COMPLETE LINEAR ALGEBRA

$$A \in R^{nxn} \quad A = A^T \quad \forall i,j \in [1,n], a_{i,j} = a_{j,i}$$

$$A = -A^T$$

$$A = XDX^{-1} \quad A = A^T \longrightarrow A = VDV^T$$

$$A = VDV^T \longrightarrow V^TAV = D$$

$$\text{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} = a_{1,2} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} = a_{1,n} & a_{n,2} = a_{2,n} & \cdots & a_{n,n} \end{bmatrix}$$

$$\text{A}_1 = \begin{bmatrix} 1 & -6 \\ -6 & 3 \end{bmatrix} \quad \text{A}_1^\text{T} = \begin{bmatrix} 1 & -6 \\ -6 & 3 \end{bmatrix}$$

$$\text{A}_2 = \begin{bmatrix} 5 & 6 & 0 \\ 6 & 4 & -3 \\ 1 & -3 & 10 \end{bmatrix} \quad \text{A}_2^\text{T} = \begin{bmatrix} 5 & 6 & 1 \\ 6 & 4 & -3 \\ 0 & -3 & 10 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{A}_3^\text{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 5 & 1 & 6 \\ 1 & 6 & 3 \end{bmatrix} \quad \text{A}_4^\text{T} = \begin{bmatrix} 5 & 1 \\ 1 & 6 \\ 6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} = X \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} X^T$$

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2} & 1 \\ 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} -1 & -\frac{1}{2} & 1 \\ 1 & -\frac{1}{2} & 1 \\ 0 & 1 & 1 \end{bmatrix}^T$$

$$\mathbf{u}_1' = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_2' = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{u}_1'}{\mathbf{u}_1' \cdot u_1'} \mathbf{u}_1' \right) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{2} \right) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$\mathbf{u}_3' = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{u}_1'}{\mathbf{u}_1' \cdot u_1'} \mathbf{u}_1' + \frac{\mathbf{u}_3 \cdot \mathbf{u}_2'}{\mathbf{u}_2' \cdot u_2'} u_2' \right)$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}}{\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

In Class Exercise 9.1

Orthogonally diagonalize A

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

In Class Exercise 9.1
SOLUTION

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & -\frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}^T$$

Quadratic Forms

$A \in R^{nxn} \quad A = A^T$

$Q = x^T A x \quad Q(x_1, \ldots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^T A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in R$

$Q(x) > 0: Positive\ Definite \quad Q(x) \geq 0: Positive\ Semi - Definite$

$Q(x) < 0: Negative\ Definite \quad Q(x) \leq 0: Negative\ Semi - Definite$

$Q(x) > 0\ or\ Q(x) < 0: Indefinite$

Find the Quadratic form of $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$

Doesn't exist

Find the Quadratic form of $B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

$$Q(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 4 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Q = \begin{bmatrix} x_1 + 4x_2 & 4x_1 - x_2 + 2x_3 & 2x_2 + 2x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1(x_1 + 4x_2) + x_2(4x_1 - x_2 + 2x_3) + x_3(2x_2 + 2x_3)]$$

$$= x_1^2 + 4x_1x_2 + 4x_2x_1 - x_2^2 + 2x_2x_3 + 2x_2x_3 + 2x_3^2$$

$$= x_1^2 - x_2^2 + 2x_3^2 + 8x_1x_2 + 4x_2x_3$$

$x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ $Q(x) = (-1)^2 - (1)^2 + 2(0)^2 + 8(-1)(1) + 4(1)(0)$

$= -8 < 0$

$x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $Q(x) = (1)^2 - (1)^2 + 2(0)^2 + 8(1)(1) + 4(1)(0)$

$= 8 > 0$

$\lambda_1 = -4.52 < 0, \lambda_2 = 4.74 > 0, \lambda_3 = 1.78 > 0$

In Class Exercise 9.2

Show that C is positive definite by evaluating its eigen values and by using the definition

$C = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

In Class Exercise 9.2 SOLUTION

$$\lambda_1 = 3 > 0, \lambda_2 = 1 > 0$$

$$Q(x) = x_1^2 + 2x_1x_2 + 3x_2^2$$

$$= x_1^2 + 2x_1x_2 + x_2^2 + 2x_2^2 \qquad (x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2$$

$$= (x_1 + x_2)^2 + 2x_2^2 \quad > 0$$