

Neutrino Oscillations in Vacuum and Matter ¹

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Notes for neutrino oscillations in vacuum and dense matter.

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Symbols and Definitions

- $\Delta = \sqrt{2}G_F n(x)$
- $\omega = \frac{\Delta m^2}{2E}$

Vacuum Oscillations

Schrodinger equation is

$$i\partial_t |\Psi\rangle = \mathbf{H} |\Psi\rangle, \quad (1)$$

where for relativistic neutrinos, the energy is²

$$\mathbf{H}^m = \begin{pmatrix} \sqrt{p^2 + m_1^2} & 0 & 0 \\ 0 & \sqrt{p^2 + m_2^2} & 0 \\ 0 & 0 & \sqrt{p^2 + m_3^2} \end{pmatrix},$$

² They all have the same momentum but different mass. The thing is we assume they have the same velocity since the mass is very small. To have an idea of the velocity difference, I can calculate the distance travelled by another neutrino in the frame of one neutrino.

Assuming the mass of a neutrino is 1eV with energy 10MeV, we will get a speed of $1 - 10^{-14}c$. This $10^{-14}c$ will make a difference about $3\mu\text{m}$ in 1s.

To Be Discussed!

Will decoherence happen due to this? For high energy neutrinos this won't be a problem however for low energy neutrinos this will definitely cause a problem for the wave function approach. Because the different mass

in which the energy terms are simplified using the relativistic condition

$$\sqrt{p^2 + m_i^2} = p \sqrt{1 + \frac{m_i^2}{p^2}} \quad (2)$$

$$\approx p \left(1 + \frac{1}{2} \frac{m_i^2}{p^2}\right). \quad (3)$$

In general the flavor eigenstates are the mixing of the mass eigenstates with a unitary matrix \mathbf{U} , that is

$$|\nu_\alpha\rangle = U_{\alpha i} |\nu_i\rangle, \quad (4)$$

where the α s are indices for flavor states while the i s are indices for mass eigenstates.

To find out the equation of motion for flavor states, plugin in the unitary transformation,

$$iU_{\alpha i}\partial_t |\nu_i\rangle = U_{\alpha i}H_{ij}^m |\nu_j\rangle. \quad (5)$$

I use index m for representation of Hamiltonian in mass eigenstates. Applying the unitary condition of the transformation,

$$\mathbf{I} = \mathbf{U}^\dagger \mathbf{U}, \quad (6)$$

I get

$$iU_{\alpha i}\partial_t |\nu_i\rangle = U_{\alpha i}H_{ij}^m U_{j\beta}^\dagger U_{\beta k} |\nu_k\rangle, \quad (7)$$

which is simplified to

$$i\partial_t |\nu_\alpha\rangle = H_{\alpha\beta}^f |\nu_\beta\rangle, \quad (8)$$

since the transformation is time independent.

The new Hamiltonian in the representations of flavor eigenstates reads

$$H_{\alpha\beta}^f = U_{\alpha i}^\dagger H_{ij}^m U_{j\beta}. \quad (9)$$

Survival Probability

The neutrino states at any time can be written as

$$|\Psi(t)\rangle = X_1 |\nu_1\rangle e^{-iE_1 t} + X_2 |\nu_2\rangle e^{-iE_2 t}, \quad (10)$$

where X_1 and X_2 are the initial conditions which are determined using the neutrino initial states.

Survival probability is the square of the projection on an flavor eigenstate,

$$P_\alpha(t) = |\langle \nu_\alpha | \Psi(t) \rangle|^2. \quad (11)$$

The calculation of this expression requires our knowledge of the relation between mass eigenstates and flavor eigenstates which we have already found out.

Recall that the transformation between flavor and mass states is

$$|\nu_i\rangle = U_{i\alpha}^{-1} |\nu_\alpha\rangle, \quad (12)$$

which leads to the inner product of mass eigenstates and flavor eigenstates,

$$\langle \nu_\alpha | \nu_i \rangle = \langle \nu_\alpha | U_{i\beta}^{-1} |\nu_\beta\rangle \quad (13)$$

$$= U_{i\beta}^{-1} \delta_{\alpha\beta} \quad (14)$$

$$U_{i\alpha} = U_{i\alpha}^{-1}. \quad (15)$$

The survival probability becomes

$$\begin{aligned} P_\alpha(t) &= |\langle \nu_\alpha | X_1 |\nu_1\rangle e^{-iE_1 t} X_2 |\nu_2\rangle e^{-iE_2 t} \rangle|^2 \\ &= |X_1 e^{-iE_1 t} \langle \nu_\alpha | \nu_1 \rangle + X_2 e^{-iE_2 t} \langle \nu_\alpha | \nu_2 \rangle|^2 \\ &= |\sum_i X_i e^{-iE_i t} U_{i\alpha}^{-1}|^2 \\ &= \sum_i X_i^* e^{iE_i t} U_{i\alpha}^{\dagger*} \sum_j X_j e^{-iE_j t} U_{j\alpha}^\dagger \\ &= |X_1|^2 U_{1\alpha}^{\dagger*} U_{1\alpha}^\dagger + |X_2|^2 U_{2\alpha}^{\dagger*} U_{2\alpha}^\dagger + X_1^* X_2 U_{1\alpha}^{\dagger*} U_{2\alpha}^\dagger e^{iE_1 t - iE_2 t} + X_2^* X_1 U_{2\alpha}^{\dagger*} U_{1\alpha}^\dagger e^{iE_2 t - iE_1 t} \end{aligned}$$

$U_{i\alpha}^{\dagger*}$ stands for the i th row and the α th column of the matrix $U^{\dagger*}$.

Two Flavor States

For 2 flavor neutrinos the Hamiltonian in the representation of propagation states,

$$\mathbf{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} = \begin{pmatrix} p_1 + \frac{1}{2} \frac{m_1^2}{p_1} & 0 \\ 0 & p_2 + \frac{1}{2} \frac{m_2^2}{p_2} \end{pmatrix}.$$

The equation of motion in matrix form is

$$i\partial_t \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} p_1 + \frac{1}{2} \frac{m_1^2}{p_1} & 0 \\ 0 & p_2 + \frac{1}{2} \frac{m_2^2}{p_2} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \quad (16)$$

The flavor eigenstate is a mixing of the propagation eigenstates,

$$\begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos \theta_v & \sin \theta_v \\ -\sin \theta_v & \cos \theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \quad (17)$$

Denote the rotation matrix using \mathbf{U} , the transformation can be written as

$$|\nu_\alpha\rangle = \mathbf{U}_{\alpha i} |\nu_i\rangle, \quad (18)$$

where α is for the flavor eigenstates and i is for the mass eigenstates.

The survival probability has been derived in previous section, which is the projection of propagation states onto flavor states.

For arbitrary initial condition,

$$\Psi(t=0) = A |\nu_a\rangle + B |\nu_b\rangle, \quad (19)$$

which can be rewritten into a matrix form,

$$\Psi(t=0) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} \quad (20)$$

To write down the projection, the relation

$$\begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos \theta_v & \sin \theta_v \\ -\sin \theta_v & \cos \theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \quad (21)$$

is needed. BTW, the inverse transformation is the transpose of \mathbf{U} since \mathbf{U} is unitary, thus we have the relation,

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \cos \theta_v & -\sin \theta_v \\ \sin \theta_v & \cos \theta_v \end{pmatrix} \begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} \quad (22)$$

Thus in the state can be written as

$$\Psi(t=0) = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \cos \theta_v & \sin \theta_v \\ -\sin \theta_v & \cos \theta_v \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \quad (23)$$

At any t , the state is

$$\Psi(t) = \begin{pmatrix} A \cos \theta_v - B \sin \theta_v & A \sin \theta_v + B \cos \theta_v \end{pmatrix} \begin{pmatrix} \nu_1 e^{-iE_1 t} \\ \nu_2 e^{-iE_2 t} \end{pmatrix} \quad (24)$$

$$= \begin{pmatrix} (A \cos \theta_v - B \sin \theta_v) e^{-iE_1 t} & (A \sin \theta_v + B \cos \theta_v) e^{-iE_2 t} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \quad (25)$$

The survival probability which is projection on a flavor state is written as

$$P(\nu_\alpha, t) = |\langle \nu_\alpha | \Psi(t) \rangle|^2. \quad (26)$$

The survival amplitude for ν_a is

$$\begin{aligned} & \langle \nu_a | \Psi(t) \rangle \\ &= \langle \nu_a | \left((A \cos \theta_v - B \sin \theta_v) e^{-iE_1 t} |\nu_1\rangle + (A \sin \theta_v + B \cos \theta_v) e^{-iE_2 t} |\nu_2\rangle \right) \\ &= (\cos \theta_v \langle \nu_1 | + \sin \theta_v \langle \nu_2 |) \left((A \cos \theta_v - B \sin \theta_v) e^{-iE_1 t} |\nu_1\rangle + (A \sin \theta_v + B \cos \theta_v) e^{-iE_2 t} |\nu_2\rangle \right) \end{aligned}$$

This is simple since the transformation matrix is real.

Applying the condition that the propagation eigenstates are orthonormal, the survival probability is

$$\begin{aligned} P(\nu_a, t) &= |\langle \nu_a | \Psi(t) \rangle|^2 \\ &= |\cos \theta_v (A \cos \theta_v - B \sin \theta_v) e^{-iE_1 t} + \sin \theta_v (A \sin \theta_v + B \cos \theta_v) e^{-iE_2 t}|^2 \\ &= |(A \cos^2 \theta_v - B \sin \theta_v \cos \theta_v) e^{-iE_1 t} + (A \sin^2 \theta_v + B \sin \theta_v \cos \theta_v) e^{-iE_2 t}|^2 \end{aligned}$$

In a special limit that $E_1 = E_2 = E$, the probability becomes

$$P(\nu_a, t) = |A|^2 \quad (27)$$

which is the same as initial probability since there is no mixing at all.

There are two kinds of initial conditions.

- The neutrinos are all in ν_a state initially, which means $A = 1, B = 0$. The survival probability simplifies to

$$\begin{aligned} P(\nu_a, t) &= |\cos^2 \theta_v e^{-iE_1 t} + \sin^2 \theta_v e^{-iE_2 t}|^2 \\ &= |\cos^2 \theta_v e^{-i(E_1 - E_2)t} + \sin^2 \theta_v|^2 \end{aligned}$$

As we have already discussed, $E_1 - E_2 = \frac{m_1^2 - m_2^2}{2p}$ assuming the neutrinos have the same momentum.³ Using the notation $\Delta m^2 = m_1^2 - m_2^2$ and the approximation that $E \approx p$, the survival

³ And here is a question.

probability can be rewritten as

$$\begin{aligned}
P(\nu_a, t) &= \cos^4 \theta_v + \sin^4 \theta_v + \cos^2 \theta_v \sin^2 \theta_v \left(e^{-i\Delta m^2 t/E} + e^{i\Delta m^2 t/E} \right) \\
&= 1 - 2 \cos^2 \theta_v \sin^2 \theta_v + 2 \cos^2 \theta_v \sin^2 \theta_v \cos \left(\frac{\Delta m^2 t}{2E} \right) \\
&= 1 - 2 \cos^2 \theta_v \sin^2 \theta_v \left(1 - \cos \left(\frac{\Delta m^2 t}{2E} \right) \right) \\
&= 1 - 4 \cos^2 \theta_v \sin^2 \theta_v \sin^2 \left(\frac{\Delta m^2 t}{4E} \right) \\
&= 1 - \sin^2(2\theta_v) \sin^2 \left(\frac{\Delta m^2 t}{4E} \right)
\end{aligned}$$

We always assuming that in the region of interest, all neutrinos are travelling with the same speed, i.e., the speed of light $c = 1$.⁴ Time is related to distance, $L = t$. Survival probability at distance L is

⁴ which is not true obviously

$$P(\nu_a, L) = 1 - \sin^2(2\theta_v) \sin^2 \left(\frac{\Delta m^2 L}{4E} \right) \quad (28)$$

- The neutrinos are all in ν_b state initially. Equivalently, we have $A = 0, B = 1$. Survival probability is

$$\begin{aligned}
P(\nu_a, t) &= |-\sin \theta_v \cos \theta_v e^{-iE_1 t} + \sin \theta_v \cos \theta_v e^{-iE_2 t}|^2 \\
&= \sin^2 \theta_v \cos^2 \theta_v |e^{-i(E_1 - E_2)t} - 1|^2 \\
&= \sin^2 \theta_v \cos^2 \theta_v \left(1 + 1 - e^{-i\Delta m^2 t/2E} - e^{i\Delta m^2 t/2E} \right) \\
&= 2 \sin^2 \theta_v \cos^2 \theta_v \left(1 - \cos \left(\frac{\Delta m^2 t}{2E} \right) \right) \\
&= \sin^2(2\theta_v) \sin^2 \left(\frac{\Delta m^2 t}{4E} \right) \\
&= \sin^2(2\theta_v) \sin^2 \left(\frac{\Delta m^2 L}{4E} \right)
\end{aligned}$$

Density Matrix

This problem can be solved using density matrix ρ and Von Neumann equation

$$i\partial_t \rho = [H, \rho]. \quad (29)$$

The initial condition for this equation is

$$\begin{aligned}
\rho(t=0) &= (A |\nu_a\rangle + B |\nu_b\rangle)(A^* \langle \nu_a| + B^* \langle \nu_b|) \\
&= AA^* |\nu_a\rangle \langle \nu_a| + BB^* |\nu_b\rangle \langle \nu_b| + AB^* |\nu_a\rangle \langle \nu_b| + A^* B |\nu_b\rangle \langle \nu_a|.
\end{aligned}$$

To calculate the propagation of the states, we need the Hamiltonian matrix in flavor basis.

This can be done by finding out how the Hamiltonian matrix transforms from one basis to another.

Using propagation basis,

$$i\partial_t |\Psi_p\rangle = H_p |\Psi_p\rangle. \quad (30)$$

The states are $|\Psi\rangle = \mathbf{U} |\Psi_p\rangle$ in flavor basis, which means we could plug in $|\Psi_p\rangle = \mathbf{U}^T |\Psi\rangle$.

$$i\partial_t \mathbf{U}^T |\Psi\rangle = H_p \mathbf{U}^T |\Psi\rangle.$$

Since $\mathbf{U}\mathbf{U}^T = \mathbf{I}$, we have a clean result by multiplying through the equation by \mathbf{U} .

$$i\partial_t |\Psi\rangle = \mathbf{U} H_p \mathbf{U}^T |\Psi\rangle.$$

So we define $H = \mathbf{U} H_p \mathbf{U}^T$ as the Hamiltonian matrix in flavor basis, which is

$$H = \left(p + \frac{m_1^2 + m_2^2}{4p} \right) \mathbf{I} - \frac{1}{4p} \begin{pmatrix} -\Delta m^2 \cos 2\theta & \Delta^2 m \sin 2\theta \\ \Delta m^2 \sin 2\theta & \Delta^2 m \cos 2\theta \end{pmatrix}. \quad (31)$$

The derivation of this is

$$\begin{aligned} \mathbf{H}_\alpha &= \mathbf{U} \hat{H}_j \mathbf{U}^T \\ &= \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \left(p \mathbf{I} + \frac{1}{2p} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \right) \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \\ &= p \mathbf{I} + \frac{1}{2p} \begin{pmatrix} \cos^2 \theta m_1^2 + \sin^2 \theta m_2^2 & -\sin \theta \cos \theta m_1^2 + \sin \theta \cos \theta m_2^2 \\ -\sin \theta \cos \theta m_1^2 + \sin \theta \cos \theta m_2^2 & \sin^2 \theta m_1^2 + \cos^2 \theta m_2^2 \end{pmatrix} \\ &= p \mathbf{I} + \frac{1}{2p} \begin{pmatrix} m_1^2 - \delta^2 m \sin^2 \theta & -\frac{1}{2} \sin 2\theta \delta m^2 \\ -\frac{1}{2} \sin 2\theta \delta m^2 & m_2^2 + \delta m^2 \sin^2 \theta \end{pmatrix} \\ &= p \mathbf{I} + \frac{1}{2p} \left(\frac{1}{2} (m_1^2 + m_2^2) \mathbf{I} - \frac{1}{2} \begin{pmatrix} -\delta m^2 \cos 2\theta & \delta^2 m \sin 2\theta \\ \delta m^2 \sin 2\theta & \delta^2 m \cos 2\theta \end{pmatrix} \right) \\ &= \left(p + \frac{m_1^2 + m_2^2}{4p} \right) \mathbf{I} - \frac{1}{4p} \begin{pmatrix} -\delta m^2 \cos 2\theta & \delta^2 m \sin 2\theta \\ \delta m^2 \sin 2\theta & \delta^2 m \cos 2\theta \end{pmatrix} \end{aligned}$$

Since identity matrix only shifts the eigenvalues we are only interested in the second term, thus the Hamiltonian we are going to use is

$$H = \frac{\Delta m^2}{4E} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}. \quad (32)$$

The equation of motion becomes

$$i\partial_t \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \frac{\Delta m^2}{4E} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \quad (33)$$

To solve this we need the eigenvalues and eigenvectors of the Hamiltonian matrix.

An Example of Survival Probability

Suppose the neutrinos are prepared in electron flavor initially, the survival probability of electron flavor neutrinos is calculated using the result I get previously.

Electron neutrinos are the lighter ones, then I have $a = e$ and denote $b = x$.⁵

The survival probability for electron neutrinos is

$$P(\nu_e, L) = 1 - \sin^2(2\theta_v) \sin^2\left(\frac{\Delta m^2 L}{4E}\right).$$

Numerical Results for 2 Flavor Oscillations

To solve a set of first order differential equations, I need the determinant of coefficient matrix. For 2 flavor neutrino oscillations,

$$\partial_t \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = -i\frac{\Delta m^2}{4E} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}.$$

To find the solutions I need the eigenvalues λ I need to find the determinant

$$\begin{aligned} & \det \left(-i\frac{\Delta m^2}{4E} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix} - \lambda \mathbf{I} \right) \\ &= \begin{vmatrix} -i\frac{\Delta m^2}{4E} \cos 2\theta - \lambda & i\frac{\Delta m^2}{4E} \sin 2\theta \\ i\frac{\Delta m^2}{4E} \sin 2\theta & i\frac{\Delta m^2}{4E} \cos 2\theta - \lambda \end{vmatrix}. \end{aligned}$$

By defining $\lambda' = \lambda / (-i\Delta m^2/4E)$, the determinant is

$$-\left(\frac{\Delta m^2}{4E}\right)^2 ((\cos 2\theta - \lambda')(-\cos 2\theta - \lambda') - \sin 2\theta \sin 2\theta).$$

The eigenvalues are the solutions to

$$-\left(\frac{\Delta m^2}{4E}\right)^2 ((\cos 2\theta - \lambda')(-\cos 2\theta - \lambda') - \sin^2 2\theta \sin 2\theta) = 0,$$

⁵ In the small mixing angle limit,

$$\begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

which is very close to an identity matrix. This implies that electron neutrino is more like mass eigenstate ν_1 . By ν_1 we mean the state with energy $\frac{\delta m^2}{4E}$ in vacuum.

whose solution is

$$\lambda' = \pm 1.$$

With the solutions

$$\lambda = \pm i \frac{\Delta m^2}{4E},$$

the eigenvectors can also be solved.

$$\begin{pmatrix} \cos 2\theta - 1 & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta - 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

gives us $\eta_2 = -\tan \theta \eta_1$, which means the eigenvectors are

$$\begin{pmatrix} 1 \\ -\tan \theta \end{pmatrix}, \begin{pmatrix} 1 \\ \cot \theta \end{pmatrix}. \quad (34)$$

The general solution of the first order differential equations is

$$\begin{pmatrix} 1 \\ -\tan \theta \end{pmatrix} e^{-i\Delta m^2 t/4E} + \begin{pmatrix} 1 \\ \cot \theta \end{pmatrix} e^{i\Delta m^2 t/4E}.$$

Initial condition is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and it determines the final solution

$$\begin{aligned} & \cos^2 \theta \begin{pmatrix} 1 \\ -\tan \theta \end{pmatrix} e^{-i\Delta m^2 t/4E} + \sin^2 \theta \begin{pmatrix} 1 \\ \cot \theta \end{pmatrix} e^{i\Delta m^2 t/4E} \\ &= \begin{pmatrix} \cos^2 \theta \\ -\sin \theta \cos \theta \end{pmatrix} e^{-i\Delta m^2 t/4E} + \begin{pmatrix} \sin^2 \theta \\ \sin \theta \cos \theta \end{pmatrix} e^{i\Delta m^2 t/4E} \end{aligned}$$

The survival probability of electron neutrino is

$$\begin{aligned} P &= |\cos^2 \theta e^{-i\Delta m^2 t/4E} + \sin^2 \theta e^{i\Delta m^2 t/4E}|^2 \\ &= |\cos^2 \theta e^{-i\Delta m^2 t/2E} + \sin^2 \theta|^2, \end{aligned}$$

which gets back to the result we had using the previous method.

Three Flavor States

For three flavor neutrinos, the oscillations matrix is 3 by 3 which is called the PMNS matrix.

$$\mathbf{U} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}. \quad (35)$$

The survival probability is given by the same derivation as the 2 flavor example.

Oscillations in Matter

The Hamiltonian should be determined first. We have already derived the Hamiltonian for vacuum oscillation,

$$H_v = \frac{\delta m^2}{2E} \frac{1}{2} \begin{pmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{pmatrix},$$

where we would like to define a new matrix,

$$\mathbf{B} = \frac{1}{2} \begin{pmatrix} -\cos 2\theta_v & \sin 2\theta_v \\ \sin 2\theta_v & \cos 2\theta_v \end{pmatrix},$$

so that the vacuum Hamiltonian can be written as

$$H_v = \frac{\delta m^2}{2E} \mathbf{B}.$$

The effect of matter, adds an extra term to this vacuum Hamiltonian which makes the electron population weighs more,

$$H_m = \sqrt{2} G_F n_e L.$$

Here we have ⁶

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Without emphasizing the self-interaction of the neutrinos, the Hamiltonian to be used is

$$H = H_v + H_m. \quad (36)$$

The equation of motion is simply the von Neumann equation

$$i\partial_t \rho = [H, \rho], \quad (37)$$

in which the ∂_t operator is actually ∂_x since we always assume neutrinos travel with speed of light c .

⁶ In principle we could shift the Hamiltonian using an identity matrix $a\mathbf{I}$ without change the eigenvectors. We will use an term $\frac{1}{2}\alpha\mathbf{e}_3$ instead in the following sections.

Analytic Solution

This Hamiltonian can be rewritten into a simple form using Pauli matrices,

$$\begin{aligned} \mathbf{H} &= \frac{\Delta m^2}{4E} (-\cos 2\theta_v \sigma_3 + \sin 2\theta_v \sigma_1) + \frac{\Delta}{2} \sigma_3 \\ &= \left(\frac{\Delta}{2} - \frac{\Delta m^2}{4E} \cos 2\theta_v \right) \sigma_3 + \frac{\Delta m^2}{4E} \sin 2\theta_v \sigma_1 \\ &= \left(\frac{\Delta}{2} - \frac{\omega}{2} \cos 2\theta_v \right) \sigma_3 + \frac{\omega}{2} \sin 2\theta_v \sigma_1, \end{aligned}$$

where $\Delta = \sqrt{2}G_F n(x)$ and $n(x)$ is the number density of the electrons.

To solve the equation of motion, this matrix should be diagonalized and its eigenvalues and eigenvectors should be identified. Since we have this Pauli matrices form, this can be done easily.

To see this effect quantitatively, we need to diagonalize this Hamiltonian (Can we actually diagonalize the equation of motion? NO!).

Equivalently, we can rewrite it in the basis of mass eigenstates $\{|\nu_L(x)\rangle, |\nu_H(x)\rangle\}$

⁷,

$$\begin{aligned} |\nu_L(x)\rangle &= \cos \theta(x) |\nu_e\rangle - \sin \theta(x) |\nu_\mu\rangle \\ |\nu_H(x)\rangle &= \sin \theta(x) |\nu_e\rangle - \cos \theta(x) |\nu_\mu\rangle. \end{aligned}$$

This new rotation in matrix form is

$$\begin{aligned} \begin{pmatrix} |\nu_L(x)\rangle \\ |\nu_H(x)\rangle \end{pmatrix} &= \begin{pmatrix} \cos \theta(x) & -\sin \theta(x) \\ \sin \theta(x) & \cos \theta(x) \end{pmatrix} \begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix} \\ &= \mathbf{U}_m^{-1} \begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{pmatrix}. \end{aligned}$$

To diagonalize it, we need to multiply on both sides the rotation matrix and its inverse, as we have done in the vacuum case,

$$\mathbf{H}_{md} = \mathbf{U}_m^{-1} \mathbf{H} \mathbf{U}_m.$$

The second step is to set the off diagonal elements to zero. By solving the equations we can find the $\sin 2\theta(x)$ and $\cos 2\theta(x)$.⁸

$$\begin{aligned} \mathbf{H}_{md} &= \mathbf{U}_m^{-1} (A_1 \sigma_1 + A_3 \sigma_3) \mathbf{U}_m \\ &= \begin{pmatrix} A_3 \cos 2\theta(x) - A_1 \sin 2\theta(x) & A_3 \sin 2\theta(x) + A_1 \cos 2\theta(x) \\ A_3 \sin 2\theta(x) + A_1 \cos 2\theta(x) & -A_3 \cos 2\theta(x) + A_1 \sin 2\theta(x) \end{pmatrix}, \end{aligned}$$

⁷ This is very different from the vacuum case since this one is local. In principle we can not use this transformation to diagonalize the Hamiltonian because the equation is a differential equation about x .

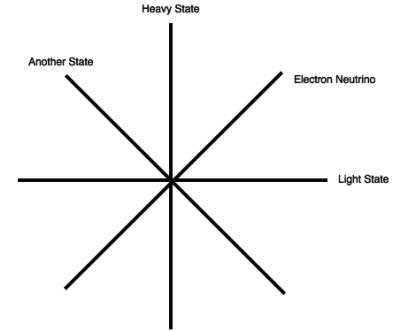


Figure 1: Due to this rotation, we can find an angle when the electron neutrinos becomes more important in Heavy state than Light state, which is $\theta = \frac{\pi}{4}$. For a mixing that has a larger angle, electron neutrinos actually take part in heavy neutrino state.

⁸ Just remember that $\Delta = \sqrt{2}G_F n(x)$ is a function of position.

where

$$A_3 = \frac{\Delta}{2} - \frac{\delta^2 m}{4E} \cos 2\theta_v = \frac{\Delta}{2} - \frac{\omega}{2} \cos 2\theta_v$$

$$A_1 = \frac{\delta^2 m}{4E} \sin 2\theta_v = \frac{\omega}{2} \sin 2\theta_v.$$

Set the off-diagonal elements to zero,

$$A_3 \sin 2\theta(x) + A_1 \cos 2\theta(x) = 0$$

So the solutions are

$$\sin 2\theta(x) = \frac{A_1}{\sqrt{A_1^2 + A_3^2}}$$

$$\cos 2\theta(x) = \frac{-A_3}{\sqrt{A_1^2 + A_3^2}}.$$

This diagonalizes the Hamiltonian LOCALLY.⁹ It's not possible to diagonalize the Hamiltonian globally if the electron number density is not a constant.

As $\Delta \rightarrow \infty$, we have $A_3 \rightarrow \infty$ and $\sin 2\theta(x)$ vanishes. Thus the neutrino will stay on the defined light and heavy eigenstates if the interaction with matter is much larger than the vacuum energy. On the other hand, $\Delta \rightarrow 0$, everything gets back to the vacuum case.

Now we can obtain an approximate solution by using this approximate diagonalization. This idea is

$$i\partial_x \Psi_m(x) = \text{ExtraMatrixFromLHS} \cdot \mathbf{H}_{md} \Psi_m(x),$$

where the **ExtraMatrixFromLHS** comes from the fact that changing from flavor basis $\Psi(x)$ to heavy-light basis $\Psi_m(x)$ using \mathbf{U}_m

$$i\partial_x (\mathbf{U}_m \Psi_m(x)) = H(\mathbf{U}_m \Psi_m(x))$$

only returns

$$i\partial_x \Psi_m(x) = \mathbf{H}_{md} \Psi_m(x) - i\mathbf{U}_m^{-1}(\partial_x \mathbf{U}_m) \Psi_m(x).$$

Combining the two terms on RHS,

$$i\partial_x \Psi_m(x) = \mathbf{H}_m \Psi_m(x),$$

where

$$\mathbf{H}_m = \mathbf{H}_{md} - i\mathbf{U}_m^{-1}(\partial_x \mathbf{U}_m).$$

⁹ The point is, for equation of motion, we have a differentiation with respect to position x ! So even we diagonalize the Hamiltonian, the equation of motion won't be diagonalized. An extra matrix will occur on the LHS and revert the diagonalization the Hamiltonian on RHS.

The only part inside $\mathbf{U}_m(x)$ that is space dependent is the number density of the electrons $n(x)$. **Thus we know immediately that the Hamiltonian is diagonalized if the number density is constant.**^{10,11,12}

Constant Electron Number Density

Suppose we have an environment with constant electron number density, the term $-i\mathbf{U}_m^{-1}(\partial_x \mathbf{U}_m)$ goes away. All we have is the diagonalized new Hamiltonian \mathbf{H}_{md} and the eigenvalues are easily obtained which are

$$\begin{aligned} E_1 &= A_3 \cos 2\theta(x) - A_1 \sin 2\theta(x) \\ E_2 &= -A_3 \cos 2\theta(x) + A_1 \sin 2\theta(x). \end{aligned}$$

The final result for these two eigenvalues are

$$\begin{aligned} E_1 &= -\sqrt{A_1^2 + A_3^2} \\ E_2 &= \sqrt{A_1^2 + A_3^2}, \end{aligned}$$

where

$$A_1^2 + A_3^2 = \frac{\Delta^2 + \omega^2}{4} - \frac{\Delta\omega}{2} \cos 2\theta_v.$$

There are two special cases. The first case is $\cos 2\theta_v = 0$ while the other one is $\cos 2\theta_v = 1$ which leads to $\sqrt{A_1^2 + A_3^2} = |\frac{\Delta-\omega}{2}|$ and the two energies becomes $\frac{\Delta-\omega}{2}$ and $\frac{\omega-\Delta}{2}$.¹³

As for the survival probability, the result has the same form as the vacuum case, which is

$$P_x(\nu_e, L) = 1 - \sin^2(2\theta_m) \sin^2\left(\frac{\omega L}{2}\right),$$

where $\theta_m = \theta(x)$ is the effective mixing angle which in fact doesn't depend on x if the matter profile is constant.¹⁴

Adiabatic Limit

In some astrophysical environments the electron number density changes very slowly which means the term $\mathbf{U}_m^{-1}\partial_x \mathbf{U}_m$ is much smaller than \mathbf{H}_{md} . By intuition we would expect that this term could be dropped to the lowest order. However to verify that we need to check the Taylor expansion of this gradient term which requires the general form of \mathbf{U}_m .

¹⁰ But even the electrons density is constant, the oscillation is different.

¹¹ It would be nice to find the expression of \mathbf{U}_m in order to play with the matter effect.

¹² In a two level system, suppose the Hamiltonian is

$$H_2 = H_{d2} + P_2,$$

where

$$\begin{aligned} H_{d2} &= \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \\ P_2 &= \begin{pmatrix} 0 & W(r) \\ W^*(r) & 0 \end{pmatrix} \end{aligned}$$

are the diagonalized Hamiltonian and the perturbation.

The diagonalized Hamiltonian will result in a system with two levels which are constant as a function of r , however the perturbation term provides the r dependent term which will cause a r dependent eigen energies.

¹³ This means the matter effect in this constant profile situation determines the difference between the eigen energies.

¹⁴ The term $\sin 2\theta_m$ has been derived previously.

$$\sin 2\theta(x) = \frac{\omega \sin 2\theta_v}{\sqrt{\omega^2 + \Delta^2 - 2\omega\Delta \cos 2\theta_v}}.$$

Before we actually do the expansion, make the quantities dimensionless would greatly help us.

$$\sin 2\theta(x) = \frac{\sin 2\theta_v}{\sqrt{\left(\frac{\Delta}{\omega}\right)^2 + 1 - 2\frac{\Delta}{\omega} \cos 2\theta_v}}$$

$$\cos 2\theta(x) = \frac{\cos 2\theta_v - \frac{\Delta}{\omega}}{\sqrt{\left(\frac{\Delta}{\omega}\right)^2 + 1 - 2\frac{\Delta}{\omega} \cos 2\theta_v}}.$$

Define $\hat{\Delta} = \frac{\Delta}{\omega}$, which represents the matter interaction strength compared to the vacuum oscillation.

$$\sin 2\theta(x) = \frac{\sin 2\theta_v}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta} \cos 2\theta_v}}$$

$$\cos 2\theta(x) = \frac{\cos 2\theta_v - \hat{\Delta}}{\sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta} \cos 2\theta_v}}.$$

$$\sin \theta(x) = \frac{\csc(2\theta_v) \sqrt{\frac{-\hat{\Delta} + \sqrt{\hat{\Delta}^2 - 2\hat{\Delta} \cos(2\theta_v) + 1} + \cos(2\theta_v)}{\sqrt{\hat{\Delta}^2 - 2\hat{\Delta} \cos(2\theta_v) + 1}}} \left(\hat{\Delta} + \sqrt{\hat{\Delta}^2 - 2\hat{\Delta} \cos(2\theta_v) + 1} - \cos(2\theta_v) \right)}{\sqrt{2}}$$

$$\cos \theta(x) = \frac{\sqrt{\frac{-\hat{\Delta} + \sqrt{\hat{\Delta}^2 - 2\hat{\Delta} \cos(2\theta_v) + 1} + \cos(2\theta_v)}{\sqrt{\hat{\Delta}^2 - 2\hat{\Delta} \cos(2\theta_v) + 1}}}}{\sqrt{2}}$$

This is extremely complicated if we take the derivative with respect to x . So we apply Taylor expansion.¹⁵

¹⁵ The i is there makes sure that $-i\partial_x \mathbf{U}_m(x)$ is real.

$$\frac{d \sin \theta(x)}{dx} = \frac{d \sin \theta(x)}{d\hat{\Delta}} \frac{d\hat{\Delta}}{dx}$$

$$= i \frac{d\hat{\Delta}}{dx} \left(-\cos \theta_v \sin^2 \theta_v - \frac{1}{2} ((1 + 5 \cos 2\theta_v) \cos \theta_v \sin^2 \theta_v) \hat{\Delta} + \mathcal{O}(\hat{\Delta}^2) \right)$$

$$\frac{d \cos \theta(x)}{dx} = \frac{d \cos \theta(x)}{d\hat{\Delta}} \frac{d\hat{\Delta}}{dx}$$

$$= i \frac{d\hat{\Delta}}{dx} \left(\cos^2 \theta_v \sin \theta_v + \frac{1}{2} \cos^2 \theta_v (-1 + 5 \cos 2\theta_v) \sin \theta_v \hat{\Delta} + \mathcal{O}(\hat{\Delta}^2) \right)$$

There is nothing in the coefficients that prohibits us from using the lowest order that $\frac{d\hat{\Delta}/dx}{\text{Energy Differences Between Two Energy Levels}} \ll 1$ since the Taylor series has finite values.¹⁶

¹⁶ No singularities possible. I assumed that $\sin \theta_v > 0$.

Solving the adiabatic limit is just the same as the constant limit but $\theta(x)$ is not constant which ensures changing energies. What would be interesting is the behavior of energies as the density profile is slowly changing.

$$E_1 = -\frac{\omega}{2} \sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta} \cos 2\theta_v}$$

$$E_2 = \frac{\omega}{2} \sqrt{\hat{\Delta}^2 + 1 - 2\hat{\Delta} \cos 2\theta_v}.$$

When the term $\hat{\Delta}$ is very small $1 - 2\hat{\Delta} \cos 2\theta_v$ will dominate and the whole term decreases. On the other hand as $\hat{\Delta}$ becomes large, $\hat{\Delta}^2$ will dominate and the whole term grows.

The survival probability for the light neutrinos would be

$$P_x(\nu_L, L) = 1 - \sin^2(2\theta(x)) \sin^2\left(\frac{\omega L}{2}\right).$$

Before we move on to higher order corrections, it would be nice to understand this phenomenon.

- The vacuum oscillation length can be extracted from vacuum oscillation survival probability. It is $L_v = \frac{2\pi}{\omega}$.
- In this problem we have another energy scale which is the interaction, Δ . Here we can define another characteristic length $l_m = \frac{2\pi}{\Delta}$.
- MSW resonance happens when the two character lengths are matching to each other. Another way to put it is that the term $\sin 2\theta(x)$ is minimized so that we have the smallest gap which leads to $\hat{\Delta} = \cos 2\theta_v$. Equivalently this is the relation

$$l_0 = l_m \cos 2\theta_v.$$

- At resonance, we have

$$\cos 2\theta(x) = 1$$

$$\sin 2\theta(x) = 0.$$

This is max mixing of the states which means that at the resonance point

$$\begin{pmatrix} \nu_L(x_{reso}) \\ \nu_H(x_{reso}) \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_x \end{pmatrix}$$

- Resonance conditions corresponds to a resonance density which is given by

$$n_e(x) = \frac{\omega}{\sqrt{2}G_F} \cos 2\theta_v \equiv n_0(E, \Delta m^2) \cos 2\theta_v,$$

where $n_0(E, \Delta m^2)$ is a characteristic number density which depends on the energy mixing angles and Δm^2 of the neutrinos.

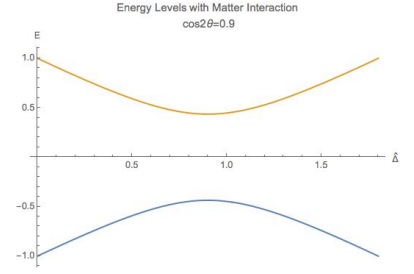


Figure 2: Energy Levels for MSW effect. We have the up-down symmetry since we shifted the energy by a constant to remove the identity matrix in the Hamiltonian.

- One should notice that to get back to the vacuum oscillation survival probability which means $\sin^2 2\theta(x) = \sin^2 2\theta_v$ we have¹⁷

$$\hat{\Delta}^2 + 1 - 2\hat{\Delta} \cos 2\theta_v = 1,$$

which leads to

$$\hat{\Delta} = 0 \quad \text{or} \quad 2 \cos 2\theta_v.$$

The first condition is trivial which corresponds to vacuum however the second condition $\Delta = 2 \cos 2\theta_v \omega$ means the interaction oscillation length is doubled compared to resonance point.

Nevertheless, we should always remember to check what survival probability the expression is describing. Here we have survival probability for $\nu_L(x)$. At $n(x) \rightarrow 0$ the oscillation becomes vacuum oscillation.

First Order Approximation

In principle we could use Taylor expansion of \mathbf{U}_m to find the series of the derivatives of $\cos \theta(x)$ and $\sin \theta(x)$, however it doesn't look like the easy way. Now we go back to find the Taylor expansion of the Hamiltonian which only involves the expression $\sin 2\theta(x)$ and $\cos 2\theta(x)$.

The series of them are

$$\begin{aligned} \sin 2\theta(x) &= \sin 2\theta_v + \cos 2\theta_v \sin 2\theta_v \hat{\Delta} + \mathcal{O}(\hat{\Delta})^2 \\ \cos 2\theta(x) &= \cos 2\theta_v - \sin^2 2\theta_v \hat{\Delta} + \mathcal{O}(\hat{\Delta})^2. \end{aligned}$$

Notice that for small θ_v the first order can be neglected since they are second or higher order small quantities.

$\sin \theta(x)$ and $\cos \theta(x)$ can be found using these results.

$$\begin{aligned} \sin \theta(x) &= \sin \theta_v + \cos^2 \theta_v \sin \theta_v \hat{\Delta} + \mathcal{O}(\hat{\Delta})^2 \\ \cos \theta(x) &= \cos \theta_v - \cos \theta_v \sin^2 \theta_v \hat{\Delta} + \mathcal{O}(\hat{\Delta})^2. \end{aligned}$$

To find out the solutions the explicit expression for \mathbf{U}_m is required. The rotation matrix is

$$\begin{aligned} \mathbf{U}_m &= \begin{pmatrix} \cos \theta(x) & \sin \theta(x) \\ -\sin \theta(x) & \cos \theta(x) \end{pmatrix} \\ &\approx \begin{pmatrix} \cos \theta_v - \cos \theta_v \sin^2 \theta_v \hat{\Delta} & \sin \theta_v + \cos^2 \theta_v \sin \theta_v \hat{\Delta} \\ -\sin \theta_v - \cos^2 \theta_v \sin \theta_v \hat{\Delta} & \cos \theta_v - \cos \theta_v \sin^2 \theta_v \hat{\Delta} \end{pmatrix}. \end{aligned}$$

¹⁷ I should check if this is still true in 3 flavor neutrino scenario. This could be interesting if it is respected in 3 flavor neutrino oscillations.

The derivative of this rotation matrix with respect to x is

$$\partial_x \mathbf{U}_{\mathbf{m}} = \frac{d\hat{\Delta}}{dx} \begin{pmatrix} -\cos \theta_v \sin^2 \theta_v & \cos^2 \theta_v \sin \theta_v \\ -\cos^2 \theta_v \sin \theta_v & -\cos \theta_v \sin^2 \theta_v \end{pmatrix}$$

The effective potential is¹⁸

$$\begin{aligned} \mathbf{V}_{\mathbf{m}} &= -i\mathbf{U}_{\mathbf{m}}^{-1}(\partial_x \mathbf{U}_{\mathbf{m}}) \\ &= -i \sin \theta_v \cos \theta_v \frac{d\hat{\Delta}}{dx} \begin{pmatrix} \cos \theta_v \sin \theta_v \hat{\Delta} & 1 \\ -1 & \cos \theta_v \sin \theta_v \hat{\Delta} \end{pmatrix}. \end{aligned}$$

¹⁸ This expression comes from the result of the first order approximation of $\sin \theta(x)$ and $\cos \theta(x)$.

Since $\frac{d\hat{\Delta}}{dx} \hat{\Delta}$ is one order higher than $\hat{\Delta}$, the effective potential can be simplified to first order, which is

$$\mathbf{V}_{\mathbf{m}} = -i \sin \theta_v \cos \theta_v \frac{d\hat{\Delta}}{dx} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The equation of motion up to first order of $\hat{\Delta}$ becomes

$$i\partial_x |\Psi_m\rangle = (\mathbf{H}_{\mathbf{md}} + \mathbf{V}_{\mathbf{m}}) |\Psi_m\rangle.$$

We have already solved

$$i\partial_x |\Psi_m\rangle = \mathbf{H}_{\mathbf{md}} |\Psi_m\rangle,$$

where the eigenstates are $|\nu_L\rangle$ and $|\nu_H\rangle$.

Decapitated - Solar Density Profile

The solar density profile can be determined using simple gravitational theory and nuclear physics models.

The mass contained in a spherical shell dr is

$$dm = 4\pi r^2 \rho dr, \tag{38}$$

which is assuming spherical symmetry.

The gravitational potential for each unit of mass can generate force as gravitational attraction, which is

$$\frac{dv}{dr} = \frac{Gm}{r^2}.$$

The balance between pressure and gravity shows that

$$\frac{dv}{dr} \rho + \frac{dp}{dr} = 0.$$

The result of these relations is a rather intuitive relation between density and pressure, which is not solvable until we find another relation between them.

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dp}{dr} \right) = -4\pi G\rho.$$

To be simple, we will use Eddington model to solve the density profile, which gives us the relation

$$p = K\rho^{4/3},$$

where

$$K = \left(\left(\frac{k}{m_p} \frac{3}{\alpha} \frac{\beta}{\mu^4(1-\beta)^4} \right)^4 \right)^{1/3}.$$

Combining the two relations between density and pressure, we can find the density profile with respect to radius.