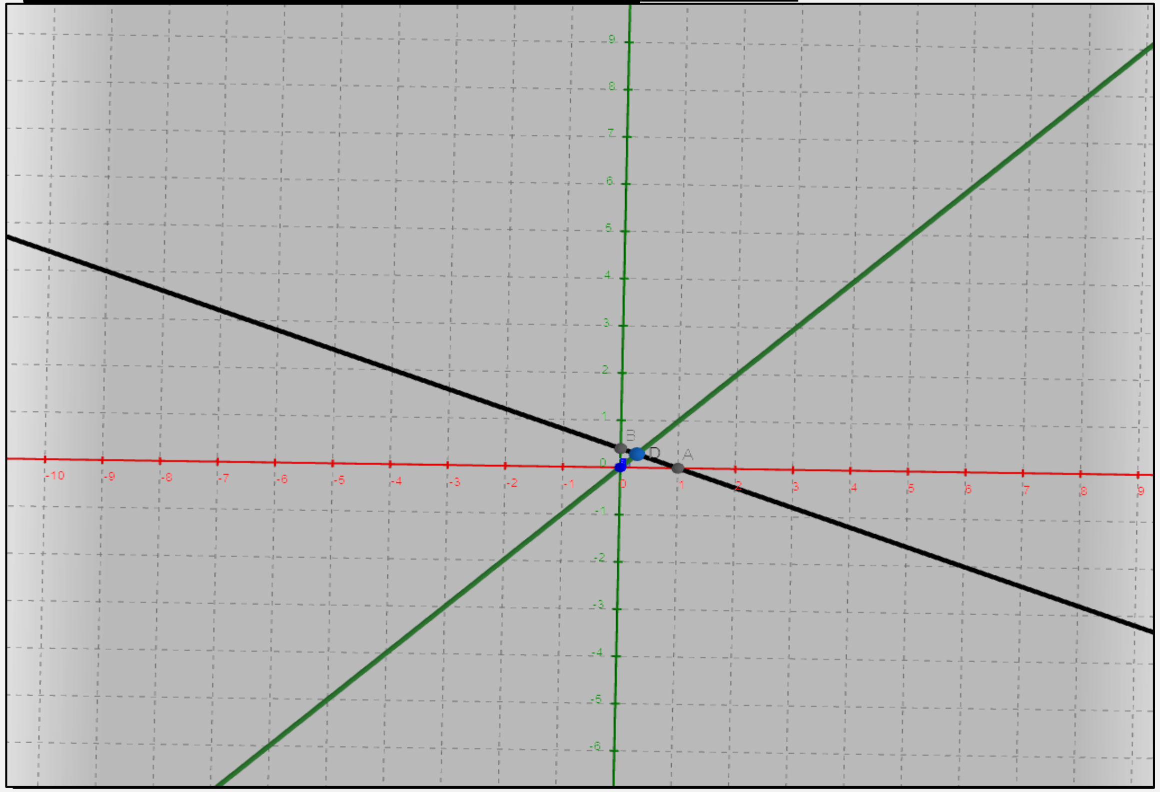
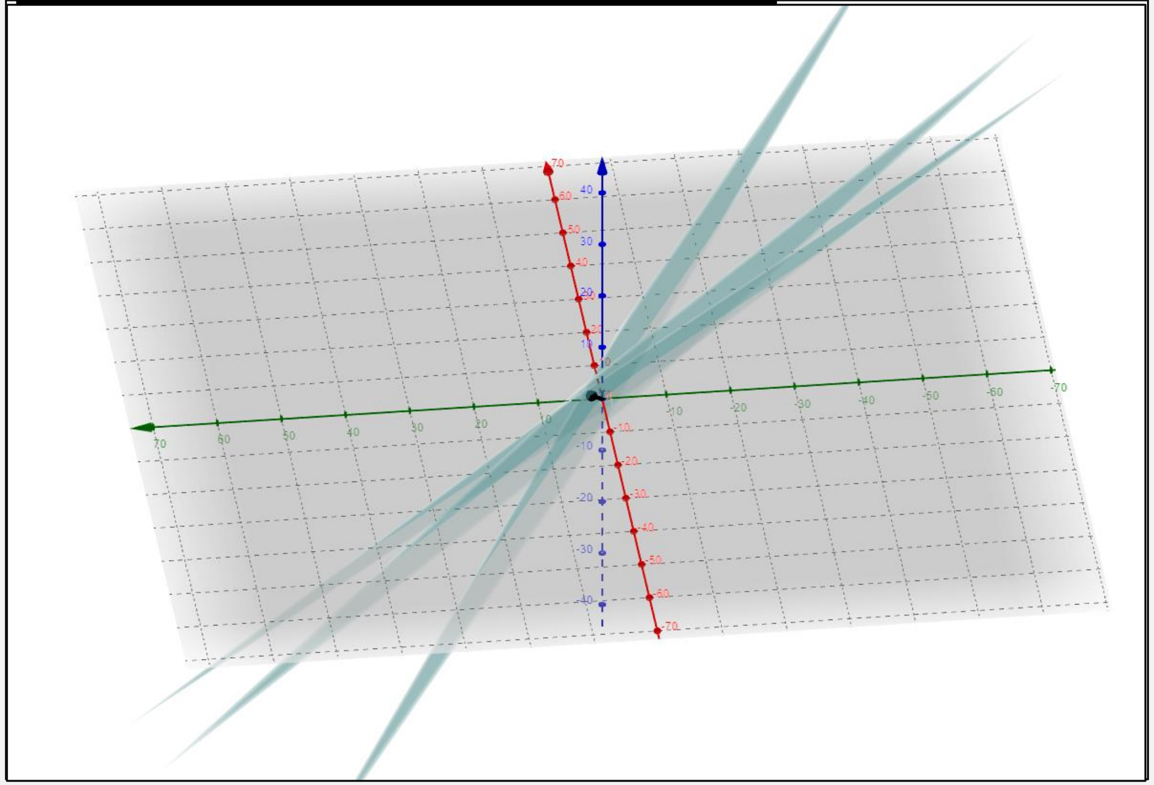
$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_{n-1}x_{n-1} + a_nx_n = b$$

$$2x_1 + 5x_2 = 2 \text{ ---(1)}$$
$$x_1 - x_2 = 0 \text{ ---(2)}$$



$$x_1 + 2x_2 + 3x_3 = 1 \text{ ---(1)}$$
$$4x_1 + 4x_2 + 5x_3 = 3 \text{ ---(2)}$$
$$6x_1 + 7x_2 + 7x_3 = -1 \text{ ---(3)}$$



$$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 4x_1 + 4x_2 + 5x_3 = 3 \\ 6x_1 + 7x_2 + 7x_3 = -1 \end{cases} \longrightarrow \begin{cases} 1x_1 + 2x_2 + 1x_3 = 1 \\ 4x_1 + 4x_2 + 5x_3 = 3 \\ 6x_1 + 7x_2 + 7x_3 = -1 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 5 \\ 6 & 7 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1x_1 + 2x_2 + 1x_3 \\ 4x_1 + 4x_2 + 5x_3 \\ 6x_1 + 6x_2 + 7x_3 \end{bmatrix}$$

$$\longrightarrow \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 5 \\ 6 & 7 & 7 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}}_{b}$$

$$\longrightarrow \quad AX = b$$

$$\longrightarrow \quad A^{-1}AX = A^{-1}b$$

$$\longrightarrow \quad X = A^{-1}b$$

$$\longrightarrow \quad X = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 5 \\ 6 & 7 & 7 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -34 \\ 6 \\ 23 \end{bmatrix}$$

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n \end{cases}$$

$$\downarrow$$

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,(n-1)} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,(n-1)} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(n-1),1} & a_{(n-1),2} & \cdots & a_{(n-1),(n-1)} & a_{(n-1),n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,(n-1)} & a_{n,n} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

$$\begin{cases} 3x_1 + 2x_2 + x_3 = 1 \\ x_1 - 2x_2 + x_3 = 0 \\ 2x_1 + x_2 - 3x_3 = -1 \end{cases}$$

$$\downarrow$$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & -1 \end{array}\right] \end{array} \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & -5 & 5 & 1 \\ 0 & -\frac{1}{3} & -\frac{11}{3} & -\frac{5}{3} \end{array}\right] \begin{array}{l} r_1 \\ r_2 = 2r_2 - r_3 \\ r_3 = r_3 - \frac{2}{3}r_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & -5 & 5 & 1 \\ 0 & -1 & -11 & -5 \end{array}\right] \begin{array}{l} r_1 \\ r_2 \\ r_3 = 3r_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & -5 & 5 & 1 \\ 0 & 0 & -60 & -26 \end{array}\right] \begin{array}{l} r_1 \\ r_2 \\ r_3 = 5r_3 - r_2 \end{array}$$

$$\sim \begin{bmatrix} 3 & 2 & 1 \\ 0 & -5 & 5 \\ 0 & 0 & -60 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -26 \end{bmatrix}$$

$$\longrightarrow \begin{cases} 3x_1 + 2x_2 + x_3 = 1 \text{ ---(1)} \\ -5x_2 + 5x_3 = 1 \text{ ---(2)} \\ -60x_3 = -26 \text{ ---(3)} \end{cases}$$

$$\longrightarrow \quad -60x_3 = -26 \quad \longrightarrow \quad x_3 = \frac{13}{30}$$

$$\longrightarrow \quad -5x_2 + 5\left(\frac{13}{30}\right) = 1 \quad \longrightarrow \quad x_2 = \frac{7}{30}$$

$$\longrightarrow \quad 3x_1 + 2\left(\frac{7}{30}\right) + \frac{13}{30} = 1 \quad \longrightarrow \quad x_1 = \frac{1}{30}$$

Number of pivots positions $= n$

Number of pivots positions $< n$

$$\begin{cases} -x_1 + 2x_2 = 1 \\ x_1 - 2x_2 = 4 \end{cases}$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} -1 & 2 & 1 \\ 1 & -2 & 4 \end{array}\right] \begin{array}{l} r_1 \\ r_2 \end{array} \sim \left[\begin{array}{cc|c} -1 & 2 & 1 \\ 0 & 0 & 5 \end{array}\right] \begin{array}{l} r_1 \\ r_2 = r_2 + r_1 \end{array}$$

$$\longrightarrow \begin{cases} -x_1 + 2x_2 = 1 \\ 0 = 5 \end{cases}$$

$$\begin{cases} 500x_1 + 800x_2 + 900x_3 = 100 \\ 500x_1 + 200x_2 + 700x_3 = 200 \\ 2500x_1 + 1000x_2 + 3500x_3 = 1000 \end{cases}$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 500 & 800 & 900 & 100 \\ 500 & 200 & 700 & 200 \\ 2500 & 1000 & 3500 & 1000 \end{array}\right] \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \sim \left[\begin{array}{ccc|c} 5 & 8 & 9 & 1 \\ 5 & 2 & 7 & 2 \\ 25 & 10 & 35 & 10 \end{array}\right] \begin{array}{l} r_1 = \frac{r_1}{100} \\ r_2 = \frac{r_2}{100} \\ r_3 = \frac{r_3}{100} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 5 & 8 & 9 & 1 \\ 0 & -6 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} r_1 \\ r_2 = r_2 - r_1 \\ r_3 = r_3 - 5r_2 \end{array}$$

$$\longrightarrow \left\{ x_1, x_2, x_3 \; such \; that \quad \begin{array}{r} 5x_1 + 8x_2 + 9x_3 = 1 \\ -6x_2 - 2x_3 = 1 \end{array} \right.$$

In Class Exercise 2.1

$$4x_1 + 2x_2 + 1x_3 - x_4 = 0$$

$$2x_2 + 7x_3 - x_4 = 0$$

$$5x_1 + 1x_2 + 35x_3 - x_4 = -1$$

$$x_1 + 1x_2 + 3x_3 - x_4 = -8$$

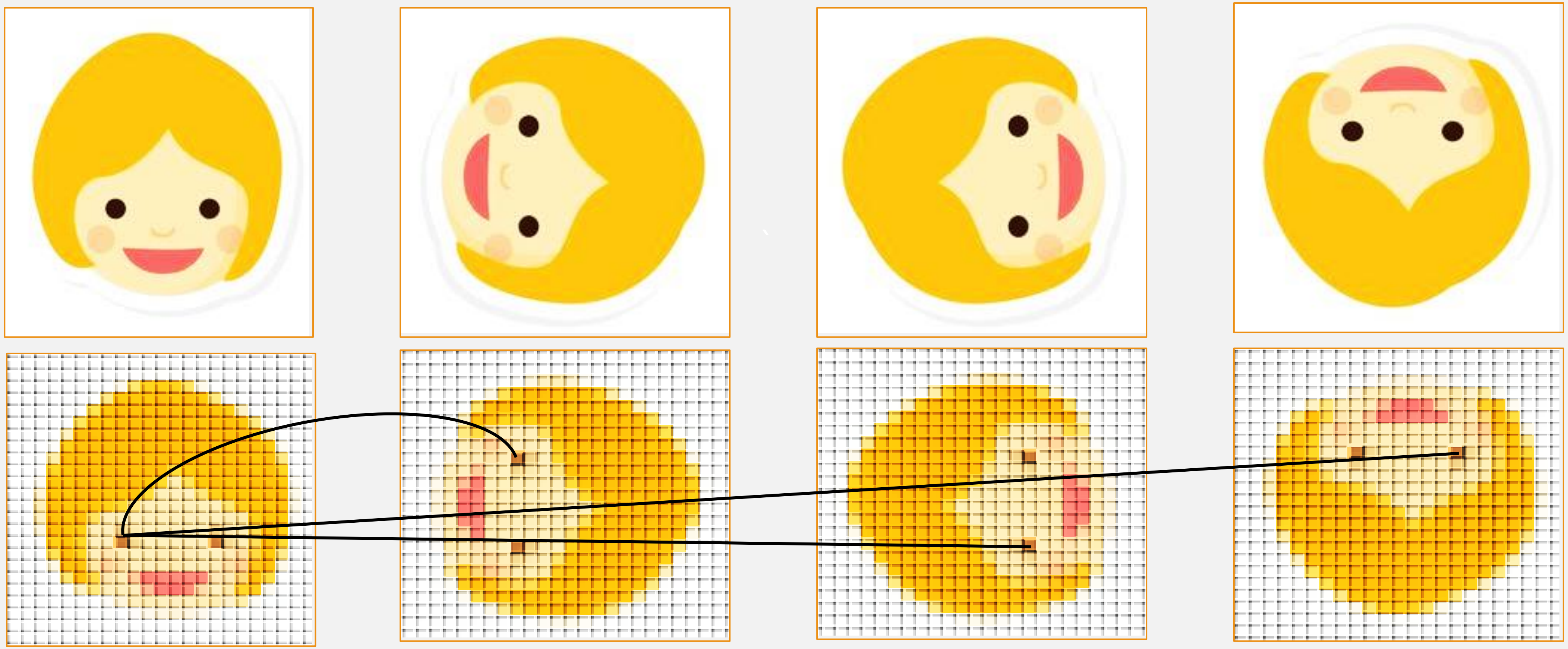
In Class Exercise 2.1
SOLUTION

$$\begin{bmatrix} 4 & 2 & 1 & -1 \\ 0 & 2 & 7 & -1 \\ 5 & 1 & 35 & -1 \\ 1 & 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -8 \end{bmatrix} \quad \dashrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{76} \begin{bmatrix} 21 \\ 573 \\ 14 \\ 1244 \end{bmatrix}$$

Transformations

$$
\begin{aligned}
T: \mathfrak{R}^{\mathrm{m}} &\rightarrow \mathfrak{R}^{n} \\
\mathbf{x} &\mapsto \mathbf{y} = \mathbf{f}(\mathbf{x})
\end{aligned}
$$

$$
\begin{aligned}
T(x_1 + x_2) &= T(X_1) + T(X_2) \\
T(ax) &= aT(x), a \in \mathfrak{R}
\end{aligned}
$$



$$T: \Re^3 \rightarrow \Re^2$$
$$\text{with } \mathrm{T}(\mathrm{x_1}, \mathrm{x_2}, \mathrm{x_3}) = (\mathrm{x_1} + \mathrm{x_2}, \mathrm{x_1} - \mathrm{x_3})$$

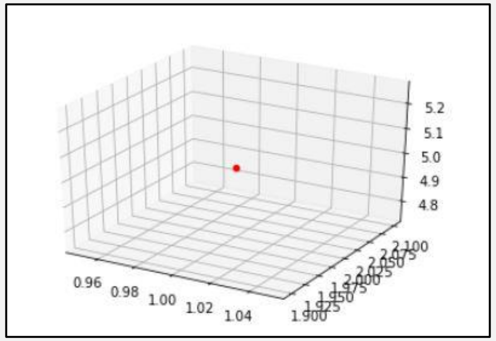
$$\mathrm{T}(\mathrm{X_1} = (\mathrm{x_1}, \mathrm{x_2}, \mathrm{x_3})) = (\mathrm{x_1} + \mathrm{x_2}, \mathrm{x_1} - \mathrm{x_3}) \qquad \mathrm{T}(\mathrm{X_2} = (\mathrm{x}'_1, \mathrm{x}'_2, \mathrm{x}'_3)) = (\mathrm{x}'_1 + \mathrm{x}'_2, \mathrm{x}'_1 - \mathrm{x}'_3)$$

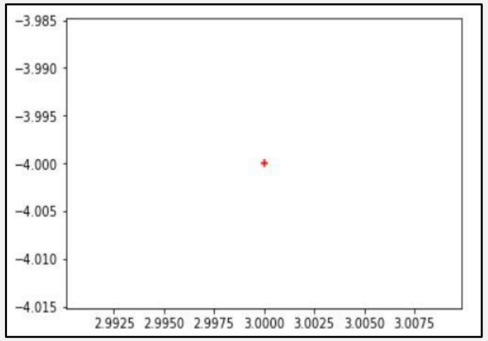
$$\mathrm{T}(\mathrm{X_1} + \mathrm{X_2}) = \mathrm{T}((\boxed{\mathrm{x_1} + \mathrm{x}'_1}, \boxed{\mathrm{x_2} + \mathrm{x}'_2}, \boxed{\mathrm{x_3} + \mathrm{x}'_3}))$$

$$= ([\mathrm{x_1} + \mathrm{x}'_1 + \mathrm{x_2} + \mathrm{x}'_2], ([\mathrm{x_1} + \mathrm{x}'_1 - \mathrm{x_3} - \mathrm{x}'_3]) = \begin{pmatrix} \mathrm{x_1} + \mathrm{x}'_1 + \mathrm{x_2} + \mathrm{x}'_2 \\ \mathrm{x_1} + \mathrm{x}'_1 - \mathrm{x_3} - \mathrm{x}'_3 \end{pmatrix}$$

$$\mathrm{T}(\mathrm{X_1}) + \mathrm{T}(\mathrm{X_2}) = \begin{pmatrix} x_1 + \mathrm{x_2} \\ x_1 - \mathrm{x_3} \end{pmatrix} + \begin{pmatrix} \mathrm{x}'_1 + \mathrm{x}'_2 \\ \mathrm{x}'_1 - \mathrm{x}'_3 \end{pmatrix} = \begin{pmatrix} \mathrm{x_1} + \mathrm{x}'_1 + \mathrm{x_2} + \mathrm{x}'_2 \\ \mathrm{x_1} + \mathrm{x}'_1 - \mathrm{x_3} - \mathrm{x}'_3 \end{pmatrix}$$

$$\mathrm{T}(\mathrm{X_1} = (1, 2, 5)) = \begin{pmatrix} 1 + 2 \\ 1 - 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$





In Class Exercise 2.2

Complete the proof of
T(X) being a linear transformation

In Class Exercise 2.2

SOLUTION

$$T(ax) = aT(x), a \in \Re$$

$$T(aX_1 = (ax_1, ax_2, ax_3)) = \begin{pmatrix} ax_1 + ax_2 \\ ax_1 - ax_3 \end{pmatrix}$$

$$aT(X_1 = (x_1, x_2, x_3)) = a\begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \end{pmatrix} = \begin{pmatrix} ax_1 + ax_2 \\ ax_1 - ax_3 \end{pmatrix}$$

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_1 + x_2 + x_3, x_1 + x_2 + x_3 + x_4, x_4)$$

$$= \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 + x_4 \\ x_4 \end{pmatrix}$$

$$= \begin{pmatrix} 1x_1 + 1x_2 + 0x_3 + 0x_4 \\ 1x_1 + 1x_2 + 1x_3 + 0x_4 \\ 1x_1 + 1x_2 + 1x_3 + 1x_4 \\ 0x_1 + 0x_2 + 0x_3 + 1x_4 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = AX$$

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \boxed{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} \boxed{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}} \boxed{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$T(1, 0, 0, 0) = (1 + 0, 1 + 0 + 0, 1 + 0 + 0 + 0, 0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$T(0, 1, 0, 0) = (0 + 1, 0 + 1 + 0, 0 + 1 + 0 + 0, 0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$T(0, 0, 1, 0) = (0 + 0, 0 + 0 + 1, 0 + 0 + 1 + 0, 0) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$T(0, 0, 0, 1) = (0 + 0, 0 + 0 + 0, 0 + 0 + 0 + 1, 1) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\dashrightarrow A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T(1, 3, 4, 0) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 8 \\ 0 \end{bmatrix}$$

$$T(1, 3, 4, 0) = (1 + 3, 1 + 3 + 4, 1 + 3 + 4 + 0, 0) = (4, 8, 8, 0)$$

In Class Exercise 2.3

Find the Transformation Matrix of the Transformation $T: \mathfrak{R}^3 \to \mathfrak{R}^2$ defined as:

$$T(x_1, x_2, x_3) = (x_1 - 6x_2, x_1 + x_2 + 4x_3)$$

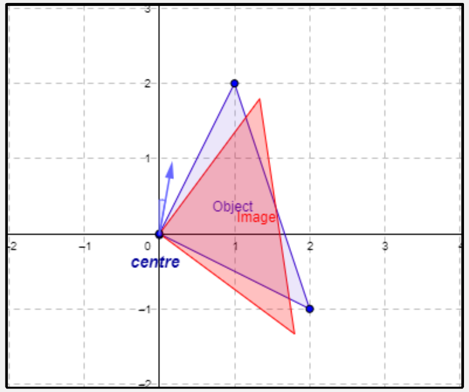
In Class Exercise 2.3 SOLUTION

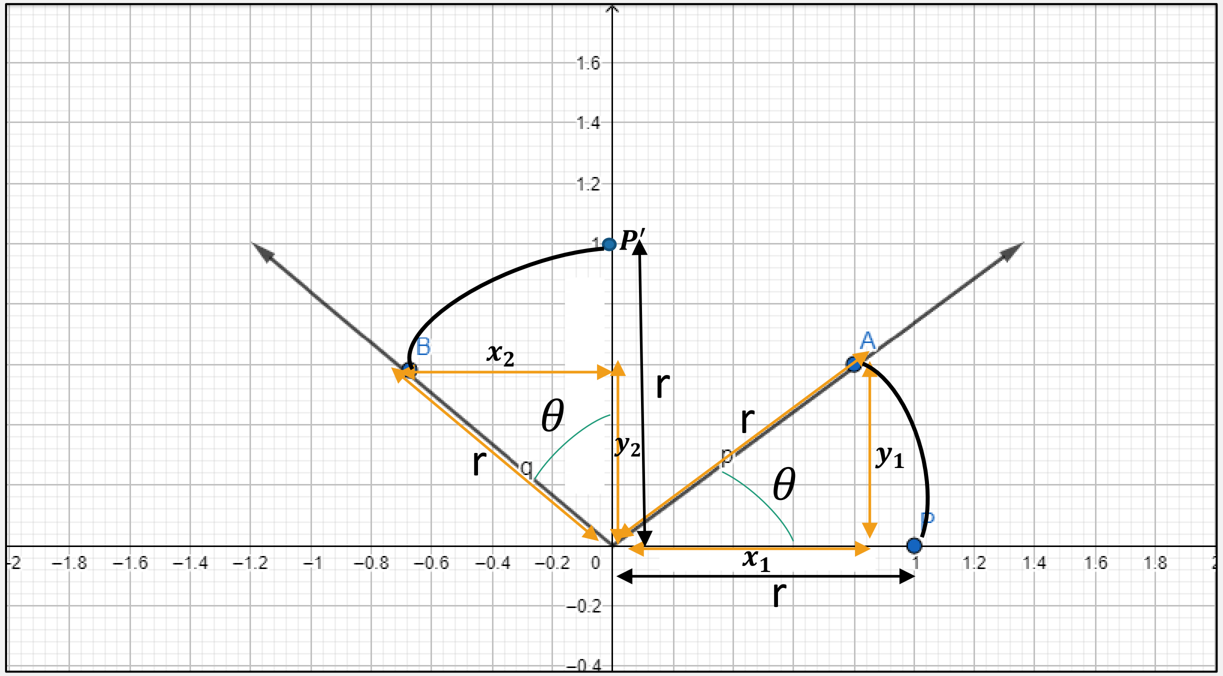
$$\mathbf{T}(1,0,0) = \ (1-6(0), 1+0+4(0)\,) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{T}(0,1,0) = \ (0-6(1), 0+1+4(0)\,) = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

$$\mathbf{T}(0,0,1) = \ (0-6(0), 0+0+4(1)\,) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$\longrightarrow A = \begin{bmatrix} 1 & -6 & 0 \\ 1 & 1 & 4 \end{bmatrix}$$





$$\left\{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right\}$$

$$\cos\theta = \frac{x_1}{r} \longrightarrow x_1 = r\cos\theta = 1\cos\theta$$

$$\sin\theta = \frac{y_1}{r} \longrightarrow y_1 = r\sin\theta = 1\sin\theta$$

$$\longrightarrow T(1,0) = (\cos\theta, \sin\theta)$$

$$\sin\theta = \frac{-x_2}{r} \longrightarrow x_2 = -r\sin\theta = -1\sin\theta$$

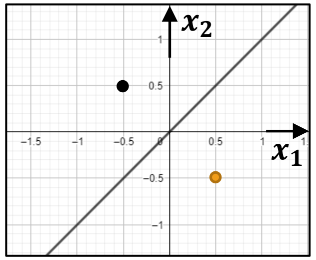
$$\cos\theta = \frac{y_2}{r} \longrightarrow y_2 = r\cos\theta = 1\cos\theta$$

$$\longrightarrow T(0,1) = (-\sin\theta, \cos\theta) \quad \longrightarrow A = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

In Class Exercise 2.4

Find the transformation matrix of a transformation which:

Takes a vector in $\Re^2$ and outputs a vector which is its reflection through the axis $x_2 = x_1$



In Class Exercise 2.4
SOLUTION

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$