COMPLETE LINEAR ALGEBRA

- Python Setup, Basic Python, numpy, pandas and matplotlib
- Matrix Algebra
- Linear equations and transformations
- Vectors
- Vector Spaces
- Metric Spaces, Normed spaces, Inner Product Spaces
- Orthogonality
- Determinant and Trace Operator
- Matrix Decompositions (Eigen, SVD and Cholesky)
- Symmetric matrices and Quadratic Forms
- Left Inverse, Right Inverse, Pseudo Inverse

Connecting the dots 1
LINEAR REGRESSION

Connecting the dots 2
PRINCIPAL COMPONENT ANALYSIS

$$d: V \; X \; V \longrightarrow R_{+}$$
$$(x, \; y) \mid\longrightarrow d(x, y) \qquad \forall x, y, z \in V$$

- $d(x, y) \geq 0 \quad \& \quad d(x, y) = 0 \quad \leftrightarrow \quad x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

COMPLETE LINEAR ALGEBRA

LINEAR ALGEBRA > **Metric Spaces, Normed spaces, Inner Product Spaces** > **Normed Spaces**

$\|\cdot\|: V \longrightarrow R_+ \qquad \forall x, y \in V$

- [ ] $\|x\| \geq 0 \quad \|x\| = 0 \leftrightarrow x = 0$
- [ ] $\|ax\| = |a|\|x\|$
- [ ] $\|x + y\| \leq \|x\| + \|y\|$

L^P norm: $$\|x\|_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}$$

L^2 norm: $$\|x\|_2 = \left(\sum_i |x_i|^2\right)^{\frac{1}{2}} = \sqrt{\sum_i |x_i|^2}$$

L^1 norm: $$\|x\|_1 = \|x\| = \sum_i |x_i|$$

L^2 norm squared: $$\|x\|^2{}_2 = \left(\left(\sum_i |x_i|^2\right)^{\frac{1}{2}}\right)^2 = \sum_i |x_i|^2$$

$$\frac{d}{dx_1}\sum_{i=1}^{2}|x_i|^2 = \frac{d}{dx_1}\left(|x_1|^2 + |x_2|^2\right) = 2x_1$$

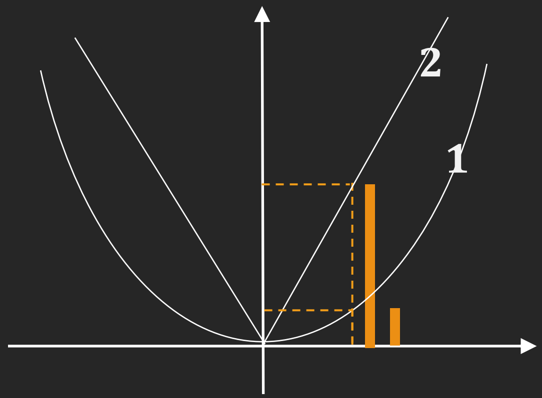
$$\frac{d}{dx_1}\left(\sum_{i=1}^{2}|x_i|^2\right)^{\frac{1}{2}} = \frac{d}{dx_1}\left(|x_1|^2 + |x_2|^2\right)^{\frac{1}{2}} = 2x_1\frac{d}{dx_1}\left(\sum_{i=1}^{2}|x_i|^2\right)$$

L^∞ norm: $$\|x\|_\infty = max(|x_i|)$$

$\|x\| = distance\ from\ Origin\ to\ x$

$\|x - y\| = distance\ from\ x\ to\ y$

$d(x, y) = \|x - y\|$



$\|x\| = |x_1|^2 = x_1^2 \ ---(1)$

$\|x\| = |x_1| \ ---(2)$

Frobenius norm:

$$\|A\|_F = \sqrt{\sum_{i,j} a_{i,j}^2}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\longrightarrow \|A\|_F = \sqrt{a^2 + b^2 + c^2 + d^2}$

$$<\cdot,\cdot>: V \ X \ V \ \longrightarrow \mathbb{R}$$

- [ ] $< x_1, x_1 > \geq 0 \quad \& \quad < x_1, x_1 > = 0 \quad \leftrightarrow \quad x_1 = 0$
- [ ] $< ax_1 + x_2, x_3 > = a < x_1, x_3 > + < x_2, x_3 >$
- [ ] $< x_1, x_2 > = < x_2, x_1 >$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad < x, y > = \sum_{\forall i} x_i y_i = x^T y$$

$$< x, y > = x_1 y_1 - x_1 y_2 - x_2 y_1 + 2x_2 y_2$$

$$< x, x > = x_1 x_1 - x_1 x_2 - x_2 x_1 + 2x_2 x_2$$

$$= x_1 x_1 - 2x_1 x_2 + 2x_2 x_2 \quad = x_1^2 - 2x_1 x_2 + 2x_2^2$$

$$(x_1 - x_2)^2 = x_1^2 - 2x_1 x_2 + x_2^2$$

$$x_1^2 - 2x_1 x_2 + 2x_2^2 = (x_1^2 - 2x_1 x_2 + x_2^2) + x_2^2$$

$$\longrightarrow x_1^2 - 2x_1 x_2 + 2x_2^2 = (x_1 - x_2)^2 + x_2^2 \ \longrightarrow < x, x > \geq 0$$

$$< x, x > = 0 \ \longrightarrow (x_1 - x_2)^2 + x_2^2 = 0 \ \longrightarrow \cancel{(x_1 - x_2)^2 = -x_2^2}$$

$$< x, x > = 0 \ \longrightarrow \ x = 0$$

$$< ax + y, z > = (ax_1 + y_1)z_1 - (ax_1 + y_1)z_2 - (ax_2 + y_2)z_1 + 2(ax_2 + y_2)z_2$$

$$= (ax_1 z_1 + y_1 z_1) - (ax_1 z_2 + y_1 z_2) - (ax_2 z_1 + y_2 z_1) + 2(ax_2 z_2 + y_2 z_2)$$

$$= (ax_1 z_1 + y_1 z_1) - (ax_1 z_2 + y_1 z_2) - (ax_2 z_1 + y_2 z_1) + 2(ax_2 z_2 + y_2 z_2)$$

$$= (ax_1 z_1 - ax_1 z_2 - ax_2 z_1 + 2ax_2 z_2) + (y_1 z_1 - y_1 z_2 - y_2 z_1 + 2y_2 z_2)$$

$$= a(x_1 z_1 - x_1 z_2 - x_2 z_1 + 2x_2 z_2) + (y_1 z_1 - y_1 z_2 - y_2 z_1 + 2y_2 z_2)$$

$$= a < x, z > + < y, z >$$

$$< y, x > = y_1 x_1 - y_1 x_2 - y_2 x_1 + 2y_2 x_2$$

$$= x_1 y_1 - x_2 y_1 - x_1 y_2 + 2x_2 y_2$$

$$= x_1 y_1 - x_1 y_2 - x_2 y_1 + 2x_2 y_2$$

$$= < x, y >$$

In Class Exercise 5.1

Show that the dot product of two vectors in R^3 is an inner product space.

In Class Exercise 5.1 SOLUTION

$$< x, y > = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$< x, x > = x_1 x_1 + x_2 x_2 + x_3 x_3 \ = x_1^2 + x_2^2 + x_3^2 \quad \dashrightarrow < x, x > \ \geq \ 0$$

$$< x, x > = 0 \quad \dashrightarrow x_1^2 + x_2^2 + x_3^2 = 0 \quad \dashrightarrow x_1^2 = \cancel{-(x_2^2 + x_3^2)}$$

$$< ax + y, z > = (ax_1 + y_1)z_1 \ + \ (ax_2 + y_2)z_2 + (ax_3 + y_3)z_3$$

$$= ax_1 z_1 + y_1 z_1 + ax_2 z_2 + y_2 z_2 + ax_3 z_3 + y_3 z_3 \ = ax_1 z_1 + ax_2 z_2 + ax_3 z_3 + (y_1 z_1 + + y_2 z_2 + y_3 z_3)$$

$$= a(x_1 z_1 + x_2 z_2 + x_3 z_3) + (y_1 z_1 + + y_2 z_2 + y_3 z_3)$$

$$= a < x, z > + < y, z >$$

$$< y, x > = y_1 x_1 + y_2 x_2 + y_3 x_3$$

$$= x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$= < x, y >$$

LINEAR ALGEBRA > METRIC SPACES

Metrics generalize the notion of distance from the Euclidean space.
But it should be noted that they're not obliged to be vector spaces.
A metric space is a place where metrics reside.
A metric can be defined as:

EXPL45

LINEAR ALGEBRA: MATRIX ALGEBRA

LINEAR ALGEBRA > NORMED SPACES

Norms generalize the notion of length from Euclidean space. They help to measure the size of a vector.

A norm can be defined as:

EXPL46

Norms are greatly used in Machine learning.

Some of the most useful norms are:

EXPL47

The squared L2 norm is more convenient to work with both mathematically and computationally, since the derivative of each element depends on it, unlike the L2 norm where the derivative depends on the whole vector.

EXPL48

Also, in situations where we want to discriminate between elements at zero or very close to zero, the L1 norm is preferred as it increases quickly as compared to the L2 norm. And also mathematical simplicity is maintained.

EXPL49

Another very useful norm is the frobenius norm, which is used to measure the size of a matrix.

LINEAR ALGEBRA > INNER PRODUCT SPACES

An inner product is defined by a function:

EXPL51

We've talked about the dot product in a previous section. The dot product happens to be the most common inner product. We shall redefine it, using inner product notation:

EXPL52

The dot product, isn't the only inner product. Another example of an inner product is:

EXPL53

$B = \{b_1, b_2, \dots, b_n\}, \quad b_1, b_2, \dots, b_n \in V \quad B = base(H \sqsubseteq V)$

$t_1 b_1 + t_2 b_2 + \cdots + t_n b_n = 0 \quad \leftrightarrow \quad t_1 = 0 = \dots = t_n$

$H = Span(B)$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$B = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

$$B =$$

$$B = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -6 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 9 \\ 1 \end{bmatrix} \right\}$$

$$B = \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{8}{10} \\ \frac{4}{10} \end{bmatrix}, \begin{bmatrix} \frac{18}{10} \\ \frac{3}{10} \\ \frac{3}{10} \end{bmatrix}, \begin{bmatrix} -\frac{22}{10} \\ -\frac{13}{10} \\ \frac{7}{2} \end{bmatrix} \right\}$$

$$B =$$

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 8 \\ -5 \\ -6 \\ 1 \end{bmatrix} \right\}$$

$$B =$$

$$B =$$

$B = \{b_1, b_2, \dots, b_n\}, \qquad b_1, b_2, \dots, b_n \in V \qquad B = base(H \sqsubseteq V)$

- [ ] $t_1 b_1 + t_2 b_2 + \cdots + t_n b_n = 0 \quad \leftrightarrow \quad t_1 = 0 = \dots = t_n$
- [ ] $H = Span(B)$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 6 & 2 \end{vmatrix}$$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \in V \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{vmatrix} +\begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} \end{vmatrix}$$

$$\begin{bmatrix} ka_{11} + pa_{21} & ka_{12} + pa_{22} & ka_{13} + pa_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} -4 & 4 & 7 \\ 1 & 1 & 9 \\ 0 & -1 & 8 \end{bmatrix} \quad \begin{vmatrix} -4 & 4 & 7 \\ 1 & 1 & 9 \\ 0 & -1 & 8 \end{vmatrix} \quad \begin{bmatrix} -22 & 0 & 0 \\ 14 & 2 & -6 \\ 5 & -4 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 & 9 \\ -4 & 4 & 7 \\ 0 & -1 & 8 \end{vmatrix} \quad \begin{vmatrix} 1 & 1 & 9 \\ 0 & 8 & 43 \\ 0 & -1 & 8 \end{vmatrix} \quad \begin{vmatrix} 1 & 1 & 9 \\ 0 & 8 & 43 \\ 0 & -8 & 64 \end{vmatrix} \quad \begin{vmatrix} 1 & 1 & 9 \\ 0 & 8 & 43 \\ 0 & 0 & 107 \end{vmatrix} \quad \begin{bmatrix} -22 & 14 & 5 \\ 0 & 2 & -4 \\ 0 & -6 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -\frac{7}{11} & -\frac{5}{22} \\ 0 & -\frac{1}{11} & \frac{2}{11} \\ 0 & -\frac{3}{11} & -\frac{1}{22} \end{bmatrix}$$