THE CONNECTION BETWEEN MUELLER AND JONES MATRICES OF POLARIZATION OPTICS

R SIMON *

Department of Physics, Indian Institute of Science, Bangalore – 560012, India

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A necessary and sufficient condition for the 4 × 4 Mueller matrix to be derivable from the 2 × 2 Jones matrix is obtained This condition allows one to determine if a given Mueller matrix describes a totally polarized system or a partially polarized (depolarizing) system The result of Barakat is analysed in the light of this condition A recently reported experimentally measured Mueller matrix is examined using this condition and is shown to represent a partially polarized system

1 Introduction

It is well known that the description of systems of interest in polarization optics in terms of the Mueller matrix is applicable to more general situations than the description in terms of the Jones matrix The latter is confined to systems which do not depolarize, i e to systems which linearly map pure (polarization) states into pure states, and for all such systems (here after called totally polarized systems) the Mueller matrix can be derived from the Jones matrix [1–5]

It has been known for long [2,3] that a totally polarized Mueller matrix can have only seven ‡ independent parameters, and consequently, there should be nine independent relations between the sixteen real elements of such a matrix Only recently Barakat [1] in an important paper has derived, using the unimodular representation of the Lorentz group, a set of such relations To be more specific, Barakat has derived a matrix equation to be satisfied by the Mueller matrix derived from any non-singular Jones matrix This non-singularity restriction excludes perfect polarizers of every sort Since polarizers are among systems of interest, it will be desirable to have a treatment without requiring the Jones matrix to be non-singular

The converse of the problem treated by Barakat is of much practical interest given a Mueller matrix (experimentally measured one, say), how do we test whether it can be derived from a Jones matrix or not, equivalently, whether the system described by the Mueller matrix is totally polarized or not? Clearly what is needed here is a set of conditions on the elements of the Mueller matrix which are both necessary and sufficient, such a set is presented in this paper

In sec 2, we derive a bilinear matrix condition to be satisfied by the Mueller matrix which by construction forms both the necessary and the sufficient condition for it to be totally polarized This is valid without any restriction on the Jones matrix, and thus solves the converse of Barakat's problem In sec 3, the matrix equation of [1] is extended to include singular Jones matrices, i e perfect polarizers It is shown that this extended equation is a necessary but in general not a sufficient condition for the Mueller matrix to be totally polarized In the subsequent section a recently reported experimentally measured Mueller matrix is analysed using the method of sec 2 and it is shown that this Mueller matrix is partially polarized

* Permanent address Department of Physics, The American College, Madurai 625002, India

‡ This reduction in the number of independent parameters from sixteen to seven is due only to the requirement that the system does not depolarize There can be a further and independent reduction due to various symmetries in the system [6]

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2 Necessary and sufficient condition for a Mueller matrix to be derivable from a Jones matrix

Consider a non-image forming optical system, which does not depolarize, and hence is described by a Jones matrix J The Mueller matrix of the system derived from J is [4,5,1]

$$\mathsf{M} = \mathsf{A}(\mathsf{J} \otimes \mathsf{J}^*)\,\mathsf{A}^{-1}\,, \tag{1}$$

where $\otimes$ denotes the Kronecker matrix product, * denotes complex conjugation and

$$\mathsf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \tag{2}$$

It is easily seen that A is unitary except for a multiplication factor

$$\mathsf{A}^{-1} = \tfrac{1}{2}\,\mathsf{A}^{+} \tag{3}$$

where $^+$ denotes hermitian adjoint operation

We can go from a pair of indices, each running over 1 and 2, to a single index running over 0 to 3 and vice versa Thus, the elements of the 2 X 2 matrix J can be written as a column with $J_0 = J_{11}, J_1 = J_{12}, J_2 = J_{21}$ and $J_3 = J_{22}$ It is convenient to work with a 4 X 4 matrix N defined through

$$N_{ij,kl} = J_{ij} J^*_{kl}, \quad i,j,k,l = 1,2\,, \tag{4a}$$

or equivalently,

$$N_{\alpha\beta} = J_\alpha J^*_\beta, \quad \alpha,\beta = 0, \ldots, 3 \tag{4b}$$

It is easily seen that the elements of N are linearly related to the elements of M in the following unique way

$$\begin{aligned} N_{\alpha\beta} &= N_{ij,kl} = Q_{ik,jl} \\ &= A^{-1}_{ik,pq} M_{pq,rs} A_{rs,jl}\,, \end{aligned} \tag{5}$$

where summation over repeated indices is implied and

$$\mathsf{Q} = \mathsf{J} \otimes \mathsf{J}^* \tag{6}$$

Using eq (3) we can rewrite eq (5) to read

$$N_{ij,kl} = \tfrac{1}{2} A^*_{pq,ik} M_{pq,rs} A_{rs,jl} \tag{7}$$

Writing out eq (7) explicitly one obtains eq (8)

And from eq (8)

$$\mathrm{tr}(\mathsf{N}) = 2m_{00} \tag{9}$$

$$\mathsf{N} = \frac{1}{2}\begin{bmatrix} m_{00}+m_{11}+m_{01}+m_{10} & m_{02}+m_{12}+i(m_{03}+m_{13}) & m_{20}+m_{21}-i(m_{30}+m_{31}) & m_{22}+m_{33}+i(m_{23}-m_{32}) \\ m_{02}+m_{12}-i(m_{03}+m_{13}) & m_{00}-m_{11}-m_{01}+m_{10} & m_{22}-m_{33}-i(m_{23}+m_{32}) & m_{20}-m_{21}-i(m_{30}-m_{31}) \\ m_{20}+m_{21}+i(m_{30}+m_{31}) & m_{22}-m_{33}+i(m_{23}+m_{32}) & m_{00}-m_{11}+m_{01}-m_{10} & m_{02}-m_{12}+i(m_{03}-m_{13}) \\ m_{22}+m_{33}-i(m_{23}-m_{32}) & m_{20}-m_{21}+i(m_{30}-m_{31}) & m_{02}-m_{12}-i(m_{03}-m_{13}) & m_{00}+m_{11}-m_{01}-m_{10} \end{bmatrix} \tag{8}$$

By construction N is hermitian, and can be easily seen from eq (4b) to be a projection operator apart from a factor

$$\mathrm{N}^2 = \mathrm{tr(N)\,N} \tag{10}$$

Eq (10) implies that N has one non-zero positive eigenvalue and all other eigenvalues of N are zero, thus implying eq (4) In other words eq (10) is both the necessary and the sufficient condition for eq (4)

Conversely, starting from a given Mueller matrix M, we define N by eq (8), and substitution in the last equation gives a set of bilinear equations in the elements of M which are both the necessary and the sufficient set of conditions for M to be totally polarized That is, those and only those Mueller matrices satisfying the bilinear matrix condition eq (10) can be derived from a Jones matrix using eq (1) Eq (10) forms the main result of the present analysis This result is analogous to the condition for hamiltonian evolution derived by Sudarshan et al [7]

The present derivation does not place any restriction on J and hence is applicable to every system of interest in polarization optics Further, eq (10) is obtained as a direct consequence of eq (1) without making use of the properties of the Lorentz group and hence is applicable to higher dimensional matrices For instance, it is applicable to the case of electromagnetic waves with arbitrary forms [8] where the Jones matrix is a 3 X 3 matrix and the Mueller matrix is a 9 X 9 one

3 Analysis of the result of ref. [1]

We can now go on to analyse the matrix equation of ref [1] to see if it forms necessary and sufficient condition for an arbitrary Mueller matrix to be totally polarized We first prove that the matrix equation of ref [1] can be extended to include Mueller matrices derived from singular Jones matrices

We start with a system described by a Jones matrix J and observe that [1,4]

$$\mathbf{\Phi} = \tfrac{1}{2}\sigma_i s_i = \tfrac{1}{2}\begin{bmatrix} s_0 + s_1 & s_2 - \mathrm{i}s_3 \\ s_2 + \mathrm{i}s_3 & s_0 - s_1 \end{bmatrix}, \tag{11}$$

$$\mathbf{\Phi}' = \mathrm{J}\mathbf{\Phi}\mathrm{J}^+ \tag{12}$$

and

$$\mathrm{S}' = \mathrm{MS}, \tag{13}$$

where $\mathbf{\Phi}$ and $\mathbf{\Phi}'$ are the input and output coherency matrices, respectively, and s_i are the components of the Stokes vector S, M is the Mueller matrix of the system

Taking determinant of both sides of eq (12) gives

$$s_0'^2 - s_1'^2 - s_2'^2 - s_3'^2 = |\det \mathrm{J}|^2 (s_0^2 - s_1^2 - s_2^2 - s_3^2) \tag{14}$$

In matrix notation the last equation reads

$$\tilde{\mathrm{S}}'\mathrm{GS}' = |\det \mathrm{J}|^2 \tilde{\mathrm{S}}\mathrm{GS}, \tag{15}$$

where G is the Minkowski metric

$$\mathrm{G} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{16}$$

and $\tilde{\mathrm{S}}$ stands for the matrix transpose of S S′ in eq (15) can be written in terms of S using eq (13) to give

$$\tilde{\mathrm{S}}\tilde{\mathrm{M}}\mathrm{G\,MS} = |\det \mathrm{J}|^2 \tilde{\mathrm{S}}\mathrm{GS} \tag{17}$$

Since $\tilde{\mathrm{G}} = \mathrm{G}$, and eq (17) is true for all S, we obtain

$$\tilde{\mathrm{M}}\mathrm{GM} = |\det \mathrm{J}|^2\, \mathrm{G} \tag{18}$$

The last matrix condition can be usefully written as one involving only M To eliminate $|\det \mathrm{J}|^2$, take the trace of both sides of eq (18) to obtain

$$|\det \mathrm{J}|^2 = -\tfrac{1}{2}\,\mathrm{tr}(\tilde{\mathrm{M}}\mathrm{GM}) \tag{19}$$

Eq (18) can now be written as

$$\tilde{\mathrm{M}}\mathrm{GM} = -\tfrac{1}{2}\,\mathrm{tr}(\tilde{\mathrm{M}}\mathrm{GM})\,\mathrm{G} \tag{20}$$

We note that eq (20) is identical with eq (17) of ref [1] (with obvious correction of a misprint there) and leads to eqs (18a)–(18f) and (19a)–(19c) of that reference when written out in component form The present derivation clearly shows that eq (20) is valid for all totally polarized Mueller matrices including the ones derived from singular J This is to be expected as can be seen by repeating the treatment of ref [1] assuming non-zero value of $|\det \mathrm{J}|$ and taking the limit $|\det \mathrm{J}| \to 0$ in the final result

Since every totally polarized Mueller matrix is shown to satisfy eq (20), it is a necessary condition for a Mueller matrix to be totally polarized But it is not a sufficient condition in general as we presently show

To analyse the sufficiency question we rewrite eq (20) as

$$\widetilde{M}GM = \gamma^2 G \tag{21}$$

Let a given M satisfy eq (21) There are two cases to be distinguished

Case (i) $\gamma \neq 0$

In this case from eq (21), which is the same as eq (17) of ref [1], M/γ is seen to be a Lorentz matrix, and $M_{00} \geqslant 0$ always Then from the unimodular representation of the Lorentz group [9] it is clear that there exists a J related to M through eq (1) with $|\det J| = \gamma$, and hence M is totally polarized

Case (ii) $\gamma = 0$

In this case we can explicitly construct a class of matrices which satisfy eq (20), but are not totally polarized

Consider the Jones matrix J_p of a perfect polarizer of some sort, and let M_p be the Mueller matrix derived from J_p Clearly M_p should obey eq (20), and since $\gamma = |\det J_p| = 0$, using eq (19) we obtain

$$\widetilde{M}_p G M_p = 0 \tag{22}$$

Now we can construct a class of Mueller matrices through

$$M = M_p M_1 \tag{23}$$

where M_1 is an arbitrary Mueller matrix By virtue of eq (22), M satisfies eq (20) with $\gamma = 0$, whatever M_1 be However, if M_1 depolarizes, M_1 and hence M cannot be derived from any Jones matrix through eq (1), and hence M will not be totally polarized This shows that eq (20) is not sufficient condition when $\gamma = 0$

As an illustration, consider the Mueller matrix

$$M_{XDP} = M_X M_{DP}\,, \tag{24}$$

where M_X is the X-polarizer and M_{DP} is the ideal perfect depolarizer [10]

$$M_{XDP} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{25}$$

M_{XDP} satisfies eq (21) with $\gamma = 0$ The N matrix can be computed from eq (8) and we obtain

$$N^2 - \mathrm{tr}(N)\,N = -\frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

The last equation shows that M_{XDP} does not satisfy eq (10) and hence is not totally polarized even though it satisfies eq (21) The reader can directly verify that there exists no Jones matrix from which M_{XDP} can be derived using eq (1)

4 Application

To apply eq (10) to a given Mueller matrix we first compute N from eq (8) and then form

$$R = N^2 - \mathrm{tr}(N)\,N = A + iB \tag{27}$$

Since N is hermitian, A will be symmetric and B skew symmetric For example, for the Howell matrix [11]

$$M_s = \begin{bmatrix} 0\,7599 & -0\,0623 & 0\,0295 & 0\,1185 \\ -0\,0573 & 0\,4687 & -0\,1811 & -0\,1863 \\ 0\,0384 & -0\,1714 & 0\,5394 & 0\,0282 \\ 0\,1240 & -0\,2168 & -0\,0120 & 0\,6608 \end{bmatrix} \tag{28}$$

we obtain

$$A = \begin{bmatrix} -0\,1613 & 0\,1251 & 0\,1253 & -0\,2026 \\ 0\,1251 & -0\,1525 & 0\,0631 & -0\,1245 \\ 0\,1253 & 0\,0631 & -0\,1524 & -0\,1179 \\ -0\,2026 & -0\,1245 & -0\,1179 & -0\,1352 \end{bmatrix} \tag{29}$$

and

$$B = \begin{bmatrix} 0\,0000 & 0\,1287 & -0\,1248 & -0\,0017 \\ -0\,1287 & 0\,0000 & -0\,0297 & 0\,1276 \\ 0\,1248 & 0\,0297 & 0\,0000 & -0\,1251 \\ 0\,0017 & -0\,1276 & 0\,1251 & 0\,0000 \end{bmatrix} \tag{30}$$

To the extent that $R = A + iB$ is different from the null matrix, the Howell system depolarizes and the Howell matrix cannot be derived from any Jones matrix

5 Conclusion

We have shown that eq (10) is the necessary and the sufficient condition for a given Mueller matrix to represent a totally polarizable optical system We have also shown that the matrix equation of Barakat can be extended to include the singular Jones matrices This extended equation, however, is only a necessary condition and not a sufficient condition as we have shown by constructing a class of Mueller matrices which satisfy this extended equation but are not totally polarized When a given Mueller matrix does not satisfy the matrix condition eq (10), it means the system depolarizes It seems desirable to have a quantitative measure of the extent to which a system depolarizes Work in this direction is in progress and will be reported in a subsequent paper

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