Neutrino Oscillations in Vacuum and Matter [1]

[1] 2015 Summer

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Notes for neutrino oscillations in vacuum and dense matter.

Contents

Symbols and Definitions

- $\Delta = \sqrt{2} G_F n(x)$
- $\omega = \frac{\Delta m^2}{2E}$

Interesting Topics

Decoherence

Not really decoherence but in the view of wave packet formalism different propagation eigenstates will be far away from each other.

Vacuum Oscillations 3 Flavor

The trick to solve the equations is to subtract the $\frac{m_1^2}{2E}\mathbf{I}$ from Hamiltonian in mass eigenbasis.

$$\begin{aligned}
&\mathbf{H} - \frac{m_1^2}{2E}\mathbf{I} \\
=&\frac{1}{2E}\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} - \frac{m_1^2}{2E}\mathbf{I} \\
=&\frac{1}{2E}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{12}^2 & 0 \\ 0 & 0 & \Delta m_{13}^2 \end{pmatrix}
\end{aligned}$$

For any kind of potential, do this first then add in the potentials.

Normal hierarchy and inverted hierarchy just changes the behavior of small scale oscillations. See caption of figure 2 for details.

Invert hierarchy of m_1 and m_2 also doesn't change anything on average sense.

Matter

Oscillation length in calculations, equation 37 and 43.

Then a discussion of the energy dependence of the vacuum oscillation length is shown in figures 7 and 8.

Matter effect can be a suppression or enhancement. See equation 38 and the discussion below.

A general discussion of solving the problem is located at equation 40. There are three different ways to solve it however the one begins with vacuum mass eigenbasis Hamiltonian is the best.

To solve the solar problem, there are three stages.

- Region with very high number density which is above resonance: $|\nu_e\rangle$ projects onto instantaneous mass eigenstates then propagate adiabatically on these states.
- Region around resonance: transition from a instantaneous mass eigenstate to another happens.

- After the transition: stays on instantaneous mass eigenstates.

A general discussion is presented in subsection *General Discussion for Neutrinos Interacting with Matter*.

Ternary Diagrams

Hierarchy of Δm_{13} changes the small oscillations, which can be seen clearly in ternary diagrams.

Vacuum Oscillations

Schrodinger equation is

$$i\partial_t |\Psi\rangle = \mathbf{H} |\Psi\rangle , \tag{1}$$

where for relativistic neutrinos, the energy is[2]

$$\mathbf{H}^m = \begin{pmatrix} \sqrt{p^2+m_1^2} & 0 & 0 \\ 0 & \sqrt{p^2+m_2^2} & 0 \\ 0 & 0 & \sqrt{p^2+m_3^2} \end{pmatrix},$$

in which the energy terms are simplified using the relativistic condition

$$\sqrt{p^2+m_i^2} = p\sqrt{1+\frac{m_i^2}{p^2}} \tag{2}$$

$$\approx p(1+\frac{1}{2}\frac{m_i^2}{p^2}). \tag{3}$$

In general the flavor eigenstates are the mixing of the mass eigenstates with a unitary matrix $\mathbf{U}$, that is

$$|\nu_\alpha\rangle = U_{\alpha i} |\nu_i\rangle , \tag{4}$$

where the αs are indices for flavor states while the is are indices for mass eigenstates.

To find out the equation of motion for flavor states, plugin in the initary tranformation,

$$iU_{\alpha i}\partial_t |\nu_i\rangle = U_{\alpha i} H_{ij}^m |\nu_j\rangle . \tag{5}$$

I use index ^{m} for representation of Hamiltonian in mass eigenstates. Applying the unitary condition of the transformation,

$$\mathbf{I} = \mathbf{U}^\dagger\mathbf{U}, \tag{6}$$

[2] They all have the same momentum but different mass. The thing is we assume they have the same velocity since the mass is very small. To have an idea of the velocity difference, I can calculate the distance travelled by another neutrino in the frame of one neutrino.

Assuming the mass of a neutrino is 1eV with energy 10MeV, we will get a speed of $1-10^{-14}$c. This 10^{-14}c will make a difference about 3μm in 1s.

To Be Discussed!

Will decoherence happen due to this? For high energy neutrinos this won't be a problem however for low energy neutrinos this will definitely cause a problem for the wave function approach.Because the different mass eigenstates will become decoherent gradually along the path.

A estimation of the decoherence length is

$$l_{\text{coh}} = \frac{v_g}{\Delta v_g}\sigma.$$

To obtain the relation,

$$\Delta x = |v_1 - v_2| t_{\text{coh}}$$
$$\frac{\hbar c}{\Delta E} = |\frac{m_1^2}{2E_1^2} - \frac{m_2^2}{2E_2^2}| t_{\text{coh}}$$
$$\frac{\hbar c}{\Delta E} = \frac{1}{2E}|\Delta m_{12}^2| t_{\text{coh}}$$

I get

$$iU_{\alpha i}\partial_t |\nu_i\rangle = U_{\alpha i} H^m_{ij} U^\dagger_{j\beta} U_{\beta k} |\nu_k\rangle , \tag{7}$$

which is simplified to

$$i\partial_t |\nu_\alpha\rangle = H^f_{\alpha\beta} |\nu_\beta\rangle , \tag{8}$$

since the transformation is time independent.

The new Hamiltonian in the representations of flavor eigenstates reads

$$H^f_{\alpha\beta} = U^\dagger_{\alpha i} H^m_{ij} U_{j\beta}. \tag{9}$$

Survival Probability

The neutrino states at any time can be written as

$$|\Psi(t)\rangle = X_1 |\nu_1\rangle e^{-iE_1 t} + X_2 |\nu_2\rangle e^{-iE_2 t}, \tag{10}$$

where X_1 and X_2 are the initial conditions which are determined using the neutrino initial states.

Survival probalility is the squrare of the projection on an flavor eigenstate,

$$P_\alpha(t) = |\langle \nu_\alpha \mid \Psi(t)\rangle|^2. \tag{11}$$

The calculation of this expression requires our knowledge of the relation between mass eigenstates and flavor eigenstates which we have already found out.

Recall that the transformation between flavor and mass states is

$$|\nu_i\rangle = U^{-1}_{i\alpha} |\nu_\alpha\rangle , \tag{12}$$

which leads to the inner product of mass eigenstates and flavor eigenstates,

$$\begin{aligned} \langle \nu_\alpha \mid \nu_i\rangle &= \langle \nu_\alpha | \, U^{-1}_{i\beta} |\nu_\beta\rangle && (13)\\ &= U^{-1}_{i\beta}\delta_{\alpha\beta} && (14)\\ &= U^{-1}_{i\alpha}. && (15) \end{aligned}$$

The survival probability becomes

$$
\begin{aligned}
P_\alpha(t) &= |\langle \nu_\alpha \mid X_1 |\nu_1\rangle e^{-iE_1 t} X_2 |\nu_2\rangle e^{-iE_2 t}\rangle|^2 \\
&= |X_1 e^{-iE_1 t}\langle \nu_\alpha \mid |\nu_1\rangle\rangle + X_2 e^{-iE_2 t}\langle \nu_\alpha \mid \nu_2\rangle|^2 \\
&= |\sum_i X_i e^{-iE_i t} U^{-1}_{i\alpha}|^2 \\
&= \sum_i X_1^* e^{iE_i t} U^{\dagger *}_{i\alpha} \sum_i X_i e^{-iE_i t} U^\dagger_{i\alpha} \\
&= |X_1|^2 U^{\dagger *}_{1\alpha} U^\dagger_{1\alpha} + |X_2|^2 U^{\dagger *}_{2\alpha} U^\dagger_{2\alpha} + X_1^* X_2 U^{\dagger *}_{1\alpha} U^\dagger_{2\alpha} e^{iE_1 t - iE_2 t} + X_2^* X_1 U^{\dagger *}_{2\alpha} U^\dagger_{1\alpha} e^{iE_2 t - iE_1 t}
\end{aligned}
$$

$U^{\dagger *}_{i\alpha}$ stands for the ith row and the αth column of the matrix $U^{\dagger *}$.

Two Flavor States

For 2 flavor neutrinos the Hamiltonian in the representation of propagation states,

$$
\mathbf{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \begin{pmatrix} p_1 + \frac{1}{2}\frac{m_1^2}{p_1} & 0 \\ 0 & p_2 + \frac{1}{2}\frac{m_1^2}{p_2} \end{pmatrix}.
$$

The equation of motion in matrix form is

$$
i\partial_t \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} p_1 + \frac{1}{2}\frac{m_1^2}{p_1} & 0 \\ 0 & p_2 + \frac{1}{2}\frac{m_1^2}{p_2} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{16}
$$

The flavor eigenstate is a mixing of the propagation eigenstates,

$$
\begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{17}
$$

Denote the rotation matrix using $\mathbf{U}$, the transofmation can be written as

$$
|\nu_\alpha\rangle = \mathbf{U}_{\alpha i} |\nu_i\rangle, \tag{18}
$$

where α is for the flavor eigenstates and i is for the mass eigenstates.

The survival probability has been derived in previous section, which is the projection of propagation states onto flavor states.

For arbitary initial condition,

$$
\Psi(t=0) = A|\nu_a\rangle + B|\nu_b\rangle, \tag{19}
$$

which can be rewritten into a matrix form,

$$
\Psi(t=0) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} \tag{20}
$$

To write down the projection, the relation

$$\begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{21}$$

is needed. BTW, the inverse transformation is the transpose of **U** since **U** is unitary, thus we have the relation,

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_v & -\sin\theta_v \\ \sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} \tag{22}$$

Thus in the state can be written as

$$\Psi(t=0) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \cos\theta_v & \sin\theta_v \\ -\sin\theta_v & \cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \tag{23}$$

At any t, the state is

$$\Psi(t) = \begin{pmatrix} A\cos\theta_v - B\sin\theta_v & A\sin\theta_v + B\cos\theta_v \end{pmatrix} \begin{pmatrix} \nu_1 e^{-iE_1 t} \\ \nu_2 e^{-iE_2 t} \end{pmatrix} \tag{24}$$

$$= \begin{pmatrix} (A\cos\theta_v - B\sin\theta_v)e^{-iE_1 t} & (A\sin\theta_v + B\cos\theta_v)e^{-iE_2 t} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{25}$$

The survival probability which is projection on a flavor state is written as

$$P(\nu_\alpha, t) = |\langle \nu_\alpha \mid \Psi(t)\rangle|^2. \tag{26}$$

The survival amplitude for ν_a is

$$\begin{aligned} &\langle \nu_a \mid \Psi(t)\rangle \\ &= \langle \nu_a| \left((A\cos\theta_v - B\sin\theta_v)e^{-iE_1 t} |\nu_1\rangle + (A\sin\theta_v + B\cos\theta_v)e^{-iE_2 t} |\nu_2\rangle \right) \\ &= (\cos\theta_v \langle \nu_1| + \sin\theta_v \langle \nu_2|) \left((A\cos\theta_v - B\sin\theta_v)e^{-iE_1 t} |\nu_1\rangle + (A\sin\theta_v + B\cos\theta_v)e^{-iE_2 t} |\nu_2\rangle \right) \end{aligned}$$

This is simple since the transformation matrix is real.

Applying the condition that the propagation eigenstates are orthonormal, the survival probability is

$$\begin{aligned} P(\nu_a, t) &= |\langle \nu_a \mid \Psi(t)\rangle|^2 \\ &= |\cos\theta_v (A\cos\theta_v - B\sin\theta_v)e^{-iE_1 t} + \sin\theta_v (A\sin\theta_v + B\cos\theta_v)e^{-iE_2 t}|^2 \\ &= |(A\cos^2\theta_v - B\sin\theta_v\cos\theta_v)e^{-iE_1 t} + (A\sin^2\theta_v + B\sin\theta_v\cos\theta_v)e^{-iE_2 t}|^2 \end{aligned}$$

In a special limit that $E_1 = E_2 = E$, the probability becomes

$$P(\nu_a, t) = |A|^2 \tag{27}$$

which is the same as initial probability since there is no mixing at all.

There are two kinds of initial conditions.

- The neutrinos are all in ν_a state initially, which means $A = 1, B = 0$. The survival probability simplifies to

$$\begin{aligned} P(\nu_a, t) &= |\cos^2\theta_v e^{-iE_1 t} + \sin^2\theta_v e^{-iE_2 t}|^2 \\ &= |\cos^2\theta_v e^{-i(E_1 - E_2)t} + \sin^2\theta_v|^2 \end{aligned}$$

As we have already discussed, $E_1 - E_2 = \frac{m_1^2 - m_2^2}{2p}$ assuming the neutrinos have the same momentum. [3] Using the notation $\Delta m^2 = m_1^2 - m_2^2$ and the approximation that $E \approx p$, the survival probability can be rewritten as

[3] And here is a question.

$$\begin{aligned} P(\nu_a, t) &= \cos^4\theta_v + \sin^4\theta_v + \cos^2\theta_v \sin^2\theta_v \left(e^{-i\Delta m^2 t/E} + e^{i\Delta m^2 t/E}\right) \\ &= 1 - 2\cos^2\theta_v \sin^2\theta_v + 2\cos^2\theta_v \sin^2\theta_v \cos\left(\frac{\Delta m^2 t}{2E}\right) \\ &= 1 - 2\cos^2\theta_v \sin^2\theta_v \left(1 - \cos\left(\frac{\Delta m^2 t}{2E}\right)\right) \\ &= 1 - 4\cos^2\theta_v \sin^2\theta_v \sin^2\left(\frac{\Delta m^2 t}{4E}\right) \\ &= 1 - \sin^2(2\theta_v) \sin^2\left(\frac{\Delta m^2 t}{4E}\right) \end{aligned}$$

We always assuming that in the region of interest, all neutrinos are travelling with the same speed, i.e., the speed of light $c = 1$. [4] Time is related to distance, $L = t$. Survival probability at distance L is

[4] which is not true obviously

$$P(\nu_a, L) = 1 - \sin^2(2\theta_v) \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \tag{28}$$

- The neutrinos are all in ν_b state initially. Equivalently, we have $A = 0, B = 1$. Survival probability is

$$\begin{aligned}
P(\nu_a, t) &= |-\sin\theta_v \cos\theta_v e^{-iE_1 t} + \sin\theta_v \cos\theta_v e^{-iE_2 t}|^2 \\
&= \sin^2\theta_v \cos^2\theta_v |e^{-i(E_1-E_2)t} - 1|^2 \\
&= \sin^2\theta_v \cos^2\theta_v \left(1 + 1 - e^{-i\Delta m^2 t/2E} - e^{i\Delta m^2 t/2E}\right) \\
&= 2\sin^2\theta_v \cos^2\theta_v \left(1 - \cos\left(\frac{\Delta m^2 t}{2E}\right)\right) \\
&= \sin^2(2\theta_v) \sin^2\left(\frac{\Delta m^2 t}{4E}\right) \\
&= \sin^2(2\theta_v) \sin^2\left(\frac{\Delta m^2 L}{4E}\right)
\end{aligned}$$

Flavor Basis

This problem can be solved using density matrix ρ and Von Neumann equation

$$i\partial_t \rho = [H, \rho]. \tag{29}$$

The initial condition for this equation is

$$\begin{aligned}
\rho(t=0) &= (A\,|\nu_a\rangle + B\,|\nu_b\rangle)(A^*\,\langle\nu_a| + B^*\,\langle\nu_b|) \\
&= AA^*\,|\nu_a\rangle\,\langle\nu_a| + BB^*\,|\nu_b\rangle\,\langle\nu_b| + AB^*\,|\nu_a\rangle\,\langle\nu_b| + A^*B\,|\nu_b\rangle\,\langle\nu_a|\,.
\end{aligned}$$

To calculate the propagation of the states, we need the Hamiltonian matrix in flavor basis.

This can be done by finding out how the Hamiltonian matrix transforms from one basis to another.

Using propagation basis,

$$i\partial_t \left|\Psi_p\right\rangle = H_p \left|\Psi_p\right\rangle. \tag{30}$$

The states are $|\Psi\rangle = \mathbf{U}\left|\Psi_p\right\rangle$ in flavor basis, which means we could plug in $\left|\Psi_p\right\rangle = \mathbf{U^T}\,|\Psi\rangle$.

$$i\partial_t \mathbf{U^T}\,|\Psi\rangle = H_p \mathbf{U^T}\,|\Psi\rangle\,.$$

Since $\mathbf{U}\mathbf{U^T} = \mathbf{I}$, we have a clean result by multiplying through the equation by $\mathbf{U}$.

$$i\partial_t\,|\Psi\rangle = \mathbf{U}H_p\mathbf{U^T}\,|\Psi\rangle\,.$$

So we define $H = \mathbf{U}H_p\mathbf{U^T}$ as the Hamiltonian matrix in flavor basis, which is

$$H = \left(p + \frac{m_1^2 + m_2^2}{4p}\right)\mathbf{I} - \frac{1}{4p}\begin{pmatrix} -\Delta m^2 \cos 2\theta & \Delta^2 m \sin 2\theta \\ \Delta m^2 \sin 2\theta & \Delta^2 m \cos 2\theta \end{pmatrix}. \tag{31}$$

The derivation of this is

$$
\begin{aligned}
\mathbf{H}_\alpha &= \mathbf{U}\hat{H}_j\mathbf{U}^T \\
&= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \left(p\mathbf{I} + \frac{1}{2p}\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \right) \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
&= p\mathbf{I} + \frac{1}{2p}\begin{pmatrix} \cos^2\theta m_1^2 + \sin^2\theta m_2^2 & -\sin\theta\cos\theta m_1^2 + \sin\theta\cos\theta m_2^2 \\ -\sin\theta\cos\theta m_1^2 + \sin\theta\cos\theta m_2^2 & \sin^2\theta m_1^2 + \cos^2\theta m_2^2 \end{pmatrix} \\
&= p\mathbf{I} + \frac{1}{2p}\begin{pmatrix} m_1^2 - \Delta m^2\sin^2\theta & -\frac{1}{2}\sin 2\theta\Delta m^2 \\ -\frac{1}{2}\sin 2\theta\Delta m^2 & m_2^2 + \Delta m^2\sin^2\theta \end{pmatrix} \\
&= p\mathbf{I} + \frac{1}{2p}\left(\frac{1}{2}(m_1^2 + m_2^2)\mathbf{I} - \frac{1}{2}\begin{pmatrix} -\Delta m^2\cos 2\theta & \Delta^2 m\sin 2\theta \\ \Delta m^2\sin 2\theta & \Delta^2 m\cos 2\theta \end{pmatrix} \right) \\
&= \left(p + \frac{m_1^2 + m_2^2}{4p} \right)\mathbf{I} - \frac{1}{4p}\begin{pmatrix} -\Delta m^2\cos 2\theta & \Delta^2 m\sin 2\theta \\ \Delta m^2\sin 2\theta & \Delta^2 m\cos 2\theta \end{pmatrix}
\end{aligned}
$$

where $\Delta m^2 = m_1^2 - m_2^2$.

Since identity matrix only shifts the eigenvalues we are only interested in the second term, thus the Hamiltonian we are going to use is

$$H = \frac{\Delta m^2}{4E}\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}. \tag{32}$$

The equation of motion becomes

$$i\partial_t \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \frac{\Delta m^2}{4E}\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \tag{33}$$

To solve this we need the eigenvalues and eigenvectors of the Hamiltonian matrix.

An Example of Survival Probability

Suppose the neutrinos are prepared in electron flavor initially, the survival probability of electron flavor neutrinos is calculated using the result I get previously.

Electron neutrinos are the lighter ones, then I have $_a = {}_e$ and denote $_b = {}_x$. [5]

In fact the dynamics of the system is very easily solved without dive into the math. Suppose we have $|\nu_e\rangle$ initially, which is

$$\Psi(x=0) = |\nu_e\rangle = \cos\theta_v\,|\nu_1\rangle - \sin\theta_v\,|\nu_2\rangle\,,$$

the state of the system at distance x is directly written down

[5] In the small mixing angle limit,

$$\begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix} \to \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

which is very close to an identity matrix. This implies that electron neutrino is more like mass eigenstate ν_1 . By ν_1 we mean the state with energy $\frac{\delta m^2}{4E}$ in vacuum.

$$\begin{aligned}\Psi(x) &= \cos\theta_v \left|\nu_1\right\rangle e^{-iE_1 x} - \sin\theta_v \left|\nu_2\right\rangle e^{-iE_2 x} \\ &= e^{-iE_1 x}(\cos\theta_v \left|\nu_1\right\rangle - \sin\theta_v \left|\nu_2\right\rangle e^{i(E_1-E_2)x}).\end{aligned}$$

Since a global phase doesn't change the detection, we write the state as

$$\Psi(x) = \cos\theta_v \left|\nu_1\right\rangle - \sin\theta_v \left|\nu_2\right\rangle e^{i(E_1-E_2)x}.$$

Notice that the period of the expression is

$$l_v = \frac{2\pi}{E_1 - E_2} = -\frac{4\pi E}{\Delta m_{12}}.$$

Then the state becomes

$$\Psi(x) = \cos\theta_v \left|\nu_1\right\rangle - \sin\theta_v \left|\nu_2\right\rangle e^{i2\pi x/l_v}.$$

The survival probability for electron neutrinos is

$$\begin{aligned}P(\nu_e, L) &= 1 - \sin^2(2\theta_v)\sin^2\left(\frac{\Delta m^2 L}{4E}\right) \\ &= 1 - \frac{1}{2}\sin^2 2\theta_v \left(1 - \cos\left(\frac{2\pi x}{l_v}\right)\right)\end{aligned}$$

Another Method for 2 Flavor Oscillations

To solve a set of first order differential equations, I need the determinant of coefficient matrix. For 2 flavor neutrino oscillations,

$$\partial_t \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = -i\frac{\Delta m^2}{4E} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}.$$

To find the solutions I need the eigenvalues λ. I need to find the determinant

$$\begin{aligned}&\det\left(-i\frac{\Delta m^2}{4E}\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix} - \lambda \mathbf{I}\right) \\ &= \begin{vmatrix} -i\frac{\Delta m^2}{4E}\cos 2\theta - \lambda & i\frac{\Delta m^2}{4E}\sin 2\theta \\ i\frac{\Delta m^2}{4E}\sin 2\theta & i\frac{\Delta m^2}{4E}\cos 2\theta - \lambda \end{vmatrix}.\end{aligned}$$

By defining $\lambda' = \lambda/(-i\Delta m^2/4E)$, the determinant is

$$-\left(\frac{\Delta m^2}{4E}\right)^2 ((\cos 2\theta - \lambda')(-\cos 2\theta - \lambda') - \sin 2\theta \sin 2\theta).$$

The eigenvalues are the solutions to

$$-\left(\frac{\Delta m^2}{4E}\right)^2((\cos 2\theta-\lambda')(-\cos 2\theta-\lambda')-\sin^2 2\theta\sin 2\theta)=0,$$

whose solution is

$$\lambda'=\pm 1.$$

With the solutions

$$\lambda=\pm i\frac{\Delta m^2}{4E},$$

the eigenvectors can also be solved.

$$\begin{pmatrix}\cos 2\theta-1 & -\sin 2\theta\\ -\sin 2\theta & -\cos 2\theta-1\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

gives us $\eta_2=-\tan\theta\eta_1$, which means the eigenvectors are

$$\begin{pmatrix}1\\ -\tan\theta\end{pmatrix},\begin{pmatrix}1\\ \cot\theta\end{pmatrix}. \tag{34}$$

The general solution of the first order differential equations is

$$\begin{pmatrix}1\\ -\tan\theta\end{pmatrix}e^{-i\Delta m^2 t/4E}$$
$$\begin{pmatrix}1\\ \cot\theta\end{pmatrix}e^{i\Delta m^2 t/4E}.$$

Initial condition is

$$\begin{pmatrix}1\\ 0\end{pmatrix},$$

and it determines the final solution

$$\cos^2\theta\begin{pmatrix}1\\ -\tan\theta\end{pmatrix}e^{-i\Delta m^2 t/4E}+\sin^2\theta\begin{pmatrix}1\\ \cot\theta\end{pmatrix}e^{i\Delta m^2 t/4E}$$
$$=\begin{pmatrix}\cos^2\theta\\ -\sin\theta\cos\theta\end{pmatrix}e^{-i\Delta m^2 t/4E}+\begin{pmatrix}\sin^2\theta\\ \sin\theta\cos\theta\end{pmatrix}e^{i\Delta m^2 t/4E}$$

The survival probability of electron neutrino is

$$\begin{aligned}P&=|\cos^2\theta e^{-i\Delta m^2 t/4E}+\sin^2\theta e^{i\Delta m^2 t/4E}|^2\\ &=|\cos^2\theta e^{-i\Delta m^2 t/2E}+\sin^2\theta|^2,\end{aligned}$$

which gets back to the result we had using the previous method.

This problem can also be solved using numerical methods. Here is a comparison between this analytical result and a numerical result.

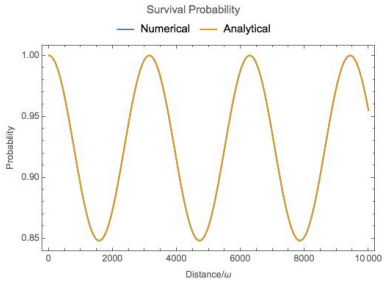


Figure 1: They overlap on all the range completely.

Three Flavor States

For three flavor neutrinos, the oscillations matrix is 3 by 3 which is called the PMNS matrix.

$$\mathbf{U} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix},$$

which acts on the mass eigenstates to give the flavor eigenstates, i.e.,

$$|\nu_\alpha\rangle = \mathbf{U}\,|\nu_i\rangle\,.$$

where

$$\mathbf{U} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}.$$

In general, a rotation for 3D with a CP violation phase δ is

$$\mathbf{U} = \begin{pmatrix} \cos(\theta_{12})\cos(\theta_{13}) & \cos(\theta_{13})\sin(\theta_{12}) & e^{-i\delta_{\text{CP}}}\sin(\theta_{13}) \\ -\cos(\theta_{23})\sin(\theta_{12}) - e^{i\delta_{\text{CP}}}\cos(\theta_{12})\sin(\theta_{13})\sin(\theta_{23}) & \cos(\theta_{12})\cos(\theta_{23}) - e^{i\delta_{\text{CP}}}\sin(\theta_{12})\sin(\theta_{13})\sin(\theta_{23}) & \cos(\theta_{13})\sin(\theta_{23}) \\ \sin(\theta_{12})\sin(\theta_{23}) - e^{i\delta_{\text{CP}}}\cos(\theta_{12})\cos(\theta_{23})\sin(\theta_{13}) & -e^{i\delta_{\text{CP}}}\cos(\theta_{23})\sin(\theta_{12})\sin(\theta_{13}) - \cos(\theta_{12})\sin(\theta_{23}) & \cos(\theta_{13})\cos(\theta_{23}) \end{pmatrix}.$$

However, with the CP violation phase δ this matrix is not unitary thus I'll choose $\delta = 0$ for simplicity.[6]

Now the matrix becomes,

[6] The effect of the phase can also be studied easily with the help of Mathematica.

$$\mathbf{U} = \begin{pmatrix} \cos(\theta_{12})\cos(\theta_{13}) & \cos(\theta_{13})\sin(\theta_{12}) & \sin(\theta_{13}) \\ -\cos(\theta_{23})\sin(\theta_{12}) - \cos(\theta_{12})\sin(\theta_{13})\sin(\theta_{23}) & \cos(\theta_{12})\cos(\theta_{23}) - \sin(\theta_{12})\sin(\theta_{13})\sin(\theta_{23}) & \cos(\theta_{13})\sin(\theta_{23}) \\ \sin(\theta_{12})\sin(\theta_{23}) - \cos(\theta_{12})\cos(\theta_{23})\sin(\theta_{13}) & -\cos(\theta_{23})\sin(\theta_{12})\sin(\theta_{13}) - \cos(\theta_{12})\sin(\theta_{23}) & \cos(\theta_{13})\cos(\theta_{23}) \end{pmatrix}.$$

The survival probability is given by the same derivation as the 2 flavor example. First of all we need to find the Hamiltonian in flavor basis from the propagation Hamiltonian, which is

$$\mathbf{H} = \mathbf{U}\mathbf{H_p}\mathbf{U}^{-1}$$

$$= \begin{pmatrix} \frac{1}{4}\left(\cos^2(\theta_{12})\,m_2^2 + \sin^2(\theta_{12})\,m_2^2\right)\cos^2(\theta_{13}) + \sin^2(\theta_{13})\,m_3^2 & \frac{1}{8}\left(\cos(\theta_{13})\cos(\theta_{23})\sin(2\theta_{12})\left(m_2^2 - m_2^2\right) + \sin(2\theta_{23})\sin(\theta_{13})\left(-\cos^2(\theta_{12})\,m_2^2 - \sin^2(\theta_{12})\,m_2^2 + m_3^2\right)\right) & \frac{1}{8}\left(\cos(\theta_{13})\sin(2\theta_{12})\left(m_2^2 - m_2^2\right) + \cos(\theta_{23})\sin(2\theta_{13})\left(-\cos^2(\theta_{12})\,m_2^2 - \sin^2(\theta_{12})\,m_2^2 + m_3^2\right)\right) \\ \frac{1}{8}\left(\cos(\theta_{13})\cos(\theta_{23})\sin(2\theta_{12})\sin(\theta_{13})\left(m_2^2 - m_2^2\right) + \cos(\theta_{23})\sin(2\theta_{13})\left(-\cos^2(\theta_{12})\,m_2^2 - \sin^2(\theta_{12})\,m_2^2 + m_3^2\right)\right) & \frac{1}{4}\left(4\cos(2\theta_{23})\sin(2\theta_{12})\sin(\theta_{13})\left(m_2^2 - m_2^2\right) - 2\cos(2\theta_{13})\sin(2\theta_{23})\left(\cos^2(\theta_{12})\,m_2^2 + \sin^2(\theta_{12})\,m_2^2 - m_3^2\right) + \sin(2\theta_{23})\left(3\cos(2\theta_{12}) - 1\right)m_2^2 - \left(3\cos(2\theta_{12}) + 1\right)m_2^2 + 2m_3^2\right)\right) & \left(\cos(\theta_{12})\cos(\theta_{23})\sin(\theta_{13}) - \sin(\theta_{12})\sin(\theta_{23})\right)^2 m_2^2 + \left(\cos(\theta_{23})\sin(\theta_{12})\sin(\theta_{13}) + \cos(\theta_{12})\sin(\theta_{23})\right)^2 m_2^2 + \cos^2(\theta_{13})\cos^2(\theta_{23})\,m_3^2 \end{pmatrix}$$

where

$$\mathbf{H_p} = \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}.$$

The next step is to define

$$\Delta m_{12}^2 = m_1^2 - m_2^2$$
$$\Delta m_{23}^2 = m_2^2 - m_3^2$$

so that they simplifies the Hermitian Hamiltonian to

$$\mathbf{H} = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix},$$

where [7]

[7] Haven't being copied from Mathematica code.

$$H_{11} = \cos^2\theta_{13}(m_2^2 - \sin^2\theta\Delta m_{12}^2) + \sin^2\theta_{13}m_3^2$$

Numerical Results for Vacuum 3 Flavor Oscillations

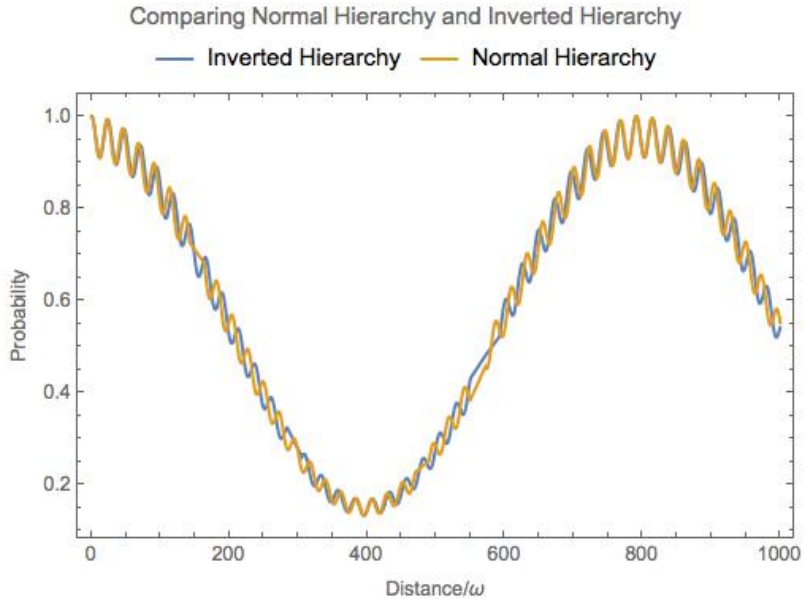


Figure 2: The overall shape is the same however they differ on small scales.

Ternary Diagram for Neutrino Flavor Oscillation

Since the probability for differential flavors of neutrinos are represented in barycentric coordinates[8], a ternary plot would be nice to representation the oscillations. An example is shown in figure 5.

[8] They sum up to 1.

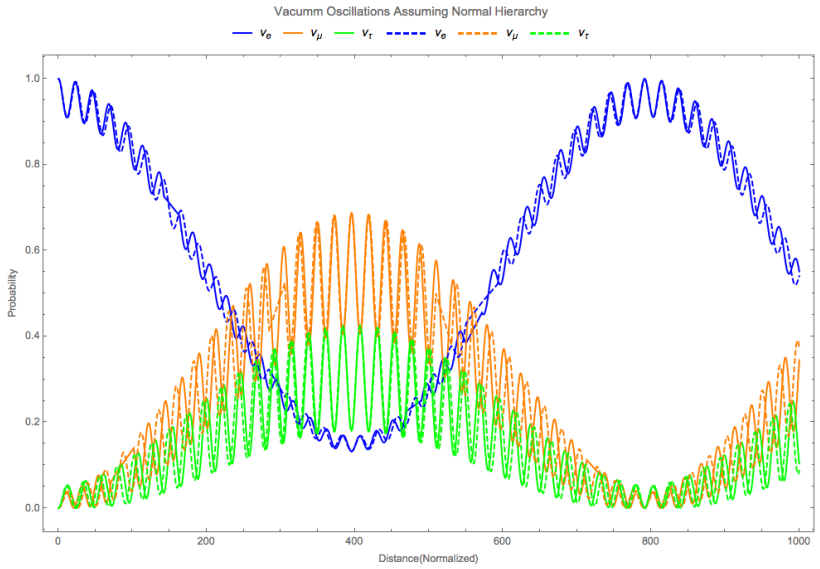


Figure 3: Comparison of normal hierarchy and inverted hierarchy.The reason that they are almost the same is that the oscillation length for Δm^2_{13} is small thus it only changes the oscillation patterns for the small oscillations. Vacuum energy scales in normal hierarchy are

$$\omega_{12} = \frac{\Delta m^2_{12}}{2E} = 3.8 \times 10^{-20}\text{GeV}$$

$$\omega_{13} = \frac{\Delta m^2_{13}}{2E} = 1.7 \times 10^{-18}\text{GeV}$$

$$\omega_{23} = \frac{\Delta m^2_{23}}{2E} \approx \omega_{13}$$

which shows that basically only two scales and the larger one determines the small oscillation.

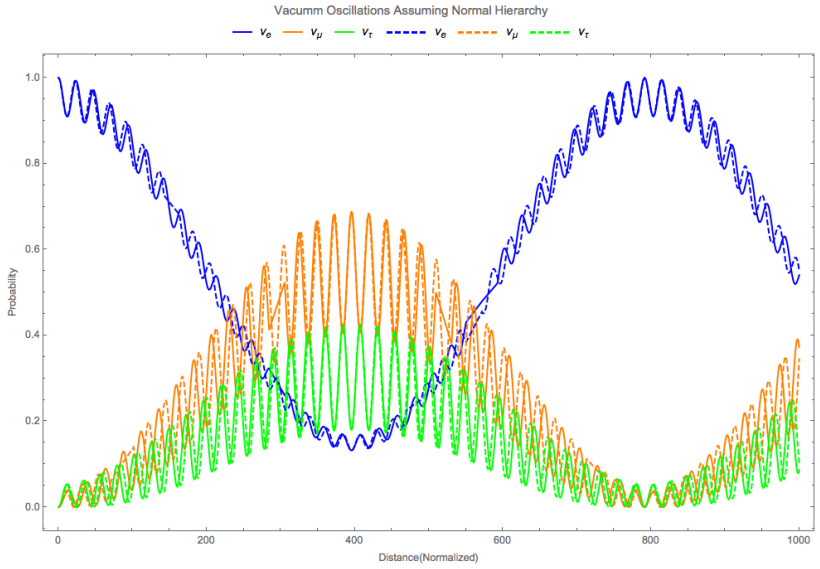


Figure 4: Comparison of normal hierarchy and inverted hierarchy but with inverted Δm^2_{12}.

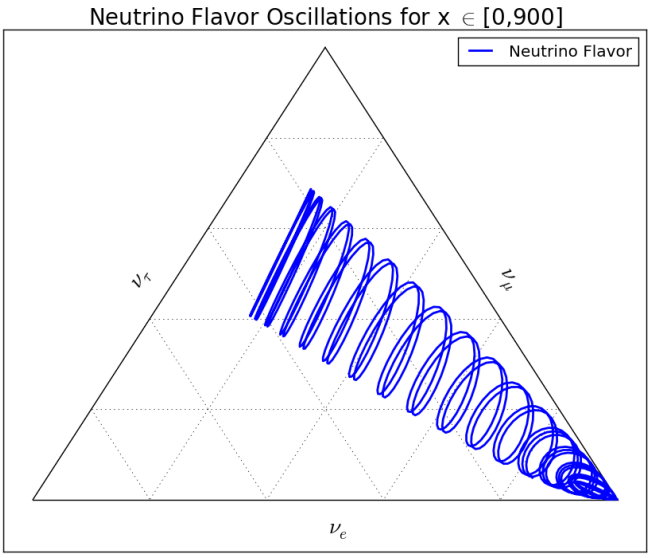


Figure 5: Ternary diagram for neutrino oscillations. The state starts from bottom left, which means that the system has only electron neutrinos. As the neutrino travels, it oscillates in curves. After one period of the beat, it reaches the far end and then oscillates backwards.

Oscillations in Matter

The Hamiltonian should be determined first. We have already derived the Hamiltonian for vacuum oscillation,

$$H_v = \frac{\delta m^2}{2E}\frac{1}{2}\begin{pmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{pmatrix},$$

where we would like to define a new matrix,

$$\mathbf{B} = \frac{1}{2}\begin{pmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{pmatrix},$$

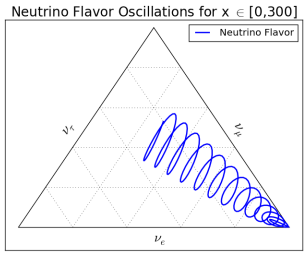
so that the vacuum Hamiltonian can be written as

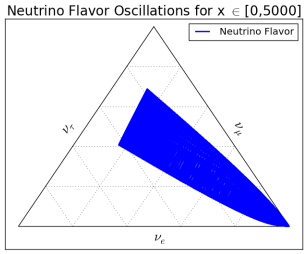
$$H_v = \frac{\delta m^2}{2E}\mathbf{B}.$$

The effect of matter, adds an extra term to this vacuum Hamiltonian which makes the electron population weighs more,

$$H_m = \sqrt{2}G_F n_e L.$$

Here we have [9]





[9] In principle we could shift the Hamiltonian using an identity matrix $\alpha\mathbf{I}$ without change the eigenvectors. We will use an term $\frac{1}{2}\boldsymbol{\alpha}_3$ instead in the following sections.

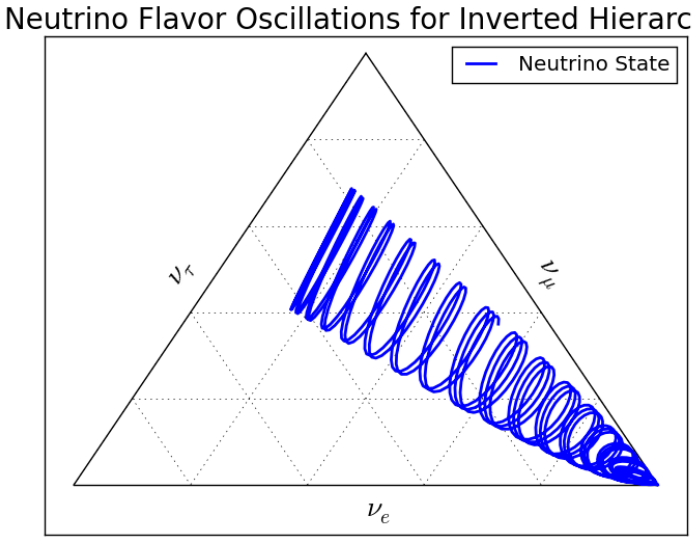


Figure 6: Ternary diagrram for inverted hierarchy.

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Without emphasizing the self-interaction of the neutrinos, the Hamiltonian to be used is

$$H = H_v + H_m. \tag{35}$$

The equation of motion is simply the von Neumann equation

$$i\partial_t \rho = [H, \rho], \tag{36}$$

in which the ∂_t operator is actually ∂_x since we always assume neutrinos travel with speed of light c.

A simple analysis before solving the system can be done rather easily. The characteristic length of the matter effect is

$$l_m = \frac{2\pi}{\sqrt{2} G_F n}. \tag{37}$$

The importance of this length is that it is a result of refractive index which causes the velocity difference thus changes the relative phase between ν_e and ν_x.[10]

With the definition of oscillation length in matter, a comparison between this length and the vacuum oscillation length can be made. Choose $\Delta m^2_{21} = 10^{-4} \times 10^{-18}\text{GeV}^2$[11], we have

[10] L. Wolfenstein. Neutrino oscillations and stellar collapse. Physical Review D, 20(10):2634–2635, November 1979. ISSN 0556-2821. DOI: 10.1103/PhysRevD.20.2634. URL http://link.aps.org/doi/10.1103/PhysRevD.20.2634

[11] Fermi constant is $G_F = 1.17 \times 10^{-5}\text{GeV}^{-2}$.

$$l_v = 4 \times 10^{19} \pi \left(\frac{E}{1\text{MeV}} \right) \text{GeV}^{-1} = 7.9 \times 10^{3} \pi \left(\frac{E}{1\text{MeV}} \right) \text{m}$$
$$l_m = 1.2 \times 10^{19} \pi \left(\frac{10^{-14}}{n} \right) \text{GeV}^{-1} = 1.2 \times 10^{4} \pi \left(\frac{10^{-24}\text{cm}^{-3}}{n} \right)$$

The two length are equal if

$$\left(\frac{10^{-14}}{n} \right) = 3.3 \left(\frac{E}{10^{-3}} \right)$$

More comparison can be found in figures 7 and 8.

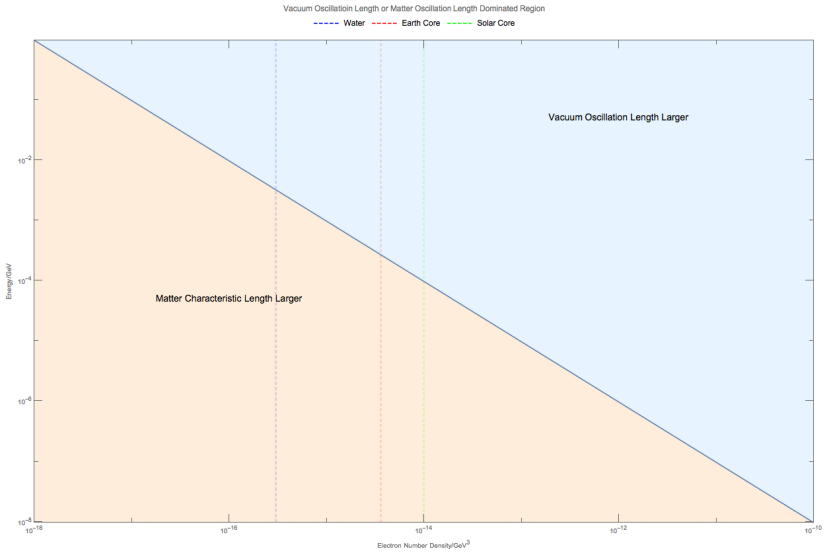


Figure 7: Comparison of the lengths

Analytic Solution

This Hamiltonian can be rewritten into a simple form using Pauli matrices,

$$\begin{aligned} \mathbf{H} &= \frac{\Delta m^2}{4E} \left(-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1 \right) + \frac{\Delta}{2} \sigma_3 \\ &= \left(\frac{\Delta}{2} - \frac{\Delta m^2}{4E} \cos 2\theta_v \right) \sigma_3 + \frac{\Delta m^2}{4E} \sin 2\theta_v \sigma_1 \\ &= \left(\frac{\Delta}{2} - \frac{\omega}{2} \cos 2\theta_v \right) \sigma_3 + \frac{\omega}{2} \sin 2\theta_v \sigma_1, \end{aligned}$$

where $\Delta = \sqrt{2} G_F n(x)$ and $n(x)$ is the number density of the electrons.

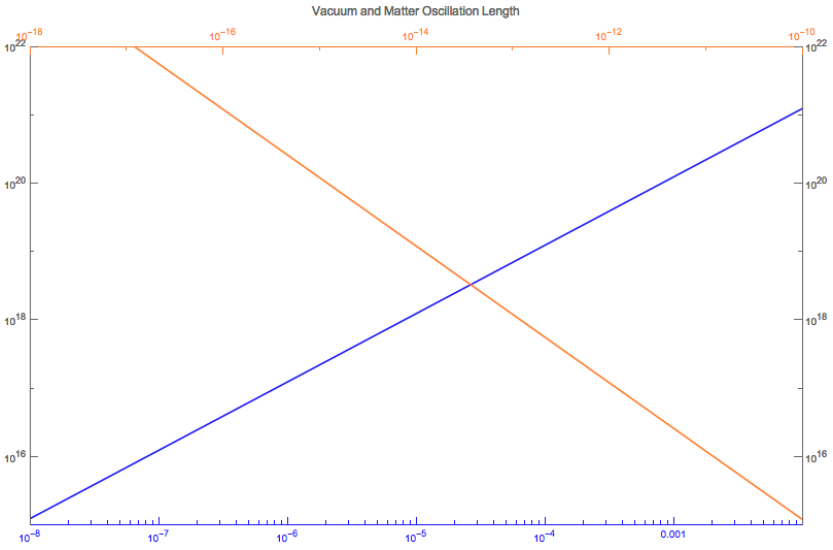


Figure 8: Comparison of the lengths

To solve the equation of motion, this matrix should be diagonalized and its eigenvalues and eigenvectors should be identified. Since we have this Pauli matrices form, this can be done easily.

To see this effect quantitatively, we need to diagonalize this Hamiltonian (Can we actually diagonalize the equation of motion? NO!). Equivalently, we can rewrite it in the basis of mass eigenstates $\{|\nu_L(x)\rangle, |\nu_H(x)\rangle\}$ [12],

$$
\begin{aligned}
|\nu_L(x)\rangle &= \cos\theta(x)\,|\nu_e\rangle - \sin\theta(x)\,|\nu_\mu\rangle \\
|\nu_H(x)\rangle &= \sin\theta(x)\,|\nu_e\rangle + \cos\theta(x)\,|\nu_\mu\rangle\,.
\end{aligned}
$$

This new rotation in matrix form is

$$
\begin{aligned}
\begin{pmatrix} |\nu_L(x)\rangle \\ |\nu_H(x)\rangle \end{pmatrix} &= \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix} \begin{pmatrix} |\nu_e\rangle \\ |\nu_x\rangle \end{pmatrix} \\
&= \mathbf{U_m^{-1}} \begin{pmatrix} |\nu_e\rangle \\ |\nu_x\rangle \end{pmatrix}.
\end{aligned}
$$

To diagonalize it, we need to multiply on both sides the rotation matrix and its inverse, as we have done in the vacuum case,

$$\mathbf{H_{md}} = \mathbf{U_m^{-1} H U_m}.$$

The second step is to set the off diagonal elements to zero. By solving the equations we can find the $\sin 2\theta(x)$ and $\cos 2\theta(x)$. [13]

[12] This is very different from the vacuum case since this one is local. In principle we can not use this transformation to diagonalize the Hamiltonian because the equation is a differential equation about x.

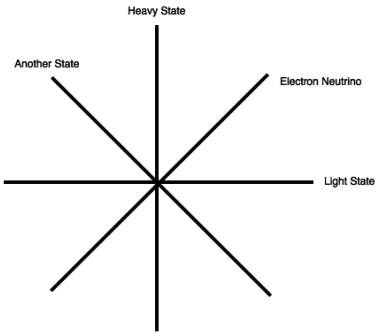


Figure 9: Due to this rotation, we can find an angle when the electron neutrinos becomes more important in Heavy state than Light state, which is $\theta = \frac{\pi}{4}$. For a mixing that has a larger angle, electron neutrinos actually take part in heavy neutrino state.

[13] Just remember that $\Delta = \sqrt{2} G_F n(x)$ is a function of position.

$$\begin{aligned}\mathbf{H_{md}} &= \mathbf{U_m^{-1}}\left(A_1\sigma_1 + A_3\sigma_3\right)\mathbf{U_m} \\ &= \begin{pmatrix} A_3\cos 2\theta(x) - A_1\sin 2\theta(x) & A_3\sin 2\theta(x) + A_1\cos 2\theta(x) \\ A_3\sin 2\theta(x) + A_1\cos 2\theta(x) & -A_3\cos 2\theta(x) + A_1\sin 2\theta(x)\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}A_3 &= \frac{\Delta}{2} - \frac{\delta^2 m}{4E}\cos 2\theta_v = \frac{\Delta}{2} - \frac{\omega}{2}\cos 2\theta_v \\ A_1 &= \frac{\delta^2 m}{4E}\sin 2\theta_v = \frac{\omega}{2}\sin 2\theta_v.\end{aligned}$$

Set the off-diagonal elements to zero,

$$A_3\sin 2\theta(x) + A_1\cos 2\theta(x) = 0$$

So the solutions are

$$\begin{aligned}\sin 2\theta(x) &= \frac{A_1}{\sqrt{A_1^2 + A_3^2}} \\ &= \frac{\sin 2\theta_v}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}} \qquad (38) \\ \cos 2\theta(x) &= \frac{-A_3}{\sqrt{A_1^2 + A_3^2}} \\ &= \frac{-(\hat{\Delta} - \cos 2\theta_v)}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}}, \qquad (39)\end{aligned}$$

where $\hat{\Delta} = \Delta/\omega$.

This diagonalizes the Hamiltonian LOCALLY. [14] It's not possible to diagonalize the Hamiltonian globally if the electron number density is not a constant.

As for a qualitative analysis, matter effect can be a suppression or enhancement. $\sin 2\theta_v$ represents the amplitude of the survivable probability of one of the flavors since larger θ_v means more mixing between the two flavors. However matter changes the mixing angle according to the expression of $\sin 2\theta_m$, which is

- larger than $\sin 2\theta_v$ if $0 < \hat{\Delta} < 2\cos 2\theta_v$.
- smaller than $\sin 2\theta_v$ if $\hat{\Delta} > 2\cos 2\theta_v$.

A smaller $\sin 2\theta$ means a smaller mixing. Matter can sometimes suppress the mixing while can also enhance the mixing.

As $\Delta \to \infty$, we have $A_3 \to \infty$ and $\sin 2\theta(x)$ vanishes. The effective mixing angle θ_m becomes $\pi/2$ thus only a phase is added to first

[14] The point is, for equation of motion, we have a differentiation with respect to position x ! So even we diagonalize the Hamiltonian, the equation of motion won't be diagonalized. An extra matrix will occur on the LHS and revert the diagonalization the Hamiltonian on RHS.

mass eigenstate, i.e., $|\nu_e\rangle = -|\nu_1\rangle$ and $|\nu_x\rangle = |\nu_2\rangle$. Thus the neutrino will stay on the defined light and heavy eigenstates if the interaction with matter is much larger than the vacuum energy. On the other hand, $\Delta \to 0$, everything gets back to the vacuum case.

Now we can obtain an approximate solution by using this approximate diagonalization. This idea is

$$i\partial_x \Psi_m(x) = \textbf{ExtraMatrixFromLHS} \cdot \mathbf{H}_{md} \Psi_m(x), \tag{40}$$

where the **ExtraMatrixFromLHS** comes from the fact that changing from flavor basis $\Psi(x)$ to heavy-light basis $\Psi_m(x)$ using $\mathbf{U_m}$

$$i\partial_x(\mathbf{U_m}\Psi_m(x)) = H(\mathbf{U_m}\Psi_m(x))$$

only returns

$$i\partial_x \Psi_m(x) = \mathbf{H_{md}}\Psi_m(x) - i\mathbf{U_m^{-1}}(\partial_x \mathbf{U_m})\Psi_m(x).$$

Combining the two terms on RHS,

$$i\partial_x \Psi_m(x) = \mathbf{H_m}\Psi_m(x),$$

where

$$\mathbf{H_m} = \mathbf{H_{md}} - i\mathbf{U_m^{-1}}(\partial_x \mathbf{U_m}).$$

The only part inside $\mathbf{U_m(x)}$ that is space dependent is the number density of the electrons $n(x)$. **Thus we know immediately that the Hamiltonian is diagonalized if the number density is constant.**[15,16,17]

In fact there are at least three different ways of solving this problem.

- Work in instantaneous eigenstates by writing down the general expression as we have done in this subsection.
- Work in flavor basis directly.
- Work in the vacuum mass eigenstates by transform the matter potential from flavor basis to vacuum mass basis. Write down it in vacuum mass eigenbasis then add in the potential.

$$\mathbf{H_{vp}} + \mathbf{U^{-1}VU},$$

where

[15] But even the electrons density is constant, the oscillation is different.

[16] **It would be nice to find the expression of $\mathbf{U_m}$ in order to play with the matter effect.**

[17] In a two level system, suppose the Hamiltonian is

$$H_2 = H_{d2} + P_2,$$

where

$$H_{d2} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$
$$P_2 = \begin{pmatrix} 0 & W(r) \\ W^*(r) & 0 \end{pmatrix}$$

are the diagonalized Hamiltonian and the perturbation.

The diagonalized Hamiltonian will result in a system with two levels which are constant as a function of r, however the perturbation term provides the r dependent term which will cause a r dependent eigen energies.

$$\mathbf{H_{vp}} = \frac{1}{2E}\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix},$$

and

$$\mathbf{V} = \begin{pmatrix} \sqrt{2}G_F n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The good thing of starting in this way is that we can eliminate one of the masses and only use mass squared differences by subtracting a $\frac{m_1^2}{2E}\mathbf{I}$ matrix.

Constant Electron Number Density

Suppose we have an environment with constant electron number density, the term $-i\mathbf{U_m^{-1}}(\partial_x \mathbf{U_m})$ goes away. All we have is the diagonalized new Hamiltonian $\mathbf{H_{md}}$ and the eigenvalues are easily obtained which are

$$E_1 = A_3 \cos 2\theta(x) - A_1 \sin 2\theta(x)$$
$$E_2 = -A_3 \cos 2\theta(x) + A_1 \sin 2\theta(x).$$

The final result for these two eigenvalues are

$$E_1 = -\sqrt{A_1^2 + A_3^2}$$
$$E_2 = \sqrt{A_1^2 + A_3^2},$$

where

$$A_1^2 + A_3^2 = \frac{\Delta^2 + \omega^2}{4} - \frac{\Delta\omega}{2}\cos 2\theta_v.$$

There are two special cases. The first case is $\cos 2\theta_v = 0$ while the other one is $\cos 2\theta_v = 1$ which leads to $\sqrt{A_1^2 + A_3^2} = |\frac{\Delta-\omega}{2}|$ and the two energies becomes $\frac{\Delta-\omega}{2}$ and $\frac{\omega-\Delta}{2}$.[18]

As for the survival probability, the result has the same form as the vacuum case, which is

$$P_x(\nu_e, L) = 1 - \sin^2(2\theta_m)\sin^2\left(\frac{\omega_m L}{2}\right),$$

where $\theta_m = \theta(x)$ is the effective mixing angle which in fact doesn't depend on x if the matter profile is constant.[19]

[18] This means the matter effect in this constant profile situation determines the difference between the eigen energies.

[19] The term $\sin 2\theta_m$ has been derived previously.

$$\sin 2\theta(x) = \frac{\omega \sin 2\theta_v}{\sqrt{\omega^2 + \Delta^2 - 2\omega\Delta\cos 2\theta_v}}.$$

As an comparison, the vacuum result is

$$P_x(\nu_e, L) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\omega L}{2}\right).$$

As the math tells us, $\sin^2 2\theta$ and $\sin^2 2\theta_m$ are the two terms that determines how much electron flavor is converted to the other flavor. The other parts determine the oscillation period.

Recall that we have defined two characteristic lengths, l_v and l_m. The overall oscillation length in matter becomes[20]

$$l = \frac{2\pi}{\omega_m} \tag{41}$$

$$= \frac{2\pi}{\omega\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}} \tag{42}$$

$$= \frac{l_v}{\sqrt{\left(\frac{l_v}{l_m}\right)^2 + 1 - 2\frac{l_v}{l_m}\cos 2\theta_v}} \tag{43}$$

[20] We know that $\omega > 0$ here since we have already determined the hierarchy for m_1 and m_2. How? There is a significant effect on the oscillations here if $\omega < 0$!

This expression shows exactly why the different length scales are important.

- $|\frac{l_v}{l_m}| \ll 1 \Rightarrow l \to l_v$, matter effect is minimal.
- $|\frac{l_v}{l_m}| \gg 1 \Rightarrow l \to 0$, matter effect kills the oscillations.
- $|\frac{l_v}{l_m}| \sim 1 \Rightarrow l \to \frac{l_v}{2\sin\theta_v}$, more interesting region.[21] To linear approximation,

  $$l \approx \frac{l_v}{2\sin\theta_v}\left(1 - \frac{l_v/l_m - 1}{2}\right) = \frac{l_v}{4\sin\theta_v}\left(1 - \frac{l_v}{l_m}\right).$$

  notice that at resonance

  $$l = \frac{l_v}{2\sin\theta_v}.$$

[21] Assume $\sin\theta_v > 0$.

Another important thing about these lengths is that l_v is a function of energy which is that higher energy means longer oscillation length as shown in figure 7 and 8. This is important because we can always find the a energy that is at resonance with the matter density, as long as the energy still makes sure the neutrinos are relativistic. So a spectral swap is possible.

Adiabatic Limit

In some astrophysical environments the electron number density changes very slowly which means the term $\mathbf{U_m^{-1}}\partial_x\mathbf{U_m}$ is much smaller than $\mathbf{H_{md}}$. By intuition we would expect that this term could

be dropped to the lowest order. However to verify that we need to check the Taylor expansion of this gradient term which requires the general form of $\mathbf{U_m}$.

Before we actually do the expansion, make the quantities dimensionless would greatly help us.

$$\sin 2\theta(x) = \frac{\sin 2\theta_v}{\sqrt{\left(\frac{\Delta}{\omega}\right)^2 + 1 - 2\frac{\Delta}{\omega}\cos 2\theta_v}}$$
$$\cos 2\theta(x) = \frac{\cos 2\theta_v - \frac{\Delta}{\omega}}{\sqrt{\left(\frac{\Delta}{\omega}\right)^2 + 1 - 2\frac{\Delta}{\omega}\cos 2\theta_v}}.$$

Define $\hat{\Delta} = \frac{\Delta}{\omega}$, which represents the matter interaction strength compared to the vacuum oscillation.

$$\sin 2\theta(x) = \frac{\sin 2\theta_v}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}}$$
$$\cos 2\theta(x) = \frac{\cos 2\theta_v - \hat{\Delta}}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}}.$$

$$\sin\theta(x) = \frac{\csc(2\theta_v)\sqrt{\frac{-\hat{\Delta}+\sqrt{\hat{\Delta}^2-2\hat{\Delta}\cos(2\theta_v)+1}+\cos(2\theta_v)}{\sqrt{\hat{\Delta}^2-2\hat{\Delta}\cos(2\theta_v)+1}}}\left(\hat{\Delta}+\sqrt{\hat{\Delta}^2-2\hat{\Delta}\cos(2\theta_v)+1}-\cos(2\theta_v)\right)}{\sqrt{2}}$$
$$\cos\theta(x) = \frac{\sqrt{\frac{-\hat{\Delta}+\sqrt{\hat{\Delta}^2-2\hat{\Delta}\cos(2\theta_v)+1}+\cos(2\theta_v)}{\sqrt{\hat{\Delta}^2-2\hat{\Delta}\cos(2\theta_v)+1}}}}{\sqrt{2}}$$

This is extremely complicated if we take the derivative with respect to x. So we apply Taylor expansion.[22]

[22] The is are there makes sure that $-i\partial_x \mathbf{U_m}(x)$ is real.

$$\begin{aligned}\frac{d\sin\theta(x)}{dx} &= \frac{d\sin\theta(x)}{d\hat{\Delta}}\frac{d\hat{\Delta}}{dx}\\ &= i\frac{d\hat{\Delta}}{dx}\left(-cos\theta_v\sin^2\theta_v - \frac{1}{2}((1+5\cos 2\theta_v)\cos\theta_v\sin^2\theta_v)\hat{\Delta} + \mathcal{O}(\hat{\Delta}^2)\right)\\ \frac{d\cos\theta(x)}{dx} &= \frac{d\cos\theta(x)}{d\hat{\Delta}}\frac{d\hat{\Delta}}{dx}\\ &= i\frac{d\hat{\Delta}}{dx}\left(cos^2\theta_v\sin\theta_v + \frac{1}{2}\cos^2\theta_v(-1+5\cos 2\theta_v)\sin\theta_v\hat{\Delta} + \mathcal{O}(\hat{\Delta}^2)\right)\end{aligned}$$

There is nothing in the coefficients that prohibits us from using the lowest order that $\frac{d\hat{\Delta}/dx}{\text{Energy Differences Between Two Energy Levels}} \ll 1$ since the Taylor series has finite values.[23]

[23] No singularities possible. I assumed that $\sin\theta_v > 0$.

Solving the adiabatic limit is just the same as the constant limit but $\theta(x)$ is not constant which ensures changing energies. What would

be interesting is the behavior of energies as the density profile is slowing changing.

$$E_1 = -\frac{\omega}{2}\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}$$
$$E_2 = \frac{\omega}{2}\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}.$$

When the term $\hat{\Delta}$ is very small $1 - 2\hat{\Delta}\cos 2\theta_v$ will dominate and the whole term decreases. On the other hand as $\hat{\Delta}$ becomes large, $\hat{\Delta}^2$ will dominate and the whole term grows.

The survival probability for the light neutrinos would be

$$P_x(\nu_L, L) = 1 - \sin^2(2\theta(x))\sin^2\left(\frac{\omega L}{2}\right).$$

The survival probability for electron flavor neutrino is

$$P_x(\nu_e, L) = \frac{1}{2} + \frac{1}{2}\cos 2\theta(x_0)\cos 2\theta_v,$$

if the neutrinos are produced in dense region and the detection happens in vacuum.

Before we move on to higher order corrections, it would be nice to understand this phenomenon.

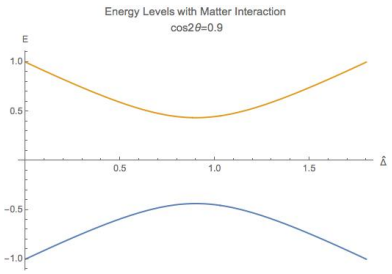


Figure 10: Energy Levels for MSW effect. We have the up-down symmetry since we shifted the energy by a constant to remove the identity matrix in the Hamiltonian.

- The vacuum oscillation length can be extracted from vacuum oscillation survival probability. It is $L_v = \frac{2\pi}{\omega}$.
- In this problem we have another energy scale which is the interaction, Δ. Here we can define another characteristic length $l_m = \frac{2\pi}{\Delta}$.
- MSW resonance happens when the two character lengths are matching to each other. Another way to put it is that the the term $\sin 2\theta(x)$ is minimized so that we have the smallest gap which leads to math $\hat{\Delta} = \cos 2\theta_v$. Equivalently this is the relation

$$l_0 = l_m \cos 2\theta_v.$$

- At resonance, we have

$$\cos 2\theta(x) = 1$$
$$\sin 2\theta(x) = 0.$$

This is max mixing of the states which means that at the resonance point

$$\begin{pmatrix} \nu_L(x_{reso}) \\ \nu_H(x_{reso}) \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix}$$

- Resonance conditions corresponds to a resonance density which is given by

$$n_e(x) = \frac{\omega}{\sqrt{2}G_F}\cos 2\theta_v \equiv n_0(E, \Delta m^2)\cos 2\theta_v,$$

where $n_0(E, \Delta m^2)$ is a characteristic number density which depends on the energy mixing angles and Δm^2 of the neutrinos.

- One should notice that to get back to the vacuum oscillation survival probability which means $\sin^2 2\theta(x) = \sin^2 2\theta_v$ we have[24]

$$\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v = 1,$$

which leads to

$$\hat{\Delta} = 0 \quad \text{or} \quad 2\cos 2\theta_v.$$

The first condition is trivial which corresponds to vacuum however the second condition $\Delta = 2\cos 2\theta_v\omega$ means the interaction oscillation length is doubled compared to resonance point.

Nevertheless, we should always remember to check what survival probability the expression is describing. Here we have survival probability for $\nu_L(x)$. At $n(x) \to 0$ the oscillation becomes vacuum oscillation.

[24] **I should check if this is still true in 3 flavor neutrino scenario. This could be interesting if it is respected in 3 flavor neutrino oscillations.**

First Order Approximation

In principle we could use Taylor expansion of $\mathbf{U_m}$ to find the series of the derivatives of $\cos\theta(x)$ and $\sin\theta(x)$, however it doesn't looks like the easy way. Now we go back to find the Taylor expansion of the Hamiltonian which only involves the expression $\sin 2\theta(x)$ and $\cos 2\theta(x)$.

Since we are dealing with resonance which is located at $\hat{\Delta} = 1$, the quantities can be expanded around $\hat{\Delta} - = 0$.

The series of them are[25]

$$\begin{aligned}
\sin 2\theta(x) &= \sin 2\theta_v \left(\frac{1}{2\sin\theta_v} - \frac{1}{4\sin\theta_v}(\hat{\Delta}-1) + \mathcal{O}(\hat{\Delta}-1)^2\right) \\
&= \cos\theta_v - \frac{\cos\theta_v}{2}(\hat{\Delta}-1) + \mathcal{O}(\hat{\Delta}-1)^2 \\
\cos 2\theta(x) &= -\sin\theta_v - \frac{\cos\theta_v}{2\tan\theta_v}(\hat{\Delta}-1) + \mathcal{O}(\hat{\Delta}-1)^2.
\end{aligned}$$

Notice that for small θ_v the first order can be neglected sine they are second or higher order small quantities.

$\sin\theta(x)$ and $\cos\theta(x)$ can be found using these results up to second order.[26]

[25] Assuming $\sin\theta$ is not negative.

[26] We need second order because we are going to take the derivative of x. However, in this special case, we can directly solve out $\sin\theta(x)$ and $\cos\theta(x)$ using by plug the approximation $\hat{\Delta}(x) = 1 + \hat{\Delta}'(x_r)(x - x_r)$ into the expression of $\sin 2\theta(x)$ and $\cos 2\theta(x)$ and WITHOUT Taylor expand $\sin 2\theta(x)$ and $\cos 2\theta(x)$.

To find out the solutions the explicit expression for $\mathbf{U_m}$ is required.[27] This can be done using Mathematica along with the derivative of this rotation matrix with respect to x where a trick

$$\partial_x \mathbf{U_m} = \frac{d\hat{\Delta}'(x_r)(x-x_r)}{dx} \partial_{\hat{\Delta}'(x_r)(x-x_r)} \mathbf{U_m}$$

has been used.

The effective potential is[28]

$$\begin{aligned}\mathbf{V_m} &= -i\mathbf{U_m^{-1}}(\partial_x \mathbf{U_m}) \\ &= -i\frac{\hat{\Delta}'(x_r)\sin 2\theta_v}{2(2+\hat{\Delta}'(x)(x-x_r)(2+\hat{\Delta}'(x)(x-x_r))) - 2(1+\hat{\Delta}'(x)(x-x_r))\cos 2\theta_v} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\end{aligned}$$

To keep only first order[29]

$$\mathbf{V_m} = -i\hat{\Delta}'(x_r)\frac{\cot\theta_v}{4}\left(-1+\hat{\Delta}'(x_r)(x-x_r)\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The equation of motion up to first order of $\hat{\Delta}$ becomes

$$i\partial_x |\Psi_m\rangle = (\mathbf{H_{md}} + \mathbf{V_m})|\Psi_m\rangle .$$

We have already solved

$$i\partial_x |\Psi_m\rangle = \mathbf{H_{md}} |\Psi_m\rangle ,$$

where the eigenstates are $|\nu_L\rangle$ and $|\nu_H\rangle$.

In general we define

$$\begin{aligned} v &= -\hat{\Delta}'(x_r)\frac{\cot\theta_v}{4}\left(-1+\hat{\Delta}'(x_r)(x-x_r)\right) \\ &\approx \hat{\Delta}'(x_r)\frac{\cot\theta_v}{4}\end{aligned}$$

so that

$$\mathbf{V_m} = \begin{pmatrix} 0 & iv \\ -iv & 0 \end{pmatrix}.$$

The general solution to the equation we need to solve can be written as

$$|\Psi_m\rangle = C_L(x)e^{-i\int \omega_{m1}dx}|\nu_L\rangle + C_H(x)e^{-i\int \omega_{m2}dx}|\nu_H\rangle ,$$

where

[27] $$\mathbf{U_m} = \begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ -\sin\theta(x) & \cos\theta(x) \end{pmatrix}$$

[28] This expression comes from the result of the second order approximation of $\sin\theta(x)$ and $\cos\theta(x)$.

[29] $$\sigma_2 = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$\begin{aligned}\omega_{m1} &= -\sqrt{\frac{\Delta^2+\omega^2}{4} - \frac{\Delta\omega}{2}\cos 2\theta_v} \\ &= -\omega\sqrt{\left(\frac{\hat{\Delta}^2+1}{4} - \frac{\hat{\Delta}}{2}\cos 2\theta_v\right)}, \\ \omega_{m2} &= -\omega_{m1} \equiv \frac{\omega_m}{2}.\end{aligned}$$

Hamiltonian applied to this state results in

$$\begin{aligned}\mathbf{H_m}\left|\Psi_m\right\rangle =& \omega_{m1}C_L(x)e^{-i\int\omega_{m1}dx}\left|\nu_L\right\rangle - ivC_L(x)e^{-i\int\omega_{m1}dx}\left|\nu_H\right\rangle \\ &+ \omega_{m2}C_H(x)e^{-i\int\omega_{m2}dx}\left|\nu_H\right\rangle + ivC_H(x)e^{-i\int\omega_{m2}dx}\left|\nu_L\right\rangle.\end{aligned}$$

Plug the state $|\Psi_m\rangle$ into the Schrödinger equation, we have

$$\begin{aligned}\dot{C}_L(x) &= vC_H(x)e^{i\int\delta\omega_m dx} \\ \dot{C}_H(x) &= -vC_L(x)e^{-i\int\delta\omega_m dx},\end{aligned}$$

in which $\delta\omega_m$ is defined as

$$\delta\omega_m = \omega_{m2} - \omega_{m1}.$$

The boundary condition for this problem **in general** is that

$$\left|\Psi_m(0)\right\rangle = C_L(0)\left|\nu_L\right\rangle + C_H(0)\left|\nu_H\right\rangle.$$

It should be made clear that the problem we will be discussing is the transition from one state to another in first order approximation. That means we will confine this system so that the initial condition is $|\Psi_m(-\infty)\rangle = |\nu_L\rangle$. In terms of C_L and C_H,

$$\begin{aligned}C_L(-\infty) &= 0, \\ |C_H(-\infty)|^2 &= 1.\end{aligned}$$

The first order differential equations of $C_L(x)$ and $C_H(x)$ can be combined and produce a second order differential equation.

$$\ddot{C}_L - \left(\frac{\dot{v}}{v} + i\delta\omega_m\right)\dot{C}_L + v^2C_L = 0.$$

If we use the approximation that $\frac{d\hat{\Delta}}{dx}$ is a constant,[30] The equation simplifies to

$$\ddot{C}_L - i\delta\omega_m\dot{C}_L + v^2C_L = 0,$$

[30] We are assuming that $n(x)$ is linearly depending on x which means $\hat{\Delta}$ is a linear function of x. Thus $\frac{d\hat{\Delta}}{dx}$ is a constant.

where $v = -\sin\theta_v \cos\theta_v \frac{d\hat{\Lambda}}{dx}$ is constant.

~~Furthermore, the eigenvalues are not varying very fast and satisfies the condition that~~

$$\delta\omega_m = \alpha x,$$

~~where α is a constant and comes from the first order of the expression.~~

~~Assuming one of the solutions is e^{rx}, we could insert it to the second order differential equation and get the characteristic equation,~~

$$r^2 - i\delta\omega_m r + v^2 = 0,$$

~~which has two solutions~~

$$r_1 = i\frac{\delta\omega_m + \sqrt{\delta\omega_m + 4v^2}}{2}$$
$$r_2 = i\frac{\delta\omega_m - \sqrt{\delta\omega_m + 4v^2}}{2}.$$

~~The solutions to $C_L(x)$ and $C_H(x)$ will be~~

$$C_L(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$
$$C_H(x) = (c_1 e^{r_1 x} + c_2 e^{r_2 x})\,\frac{e^{i\int \delta\omega_m dx}}{v}.$$

~~These constants are determined by the initial condition. The state of the system at any x is~~

$$|\Psi_m(x)\rangle = (c_1 e^{r_1 x} + c_2 e^{r_2 x})\, e^{-i\int_0^x \omega_{m1}(x')dx'} |\nu_L\rangle + (c_1 e^{r_1 x} + c_2 e^{r_2 x})\,\frac{e^{i\int \delta\omega_m dx}}{v} e^{-i\int_0^x \omega_{m2}(x')dx'} |\nu_H\rangle .$$

~~Doesn't make sense.~~

In the paper by Zener, we need to do substitution of the function C_L and reduce the equation to Weber equation.[31]

Define a new variable W which is determined by[32]

$$C_L = e^{\frac{i}{2}\int \delta\omega_m dx} W.$$

Then we get a simple equation about W,

$$\ddot{W} + \left(v^2 + \frac{i\alpha}{2} + \frac{\alpha^2 x^2}{4}\right) W = 0,$$

which can be reduced to the standard form of Weber equation with the new parameters[33]

[31] C. Zener. Non-Adiabatic Crossing of Energy Levels. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 137(833): 696–702, 1932. ISSN 1364-5021. DOI: 10.1098/rspa.1932.0165

[32] This is done by assuming $C_L = f(x)W$ and plugging it back to the equation then set the coefficient of $\dot{C}_L$ to 0.

[33] They are found using a single assumption that $z = gx$.

$$z = gx$$
$$\nu = i\frac{v^2}{\alpha},$$

where $g^2 \equiv -i\alpha$ ($g = (1-i)\sqrt{\alpha}/\sqrt{2} = \sqrt{\alpha}e^{-i\pi/4}x$). The equation we need to solve becomes

$$\frac{d^2W(z)}{dz^2} + \left(\nu + \frac{1}{2} - \frac{1}{4}z^2\right)W(z) = 0.$$

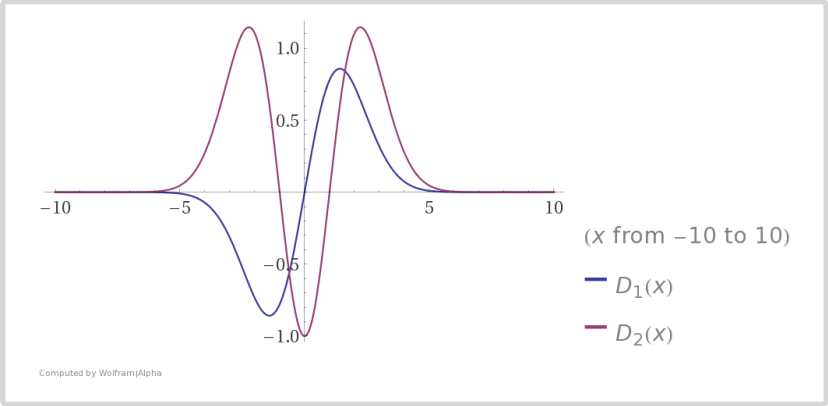


Figure 11: The parabolic cylinder function $D_\nu(z)$ for $\nu = 1$ (blue) and $\nu = 2$ (red). But for imaginary z the function blows up.

The Weber equation has two independent solutions $D_\nu(z)$ and $D_{-\nu-1}(iz)$.[34]

Since $D_\nu(z)$ blows up for complex $z \propto e^{-\pi i/4}$, the solution that works is $D_{-\nu-1}(iz)$. Then the solution to U_L is

$$U_L(x) = u_+ D_{-iv^2/\alpha-1}(\frac{1-i}{\sqrt{2}}x),$$

or

$$U_L(x) = u_- D_{-iv^2/\alpha-1}(-\frac{1-i}{\sqrt{2}}x).$$

The asymptotic expression for $D_{-\nu-1}$ on the line of $e^{-i\pi/4}$ and $e^{-3i\pi/4}$ at infinite contour radius on complex plane are

$$D_{-\nu-1}(ixe^{-3i\pi/4}) \to e^{i(\nu+1)\pi/4}e^{ix^2/4}x^{-\nu-1}$$
$$D_{-\nu-1}(ixe^{-i\pi/4}) \to e^{-i(\nu+1)\pi/4}e^{-ix^2/4}x^{-\nu-1}.$$

So the real part of these asymptotic expressions are

$$e^{i\nu\pi/4}x^{-\nu-1} = e^{-v^2\pi/4\alpha}x^{-\nu-1}e^{-i\nu\pi/4}x^{-\nu-1} \qquad = e^{v^2\nu\pi/4\alpha}x^{-\nu-1}$$

[34] They are alled Parabolic Cylinder Function on wolfram mathworld. http://mathworld.wolfram.com/ParabolicCylinderFunction.html

Apply the boundary condition we have the results of the coefficients[35]

[35] The details depends on the signs of α.

$$|u_+| = |u_-| = e^{\gamma\pi/4}\sqrt{\gamma},$$

where $\gamma = \frac{v^2}{|\alpha|}$.

What we need to find out is the state at $t \to \infty$, which depends on the asymptotic values of $D_{-\nu-1}$,

$$C_L(x) \to \sqrt{\gamma}e^{-\gamma\pi/4}\left(e^{3\pi(\nu+1)i/4}e^{-ix^2/4}x^{-\nu-1} + \frac{\sqrt{2\pi}}{\Gamma(\nu+1)}e^{\pi\nu/4}e^{ix^2/4}x^{\nu}\right),$$

or

$$C_L \to \sqrt{\gamma}e^{-\gamma\pi/4}\left(e^{-3\pi(\nu+1)i/4}e^{ix^2/4}x^{-\nu-1} + \frac{\sqrt{2\pi}}{\Gamma(\nu+1)}e^{\pi\nu/4}e^{-ix^2/4}x^{\nu}\right).$$

The transition rate is determined by $|C_L|^2$,[36]

[36] C. Zener. Non-Adiabatic Crossing of Energy Levels. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 137(833): 696–702, 1932. ISSN 1364-5021. DOI: 10.1098/rspa.1932.0165

$$|C_L(\infty)| = \gamma e^{-\pi\gamma}\frac{2\pi}{\Gamma(i\gamma+1)\Gamma(-i\gamma+1)} = 2e^{-\pi\gamma}\sinh\pi\gamma = 1 - e^{-2\pi\gamma}.$$

Now we understand the transition probability is given by

$$P_{tran} = e^{-2\pi\gamma}.$$

Suppose we have the initial condition as $|\Psi_m(x=-\infty)\rangle = |\nu_L\rangle$, the system can jump to $|\nu_H\rangle$ since the state at arbitrary position x is a mixing of the two states.[37] The rate of jumping is given by[38]

[37] In fact this approximation is valid at the level crossing.

[38] Amar C Vutha. A simple approach to the Landau-Zener formula. 389: 4, 2010. ISSN 0143-0807. DOI: 10.1088/0143-0807/31/2/016. URL http://arxiv.org/abs/1001.3322; C. Zener. Non-Adiabatic Crossing of Energy Levels. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 137(833):696–702, 1932. ISSN 1364-5021. DOI: 10.1098/rspa.1932.0165; and Jan R. Rubbmark, Michael M. Kash, Michael G. Littman, and Daniel Kleppner. Dynamical effects at avoided level crossings: A study of the Landau-Zener effect using Rydberg atoms. Physical Review A, 23(6):3107–3117, 1981. ISSN 10502947. DOI: 10.1103/PhysRevA.23.3107

$$\begin{aligned}
&P(x\to\infty, |\nu_L\rangle \to |\nu_H\rangle)\\
=&e^{-2\pi\gamma}\\
=&\exp\left(-2\pi\frac{v^2}{|\frac{d\omega_m}{xdx}|_{x_r}}\right)\\
=&\exp\left(-2\pi\frac{\left(-\sin\theta_v\cos\theta_v\frac{d\hat{\Delta}}{dx}\omega\right)^2}{|\frac{d\omega_m}{xdx}|_{x_r}}\right)\\
=&\exp\left(-\frac{\pi}{2}\frac{\sin^2 2\theta_v\omega}{\cos 2\theta_v|\frac{d\hat{\Delta}}{dx}/\hat{\Delta}|_{x_r}}\right)
\end{aligned}$$

The final result is

$$P(x \to \infty, |\nu_L\rangle \to |\nu_H\rangle) = \exp\left(-\frac{\pi}{2}\frac{\sin^2 2\theta_v}{\cos 2\theta_v}\frac{\omega}{|\frac{1}{\hat{\Delta}}\frac{d\hat{\Delta}}{dx}|_{x_r}|}\right)$$

The survival probability can be calculated by counting the probability left on the initial state.

To be clear, if electron neutrinos are produced inside core of our sun, it will be the heavy state. Since the interaction with matter is very strong, it transfers to $|\nu_L\rangle$ with probability $P(x \to \infty, |\nu_L\rangle \to |\nu_H\rangle)$ due to the gradient of the matter profile which works as the perturbation. Thus the final state will be a mixing of $|\nu_L\rangle$ and $|\nu_H\rangle$.

General Discussion for Neutrinos Interacting with Matter

This part is a very general discussion of the matter effect.[39]

To work in flavor basis, we use the subscript $_{mf}$ to denote the flavor basis representation with mass effect. The equation of motion in flavor basis can be written down as[40]

$$i\partial_x \Psi_{mf}(x) = \mathbf{H_{mf}}\Psi_{mf}(x)$$

where

$$\mathbf{H_{mf}} = \left(\frac{\Delta}{2} - \frac{\omega}{2}\cos 2\theta_v\right)\boldsymbol{\sigma_3} + \frac{\omega}{2}\sin 2\theta_v \boldsymbol{\sigma_1}.$$

As we have seen in adiabatic situation, the states will stay in heavy and light states all along the evolution if the system starts from heavy or light state,

$$|\nu_{a1}(x)\rangle = \exp(-i\int_0^x \frac{\omega_m(x)}{2}dx)\,|\nu_L(x)\rangle$$
$$|\nu_{a2}(x)\rangle = \exp(i\int_0^x \frac{\omega_m(x)}{2}dx)\,|\nu_H(x)\rangle\,,$$

where the heavy and light states are defined in the adiabatic situation previously. This is what happens before the passing through of the resonance.

However, when it comes to the non-adiabatic transitions, the evolution of the states will be the mixture of the heavy and light state since transitions occurs $|\nu_{a1}\rangle \to |\nu_1(x)\rangle$ and $|\nu_{a2}\rangle \to |\nu_2(x)\rangle$,

$$|\nu_1(x)\rangle = a_L \exp(-i\int_{x_r}^x \omega_m(x')/2dx')\,|\nu_L(x)\rangle + a_H \exp(i\int_{x_r}^x \omega_m(x')/2dx')\,|\nu_H(x)\rangle$$
$$|\nu_2(x)\rangle = b_L \exp(-i\int_{x_r}^x \omega_m(x')/2dx')\,|\nu_L(x)\rangle + b_H \exp(i\int_{x_r}^x \omega_m(x')/2dx')\,|\nu_H(x)\rangle\,,$$

[39] Stephen J. Parke. Nonadiabatic Level Crossing in Resonant Neutrino Oscillations. Physical Review Letters, 57(10):1275–1278, September 1986. ISSN 0031-9007. DOI: 10.1103/PhysRevLett.57.1275. URL http://link.aps.org/doi/10.1103/PhysRevLett.57.1275

[40] Writing down the dimensionless equation, I have

$$i\partial_{\hat{x}}\Psi_{mf} = \frac{R_S\omega}{2}((\hat{\Delta} - \cos 2\theta_v)\boldsymbol{\sigma_3} + \sin 2\theta_v \boldsymbol{\sigma_1})\Psi_{mf}.$$

where the relations between the constants are determined using the condition that $|\nu_1(x)\rangle$ and $|\nu_2(x)\rangle$ are orthonormal[41], which leads to the conclusion that[42]

[41] I don't have a good argument for this requirement.

[42] It is easy to verify the results.

$$\begin{aligned} b_L &= -a_H^* \\ b_H &= a_L^* \\ |a_L|^2 &= -|a_H|^2. \end{aligned}$$

Electron neutrinos are produced in a dense region as $|\nu_e\rangle$, which are partially transformed to other the other neutrinos due to matter and the resonance then it propagates as if it satisfies the adiabatic condition again. The initial state in terms of light and heavy state is[43]

[43] The relation between θ_m and θ_v is given by

$$\omega_m \sin 2\theta_m = \omega \sin 2\theta_v.$$

$$|\Psi_m(x_0)\rangle = |\nu_e\rangle = \cos\theta_m(x_0)\,|\nu_L(x_0)\rangle + \sin\theta_m(x_0)\,|\nu_H(x_0)\rangle\,.$$

The final state right before the resonance is

$$|\Psi_m(x_{r-})\rangle = \cos\theta_m(x_0)\exp\left(-i\int_{x_0}^{x_{r-}}\frac{\omega_m(x)}{2}dx\right)|\nu_L(x_{r-})\rangle + \sin\theta_m(x_0)\exp\left(i\int_{x_0}^{x_{r-}}\frac{\omega_m(x)}{2}dx\right)|\nu_H(x_{r-})\rangle$$

After the resonance the state is described by the general jumping

$$\begin{aligned} |\Psi_m(x)\rangle =& \cos\theta_m(x_0)\exp\left(-i\int_{x_0}^{x_{r-}}\frac{\omega_m(x)}{2}dx\right)\left(a_L\exp(-i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\,|\nu_L(x)\rangle + a_H\exp(i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\,|\nu_H(x)\rangle\right) \\ &+ \sin\theta_m(x_0)\exp\left(i\int_{x_0}^{x_{r-}}\frac{\omega_m(x)}{2}dx\right)\left(-a_H^*\exp(-i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\,|\nu_L(x)\rangle + a_L^*\exp(i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\,|\nu_H(x)\rangle\right) \end{aligned}$$

in which the x_{r-} is actually x_r thus

$$\begin{aligned} |\Psi_m(x)\rangle =& \cos\theta_m(x_0)\exp\left(-i\int_{x_0}^{x_{r}}\frac{\omega_m(x)}{2}dx\right)\left(a_L\exp(-i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\,|\nu_L(x)\rangle + a_H\exp(i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\,|\nu_H(x)\rangle\right) \\ &+ \sin\theta_m(x_0)\exp\left(i\int_{x_0}^{x_{r-}}\frac{\omega_m(x)}{2}dx\right)\left(-a_H^*\exp(-i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\,|\nu_L(x)\rangle + a_L^*\exp(i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\,|\nu_H(x)\rangle\right) \end{aligned}$$

To calculate the survival probability it is easier to use flavor basis, hence we have another form of $|\Psi_m(x)\rangle$ which is

$$
\begin{aligned}
|\Psi_m(x)\rangle = &\left[\cos\theta_m(x_0)\exp\left(-i\int_{x_0}^{x_r}\frac{\omega_m(x')}{2}dx'\right)a_L\exp(-i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\right.\\
&\left.-\sin\theta_m(x_0)\exp\left(i\int_{x_0}^{x_{r-}}\frac{\omega_m(x')}{2}dx'\right)a_H^*\exp(-i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\right]|\nu_L(x)\rangle\\
&+\left[\cos\theta_m(x_0)\exp\left(-i\int_{x_0}^{x_r}\frac{\omega_m(x)}{2}dx\right)a_H\exp(i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\right.\\
&\left.+\sin\theta_m(x_0)\exp\left(i\int_{x_0}^{x_{r-}}\frac{\omega_m(x)}{2}dx\right)a_L^*\exp(i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\right]|\nu_H(x)\rangle\\
=&\left[\cos\theta_m(x_0)\exp\left(-i\int_{x_0}^{x_r}\frac{\omega_m(x)}{2}dx\right)a_L\exp(-i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\right.\\
&\left.-\sin\theta_m(x_0)\exp\left(i\int_{x_0}^{x_{r-}}\frac{\omega_m(x)}{2}dx\right)a_H^*\exp(-i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\right](\cos\theta_m(x)|\nu_e\rangle-\sin\theta_m(x)|\nu_x\rangle)\\
&+\left[\cos\theta_m(x_0)\exp\left(-i\int_{x_0}^{x_r}\frac{\omega_m(x)}{2}dx\right)a_H\exp(i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\right.\\
&\left.+\sin\theta_m(x_0)\exp\left(i\int_{x_0}^{x_{r-}}\frac{\omega_m(x)}{2}dx\right)a_L^*\exp(i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\right](\sin\theta_m(x)|\nu_e\rangle+\cos\theta_m(x)|\nu_x\rangle)
\end{aligned}
$$

Survival amplitude of electron neutrinos is given by[44]

[44] $\cos\theta_m$, $\sin\theta_m$ and ω_m are real but a_L and a_H are complex.

$$
\begin{aligned}
&\langle\Psi_m(0)\mid\Psi_m(x)\rangle\\
=&\langle\nu_e\mid\Psi_m(x)\rangle\\
=&\left[\cos\theta_m(x_0)\exp\left(-i\int_{x_0}^{x_r}\frac{\omega_m(x')}{2}dx'\right)a_L\exp(-i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\right.\\
&\left.-\sin\theta_m(x_0)\exp\left(i\int_{x_0}^{x_r}\frac{\omega_m(x')}{2}dx'\right)a_H^*\exp(-i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\right]\cos\theta_m(x)\\
&+\left[\cos\theta_m(x_0)\exp\left(-i\int_{x_0}^{x_r}\frac{\omega_m(x')}{2}dx'\right)a_H\exp(i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\right.\\
&\left.+\sin\theta_m(x_0)\exp\left(i\int_{x_0}^{x_r}\frac{\omega_m(x')}{2}dx'\right)a_L^*\exp(i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx')\right]\sin\theta_m(x)\\
=&A_L\exp\left(-i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx'\right)+A_H\exp\left(i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx'\right),
\end{aligned}
$$

where the coefficients are

$$
\begin{aligned}
A_L(x) &= \cos\theta_m(x)\left[a_L\cos\theta_m(x_0)\exp\left(-i\int_{x_0}^{x_r}\frac{\omega_m(x')}{2}dx'\right)-a_H^*\sin\theta_m(x_0)\exp\left(i\int_{x_0}^{x_r}\frac{\omega_m(x')}{2}dx'\right)\right]\\
A_H(x) &= \sin\theta_m(x)\left[a_H\cos\theta_m(x_0)\exp\left(-i\int_{x_0}^{x_r}\frac{\omega_m(x')}{2}dx'\right)+a_L^*\sin\theta_m(x_0)\exp\left(i\int_{x_0}^{x_r}\frac{\omega_m(x')}{2}dx'\right)\right].
\end{aligned}
$$

The detection is in a region where matter density is very small, thus we use $x\to\infty$ which means the effective mixing angle becomes vacuum mixing angle. The probability is the square of the amplitude,

$$\begin{aligned}P(\nu_e, x) &= |\langle \Psi_m(0) \mid \Psi_m(x)\rangle|^2 \\ &= |A_L(x)\exp\left(-i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx'\right) + A_H(x)\exp\left(i\int_{x_r}^{x}\frac{\omega_m(x')}{2}dx'\right)|^2 \\ &= |A_L(x)|^2 + |A_H(x)|^2 + A_L^*(x)A_H(x)\exp(2i\phi) + A_H^*(x)A_L(x)\exp(-2i\phi) \\ &= |A_L(x)|^2 + |A_H(x)|^2 + 2\mathbf{Re}(A_L^*(x)A_H(x)\exp(2i\phi)),\end{aligned}$$

where ϕ is defined as

$$\phi = \int_{x_r}^{x}\frac{\omega_m(x')}{2}dx'$$

Note that for any complex number $(a+ib)e^{i\phi} \equiv \rho e^{i\psi}$,

$$(a+ib)e^{i\phi} + c.c. = 2\rho\cos(\psi+\phi),$$

which means that the previous result can be simplified to

$$\begin{aligned}P(\nu_e, x) &= |A_L(x)|^2 + |A_H(x)|^2 + 2\mathbf{Re}(A_L^*(x)A_H(x)\exp(2i\phi)) \\ &= |A_L(x)|^2 + |A_H(x)|^2 + 2|A_L^*(x)A_H(x)|\cos\left(2\phi + \psi_{LH}\right),\end{aligned}$$

with the definition that $\psi_{LH}(x)$ is the argument of $A_L^*(x)A_H(x)$.

However the coefficients a_L and a_H are still not known yet. The trick is to average over the detection and production. The average over x removes the cos term and averages $\cos^2\theta_m(x)$ to $\frac{1}{2}$, which results in

$$\begin{aligned}\langle P(\nu_e, x)\rangle_x =& \cos^2\theta_m(x)(|a_H|^2\cos^2\theta_m(x_0) + |a_L|^2\sin^2\theta_m(x_0)) \\ &+ \sin^2\theta_m(x)(|a_H|^2\cos^2\theta_m(x_0) + |a_L|^2\sin^2\theta_m(x_0)) \\ &+ (-\cos^2\theta_m(x) + \sin^2\theta_m(x))\cos\theta_m(x_0)\sin\theta_m(x_0)(a_Ha_Le^{-2i\phi} + \text{c.c}).\end{aligned}$$

Applying the condition that $|a_L|^2 + |a_H|^2 = 1$, the probability becomes

$$\langle P(\nu_e, x)\rangle_x = \frac{1}{2} + \frac{1}{2}(1-2|a_H|^2)\cos 2\theta_m(x_0)\cos 2\theta_v - |a_Ha_L|\sin 2\theta_m(x_0)\cos 2\theta_v\cos(2\phi+\psi_{LH}),$$

where ψ_{LH} is the argument of a_Ha_L.

Notice that in fact the detection happens in vacuum, which means $\theta_m(x) = \theta_v$. The average over production removes the last part,[45]

$$\langle\langle P(\nu_e, x)\rangle_x\rangle_{x_0} = \frac{1}{2} + \frac{1}{2}(1-2|a_H|^2)\cos 2\theta_m(x_0)\cos 2\theta_v.$$

[45] In these averages it is important to keep the angles not averaged because these are the densities around the production and detection. The average only gets rid of the integral formed phase.

This means that the adiabatic result is of the form

$$P(\nu_e, x)_{\text{adiabatic}} = \frac{1}{2}(1 + \cos 2\theta_m \cos 2\theta_v).$$

Define a transition probability at resonance

$$P_r(\nu_L \to \nu_H) = |a_2|^2,$$

which is determined by the Landau-Zener transition to the first order.[46]

[46] As paper by Parke, the probability is

$$P_r(\nu_L \to \nu_H) = \exp\left(-\frac{\pi}{2}\frac{\sin^2 2\theta_v}{\cos 2\theta_v}\frac{\omega}{|\frac{1}{N}\frac{dN}{dx}x_r|}\right)$$

Numerical Method and Results

To simplify the codes, I prefer to write down all quantities dimensionless. In this spirit, the Hamiltonian becomes

$$\mathbf{H_m d} = \begin{pmatrix} \omega_{m1} & 0 \\ 0 & \omega_{m2} \end{pmatrix} + \mathbf{V_m},$$

where

$$\omega_{m2} = -\omega_{m1} \equiv \frac{\omega_m}{2} = \omega\sqrt{\frac{\hat{\Delta}^2 + 1}{4} - \frac{\hat{\Delta}}{2}\cos 2\theta_v}$$

while the effective potential $\mathbf{V_m}$ is

$$\begin{aligned}\mathbf{V_m} &= -i\mathbf{U_m^{-1}}\partial_x\mathbf{U_m} \\ &= -i\begin{pmatrix} \cos\theta(x) & -\sin\theta(x) \\ \sin\theta(x) & \cos\theta(x) \end{pmatrix}\partial_x\begin{pmatrix} \cos\theta(x) & \sin\theta(x) \\ -\sin\theta(x) & \cos\theta(x) \end{pmatrix}\end{aligned}$$

in which

$$\begin{aligned}\sin 2\theta(x) &= \frac{\sin 2\theta_v}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}} \\ \cos 2\theta(x) &= \frac{\cos 2\theta_v - \hat{\Delta}}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos 2\theta_v}}.\end{aligned}$$

and

$$\hat{\Delta} = \frac{\sqrt{2}G_F n(x)}{\omega}.$$

To compare with the theory, the first order effective potential is

$$\begin{aligned}\mathbf{V_m} &= -i\sin\theta_v\cos\theta_v\frac{d\hat{\Delta}}{dx}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= -i\sin\theta_v\cos\theta_v\sqrt{2}\frac{G_F}{\omega}\frac{dn(\hat{x})}{d\hat{x}}\frac{1}{R_S}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.\end{aligned}$$

Now we could write down the dimensionless equation

$$i\partial_{\hat{x}}\Psi(x) = (R_S\mathbf{H_m} + R_S\mathbf{V_m})\Psi(x),$$

which becomes a matrix equation given a set of basis

$$i\partial_{\hat{x}}\begin{pmatrix}\nu_L\\ \nu_H\end{pmatrix} = \left(\frac{R_S\omega}{2}\begin{pmatrix}-\hat{\omega}_m & 0\\ 0 & \hat{\omega}_m\end{pmatrix} - i\sin\theta_v\cos\theta_v\sqrt{2}\frac{G_F}{\omega}\frac{dn(\hat{x})}{d\hat{x}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\begin{pmatrix}\nu_L\\ \nu_H\end{pmatrix}$$

The solar electron number density from standard solar model is given in arXiv:astro-ph/001034.[47]

The data file[48] is listed in units of R_S and cm^{-3} while the density is actually $\log(n(x)/N_A)$ where N_A is Avogadro number.

Before any numerical calculations, I would like to do a simple analysis of the equation. First thing to do is to see if the second term in Hamiltonian is too small.

The radius of the sun is about $R_S = 695800\text{km}$ which can be approximated as[49]

$$R_S = 7\times10^{23}\text{fm} = \frac{7\times10^{23}}{0.2\text{GeV}} = 3.5\times10^{24}\text{GeV}^{-1}.$$

The frequency $\omega = \Delta m^2/2E$ is approximately

$$\omega = \frac{10^{-4}\text{eV}^2}{10^{-3}\text{GeV}} = 10^{-19}\text{GeV}.$$

Thus

$$R_S\omega \approx 10^5$$

Notice that the dimensionless frequency ω_m is

$$\hat{\omega}_m = \sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta}\cos2\theta_v}.$$

In the situation of the sun, where $\hat{\Delta} \ll 1$[50], we keep the first order of this quanitity,

$$\begin{aligned}\frac{\hat{\omega}_m}{2} &= \sqrt{\frac{\hat{\Delta}^2+1}{4} - \frac{\hat{\Delta}}{2}\cos2\theta_v}\\ &= \frac{1}{2} - \frac{1}{2}\cos2\theta_v\hat{\Delta} + \mathcal{O}(\hat{\Delta})^3.\end{aligned}$$

The derivative of number density of the sun $\frac{dn(\hat{x})}{d\hat{x}}$ is shown to be of the order -10^{26}cm^{-3}[51], i.e.,

$$\frac{dn(\hat{x})}{d\hat{x}} \approx -10^{26}\text{cm}^{-3} \approx -10^{-13}\text{GeV}^3.$$

[47] Data from this work is listed http://www.sns.ias.edu/ jnb/SNdata/sndata.html while the data file is actually in this file here http://www.sns.ias.edu/ jnb/SNdata/Export/BP2000/neordered.output .

[48] http://www.sns.ias.edu/ jnb/SNdata/Export/BP2000/neordered.output

[49] The conversion between SI and natrual untis is done by the relation $1\text{fm} = 1/0.2\text{GeV}$.

[50] Put in the numbers, $\hat{\Delta} \approx 10^{-19} \sim 10^{-11}$

[51] $1\text{cm}^{-3} = 2\times10^{-39}\text{GeV}^3$

We also have

$$\frac{G_F}{\omega} \approx \frac{10^{-5}\text{GeV}^{-2}}{10^{-19}\text{GeV}} = 10^{14}\text{GeV}^{-3}$$

The second term is of order[52]

$$\frac{G_F}{\omega}\frac{dn(\hat{x})}{d\hat{x}} \approx -10^{14}\text{GeV}^{-3} \times 10^{-23}\text{GeV}^3 = 10^{-9}.$$

So this means that I can not use the data from this paper since the data starts from too far away from the center.

Another way of doing the estimation is to find the resonance density, which means a maximum mixing of the two states, as the diagonal elements disappears,[53]

$$\Delta - \omega\cos 2\theta_v = 0.$$

By solving this equation, the number density of electrons should be of the order

$$n \approx \frac{\omega}{G_F} \approx 10^{-14}\text{GeV}^3.$$

In terms of cm^{-3}, the results would be of the order 10^{25}cm^{-3}.

Combining the two relations between density and pressure, we can find the density profile with respect to radius.

Numerical Calculations for Matter Effect in Flavor Basis

The previous method is somewhat more complicated in the calculations. As we have mentioned, another method is that we start from working in flavor basis.

The equation of motion in flavor basis is

$$i\partial_x \Psi_{mf}(x) = \mathbf{H_{mf}}\Psi_{mf}(x)$$

where

$$\mathbf{H_{mf}} = \left(\frac{\Delta}{2} - \frac{\omega}{2}\cos 2\theta_v\right)\boldsymbol{\sigma_3} + \frac{\omega}{2}\sin 2\theta_v\boldsymbol{\sigma_1}.$$

Writing down the dimensionless equation, I have

$$i\partial_{\hat{x}}\Psi_{mf} = \frac{R_S\omega}{2}((\hat{\Delta} - \cos 2\theta_v)\boldsymbol{\sigma_3} + \sin 2\theta_v\boldsymbol{\sigma_1})\Psi_{mf}.$$

As for the data of the sun I use a simple exponential distribution. The data is also from the paper by Bahcall which is shown in figure 12.

[52] $G_F = 1.17 \times 10^{-5}\text{GeV}^{-2}$

[53] The Hamiltonian in flavor basis is

$$\mathbf{H} = \frac{\omega}{2}(-\cos 2\theta_v\sigma_3 + \sin 2\theta_v\sigma_1) + \frac{\Delta}{2}\boldsymbol{\sigma_3} + \Delta\mathbf{I}$$

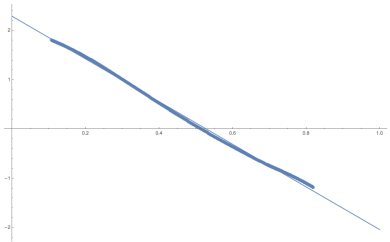


Figure 12: Solar electron density from Bahcall. Horizontal axis is the distance from the core of the sun normalized by the radius of the sun while the vertical axis is the number density of electrons in $\log_{10}(n/N_A)$. The best fit for the line is

$$2.3 - 4.3\hat{x}.$$

So the equation of the number density distribution is

$$n = N_A 10^{2.3-4.3\hat{x}}.$$

The model using just exponential is not accurate however it is enough to make the point in MSW resonance.

So I choose a solar model in which the core density is $n(0) = 10^{-13}\text{GeV}^3$. The distribution is

$$n = 10^{-13-4.3\hat{x}}\text{GeV}^3.$$

The numerical results can be obtains by plugging this result into the differential equation solver. The result is shown in figure 13.

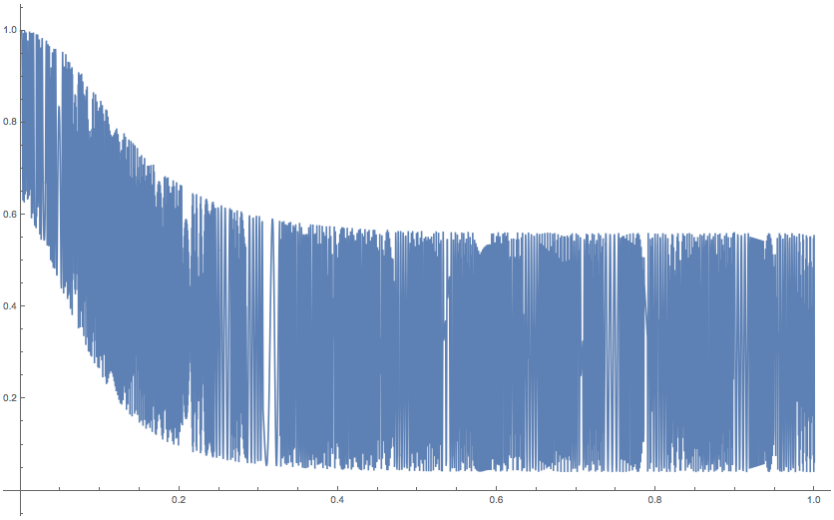


Figure 13: Numerical results for electron flavor neutrino when the electron density profile is $10^{-14-4.3\hat{x}}\text{GeV}^3$.

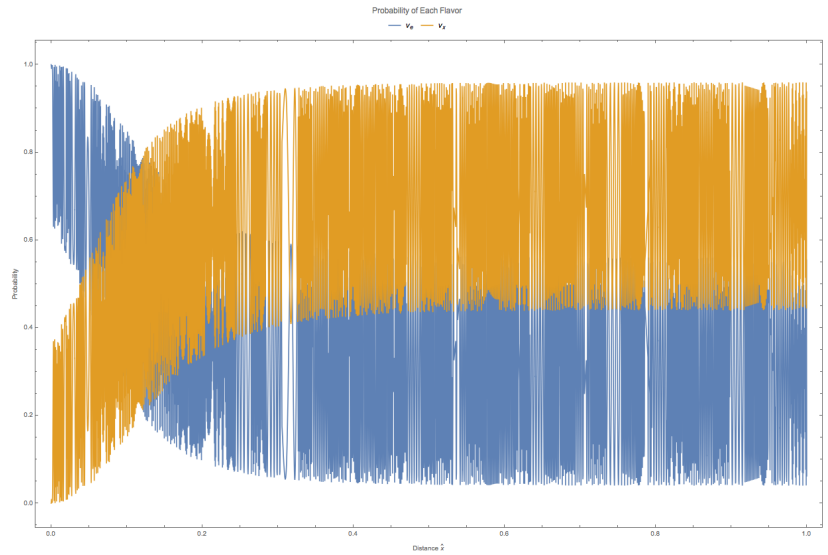


Figure 14: Numerical results for electron flavor neutrino probability and the other flavor neutrino probability when the electron density profile is $10^{-14-4.3\hat{x}}\text{GeV}^3$.

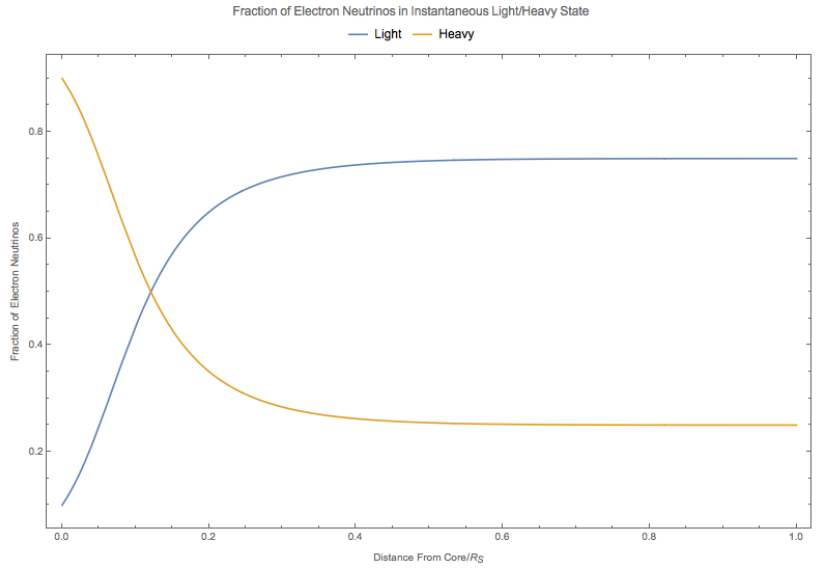


Figure 15: The fraction of electron flavor in a $|\nu_L\rangle$ or $|\nu_H\rangle$ state as the neutrino passing through the sun from the core. This clearly shows a conversion of neutrino flavor.

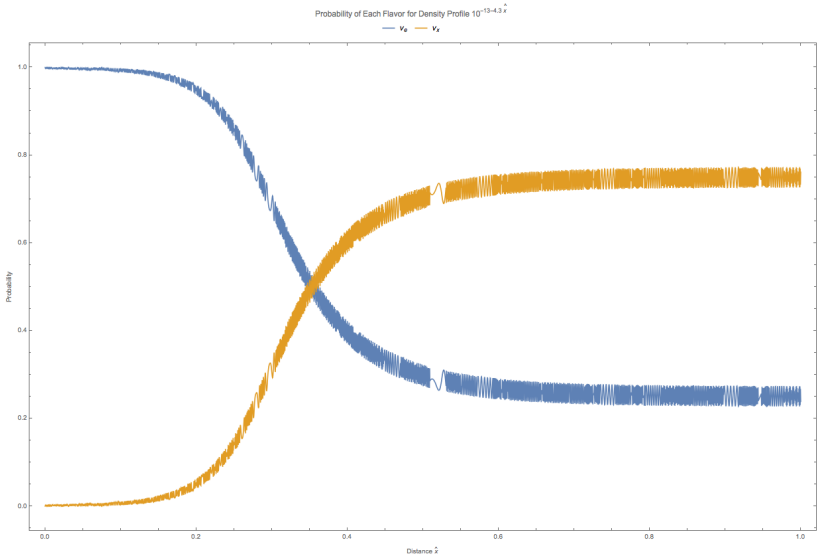


Figure 16: Number density profile $n(\hat{x}) = 10^{-13-4.3\hat{x}}\text{GeV}^3$.

Working in Vacuum Mass Eigenstates

The third method is to work in vacuum mass eigenstates.

Vacuum part of the Hamiltonian is

$$\mathbf{H_{mvv}} = \frac{1}{2E}\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix}$$

The matter interaction in flavor basis is

$$\mathbf{V_{mf}} = \sqrt{2}G_F n \text{diag} 1, 0, 0.$$

Thus to work in vacuum mass eigenstates, we need a transformation

$$\mathbf{V_{mv}} = \mathbf{U^{-1}V_{mf}U}.$$

Then the Hamiltonian becomes

$$\mathbf{H_m} = \begin{pmatrix} \frac{m_1^2}{2E} + \Delta U_{e1}^2 & \Delta U_{e1}U_{e2} & \Delta U_{e1}U_{e3} \\ \Delta U_{e2}U_{e1} & \frac{m_2^2}{2E} + \Delta U_{e2}^2 & \Delta U_{e2}U_{e3} \\ \Delta U_{e3}U_{e1} & \Delta U_{e3}U_{e2} & \frac{m_3^2}{2E} + \Delta U_{e3}^2 \end{pmatrix}$$

Trace of this Hamiltonian is $\text{Tr}(\mathbf{H_m}) = \frac{m_1^2+m_2^2+m_3^2}{2E} + \Delta$. To find the traceless part, we can use the relation**??**[54]

[54] arXiv:hep-ph/9910546

$$M = M_{traceless} + \frac{1}{N}\text{Tr}(M)I,$$

where N is the rank.

The traceless part of Hamiltonian becomes

$$\mathbf{H_m} = \begin{pmatrix} \Delta U_{e1}^2 - \frac{1}{3}\Delta + \frac{1}{3}(\frac{m_1^2-m_2^2+m_1^2-m_3^2}{2E}) & \Delta U_{e1}U_{e2} & \Delta U_{e1}U_{e3} \\ \Delta U_{e2}U_{e1} & \Delta U_{e2}^2 - \frac{1}{3}\Delta + \frac{1}{3}\frac{m_2^2-m_1^2+m_2^2-m_3^2}{2E} & \Delta U_{e2}U_{e3} \\ \Delta U_{e1}U_{e3} & \Delta U_{e2}U_{e3} & \Delta U_{e3}^2 - \frac{1}{3}\Delta + \frac{1}{3}\frac{m_3^2-m_1^2+m_3^2-m_2^2}{2E} \end{pmatrix}$$

Define the following quantities[55]

[55] However only two of them are linearly independent.

$$\begin{aligned} \Delta m_{12}^2 &= m_2^2 - m_1^2 \\ \Delta m_{23}^2 &= m_3^2 - m_2^2 \\ \Delta m_{13}^2 &= m_3^2 - m_1^2. \end{aligned}$$

We define an energy scale related to the radius of the sun

$$\epsilon_S = \frac{1}{R_S}.$$

The EoM can be written in a dimensionless manner,

$$i\partial_{\hat{x}}\Psi_m = \begin{pmatrix} \hat{\Delta}U_{e1}^2 - \frac{1}{3}\hat{\Delta} + \frac{1}{3}(\frac{\Delta m_{12}^2+\Delta m_{13}^2}{2E\epsilon_S}) & \hat{\Delta}U_{e1}U_{e2} & \hat{\Delta}U_{e1}U_{e3} \\ \hat{\Delta}U_{e2}U_{e1} & \hat{\Delta}U_{e2}^2 - \frac{1}{3}\hat{\Delta} + \frac{1}{3}\frac{\Delta m_{12}^2+\Delta m_{23}^2}{2E\epsilon_S} & \hat{\Delta}U_{e2}U_{e3} \\ \hat{\Delta}U_{e1}U_{e3} & \hat{\Delta}U_{e2}U_{e3} & \hat{\Delta}U_{e3}^2 - \frac{1}{3}\hat{\Delta} + \frac{1}{3}\frac{\Delta m_{13}^2+\Delta m_{23}^2}{2E\epsilon_S} \end{pmatrix},$$

where $\hat{\Delta} = \Delta/\epsilon_S$.

Numerical Results of 3 Flavor

The parameters for this calculation in units of $GeV^{whatever}$ are

$$\begin{aligned} numDen2(x) &= 10^{-12-4.3x} \\ epSun &= 10^{-24} \\ deltaH(x) &= \sqrt[2]{G_F}numDen2(x)/epSun \\ deltam12sq &= 7.6 \times 10^{-5} \times 10^{-18} \\ deltam13sq &= 2.3 \times 10^{-3} \times 10^{-18} \\ deltam23sq &= deltam13sq - deltam12sq \\ ene &= 10^{-3} \end{aligned}$$

For these parameters there is only resonance for $\Delta m_{13}^2 + \Delta m_{23}^2$.
A quick check over the different energy scales.

- Vacuum energy scales in normal hierarchy

$$\begin{aligned} \omega_{12} &= \frac{\Delta m_{12}^2}{2E} = 3.8 \times 10^{-20}\text{GeV} \\ \omega_{13} &= \frac{\Delta m_{13}^2}{2E} = 1.7 \times 10^{-18}\text{GeV} \\ \omega_{23} &= \frac{\Delta m_{23}^2}{2E} \approx \omega_{13} \end{aligned}$$

- Matter related scale for density profile $10^{-14-4.3\hat{x}}$

$$\Delta_1 = 1.6 \times 10^{-19-4.3\hat{x}} \in [1.6 \times 10^{-23.3}, 1.6 \times 10^{-19}]$$

- Matter related scale for density profile $10^{-13-4.3\hat{x}}$

$$\Delta_1 = 1.6 \times 10^{-18-4.3\hat{x}} \in [1.6 \times 10^{-22.3}, 1.6 \times 10^{-18}]$$

Applying a number density function $numDen2(x) = 10^{-13-4.3x}$ to the system, the small scale oscillations are revived,

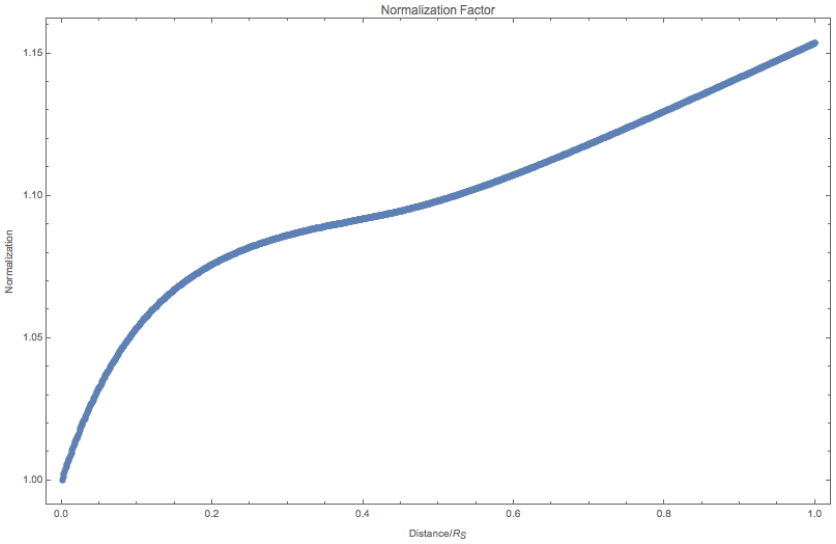


Figure 17: Normalization factor as a function of distance.

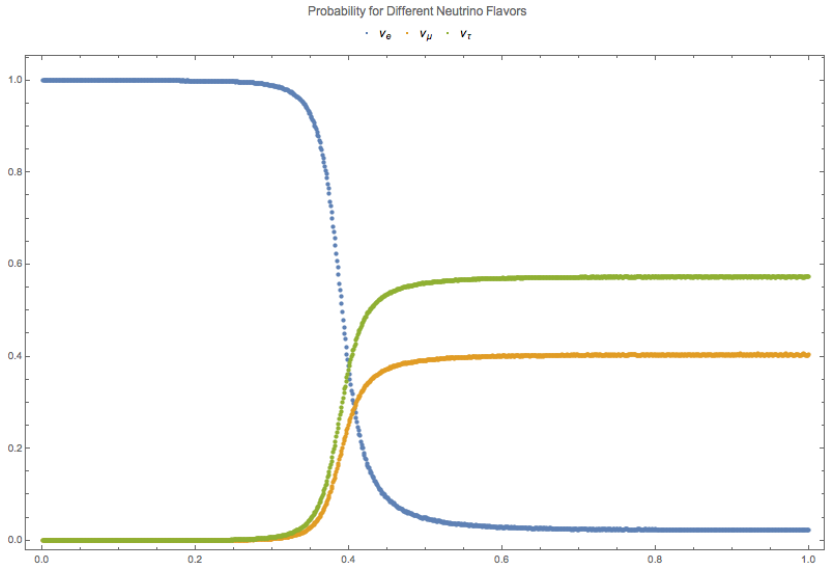


Figure 18: Probability for each flavor of neutrinos.

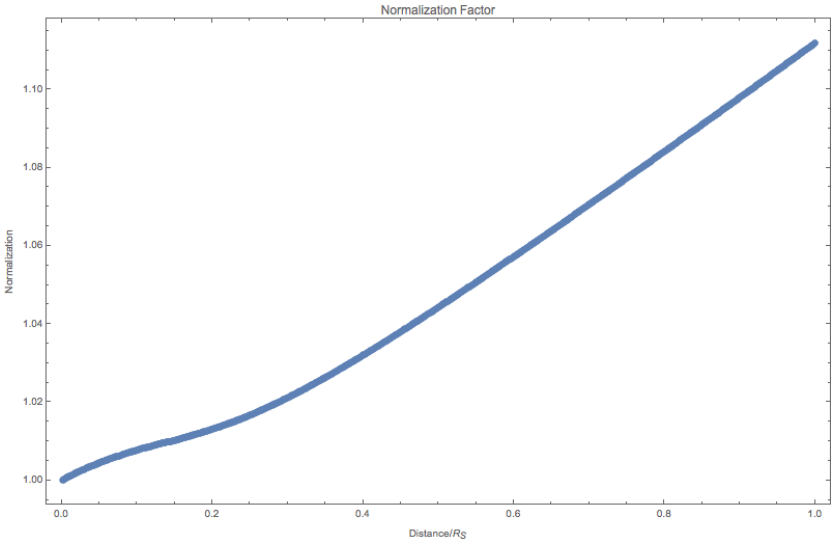


Figure 19: Normalization of the states for numerical 3 flavor oscillation in the sun with density profile $10^{-13-4.3x}$.

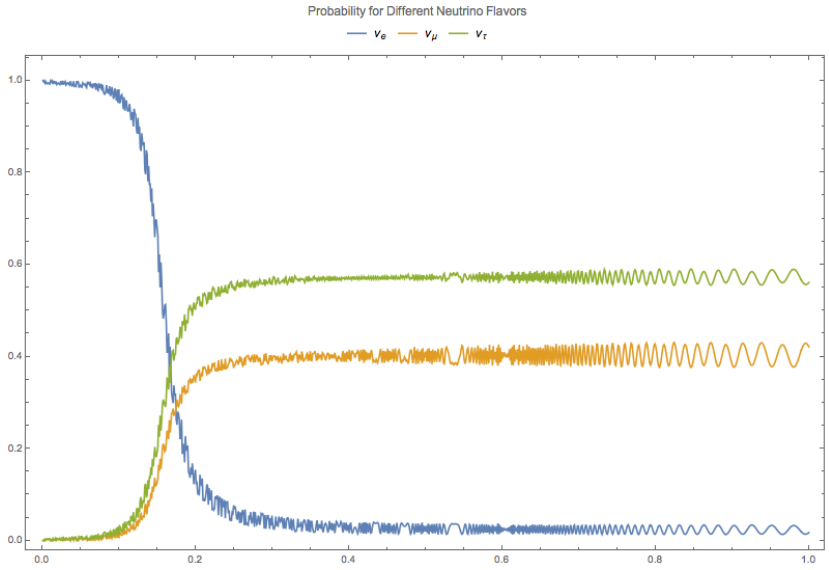


Figure 20: Numerical results for 3 flavor oscillation in the sun with density profile $10^{-13-4.3x}$.

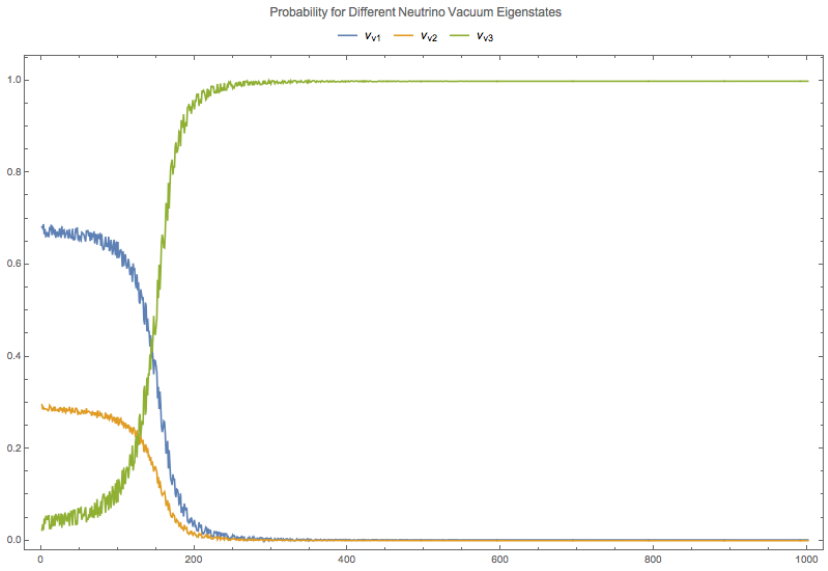


Figure 22: Survival probabilities for different vacuum mass eigenstates for 3 flavor oscillation in the sun with density profile $10^{-13-4.3x}$.

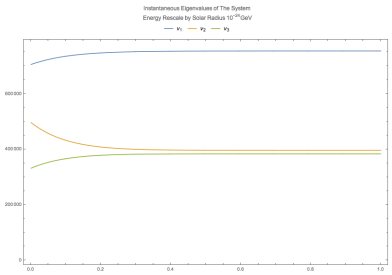


Figure 21: Eigenenergies for density profile $10^{-13-4.3x}$.

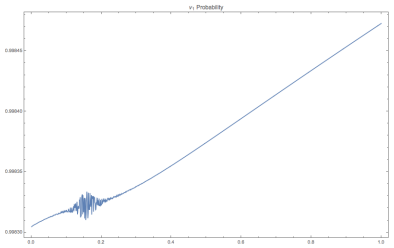


Figure 23: Probability for the first instantaneous eigenstate for matter profile $10^{-13-4.3x}$.

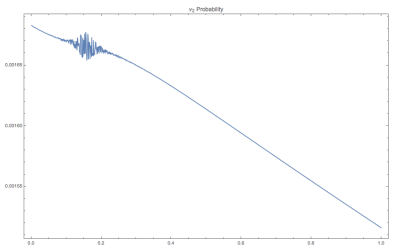


Figure 24: Probability for the second instantaneous eigenstate for matter profile $10^{-13-4.3x}$.

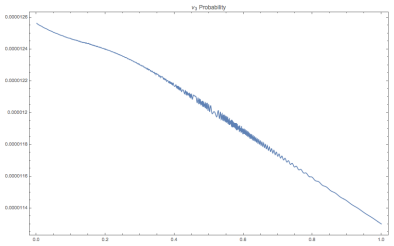


Figure 25: Probability for the third instantaneous eigenstate for matter profile $10^{-13-4.3x}$.

Ternary Diagrams

Decapitated - Solar Density Profile

The solar density profile can be determined using simple gravitational theory and nuclear physics models.

The mass contained in a spherical shell dr is

$$dm = 4\pi r^2 \rho dr, \tag{44}$$

which is assuming spherical symmetry.

The gravitational potential for each unit of mass can generate force as gravitational attraction, which is

$$\frac{dv}{dr} = \frac{Gm}{r^2}.$$

The balance between pressure and gravity shows that

$$\frac{dv}{dr}\rho + \frac{dp}{dr} = 0.$$

The result of these relations is a rather intuitive relation between density and pressure, which is not solvable until we find another relation between them.

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho.$$

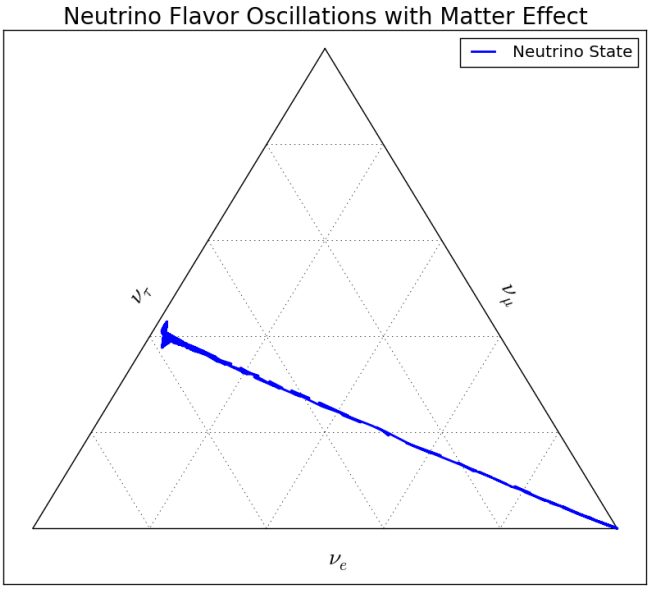


Figure 26: Ternary diagram for MSW effect.

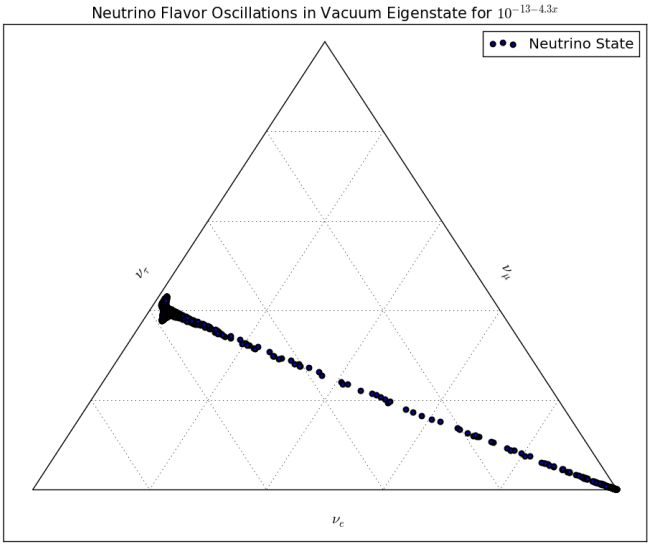


Figure 27: Ternary diagram for MSW effect.

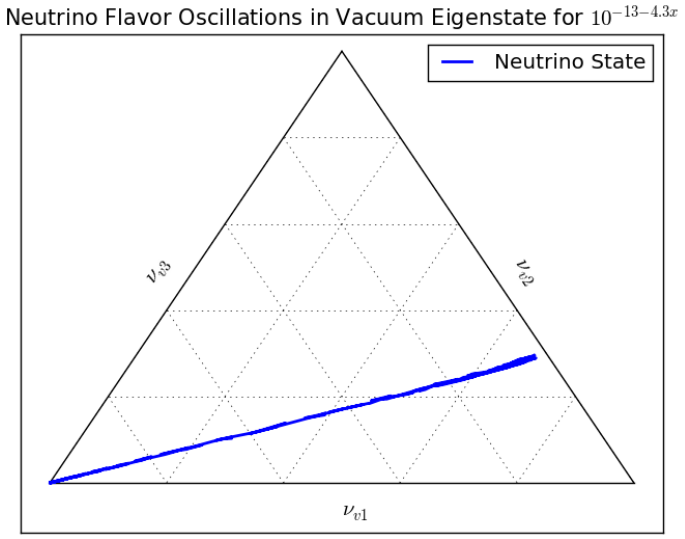


Figure 28: Ternary diagram for vacuum eigenstates

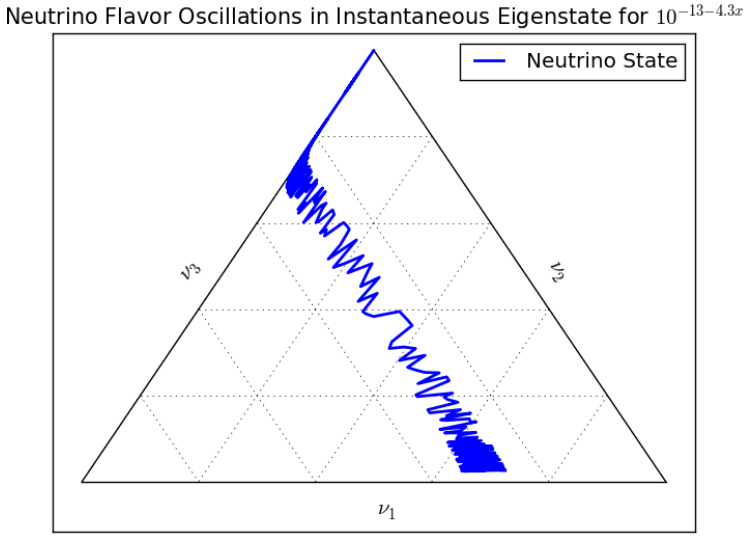


Figure 29: Ternary diagram for instantaneous eigenstates

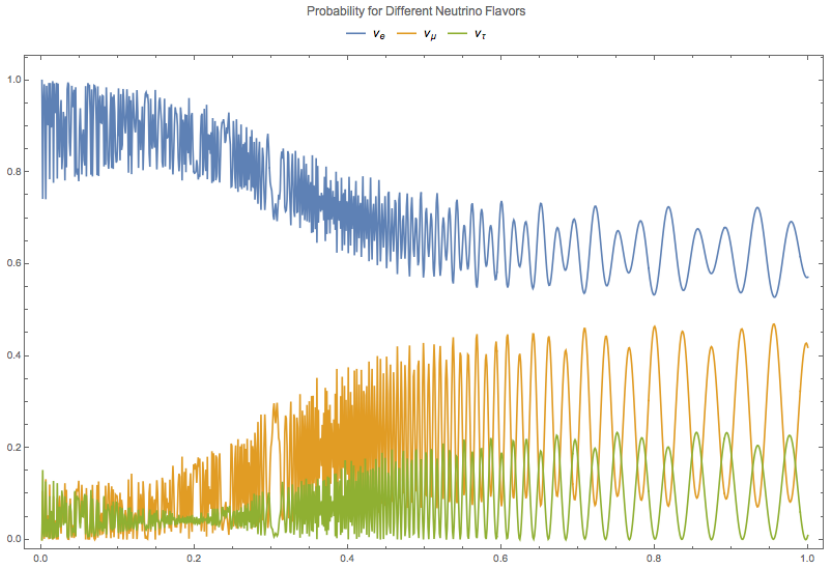


Figure 30: Numerical results for 3 flavor oscillation in the sun with density profile $10^{-14-4.3x}$.

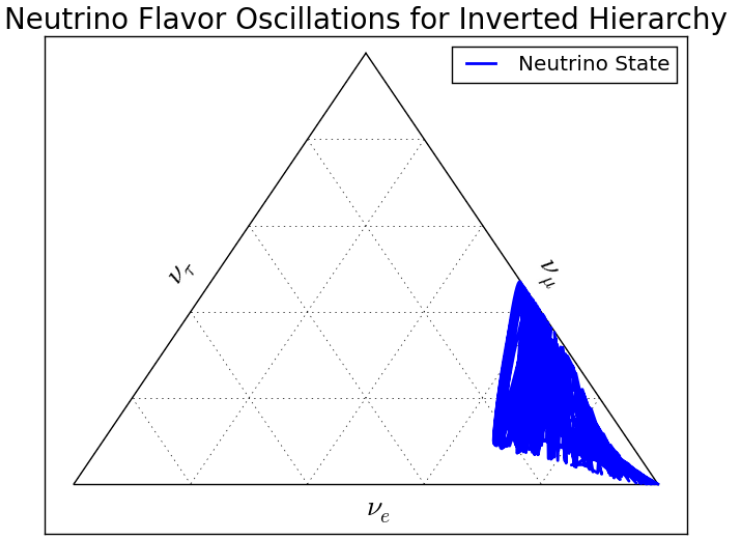


Figure 31: Ternary diagram for MSW effect with matter density profile $10^{-14-4.3x}$.

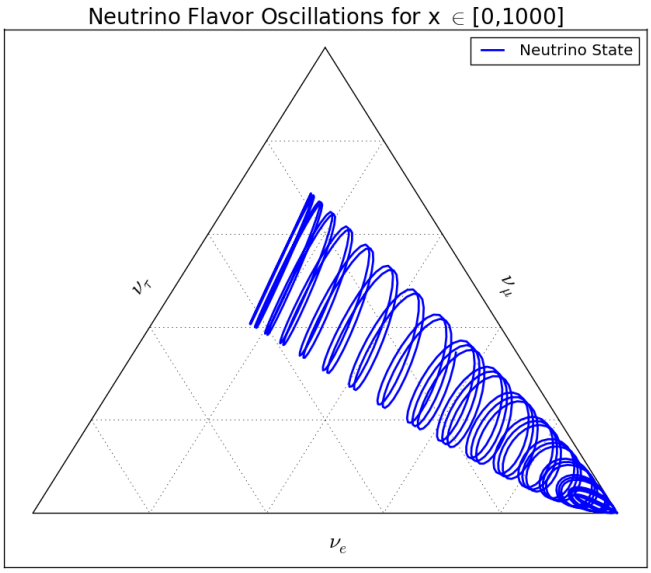


Figure 32: theta12 = 33.36/180*Pi;
theta13 = 8.66/180*Pi;
theta23 = 40/180*Pi;
deltacp = 0;
m1sq = 0.01;
m2sq = m1sq + 0.000079;
oneFourE = 100;

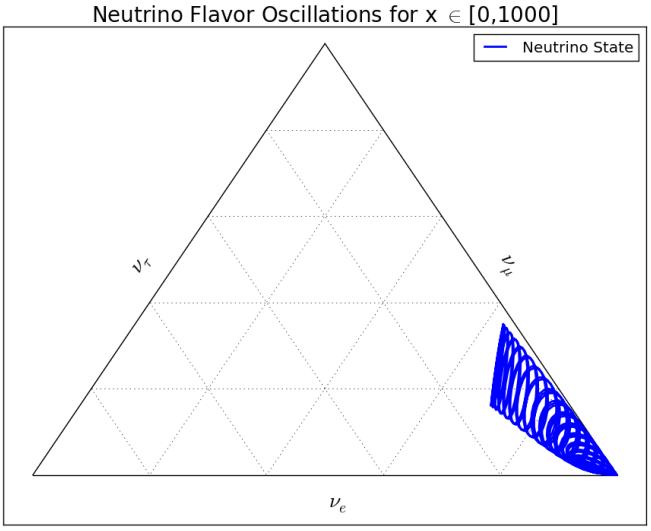


Figure 33: Reduce theta12 to half of the original value
theta12 = 33.36/180*Pi*1/2;
theta13 = 8.66/180*Pi;
theta23 = 40/180*Pi;
deltacp = 0;
m1sq = 0.01;
m2sq = m1sq + 0.000079;
oneFourE = 100;

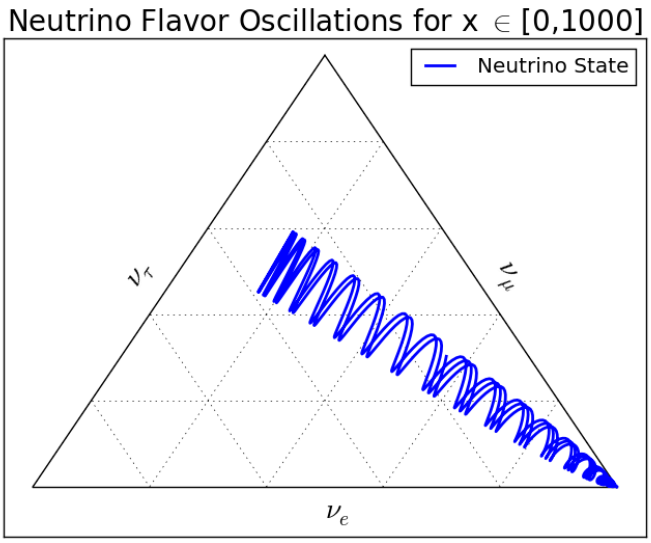


Figure 34: Reduce theta13
theta12 = 33.36/180*Pi;
theta13 = 8.66/180*Pi*1/2;
theta23 = 40/180*Pi;
deltacp = 0;
m1sq = 0.01;
m2sq = m1sq + 0.000079;
oneFourE = 100;

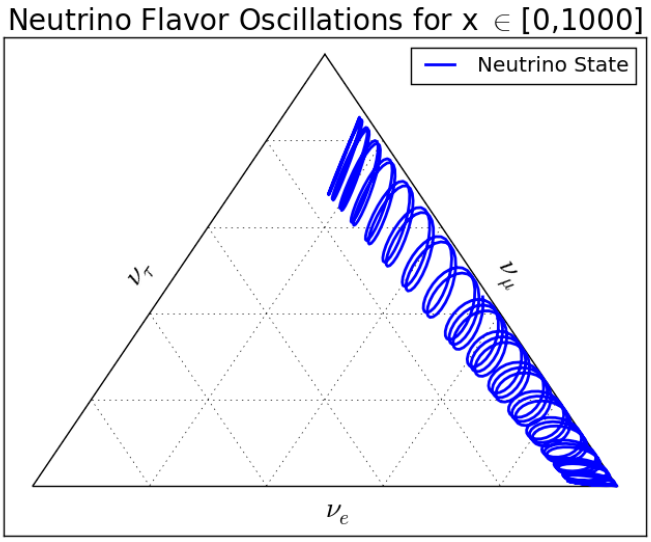


Figure 35: Reduce theta23
theta12 = 33.36/180*Pi;
theta13 = 8.66/180*Pi;
theta23 = 40/180*Pi*1/2;
deltacp = 0;
m1sq = 0.01;
m2sq = m1sq + 0.000079;
oneFourE = 100;

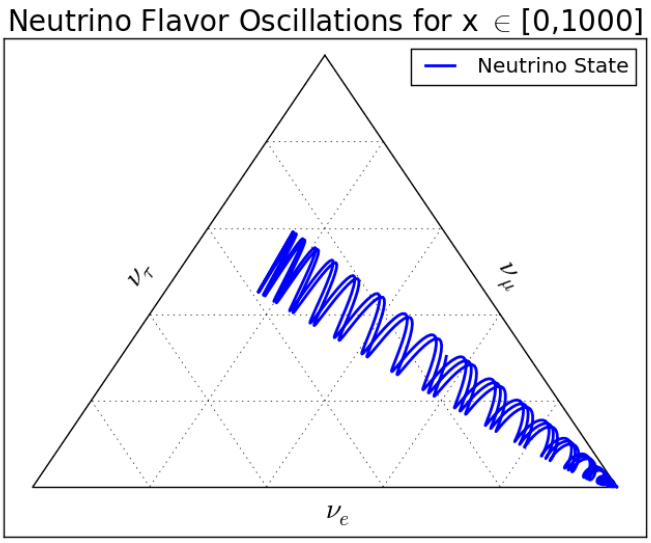


Figure 36: Increase $deltam2^2 - deltam1^2$
theta12 = 33.36/180*Pi;
theta13 = 8.66/180*Pi;
theta23 = 40/180*Pi;
deltacp = 0;
m1sq = 0.01;
m2sq = m1sq + 0.000079*10;
oneFourE = 100;

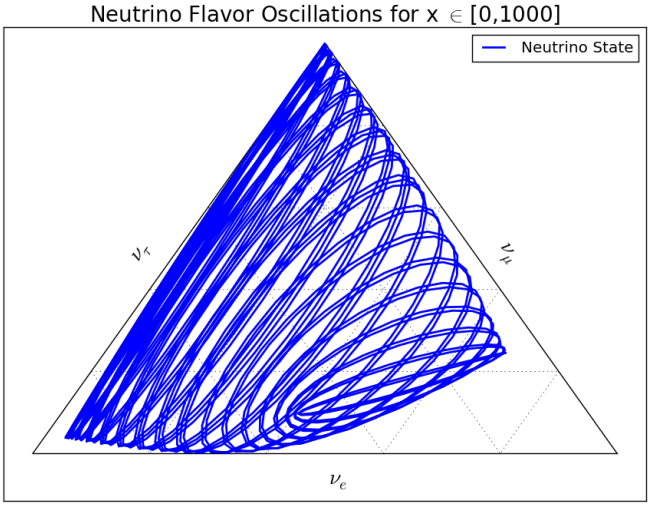


Figure 37: Starting from all ν_μ
theta12 = 33.36/180*Pi;
theta13 = 8.66/180*Pi;
theta23 = 40/180*Pi;
deltacp = 0;
m1sq = 0.01;
m2sq = m1sq + 0.000079;
oneFourE = 100;

To be simple, we will use Eddington model to solve the density profile, which gives us the relation

$$p = K\rho^{4/3},$$

where

$$K = \left(\left(\frac{k}{m_p} \frac{3}{\alpha} \frac{\beta}{\mu^4 (1-\beta)^4} \right)^4 \right)^{1/3}.$$

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