

Neutrino Oscillations in Vacuum and Matter ¹

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Notes for neutrino oscillations in vacuum and dense matter.

Vacuum Oscillations

Schrodinger equation is

$$i\partial_t |\Psi\rangle = \mathbf{H} |\Psi\rangle, \quad (1)$$

where for relativistic neutrinos, the energy is²

$$\mathbf{H}^m = \begin{pmatrix} \sqrt{p^2 + m_1^2} & 0 & 0 \\ 0 & \sqrt{p^2 + m_2^2} & 0 \\ 0 & 0 & \sqrt{p^2 + m_3^2} \end{pmatrix},$$

in which the energy terms are simplified using the relativistic condition

$$\sqrt{p^2 + m_i^2} = p \sqrt{1 + \frac{m_i^2}{p^2}} \quad (2)$$

$$\approx p \left(1 + \frac{1}{2} \frac{m_i^2}{p^2}\right). \quad (3)$$

In general the flavor eigenstates are the mixing of the mass eigenstates with a unitary matrix \mathbf{U} , that is

$$|v_\alpha\rangle = U_{\alpha i} |v_i\rangle, \quad (4)$$

where the α s are indices for flavor states while the i s are indices for mass eigenstates.

To find out the equation of motion for flavor states, plugin in the unitary transformation,

$$iU_{\alpha i}\partial_t |v_i\rangle = U_{\alpha i}H_{ij}^m |v_j\rangle. \quad (5)$$

I use index m for representation of Hamiltonian in mass eigenstates. Applying the unitary condition of the transformation,

$$\mathbf{I} = \mathbf{U}^\dagger \mathbf{U}, \quad (6)$$

I get

$$iU_{\alpha i}\partial_t |v_i\rangle = U_{\alpha i}H_{ij}^m U_{j\beta}^\dagger U_{\beta k} |v_k\rangle, \quad (7)$$

² They all have the same momentum but different mass. The thing is we assume they have the same velocity since the mass is very small. To have an idea of the velocity difference, I can calculate the distance travelled by another neutrino in the frame of one neutrino.

To Be Discussed!

Will decoherence happen due to this?

which is simplified to

$$i\partial_t |\nu_\alpha\rangle = H_{\alpha\beta}^f |\nu_\beta\rangle, \quad (8)$$

since the transformation is time independent.

The new Hamiltonian in the representations of flavor eigenstates reads

$$H_{\alpha\beta}^f = U_{\alpha i}^\dagger H_{ij}^m U_{j\beta}. \quad (9)$$

Survival Probability

The neutrino states at any time can be written as

$$|\Psi(t)\rangle = X_1 |\nu_1\rangle e^{-iE_1 t} + X_2 |\nu_2\rangle e^{-iE_2 t}, \quad (10)$$

where X_1 and X_2 are the initial conditions which are determined using the neutrino initial states.

Survival probability is the square of the projection on an flavor eigenstate,

$$P_\alpha(t) = |\langle \nu_\alpha | \Psi(t) \rangle|^2. \quad (11)$$

The calculation of this expression requires our knowledge of the relation between mass eigenstates and flavor eigenstates which we have already found out.

Recall that the transformation between flavor and mass states is

$$|\nu_i\rangle = U_{i\alpha}^{-1} |\nu_\alpha\rangle, \quad (12)$$

which leads to the inner product of mass eigenstates and flavor eigenstates,

$$\langle \nu_\alpha | \nu_i \rangle = \langle \nu_\alpha | U_{i\beta}^{-1} |\nu_\beta\rangle \quad (13)$$

$$= U_{i\beta}^{-1} \delta_{\alpha\beta} \quad (14)$$

$$U = U_{i\alpha}^{-1}. \quad (15)$$

The survival probability becomes

$$\begin{aligned} P_\alpha(t) &= |\langle \nu_\alpha | X_1 |\nu_1\rangle e^{-iE_1 t} + X_2 |\nu_2\rangle e^{-iE_2 t} \rangle|^2 \\ &= |X_1 e^{-iE_1 t} \langle \nu_\alpha | \nu_1 \rangle + X_2 e^{-iE_2 t} \langle \nu_\alpha | \nu_2 \rangle|^2 \\ &= \left| \sum_i X_i e^{-iE_i t} U_{i\alpha}^{-1} \right|^2 \\ &= \sum_i X_i^* e^{iE_i t} U_{i\alpha}^{\dagger*} \sum_j X_j e^{-iE_j t} U_{j\alpha}^\dagger \\ &= |X_1|^2 U_{1\alpha}^{\dagger*} U_{1\alpha}^\dagger + |X_2|^2 U_{2\alpha}^{\dagger*} U_{2\alpha}^\dagger + X_1^* X_2 U_{1\alpha}^{\dagger*} U_{2\alpha}^\dagger e^{iE_1 t - iE_2 t} + X_2^* X_1 U_{2\alpha}^{\dagger*} U_{1\alpha}^\dagger e^{iE_2 t - iE_1 t} \end{aligned}$$

$U_{i\alpha}^{\dagger*}$ stands for the i th row and the α th column of the matrix $U^{\dagger*}$.

Two Flavor States

For 2 flavor neutrinos the Hamiltonian in the representation of propagation states,

$$\mathbf{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \begin{pmatrix} p_1 + \frac{1}{2} \frac{m_1^2}{p_1} & 0 \\ 0 & p_2 + \frac{1}{2} \frac{m_1^2}{p_2} \end{pmatrix}.$$

The equation of motion in matrix form is

$$i\partial_t \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} p_1 + \frac{1}{2} \frac{m_1^2}{p_1} & 0 \\ 0 & p_2 + \frac{1}{2} \frac{m_1^2}{p_2} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \quad (16)$$

The flavor eigenstate is a mixing of the propagation eigenstates,

$$\begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos \theta_v & \sin \theta_v \\ -\sin \theta_v & \cos \theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \quad (17)$$

Denote the rotation matrix using \mathbf{U} , the transformation can be written as

$$|\nu_\alpha\rangle = \mathbf{U}_{\alpha i} |\nu_i\rangle, \quad (18)$$

where α is for the flavor eigenstates and i is for the mass eigenstates.

The survival probability has been derived in previous section, which is the projection of propagation states onto flavor states.

For arbitrary initial condition,

$$\Psi(0) = A |\nu_a\rangle + B |\nu_b\rangle, \quad (19)$$

which can be rewritten into a matrix form,

$$\Psi(t=0)_i = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} |\nu_a\rangle \\ |\nu_b\rangle \end{pmatrix} \quad (20)$$

To write down the projection, the relation

$$\begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos \theta_v & \sin \theta_v \\ -\sin \theta_v & \cos \theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \quad (21)$$

is needed. The inverse transformation is

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \quad (22)$$

Three Flavor States

For three flavor neutrinos, the oscillations matrix is 3 by 3. To find it, I can do more about something.

Oscillations in Dense Medium