OPTICAL ALGEBRA*

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There have been three major lines of attack on the problem of obtaining a useful and satisfying mathematical representation of light and the instruments through which it passses. Wiener[1] developed a generalized harmonic analysis and a statistical description of light. He introduced the coherency matrix. His work follows the lead of Rayleigh, Schuster and others. Jones[2] developed a matrix-vector calculus of monochromatic plane waves based on the electric vector. Mueller[3] developed a phenomenological matrix-vector calculus based on the Stokes[4] vector which is defined in terms of observable light intensities. The work of Jones and Mueller is characterized by the explicit recognition of the role of the instrument and its representation by a matrix. All these researches are primarily concerned with polarization or coherence, not with images.

The central purpose of the present research was to determine the mathematical relation between the work of Wiener, Jones and Mueller. The interrelation was discovered. The result is an algebraic formulation of optical statistics in which the results of Wiener, Jones and Mueller play natural and essential roles. As a result of the connection it is now possible to compute theoretically the expected values of the observables defined phenomenologically by Mueller. Indeed, one has what appears to be the beginning of a statistical optics which bears the same relation to phenomenological optics that statistical mechanics bears to thermodynamics.[4]

The overall structure of optical algebra is most readily apparent from Fig. 1. The algebras have been associated with the names of Jones and Mueller, the original investigators. However, at the beginning of the present research, only the Jones frequency algebra and the elementary Mueller frequency algebra existed. The remainder of the structure is new.

Jones Frequency Algebra. Jones frequency algebra is a mathematical system in which the complex vector function

(1) $$E(\omega) = (F_1(\omega), F_2(\omega)) \;.=.\; \text{Maxwell vector}^5$$

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[1] Wiener, N.: Acta Math. **55:** 117 (1930); Extensive bibliography.

[2] Jones, R. C.: Jour. Opt. Soc. Am., **31:** 488 (1941); **37:** 707 (1947); **38:** 671 (1948).

[3] Mueller, H.: M.I.T., course 8.262; Fall 1946, and an unpublished manuscript.

[4] The subject is treated in full in: Parke, N. G.: "Matrix Optics," Ph.D. Thesis, Course VIII, M.I.T., 1948. 200 pp. Ch. I, Generalized Optical Algebra; Ch. II, Illustrations and Examples; Ch. III, Statistical Harmonic Analysis; Ch. IV, Quasi-Stationary Scattering. The present paper summarizes the mathematical aspects of the algebraic structure. Space limitations preclude any motivation or interpretation. A slightly more extensive summary will be found in Report No. 70, Research Laboratory of Electronics, M.I.T., 1948.

[5] The symbol ".=." is read "called" or "represents."

characterizes the radiation. The complex matrix function

$$J(\omega) = [J_{ij}(\omega)] \ .=. \ \text{Jones matrix} \tag{2}$$

characterizes the instrument. The definition of J is the basic transfer relation,

$$E' = 2\pi JE, \text{ for all } E; \ E' \ .=. \ \text{outgoing radiation.} \tag{3}$$

The following operations occur,

$$E = E_1 + E_2 \ .=. \ \text{vector addition} \ .=. \ \text{coherent superposition} \tag{4}$$

			ELEMENTARY ALGEBRAS	GENERALIZED ALGEBRAS
JONES ALGEBRAS	Theoretical Quantities	ω-Algebra	Jones Frequency Algebra Eq. (3) $E' = 2\pi JE$	Symbols Maxwell Quantities .=. E, e, f .=. Radiat Stokes Quantities .=. L, ϕ, S .=. Radiat Jones Quantities .=. J, j .=. Instrumen Mueller Quantities .=. M, m .=. Instrum
		t-Algebra	Jones Time Algebra Eq. (25) $e'(t) = \int_{-\infty}^{\infty} j(t_0)e(t - t_0)\, dt_0$ Eq. (34) $f'_\beta(t) = j_\beta^{\alpha}(\underline{t_0})f_\alpha(t - \underline{t_0})$	Generalized Jones Time Algebra Eq. (30) $f'^{ij}_\alpha(t) = j_\alpha^{i\beta}(t_0)f_\beta^j(t - \underline{t_0})$
			Wiener's Generalized Harmonic Analysis	
MUELLER ALGEBRAS	Observables	t-Algebra	Mueller Time Algebra Eq. (38) $\varphi'_{\alpha\beta}(t) = m_{\alpha\beta}^{\gamma\delta}(\underline{t_1}, \underline{t_2})\varphi_{\gamma\delta}(t - \underline{t_1} + \underline{t_2})$	Generalized Mueller Time Algebra Eq. (50) $\varphi'_{\alpha\beta}(t) = m_{\alpha\beta}^{jk\gamma\delta}(\underline{t_1}, \underline{t_2})\varphi_{jk\gamma\delta}(t - \underline{t_1} +$
		ω-Algebra	Mueller Frequency Algebra Eq. (40) $S'_{\alpha\beta}(\omega) = M_{\alpha\beta}^{\gamma\delta}(\omega)S_{\gamma\delta}(\omega)$	Generalized Mueller Frequency Algeb Eq. (52) $S'_{\alpha\beta}(\omega) = M_{\alpha\beta}^{jk\gamma\delta}(\omega)S_{jk\gamma\delta}(\omega)$

FIG. 1. THE STRUCTURE OF OPTICAL ALGEBRA

$$J = J_2 J_1 \ .=. \ \text{matrix multiplication} \ .=. \ \text{instruments in series} \tag{5}$$

$$J = J_1 + J_2 \ .=. \ \text{instruments in parallel.} \tag{6}$$

Mueller Frequency Algebra. Mueller frequency algebra is a mathematical system in which the complex vector function

$$L(\omega) = (I(\omega), M(\omega), C(\omega), S(\omega)) \ .=. \ \text{Stokes vector} \tag{7}$$

characterizes the radiation. The complex matrix function

$$M(\omega) = [M_{ij}(\omega)] \ .=. \ \text{Mueller matrix} \tag{8}$$

characterizes the instrument. The definition of M is the basic transfer relation,

$$L' = ML, \text{ for all } L; \ L' \ .=. \ \text{outgoing radiation.} \tag{9}$$

The following operations occur,

(10) $L = L_1 + L_2$.=. vector addition .=. incoherent superposition

(11) $M = M_2 M_1$.=. matrix multiplication .=. instruments in series

(12) $M = M_1 + M_2$.=. matrix addition .=. instruments in parallel.

Wiener's Generalized Harmonic Analysis. Wiener has pointed out that, in the electromagnetic theory of light, the field vectors E and B are not observables as they appear to be at lower frequencies.[6] Optical observations always end in intensity measurements using the eye, a photographic plate, the photoelectric effect, a bolometer, etc. The quantities of the Maxwell theory most nearly corresponding with these observations are,

$$\tfrac{1}{2}(E.D + B.H) \text{ .=. energy density}$$

$$E \times H \text{ .=. Poynting vector .=. energy flow}$$

quantities depending "quadratically" on the field quantities E, B, D, H. This linear-quadratic duality is exactly the relation between Jones and Mueller algebra. The correspondence is supplied by Wiener's generalized harmonic analysis.

As introduced by Wiener, the generalized harmonic analysis of a function $f(t)$, takes place in two steps,

$$\begin{aligned}\varphi(t) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t + t_0)f^*(t_0)\, dt_0 \cdot \\ &= \langle f(t + t_0)f^*(t_0)\rangle_{\text{Avg.}} \text{ .=. auto-correlation}\end{aligned} \tag{13}$$

$$\begin{aligned}S(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t)e^{i\omega t}\, dt \\ &= F\varphi(t) \text{ .=. Fourier transformation.}\end{aligned} \tag{14}$$

If this is applied to

(15) $f(t) = a_n f_n(t)$; a_n complex; (Summation convention),

then

$$\varphi(t) = a_n \varphi_{nm}(t) a_m^* \tag{16}$$

where

$$\varphi_{nm}(t) = \langle f_n(t + t_0)f_m^*(t_0)\rangle \text{ .=. Interference matrix.} \tag{17}$$

The Fourier transformation of eq. (16) yields,

$$S(\omega) = a_n S_{nm}(\omega) a_m^* \tag{18}$$

where

$$S_{nm}(\omega) = F\varphi_{nm}(t) \text{ .=. Spectral matrix.} \tag{19}$$

[6] Wiener, N.: Jour. Franklin Institute **207:** 525 (1929).

Wiener prefers the matrix

$$C_{nm}(\omega) = \frac{1}{2\pi}\int_{\infty}^{\infty} \varphi_{nm}(t)\,\frac{e^{i\omega t}-1}{it}\,dt \;.=.\; \text{coherency matrix.} \tag{20}$$

When differentiation is meaningful,

$$S_{nm}(\omega) = (d/d\omega)C_{nm}(\omega). \tag{21}$$

For a monochromatic plane wave of frequency ω, it turns out that the spectral matrix is

$$S_{nm}(\omega) = \frac{1}{2}\begin{bmatrix} I+M & C-iS \\ C+iS & I-M \end{bmatrix} \tag{22}$$

$$\delta \;.=.\; \text{unit impulse function}$$

$$(I, M, C, S) \;.=.\; \text{monochromatic Stokes vector.}$$

Jones Time Algebra. Jones time algebra is the more fundamental. Its Fourier transform (when defined on a finite time interval) is Jones frequency algebra. Briefly,

$$e(t) = (f_1(t), f_2(t)) \;.=.\; \text{Maxwell vector} \;.=.\; \text{radiation,} \tag{23}$$

$$E(\omega) = Fe(t), \tag{24}$$

$$e'(t) = \int_{-\infty}^{\infty} j(t_0)e(t-t_0)\,dt_0 \;.=.\; \text{convolution} \;.=.\; \text{transfer relation,} \tag{25}$$

$$E'(\omega) = Fe'(t) \tag{26}$$

$$J(\omega) \doteq Fj(t) \tag{27}$$

$$e(t) = e_1(t) + e_2(t) \;.=.\; \text{vector addition} \;.=.\; \text{superposition.} \tag{28}$$

J is the response of an optical instrument to polarized monochromatic waves. j is the response of an optical instrument to polarized impulses. The correlation applies to functions of time; hence the introduction of Jones time algebra. The quantum theory of scattering, or the Mie theory of scattering, refers to monochromatic plane waves and gives the J of Jones frequency algebra. It turns out that it is only necessary to make conceptual use of Jones time algebra. The J's can be used directly to compute the M's of Mueller frequency algebra, e.g., eq. (53).

Generalized Jones Algebra. The problem which motivates the generalization of Jones algebra is scattering from a statistical assemblage of centers, Fig. 2. The following notation is useful,

$$f_\beta^j(t) \;.=.\; \text{set of } N \text{ incoming wavelets; } j = 1, 2, \cdots, N$$

$$j_\alpha^{i\beta}(t) \;.=.\; \text{set of } M \text{ scattering centers; } i = 1, 2, \cdots, M$$

These multi-index quantities are direct sums of the corresponding elementary quantities. Using the summation convention one has,

$$f'^{ij}_{\alpha}(t) = \int_{-\infty}^{\infty} j^{i\beta}_{\alpha}(t_0) f^j_\beta(t - t_0)\, dt_0 \mathrel{.=.} MN \text{ outgoing wavelets.} \tag{29}$$

The last operation is a combined Kronecker product, summation and convolution. In order to free the calculation of integral signs one introduces the "convolution convention": *Repetition of a time variable with a single bar under both occurrences implies convolution.* Thus,

$$f'^{ij}_{\alpha}(t) = j^{i\beta}_{\alpha}(\underline{t_0}) f^j_\beta(t - \underline{t_0}). \tag{30}$$

If only a single wave comes in,

$$f'^{i}_{\alpha}(t) = j^{i\beta}_{\alpha}(t_0) f_\beta(t - t_0). \tag{31}$$

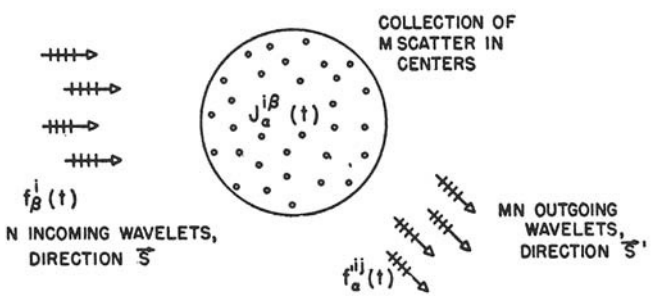

FIG. 2. SCATTERING, A BASIC OPTICAL PATTERN

The geometry of the relative retardations of the various scattered wavelets $f'^{i}_{\alpha}(t)$ must be considered before summation on i is carried out. Let $r_i(t)$ be the position of the M scattering centers, Fig. 3.
At the point of observation P, eq. (31) becomes

$$f'^{i}_{\alpha}\left(t - \frac{s' \cdot r}{v}\right) = j^{i\beta}_{\alpha}(\underline{t_0}) f_\beta\left(t - \frac{s' \cdot (r - r_i(t))}{v} - \frac{s \cdot r_i(t)}{v} - \underline{t_0}\right), \tag{32}$$

i not summed, v .=. velocity of radiation. After change of epoch, eq. (32) becomes,

$$f'_{\alpha}(t) = j^{i\beta}_{\alpha}(t_0) f_\beta\left(t - \frac{(s - s') \cdot r_i(t - \underline{t_0})}{v}\right); \qquad (i \text{ summed}). \tag{33}$$

This result involves a quasi-stationary assumption on the scattering centers but leads to many interesting results in agreement with a widely scattered literature, including scattering of x-rays by crystals, liquids and gases. Elementary Jones algebra could not describe depolarizing instruments and partially polarized radiation, the generalized algebra can.

Elementary Mueller Algebra. The relation between Jones and Mueller algebra is most easily understood if one restricts attention to the effect of a single elementary instrument,

$$f'_\beta(t) = j^\alpha_\beta(\underline{t_0})f_\alpha(t - \underline{t_0}). \tag{34}$$

The Wiener correlation of the incoming radiation is

$$\varphi_{\alpha\beta}(t) = \langle f_\alpha(t + t_0)f^*_\beta(t_0)\rangle. \tag{35}$$

One can introduce a "correlation convention": *Repetition of a time variable with a double bar under both occurrences implies correlation,* and rewrite eq. (35),

$$\varphi_{\alpha\beta}(t) = f_\alpha(t + \underline{\underline{t_0}})f^*_\beta(\underline{\underline{t_0}}). \tag{36}$$

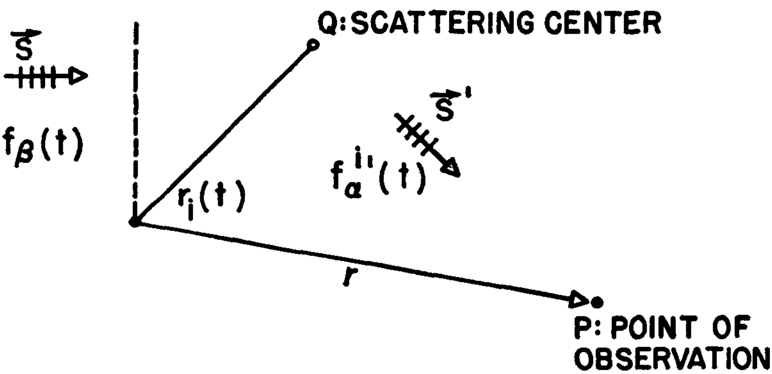

Fig. 3. Retardation Geometry

Also,

$$\varphi'_{\alpha\beta}(t) = f'_\alpha(t + \underline{\underline{t_0}})f'^*_\beta(\underline{\underline{t_0}}) = j^\gamma_\alpha(\underline{t_1})j^{\delta*}_\beta(\underline{t_2})f_\gamma(t - \underline{t_1} + \underline{t_2} + \underline{\underline{t_0}})f^*_\delta(\underline{\underline{t_0}}). \tag{37}$$

From this one obtains the important formula,

$$\varphi'_{\alpha\beta}(t) = m^{\gamma\delta}_{\alpha\beta}(t_1, \underline{t_2})\varphi_{\gamma\delta}(t - t_1 + \underline{t_2}) \text{ .=. transfer relation,} \tag{38}$$

in which

$$m^{\gamma\delta}_{\alpha\beta}(t_1, t_2) = j^\gamma_\alpha(t_1)j^{\delta*}_\beta(t_2) \text{ .=. Mueller function,} \tag{39}$$

the Kronecker product of two Jones matrices. The Fourier transform of eq. (39) gives

$$S'_{\alpha\beta}(\omega) = M^{\gamma\delta}_{\alpha\beta}(\omega)S_{\gamma\delta}(\omega) \tag{40}$$

where,

$$M^{\gamma\delta}_{\alpha\beta}(\omega) = (2\pi J^\gamma_\delta(\omega))(2\pi J^\delta_\beta(\omega))^* \text{ .=. Mueller function.} \tag{41}$$

If one introduces the "index reduction transformation,"

(42) $$i = \alpha + 2(\beta - 1);\ j = \gamma + 2(\delta - 1)$$

one obtains,

(43) $$M_{\alpha\beta}^{\gamma\delta}(\omega) = M_{ij} \ .=.\ \text{complex Mueller matrix,}$$

(44) $$S_{\gamma\delta}(\omega) = L_j \ .=.\ \text{complex Stokes vector,}$$

(45) $$S'_{\alpha\beta}(\omega) = L'_i \ .=.\ \text{complex Stokes vector.}$$

On dropping indices and using matrix notation,

(46) $$L'(\omega) = M(\omega)L(\omega).$$

These matrices and vectors differ from those used by Mueller. In particular, the complex Stokes vector turns out to be,

(47) $$L = (I + M, C + iS, C - iS, I - M).$$

The complex Mueller matrix differs correspondingly from the real Mueller matrix used by Mueller. The advantage of the complex Mueller matrix is its simple relation to the Jones matrix, eq. (41). The real Stokes vector and Mueller matrix have easier phenomenological definitions.

Generalized Mueller Algebra. One begins with the transfer relation for generalized Jones algebra,

(33) $$f'_\alpha(t) = j_\alpha^{j\beta}(\underline{t_0})f_{j\beta}(t - \underline{t_0})$$

where,

(48) $$f_{j\beta}(t) = f_\beta\left(t - \frac{(s - s')\cdot r_j(t)}{v}\right).$$

Wiener correlation gives,

(49) $$\varphi'_{\alpha\beta}(t) = [j_\alpha^{j\gamma}(\underline{t_1})j_\beta^{k\delta *}(\underline{t_2})]f_{j\gamma}(t + \underline{\underline{t_0}} - t_1)f^*_{k\delta}(\underline{\underline{t_0}} - t_2).$$

This reduces to,

(50) $$\varphi'_{\alpha\beta}(t) = m_{\alpha\beta}^{jk\gamma\delta}(\underline{t_1}, \underline{t_2})\varphi_{jk\gamma\delta}(t - t_1 + \underline{t_2}),$$

where

(51) $$m_{\alpha\beta}^{jk\gamma\delta}(t_1, t_2) = j_\alpha^{j\gamma}(t_1)j_\beta^{k\delta *}(t_2) \ .=.\ \text{Mueller function.}$$

The Fourier transform of eq. (5) yields,

(52) $$S'_{\alpha\beta}(\omega) = M_{\alpha\beta}^{jk\gamma\delta}(\omega)S_{jk\gamma\delta}(\omega),$$

where,

(53) $$M_{\alpha\beta}^{jk\gamma\delta}(\omega) = (2\pi J_\alpha^{j\gamma}(\omega))(2\pi J_\beta^{k\delta}(\omega))^* \ .=.\ \text{Mueller function.}$$

When $j = k$, the M's are called auto-functions, when $j \neq k$ they are called cross-functions.

The difficulties of the scattering problem are reduced to the evaluation of the interference,

$$\varphi_{jk\gamma\delta}(\omega) = \underline{f_{j\gamma}(t + t_0)f^*_{k\delta}(t_0)}. \tag{54}$$

For an important class of cases, the Fourier transform of eq. (54) separates, i.e.,

$$S_{jk\gamma\delta}(\omega) = N_{jk}(\omega)S_{\gamma\delta}(\omega). \tag{55}$$

This leads to a simplification in which $N_{jk}(\omega)$ is characteristic of the statistical distribution of scattering centers. In this case,

$$M^{\gamma\delta}_{\alpha\beta}(\omega) = N_{jk}(\omega)M^{jk\gamma\delta}_{\alpha\beta}(\omega), \tag{56}$$

a general law for the composition of Mueller matrices, for instruments in parallel. Each scattering center is thought of as an elementary instrument.

The Form of $N_{jk}(\omega)$ in Typical Cases. The importance of $N_{jk}(\omega)$ is well illustrated by three examples which embody a unification of the treatment of scattering from liquids, solids and gases, i.e., coherent scattering (solids), partially coherent scatterings (liquids, electrolytes), and incoherent scattering (gases).

1. Coherent Scattering,

$$N_{jk}(\omega) = e^{i(\mathbf{s}-\mathbf{s}')\cdot(\mathbf{r}_j-\mathbf{r}_k)\omega/v}. \tag{57}$$

2. Incoherent Scattering,

$$N_{jk} = \delta_{jk}. \tag{58}$$

3. Partially Coherent Scattering (Radial Symmetry),

$$N_{jk}(s) = \begin{cases} 1, & \text{when } j = k \\ \dfrac{i(s)}{nV} & \text{when } j \neq k, \end{cases} \tag{59}$$

$$s = \frac{\omega\,|\,\mathbf{s} - \mathbf{s}'\,|}{v}; \qquad i(s) = F(\rho(r) - n) \tag{60}$$

n .=. concentration

$\rho(r)$.=. radial concentration density.

4. Special Case of an Electrolyte

$$i(s) = \pm\frac{1}{2(1 + (s/\kappa)^2)} \tag{61}$$

$+$.=. unlike-signed ions

$-$.=. like-signed ions

$$\kappa^2 = \frac{2nZ^2\,|\,e\,|}{\epsilon kT} \tag{62}$$

$N_{jk}(\omega)$ is called a Debye statistical matrix because it is closely related to some of Debye's results in electrolyte theory and in x-ray scattering.

Composition of Mueller Matrices. Written in full, eq. (54) becomes,

$$\varphi_{jk\gamma\delta}(t) = f_\gamma\left(t + \underline{\underline{t_0}} - \frac{(s - s'\cdot r_j(t - \underline{\underline{t_0}})}{v}\right) f_\delta^*\left(\underline{\underline{t_0}} - \frac{(s - s')\cdot r_k(\underline{\underline{t_0}})}{v}\right). \tag{63}$$

Making the quasi-stationary approximation,

$$\begin{aligned} \varphi_{jk\gamma\delta}(t) &= f_\gamma\left(t + \underline{\underline{t_1}} - \frac{(s - s')\cdot(r_j(\underline{\underline{t_2}}) - r_k(\underline{\underline{t_2}}))}{v}\right) f_\delta^*(\underline{t_1}) \\ &= \varphi_{\gamma\delta}\left(t - \frac{(s - s')\cdot(r_j(\underline{\underline{t_2}}) - r_k(\underline{\underline{t_2}}))}{v}\right). \end{aligned} \tag{64}$$

The Fourier transform of the last expression gives,

$$S_{jk\gamma\delta}(\omega) = S_{\gamma\delta}(\omega)e^{i\omega(s-s')(r_j(\underline{\underline{t_2}})-r_k(\underline{\underline{t_2}}))/v} \tag{65}$$

If one defines,

$$N_{jk}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{i\omega(s-s')\cdot(r_j(t)-r_k(t))/v}\, dt \tag{66}$$

then

$$S_{ij\gamma\delta}(\omega) = N_{jk}(\omega)S_{\gamma\delta}(\omega) \tag{67}$$

which is eq. (55), as required. Under these circumstances one has a general rule for the composition of Mueller matrices, eq. (56).

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