Neutrino Oscillations in Vacuum and Matter [1]

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Notes for neutrino oscillations in vacuum and dense matter.

Contents

Symbols and Definitions

- $\Delta = \sqrt{2} G_F n(x)$
- $\omega = \frac{\Delta m^2}{2E}$

Vacuum Oscillations

Schrodinger equation is

$$i\partial_t \left|\Psi\right\rangle = \mathbf{H} \left|\Psi\right\rangle, \tag{1}$$

where for relativistic neutrinos, the energy is[2]

$$\mathbf{H}^m = \begin{pmatrix} \sqrt{p^2 + m_1^2} & 0 & 0 \\ 0 & \sqrt{p^2 + m_2^2} & 0 \\ 0 & 0 & \sqrt{p^2 + m_3^2} \end{pmatrix},$$

in which the energy terms are simplified using the relativistic condition

$$\sqrt{p^2 + m_i^2} = p\sqrt{1 + \frac{m_i^2}{p^2}} \tag{2}$$

$$\approx p(1 + \frac{1}{2}\frac{m_i^2}{p^2}). \tag{3}$$

In general the flavor eigenstates are the mixing of the mass eigenstates with a unitary matrix $\mathbf{U}$, that is

$$\left|\nu_\alpha\right\rangle = U_{\alpha i} \left|\nu_i\right\rangle, \tag{4}$$

where the αs are indices for flavor states while the is are indices for mass eigenstates.

To find out the equation of motion for flavor states, plugin in the initary tranformation,

$$iU_{\alpha i}\partial_t \left|\nu_i\right\rangle = U_{\alpha i}H_{ij}^m \left|\nu_j\right\rangle. \tag{5}$$

I use index ^{m} for representation of Hamiltonian in mass eigenstates. Applying the unitary condition of the transformation,

$$\mathbf{I} = \mathbf{U}^\dagger\mathbf{U}, \tag{6}$$

I get

$$iU_{\alpha i}\partial_t \left|\nu_i\right\rangle = U_{\alpha i}H_{ij}^m U_{j\beta}^\dagger U_{\beta k} \left|\nu_k\right\rangle, \tag{7}$$

which is simplified to

$$i\partial_t \left|\nu_\alpha\right\rangle = H_{\alpha\beta}^f \left|\nu_\beta\right\rangle, \tag{8}$$

since the transformation is time independent.

[2] They all have the same momentum but different mass. The thing is we assume they have the same velocity since the mass is very small. To have an idea of the velocity difference, I can calculate the distance travelled by another neutrino in the frame of one neutrino.

Assuming the mass of a neutrino is 1eV with energy 10MeV, we will get a speed of $1 - 10^{-14}$c. This 10^{-14}c will make a difference about 3μm in 1s.

To Be Discussed!

Will decoherence happen due to this? For high energy neutrinos this won't be a problem however for low energy neutrinos this will definitely cause a problem for the wave function approach.Because the different mass eigenstates will decoherent gradually along the path.

The new Hamiltonian in the representations of flavor eigenstates reads

$$H^f_{\alpha\beta} = U^\dagger_{\alpha i} H^m_{ij} U_{j\beta}. \tag{9}$$

Survival Probability

The neutrino states at any time can be written as

$$|\Psi(t)\rangle = X_1 |\nu_1\rangle e^{-iE_1 t} + X_2 |\nu_2\rangle e^{-iE_2 t}, \tag{10}$$

where X_1 and X_2 are the initial conditions which are determined using the neutrino initial states.

Survival probalility is the squrare of the projection on an flavor eigenstate,

$$P_\alpha(t) = |\langle \nu_\alpha | \Psi(t)\rangle|^2. \tag{11}$$

The calculation of this expression requires our knowledge of the relation between mass eigenstates and flavor eigenstates which we have already found out.

Recall that the transformation between flavor and mass states is

$$|\nu_i\rangle = U^{-1}_{i\alpha} |\nu_\alpha\rangle, \tag{12}$$

which leads to the inner product of mass eigenstates and flavor eigenstates,

$$\langle \nu_\alpha | \nu_i \rangle = \langle \nu_\alpha | U^{-1}_{i\beta} |\nu_\beta\rangle \tag{13}$$

$$= U^{-1}_{i\beta} \delta_{\alpha\beta} \tag{14}$$

$$7 = U^{-1}_{i\alpha}. \tag{15}$$

The survival probability becomes

$$
\begin{aligned}
P_\alpha(t) &= |\langle \nu_\alpha | X_1 |\nu_1\rangle e^{-iE_1 t} X_2 |\nu_2\rangle e^{-iE_2 t}\rangle|^2 \\
&= |X_1 e^{-iE_1 t} \langle \nu_\alpha | |\nu_1\rangle\rangle + X_2 e^{-iE_2 t} \langle \nu_\alpha | \nu_2\rangle|^2 \\
&= |\sum_i X_i e^{-iE_i t} U^{-1}_{i\alpha}|^2 \\
&= \sum_i X_1^* e^{iE_i t} U^{\dagger*}_{i\alpha} \sum_i X_i e^{-iE_i t} U^\dagger_{i\alpha} \\
&= |X_1|^2 U^{\dagger*}_{1\alpha} U^\dagger_{1\alpha} + |X_2|^2 U^{\dagger*}_{2\alpha} U^\dagger_{2\alpha} + X_1^* X_2 U^{\dagger*}_{1\alpha} U^\dagger_{2\alpha} e^{iE_1 t - iE_2 t} + X_2^* X_1 U^{\dagger*}_{2\alpha} U^\dagger_{1\alpha} e^{iE_2 t - iE_1 t}
\end{aligned}
$$

$U^{\dagger*}_{i\alpha}$ stands for the ith row and the αth column of the matrix $U^{\dagger*}$.

Two Flavor States

For 2 flavor neutrinos the Hamiltonian in the representation of propagation states,

$$\mathbf{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \begin{pmatrix} p_1 + \frac{1}{2}\frac{m_1^2}{p_1} & 0 \\ 0 & p_2 + \frac{1}{2}\frac{m_1^2}{p_2} \end{pmatrix}.$$

The equation of motion in matrix form is

$$i\partial_t \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} p_1 + \frac{1}{2}\frac{m_1^2}{p_1} & 0 \\ 0 & p_2 + \frac{1}{2}\frac{m_1^2}{p_2} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{16}$$

The flavor eigenstate is a mixing of the propagation eigenstates,

$$\begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{17}$$

Denote the rotation matrix using $\mathbf{U}$, the transofmation can be written as

$$|\nu_\alpha\rangle = \mathbf{U}_{\alpha i} |\nu_i\rangle, \tag{18}$$

where α is for the flavor eigenstates and i is for the mass eigenstates.

The survival probability has been derived in previous section, which is the projection of propagation states onto flavor states.

For arbitary initial condition,

$$\Psi(t = 0) = A|\nu_a\rangle + B|\nu_b\rangle, \tag{19}$$

which can be rewritten into a matrix form,

$$\Psi(t = 0) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} \tag{20}$$

To write down the projection, the relation

$$\begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{21}$$

is needed. BTW, the inverse transformation is the transpose of $\mathbf{U}$ since $\mathbf{U}$ is unitary, thus we have the relation,

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_v & -\sin\theta_v \\ \sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} \tag{22}$$

Thus in the state can be written as

$$\Psi(t=0) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \qquad (23)$$

At any t, the state is

$$\Psi(t) = \begin{pmatrix} A\cos\theta_v - B\sin\theta_v & A\sin\theta_v + B\cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 e^{-iE_1 t} \\ \nu_2 e^{-iE_2 t} \end{pmatrix} \qquad (24)$$

$$= \begin{pmatrix} (A\cos\theta_v - B\sin\theta_v)e^{-iE_1 t} & (A\sin\theta_v + B\cos\theta_v)e^{-iE_2 t} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \qquad (25)$$

The survival probability which is projection on a flavor state is written as

$$P(\nu_\alpha, t) = |\langle \nu_\alpha \mid \Psi(t) \rangle|^2. \qquad (26)$$

The survival amplitude for ν_a is

$$\langle \nu_a \mid \Psi(t) \rangle$$
$$= \langle \nu_a | \left((A\cos\theta_v - B\sin\theta_v)e^{-iE_1 t} |\nu_1\rangle + (A\sin\theta_v + B\cos\theta_v)e^{-iE_2 t} |\nu_2\rangle \right)$$
$$= (\cos\theta_v \langle \nu_1| + \sin\theta_v \langle \nu_2|) \left((A\cos\theta_v - B\sin\theta_v)e^{-iE_1 t} |\nu_1\rangle + (A\sin\theta_v + B\cos\theta_v)e^{-iE_2 t} |\nu_2\rangle \right)$$

This is simple since the transformation matrix is real.

Applying the condition that the propagation eigenstates are orthonormal, the survival probability is

$$P(\nu_a, t) = |\langle \nu_a \mid \Psi(t) \rangle|^2$$
$$= |\cos\theta_v (A\cos\theta_v - B\sin\theta_v)e^{-iE_1 t} + \sin\theta_v (A\sin\theta_v + B\cos\theta_v)e^{-iE_2 t}|^2$$
$$= |(A\cos^2\theta_v - B\sin\theta_v\cos\theta_v)e^{-iE_1 t} + (A\sin^2\theta_v + B\sin\theta_v\cos\theta_v)e^{-iE_2 t}|^2$$

In a special limit that $E_1 = E_2 = E$, the probability becomes

$$P(\nu_a, t) = |A|^2 \qquad (27)$$

which is the same as initial probability since there is no mixing at all.

There are two kinds of initial conditions.

- The neutrinos are all in ν_a state initially, which means $A = 1, B = 0$. The survival probability simplifies to

$$P(\nu_a, t) = |\cos^2\theta_v e^{-iE_1 t} + \sin^2\theta_v e^{-iE_2 t}|^2$$
$$= |\cos^2\theta_v e^{-i(E_1 - E_2)t} + \sin^2\theta_v|^2$$

As we have already discussed, $E_1 - E_2 = \frac{m_1^2 - m_2^2}{2p}$ assuming the neutrinos have the same momentum. [3] Using the notation $\Delta m^2 = m_1^2 - m_2^2$ and the approximation that $E \approx p$, the survival probability can be rewritten as

$$P(\nu_a, t) = \cos^4 \theta_v + \sin^4 \theta_v + \cos^2 \theta_v \sin^2 \theta_v \left(e^{-i\Delta m^2 t/E} + e^{i\Delta m^2 t/E} \right)$$

$$= 1 - 2\cos^2 \theta_v \sin^2 \theta_v + 2\cos^2 \theta_v \sin^2 \theta_v \cos \left(\frac{\Delta m^2 t}{2E} \right)$$

$$= 1 - 2\cos^2 \theta_v \sin^2 \theta_v \left(1 - \cos \left(\frac{\Delta m^2 t}{2E} \right) \right)$$

$$= 1 - 4\cos^2 \theta_v \sin^2 \theta_v \sin^2 \left(\frac{\Delta m^2 t}{4E} \right)$$

$$= 1 - \sin^2(2\theta_v) \sin^2 \left(\frac{\Delta m^2 t}{4E} \right)$$

We always assuming that in the region of interest, all neutrinos are travelling with the same speed, i.e., the speed of light $c = 1$. [4] Time is related to distance, $L = t$. Survival probability at distance L is

$$P(\nu_a, L) = 1 - \sin^2(2\theta_v) \sin^2 \left(\frac{\Delta m^2 L}{4E} \right) \tag{28}$$

- The neutrinos are all in ν_b state initially. Equivalently, we have $A = 0, B = 1$. Survival probability is

$$P(\nu_a, t) = |-\sin \theta_v \cos \theta_v e^{-iE_1 t} + \sin \theta_v \cos \theta_v e^{-iE_2 t}|^2$$

$$= \sin^2 \theta_v \cos^2 \theta_v |e^{-i(E_1 - E_2)t} - 1|^2$$

$$= \sin^2 \theta_v \cos^2 \theta_v \left(1 + 1 - e^{-i\Delta m^2 t/2E} - e^{i\Delta m^2 t/2E} \right)$$

$$= 2\sin^2 \theta_v \cos^2 \theta_v \left(1 - \cos \left(\frac{\Delta m^2 t}{2E} \right) \right)$$

$$= \sin^2(2\theta_v) \sin^2 \left(\frac{\Delta m^2 t}{4E} \right)$$

$$= \sin^2(2\theta_v) \sin^2 \left(\frac{\Delta m^2 L}{4E} \right)$$

Density Matrix

This problem can be solved using density matrix ρ and Von Neumann equation

$$i\partial_t \rho = [H, \rho]. \tag{29}$$

The initial condition for this equation is

$$\rho(t=0) = (A\,|\nu_a\rangle + B\,|\nu_b\rangle)(A^*\,\langle\nu_a| + B^*\,\langle\nu_b|)$$
$$= AA^*\,|\nu_a\rangle\langle\nu_a| + BB^*\,|\nu_b\rangle\langle\nu_b| + AB^*\,|\nu_a\rangle\langle\nu_b| + A^*B\,|\nu_b\rangle\langle\nu_a|.$$

To calculate the propagation of the states, we need the Hamiltonian matrix in flavor basis.

This can be done by finding out how the Hamiltonian matrix transforms from one basis to another.

Using propagation basis,

$$i\partial_t\,|\Psi_p\rangle = H_p\,|\Psi_p\rangle. \qquad (30)$$

The states are $|\Psi\rangle = \mathbf{U}\,|\Psi_p\rangle$ in flavor basis, which means we could plug in $|\Psi_p\rangle = \mathbf{U^T}\,|\Psi\rangle$.

$$i\partial_t\mathbf{U^T}\,|\Psi\rangle = H_p\mathbf{U^T}\,|\Psi\rangle.$$

Since $\mathbf{UU^T} = \mathbf{I}$, we have a clean result by multiplying through the equation by $\mathbf{U}$.

$$i\partial_t\,|\Psi\rangle = \mathbf{U}H_p\mathbf{U^T}\,|\Psi\rangle.$$

So we define $H = \mathbf{U}H_p\mathbf{U^T}$ as the Hamiltonian matrix in flavor basis, which is

$$H = \left(p + \frac{m_1^2 + m_2^2}{4p}\right)\mathbf{I} - \frac{1}{4p}\begin{pmatrix} -\Delta m^2\cos 2\theta & \Delta^2 m\sin 2\theta \\ \Delta m^2\sin 2\theta & \Delta^2 m\cos 2\theta \end{pmatrix}. \qquad (31)$$

The derivation of this is

$$\mathbf{H}_\alpha = \mathbf{U}\hat{H}_j\mathbf{U}^T$$
$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\left(p\mathbf{I} + \frac{1}{2p}\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}\right)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$
$$= p\mathbf{I} + \frac{1}{2p}\begin{pmatrix} \cos^2\theta m_1^2 + \sin^2\theta m_2^2 & -\sin\theta\cos\theta m_1^2 + \sin\theta\cos\theta m_2^2 \\ -\sin\theta\cos\theta m_1^2 + \sin\theta\cos\theta m_2^2 & \sin^2\theta m_1^2 + \cos^2\theta m_2^2 \end{pmatrix}$$
$$= p\mathbf{I} + \frac{1}{2p}\begin{pmatrix} m_1^2 - \Delta m^2\sin^2\theta & -\frac{1}{2}\sin 2\theta\Delta m^2 \\ -\frac{1}{2}\sin 2\theta\Delta m^2 & m_2^2 + \Delta m^2\sin^2\theta \end{pmatrix}$$
$$= p\mathbf{I} + \frac{1}{2p}\left(\frac{1}{2}(m_1^2 + m_2^2)\mathbf{I} - \frac{1}{2}\begin{pmatrix} -\Delta m^2\cos 2\theta & \Delta^2 m\sin 2\theta \\ \Delta m^2\sin 2\theta & \Delta^2 m\cos 2\theta \end{pmatrix}\right)$$
$$= \left(p + \frac{m_1^2 + m_2^2}{4p}\right)\mathbf{I} - \frac{1}{4p}\begin{pmatrix} -\Delta m^2\cos 2\theta & \Delta^2 m\sin 2\theta \\ \Delta m^2\sin 2\theta & \Delta^2 m\cos 2\theta \end{pmatrix}$$

where $\Delta m^2 = m_1^2 - m_2^2$.

Since identity matrix only shifts the eigenvalues we are only interested in the second term, thus the Hamiltonian we are going to use is

$$H = \frac{\Delta m^2}{4E} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}. \tag{32}$$

The equation of motion becomes

$$i\partial_t \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \frac{\Delta m^2}{4E} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \tag{33}$$

To solve this we need the eigenvalues and eigenvectors of the Hamiltonian matrix.

An Example of Survival Probability

Suppose the neutrinos are prepared in electron flavor initially, the survival probability of electron flavor neutrinos is calculated using the result I get previously.

Electron neutrinos are the lighter ones, then I have $a = e$ and denote $b = x$. [5]

The survival probability for electron neutrinos is

$$P(\nu_e, L) = 1 - \sin^2(2\theta_v) \sin^2\left(\frac{\Delta m^2 L}{4E}\right).$$

Numerical Results for 2 Flavor Oscillations

To solve a set of first order differential equations, I need the determinant of coefficient matrix. For 2 flavor neutrino oscillations,

$$\partial_t \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = -i\frac{\Delta m^2}{4E} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}.$$

To find the solutions I need the eigenvalues λ I need to find the determinant

$$\det\left(-i\frac{\Delta m^2}{4E} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix} - \lambda \mathbf{I}\right)$$

$$= \begin{vmatrix} -i\frac{\Delta m^2}{4E}\cos 2\theta - \lambda & i\frac{\Delta m^2}{4E}\sin 2\theta \\ i\frac{\Delta m^2}{4E}\sin 2\theta & i\frac{\Delta m^2}{4E}\cos 2\theta - \lambda \end{vmatrix}.$$

By defining $\lambda' = \lambda/(-i\Delta m^2/4E)$, the determinant is

$$-\left(\frac{\Delta m^2}{4E}\right)^2 \left((\cos 2\theta - \lambda')(-\cos 2\theta - \lambda') - \sin 2\theta \sin 2\theta\right).$$

[5] In the small mixing angle limit,

$$\begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix} \to \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

which is very close to an identity matrix. This implies that electron neutrino is more like mass eigenstate ν_1. By ν_1 we mean the state with energy $\frac{\delta m^2}{4E}$ in vacuum.

The eigenvalues are the solutions to

$$-\left(\frac{\Delta m^2}{4E}\right)^2 \left((\cos 2\theta - \lambda')(-\cos 2\theta - \lambda') - \sin^2 2\theta \sin 2\theta\right) = 0,$$

whose solution is

$$\lambda' = \pm 1.$$

With the solutions

$$\lambda = \pm i\frac{\Delta m^2}{4E},$$

the eigenvectors can also be solved.

$$\begin{pmatrix} \cos 2\theta - 1 & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta - 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

gives us $\eta_2 = -\tan\theta\eta_1$, which means the eigenvectors are

$$\begin{pmatrix} 1 \\ -\tan\theta \end{pmatrix}, \begin{pmatrix} 1 \\ \cot\theta \end{pmatrix}. \tag{34}$$

The general solution of the first order differential equations is

$$\begin{pmatrix} 1 \\ -\tan\theta \end{pmatrix} e^{-i\Delta m^2 t/4E}$$

$$\begin{pmatrix} 1 \\ \cot\theta \end{pmatrix} e^{i\Delta m^2 t/4E}.$$

Initial condition is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and it determines the final solution

$$\cos^2\theta \begin{pmatrix} 1 \\ -\tan\theta \end{pmatrix} e^{-i\Delta m^2 t/4E} + \sin^2\theta \begin{pmatrix} 1 \\ \cot\theta \end{pmatrix} e^{i\Delta m^2 t/4E}$$

$$= \begin{pmatrix} \cos^2\theta \\ -\sin\theta\cos\theta \end{pmatrix} e^{-i\Delta m^2 t/4E} + \begin{pmatrix} \sin^2\theta \\ \sin\theta\cos\theta \end{pmatrix} e^{i\Delta m^2 t/4E}$$

The survival probability of electron neutrino is

$$P = |\cos^2\theta e^{-i\Delta m^2 t/4E} + \sin^2\theta e^{i\Delta m^2 t/4E}|^2$$

$$= |\cos^2\theta e^{-i\Delta m^2 t/2E} + \sin^2\theta|^2,$$

which gets back to the result we had using the previous method.

This problem can also be solved using numerical methods. Here is a comparison between this analytical result and a numerical result.

Three Flavor States

For three flavor neutrinos, the oscillations matrix is 3 by 3 which is called the PMNS matrix.

$$
\mathbf{U} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix},
$$

Figure 1: They overlap on all the range completely.

which acts on the mass eigenstates to give the flavor eigenstates, i.e.,

$$
|\nu_\alpha\rangle = \mathbf{U}\,|\nu_i\rangle .
$$

In general, a rotation for 3D with a CP violation phase δ is

$$
\mathbf{U} = \begin{pmatrix} \cos(\theta_{12})\cos(\theta_{13}) & \cos(\theta_{13})\sin(\theta_{12}) & e^{-i\delta_{CP}}\sin(\theta_{13}) \\ -\cos(\theta_{23})\sin(\theta_{12}) - e^{i\delta_{CP}}\cos(\theta_{12})\sin(\theta_{13})\sin(\theta_{23}) & \cos(\theta_{12})\cos(\theta_{23}) - e^{i\delta_{CP}}\sin(\theta_{12})\sin(\theta_{13})\sin(\theta_{23}) & \cos(\theta_{13})\sin(\theta_{23}) \\ \sin(\theta_{12})\sin(\theta_{23}) - e^{i\delta_{CP}}\cos(\theta_{12})\cos(\theta_{23})\sin(\theta_{13}) & -e^{i\delta_{CP}}\cos(\theta_{23})\sin(\theta_{12})\sin(\theta_{13}) - \cos(\theta_{12})\sin(\theta_{23}) & \cos(\theta_{13})\cos(\theta_{23}) \end{pmatrix}.
$$

However, with the CP violation phase δ this matrix is not unitary thus I'll choose $\delta = 0$ for simplicity.[6]

Now the matrix becomes,

[6] The effect of the phase can also be studied easily with the help of Mathematica.

$$
\mathbf{U} = \begin{pmatrix} \cos(\theta_{12})\cos(\theta_{13}) & \cos(\theta_{13})\sin(\theta_{12}) & \sin(\theta_{13}) \\ -\cos(\theta_{23})\sin(\theta_{12}) - \cos(\theta_{12})\sin(\theta_{13})\sin(\theta_{23}) & \cos(\theta_{12})\cos(\theta_{23}) - \sin(\theta_{12})\sin(\theta_{13})\sin(\theta_{23}) & \cos(\theta_{13})\sin(\theta_{23}) \\ \sin(\theta_{12})\sin(\theta_{23}) - \cos(\theta_{12})\cos(\theta_{23})\sin(\theta_{13}) & -\cos(\theta_{23})\sin(\theta_{12})\sin(\theta_{13}) - \cos(\theta_{12})\sin(\theta_{23}) & \cos(\theta_{13})\cos(\theta_{23}) \end{pmatrix}.
$$

The survival probability is given by the same derivation as the 2 flavor example. First of all we need to find the Hamiltonian in flavor basis from the propagation Hamiltonian, which is

$$
\mathbf{H} = \mathbf{U}\mathbf{H_p}\mathbf{U}^{-1}
$$

where

$$
\mathbf{H_p} = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}.
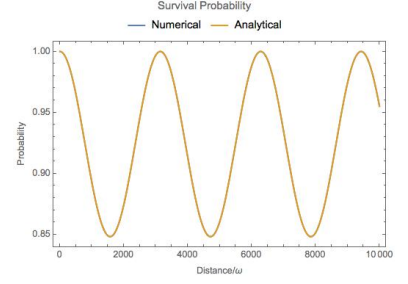$$

The next step is to define

$$\Delta m_{12}^2 = m_1^2 - m_2^2$$
$$\Delta m_{23}^2 = m_2^2 - m_3^2$$

so that they simplifies the Hermitian Hamiltonian to

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix},$$

where [7]

[7] Haven't being copied from Mathematica code.

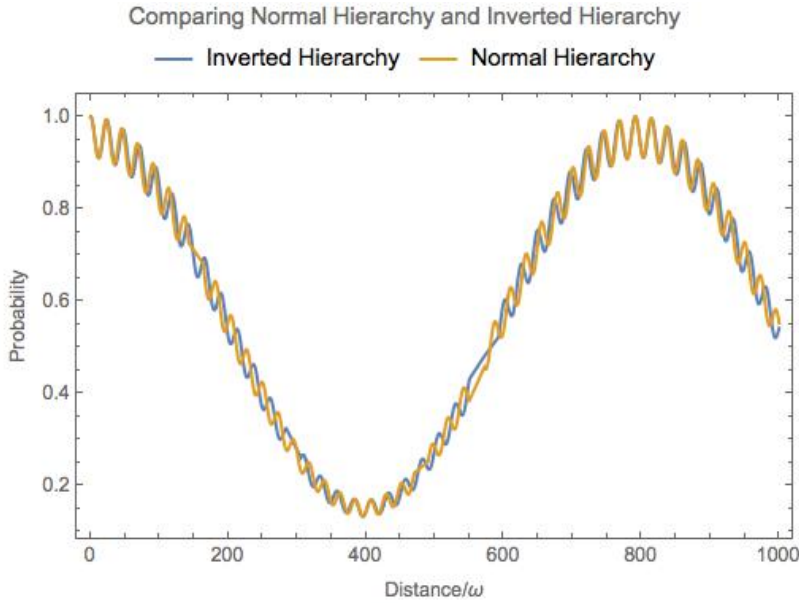$$H_{11} = \cos^2 \theta_{13}(m_2^2 - \sin^2 \theta \Delta m_{12}^2) + \sin^2 \theta_{13} m_3^2$$

Figure 2: The overall shape is the same however they differ on small scales.

Ternary Diagram for Neutrino Flavor Oscillation

Since the probability for differential flavors of neutrinos are represented in barycentric coordinates[8], a ternary plot would be nice to representation the oscillations. An example is shown in figure 3.
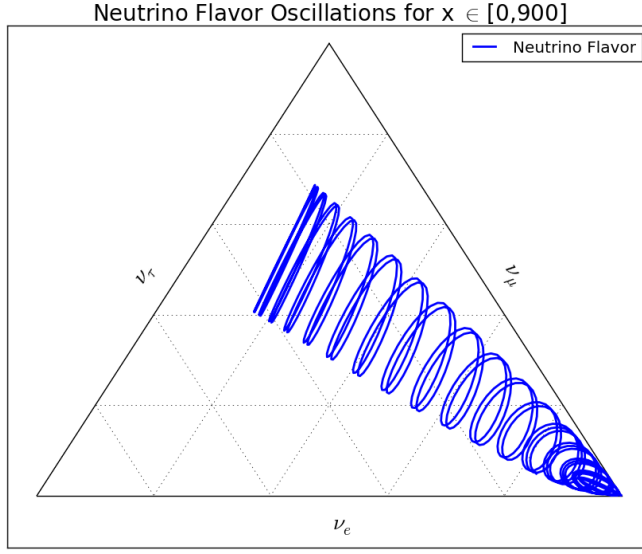
[8] They sum up to 1.

Figure 3: Ternary diagram for neutrino oscillations. The state starts from bottom left, which means that the system has only electron neutrinos. As the neutrino travels, it oscillates in curves. After one period of the beat, it reaches the far end and then oscillates backwards.

Oscillations in Matter

The Hamiltonian should be determined first. We have already derived the Hamiltonian for vacuum oscillation,

$$H_v = \frac{\delta m^2}{2E} \frac{1}{2} \begin{pmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{pmatrix},$$

where we would like to define a new matrix,

$$\mathbf{B} = \frac{1}{2} \begin{pmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{pmatrix},$$
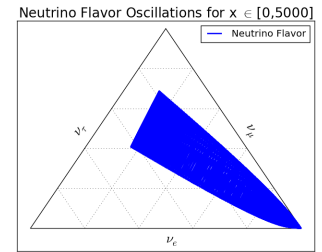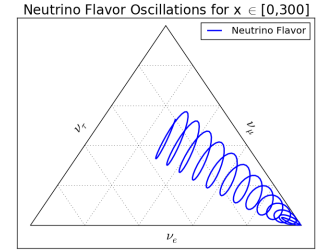
so that the vacuum Hamiltonian can be written as

$$H_v = \frac{\delta m^2}{2E} \mathbf{B}.$$

The effect of matter, adds an extra term to this vacuum Hamiltonian which makes the electron population weighs more,

$$H_m = \sqrt{2} G_F n_e L.$$

Here we have [9]

[9] In principle we could shift the Hamiltonian using an identity matrix $\alpha \mathbf{I}$ without change the eigenvectors. We will use an term $\frac{1}{2} \boldsymbol{\alpha}_3$ instead in the following sections.

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Without emphasizing the self-interaction of the neutrinos, the Hamiltonian to be used is

$$H = H_v + H_m. \tag{35}$$

The equation of motion is simply the von Neumann equation

$$i\partial_t \rho = [H, \rho], \tag{36}$$

in which the ∂_t operator is actually ∂_x since we always assume neutrinos travel with speed of light c.

Analytic Solution

This Hamiltonian can be rewritten into a simple form using Pauli matrices,

$$\begin{aligned} \mathbf{H} &= \frac{\Delta m^2}{4E} \left(-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1 \right) + \frac{\Delta}{2}\sigma_3 \\ &= \left(\frac{\Delta}{2} - \frac{\Delta m^2}{4E}\cos 2\theta_v \right)\sigma_3 + \frac{\Delta m^2}{4E}\sin 2\theta_v \sigma_1 \\ &= \left(\frac{\Delta}{2} - \frac{\omega}{2}\cos 2\theta_v \right)\sigma_3 + \frac{\omega}{2}\sin 2\theta_v \sigma_1, \end{aligned}$$

where $\Delta = \sqrt{2}G_F n(x)$ and $n(x)$ is the number density of the electrons.

To solve the equation of motion, this matrix should be diagonalized and its eigenvalues and eigenvectors should be identified. Since we have this Pauli matrices form, this can be done easily.

To see this effect quantitatively, we need to diagonalize this Hamiltonian (Can we actually diagonalize the equation of motion? NO!). Equivalently, we can rewrite it in the basis of mass eigenstates $\{|\nu_L(x)\rangle, |\nu_H(x)\rangle\}$ [10],

$$\begin{aligned} |\nu_L(x)\rangle &= \cos\theta(x)|\nu_e\rangle - \sin\theta(x)|\nu_\mu\rangle \\ |\nu_H(x)\rangle &= \sin\theta(x)|\nu_e\rangle + \cos\theta(x)|\nu_\mu\rangle. \end{aligned}$$

This new rotation in matrix form is

$$\begin{aligned} \begin{pmatrix} |\nu_L(x)\rangle \\ |\nu_H(x)\rangle \end{pmatrix} &= \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix} \begin{pmatrix} |\nu_e\rangle \\ |\nu_x\rangle \end{pmatrix} \\ &= \mathbf{U_m^{-1}} \begin{pmatrix} |\nu_e\rangle \\ |\nu_x\rangle \end{pmatrix}. \end{aligned}$$

[10] This is very different from the vacuum case since this one is local. In principle we can not use this transformation to diagonalize the Hamiltonian because the equation is a differential equation about x.
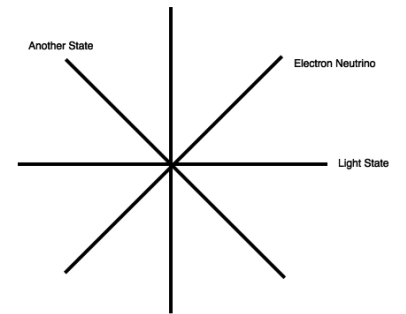

Figure 4: Due to this rotation, we can find an angle when the electron neutrinos becomes more important in Heavy state than Light state, which is $\theta = \frac{\pi}{4}$. For a mixing that has a larger angle, electron neutrinos actually take part in heavy neutrino state.

To diagonalize it, we need to multiply on both sides the rotation matrix and its inverse, as we have done in the vacuum case,

$$\mathbf{H_{md}} = \mathbf{U_m^{-1} H U_m}.$$

The second step is to set the off diagonal elements to zero. By solving the equations we can find the $\sin 2\theta(x)$ and $\cos 2\theta(x)$. [11]

$$\mathbf{H_{md}} = \mathbf{U_m^{-1}} \left(A_1 \sigma_1 + A_3 \sigma_3 \right) \mathbf{U_m}$$
$$= \begin{pmatrix} A_3 \cos 2\theta(x) - A_1 \sin 2\theta(x) & A_3 \sin 2\theta(x) + A_1 \cos 2\theta(x) \\ A_3 \sin 2\theta(x) + A_1 \cos 2\theta(x) & -A_3 \cos 2\theta(x) + A_1 \sin 2\theta(x) \end{pmatrix},$$

where

$$A_3 = \frac{\Delta}{2} - \frac{\delta^2 m}{4E} \cos 2\theta_v = \frac{\Delta}{2} - \frac{\omega}{2} \cos 2\theta_v$$
$$A_1 = \frac{\delta^2 m}{4E} \sin 2\theta_v = \frac{\omega}{2} \sin 2\theta_v.$$

Set the off-diagonal elements to zero,

$$A_3 \sin 2\theta(x) + A_1 \cos 2\theta(x) = 0$$

So the solutions are

$$\sin 2\theta(x) = \frac{A_1}{\sqrt{A_1^2 + A_3^2}}$$
$$\cos 2\theta(x) = \frac{-A_3}{\sqrt{A_1^2 + A_3^2}}.$$

This diagonalizes the Hamiltonian LOCALLY. [12] It's not possible to diagonalize the Hamiltonian globally if the electron number density is not a constant.

As $\Delta \to \infty$, we have $A_3 \to \infty$ and $\sin 2\theta(x)$ vanishes. Thus the neutrino will stay on the defined light and heavy eigenstates if the interaction with matter is much larger than the vacuum energy. On the other hand, $\Delta \to 0$, everything gets back to the vacuum case.

Now we can obtain an approximate solution by using this approximate diagonalization. This idea is

$$i \partial_x \Psi_m(x) = \mathbf{ExtraMatrixFromLHS} \cdot \mathbf{H}_{md} \Psi_m(x),$$

where the **ExtraMatrixFromLHS** comes from the fact that changing from flavor basis $\Psi(x)$ to heavy-light basis $\Psi_m(x)$ using $\mathbf{U_m}$

$$i \partial_x (\mathbf{U_m} \Psi_m(x)) = H(\mathbf{U_m} \Psi_m(x))$$

[12] The point is, for equation of motion, we have a differentiation with respect to position x ! So even we diagonalize the Hamiltonian, the equation of motion won't be diagonalized. An extra matrix will occur on the LHS and revert the diagonalization the Hamiltonian on RHS.

only returns

$$i\partial_x \Psi_m(x) = \mathbf{H_{md}}\Psi_m(x) - i\mathbf{U_m^{-1}}(\partial_x \mathbf{U_m})\Psi_m(x).$$

Combining the two terms on RHS,

$$i\partial_x \Psi_m(x) = \mathbf{H_m}\Psi_m(x),$$

where

$$\mathbf{H_m} = \mathbf{H_{md}} - i\mathbf{U_m^{-1}}(\partial_x \mathbf{U_m}).$$

The only part inside $\mathbf{U_m}(\mathbf{x})$ that is space dependent is the number density of the electrons $n(x)$. **Thus we know immediately that the Hamiltonian is diagonalized if the number density is constant.**[13],[14],[15]

Constant Electron Number Density

Suppose we have an environment with constant electron number density, the term $-i\mathbf{U_m^{-1}}(\partial_x \mathbf{U_m})$ goes away. All we have is the diagonalized new Hamiltonian $\mathbf{H_{md}}$ and the eigenvalues are easily obtained which are

$$E_1 = A_3 \cos 2\theta(x) - A_1 \sin 2\theta(x)$$
$$E_2 = -A_3 \cos 2\theta(x) + A_1 \sin 2\theta(x).$$

The final result for these two eigenvalues are

$$E_1 = -\sqrt{A_1^2 + A_3^2}$$
$$E_2 = \sqrt{A_1^2 + A_3^2},$$

where

$$A_1^2 + A_3^2 = \frac{\Delta^2 + \omega^2}{4} - \frac{\Delta\omega}{2}\cos 2\theta_v.$$

There are two special cases. The first case is $\cos 2\theta_v = 0$ while the other one is $\cos 2\theta_v = 1$ which leads to $\sqrt{A_1^2 + A_3^2} = |\frac{\Delta-\omega}{2}|$ and the two energies becomes $\frac{\Delta-\omega}{2}$ and $\frac{\omega-\Delta}{2}$.[16]

As for the survival probability, the result has the same form as the vacuum case, which is

$$P_x(\nu_e, L) = 1 - \sin^2(2\theta_m)\sin^2\left(\frac{\omega L}{2}\right),$$

where $\theta_m = \theta(x)$ is the effective mixing angle which in fact doesn't depend on x if the matter profile is constant.[17]

[13] But even the electrons density is constant, the oscillation is different.

[14] **It would be nice to find the expression of $\mathbf{U_m}$ in order to play with the matter effect.**

[15] In a two level system, suppose the Hamiltonian is

$$H_2 = H_{d2} + P_2,$$

where

$$H_{d2} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & W(r) \\ W^*(r) & 0 \end{pmatrix}$$

are the diagonalized Hamiltonian and the perturbation.

The diagonalized Hamiltonian will result in a system with two levels which are constant as a function of r, however the perturbation term provides the r dependent term which will cause a r dependent eigen energies.

[16] This means the matter effect in this constant profile situation determines the difference between the eigen energies.

[17] The term $\sin 2\theta_m$ has been derived previously.

$$\sin 2\theta(x) = \frac{\omega \sin 2\theta_v}{\sqrt{\omega^2 + \Delta^2 - 2\omega\Delta\cos 2\theta_v}}.$$

Adiabatic Limit

In some astrophysical environments the electron number density changes very slowly which means the term $\mathbf{U_m^{-1}}\partial_x\mathbf{U_m}$ is much smaller than $\mathbf{H_{md}}$. By intuition we would expect that this term could be dropped to the lowest order. However to verify that we need to check the Taylor expansion of this gradient term which requires the general form of $\mathbf{U_m}$.

Before we actually do the expansion, make the quantities dimensionless would greatly help us.

$$\sin 2\theta(x) = \frac{\sin 2\theta_v}{\sqrt{\left(\frac{\Delta}{\omega}\right)^2 + 1 - 2\frac{\Delta}{\omega}\cos 2\theta_v}}$$

$$\cos 2\theta(x) = \frac{\cos 2\theta_v - \frac{\Delta}{\omega}}{\sqrt{\left(\frac{\Delta}{\omega}\right)^2 + 1 - 2\frac{\Delta}{\omega}\cos 2\theta_v}}.$$

Define $\hat{\Delta} = \frac{\Delta}{\omega}$, which represents the matter interaction strength compared to the vacuum oscillation.

$$\sin 2\theta(x) = \frac{\sin 2\theta_v}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}}$$

$$\cos 2\theta(x) = \frac{\cos 2\theta_v - \hat{\Delta}}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}}.$$

$$\sin\theta(x) = \frac{\csc(2\theta_v)\sqrt{\frac{-\hat{\Delta}+\sqrt{\hat{\Delta}^2-2\hat{\Delta}\cos(2\theta_v)+1}+\cos(2\theta_v)}{\sqrt{\hat{\Delta}^2-2\hat{\Delta}\cos(2\theta_v)+1}}}\left(\hat{\Delta}+\sqrt{\hat{\Delta}^2-2\hat{\Delta}\cos(2\theta_v)+1}-\cos(2\theta_v)\right)}{\sqrt{2}}$$

$$\cos\theta(x) = \frac{\sqrt{\frac{-\hat{\Delta}+\sqrt{\hat{\Delta}^2-2\hat{\Delta}\cos(2\theta_v)+1}+\cos(2\theta_v)}{\sqrt{\hat{\Delta}^2-2\hat{\Delta}\cos(2\theta_v)+1}}}}{\sqrt{2}}$$

This is extremely complicated if we take the derivative with respect to x. So we apply Taylor expansion.[18]

[18] The *i*s are there makes sure that $-i\partial_x\mathbf{U_m}(x)$ is real.

$$\frac{d\sin\theta(x)}{dx} = \frac{d\sin\theta(x)}{d\hat{\Delta}}\frac{d\hat{\Delta}}{dx}$$

$$= i\frac{d\hat{\Delta}}{dx}\left(-\cos\theta_v\sin^2\theta_v - \frac{1}{2}((1+5\cos 2\theta_v)\cos\theta_v\sin^2\theta_v)\hat{\Delta} + \mathcal{O}(\hat{\Delta}^2)\right)$$

$$\frac{d\cos\theta(x)}{dx} = \frac{d\cos\theta(x)}{d\hat{\Delta}}\frac{d\hat{\Delta}}{dx}$$

$$= i\frac{d\hat{\Delta}}{dx}\left(\cos^2\theta_v\sin\theta_v + \frac{1}{2}\cos^2\theta_v(-1+5\cos 2\theta_v)\sin\theta_v\hat{\Delta} + \mathcal{O}(\hat{\Delta}^2)\right)$$

There is nothing in the coefficients that prohibits us from using the lowest order that $\frac{d\hat{\Delta}/dx}{\text{Energy Differences Between Two Energy Levels}} \ll 1$ since the Taylor series has finite values.[19]

Solving the adiabatic limit is just the same as the constant limit but $\theta(x)$ is not constant which ensures changing energies. What would be interesting is the behavior of energies as the density profile is slowing changing.

$$E_1 = -\frac{\omega}{2}\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}$$
$$E_2 = \frac{\omega}{2}\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}.$$

When the term $\hat{\Delta}$ is very small $1 - 2\hat{\Delta}\cos 2\theta_v$ will dominate and the whole term decreases. On the other hand as $\hat{\Delta}$ becomes large, $\hat{\Delta}^2$ will dominate and the whole term grows.

The survival probability for the light neutrinos would be

$$P_x(\nu_L, L) = 1 - \sin^2(2\theta(x))\sin^2\left(\frac{\omega L}{2}\right).$$

Before we move on to higher order corrections, it would be nice to understand this phenomenon.
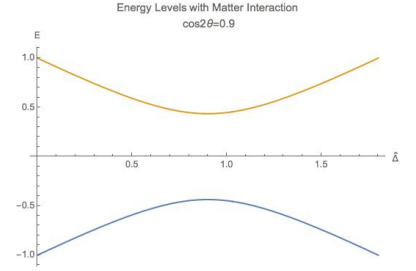

Figure 5: Energy Levels for MSW effect. We have the up-down symmetry since we shifted the energy by a constant to remove the identity matrix in the Hamiltonian.

- The vacuum oscillation length can be extracted from vacuum oscillation survival probability. It is $L_v = \frac{2\pi}{\omega}$.

- In this problem we have another energy scale which is the interaction, Δ. Here we can define another characteristic length $l_m = \frac{2\pi}{\Delta}$.

- MSW resonance happens when the two character lengths are matching to each other. Another way to put it is that the the term $\sin 2\theta(x)$ is minimized so that we have the smallest gap which leads to math $\hat{\Delta} = \cos 2\theta_v$. Equivalently this is the relation

$$l_0 = l_m \cos 2\theta_v.$$

- At resonance, we have

$$\cos 2\theta(x) = 1$$
$$\sin 2\theta(x) = 0.$$

This is max mixing of the states which means that at the resonance point

$$\begin{pmatrix} \nu_L(x_{reso}) \\ \nu_H(x_{reso}) \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix}$$

- Resonance conditions corresponds to a resonance density which is given by

$$n_e(x) = \frac{\omega}{\sqrt{2}G_F} \cos 2\theta_v \equiv n_0(E, \Delta m^2) \cos 2\theta_v,$$

where $n_0(E, \Delta m^2)$ is a characteristic number density which depends on the energy mixing angles and Δm^2 of the neutrinos.

- One should notice that to get back to the vacuum oscillation survival probability which means $\sin^2 2\theta(x) = \sin^2 2\theta_v$ we have[20]

$$\hat{\Delta}^2 + 1 - 2\hat{\Delta} \cos 2\theta_v = 1,$$

which leads to

$$\hat{\Delta} = 0 \quad \text{or} \quad 2\cos 2\theta_v.$$

The first condition is trivial which corresponds to vacuum however the second condition $\Delta = 2\cos 2\theta_v \omega$ means the interaction oscillation length is doubled compared to resonance point.

Nevertheless, we should always remember to check what survival probability the expression is describing. Here we have survival probability for $\nu_L(x)$. At $n(x) \to 0$ the oscillation becomes vacuum oscillation.

First Order Approximation

In principle we could use Taylor expansion of $\mathbf{U_m}$ to find the series of the derivatives of $\cos\theta(x)$ and $\sin\theta(x)$, however it doesn't looks like the easy way. Now we go back to find the Taylor expansion of the Hamiltonian which only involves the expression $\sin 2\theta(x)$ and $\cos 2\theta(x)$.

The series of them are

$$\sin 2\theta(x) = \sin 2\theta_v + \cos 2\theta_v \sin 2\theta_v \hat{\Delta} + \mathcal{O}(\hat{\Delta})^2$$
$$\cos 2\theta(x) = \cos 2\theta_v - \sin^2 2\theta_v \hat{\Delta} + \mathcal{O}(\hat{\Delta})^2.$$

Notice that for small θ_v the first order can be neglected sine they are second or higher order small quantities.

$\sin\theta(x)$ and $\cos\theta(x)$ can be found using these results.

$$\sin\theta(x) = \sin\theta_v + \cos^2\theta_v \sin\theta_v \hat{\Delta} + \mathcal{O}(\hat{\Delta}^2)$$
$$\cos\theta(x) = \cos\theta_v - \cos\theta_v \sin^2\theta_v \hat{\Delta} + \mathcal{O}(\hat{\Delta}^2).$$

[20] **I should check if this is still true in 3 flavor neutrino scenario. This could be interesting if it is respected in 3 flavor neutrino oscillations.**

To find out the solutions the explicit expression for $\mathbf{U_m}$ is required. The rotation matrix is

$$\mathbf{U_m} = \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ -\sin\theta(x) & \cos\theta(x) \end{pmatrix}$$

$$\approx \begin{pmatrix} \cos\theta_v - \cos\theta_v \sin^2\theta_v\hat{\Delta} & \sin\theta_v + \cos^2\theta_v \sin\theta_v\hat{\Delta} \\ -\sin\theta_v - \cos^2\theta_v \sin\theta_v\hat{\Delta} & \cos\theta_v - \cos\theta_v \sin^2\theta_v\hat{\Delta} \end{pmatrix}.$$

The derivative of this rotation matrix with respect to x is

$$\partial_x\mathbf{U_m} = \frac{d\hat{\Delta}}{dx} \begin{pmatrix} -\cos\theta_v\sin^2\theta_v & \cos^2\theta_v\sin\theta_v \\ -\cos^2\theta_v\sin\theta_v & -\cos\theta_v\sin^2\theta_v \end{pmatrix}$$

The effective potential is[21]

[21] This expression comes from the result of the first order approximation of $\sin\theta(x)$ and $\cos\theta(x)$.

$$\mathbf{V_m} = -i\mathbf{U_m^{-1}}(\partial_x\mathbf{U_m})$$

$$= -i\sin\theta_v\cos\theta_v\frac{d\hat{\Delta}}{dx} \begin{pmatrix} \cos\theta_v\sin\theta_v\hat{\Delta} & 1 \\ -1 & \cos\theta_v\sin\theta_v\hat{\Delta} \end{pmatrix}.$$

Since $\frac{d\hat{\Delta}}{dx}\hat{\Delta}$ is one order higher than $\hat{\Delta}$, the effective potential can be simplified to first order, which is

$$\mathbf{V_m} = -i\sin\theta_v\cos\theta_v\frac{d\hat{\Delta}}{dx} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The equation of motion up to first order of $\hat{\Delta}$ becomes

$$i\partial_x\ket{\Psi_m} = (\mathbf{H_{md}} + \mathbf{V_m})\ket{\Psi_m}.$$

We have already solved

$$i\partial_x\ket{\Psi_m} = \mathbf{H_{md}}\ket{\Psi_m},$$

where the eigenstates are $\ket{\nu_L}$ and $\ket{\nu_H}$.
In general we define

$$v = -\sin\theta_v\cos\theta_v\frac{d\hat{\Delta}}{dx}$$

so that

$$\mathbf{V_m} = \begin{pmatrix} 0 & iv \\ -iv & 0 \end{pmatrix}.$$

The general solution to the equation we need to solve can be written as

$$|\Psi_m\rangle = C_L(x)e^{-i\int \omega_{m1}dx}|\nu_L\rangle + C_H(x)e^{-i\int \omega_{m2}dx}|\nu_H\rangle,$$

where

$$\omega_{m1} = -\sqrt{\frac{\Delta^2 + \omega^2}{4} - \frac{\Delta\omega}{2}\cos 2\theta_v}$$

$$= -\omega\sqrt{\left(\frac{\hat{\Delta}^2 + 1}{4} - \frac{\hat{\Delta}}{2}\cos 2\theta_v\right)},$$

$$\omega_{m2} = -\omega_{m1}.$$

Hamiltonian applied to this state results in

$$\mathbf{H_m}|\Psi_m\rangle = \omega_{m1}C_L(x)e^{-i\int \omega_{m1}dx}|\nu_L\rangle - ivC_L(x)e^{-i\int \omega_{m1}dx}|\nu_H\rangle$$
$$+ \omega_{m2}C_H(x)e^{-i\int \omega_{m2}dx}|\nu_H\rangle + ivC_H(x)e^{-i\int \omega_{m2}dx}|\nu_L\rangle.$$

Plug the state $|\Psi_m\rangle$ into the SchrÃűdinger equation, we have

$$\dot{C}_L(x) = vC_H(x)e^{-i\int \delta\omega_m dx}$$
$$\dot{C}_H(x) = -vC_L(x)e^{i\int \delta\omega_m dx},$$

in which $\delta\omega_m$ is defined as

$$\delta\omega_m = \omega_{m2} - \omega_{m1}.$$

The first order differential equations of $C_L(x)$ and $C_H(x)$ can be combined and produce a second order differential equation.

$$\ddot{C}_L - \left(\frac{\dot{v}}{v} - i\delta\omega_m\right)\dot{C}_L + v^2 C_L = 0.$$

In our linear approximation that $\frac{d\hat{\Delta}}{dx}$ is a constant.[22] The equation simplifies to

$$\ddot{C}_L + i\delta\omega_m\dot{C}_L + v^2 C_L = 0,$$

where $v = -\sin\theta_v\cos\theta_v\frac{d\hat{\Delta}}{dx}$ is constant.

Assuming one of the solutions is e^{rx}, we could insert it to the second order differential equation and get the characteristic equation,

$$r^2 + i\delta\omega_m r + v^2 = 0,$$

which has two solutions

[22] We are assuming that $n(x)$ is linearly depending on x which means $\hat{\Delta}$ is a linear function of x. Thus $\frac{d\hat{\Delta}}{dx}$ is a constant.

$$r_1 = i\frac{-\delta\omega_m + \sqrt{\delta\omega_m + 4v^2}}{2}$$

$$r_2 = i\frac{-\delta\omega_m - \sqrt{\delta\omega_m + 4v^2}}{2}.$$

The solutions to $C_L(x)$ and $C_H(x)$ will be

$$C_L(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$C_H(x) = (c_1 e^{r_1 x} + c_2 e^{r_2 x})\frac{e^{i\int \delta\omega_m dx}}{v}.$$

These constants are determined by the initial condition. The state of the system at any x is

$$|\Psi_m(x)\rangle = (c_1 e^{r_1 x} + c_2 e^{r_2 x})\, e^{-i\int_0^x \omega_{m1}(x')dx'}\, |\nu_L\rangle$$

$$+ (c_1 e^{r_1 x} + c_2 e^{r_2 x})\frac{e^{i\int \delta\omega_m dx}}{v} e^{-i\int_0^x \omega_{m2}(x')dx'}\, |\nu_H\rangle.$$

Suppose we have the initial condition as $|\Psi_m(x=0)\rangle = |\nu_L\rangle$, the system can jump to $|\nu_H\rangle$ since the state at arbitrary position x is a mixing of the two states.[23] The rate of jumping is given by[24]

$$\Gamma = \frac{(2v)^2}{4\left(\frac{d\omega_m}{dx}\right)} = \frac{v^2}{\frac{d\omega_m}{dx}}.$$

The probability jumping to state $|\nu_H\rangle$ after the level crossing (at infinite x) is

$$P(x \to \infty, |\nu_L\rangle \to |\nu_H\rangle) = e^{-2\pi\Gamma}$$

$$= \exp\left(-2\pi\frac{v^2}{\frac{d\omega_m}{dx}}\right)$$

$$= \exp\left(-2\pi\frac{\left(-\sin\theta_v\cos\theta_v\frac{d\hat\Delta}{dx}\right)^2}{\frac{d\omega_m}{dx}}\right).$$

The survival probability can be calculated by counting the probability left on the initial state.

To be clear, if electron neutrinos are produced inside core of our sun, it will be the light state. Since the interaction with matter is very strong, it transfers to $|\nu_L\rangle$ with probability $P(x \to \infty, |\nu_L\rangle \to |\nu_H\rangle)$ due to the gradient of the matter profile which works as the perturbation. Thus the final state will be a mixing of $|\nu_L\rangle$ and $|\nu_H\rangle$.

[23] In fact this approximation is valid at the level crossing.

[24] Amar C Vutha. A simple approach to the Landau-Zener formula. 389: 4, 2010. ISSN 0143-0807. DOI: 10.1088/0143-0807/31/2/016. URL http://arxiv.org/abs/1001.3322; C. Zener. Non-Adiabatic Crossing of Energy Levels. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 137 (833):696–702, 1932. ISSN 1364-5021. DOI: 10.1098/rspa.1932.0165; and Jan R. Rubbmark, Michael M. Kash, Michael G. Littman, and Daniel Kleppner. Dynamical effects at avoided level crossings: A study of the Landau-Zener effect using Rydberg atoms. *Physical Review A*, 23(6):3107–3117, 1981. ISSN 10502947. DOI: 10.1103/PhysRevA.23.3107

Numerical Method and Results

To simplify the codes, I prefer to write down all quantities dimensionless. In this spirit, the Hamiltonian becomes

$$\mathbf{H_m d} = \begin{pmatrix} \omega_{m1} & 0 \\ 0 & \omega_{m2} \end{pmatrix} + \mathbf{V_m},$$

where

$$\omega_{m2} = -\omega_{m1} \equiv \omega_m = \omega \sqrt{\frac{\hat{\Delta}^2 + 1}{4} - \frac{\hat{\Delta}}{2} \cos 2\theta_v}$$

while the effective potential $\mathbf{V_m}$ is

$$\mathbf{V_m} = -i\mathbf{U_m^{-1}}\partial_x \mathbf{U_m}$$
$$= -i \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix} \partial_x \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ -\sin\theta(x) & \cos\theta(x) \end{pmatrix}$$

in which

$$\sin 2\theta(x) = \frac{\sin 2\theta_v}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}}$$
$$\cos 2\theta(x) = \frac{\cos 2\theta_v - \hat{\Delta}}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}}.$$

and

$$\hat{\Delta} = \frac{\sqrt{2}G_F n(x)}{\omega}.$$

To compare with the theory, the first order effective potential is

$$\mathbf{V_m} = -i\sin\theta_v \cos\theta_v \frac{d\hat{\Delta}}{dx} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= -i\sin\theta_v \cos\theta_v \sqrt{2}\frac{G_F}{\omega} \frac{dn(\hat{x})}{d\hat{x}} \frac{1}{R_S} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now we could write down the dimensionless equation

$$i\partial_{\hat{x}} \Psi(x) = (R_S \mathbf{H_m} + R_S \mathbf{V_m})\Psi(x),$$

which becomes a matrix equation given a set of basis

$$i\partial_{\hat{x}} \begin{pmatrix} v_L \\ v_H \end{pmatrix} = \left(R_S\omega \begin{pmatrix} -\hat{\omega}_m & 0 \\ 0 & \hat{\omega}_m \end{pmatrix} - i\sin\theta_v \cos\theta_v \sqrt{2}\frac{G_F}{\omega} \frac{dn(\hat{x})}{d\hat{x}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \begin{pmatrix} v_L \\ v_H \end{pmatrix}$$

The solar electron number density from standard solar model is given in arXiv:astro-ph/001034.[25]

The data file[26] is listed in units of R_S and cm^{-3} while the density is actually $\log(n(x)/N_A)$ where N_A is Avogadro number.

Before any numerical calculations, I would like to do a simple analysis of the equation. First thing to do is to see if the second term in Hamiltonian is too small.

The radius of the sun is about $R_S = 695800$km which can be approximated as[27]

$$R_S = 7 \times 10^{23}\text{fm} = \frac{7 \times 10^{23}}{0.2\text{GeV}} = 3.5 \times 10^{24}\text{GeV}^{-1}.$$

The frequency $\omega = \Delta m^2/2E$ is approximately

$$\omega = \frac{10^{-4}\text{eV}^2}{10^{-3}\text{GeV}} = 10^{-19}\text{GeV}.$$

Thus

$$R_S\omega \approx 10^5$$

Notice that the dimensionless frequency ω_m is

$$\hat{\omega}_m = \sqrt{\frac{\hat{\Delta}^2 + 1}{4} - \frac{\hat{\Delta}}{2}\cos 2\theta_v}.$$

In the situation of the sun, where $\hat{\Delta} \ll 1$[28], we keep the first order of this quanitity,

$$\hat{\omega}_m = \sqrt{\frac{\hat{\Delta}^2 + 1}{4} - \frac{\hat{\Delta}}{2}\cos 2\theta_v}$$
$$= \frac{1}{2} - \frac{1}{2}\cos 2\theta_v\hat{\Delta} + \mathcal{O}(\hat{\Delta})^3.$$

The derivative of number density of the sun $\frac{dn(\hat{x})}{d\hat{x}}$ is shown to be of the order -10^{26}cm^{-3}[29], i.e.,

$$\frac{dn(\hat{x})}{d\hat{x}} \approx -10^{26}\text{cm}^{-3} \approx -10^{-23}\text{GeV}^3.$$

We also have

$$\frac{G_F}{\omega} \approx \frac{10^{-5}\text{GeV}^{-2}}{10^{-19}\text{GeV}} = 10^{14}\text{GeV}^{-3}$$

The second term is of order[30]

$$\frac{G_F}{\omega}\frac{dn(\hat{x})}{d\hat{x}} \approx -10^{14}\text{GeV}^{-3} \times 10^{-23}\text{GeV}^3 = 10^{-9}.$$

[25] Data from this work is listed http://www.sns.ias.edu/ jnb/S-Ndata/sndata.html while the data file is actually in this file here http://www.sns.ias.edu/ jnb/SNdata/-Export/BP2000/neordered.output .

[26] http://www.sns.ias.edu/ jn-b/SNdata/Ex-port/BP2000/neordered.output

[27] The conversion between SI and natrual untis is done by the relation 1fm = 1/0.2GeV.

[28] Put in the numbers,
$$\hat{\Delta} \approx 10^{-19} \sim 10^{-11}$$

[29] 1cm^{-3} = 8 × 10^{-51}GeV^3

[30] $G_F = 1.17 \times 10^{-5}$GeV^{-2}

Decapitated - Solar Density Profile

The solar density profile can be determined using simple gravitational theory and nuclear physics models.

The mass contained in a spherical shell dr is

$$dm = 4\pi r^2 \rho dr, \tag{37}$$

which is assuming spherical symmetry.

The gravitational potential for each unit of mass can generate force as gravitational attraction, which is

$$\frac{dv}{dr} = \frac{Gm}{r^2}.$$

The balance between pressure and gravity shows that

$$\frac{dv}{dr}\rho + \frac{dp}{dr} = 0.$$

The result of these relations is a rather intuitive relation between density and pressure, which is not solvable until we find another relation between them.

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho.$$

To be simple, we will use Eddington model to solve the density profile, which gives us the relation

$$p = K\rho^{4/3},$$

where

$$K = \left(\left(\frac{k}{m_p}\frac{3}{\alpha}\frac{\beta}{\mu^4(1-\beta)^4}\right)^4\right)^{1/3}.$$

Combining the two relations between density and pressure, we can find the density profile with respect to radius.

Ternary Diagrams

References

Jan R. Rubbmark, Michael M. Kash, Michael G. Littman, and Daniel Kleppner. Dynamical effects at avoided level crossings: A study of the Landau-Zener effect using Rydberg atoms. *Physical Review A*, 23(6):3107–3117, 1981. ISSN 10502947. DOI: 10.1103/PhysRevA.23.3107.
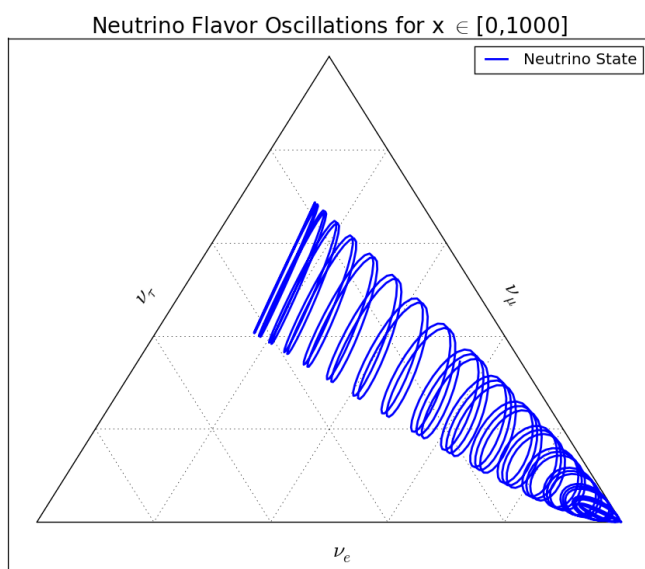
Neutrino Flavor Oscillations for x $\in$ [0,1000]

Figure 6: theta12 = 33.36/180*Pi;
  theta13 = 8.66/180*Pi;
  theta23 = 40/180*Pi;
  deltacp = 0;
  m1sq = 0.01;
  m2sq = m1sq + 0.000079;
  oneFourE = 100;

Neutrino Flavor Oscillations for x $\in$ [0,1000]

Figure 7: Change theta12 to half
  theta12 = 33.36/180*Pi*1/2;
  theta13 = 8.66/180*Pi;
  theta23 = 40/180*Pi;
  deltacp = 0;
  m1sq = 0.01;
  m2sq = m1sq + 0.000079;
  oneFourE = 100;
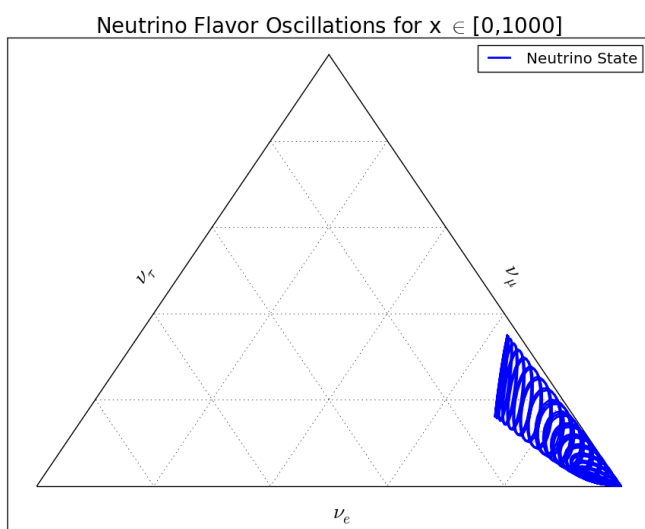
Neutrino Flavor Oscillations for x $\in$ [0,1000]

Figure 8: Change theta13 to half
    theta12 = 33.36/180*Pi;
    theta13 = 8.66/180*Pi*1/2;
    theta23 = 40/180*Pi;
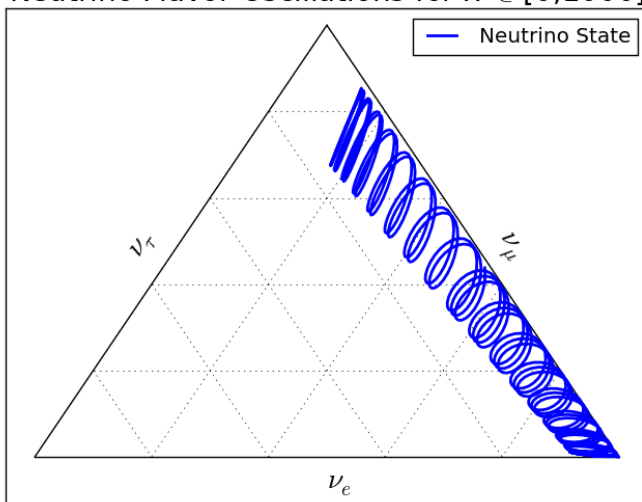   deltacp = 0;
    m1sq = 0.01;
    m2sq = m1sq + 0.000079;
    oneFourE = 100;

Neutrino Flavor Oscillations for x $\in$ [0,1000]

Figure 9: Change theta23 to half
    theta12 = 33.36/180*Pi;
    theta13 = 8.66/180*Pi;
    theta23 = 40/180*Pi*1/2;
    deltacp = 0;
    m1sq = 0.01;
    m2sq = m1sq + 0.000079;
    oneFourE = 100;

Neutrino Flavor Oscillations for x $\in$ [0,1000]
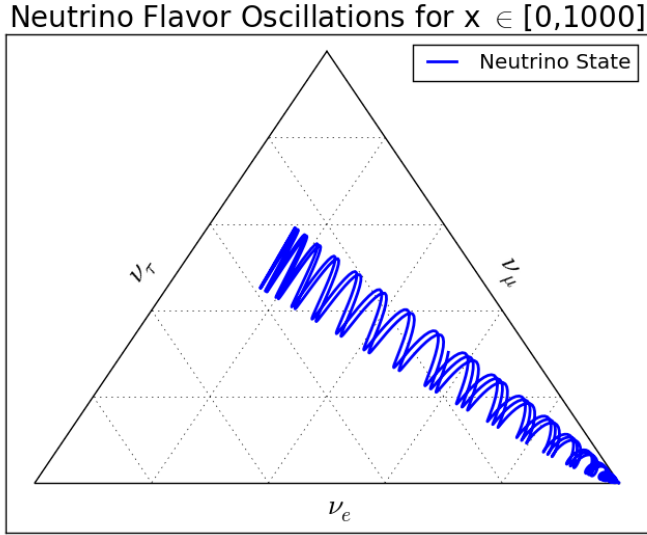
Figure 10: Increase $deltam2^2 - deltam1^2$
  theta12 = 33.36/180*Pi;
  theta13 = 8.66/180*Pi;
  theta23 = 40/180*Pi;
  deltacp = 0;
  m1sq = 0.01;
  m2sq = m1sq + 0.000079*10;
  oneFourE = 100;

Neutrino Flavor Oscillations for x $\in$ [0,1000]
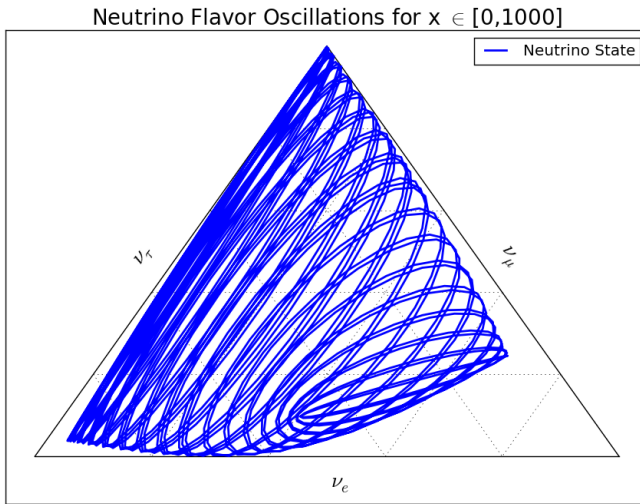
Figure 11: Starting from all ν_μ
  theta12 = 33.36/180*Pi;
  theta13 = 8.66/180*Pi;
  theta23 = 40/180*Pi;
  deltacp = 0;
  m1sq = 0.01;
  m2sq = m1sq + 0.000079;
  oneFourE = 100;

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C. Zener. Non-Adiabatic Crossing of Energy Levels. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 137(833):696–702, 1932. ISSN 1364-5021. DOI: 10.1098/rspa.1932.0165.