Laplacian Operator on Bundled Networks

NETWORK DAYS
Bridging micro to macro
Padua 24-25 October 2024

ALESSANDRO VEZZANI

ALMA · UNIVERSITAS · STUDIORUM · PARMENSIS
A.D. 962

CNR
IMEM

Laplacian Operator on Bundled Networks

NETWORKS PHYSICS in Parma

Adaptive temporal network for epidemic propagation: M. Mancastroppa, C. Castellano et all.. Nat. Commun. 12, 1919 (2021), M. Mancastroppa, V. Colizza et all. Phys. Rev. Research 6, 033159 (2024),

Network reconstruction for neural systems: Phys. Rev. Lett. 118(9), 098102 (2017)

Neural Network for machine learning R. Aiudi, P. Rotondo et all. Phys. Rev. Lett. 133, 027301 (2024)

Network for human mobility in progress

Laplacian Operator on Bundled Networks

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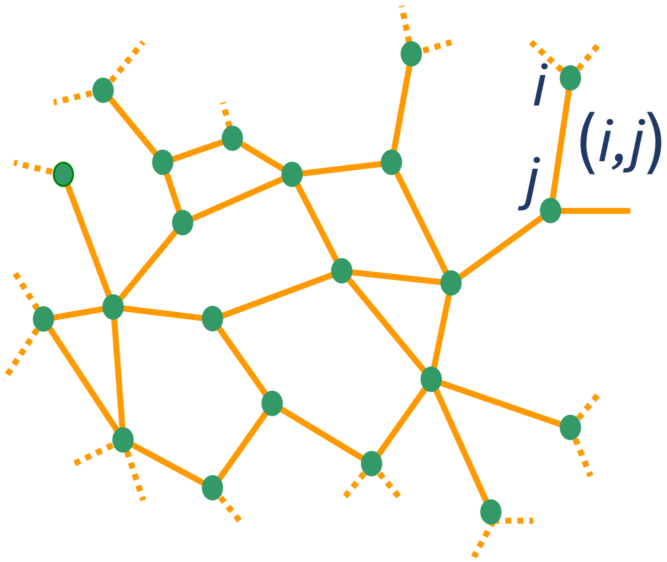
Lorenzo Grimaldi (CREF), Pablo Villegas (CREF), Andrea Gabrielli (UniRoma3)
Alessandro Vezzani (IMEM-CNR Parma), Raffaella Burioni (Unipr), Davide Cassi (Unipr)

Work-in-progress bridging
recent results on Laplacian renormalization (Rome)
to old results on Laplacian spectrum and bundled networks (Parma)

Laplacian Operator on Bundled Networks



Graph/network general discrete structure composed by nodes and edges

d_f fractal dimension $\quad N(R) \sim R^{d_f}$

$N(R)$ volume (number of sites) in a network of radius R

Adjacency matrix

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

Laplacian matrix

Positive defined

$$L_{ij} = A_{ij} - k_i \delta_{ij}$$

Node degree

$$k_i = \sum_j A_{ij}$$

Elastic model where nodes are masses and link are harmonic interactions

$$H_{el} = K \frac{1}{2} \sum_{ij} A_{ij} (x_i - x_j)^2 = K \sum_{ij} L_{ij} x_i x_j$$

Spectrum of L corresponds to normal modes

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Laplacian renormalization group

De Domenico and Biamonte Phys. Rev. X 6, 041062 (2016)
Villegas, Gili, Caldarelli, Gabrielli Nature Physics 19 445–450 (2023)

Density operator

$$\tilde{\rho}(\tau) = \frac{e^{-\tau L}}{Z}$$

Partition function: λ_k eigenvectors of L_{ij}

$$Z = \mathrm{Tr}(e^{-\tau L}) = \sum_{k=0}^{N-1} e^{-\tau \lambda_k}$$

Average energy

$$\langle \lambda \rangle = Z^{-1} \sum_{k=0}^{N-1} \lambda_k e^{-\tau \lambda_k} = -\frac{d \log(Z)}{d\tau}$$

Specific heat (τ inverse temperature)

$$C = \frac{d\langle \lambda \rangle}{d(1/\tau)} = \tau^2 \frac{d^2 \log(Z)}{d\tau^2} = \tau^2(\langle \lambda^2 \rangle - \langle \lambda \rangle^2)$$

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Spectral dimension $\bar{d}$

$$\rho(\lambda) \sim \lambda^{\bar{d}/2-1} \quad \text{for } \lambda \to 0$$

$\rho(\lambda)$ is the spectral density of L_{ij}
Density, of long range, low frequency modes of oscillations

Filed theory on networks — Hattori, Hattori, Watanabe Progr. of Theor. Phys. Suppl. 92, 108 (1987)

Phase transition on networks — Cassi PRL. 76 2941 (1996) Burioni, Cassi, Vezzani PRE 60, 1500 (1999)

In general, $\bar{d} \neq d_f$

Singularity of the average of the propagator vs Singularity of the propagator

$$\frac{1}{N}\sum_{i=1}^{N}(L+m^2\mathbb{1})^{-1}_{ii} = \int d\lambda \frac{\rho(\lambda)}{\lambda+m^2} \sim m^{\bar{d}-1} \qquad (L+m^2\mathbb{1})^{-1}_{ii} \sim m^{d_s-1}$$

d_s random walks and
anomalous diffusion

For homogeneous structures
(e.g. lattices and fractals) $\bar{d} = d_s$

Alexander, Orbach J. Phys. Lett. 43, L635 (1982).

Review: Cassi, Burioni J. Phys. A: Math. Gen. 38 R45 (2005)

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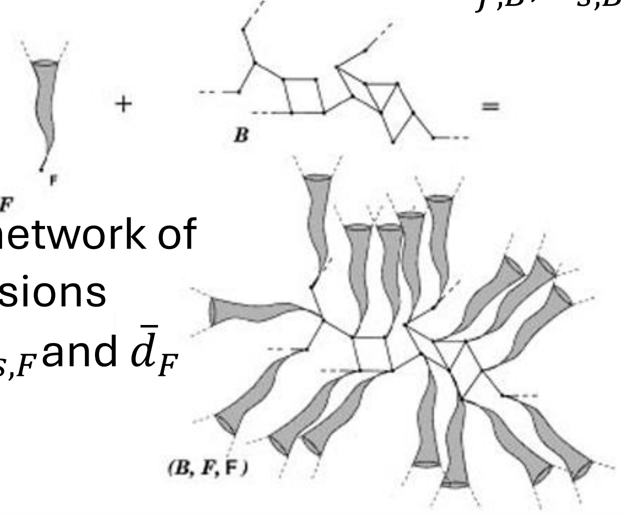
Bundled networks

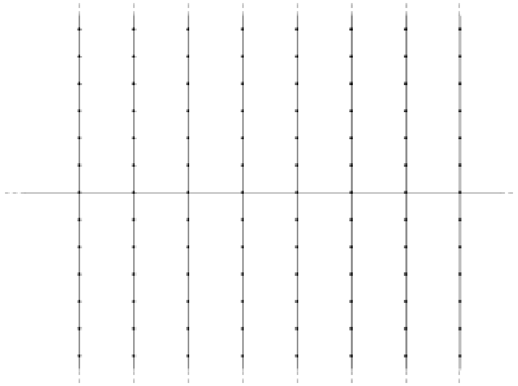
Typical inhomogeneous structure where $d_s \neq \bar{d}$

Base network of dimensions $d_{f,B}, d_{s,B}$ and $\bar{d}_B$

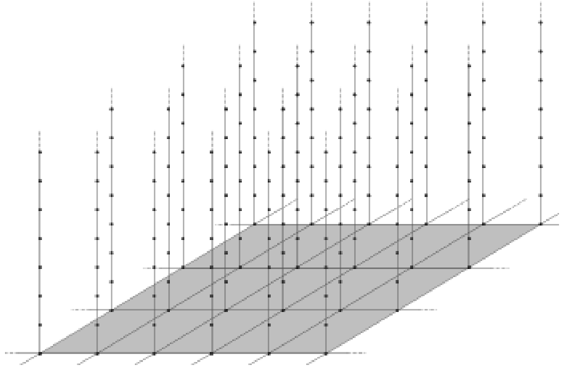
Fiber network of dimensions $d_{f,F}, d_{s,F}$ and $\bar{d}_F$



Comb Lattice



Brush Lattice



Models for polymers

$$d_s = \begin{cases} d_{s,F} & \text{if } d_{s,F} > 2 \\ 4 - d_{s,F} & \text{if } d_{s,F} < 2 \text{ and } d_{s,B} > 4 \\ d_{s,B} + d_{s,F} - \frac{d_{s,F} d_{s,B}}{2} & \text{if } d_{s,F} < 2 \text{ and } d_{s,B} < 4 \end{cases}$$

$$\bar{d} = \bar{d}_F$$

Cassi, Regina
Phys. Rev. Lett. 76 2914 (1996)

Laplacian Operator on Bundled Networks

Fiedler spectral gap exponent

Further exponent for the gap
$$\lambda_1 \sim N^{-2/d_g}$$

For lattices and fractals $d_g = \bar{d}$ but in general, $d_g \neq \bar{d}$

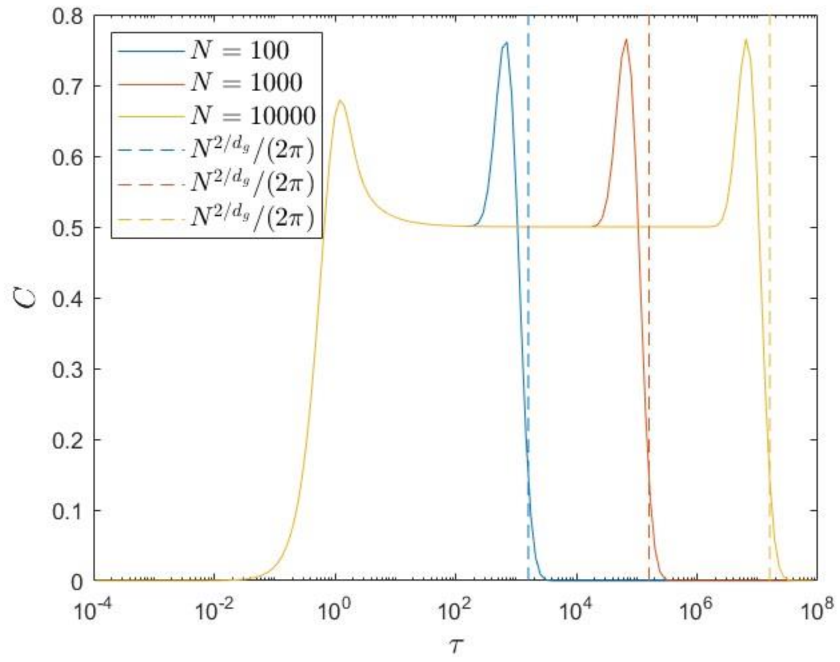
Laplacian specific heat and spectral dimension

Let us consider the spectral density in the continuous limit $\rho(\lambda) = A\lambda^{-\bar{d}/2}$ for all $\lambda > 0$

$$Z = \mathrm{Tr}(e^{-\tau L}) = \sum_{k=0}^{N-1} e^{-\tau\lambda_k} = N\int_0^\infty \rho(\lambda)e^{-t\lambda}d\lambda = NA\int_0^\infty \lambda^{\bar{d}/2-1}e^{-t\lambda}d\lambda = BNt^{-\bar{d}/2}$$

$$C = \tau^2 \frac{d^2\log(Z)}{d\tau^2} = \bar{d}/2$$

Specific heat of a linear chain with N sites
plateau at $C = \bar{d}/2$ but
At small τ cut off due to the discrete structure
At large τ the cut off is related to the spectral gap
and τ scales with N as the inverse of λ_1 i.e. as N^{2/d_g}



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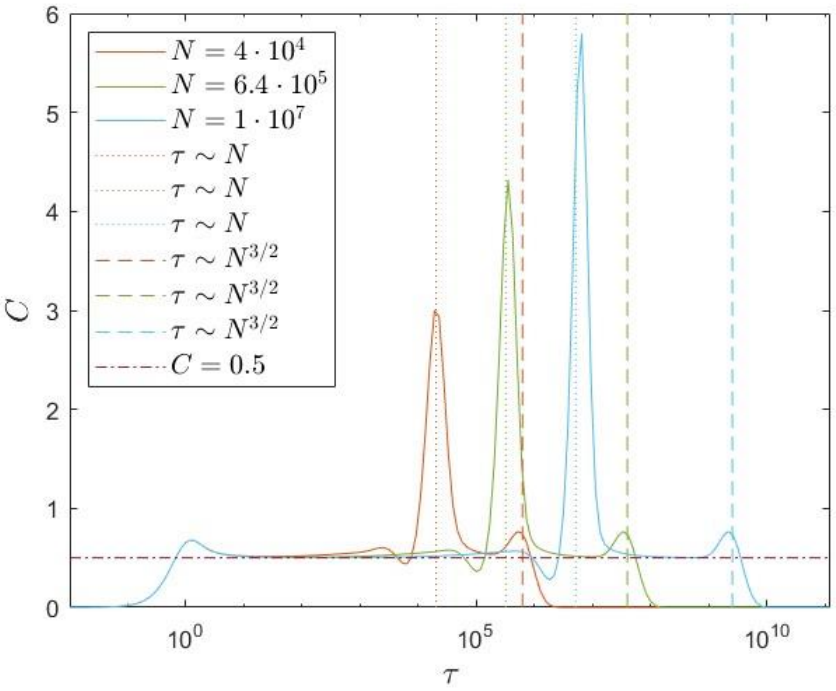
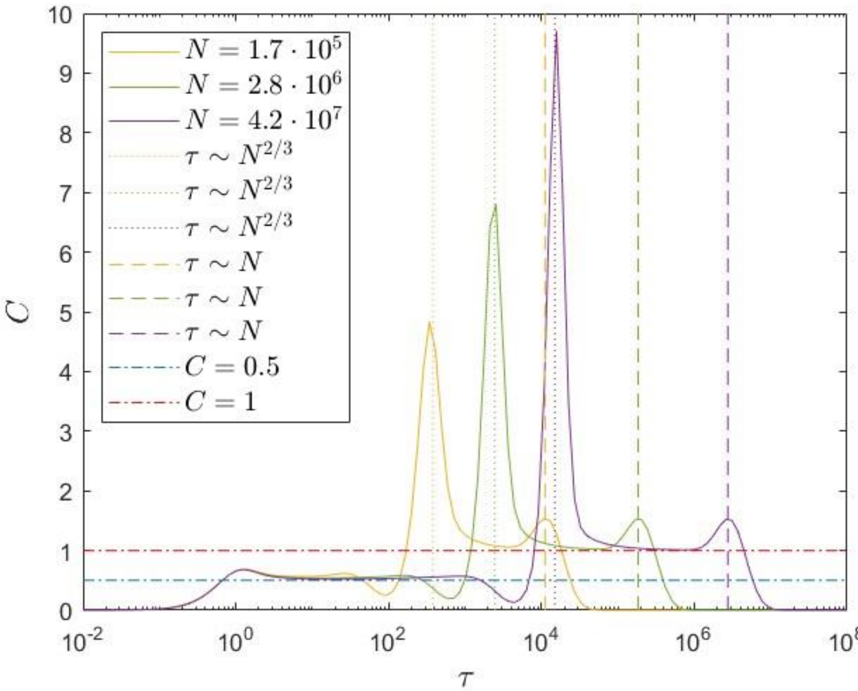
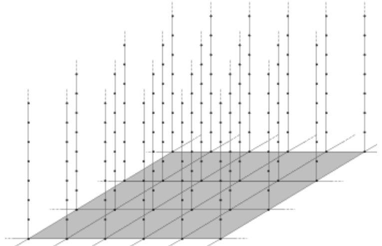
Laplacian specific heat on bundled structures

Comb Lattice





Brush Lattice



First plateau $C = \bar{d}/2$ the only feature visible at fixed (large) τ in the thermodynamic limit

Diverging peaks at $\tau \sim N^{2/d_f}$.

Second plateau $C = \bar{d}_B/2$?

Final drop at $\tau \sim N^{2/d_g}$ $d_g = 4/3$ for comb and $d_g = 2$ for brush?

Laplacian Operator on Bundled Networks

Laplacian diagonalization on bundled structures and quantum particles on networks

Spectrum of L_{ij}:free quantum particles.

Burioni, Cassi, Rasetti, Sodano, Vezzani J. Phys. B 34 4697 (2001)

Bundled network:

Base generic network of size N^B Laplacian $L^B_{i_1 j_1}$ indices i_1 , j_1

Fiber ring of length L indices i_2

$$2\psi_{i_1,i_2} - \psi_{i_1,i_2-1} - \psi_{i_1,i_2+1} + \delta_{i_2,0} \sum_{j_1} L^B_{i_1,j_1} \psi_{j_1,0} = E\psi_{i_1,i_2}$$

$$(TL^BT^+)_{n_1,m_1} = \ell^B_{n_1} \delta_{n_1,m_1} \quad \tilde{\psi}_{n_1,i_2} = \sum T_{n_1,i_1} \psi_{i_1,i_2}$$

$\ell^B_{n_1}$ eigenvalues of $L^B_{i_1 j_1}$

$$2\tilde{\psi}_{n_1,i_2} - \tilde{\psi}_{n_1,i_2-1} - \tilde{\psi}_{n_1,i_2+1} + \ell^B_{n_1} \delta_{0,i_2} \tilde{\psi}_{n_1,i_2} = E\tilde{\psi}_{n_1,0}$$

Quantum Particle on a chain with a repulsive potential $V = \ell^B_{n_1}$ in the origin

Odd solutions $\tilde{\psi}_{n_1,i_2} = \sin(k_2 i_2)$ $\quad k_2 = 2n_2\pi/L$ with $n_2 = 1, \ldots, L/2-1$ $\quad E^{n_1,n_2} = 2(1-\cos(k_2))$ Degeneracy N^B

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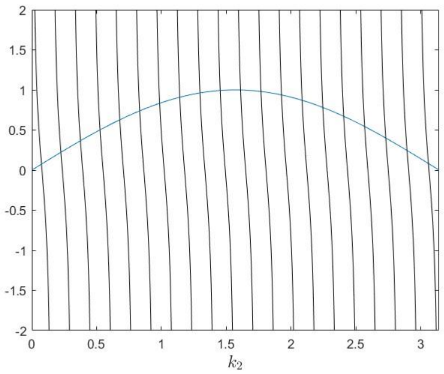
Even solutions

$$\tilde{\psi}_{n_1,i_2} = \cos(k_2(L/2 - i_2))$$

$$E = 2(1 - \cos(k_2))$$

Periodic boundaries and consistency at $i_2 = 0$:

$$2\sin(k_2) = \ell^B_{n_1}\cot(k_2 L/2)$$



A solution for each vertical asymptote approximatively

$$k_2 \simeq 2\pi n_2/L \text{ with } n_2 = 0,\ldots,L/2-1$$

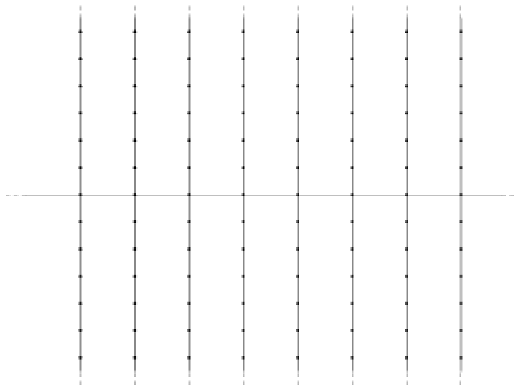
Quasi Degeneracy N^B

Localized solutions $\psi_{n_1,i_2} = (-1)^{i_2} e^{-A(n_1)|i_2|}$

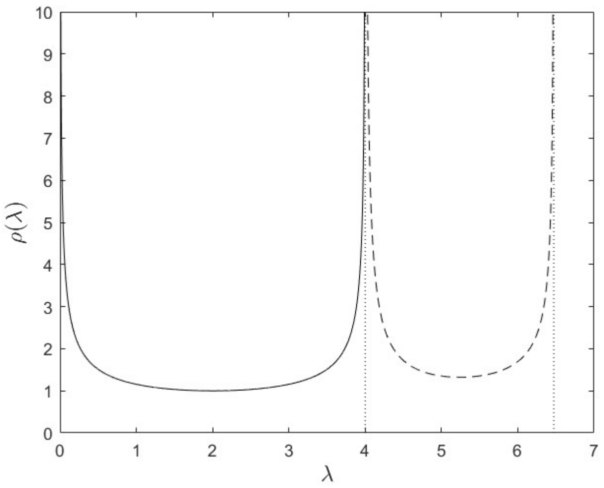
A single eigenstate for each value n_1 for large eigenvalues hidden spectrum

$$E^{n_1} = 2 + \sqrt{4 + (\ell_{n_1})^2}$$

Approximate even eigenvalues with error of order L^{-2}: exact spectral density in the thermodynamic limit



Main Spectrum same spectrum of the fiber network ($\bar{d} = \bar{d}_F$); N states.



Hidden spectrum. N^B states Number of sites in the base

Burioni, Cassi, Meccoli, Rasetti, Regina, Sodano, Vezzani 2000 *Europhys. Lett.* **52** 251 (2000)

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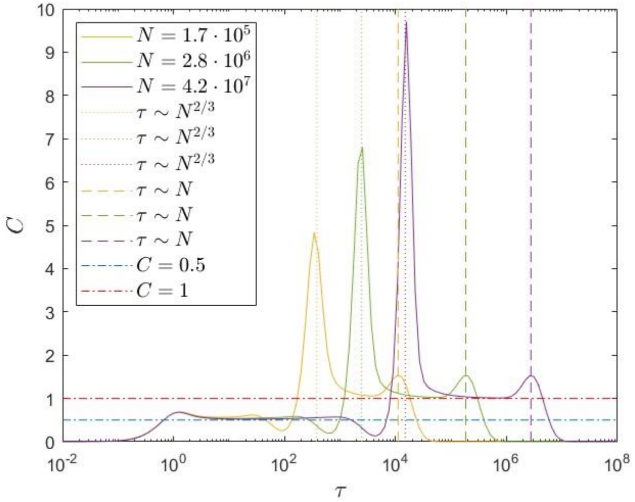
Exact spectral density but $Z = N \int \rho(\lambda) e^{-t\lambda} d\lambda$ Exact specific heat only in the thermodynamic limit

Approximation can be relevant for finite size effects; in particular, in mesoscopic properties at large finite N
We consider the ground state to be degenerate: gap exists at finite N, gap exponent?
Significant error on small eigenvalues $E \lesssim L^{-2} \sim N^{-2/d_f}$ i.e. error in specific heat for $\tau \gtrsim N^{2/d_f}$

Solve for small k_2

$$2\sin(k_2) = \ell^B_{n_1} \cot(k_2 L/2) \qquad k_2 = \frac{(\ell^B_{n_1})^{1/2}}{L^{1/2}} \qquad E = \frac{\ell^B_{n_1}}{L}$$

Low energy states $E \ll L^{-2}$, with same spectral density of the base specific heat at large τ.



First excited state E_1

$$E_1 = \frac{\ell^B_1}{L} \qquad \ell^B_1 \sim (N^B)^{-2/d_{g,B}} \qquad N^B \sim L^{d_{f,B}} \qquad N \sim L^{d_{f,B}+1} = L^{d_f}$$

$$E_1 \sim \frac{1}{N^{\frac{1+2d_{f,B}/d_{g,B}}{d_{f,B}+1}}} \qquad d_g = \frac{2(d_{f,B}+1)}{1+2d_{f,B}/d_{g,B}}$$

$d_g = 4/3$ for comb and $d_g = 2$ for brush numerically observed in specific heat

Calculation for $k_2 \ll L^{-1}$ (i.e. $E \ll L^{-2}$ and $\tau \gg N^{2/d_F}$) no information on the singularity at $\tau \sim N^{2/d_F}$, at this energies ($E \approx L^{-2}$) however we still expect the approximation we introduce in the spectrum to be important

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Conclusions

Bundled network tools to study of effects of inhomogeneity

The specific heat defined from the Laplacian density provides information on important topological feature of the network: e.g. average spectral dimension, gap induced dimension.

It is particularly relevant for mesoscopic properties i.e. properties emerging on large finite size networks when scaling with the system size is considered.

Novel kind of hidden eigen states that occupy a small region in the low energy spectrum, whose size vanishes in the thermodynamic limit. These states determine the specific heat and d_g at large τ.

Future work

Understand the divergence/singularity observed in the specific heat

Study different inhomogeneous networks

Study possible interesting effect of the hidden low energy states in physical phenomena as Debay specific heat, thermal instability or Bose Einstein condensation.