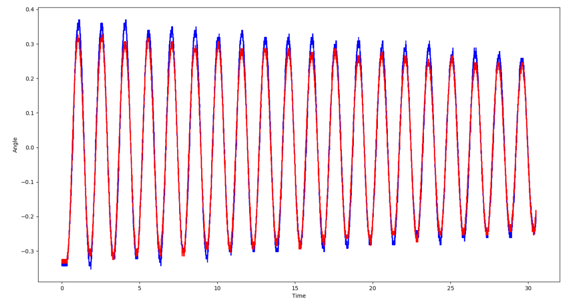

Fig. 13. Spring 1 Level 2 Even

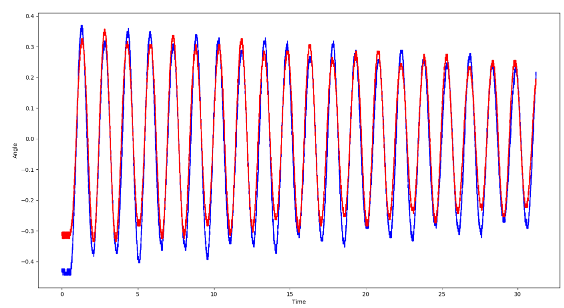

Fig. 14. Spring 1 Level 3 Even

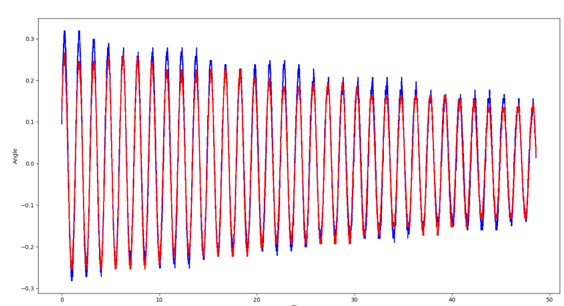

Fig. 15. Spring 1 Level 4 Even

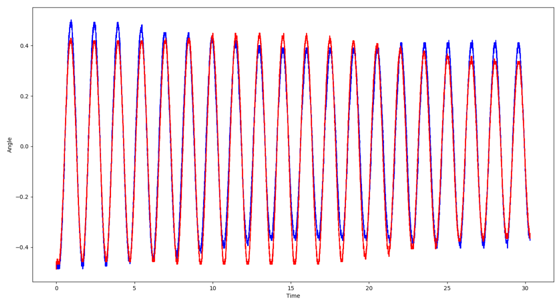

Fig. 16. Spring 2 Level 1 Even

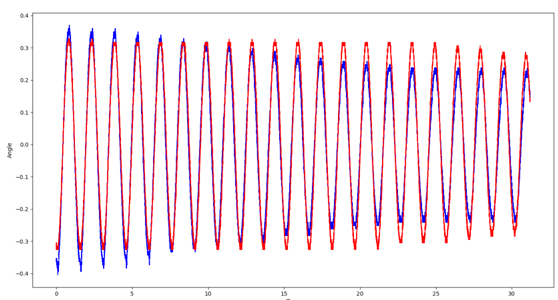

Fig. 17. Spring 2 Level 2 Even

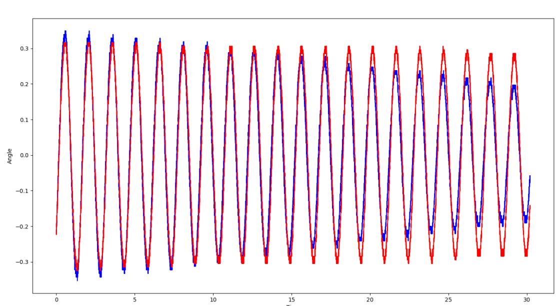

Fig. 18. Spring 2 Level 3 Even

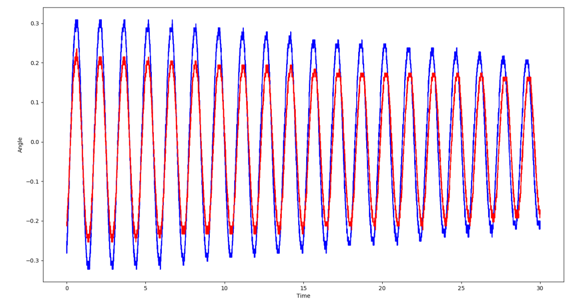

Fig. 19. Spring 2 Level 4 Even

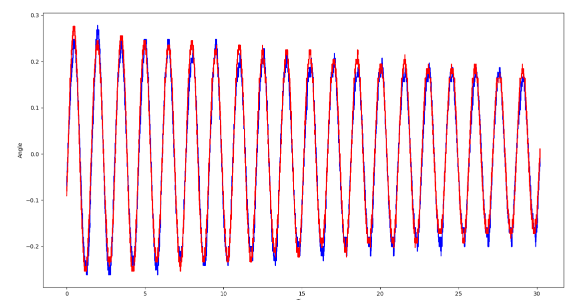

Fig. 20. Spring 3 Level 1 Even

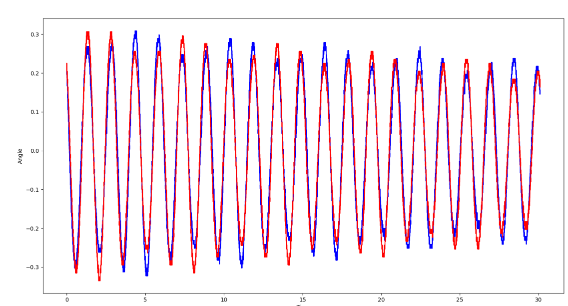

Fig. 21. Spring 3 Level 2 Even

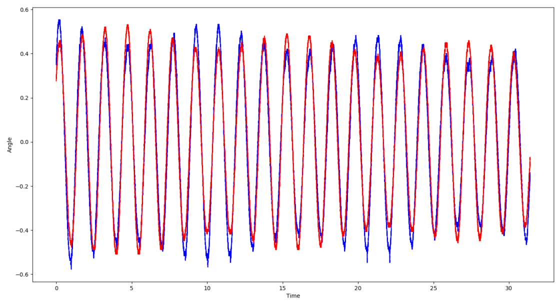

Fig. 22. Spring 3 Level 3 Even

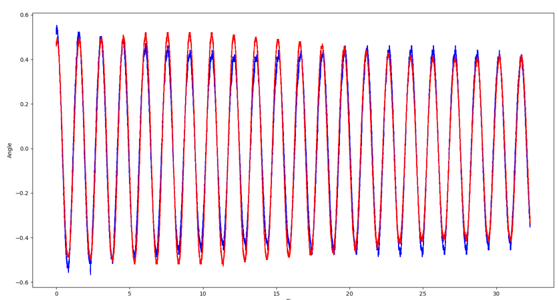

Fig. 23. Spring 3 Level 4 Even

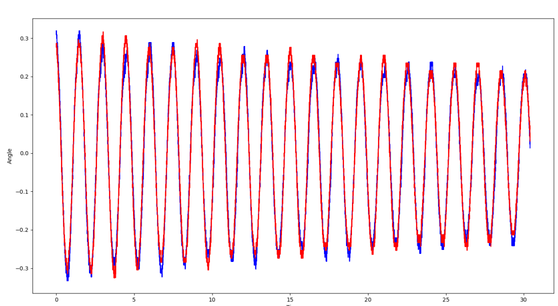

Fig. 24. Spring 4 Level 1 Even

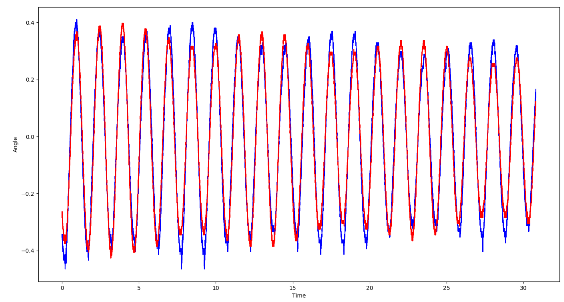

Fig. 25. Spring 4 Level 2 Even

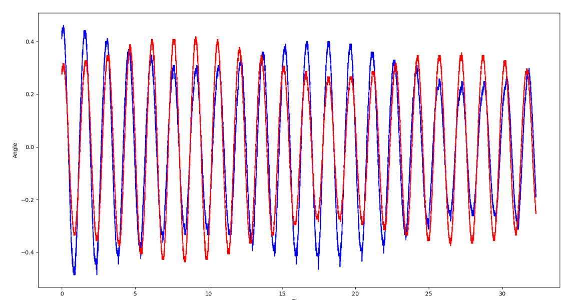

Fig. 26. Spring 4 Level 3 Even

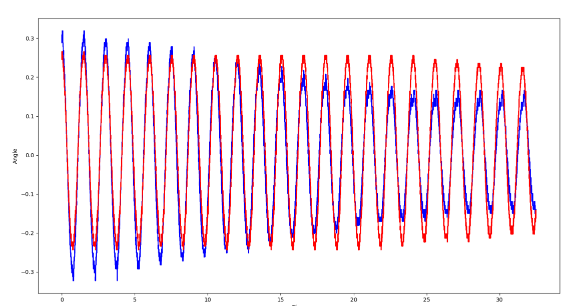

Fig. 27. Spring 4 Level 4 Even

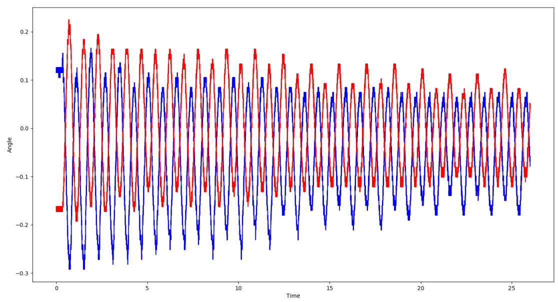

Fig. 28. Spring 1 Level 1 Odd

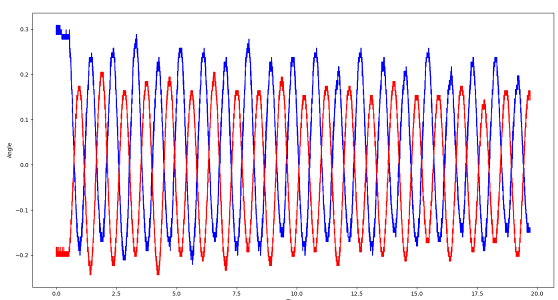

Fig. 29. Spring 1 Level 2 Odd

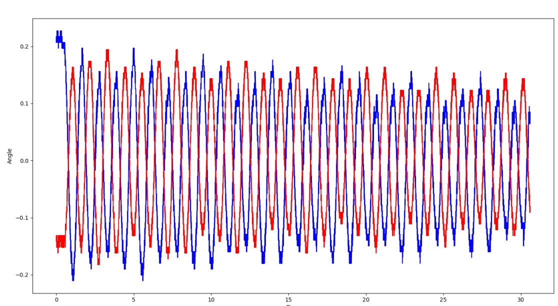

Fig. 30. Spring 1 Level 3 Odd

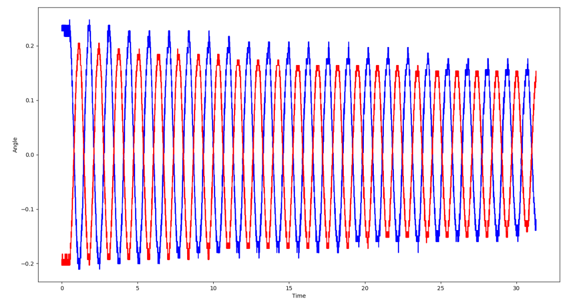

Fig. 31. Spring 1 Level 4 Odd

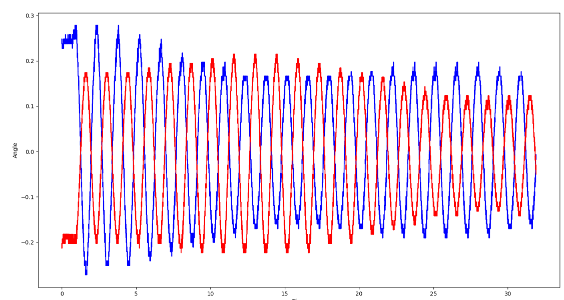

Fig. 32. Spring 2 Level 1 Odd

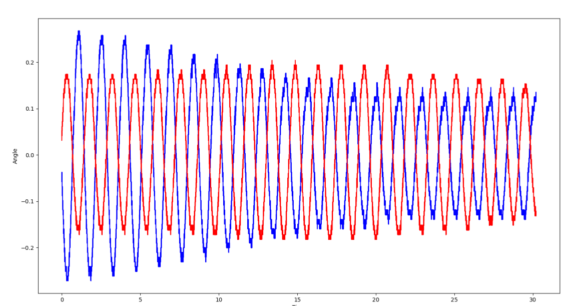

Fig. 33. Spring 2 Level 2 Odd

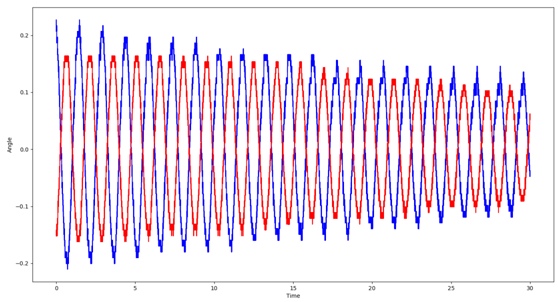

Fig. 34. Spring 2 Level 3 Odd

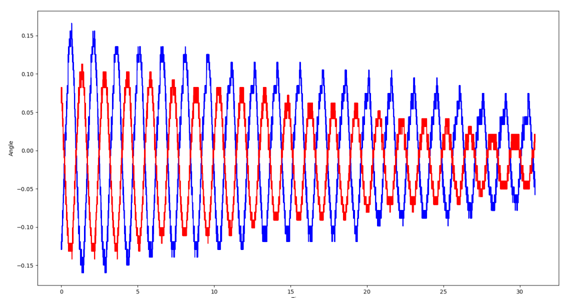

Fig. 35. Spring 2 Level 4 Odd

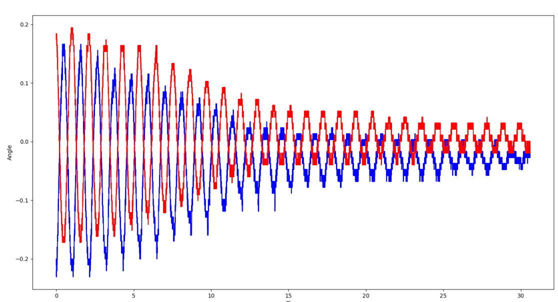

Fig. 36. Spring 3 Level 1 Odd

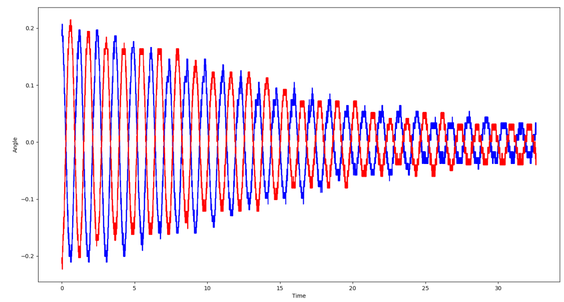

Fig. 37. Spring 3 Level 2 Odd

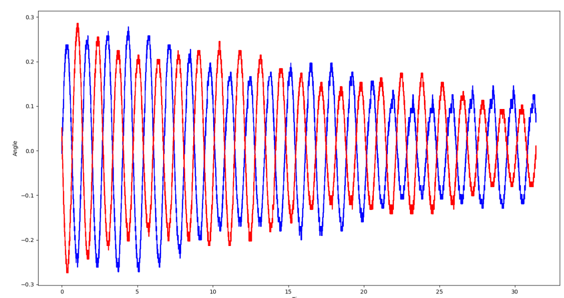

Fig. 38. Spring 3 Level 3 Odd

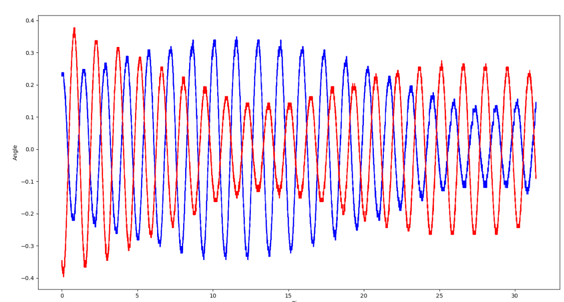

Fig. 39. Spring 3 Level 4 Odd

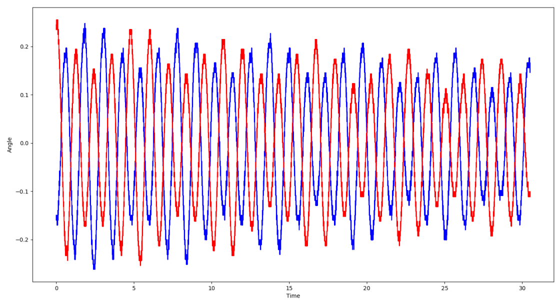

Fig. 40. Spring 4 Level 1 Odd

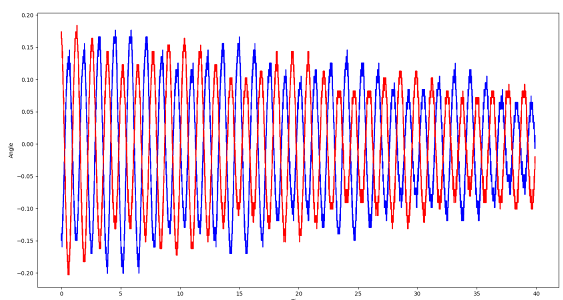

Fig. 41. Spring 4 Level 2 Odd

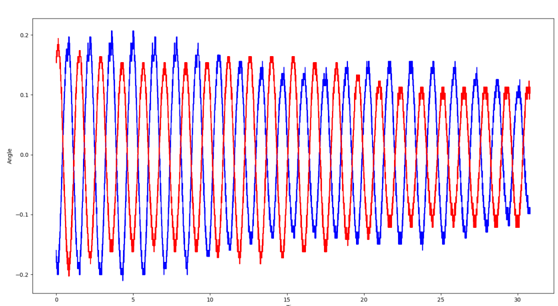

Fig. 42. Spring 4 Level 3 Odd

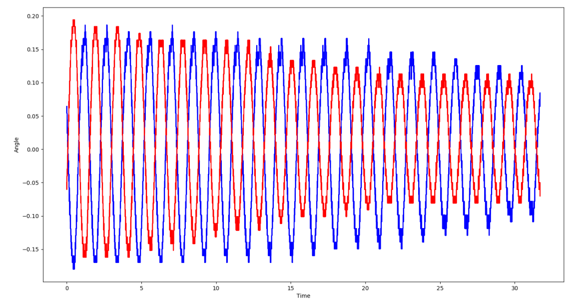

Fig. 43. Spring 4 Level 4 Odd

B. Discussion of results

1) The Spring Constants: Determination of the spring constants were fairly straightforward and without issue

2) Even and Odd Mode Frequencies: Fourier analysis proved to be a fairly reliable method of finding the peak frequency of the wave of even modes. For odd modes however, the peak frequencies shown using a fast Fourier transform were very unreliable, as well as hidden in substantial noise in lower frequencies. As an example, consider Figure 44, which is the fourier analysis of Beat 1. Frequencies below 1 Hz (which the beat frequency is) are entirely covered in noise. Instead, the frequencies were hand calculated using the number of peaks divided by the time on the raw data.

While these gave far better results that seem to line up with the theoretical value, there is a categorical increase of slope for the ω^2 vs. l^2 and ω^2 vs. k plots, indicating a reduction of frequency from the theoretical value. My best hypothesis for this behaviour is that friction, and other resisting mechanisms cause the effective spring constant of the system to be different from the spring constant of the spring itself, causing Hooke's Law to be incomplete.

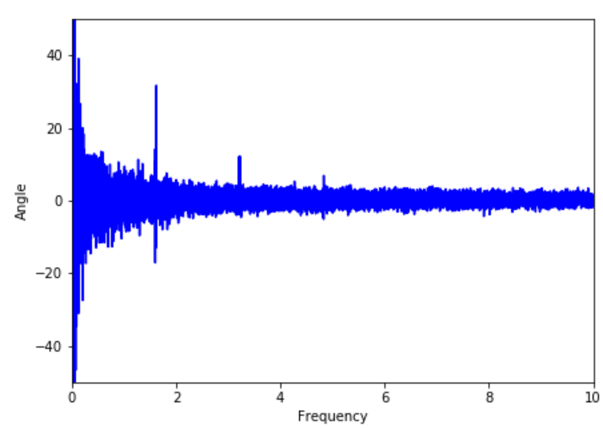

Fig. 44. Fourier Analysis Example

3) Beat Frequencies: Due to the fact that low frequencies are lost in noise to the Fourier analysis, this too was hand calculated using the data. The calculated experimental beat frequencies still seem to be categorically lower than the theoretical value. I believe this is due to the same mechanism that caused the increase in slope in Figures 11 and 10.

V. CONCLUSION

This lab effectively showed the phenomenon of coupled oscillations using Fourier analysis, as well as other methods to measure the frequency of oscillation for various initial conditions of a coupled oscillator.

REFERENCES

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