Neutrino Oscillations in Vacuum and Matter [1]

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Notes for neutrino oscillations in vacuum and dense matter.

Vacuum Oscillations

Schrodinger equation is

$$i\partial_t |\Psi\rangle = \mathbf{H} |\Psi\rangle , \tag{1}$$

where for relativistic neutrinos, the energy is[2]

[2] They all have the same momentum but different mass. The thing is we assume they have the same velocity since the mass is very small. To have an idea of the velocity difference, I can calculate the distance travelled by another neutrino in the frame of one neutrino.
To Be Discussed!
Will decoherence happen due to this?

$$\mathbf{H}^m = \begin{pmatrix} \sqrt{p^2+m_1^2} & 0 & 0 \\ 0 & \sqrt{p^2+m_2^2} & 0 \\ 0 & 0 & \sqrt{p^2+m_3^2} \end{pmatrix},$$

in which the energy terms are simplified using the relativistic condition

$$\sqrt{p^2+m_i^2} = p\sqrt{1+\frac{m_i^2}{p^2}} \tag{2}$$

$$\approx p(1+\frac{1}{2}\frac{m_i^2}{p^2}). \tag{3}$$

In general the flavor eigenstates are the mixing of the mass eigenstates with a unitary matrix $\mathbf{U}$, that is

$$|\nu_\alpha\rangle = U_{\alpha i} |\nu_i\rangle , \tag{4}$$

where the αs are indices for flavor states while the is are indices for mass eigenstates.

To find out the equation of motion for flavor states, plugin in the initary tranformation,

$$iU_{\alpha i}\partial_t |\nu_i\rangle = U_{\alpha i} H^m_{ij} |\nu_j\rangle . \tag{5}$$

I use index ^{m} for representation of Hamiltonian in mass eigenstates. Applying the unitary condition of the transformation,

$$\mathbf{I} = \mathbf{U}^\dagger \mathbf{U}, \tag{6}$$

I get

$$iU_{\alpha i}\partial_t |\nu_i\rangle = U_{\alpha i} H^m_{ij} U^\dagger_{j\beta} U_{\beta k} |\nu_k\rangle , \tag{7}$$

which is simplified to

$$i\partial_t \left|\nu_\alpha\right\rangle = H^f_{\alpha\beta} \left|\nu_\beta\right\rangle , \tag{8}$$

since the transformation is time independent.

The new Hamiltonian in the representations of flavor eigenstates reads

$$H^f_{\alpha\beta} = U^\dagger_{\alpha i} H^m_{ij} U_{j\beta}. \tag{9}$$

Survival Probability

The neutrino states at any time can be written as

$$\left|\Psi(t)\right\rangle = X_1 \left|\nu_1\right\rangle e^{-iE_1 t} + X_2 \left|\nu_2\right\rangle e^{-iE_2 t}, \tag{10}$$

where X_1 and X_2 are the initial conditions which are determined using the neutrino initial states.

Survival probalility is the squrare of the projection on an flavor eigenstate,

$$P_\alpha(t) = |\langle\nu_\alpha \mid \Psi(t)\rangle|^2. \tag{11}$$

The calculation of this expression requires our knowledge of the relation between mass eigenstates and flavor eigenstates which we have already found out.

Recall that the transformation between flavor and mass states is

$$\left|\nu_i\right\rangle = U^{-1}_{i\alpha} \left|\nu_\alpha\right\rangle , \tag{12}$$

which leads to the inner product of mass eigenstates and flavor eigenstates,

$$\begin{aligned} \langle\nu_\alpha \mid \nu_i\rangle &= \left\langle\nu_\alpha\right| U^{-1}_{i\beta} \left|\nu_\beta\right\rangle & (13)\\ &= U^{-1}_{i\beta}\delta_{\alpha\beta} & (14)\\ 7 &= U^{-1}_{i\alpha}. & (15) \end{aligned}$$

The survival probability becomes

$$\begin{aligned} P_\alpha(t) &= |\langle\nu_\alpha \mid X_1 \left|\nu_1\right\rangle e^{-iE_1 t} X_2 \left|\nu_2\right\rangle e^{-iE_2 t}\rangle|^2 \\ &= |X_1 e^{-iE_1 t}\langle\nu_\alpha \mid \left|\nu_1\right\rangle\rangle + X_2 e^{-iE_2 t}\langle\nu_\alpha \mid \nu_2\rangle|^2 \\ &= |\sum_i X_i e^{-iE_i t} U^{-1}_{i\alpha}|^2 \\ &= \sum_i X_1^* e^{iE_i t} U^{\dagger *}_{i\alpha} \sum_i X_i e^{-iE_i t} U^\dagger_{i\alpha} \\ &= |X_1|^2 U^{\dagger *}_{1\alpha} U^\dagger_{1\alpha} + |X_2|^2 U^{\dagger *}_{2\alpha} U^\dagger_{2\alpha} + X_1^* X_2 U^{\dagger *}_{1\alpha} U^\dagger_{2\alpha} e^{iE_1 t - iE_2 t} + X_2^* X_1 U^{\dagger *}_{2\alpha} U^\dagger_{1\alpha} e^{iE_2 t - iE_1 t} \end{aligned}$$

$U^{\dagger *}_{i\alpha}$ stands for the ith row and the αth column of the matrix $U^{\dagger *}$.

Two Flavor States

For 2 flavor neutrinos the Hamiltonian in the representation of propagation states,

$$\mathbf{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \begin{pmatrix} p_1 + \frac{1}{2}\frac{m_1^2}{p_1} & 0 \\ 0 & p_2 + \frac{1}{2}\frac{m_1^2}{p_2} \end{pmatrix}.$$

The equation of motion in matrix form is

$$i\partial_t \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} p_1 + \frac{1}{2}\frac{m_1^2}{p_1} & 0 \\ 0 & p_2 + \frac{1}{2}\frac{m_1^2}{p_2} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{16}$$

The flavor eigenstate is a mixing of the propagation eigenstates,

$$\begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{17}$$

Denote the rotation matrix using $\mathbf{U}$, the transofmation can be written as

$$|\nu_\alpha\rangle = \mathbf{U}_{\alpha i} |\nu_i\rangle\,, \tag{18}$$

where α is for the flavor eigenstates and i is for the mass eigenstates.

The survival probability has been derived in previous section, which is the projection of propagation states onto flavor states.

For arbitary initial condition,

$$\Psi(t=0) = A\,|\nu_a\rangle + B\,|\nu_b\rangle\,, \tag{19}$$

which can be rewritten into a matrix form,

$$\Psi(t=0) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} \tag{20}$$

To write down the projection, the relation

$$\begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{21}$$

is needed. BTW, the inverse transformation is the transpose of $\mathbf{U}$ since $\mathbf{U}$ is unitary, thus we have the relation,

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_v & -\sin\theta_v \\ \sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} \tag{22}$$

Thus in the state can be written as

$$\Psi(t=0) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \tag{23}$$

At any t, the state is

$$\Psi(t) = \begin{pmatrix} A\cos\theta_v - B\sin\theta_v & A\sin\theta_v + B\cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 e^{-iE_1 t} \\ \nu_2 e^{-iE_2 t} \end{pmatrix} \tag{24}$$

$$= \begin{pmatrix} (A\cos\theta_v - B\sin\theta_v)e^{-iE_1 t} & (A\sin\theta_v + B\cos\theta_v)e^{-iE_2 t} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{25}$$

The survival probability which is projection on a flavor state is written as

$$P(\nu_\alpha, t) = |\langle \nu_\alpha \mid \Psi(t)\rangle|^2. \tag{26}$$

The survival amplitude for ν_a is

$$\begin{aligned} &\langle \nu_a \mid \Psi(t)\rangle \\ &= \langle \nu_a| \left((A\cos\theta_v - B\sin\theta_v)e^{-iE_1 t} |\nu_1\rangle + (A\sin\theta_v + B\cos\theta_v)e^{-iE_2 t} |\nu_2\rangle \right) \\ &= (\cos\theta_v \langle \nu_1| + \sin\theta_v \langle \nu_2|) \left((A\cos\theta_v - B\sin\theta_v)e^{-iE_1 t} |\nu_1\rangle + (A\sin\theta_v + B\cos\theta_v)e^{-iE_2 t} |\nu_2\rangle \right) \end{aligned}$$

THis is simple since the transformation matrix is real.

Applying the condition that the propagation eigenstates are orthonormal, the survival probability is

$$\begin{aligned} P(\nu_a, t) &= |\langle \nu_a \mid \Psi(t)\rangle|^2 \\ &= |\cos\theta_v (A\cos\theta_v - B\sin\theta_v)e^{-iE_1 t} + \sin\theta_v (A\sin\theta_v + B\cos\theta_v)e^{-iE_2 t}|^2 \\ &= |(A\cos^2\theta_v - B\sin\theta_v\cos\theta_v)e^{-iE_1 t} + (A\sin^2\theta_v + B\sin\theta_v\cos\theta_v)e^{-iE_2 t}|^2 \end{aligned}$$

In a special limit that $E_1 = E_2 = E$, the probability becomes

$$P(\nu_a, t) = |A|^2 \tag{27}$$

which is the same as initial probability since there is no mixing at all.

There are two kinds of initial conditions.

- The neutrinos are all in ν_a state initially, which means $A = 1, B = 0$. The survival probability simplifies to

$$\begin{aligned} P(\nu_a, t) &= |\cos^2\theta_v e^{-iE_1 t} + \sin^2\theta_v e^{-iE_2 t}|^2 \\ &= |\cos^2\theta_v e^{-i(E_1-E_2)t} + \sin^2\theta_v|^2 \end{aligned}$$

As we have already discussed, $E_1 - E_2 = \frac{m_1^2 - m_2^2}{2p^2}$ assuming the neutrinos have the same momentum. [3] Using the notation $\Delta m^2 = m_1^2 - m_2^2$ and the approximation that $E \approx p$, the survival probability can be rewritten as

[3] And here is a question.

$$\begin{aligned} P(\nu_a, t) &= \cos^4\theta_v + \sin^4\theta_v + \cos^2\theta_v \sin^2\theta_v \left(e^{-i\Delta m^2 t/E} + e^{i\Delta m^2 t/E}\right) \\ &= 1 - 2\cos^2\theta_v \sin^2\theta_v + 2\cos^2\theta_v \sin^2\theta_v \cos\left(\frac{\Delta m^2 t}{2E}\right) \\ &= 1 - 2\cos^2\theta_v \sin^2\theta_v \left(1 - \cos\left(\frac{\Delta m^2 t}{2E}\right)\right) \\ &= 1 - 4\cos^2\theta_v \sin^2\theta_v \sin^2\left(\frac{\Delta m^2 t}{4E}\right) \\ &= 1 - \sin^2(2\theta_v)\sin^2\left(\frac{\Delta m^2 t}{4E}\right) \end{aligned}$$

We always assuming that in the region of interest, all neutrinos are travelling with the same speed, i.e., the speed of light $c = 1$. [4] Time is related to distance, $L = t$. Survival probability at distance L is

[4] which is not true obviously

$$P(\nu_a, L) = 1 - \sin^2(2\theta_v)\sin^2\left(\frac{\Delta m^2 L}{4E}\right) \tag{28}$$

- The neutrinos are all in ν_b state initially. Equivalently, we have $A = 0, B = 1$. Survival probability is

$$\begin{aligned} P(\nu_a, t) &= |-\sin\theta_v \cos\theta_v e^{-iE_1 t} + \sin\theta_v \cos\theta_v e^{-iE_2 t}|^2 \\ &= \sin^2\theta_v \cos^2\theta_v |e^{-i(E_1-E_2)t} - 1|^2 \\ &= \sin^2\theta_v \cos^2\theta_v \left(1 + 1 - e^{-i\Delta m^2 t/2E} - e^{i\Delta m^2 t/2E}\right) \\ &= 2\sin^2\theta_v \cos^2\theta_v \left(1 - \cos\left(\frac{\Delta m^2 t}{2E}\right)\right) \\ &= \sin^2(2\theta_v)\sin^2\left(\frac{\Delta m^2 t}{4E}\right) \\ &= \sin^2(2\theta_v)\sin^2\left(\frac{\Delta m^2 L}{4E}\right) \end{aligned}$$

Density Matrix

This problem can be solved using density matrix ρ and Von Neumann equation

$$i\partial_t \rho = [H, \rho]. \tag{29}$$

The initial condition for this equation is

$$\begin{aligned}\rho(t=0) &= (A\,|\nu_a\rangle + B\,|\nu_b\rangle)(A^*\,\langle\nu_a| + B^*\,\langle\nu_b|) \\ &= AA^*\,|\nu_a\rangle\,\langle\nu_a| + BB^*\,|\nu_b\rangle\,\langle\nu_b| + AB^*\,|\nu_a\rangle\,\langle\nu_b| + A^*B\,|\nu_b\rangle\,\langle\nu_a|\,.\end{aligned}$$

To calculate the propagation of the states, we need the Hamiltonian matrix in flavor basis.

This can be done by finding out how the Hamiltonian matrix transforms from one basis to another.

Using propagation basis,

$$i\partial_t \left|\Psi_p\right\rangle = H_p \left|\Psi_p\right\rangle. \tag{30}$$

The states are $|\Psi\rangle = \mathbf{U}\left|\Psi_p\right\rangle$ in flavor basis, which means we could plug in $\left|\Psi_p\right\rangle = \mathbf{U^T}\,|\Psi\rangle$.

$$i\partial_t \mathbf{U^T}\,|\Psi\rangle = H_p \mathbf{U^T}\,|\Psi\rangle\,.$$

Since $\mathbf{U}\mathbf{U^T} = \mathbf{I}$, we have a clean result by multiplying through the equation by $\mathbf{U}$.

$$i\partial_t\,|\Psi\rangle = \mathbf{U}H_p\mathbf{U^T}\,|\Psi\rangle\,.$$

So we define $H = \mathbf{U}H_p\mathbf{U^T}$ as the Hamiltonian matrix in flavor basis, which is

$$H = \left(p + \frac{m_1^2 + m_2^2}{4p}\right)\mathbf{I} - \frac{1}{4p}\begin{pmatrix} -\Delta m^2 \cos 2\theta & \Delta^2 m \sin 2\theta \\ \Delta m^2 \sin 2\theta & \Delta^2 m \cos 2\theta \end{pmatrix}. \tag{31}$$

Since identity matrix only shifts the eigenvalues we are only interested in the second term, thus the Hamiltonian we are going to use is

$$H = \frac{\Delta m^2}{4E}\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}. \tag{32}$$

The equation of motion becomes

$$i\partial_t \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \frac{\Delta m^2}{4E}\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \tag{33}$$

To solve this we need the eigenvalues and eigenvectors of the Hamiltonian matrix.

An Example of Survival Probability

Suppose the neutrinos are prepared in electron flavor initially, the survival probability of electron flavor neutrinos is calculated using the result I get previously.

Electron neutrinos are the lighter ones, then I have $_a = _e$ and denote $_b = _x$.

The survival probability for electron neutrinos is

$$P(\nu_e, L) = 1 - \sin^2(2\theta_v)\sin^2\left(\frac{\Delta m^2 L}{4E}\right).$$

Numerical Results for 2 Flavor Oscillations

Three Flavor States

For three flavor neutrinos, the oscillations matrix is 3 by 3 which is called the PMNS matrix.

$$\mathbf{U} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}. \tag{34}$$

The survival probability is given by the same derivation as the 2 flavor example.

Oscillations in Dense Medium