COMPLETE LINEAR ALGEBRA

- Python Setup, Basic Python, numpy, pandas and matplotlib
- Matrix Algebra
- Linear equations and transformations
- Vectors
- Vector Spaces
- Metric Spaces, Normed spaces, Inner Product Spaces
- Orthogonality
- Determinant and Trace Operator
- Matrix Decompositions (Eigen, SVD and Cholesky)
- Symmetric matrices and Quadratic Forms
- Left Inverse, Right Inverse, Pseudo Inverse

Connecting the dots 1
LINEAR REGRESSION

Connecting the dots 2
PRINCIPAL COMPONENT ANALYSIS

COMPLETE LINEAR ALGEBRA

LINEAR ALGEBRA > Orthogonality > **Orthogonal and Orthonormal Vectors**

$u, v \in R^n \ < u, v > = 0 \quad \|u\| = \|v\| = 1$

$u_1, u_2, \ldots, u_n \in R^n \ \forall i \neq j, < u_i, u_j > = 0 \ \|u_1\| = \cdots = \|v_n\| = 1$

$A = [u_1, u_2, \ldots, u_n] \longrightarrow A^{-1} = A^T$

$H \sqsubseteq R^n \quad U = \{u_1, \ldots, u_n\} \ / \ \forall i \neq j, < u_i, u_j > = 0 \ \ \& \ U = base(H)$

$\forall x \in H, x = k_1 u_1 + \cdots + k_n u_n \quad k_i = \frac{x \cdot u_i}{u_i \cdot u_i} \quad i = \overline{1, \ldots, n}$

$u_1 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, u_2 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$ Are they orthogonal vectors?

$< u_1, u_2 > = u_1 \cdot u_2 = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} = (-1 * 5) + (2 * 1) + (3 * 1) = 0$

$\|u\| \neq 1 \ \ \& \ \ \|v\| \neq 1$

$u_1 = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}, u_2 = \begin{bmatrix} 7 \\ 1 \\ 4 \\ 8 \end{bmatrix}, u_3 = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 1 \end{bmatrix}, u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$ Do they form an orthogonal set?

$< u_1, u_2 > = u_1 \cdot u_2 = \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = (5 * 0) + (4 * 0) + (3 * 0) + (2 * 1) = 2$

$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$ Do they form and orthogonal set?

$< u_1, u_2 > = u_1 \cdot u_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad < u_2, u_3 > = u_2 \cdot u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$< u_1, u_3 > = u_1 \cdot u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \|u_1\| = \|u_2\| = \|u_2\| = 1$

$x = k_1 u_1 + k_2 u_2 + \cdots + k_i u_i + \cdots + k_n u_n$

$\longrightarrow x \cdot u_i = (k_1 u_1 + k_2 u_2 + \cdots + k_i u_i + \cdots + k_n u_n) \cdot u_i$

$\longrightarrow x \cdot u_i = (k_1 u_1 \cdot u_i + k_2 u_2 \cdot u_i + \cdots + k_i u_i \cdot u_i + \cdots + k_n u_n \cdot u_i)$

$\longrightarrow x \cdot u_i = (k_1(0) + k_2(0) + \cdots + k_i(u_i \cdot u_i) + \cdots + k_n(0)) \longrightarrow x \cdot u_i = k_i(u_i \cdot u_i)$

$\longrightarrow k_i = \frac{x \cdot u_i}{u_i \cdot u_i}$

$B = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \ B = basis(R^2), x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$x = k_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad k_1 = \frac{x \cdot u_1}{u_1 \cdot u_1} = \frac{\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix}} = 1 \quad k_2 = \frac{x \cdot u_2}{u_2 \cdot u_2} = \frac{\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}} = 0$

$E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \ E = basis(R^2), x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$x = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$c_1 = \frac{x \cdot u_1}{u_1 \cdot u_1} = \frac{\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}} = 2 \quad c_2 = \frac{x \cdot u_2}{u_2 \cdot u_2} = \frac{\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}} = -1$

In Class Exercise 6.1

– *Find* $P_{B \leftarrow E}$ & $[x]_B$

– *Show that the angle between 2 vectors is* 90^0

In Class Exercise 6.1 SOLUTION

$$E = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad B = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = s_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow \begin{cases} 1 = 2s_1 + s_2 \\ 0 = -s_1 + 2s_2 \end{cases}$$

$$\longrightarrow \begin{cases} s_1 = \frac{2}{5} \\ s_2 = \frac{1}{5} \end{cases}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = t_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow \begin{cases} 0 = 2t_1 + t_2 \\ 1 = -t_1 + 2t_2 \end{cases}$$

$$\longrightarrow \begin{cases} t_1 = -\frac{1}{5} \\ t_2 = \frac{2}{5} \end{cases}$$

$$\longrightarrow P_{B \leftarrow E} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

$$\longrightarrow [x]_B = P_{B \leftarrow E}$$

$$\longrightarrow [x]_B = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$< u, v > = 0 \qquad u \cdot v = 0$$

$$\longrightarrow \|u\| \|v\| cos\theta = 0$$

$$\longrightarrow cos\theta = 0 \quad \longrightarrow \theta = cos^{-1}(0) = 90^0$$

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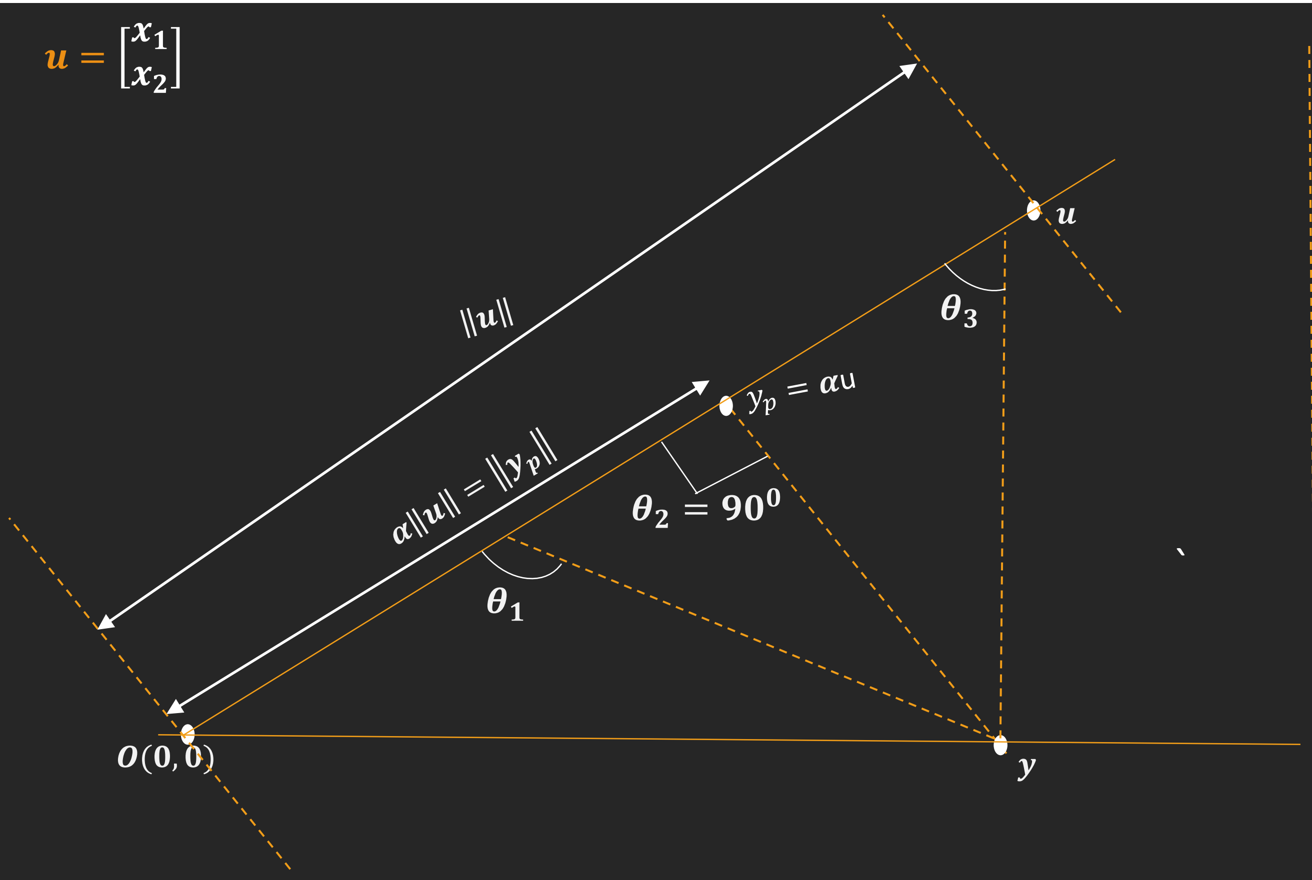
Orthogonal projection onto a one-dimensional space



$$(y - y_p) \cdot u = 0 \quad \longrightarrow (y - \alpha u) \cdot u = 0$$

$$< ax_1 + x_2, x_3 > = a < x_1, x_3 > + < x_2, x_3 >$$

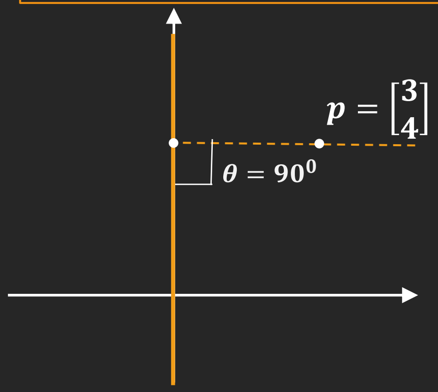
$$\longrightarrow (ax_1 + x_2) \cdot x_3 = ax_1 \cdot x_3 + x_2 \cdot x_3$$

$$\longrightarrow (y \cdot u) - \alpha(u \cdot u) = 0 \quad \longrightarrow (y \cdot u) = \alpha(u \cdot u)$$

$$\longrightarrow \alpha = \frac{y \cdot u}{u \cdot u} \qquad \boxed{\longrightarrow y_p = Proj_u(y) = \left(\frac{y \cdot u}{u \cdot u}\right) u}$$

Find the orthogonal projection of $p = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

on the line spanned by a vector $u = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$



$$\alpha = \left(\frac{y \cdot u}{u \cdot u}\right) = \left(\frac{\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 5 \end{bmatrix}}{\begin{bmatrix} 0 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 5 \end{bmatrix}}\right) = \frac{4}{5}$$

$$[y_p]_u = \frac{4}{5} \qquad [y_p]_{R^2} = \frac{4}{5}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

COMPLETE LINEAR ALGEBRA

LINEAR ALGEBRA > Orthogonality > Orthogonal projection onto an n-dimensional space

$U \sqsubseteq R^n \quad B = base(U)$

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix}$$

$b_1 \quad b_2 \quad b_n$

$$= [b_1 \quad b_2 \quad \ldots \quad b_n]$$

$$Proj_U(y) = \sum_{i=1}^{n} \propto_i b_i \qquad \propto = \begin{bmatrix} \propto_1 \\ \vdots \\ \propto_n \end{bmatrix}$$

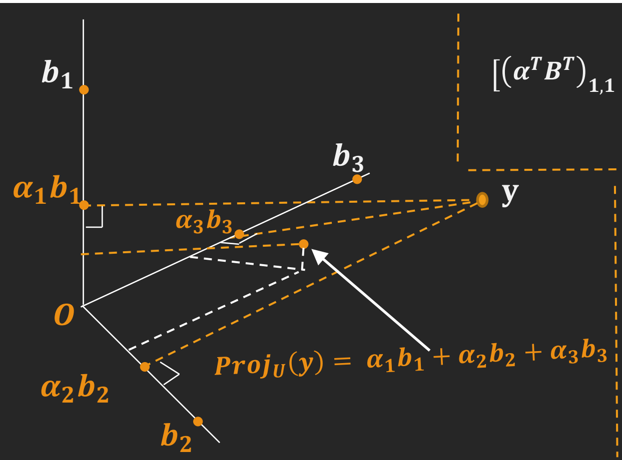
$$Proj_U(y) = \propto_1 b_1 + \propto_2 b_2 + \cdots + \propto_n b_n$$

$$= \propto_1 \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ \vdots \\ b_{n,1} \end{bmatrix} + \propto_2 \begin{bmatrix} b_{1,2} \\ b_{2,2} \\ \vdots \\ b_{n,2} \end{bmatrix} + \cdots + \propto_n \begin{bmatrix} b_{1,n} \\ b_{2,n} \\ \vdots \\ b_{n,n} \end{bmatrix}$$

$$= \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} \begin{bmatrix} \propto_1 \\ \vdots \\ \propto_n \end{bmatrix}$$

$$= B\alpha$$

$$[Proj_U(y)]_B = \begin{bmatrix} \propto_1 \\ \vdots \\ \propto_n \end{bmatrix}$$



$$\begin{cases} (Proj_U(y) - y) \cdot b_1 = 0 \\ (Proj_U(y) - y) \cdot b_2 = 0 \\ \vdots \\ (Proj_U(y) - y) \cdot b_n = 0 \end{cases} \longrightarrow \begin{cases} (B\alpha - y) \cdot b_1 = 0 \\ (B\alpha - y) \cdot b_2 = 0 \\ \vdots \\ (B\alpha - y) \cdot b_n = 0 \end{cases} \longrightarrow \begin{cases} B\alpha \cdot b_1 - y \cdot b_1 = 0 \\ B\alpha \cdot b_2 - y \cdot b_2 = 0 \\ \vdots \\ B\alpha \cdot b_n - y \cdot b_n = 0 \end{cases}$$

$$\longrightarrow \begin{cases} (B\alpha)^T b_1 - y^T b_1 = 0 \\ (B\alpha)^T b_2 - y^T b_2 = 0 \\ \vdots \\ (B\alpha)^T b_n - y^T b_n = 0 \end{cases} \longrightarrow \begin{cases} \alpha^T B^T b_1 - y^T b_1 = 0 \\ \alpha^T B^T b_2 - y^T b_2 = 0 \\ \vdots \\ \alpha^T B^T b_n - y^T b_n = 0 \end{cases}$$

$$\longrightarrow \begin{cases} [(\alpha^T B^T)_{1,1} \quad \ldots \quad (\alpha^T B^T)_{1,n}] \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ \vdots \\ b_{n,1} \end{bmatrix} - [y_{1,1} \quad \ldots \quad y_{1,n}] \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ \vdots \\ b_{n,1} \end{bmatrix} = 0 \\ [(\alpha^T B^T)_{1,1} \quad \ldots \quad (\alpha^T B^T)_{1,n}] \begin{bmatrix} b_{1,2} \\ b_{2,2} \\ \vdots \\ b_{n,2} \end{bmatrix} - [y_{1,1} \quad \ldots \quad y_{1,n}] \begin{bmatrix} b_{1,2} \\ b_{2,2} \\ \vdots \\ b_{n,2} \end{bmatrix} = 0 \\ \vdots \\ [(\alpha^T B^T)_{1,1} \quad \ldots \quad (\alpha^T B^T)_{1,n}] \begin{bmatrix} b_{1,n} \\ b_{2,n} \\ \vdots \\ b_{n,n} \end{bmatrix} - [y_{1,1} \quad \ldots \quad y_{1,n}] \begin{bmatrix} b_{1,n} \\ b_{2,n} \\ \vdots \\ b_{n,n} \end{bmatrix} = 0 \end{cases}$$

$$[(\alpha^T B^T)_{1,1} \quad \ldots \quad (\alpha^T B^T)_{1,n}] \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} - [y_{1,1} \quad \ldots \quad y_{1,n}] \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} = 0$$

$$\longrightarrow \alpha^T B^T B - y^T B = 0 \quad \longrightarrow \alpha^T B^T B = y^T B \quad \longrightarrow \alpha^T (B^T B)(B^T B)^{-1} = y^T B (B^T B)^{-1}$$

$$\longrightarrow \alpha^T = y^T B (B^T B)^{-1} \quad \longrightarrow \alpha = (\alpha^T)^T = \left(y^T B (B^T B)^{-1}\right)^T$$

$$(A_1 * A_2 * \cdots * A_n)^T = A_n * A_{n-1} * \cdots * A_1 \; \& \; (ABC)^T = C^T B^T A^T$$

$$\longrightarrow \alpha = \left((B^T B)^{-1}\right)^T B^T (y^T)^T \qquad (A^{-1})^T = (A^T)^{-1} \qquad \longrightarrow \alpha = \left((B^T B)^T\right)^{-1} B^T y$$

$$(B^T B)^T = B^T (B^T)^T = B^T B \qquad \longrightarrow \alpha = (B^T B)^{-1} B^T y \qquad Proj_U(y) = B\alpha$$

$$\longrightarrow [Proj_U(y)]_E = B (B^T B)^{-1} B^T y = Proj_U(y)$$

$$[Proj_U(y)]_B = (B^T B)^{-1} B^T y$$

$$\mathrm{B}^T = B^{-1} \quad \longrightarrow \mathrm{B}^T \mathrm{B} = B^{-1} B = I_n$$

$$\longrightarrow [Proj_U(y)]_E = B (I_n)^{-1} B^T y = B B^T y$$

$$y = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C^T C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$(C^T C)^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 6 & -1 & -2 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$(C^T C)^{-1} C^T = \begin{bmatrix} 6 & -1 & -2 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha = \left((C^T C)^T\right)^{-1} C^T y = \begin{bmatrix} -1 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 4 \end{bmatrix}$$

$$\longrightarrow [Proj_U(y)]_C = \begin{bmatrix} -6 \\ 0 \\ 4 \end{bmatrix} \qquad \longrightarrow [Proj_U(y)]_E = C\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

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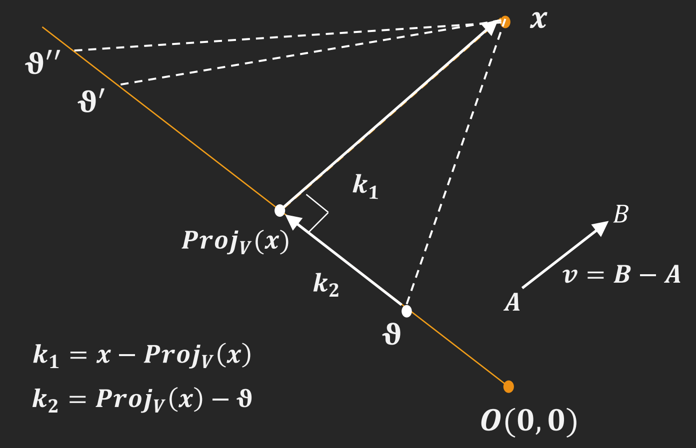
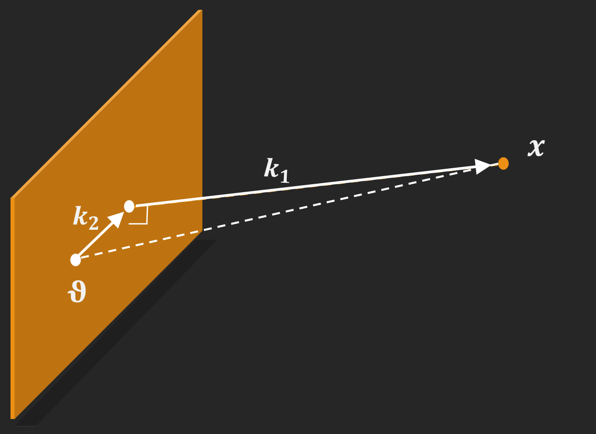
LINEAR ALGEBRA > Orthogonality > **Minimal distance from point to an n-dimensional space**

$V \sqsubseteq R^n \quad \vartheta \epsilon V \quad \vartheta' \epsilon V \quad \vartheta'' \epsilon V$

$\forall x \epsilon \mathrm{R}^n \quad \|x - Proj_V(x)\| \leq \|x - \vartheta\|$





$$k_1 = x - Proj_V(x) \quad \longrightarrow x = k_1 + Proj_V(x)$$

$$k_2 = Proj_V(x) - \vartheta \quad \longrightarrow \vartheta = Proj_V(x) - k_2$$

$$\|x - \vartheta\|^2 = \|k_1 + Proj_V(x) - (Proj_V(x) - k_2)\|^2$$

$$= \|k_1 + Proj_V(x) + k_2 - Proj_V(x)\|^2$$

$$= \|k_1 + k_2\|^2$$

$$\|x\|^2 = x \cdot x \quad \longrightarrow \|(k_1 + k_2)\|^2 = (k_1 + k_2) \cdot (k_1 + k_2)$$

$$\longrightarrow \|x - \vartheta\|^2 = k_1 \cdot k_1 + k_1 \cdot k_2 + k_2 \cdot k_1 + k_2 \cdot k_2$$

$$= \|k_1\|^2 + 2k_1 \cdot k_2 + \|k_2\|^2$$

$$k_1 \perp k_2 \quad \longrightarrow \quad k_1 \cdot k_2 = 0$$

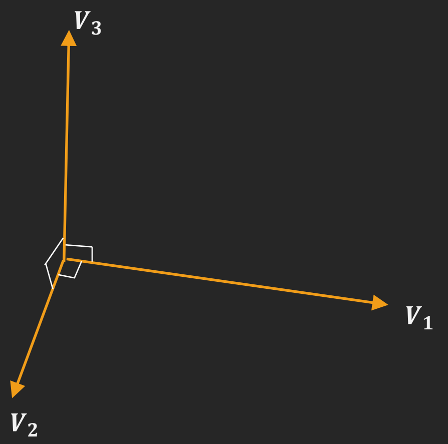
$$= \|k_1\|^2 + \|k_2\|^2$$

$$\geq \|k_1\|^2$$

$$\longrightarrow \|x - \vartheta\|^2 \geq \|x - Proj_V(x)\|^2$$

$$\longrightarrow \|x - Proj_V(x)\|^2 \leq \|x - \vartheta\|^2$$

$$\boxed{\longrightarrow \|x - Proj_V(x)\| \leq \|x - \vartheta\|}$$



$V_1 = u_1 \qquad Proj_u(y) = \left(\frac{y \cdot u}{u \cdot u}\right) u$

$$V_2 = u_2 - Proj_{V_1}(u_2) \quad = u_2 - \frac{u_2 \cdot V_1}{V_1 \cdot V_1} V_1$$

$$\dashrightarrow V_2 = u_2 - \frac{u_2 \cdot V_1}{V_1 \cdot V_1} V_1$$

$$V_3 = u_3 - \left(Proj_{V_1}(u_3) + Proj_{V_2}(u_3)\right) \quad = u_3 - \left(\frac{u_3 \cdot V_1}{V_1 \cdot V_1} V_1 + \frac{u_3 \cdot V_2}{V_2 \cdot V_2} V_2\right)$$

$$\dashrightarrow V_3 = u_3 - \left(\frac{u_3 \cdot V_1}{V_1 \cdot V_1} V_1 + \frac{u_3 \cdot V_2}{V_2 \cdot V_2} V_2\right)$$

$$\dashrightarrow V_n = u_n - \left(\frac{u_n \cdot V_1}{V_1 \cdot V_1} V_1 + \frac{u_n \cdot V_2}{V_2 \cdot V_2} V_2 + \cdots + \frac{u_n \cdot V_{n-1}}{V_{n-1} \cdot V_{n-1}} V_{n-1}\right)$$

$$\mathrm{B} = \left\{\begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix}\right\} = \{b_1, b_2, b_3\}$$

$B = base(H), H \sqsubseteq R^4$

$Find\ \{B_1, B_2, B_3\}$

$$B_3 = b_3 - \left(\frac{b_3 \cdot B_1}{B_1 \cdot B_1} B_1 + \frac{b_3 \cdot B_2}{B_2 \cdot B_2} B_2\right)$$

$$B_1 = b_1$$

$$B_2 = b_2 - \frac{b_2 \cdot B_1}{B_1 \cdot B_1} B_1$$

$$= \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}}{\begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}} \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix} - \frac{12 + 6 + 36 - 6}{4 + 4 + 16 + 4} \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{24}{7} \\ -\frac{24}{7} \\ \frac{48}{7} \\ -\frac{24}{7} \end{bmatrix} = \begin{bmatrix} \frac{18}{7} \\ \frac{3}{7} \\ \frac{15}{7} \\ \frac{45}{7} \end{bmatrix}$$

$$\dashrightarrow B_3 = \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} - \left(\frac{\begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}}{\begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}} \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix} + \frac{\begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} \frac{18}{7} \\ \frac{3}{7} \\ \frac{15}{7} \\ \frac{45}{7} \end{bmatrix}}{\begin{bmatrix} \frac{18}{7} \\ \frac{3}{7} \\ \frac{15}{7} \\ \frac{45}{7} \end{bmatrix} \cdot \begin{bmatrix} \frac{18}{7} \\ \frac{3}{7} \\ \frac{15}{7} \\ \frac{45}{7} \end{bmatrix}} \begin{bmatrix} \frac{18}{7} \\ \frac{3}{7} \\ \frac{15}{7} \\ \frac{45}{7} \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} - \left(\begin{bmatrix} 2.72 \\ -2.72 \\ 5.44 \\ -2.72 \end{bmatrix} + \begin{bmatrix} 0.54 \\ 0.09 \\ 0.45 \\ 1.35 \end{bmatrix}\right) = \begin{bmatrix} -1.26 \\ -0.37 \\ -0.89 \\ -2.63 \end{bmatrix}$$

$$B_1' = \frac{b_1}{\|b_1\|} = \frac{b_1}{\sqrt{2^2 + (-2)^2 + 4^2 + (-2)^2}} = 0.189 b_1$$

$$B_2' = \frac{B_2}{\|B_2\|} = \frac{B_2}{\sqrt{(\frac{18}{7})^2 + (\frac{3}{7})^2 + (\frac{15}{7})^2 + (\frac{45}{7})^2}} = 0.137 B_2$$

$$B_3' = \frac{B_3}{\|B_3\|} = \frac{B_3}{\sqrt{(-1.26)^2 + (-0.37)^2 + (-0.89)^2 + (-2.63)^2}} = 0.106 B_3$$

In Class Exercise 6.2

$Orthonormalize\ U$

$$U = \left\{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right\}$$

In Class Exercise 6.2
SOLUTION

$$V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$$V' = \left\{ \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{2}{\sqrt{2}} \end{bmatrix} \right\}$$

- $U \perp V \dashrightarrow \|U+V\|^2 = \|U\|^2 + \|V\|^2$
- $\|U+V\| \leq \|U\|\|V\|$

$\|x\|^2 = < x, x >$

$\dashrightarrow \|U+V\|^2 = < U+V, U+V >$

$\dashrightarrow \|U+V\|^2 = < U,U > + < U,V > + < V,U > + < V,V >$

$U \perp V \dashrightarrow < U,V \geq 0$

$= < U,U > + < V,V >$

$= \|U\|^2 + \|V\|^2$

$\dashrightarrow \|U+V\|^2 = \|U\|^2 + \|V\|^2$

$\|U+V\|^2 = < U,U > + < U,V > + < V,U > + < V,V >$

$\|U+V\|^2 = < U,U > +2 < U,V > + < V,V >$

$\leq < U,U > +2| < U,V > | + < V,V >$

$= \|U\|^2 + 2| < U,V > | + \|V\|^2$

$< U,V > = \|U\|\|V\| cos \propto$

$\dashrightarrow < U,V > \leq \|U\|\|V\|$

$\leq \|U\|^2 + 2\|U\|\|V\| + \|V\|^2$

$= \|U\|^2 + \|V\|^2$

$\dashrightarrow \|U+V\|^2 \leq \|U\|^2 + \|V\|^2$

LINEAR ALGEBRA > ORTHOGONALITY

ORTHOGONAL VECTORS:
Two vectors in Rn, ui and uj are said to be orthogonal if:
EXPL54
ORTHOGONAL SET:
A set of vectors {u1,…un} are said to form an orthogonal set if for each distinct pair of different vectors, their inner product is zero:
ORTHOGONAL MATRIX:
An orthogonal matrix is a square matrix, made up of an orthogonal set of vectors.
EXPL55:

ORTHONORMAL VECTORS:
Two vectors in Rn, ui and uj are said to be orthonormal if:
They are orthogonal and their norms equals 1.
EXPL58

Considering an orthogonal set,
EXPL56:
ORTHOGONAL PROJECTION onto a one dimensional subspace
The orthogonal projection of a point, P on a line, is that point, Q on the line which when connected to the point P, is perpendicular to the line.
The orthogonal projection of the point y to a line u is derived as such:
EXPL57

LINEAR ALGEBRA: MATRIX ALGEBRA

LINEAR ALGEBRA > ORTHOGONALITY

ORTHOGONAL PROJECTION ONTO an n-dimensional subspace.
The orthogonal projection of a point, y to a subspace, U in Rn is the sum of all the projections from that point to each vector which is in the bases, B of U.
Also, the difference between the orthogonal projection and the point y is perpendicular to each and every vector which is part of the Bases, B of U.

EXPL 88

The orthogonal projection from a point to a subspace is the shortest distance from that point to the subspace. This is a very useful property.

EXPL95

GRAM-SCHMIDT PROCESS
With the help of this process, we can convert the basis {u1,…,un} of a subspace of Rn into an orthogonal or orthonormal base{v1,…,vn}. Let us look at a concrete example.

EXPL59

PYTHAGORAN THEOREM

EXPL60

CAUCHY SCHWARZ INEQUALITY

EXPL61

TRIANGULAR INEQUALITY

EXPL62

EXPL63
ORTHOGONAL COMPLEMENTS
If H is a subspace of a vector space, V, then the complement of H is:
EXPL64
D

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QR FACTORIZATION OF MATRICES
A method of solving linear equations of type AX = b, where A is made of linearly independent columns, is by factorizing A (m by n matrix) into a matrix Q (and m by n orthonormal matrix) and R (an n by n upper triangular invertible matrix with all its diagonal elements greater than zero)