$K \in \mathcal{K}^d$ DECENTRALIZED $\Rightarrow$ IN ORDER TO DEFINE K WITH A GIVEN STRUCTURE WE CAN USE LMIs BASED DESIGN 20231024

$K \in \mathcal{K}^m$

$\boxed{\text{LMIs}}$

$$\left.\begin{array}{l} P > 0 \\ A^T P + PA < 0 \end{array}\right\} A \text{ IS HURWITZ} \qquad \left\{\begin{array}{l} P > 0 \\ F^T P F - P < 0 \end{array}\right. \Rightarrow F \text{ IS SCHUR STABLE}$$

LMIs FOR STABILITY

CENTRALIZED STATE-FEEDBACK CONTROLLER DESIGN

$$\begin{array}{l} \dot{x} = Ax + Bu \\ u = Kx \qquad K \in \mathbb{R}^{n \times n} \end{array} \longrightarrow \dot{x} = \underbrace{(A+BK)}_{A_{CL}} x$$

LYAPUNOV: A_{cl} IS HURWITZ $\Leftrightarrow \exists P = P^T$ s.t. AND K s.t.

$$\begin{cases} P > 0 \\ A_{cl}^T P + P A_{cl} < 0 \rightarrow (A+KB)^T P + P(A+KB) < 0 \end{cases}$$

$$\underbrace{A^T P + PA}_{\text{LINEAR IN } P} + \underbrace{K^T B^T P + PBK}_{\text{NON LINEAR IN } P \text{ AND } K} < 0 \Longrightarrow \text{REFORMULATE IT AS AN LMI}$$

$Y = P^{-1}$ $\quad Y(A^T P + PA + K^T B^T P + PBK) Y < 0$

$$\underbrace{YA^T + AY}_{\text{LINEAR}} + \underbrace{YK^T}_{L^T} B^T + B \underbrace{KY}_{L} < 0 \Rightarrow \underbrace{YA^T + AY}_{\text{LINEAR}} + \underbrace{L^T B^T + BL}_{\text{LINEAR IN } L} < 0$$

$\rightarrow$ SOLUTION TO LMI PROBLEM $\longrightarrow Y; L \Rightarrow P = Y^{-1} \Rightarrow K = LY^{-1} \Rightarrow A + BK$ IS HURWITZ

DISCRETE TIME

$$\begin{array}{l} x_{k+1} = F x_k + G u_k \\ u_k = K x_k \end{array} \longrightarrow x_{k+1} = \underbrace{(F+GK)}_{F_{CL}} x_k$$

LYAPUNOV: F_{cl} IS SCHUR STABLE IFF $\exists P = P^T$ s.t. AND K s.t.

$$\begin{cases} P > 0 \\ F_{cl} P F_{cl}^T - P < 0 \rightarrow (F+GK)^T P (F+GK) - P < 0 \end{cases}$$

$$\underbrace{FPF - P}_{\text{LINEAR IN } P} + \underbrace{FP\overbrace{K^T}^{L^T} G^T + G\overbrace{KP}^{L} F^T}_{\text{NL IN } P \text{ AND } K} + \underbrace{GKPK^T G^T}_{\text{NL IN } P,K} < 0 \longrightarrow FPF^T - P + FL^T G^T + GLF^T + \underbrace{GLP^{-1}L^T G^T}_{\text{HEAVILY NL}} < 0$$

$\rightarrow GKPK^T G^T = GLP^{-1}PK^T G^T = GLP^{-1}L^T G^T$

$$-(FPF^T - P + FL^T G^T + GLF' + GLP^{-1}L^T G^T) < 0 \quad \underbrace{P - FPF^T - FL^T G^T - GLF^T}_{\text{LINEAR}} - \underbrace{GL}_{S} P^{-1} \underbrace{L^T G^T}_{S^T} > 0 \quad (*)$$

SCHUR COMPLEMENT

$$\begin{cases} P > 0 \\ Q - S^{-1} P S > 0 \end{cases} \Leftrightarrow \begin{bmatrix} Q & S \\ S^{-1} & P \end{bmatrix} > 0$$

THANKS TO THE SCHUR COMPLEMENT $(*)$ IS EQUIVALENT TO

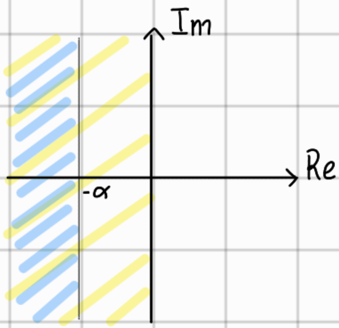
$$\begin{bmatrix} P - FPF^T - FL^T G^T - GLF^T & GL \\ L^T G^T & P \end{bmatrix} > 0$$

$\Rightarrow$ LMI ! $\Rightarrow P, L \Rightarrow K = LP^{-1}$

$\Rightarrow F + GK$ IS SCHUR STABLE !

LMIs FOR PERFORMANCE (HOW TO PLACE EIGENVALUES IN A REGION?)

CONTINUOUS TIME $\quad A+BK$, FOR FAST RESPONSE



$$\Rightarrow Re(eig(A+BK)) < -\alpha \Rightarrow Re(eig(A+BK)) + \alpha < 0$$

$\alpha > 0$

$$Re(eig(\underbrace{A+BK+\alpha I}_{})) < 0$$

$\hookrightarrow A+BK+\alpha I$ HURVITZ STABLE

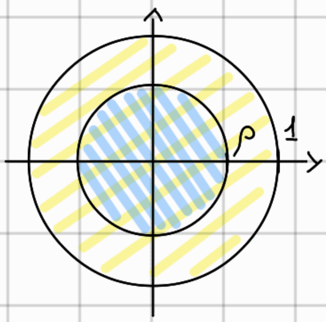
$\Rightarrow$ PLUG THIS IN THE LMI!

$$(A+BK+\alpha I)^T P + P(A+BK+\alpha I) < 0 \qquad A^TP + PA + K^TB^TP + PBK + 2\alpha P < 0$$

$$Y(A^TP+PA+K^TB^TP+PBK+2\alpha P)Y < 0 \qquad YA^T + AY + \underbrace{YK^T}_{L^T}B^T + B\underbrace{KY}_{L} + 2\alpha Y < 0$$

$$YA^T + YA + L^TB^T + BL + 2\alpha Y < 0 \quad \text{LMI!}$$

DISCRETE TIME



FOR FASTER RESPONSE, $\rho < 1$

$$|eig(A+BK)| < \rho \Rightarrow \frac{|eig(A+BK)|}{\rho} < 1$$

$$\Rightarrow \left|eig\left(\frac{A+BK}{\rho}\right)\right| < 1 \qquad \frac{A+BK}{\rho} \text{ SCHUR STABLE}$$

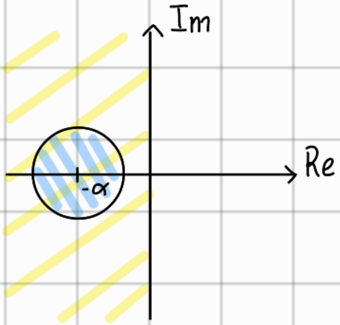
$$\frac{(F+GK)}{\rho} P \frac{(F+GK)^T}{\rho} - P < 0 \qquad (F+GK)P(F+GK)^T - \rho^2 P < 0$$

ENDING UP WITH
$$\begin{bmatrix} \rho^2 P - FPF^T - FL^TG^T - GLF^T & GL \\ L^TG^T & P \end{bmatrix} > 0 \quad \text{LMI!}$$

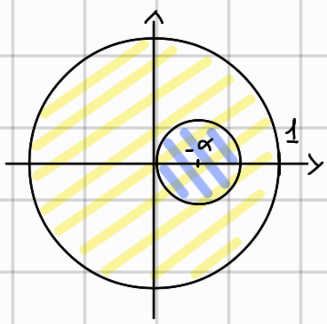
CONFINE EIGS IN A CIRCLE WITH CENTER IN $-\alpha$ AND RADIUS ρ (LIMIT OSCILLATIONS)

20231026

CONTINUOUS TIME



DISCRETE TIME



e.g. $x_{k+1} = -0{,}5x_k$

$\Rightarrow x_{k+1}$ WILL CHANGE SIGN EACH STEP

$$\longrightarrow \frac{(A+BK+\alpha I)}{\rho} P \frac{(A+BK+\alpha I)^T}{\rho} - P<0 \qquad (A+BK+\alpha I)P(A+BK+\alpha I)^T-\rho^2 P<0$$

$$(A+BK)P(A+BK)^T+\alpha P(A+BK)^T+\alpha(A+BK)P+\alpha^2 P-\rho^2 P<0$$

$$(AP+BKP)P^{-1}(AP^{-1}+BKP)^T=(AP+BL)P^{-1}(AP+BL)^{-1}$$

$$\alpha PA^T+\alpha PK^TB+\alpha AP+\alpha BKP=\alpha PA^T+\alpha L^TB+\alpha AP+\alpha BL$$

$$\underbrace{(\rho^2-\alpha^2)P-\alpha(PA^T+AP+L^TB+BL)}_{Q}-\underbrace{(AP+BL)}_{S}P^{-1}\underbrace{(AP+BL^{-1})}_{S^T}>0$$

USING THE SCHUR COMPLEMENT

$$\begin{bmatrix} Q & S \\ S^T & P \end{bmatrix}>0 \quad \Rightarrow \quad \begin{bmatrix} (\rho^2-\alpha^2)P-\alpha(PA^T+AP+L^TB+BL) & (AP+BL) \\ (AP+BL) & P \end{bmatrix}>0$$

CONFINE EIGS IN A REGION D

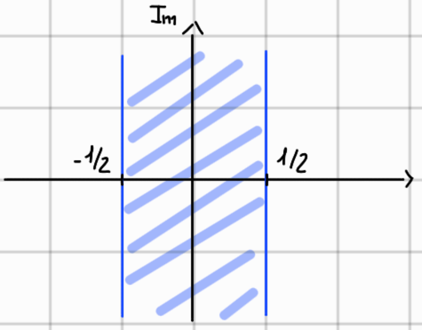
$$D:=\{z\in\mathbb{C} \text{ s.t. } f_D(z)<0\},\quad f_D(z)=\underbrace{\Lambda}_{\substack{\text{REAL}\\\text{SYMMETRIC}\\\text{SQUARE}\\\text{MATRIX}}}+\underbrace{\vartheta}_{\substack{\text{REAL}\\\text{SQUARE}}} z+\vartheta^T \underbrace{z^*}_{\text{COMPLEX CONJUGATE}}$$

EXAMPLE 1

$$\Lambda=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \vartheta=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$f_D(z)=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}+\begin{bmatrix} z & 0 \\ 0 & -z \end{bmatrix}+\begin{bmatrix} z^* & 0 \\ 0 & z^* \end{bmatrix}=\begin{bmatrix} -1+z+z^* & 0 \\ 0 & -1-(z+z^*) \end{bmatrix}<0$$

$$=\begin{bmatrix} -1+2Re(z) & 0 \\ 0 & -1-2Re(z) \end{bmatrix} \Rightarrow \begin{cases} -1+2Re(z)<0 \Rightarrow Re(z)<1/2 \\ -1-2Re(z)<0 \Rightarrow Re(z)>-1/2 \end{cases}$$



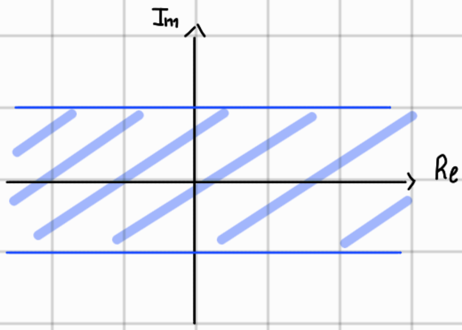
$$z+z^*=Re(z)+jIm(z)+Re(z)-jIm(z)=2Re(z)$$

EXAMPLE 2

$$\Lambda=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \vartheta=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$f_D(z)=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}+\begin{bmatrix} 0 & z \\ -z & 0 \end{bmatrix}+\begin{bmatrix} 0 & -z^* \\ z^* & 0 \end{bmatrix}=\begin{bmatrix} -1 & z-z^* \\ -(z-z^*) & -1 \end{bmatrix}<0$$

$$=\begin{bmatrix} -1 & 2jIm(z) \\ -2jIm(z) & -1 \end{bmatrix}<0 \quad \text{BOTH EIGS OF } f_D(z)<0$$



$$tr(f_D(z))=\lambda_1+\lambda_2<0 \qquad tr(f_D(z))=-2<0$$

$$det(f_D(z))=\lambda_1\lambda_2>0 \qquad det(f_D(z))=1-4jIm(z)>0$$

$$\Rightarrow -1/2<Im(z)<1/2$$

COMBINING THE 2 $\left(\vartheta = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\right)$ YOU OBTAIN THE INTERSECTION OF THE REGIONS.
OBVIOUSLY THE REGIONS CAN BE SHIFTED AND SCALED

EXAMPLE 3 (SECTOR)

$\Lambda = 0$

$$\vartheta = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix} \qquad f_D(z) = \begin{bmatrix} S_\alpha z & C_\alpha z \\ -C_\alpha z & S_\alpha z \end{bmatrix} + \begin{bmatrix} S_\alpha z^* & -C_\alpha z^* \\ C_\alpha z^* & S_\alpha z^* \end{bmatrix} = \begin{bmatrix} S_\alpha (z+z^*) & C_\alpha (z-z^*) \\ -C_\alpha (z-z^*) & S_\alpha (z+z^*) \end{bmatrix} =$$

$$= 2\begin{bmatrix} S_\alpha Re(z) & j C_\alpha Im(z) \\ -j C_\alpha Im(z) & S_\alpha Re(z) \end{bmatrix}$$

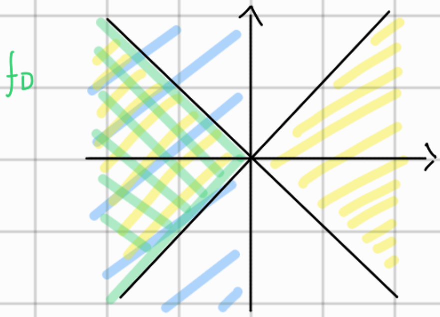
THIS IS GUARANTEED BY
- $tr(f_D(z)) = 4\sin\alpha\, Re(z) < 0$
- $\det(f_D(z)) = \sin^2(\alpha) Re^2(z) - \cos^2(\alpha) Im(z) > 0$

$\alpha \in (0; \pi/2)$

$\Rightarrow \sin(\alpha) Re(z) > 0 \Leftrightarrow Re(z) < 0$

$\Rightarrow \sin^2(\alpha) Re^2(z) - \cos^2(\alpha) Im(z) > 0$

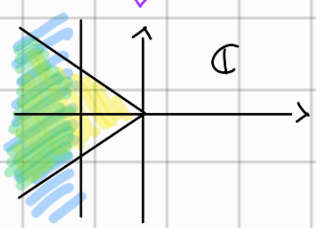
$$\left|\frac{Im(z)}{Re(z)}\right| < \left|\frac{\sin\alpha}{\cos\alpha}\right| \Rightarrow \xi \in (\cos(\alpha), 1)$$

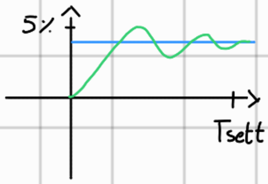


e.g. REQUIREMENTS

$\boxed{\begin{matrix} T_{sett} < 10s \\ S\% < 5 \end{matrix}}$

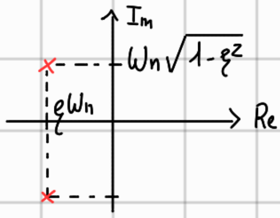
$\Downarrow$





$$T_{sett} = \frac{5}{|Re(\lambda)|} \qquad S\% = 100 e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$$

$$\frac{\xi}{\sqrt{1-\xi^2}} = \frac{Re(\lambda)}{Im(\lambda)}$$



THEOREM

$\lambda_i \in D$, $\forall i = 1, \dots, n$ IF $\exists Y = Y^T$ s.t. $M_D < 0$

THE BLOCKS OF M_D ARE $m_{ij} = \lambda_{ij} I + \vartheta_{ij}(A+BK)Y + \vartheta_{ji} Y (A+BK)^T$

EXAMPLE 3

$\Lambda = 0$

$$\vartheta = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix} \qquad M_D = \begin{bmatrix} \sin\alpha (A+BK)Y + \sin Y (A+BK)^T & \cos\alpha (A+BK)Y - \cos\alpha\, Y(A+BK) \\ -\cos\alpha (A+BK)Y + \cos\alpha Y (A+BK)^T & \sin\alpha (A+BK)Y + \sin Y (A+BK)^T \end{bmatrix} < 0$$

$\rightarrow (A+BK)Y = AY + BKY = AY + BL \Rightarrow$ ALWAYS LMIs!

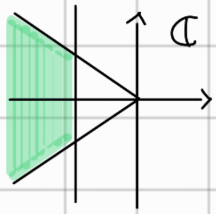
COMPUTATIONAL COMPLEXITY DEPENDS ON:

- # CONSTRAINTS (CAN'T BE REDUCED)
- # FREE VARIABLE $\longrightarrow Y \in \mathbb{R}^{n\times n}$ SYMMETRIC $\longrightarrow 1+2+3+\dots+n = \frac{n(n+1)}{2}$ FREE VARIABLES

  $L \in \mathbb{R}^{m\times n}$ $\longrightarrow m\times n$ FREE VARIABLES

LMI OPTIMIZATION PROBLEMS

① MINIMIZE CONTROL EFFORT



$u = Kx$ minimize $\|K\|$

$\hookrightarrow$ $\boxed{\begin{array}{l}\text{minimize } \alpha_K \\ \text{SUBJECT TO } \alpha_K \geq \|K\|\end{array}}$ $\Longrightarrow$ $\boxed{\begin{array}{ll}\text{minimize } \alpha_L + \alpha_Y & (1) \\ \text{SUBECT TO } \alpha_L \geq \|L\|, \alpha_Y \geq \|Y\| & (2)\end{array}}$

FREE VARIABLES ARE $Y(P)$ AND L

$\|K\| = \|LY^{-1}\| \leq \|L\| \cdot \|Y^{-1}\|$

LET'S REWRITE (1) AND (2) AS LMIs

(1) $\alpha_L \geq \|L\|$ $\quad \alpha_L^2 I \geq L^T L$ $\quad \gamma_L I - L^T I L \geq 0$

$\gamma_L = \alpha_L^2$ $\quad$ SCHUR COMPL $\longrightarrow \begin{bmatrix} \gamma_L I & L^T \\ L & I \end{bmatrix} > 0$

(2) $\alpha_Y I - I Y^{-1} I$ SCHUR COMPL $\longrightarrow \begin{bmatrix} \alpha_L I & I \\ I & Y^{-1} \end{bmatrix} > 0$

LMIs ALLOW TO DEFINE A STRUCTURE FOR THE FREE MATRICES

Y --- DEC $K \in \mathcal{K}^d$ BLOCK-DIAGONAL

DIST $K \in \mathcal{K}^{M}$ STRUCTURED

$\rightarrow$ HOW TO FIX A STRUCTURE TO $K = LY^{-1}$ BY FIXING A STRUCTURE TO Y AND L?

e.g. $K = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ IMPOSING $L = \begin{bmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{bmatrix}$ ISN'T ENOUGH, BECAUSE IF $Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \Rightarrow K = LY^{-1} = \begin{bmatrix} \ell_1 y_{11} & \ell_1 y_{12} \\ \ell_2 y_{21} & \ell_2 y_{22} \end{bmatrix}$ NOT DIAGONAL

BUT IF $Y = \begin{bmatrix} y_{11} & 0 \\ 0 & y_{22} \end{bmatrix} \Rightarrow K = LY^{-1} = \begin{bmatrix} \ell_1 y_{11} & 0 \\ 0 & \ell_2 y_{22} \end{bmatrix}$ DIAGONAL

THE RULE IS:

- L MUST HAVE THE SAME BLOCK STRUCTURE AS K
- Y MUST BE BLOCK-DIAGONAL!

IMPLICATIONS

- REDUCTION OF # OF FREE VARIABLES $Y = \begin{bmatrix} Y_1 & 0 & \cdots & 0 \\ 0 & Y_2 & & \vdots \\ 0 & 0 & \cdots & Y_n \end{bmatrix}$ $\cancel{\frac{n(n+1)}{2}}$ FREE VARIABLES

$\Rightarrow$ FOR EACH BLOCK $\frac{n_i(n_i+1)}{2}$ FREE VARIABLES

SAME HAPPENS FOR L WHEN STRUCTURED ACCORDINGLY TO THE CASE.

- WE LOOSE THEORETICAL GUARANTEES

IN THE LYAPUNOV THEOREM $P = P^T > 0 \longrightarrow$ IN DISTRIBUTED AND DECENTRALIZED DESIGN WE FIX A STRUCTURE TO $P \Rightarrow$ IF A SOLUTION $\exists \Rightarrow$ THE CORRESPONDING CONTROLLER IS STABILIZING

~~IF A CONTROLLER $\exists \Rightarrow$ WE CAN FIND IT WITH LMIs~~

H_2-NORM MINIMIZATION CONTROL

20231102

CONT. TIME

$$\dot{x} = Ax + Bu + B_w w$$

NOISE/DISTURBANCE

$$z = Cx + Du$$

PERFORMANCE OUTPUT

$\boxed{u = Kx}$ s.t. THE H_2-NORM OF $G_{zw}(s)$ (THE TR. FUN. BETWEEN w AND z) IS MINIMIZED.

E.G. IF WE DEFINE $z = \begin{bmatrix} \sqrt{Q}\,x \\ \sqrt{R}\,u \end{bmatrix} = \begin{bmatrix} \sqrt{Q} \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \sqrt{R} \end{bmatrix} u$ $\Rightarrow$ WE DERIVE LQ CONTROLLER

IN CLOSED LOOP

$$\begin{cases} \dot{x} = (A+BK)x + B_w w \\ z = (C+DK)x \end{cases}$$

THEOREM

$$\|G_{zw}(s)\|_2^2 = \inf_{Y} \left\{ \mathrm{trace}\left[(C+DK_y)Y(C+DK)^T\right] \right.$$

$$\left. \text{WHERE } (A+BK)Y + Y(A+BK)^T + \underline{B_w B_w^T} < 0 \right\}$$

$\Rightarrow$ NOT VITAL IF NOT WELL KNOWN, JUST $B_w = I$

LET'S MAKE THIS AN LMI

$\min_K \mathrm{tr}\left[(C+DK)Y(C+DK)^T\right]$ WHERE $(A+BK)Y + Y(A+BK)^T + B_w B_w' < 0$

THIS IS ALSO STABILIZING, BECAUSE IF $(A+BK)Y + Y(A+BK)^T + B_w B_w' < 0$ THEN $(A+BK)Y + Y(A+BK)^T < 0$

$\hookrightarrow$ LYAPUNOV THEOREM

LET'S DEFINE A NEW MATRIX

$\min_{Y,L,S} \mathrm{trace}(S)$ SUBJECT TO $(A+BK)Y + Y(A+BK)^T + B_w B_w' < 0$

$S \geq (C+DK)Y(C+DK)^T \Rightarrow S$ IS AN UPPER BOUND

BOTH LMI UNSUITABLE

$\Rightarrow S - (C+DK)Y(C+DK)^T \geq 0 \longrightarrow (CY + D\underbrace{KY}_{L})Y^{-1}(CY + D\underbrace{KY}_{L})^T$

$\Rightarrow$ SCHUR COMPONENT

$\begin{bmatrix} S & CY+DL \\ (CY+DL)^T & Y \end{bmatrix} \geq 0$ AND $(A+BK)Y + Y(A+BK)^T + B_w B_w' < 0 \Rightarrow AY + BL + YA^T + L^T B^T + B_w^T B_w < 0$

DISCR. TIME

$$\begin{cases} x_{k+1} = F x_k + G u_k + B_w w_k \\ z = C x_k + D u_k \end{cases}$$

$u_k = K x_k$ s.t. THE $\mathcal{H}_2$-NORM OF $G_{zw}(z)$ (THE TR. FUN. BETWEEN w AND z) IS MINIMIZED.

THEOREM

$$\|G_{zw}(s)\|_2^2 = \inf_{Y} \left\{ \text{trace}\left[(C+DK_y)Y(C+DK)^T \text{ WHERE } (F+GK)Y(F+GK)^T - Y + B_w B_w^T < 0\right\}\right.$$

$$\min_K tr\left[(C+DK)Y(C+DK)^T\right] \quad \text{WHERE } (F+GK)Y(F+GK)^T - Y + B_w B_w^T < 0$$

FOLLOWING THE SAME STEPS

$\min \text{trace}(S)$ SUBJECT TO SCHUR COMPONENT $\begin{bmatrix} S & CY+DL \\ (CY+DL)^T & Y \end{bmatrix} \geq 0$

$$(F+GK)Y+Y(F+GK)^T - Y + B_w B_w^T < 0 \longrightarrow \text{UNSUITABLE FOR AN LMI}$$

$$(F+GK)Y(F+GK)^T - Y + B_w B_w^T < 0 \rightarrow Y - B_w B_w - (F+GK)Y(F+GK)^T > 0$$

$$Y - B_w B_w - (FY + G\underbrace{KY}_{L})Y^{-1}(FY + G\underbrace{KY}_{L})^T > 0 \Rightarrow \text{SCHUR COMPLEMENT} \begin{bmatrix} Y - B_w B_w^T & (FY+GL) \\ (FY+GL)^T & Y \end{bmatrix} > 0$$