Tangent Lines, Instantaneous Rate of Change, and Applications

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Summary

This lecture introduces tangent lines and instantaneous rate of change, contrasting it with average rate of change. It explores the properties of tangent lines and demonstrates how to approximate their slopes using computational tools like Desmos. The concepts are applied to a physics problem involving a rock thrown on Mars to calculate average and instantaneous velocities.

4.2 Tangent Lines, Instantaneous Rate of Change, and Applications

Overview of objectives for understanding tangent lines and their applications.

Objectives

- Be able to visualize tangent lines.
- Approximate the slope of tangent lines.

Instantaneous Rate of Change

Understanding the concept of change at a specific moment in time.

Concept Introduction

- A speeding incident: A police officer used a radar gun to determine a car was going 56 miles per hour at a particular instant.
    - This is not an average rate of change but the rate at one specific time.
- Instantaneous rate of change is how a function is changing at a particular instant of time (e.g., velocity at a specific moment).
- Secant lines were used in the previous video to visualize average rates of change.
- Tangent lines will be used to visualize instantaneous rate of change.

Tangent Line Properties

Exploring the characteristics and visualization of a tangent line.

Definition and Characteristics

- A tangent line requires a function.
- It needs one specified point (P) on the function.
- It just barely skims the graph of the function at that point.
- It's a line very close to the secant line through two very close points.

[Diagram]: A graph showing a function f(x) with a point P on it. A purple line is drawn tangent to the function at point P, skimming the graph. Another point Q is shown very close to P, and a pink secant line is drawn through P and Q, indicating its closeness to the tangent line.

Example 4.5: Approximating the Slope of a Tangent Line

Using Desmos to numerically approximate the slope of a tangent line.

Approximation Method with Desmos

- The goal is to approximate the slope of the tangent line of the function at x=2.

$$f(x) = x^2 - 3$$

[Diagram]: A Desmos graph showing the function y = x^2 - 3 (blue curve), a green secant line, and a purple tangent line. Sliders are present to adjust x1 and x2 for the secant line, and 'c' for the tangent line's point of tangency.

- The secant line's slope is calculated as:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- To approximate the tangent line's slope at x=2, we select one point as 2 and another point very close to 2. By adjusting the 'x1' and 'x2' values in Desmos, we can see the secant line getting closer to the tangent line, and its slope approaching a certain value.

x1	x2	Slope of Secant Line
1.9	2	3.9
1.99	2	3.99
2.1	2	4.1
2.01	2	4.01

- From the approximations, the slope of the tangent line at x=2 is approximately 4.

Example 4.7: Approximating Instantaneous Velocity

Applying average rate of change to approximate instantaneous velocity in a physics context.

Problem Statement

- A rock is thrown upward on the planet Mars with an initial velocity of 10 meters per second. Its height (in meters) t seconds later is well approximated by the function:

$$h(t) = 10t - 2t^2$$

Part (a): Find the average velocity over the time interval [1, 2]

- The formula for average velocity (average rate of change) is the change in height divided by the change in time.

$$\frac{h(2) - h(1)}{2 - 1} \text{ m/s}$$

- Calculate h(2) and h(1):

$$h(2) = 10(2) - 2(2^2) = 20 - 8 = 12$$

$$h(1) = 10(1) - 2(1^2) = 10 - 2 = 8$$

$$\frac{12 - 8}{2 - 1} = \frac{4}{1} = 4 \text{ m/s}$$

- The average velocity over the time interval [1, 2] is 4 meters per second.

Part (b): Find the average velocity over the time interval [0, 1]

$$\frac{h(1) - h(0)}{1 - 0} \text{ m/s}$$

- Calculate h(1) and h(0):

$$h(1) = 10(1) - 2(1^2) = 10 - 2 = 8$$

$$h(0) = 10(0) - 2(0^2) = 0 - 0 = 0$$

$$\frac{8 - 0}{1 - 0} = \frac{8}{1} = 8 \text{ m/s}$$

- The average velocity over the time interval [0, 1] is 8 meters per second.

Part (c): Using part (a) and (b) approximate the instantaneous velocity at t=1

- To approximate instantaneous velocity at t=1, we can use the average velocities from time intervals close to t=1.

- The interval [0, 1] ends at t=1 (approaching from the left).

- The interval [1, 2] starts at t=1 (approaching from the right).

- We can average the two average velocities calculated in parts (a) and (b) for a better approximation.

$$\frac{4+8}{2} = \frac{12}{2} = 6 \text{ m/s}$$

- The instantaneous velocity at t=1 is approximately 6 meters per second.

Remark 4.8: Units of Instantaneous Rate of Change

General units for the instantaneous rate of change of a function.

Unit Derivation

- Just like the average rate of change, the instantaneous rate of change has units of:

$$\frac{\text{units of f}}{\text{units of x}}$$