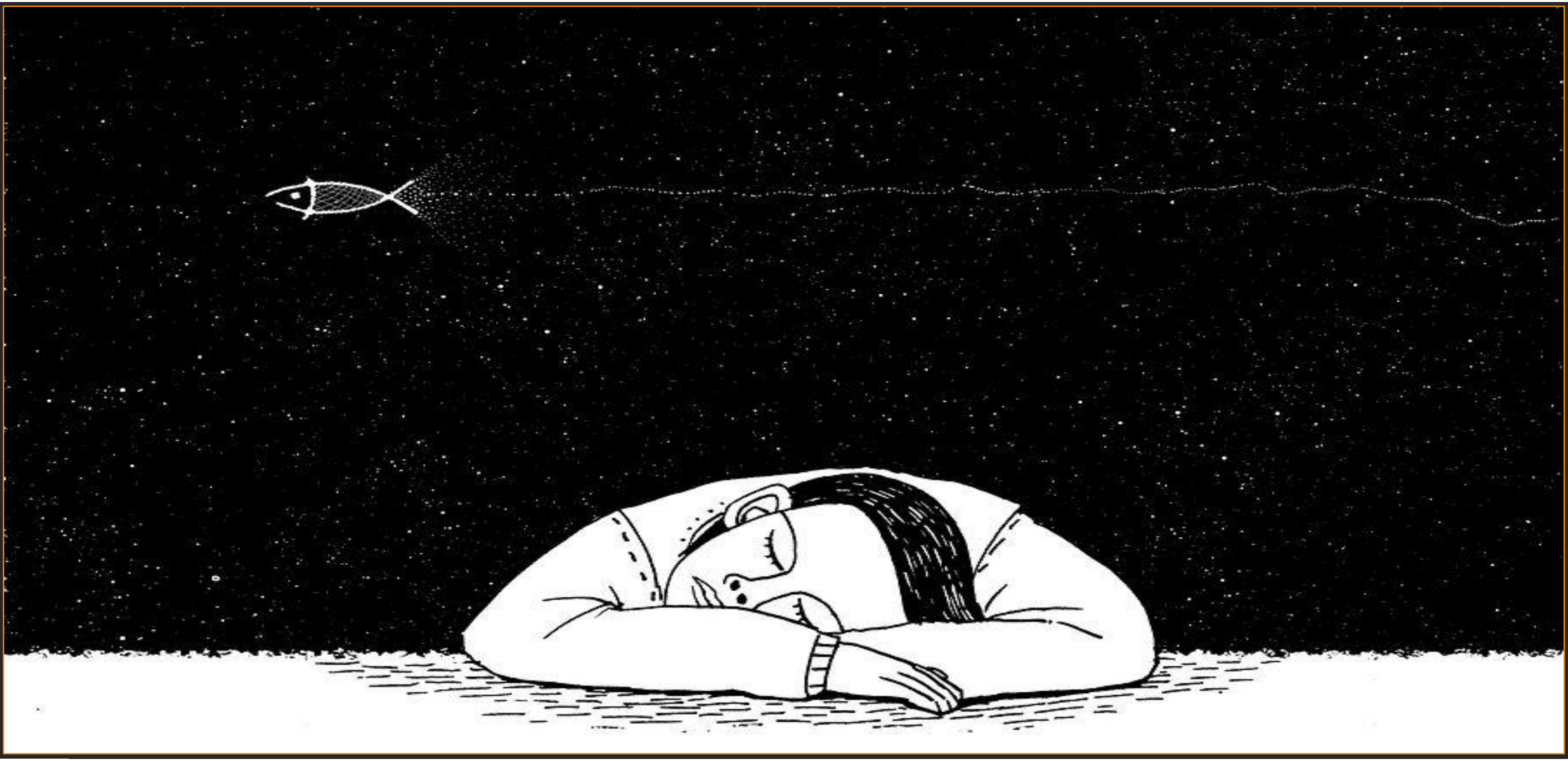
COMPLETE LINEAR ALGEBRA

LINEAR ALGEBRA > INTRODUCTION > About This Course

- Python Setup, Basic Python, numpy, pandas and matplotlib
- Matrix Algebra
- Linear equations and transformations
- Vectors
- Vector Spaces
- Metric Spaces, Normed spaces, Inner Product Spaces
- Orthogonality
- Determinant and Trace Operator
- Matrix Decompositions (Eigen, SVD and Cholesky)
- Symmetric matrices and Quadratic Forms
- Left Inverse, Right Inverse, Pseudo Inverse

Connecting the dots 1
LINEAR REGRESSION

Connecting the dots 2
PRINCIPAL COMPONENT ANALYSIS



COMPLETE LINEAR ALGEBRA

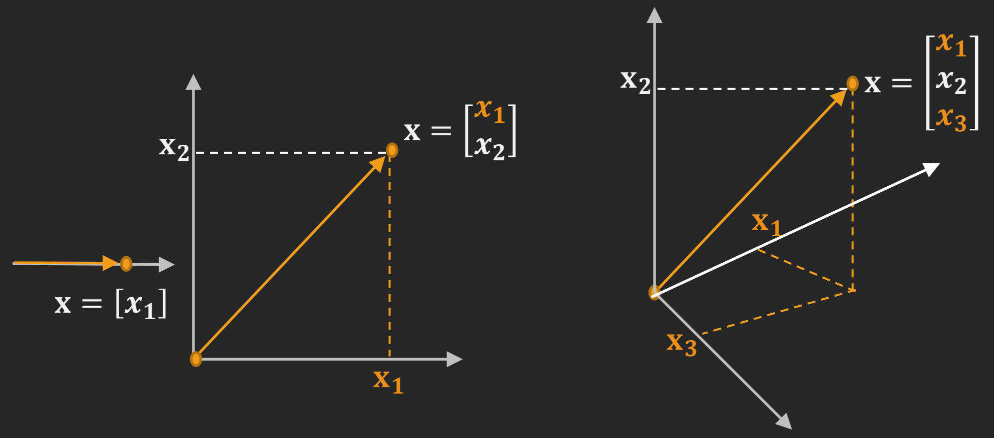
LINEAR ALGEBRA > **VECTOR SPACES** > Definition

$+: V X V \longrightarrow V$ $\quad \cdot : \Re X V \longrightarrow V$

$x, y, z \in V \quad \propto, \beta \in \Re$

- $\exists e_+ \;/\; x + e_+ = x, \forall x \in V$
- $\forall x \in V \;\; \exists x^{-1} \;/\; x + x^{-1} = e_+$
- $\exists e_. \;/\; x \cdot e_. = e_. \cdot x = x$
- $\forall x, y \in V \;, x + y = y + x$
- $\forall x, y, z \in V \;\&\; \propto, \beta \in \Re$
  - $(x + y) + z = x + (y + z)$
  - $\propto \cdot (\beta \cdot x) = (\propto \cdot \beta) \cdot x$
- $\forall x, y \in V \;\&\; \propto, \beta \in \Re$
  - $\propto \cdot (x + y) = \propto \cdot x + \propto \cdot y$
  - $(\propto + \beta) x = \propto \cdot x + \beta \cdot x$

$V = \Re^n \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$



$V = \Re^{mxn}, m, n \geq 1 \quad x = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,(n-1)} & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,(n-1)} & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(m-1),1} & x_{(m-1),2} & \cdots & x_{(m-1),(n-1)} & x_{(m-1),n} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,(n-1)} & x_{m,n} \end{bmatrix}$

$V = P_n \quad x = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \quad a_0, .., a_n \in R, t \in R$

COMPLETE LINEAR ALGEBRA

$$\forall x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \Re^n \quad \exists e_+ = 0_{R^n} = \begin{bmatrix} x_1 = 0 \\ x_2 = 0 \\ \vdots \\ x_n = 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} / \; x + 0_{R^n} = x$$

$$\forall x \in \Re^n \quad \exists x^{-1} = \begin{bmatrix} x_1 = -x_1 \\ x_2 = -x_2 \\ \vdots \\ x_n = -x_n \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix} \; / \; x + x^{-1} = 0_{R^n}$$

$$\forall x \in \Re^n \quad \exists e_. = 1 \in \Re = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} * 1 = 1 * \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\forall x, y \in \Re^n \quad (x + y) = \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right) = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$(y + x) = \left(\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right) = \begin{bmatrix} y_1 + x_1 \\ y_2 + x_2 \\ \vdots \\ y_n + x_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$\forall x, y, z \in \Re^n \quad (x + y) + z = \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \right) + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + z_2 \\ \vdots \\ x_n + y_n + z_n \end{bmatrix}$$

$$x + (y + z) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \left(\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \right) = \begin{bmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + z_2 \\ \vdots \\ x_n + y_n + z_n \end{bmatrix}$$

$$\forall x \in \Re^n, \propto, \beta \in \Re \quad (\propto * \beta) * x = (\propto * \beta) * \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \propto \beta x_1 \\ \propto \beta x_2 \\ \vdots \\ \propto \beta x_n \end{bmatrix} \qquad \propto * (\beta * x) = \propto \begin{bmatrix} \beta x_1 \\ \beta x_2 \\ \vdots \\ \beta x_n \end{bmatrix} = \begin{bmatrix} \propto \beta x_1 \\ \propto \beta x_2 \\ \vdots \\ \propto \beta x_n \end{bmatrix}$$

$$\forall x, y \in \Re^n, \propto \in \Re \quad \propto (x + y) = \propto * \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} \propto (x_1 + y_1) \\ \propto (x_2 + y_2) \\ \vdots \\ \propto (x_n + y_n) \end{bmatrix} = \begin{bmatrix} \propto x_1 + \propto y_1 \\ \propto x_2 + \propto y_2 \\ \vdots \\ \propto x_n + \propto y_n \end{bmatrix}$$

$$\propto * (x) + \propto * (y) = \propto \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \propto \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \propto x_1 + \propto y_1 \\ \propto x_2 + \propto y_2 \\ \vdots \\ \propto x_n + \propto y_n \end{bmatrix}$$

$$\forall x \in \Re^n, \propto, \beta \in \Re \quad (\propto + \beta) * x = (\propto + \beta) * \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (\propto + \beta) x_1 \\ (\propto + \beta) x_2 \\ \vdots \\ (\propto + \beta) x_n \end{bmatrix} = \begin{bmatrix} \propto * x_1 + \beta * x_1 \\ \propto * x_2 + \beta * x_2 \\ \vdots \\ \propto * x_n + \beta * x_n \end{bmatrix}$$

$$\propto * (x) + \propto * (y) = \propto \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \propto \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \propto x_1 + \propto y_1 \\ \propto x_2 + \propto y_2 \\ \vdots \\ \propto x_n + \propto y_n \end{bmatrix}$$

In Class Exercise 4.1

Show that $V = R^{mxn}$, is a vector space

In Class Exercise 4.1 SOLUTION

$\boldsymbol{x} + \boldsymbol{e}_{+} = \boldsymbol{x}, \forall \boldsymbol{x} \in \boldsymbol{V}$ $\quad \boldsymbol{x} + \boldsymbol{x}^{-1} = \boldsymbol{e}_{+}$ $\quad \boldsymbol{x} \cdot \boldsymbol{e}_{.} = \boldsymbol{e}_{.} \cdot \boldsymbol{x} = \boldsymbol{x}$ $\quad \boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$

$(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$ $\quad \propto \cdot (\boldsymbol{\beta} \cdot \boldsymbol{x}) = (\propto \cdot \boldsymbol{\beta}) \cdot \boldsymbol{x}$

$\propto \cdot (\boldsymbol{x} + \boldsymbol{y}) = \propto \cdot \boldsymbol{x} + \propto \cdot \boldsymbol{y}$ $\quad (\propto + \boldsymbol{\beta})\boldsymbol{x} = \propto \cdot \boldsymbol{x} + \boldsymbol{\beta} \cdot \boldsymbol{x}$

- $$e_{+} = 0_{R^{mxn}} = \begin{bmatrix} x_{1,1} = 0 & x_{1,2} = 0 & \cdots & x_{1,(n-1)} = 0 & x_{1,n} = 0 \\ x_{2,1} = 0 & x_{2,2} = 0 & \cdots & x_{2,(n-1)} = 0 & x_{2,n} = 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(m-1),1} = 0 & x_{(m-1),2} = 0 & \cdots & x_{(m-1),(n-1)} = 0 & x_{(m-1),n} = 0 \\ x_{m,1} = 0 & x_{m,2} = 0 & \cdots & x_{m,(n-1)} = 0 & x_{m,n} = 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

- $e_{.} = 1$

- $$\forall \boldsymbol{x} \in \Re^{mxn}, \quad \boldsymbol{x}^{-1} = \begin{bmatrix} -x_{1,1} & -x_{1,2} & \cdots & -x_{1,(n-1)} & -x_{1,n} \\ -x_{2,1} & -x_{2,2} & \cdots & -x_{2,(n-1)} & -x_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{(m-1),1} & -x_{(m-1),2} & \cdots & -x_{(m-1),(n-1)} & -x_{(m-1),n} \\ -x_{m,1} & -x_{m,2} & \cdots & -x_{m,(n-1)} & -x_{m,n} \end{bmatrix}$$

$\mathrm{H} \subseteq V \quad x,y \epsilon \mathrm{H} \quad c \; \epsilon \Re$

- $e_{+} = 0_V \in H$
- $\forall x, y \in H \quad x + y \in H$
- $\forall x \in H, c \epsilon \Re \;\; c \cdot x \in H$

Is $\Re^2$ a subspace of $\Re^3$?

$0_{\Re^3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad x \in \Re^2 \longrightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_1, x_2 \in \Re$

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_1, x_2 \in \Re \quad \longrightarrow 0_{\Re^3} \notin \Re^2$

Is $H = \left[\begin{bmatrix} t^2 \\ 0 \\ 0 \end{bmatrix}, t \in \Re \right]$ a subspace of $\Re^3$?

$x \in H \longrightarrow x = \begin{bmatrix} x_1^2 \\ 0 \\ 0 \end{bmatrix}, x_1 \in \Re \quad \longrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathrm{H}$

$0_{\Re^3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \longrightarrow 0_{\Re^3} \in H$

$x = \begin{bmatrix} x_1^2 \\ 0 \\ 0 \end{bmatrix}, x_1 \in \Re, \quad y = \begin{bmatrix} y_1^2 \\ 0 \\ 0 \end{bmatrix}, y_1 \in \Re$

$x + y = \begin{bmatrix} x_1^2 + y_1^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{x_1^2 + y_1^2}^{\,2} \\ 0 \\ 0 \end{bmatrix}, \sqrt{x_1^2 + y_1^2} \in \Re \quad \longrightarrow x + y \in H$

$x = \begin{bmatrix} x_1^2 \\ 0 \\ 0 \end{bmatrix}, x_1 \in \Re, \quad c \in \Re$

$\mathrm{c}x = \begin{bmatrix} cx_1^2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \left(\sqrt{C}x_1\right)^2 \\ 0 \\ 0 \end{bmatrix}, \sqrt{C}x_1 \in \Re \quad \longrightarrow \mathrm{c}x \in \mathrm{H}$

In Class Exercise 4.2

Is $H = \left[\begin{bmatrix} x_1 & 1 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}, x_1, x_2, x_3 \in \Re \right]$ a subspace of R^{3x3} ?

In Class Exercise 4.2 SOLUTION

$$\boldsymbol{H} = \left[\begin{bmatrix} x_1 & 1 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}, \boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3} \in \Re \right] \qquad 0_V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad No$$

$$\boldsymbol{H} = \left[\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}, \boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3} \in \Re \right] \qquad Yes$$

$v_1, v_2, \dots, v_n \in \mathrm{V} \qquad a_1, a_2, \dots, a_n \in \Re$

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \in \mathrm{V}$$

$$x_1 = \begin{bmatrix}1\\0\end{bmatrix}, x_2 = \begin{bmatrix}-3\\6\end{bmatrix} \qquad 2x_1 + 3x_2 = 2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}-3\\6\end{bmatrix}$$

$$x_1 = \begin{bmatrix}5\\0\\5\end{bmatrix}, x_2 = \begin{bmatrix}-6\\1\\2\end{bmatrix}, x_3 = \begin{bmatrix}-9\\2\\0\end{bmatrix}, x_4 = \begin{bmatrix}1\\2\\3\end{bmatrix}, x_5 = \begin{bmatrix}5\\0\\5\end{bmatrix} \quad 3x_1 + 0x_2 + 1x_3 + 4x_4 + 7x_5 = 3\begin{bmatrix}5\\0\\5\end{bmatrix} + 0\begin{bmatrix}-6\\1\\2\end{bmatrix} 1 \begin{bmatrix}-9\\2\\0\end{bmatrix} + 4\begin{bmatrix}1\\2\\3\end{bmatrix} + 7\begin{bmatrix}5\\0\\5\end{bmatrix}$$

$$x_1 = \begin{bmatrix}2\\2\\-3\\0\\0\\1\end{bmatrix}, x_2 = \begin{bmatrix}2\\12\\3\\1\\4\\8\end{bmatrix}, x_3 = \begin{bmatrix}0\\1\\3\\11\\4\\18\end{bmatrix} \quad 3x_1 + 0x_2 + 1x_3 = 3\begin{bmatrix}2\\2\\-3\\0\\0\\1\end{bmatrix} + 0\begin{bmatrix}2\\12\\3\\1\\4\\8\end{bmatrix} + 1\begin{bmatrix}0\\1\\3\\11\\4\\18\end{bmatrix}$$

COMPLETE LINEAR ALGEBRA

LINEAR ALGEBRA > **VECTOR SPACES** > Spanning set

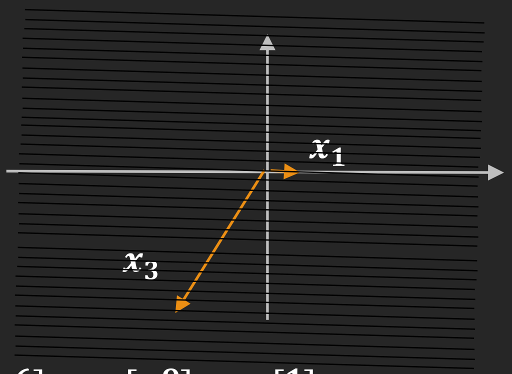
$v_1, v_2, \dots, v_n, x \in V \quad S = \text{Span}\{v_1, v_2, \dots, v_n\} \longrightarrow \forall x \in S, \exists a_1, a_2, \dots, a_n \in \Re \;/\; x = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$

$x \in \text{Span}\{v_1, v_2, \dots, v_n\} \longrightarrow \exists a_1, a_2, \dots, a_n \in \Re \;/\; x = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$

$x_1 = \begin{bmatrix}1\\0\end{bmatrix}, x_2 = \begin{bmatrix}-5\\0\end{bmatrix} \qquad Span\{x_1, x_2\} = \{x \;/ x = a_1 \begin{bmatrix}1\\0\end{bmatrix} + a_2 \begin{bmatrix}-5\\0\end{bmatrix}, a_1, a_2 \in \text{R}\}$

$x_3 = \begin{bmatrix}-3\\6\end{bmatrix} \qquad Span\{x_3\} = \{x \;/ x = a_3 \begin{bmatrix}-3\\6\end{bmatrix} \; a_3 \in \text{R}\}$

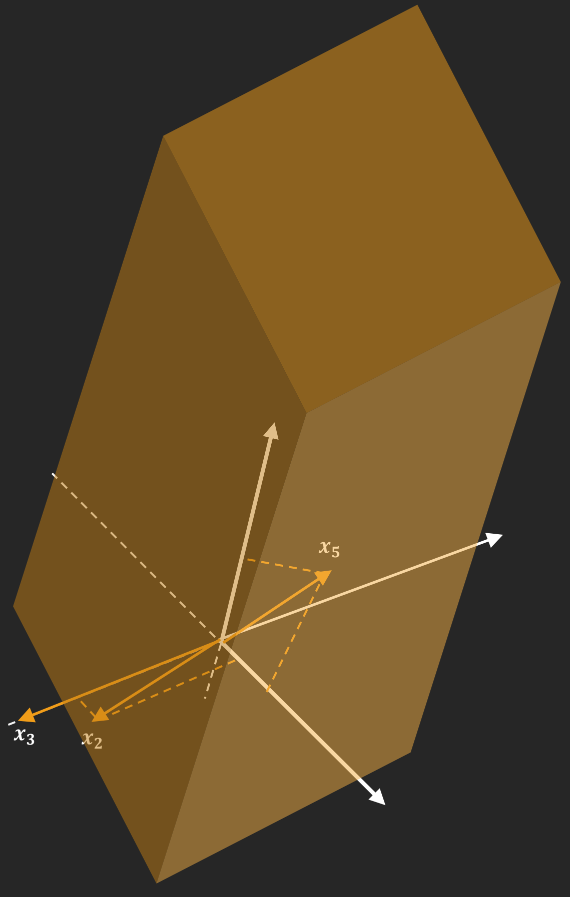
$x_1 = \begin{bmatrix}1\\0\end{bmatrix}, x_3 = \begin{bmatrix}-3\\6\end{bmatrix} \qquad Span\{x_1, x_2\} = \{x \;/ x = a_1 \begin{bmatrix}1\\0\end{bmatrix} + a_3 \begin{bmatrix}-3\\6\end{bmatrix}, a_1, a_3 \in \text{R}\}$



$x_1 = \begin{bmatrix}5\\0\\0\end{bmatrix}, x_2 = \begin{bmatrix}-6\\1\\0\end{bmatrix}, x_3 = \begin{bmatrix}-9\\0\\0\end{bmatrix}, x_4 = \begin{bmatrix}1\\2\\0\end{bmatrix} \quad Span\{x_1, x_2, x_3, x_4\} = \{x \;/ x = a_1 \begin{bmatrix}5\\0\\0\end{bmatrix} + a_2 \begin{bmatrix}-6\\1\\0\end{bmatrix} + a_3 \begin{bmatrix}-9\\0\\0\end{bmatrix} + a_4 \begin{bmatrix}1\\2\\0\end{bmatrix} a_1, \dots, a_4 \in \text{R}\}$

$x_5 = \begin{bmatrix}5\\0\\7\end{bmatrix}, x_2 = \begin{bmatrix}-6\\1\\0\end{bmatrix}, \qquad Span\{x_5, x_2\} = \{x \;/ x = a_5 \begin{bmatrix}5\\0\\7\end{bmatrix} + a_2 \begin{bmatrix}-6\\1\\0\end{bmatrix} a_5, a_2 \in \text{R}\}$ (0,9,0) can't be obtained

$x_5 = \begin{bmatrix}5\\0\\7\end{bmatrix}, x_2 = \begin{bmatrix}-6\\1\\0\end{bmatrix}, x_3 = \begin{bmatrix}-9\\0\\0\end{bmatrix} \quad Span\{x_5, x_2, x_3\} = \{x \;/ x = a_5 \begin{bmatrix}5\\0\\7\end{bmatrix} + a_2 \begin{bmatrix}-6\\1\\0\end{bmatrix} + a_3 \begin{bmatrix}-9\\0\\0\end{bmatrix} a_5, a_2, a_3 \in \text{R}\}$



$v_1, v_2, \dots, v_n \in \mathrm{V}$ $\quad$ $\mathrm{Span}\{v_1, v_2, \dots, v_n\}$ **is a subspace of V**

$When\ a_1 = 0 = \cdots = a_n \quad 0_V = a_1 v_1 + \cdots + a_n v_n \in \mathrm{Span}\{v_1, v_2, \dots, v_n\}$

$\longrightarrow 0_v \in \mathrm{Span}\{v_1, v_2, \dots, v_n\}$

$x, y \in \mathrm{Span}\{v_1, v_2, \dots, v_n\} \quad \longrightarrow x = a_1 v_1 + \cdots + a_n v_n \quad \& \quad y = b_1 v_1 + \cdots + b_n v_n$

$x + y = (a_1 + b_1) v_1 + \cdots + (a_n + b_n) v_n, (a_1 + b_1), \dots, (a_n + b_n) \in \mathrm{R}$

$\longrightarrow x + y \in Span\{v_1, .., v_n\}$

$x \in \mathrm{Span}\{v_1, v_2, \dots, v_n\}, c \in \mathrm{R} \quad \longrightarrow x = a_1 v_1 + \cdots + a_n v_n$

$\mathrm{cx} = (ca_1) v_1 + \cdots + (ca_n) v_n, (ca_1), \dots, (ca_n) \in \mathrm{R}$

$\longrightarrow cx \in Span\{v_1, .., v_n\}$

$$v_1 = \begin{bmatrix}1\\0\\5\end{bmatrix}, v_2 = \begin{bmatrix}-1\\ \frac{1}{2}\\0\end{bmatrix}, v_3 = \begin{bmatrix}3\\-1\\5\end{bmatrix} \quad v_3 = 1 * \begin{bmatrix}1\\0\\5\end{bmatrix} - 2 * \begin{bmatrix}-1\\ \frac{1}{2}\\0\end{bmatrix}$$

$\longrightarrow v_3 = 1v_1 - 2v_2$ $\quad\longrightarrow v_1 = 2v_2 + v_3$

$\longrightarrow v_2 = \frac{1}{2}v_1 - \frac{1}{2}v_3$ $\quad\longrightarrow 1v_3 - 1v_1 + 2v_2 = 0$

$a_1, a_2, \ldots, a_n \in \Re \qquad v_1, v_2, \ldots, v_n \in \mathrm{V}$

$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \quad \leftrightarrow \quad a_1 = 0 = \ldots = a_n$

$$v_i = \sum_{j=1 (j\neq i)}^{n} a_jv_j \quad \leftrightarrow \quad a_1 = 0 = \ldots = a_n$$

$v = [v_1, v_2, \ldots, v_n]$ *has n pivot columns*

$$\begin{bmatrix}1 & -1 & 3\\0 & \frac{1}{2} & -1\\5 & 0 & 5\end{bmatrix}\begin{matrix}r_1\\r_2\\r_3\end{matrix} \sim \begin{bmatrix}1 & -1 & 3\\0 & \frac{1}{2} & -1\\0 & 5 & -10\end{bmatrix}\begin{matrix}r_1\\r_2\\r_3 = r_3 - 5r_1\end{matrix}$$

$$\sim \begin{bmatrix}1 & -1 & 3\\0 & \frac{1}{2} & -1\\0 & 0 & 0\end{bmatrix}\begin{matrix}r_1\\r_2\\r_3 = r_3 - 10r_2\end{matrix}$$

$\longrightarrow$ ***Linear Dependence***

$$\begin{bmatrix}1 & 2 & 3\\0 & 1 & 4\\0 & 6 & 2\end{bmatrix}\begin{matrix}r_1\\r_2\\r_3\end{matrix} \sim \begin{bmatrix}1 & 2 & 3\\0 & 1 & 4\\0 & 0 & -22\end{bmatrix}\begin{matrix}r_1\\r_2\\r_3 = r_3 - 6r_2\end{matrix}$$

$\longrightarrow$ ***Linear Independence***

In Class Exercise 4.3

Are these vectors linearly independent?

$$v_1 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, v_2 = \begin{bmatrix}-1\\0\\-1\\0\end{bmatrix}, v_3 = \begin{bmatrix}-1\\0\\0\\1\end{bmatrix}, v_4 = \begin{bmatrix}9\\8\\2\\3\end{bmatrix}$$

In Class Exercise 4.3
SOLUTION

Yes

$A\epsilon R^{mxn}$ $\text{Nul}(A) = \{x\epsilon R^n \ / \ Ax = 0_{R^m}\}$

$\text{Nul}(A)$ *is a subspace of* R^n

$$0_{R^n} = \begin{bmatrix} x_1 = 0 \\ \vdots \\ x_n = 0 \end{bmatrix} \quad A * 0_{R^n} = 0_{R^m} \quad \longrightarrow 0_{R^n}\ \epsilon \text{Nul(A)}$$

$$x, y\ \epsilon\ Nul(A) \quad \longrightarrow Ax = 0\ \ \&\ \ Ay = 0$$
$$\longrightarrow Ax + Ay = 0 \quad \longrightarrow A(x+y) = 0$$
$$\longrightarrow x + y\epsilon\ Nul(A)$$

$$x\ \epsilon\ Nul(A), c\epsilon \text{R} \quad \longrightarrow Ax = 0 \longrightarrow (cA)x = 0$$
$$\longrightarrow A(cx) = 0$$
$$\longrightarrow cx\ \epsilon Nul(A)$$

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \epsilon Nul(A) \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 0 \\ 2 & 4 & 6 & 8 & 0 \\ 3 & 6 & 8 & 10 & 0 \end{array}\right] \begin{array}{l} r_1 \\ r_2 \\ r_3 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 2 & 4 & 0 \end{array}\right] \begin{array}{l} r_1 \\ r_2 = r_2 - 2r_1 \\ r_3 = r_3 - 3r_1 \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} r_1 \\ r_2 = \frac{r_2}{2} \\ r_3 - r_2 \end{array}$$

$$\longrightarrow \begin{cases} 1x_1 + 2x_2 + 2x_3 + 2x_4 = 0 --(1) \\ x_3 + x_4 = 0 --(2) \end{cases}$$

$$\longrightarrow \begin{cases} x_1 = -2x_2 - 2x_3 - 2x_4 = -2x_2 - 2(x_3 + x_4) \\ x_2 = x_2 \\ x_3 = -x_4 \\ x_4 = x_4 \end{cases}$$

$$\longrightarrow \begin{cases} x_1 = -2x_2 \\ x_2 = x_2 \\ x_3 = -x_4 \\ x_4 = x_4 \end{cases} \longrightarrow Nul(A) = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle/ \begin{array}{l} x_1 = -2x_2 \\ x_2 = 1x_2 \\ x_3 = -x_4 \\ x_4 = 1x_4 \end{array} \right.$$

$$\longrightarrow Nul(A) = \left[x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right], x_2, x_4 \epsilon \text{R}$$

$$\longrightarrow Nul(A) = Span \left[\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right]$$

In Class Exercise 4.4

$$H = \begin{bmatrix} 2 & 4 & 7 \\ 1 & 5 & 2 \\ 3 & 1 & 2 \end{bmatrix}$$

- *Is* $\begin{bmatrix} 2 \\ 3 \\ 4 \\ 0 \end{bmatrix} \epsilon Nul(H)$
- *Is* $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \epsilon Nul(H)$
- *Find* $Nul(H)$

In Class Exercise 4.4
SOLUTION

$- No$

$- No$

$$-Nul(A) = \left\{ \xcancel{x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Big/ \begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix}} \right. \qquad \boldsymbol{Nul(A)} = \{\,\}$$

$A \epsilon R^{mxn}, x \epsilon R^n \;\; Col(A) = Span\{a_1, \dots, a_n\} \;\; a_1, \dots, a_n = Linearly\ independent\ Columns\ (A), a_1, \dots, a_n\ \epsilon R^m$
$= Pivot\ Columns\ (A), a_1, \dots, a_n\ \epsilon R^m$

$Span\{a_1, \dots, a_n\} is\ a\ subspace\ of\ R^m \;\; \longrightarrow Col(A) is\ a\ subspace\ of\ R^m$

$b \epsilon\ Col(A) \;\; \exists x_1, x_2, \dots, x_n \epsilon R \; / \; b = x_1 a_1 + x_2 a_2 + \dots + x_n a_n = Ax$

$$A = \begin{array}{c} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \end{array} \begin{array}{l} r_1 \\ r_2 \end{array} \sim \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 3 \\ -2 \end{matrix}} & \begin{matrix} 5 \\ -4 \end{matrix} \end{bmatrix} \begin{array}{l} r_1 \\ r_2 = r_2 - 2r_1 \end{array}$$

$$\longrightarrow Col(A) = Span\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\}$$

$$\exists x_1 = -1, x_2 = 2 \;\; / \;\; \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = Ax, x = \begin{bmatrix} x_1 = -1 \\ x_2 = 2 \end{bmatrix}$$

$$= -1 * \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 * \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

In Class Exercise 4.5

Find the COlumn space of A

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 3 & 8 \\ 2 & 3 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

In Class Exercise 4.5 SOLUTION

$$Col(A) = Span\left\{\begin{bmatrix}5\\0\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\3\\2\end{bmatrix}, \begin{bmatrix}1\\8\\1\\0\end{bmatrix}\right\}$$

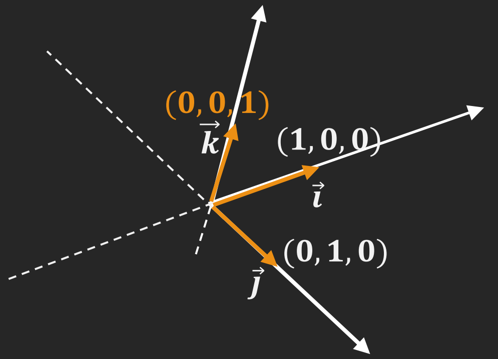
$$Row(A) = Col(A^T) \qquad LeftNull(A) = Null(A^T)$$

$B = \{b_1, b_2, \dots, b_n\}, \qquad b_1, b_2, \dots, b_n \in V \qquad B = basis(H \sqsubseteq V)$

- $t_1 b_1 + t_2 b_2 + \cdots + t_n b_n = 0 \quad \leftrightarrow \quad t_1 = 0 = \ \dots = \ t_n$
- $H = Span(B)$

$If\ B = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\} \qquad Is\ B = base(R^3)\ ?$

- $B = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \dashrightarrow B\ is\ linearly\ independent$
- $R^3 = Span(B)$



$If\ B = \left\{ \begin{bmatrix}1\\2\\0\\4\end{bmatrix}, \begin{bmatrix}2\\-1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\0\\-4\end{bmatrix} \right\} \qquad Is\ B = base(R^4)\ ?$

- $B = \begin{array}{c} \begin{bmatrix}1&2&1\\2&-1&1\\0&0&0\\4&2&-4\end{bmatrix} \end{array} \begin{array}{l} r_1\\r_2\\r_3\\r_4 \end{array} \sim \begin{bmatrix}1&2&1\\0&-5&-1\\4&2&-4\\0&0&0\end{bmatrix} \begin{array}{l} r_1\\r_2 = r_2 - 2r_1\\r_3 = r_4\\r_4 = r_3 \end{array}$

$\sim \begin{bmatrix}1&2&1\\0&-5&-1\\0&-6&-8\\0&0&0\end{bmatrix} \begin{array}{l} r_1\\r_2\\r_3 = r_3 - 4r_1\\r_4 \end{array}$

$\sim \begin{bmatrix}1&2&1\\0&-5&-1\\0&0&-\frac{34}{5}\\0&0&0\end{bmatrix} \begin{array}{l} r_1\\r_2\\r_3 = r_3 - \frac{6}{5}r_2\\r_4 \end{array}$

$\dashrightarrow B\ is\ linearly\ independent$

- $R^4 \neq Span(B)$

$B = \left\{ \begin{bmatrix}1\\2\\-1\\-1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\2\\-2\end{bmatrix}, \begin{bmatrix}3\\-4\\3\\5\\-3\end{bmatrix}, \begin{bmatrix}-1\\8\\-5\\-6\\1\end{bmatrix} \right\} \qquad If\ H \sqsubseteq R^5\ \&\ H = Span(B) \qquad Find\ B' = base(H)$

$\begin{bmatrix}1&2&3&-1\\2&-1&-4&8\\-1&1&3&-5\\-1&2&5&-6\\-1&-2&-3&1\end{bmatrix} \begin{array}{l} r_1\\r_2\\r_3\\r_4\\r_5 \end{array} \sim \begin{bmatrix}1&2&3&-1\\0&-5&-10&10\\0&3&6&-6\\0&4&8&-7\\0&0&0&0\end{bmatrix} \begin{array}{l} r_1\\r_2 = r_2 - 2r_1\\r_3 = r_3 + r_1\\r_4 = r_4 + r_1\\r_5 = r_5 + r_1 \end{array}$

$\sim \begin{bmatrix}1&2&3&-1\\0&1&2&-2\\0&1&2&-2\\0&1&2&-\frac{7}{4}\\0&0&0&0\end{bmatrix} \begin{array}{l} r_1\\r_2 = -\frac{r_2}{5}\\r_3 = \frac{r_3}{3}\\r_4 = \frac{r_4}{4}\\r_5 \end{array} \sim \begin{bmatrix}1&2&3&-1\\0&1&2&-2\\0&0&0&0\\0&0&0&\frac{1}{4}\\0&0&0&0\end{bmatrix} \begin{array}{l} r_1\\r_2\\r_3 = r_3 - r_2\\r_4 = r_4 - r_2\\r_5 \end{array}$

$\sim \begin{bmatrix}1&2&3&-1\\0&1&2&-2\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix} \begin{array}{l} r_1\\r_2\\r_3 = r_4\\r_4 = r_3\\r_5 \end{array} \dashrightarrow B' = \left\{ \begin{bmatrix}1\\2\\-1\\-1\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\2\\-2\end{bmatrix}, \begin{bmatrix}-1\\8\\-5\\-6\\1\end{bmatrix} \right\}$

In Class Exercise 4.6

$B = \left\{ \begin{bmatrix}3\\2\\3\end{bmatrix}, \begin{bmatrix}1\\5\\4\end{bmatrix} \right\} Is\ B = base(R^3)\ ?$

In Class Exercise 4.6
SOLUTION

$$\boldsymbol{No}$$

$$x \in R^n \qquad B = \{b_1, b_2, \dots, b_n\}$$

$$\exists\ a_1, \dots, a_n \in R / x = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$B = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \text{ and } x = \begin{bmatrix}2\\-3\\-4\end{bmatrix}$$

$$x = 2\begin{bmatrix}1\\0\\0\end{bmatrix} + -3\begin{bmatrix}0\\1\\0\end{bmatrix} + -4\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\-3\\-4\end{bmatrix}$$

In Class Exercise 4.7

$$[x]_E = \begin{bmatrix}3\\-5\end{bmatrix} \in R^2 \quad B = \left\{\begin{bmatrix}3\\0\end{bmatrix}, \begin{bmatrix}0\\5\end{bmatrix}\right\}$$

Find $[x]_B$, Sketch the new Base and confirm your answer

$$E = \{e_1, e_2, \dots, e_n\} = \left\{\begin{bmatrix}1\\0\\\vdots\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}\right\} \qquad x \in R^n \qquad B = \{b_1, b_2, \dots, b_n\}$$

$$\exists\ a_1, \dots, a_n / \quad x = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \qquad \text{Let } x = [x]_E$$

$$\exists\ a'_1, \dots, a'_n / \quad [x]_E = a'_1 b_1 + a'_2 b_2 + \cdots + a'_n b_n \ \&\ [x]_B = (a'_1, \dots, a'_n)$$

$$[x]_E = \begin{bmatrix}0\\-8\end{bmatrix} \in R^2 \quad B = \left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-2\\2\end{bmatrix}\right\} \quad \textit{Find } [x]_B$$

$$[x]_E = \begin{bmatrix}0\\-8\end{bmatrix} = a'_1 b_1 + a'_2 b_2 = a'_1\begin{bmatrix}1\\3\end{bmatrix} + a'_2\begin{bmatrix}-2\\2\end{bmatrix}$$

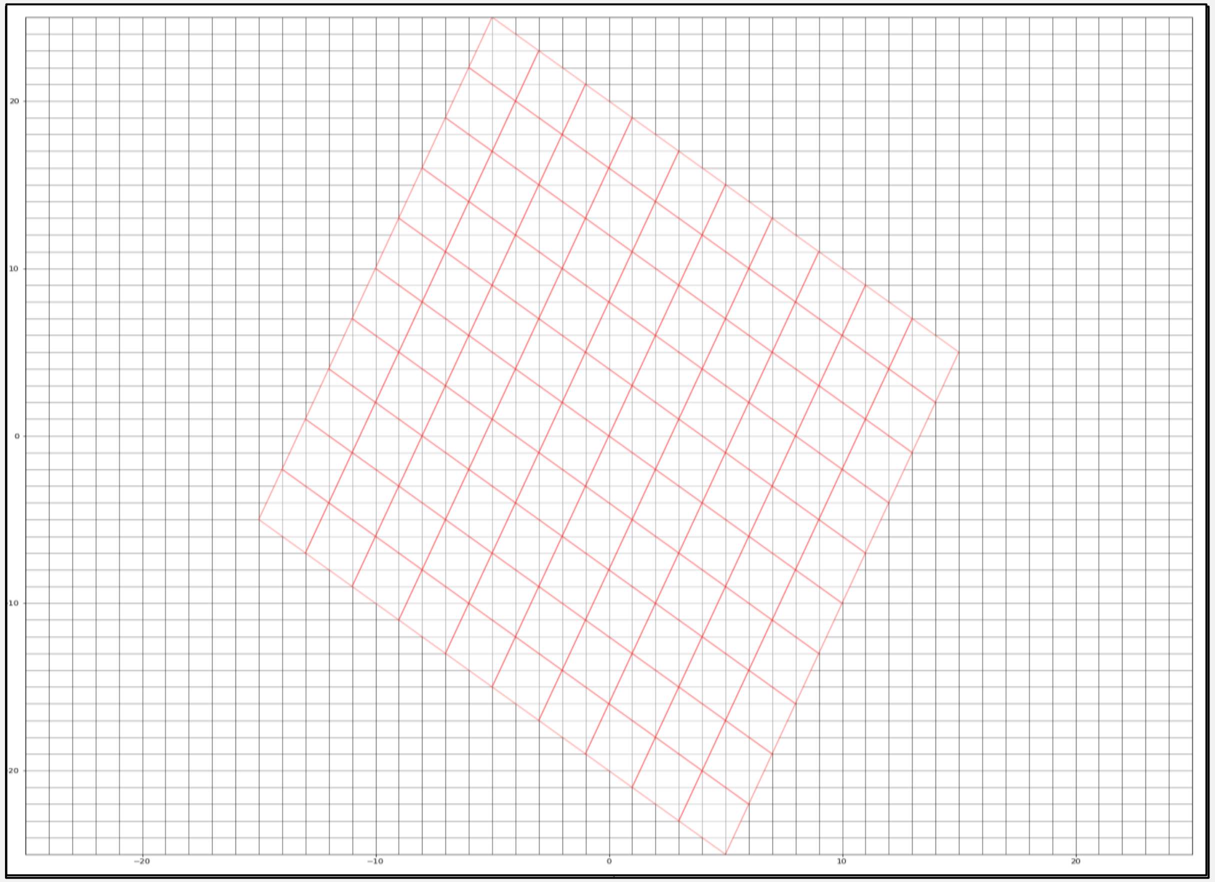
$$\longrightarrow \begin{bmatrix}0\\-8\end{bmatrix} = \begin{bmatrix}a'_1 - 2a'_2\\3a'_1 + 2a'_2\end{bmatrix} \longrightarrow \begin{cases}a'_1 - 2a'_2 = 0\\3a'_1 + 2a'_2 = -8\end{cases} \longrightarrow \begin{cases}a'_1 = -2\\a'_2 = -1\end{cases}$$

$$\longrightarrow [x]_B = \begin{bmatrix}-2\\-1\end{bmatrix}$$

$$[x]_E = a'_1 b_1 + a'_2 b_2 = a'_1\begin{bmatrix}1\\3\end{bmatrix} + a'_2\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}\begin{bmatrix}a'_1\\a'_2\end{bmatrix}$$

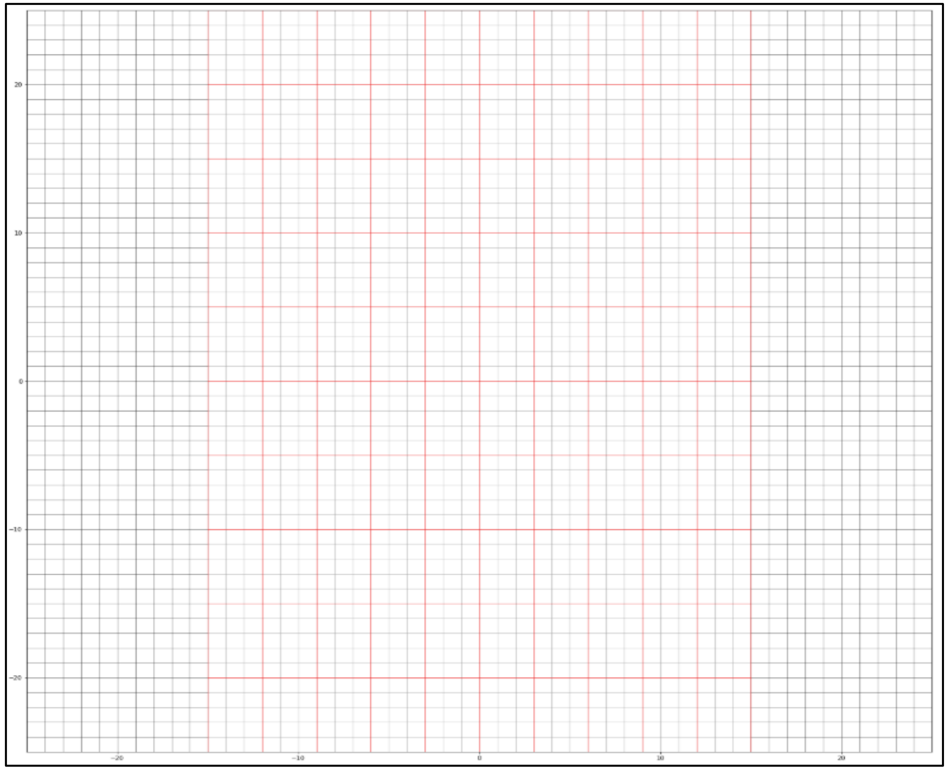
$$[x]_B = \begin{bmatrix}0\\0\end{bmatrix} \longrightarrow [x]_E = \begin{bmatrix}0\\0\end{bmatrix} \qquad [x]_B = \begin{bmatrix}0\\1\end{bmatrix} \longrightarrow [x]_E = \begin{bmatrix}-2\\2\end{bmatrix}$$

$$[x]_B = \begin{bmatrix}1\\1\end{bmatrix} \longrightarrow [x]_E = \begin{bmatrix}-1\\5\end{bmatrix} \qquad [x]_B = \begin{bmatrix}1\\0\end{bmatrix} \longrightarrow [x]_E = \begin{bmatrix}1\\3\end{bmatrix}$$



In Class Exercise 4.7
SOLUTION

$$\longrightarrow [x]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



$B = \{b_1, b_2, \dots, b_n\}$ $\quad C = \{c_1, c_2, \dots, c_n\}$ $\quad P_{C\leftarrow B} = [[b_1]_C, \dots, [b_n]_C]$

$[x]_C, [x]_B \in R_n$ $\quad [x]_C = P_{C\leftarrow B}[x]_B$ $\quad P^{-1}_{C\leftarrow B}$ *exists* $\quad P^{-1}_{C\leftarrow B} = P_{B\leftarrow C}$ $\quad [x]_B = P^{-1}_{C\leftarrow B}[x]_C$

$B = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ $\quad C = \left\{\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-2\\2\end{bmatrix}\right\}$ $\quad$ *Find* $P_{C\leftarrow B}$

$[x]_B = \begin{bmatrix}0\\-8\end{bmatrix} \in R^2$ $\quad$ *Find* $[x]_C$

$$\begin{bmatrix}1\\0\end{bmatrix} = s_1\begin{bmatrix}1\\3\end{bmatrix} + s_2\begin{bmatrix}-2\\2\end{bmatrix} \longrightarrow \begin{cases} 1 = s_1 - 2s_2 \\ 0 = 3s_1 + 2s_2 \end{cases}$$

$$\longrightarrow \begin{cases} s_1 = \dfrac{1}{4} \\ s_2 = -\dfrac{3}{8} \end{cases}$$

$$\begin{bmatrix}0\\1\end{bmatrix} = t_1\begin{bmatrix}1\\3\end{bmatrix} + t_2\begin{bmatrix}-2\\2\end{bmatrix} \longrightarrow \begin{cases} 0 = t_1 - 2t_2 \\ 1 = 3t_1 + 2t_2 \end{cases}$$

$$\longrightarrow \begin{cases} t_1 = \dfrac{1}{4} \\ t_2 = \dfrac{1}{8} \end{cases}$$

$$\longrightarrow P_{C\leftarrow B} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & \frac{1}{8} \end{bmatrix}$$

$$\longrightarrow [x]_C = P_{C\leftarrow B}[x]_B = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{8} & \frac{1}{8} \end{bmatrix}\begin{bmatrix}0\\-8\end{bmatrix}$$

$$= \begin{bmatrix}-2\\-1\end{bmatrix}$$

In Class Exercise 4.8

$B = \left\{\begin{bmatrix}1\\-3\end{bmatrix}, \begin{bmatrix}-2\\4\end{bmatrix}\right\}$ $\quad C = \left\{\begin{bmatrix}-7\\9\end{bmatrix}, \begin{bmatrix}-5\\7\end{bmatrix}\right\}$

Find $P_{B\leftarrow C}$

In Class Exercise 4.8 SOLUTION

$$B = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix} \right\} \qquad C = \left\{ \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \end{bmatrix} \right\}$$

$$\begin{bmatrix} -7 \\ 9 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + 6 \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$\longrightarrow P_{B \leftarrow C} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -5 \\ 7 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + 4 \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

COMPLETE LINEAR ALGEBRA

LINEAR ALGEBRA > VECTOR SPACES > Dimension of Vector Space and Matrix Rank

$$B = \{b_1, \dots, b_n\} \ \& \ V$$

$$V = Span(B) \longrightarrow dim(V) = n$$

$$M_B = [b_1 \quad b_2 \quad \dots \quad b_n] \longrightarrow Rank(M_B) = n$$

$$x = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,(n-1)} & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,(n-1)} & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{(m-1),1} & x_{(m-1),2} & \cdots & x_{(m-1),(n-1)} & x_{(m-1),n} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,(n-1)} & x_{m,n} \end{bmatrix}$$

$$Rank(x) = N^o \ of \ Pivot \ Columns$$

$$\longrightarrow Rank(x) = dim(Col(x)) = dim(Row(x))$$

$$m = dim(Col(x)) + dim(Nul(A^T))$$

$$n = dim(Row(x)) + dim(Nul(A))$$

$$B = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -6 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 9 \\ 1 \end{bmatrix} \right\}$$

Find the rank of its corresponding Matrix, its fundamental spaces & deduce the relation between their dimensions

$$M_B = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} \sim \begin{array}{cl} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{matrix} \end{array}$$

$$\sim \begin{array}{cl} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 = r_2 + r_1 \\ r_3 = r_3 - 2r_1 \\ r_4 = r_4 - r_2 \end{matrix} \end{array} \sim \begin{array}{cl} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 = r_3 + r_2 \\ r_4 = r_4 - 3r_2 \end{matrix} \end{array}$$

$$\sim \begin{array}{cl} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 = r_4 \\ r_4 = r_3 \end{matrix} \end{array} \longrightarrow Col(M_B) = Span\left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\}$$

$$M_B^T = \begin{array}{cl} \begin{bmatrix} 2 & -2 & 4 & -2 \\ -3 & 3 & -6 & 3 \\ 6 & -3 & 9 & 3 \\ 2 & -3 & 5 & -4 \\ 5 & -4 & 9 & 1 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{matrix} \end{array} \sim \begin{array}{cl} \begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 9 \\ 0 & -1 & 1 & -2 \\ 0 & 1 & -1 & 6 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 = r_2 + \frac{3}{2} r_1 \\ r_3 = r_3 - 3r_1 \\ r_4 = r_4 - r_1 \\ r_5 = r_5 - \frac{5}{2} r_1 \end{matrix} \end{array}$$

$$\sim \begin{array}{cl} \begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & -3 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 = r_4 + \frac{1}{3} r_3 \\ r_5 = r_5 - \frac{1}{3} r_3 \end{matrix} \end{array} \sim \begin{array}{cl} \begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & -3 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 = r_3 \\ r_3 = r_2 \\ r_4 \\ r_5 = r_5 - 3r_4 \end{matrix} \end{array}$$

$$\sim \begin{array}{cl} \begin{bmatrix} 2 & -2 & 4 & -2 \\ 0 & 3 & -3 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} r_1 \\ r_2 = r_3 \\ r_3 = r_4 \\ r_4 = r_3 \\ r_5 \end{matrix} \end{array} \longrightarrow Row(M_B) = Span\left\{ \begin{bmatrix} 2 \\ -3 \\ 6 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -3 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 3 \\ -4 \\ 1 \end{bmatrix} \right\}$$

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$$M_B = \left[\begin{array}{ccccc|c} 2 & -3 & 6 & 2 & 5 & 0 \\ 0 & 0 & 3 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{cases} 2x_1 - 3x_2 + 6x_3 + 2x_4 + 5x_5 = 0 --(1) \\ 3x_3 - x_4 + x_5 = 0 --(2) \\ x_4 + 3x_5 = 0 --(3) \end{cases} \longrightarrow \begin{cases} 2x_1 = 3x_2 - 6x_3 - 2x_4 - 5x_5 \\ x_2 = x_2 \\ 3x_3 = x_4 - x_5 \\ x_4 = -3x_5 \\ x_5 = x_5 \end{cases}$$

$$\longrightarrow \begin{cases} 2x_1 = 3x_2 - 6x_3 - 2x_4 - 5x_5 = 3x_2 + 8x_5 + 6x_5 - 5x_5 = 3x_2 + 9x_5 \\ x_2 = x_2 \\ 3x_3 = x_4 - x_5 = -3x_5 - x_5 = -4x_5 \\ x_4 = -3x_5 \\ x_5 = x_5 \end{cases} \longrightarrow \begin{cases} x_1 = \frac{3}{2}x_2 + \frac{9}{2}x_5 \\ x_2 = x_2 \\ x_3 = -\frac{4}{3}x_5 \\ x_4 = -3x_5 \\ x_5 = x_5 \end{cases}$$

$$\longrightarrow Nul(M_B) = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \Bigg/ \begin{array}{c} x_1 = \frac{3}{2}x_2 + \frac{9}{2}x_5 \\ x_2 = x_2 \\ x_3 = -\frac{4}{3}x_5 \\ x_4 = -3x_5 \\ x_5 = x_5 \end{array} \right. = \left[x_2 \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} \frac{9}{2} \\ 0 \\ -\frac{4}{3} \\ -3 \\ 1 \end{bmatrix} \right], x_2, x_5 \epsilon R \quad = Span \left[\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{9}{2} \\ 0 \\ -\frac{4}{3} \\ -3 \\ 1 \end{bmatrix} \right],$$

$$M_B^T = \left[\begin{array}{cccc|c} 2 & -2 & 4 & -2 & 0 \\ 0 & 3 & -3 & 9 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{cases} 2x_1 - 2x_2 + 4x_3 - 2x_4 = 0 --(1) \\ 3x_2 - 3x_3 + 9x_4 = 0 --(2) \\ x_4 = 0 --(3) \end{cases} \longrightarrow \begin{cases} 2x_1 = 2x_2 - 4x_3 + 2x_4 \\ 3x_2 = 3x_3 - 9x_4 \\ x_3 = x_3 \\ x_4 = 0 \end{cases}$$

$$\longrightarrow \begin{cases} 2x_1 = 2x_2 - 4x_3 + 2x_4 = 2x_3 - 4x_3 = -2x_3 \\ 3x_2 = 3x_3 - 9x_4 = 3x_3 \\ x_3 = x_3 \\ x_4 = 0 \end{cases} \longrightarrow \begin{cases} 2x_1 = -2x_3 \\ 3x_2 = 3x_3 \\ x_3 = x_3 \\ x_4 = 0 \end{cases}$$

$$\longrightarrow Nul(M_B^T) = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Bigg/ \begin{array}{c} x_1 = -x_3 \\ x_2 = x_3 \\ x_3 = x_3 \\ x_4 = 0 \end{array} \right. = \left[x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right], x_3 \epsilon R \ = Span \left[\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right]$$

$$Col(M_B) = Span \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\} \quad \begin{array}{l} \longrightarrow \dim(Col(M_B)) = 3 \\ \longrightarrow Rank(M_B) = 3 \end{array}$$

$$Row(M_B) = Span \left\{ \begin{bmatrix} 2 \\ -3 \\ 6 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -3 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 3 \\ -4 \\ 1 \end{bmatrix} \right\} \longrightarrow \dim(Row(M_B)) = 3$$

$$Null(M_B) = Span \left[\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{9}{2} \\ 0 \\ -\frac{4}{3} \\ -3 \\ 1 \end{bmatrix} \right], \longrightarrow \dim(Nul(M_B)) = 2$$

$$Null(M_B^T) = Span \left[\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right] \longrightarrow \dim(Nul(M_B)) = 1$$

$$\boldsymbol{m = dim(Col(M_B)) + dim(Nul(M_B^T))}$$

$$4 = \quad 3 \quad + \quad 1 \ ✓$$

$$\boldsymbol{n = dim(Row(M_B)) + dim(Nul(M_B))}$$

$$5 = \quad 3 \quad + \quad 2 \ ✓$$

In Class Exercise 4.9

Find the Fundamental spaces of the following Matrix

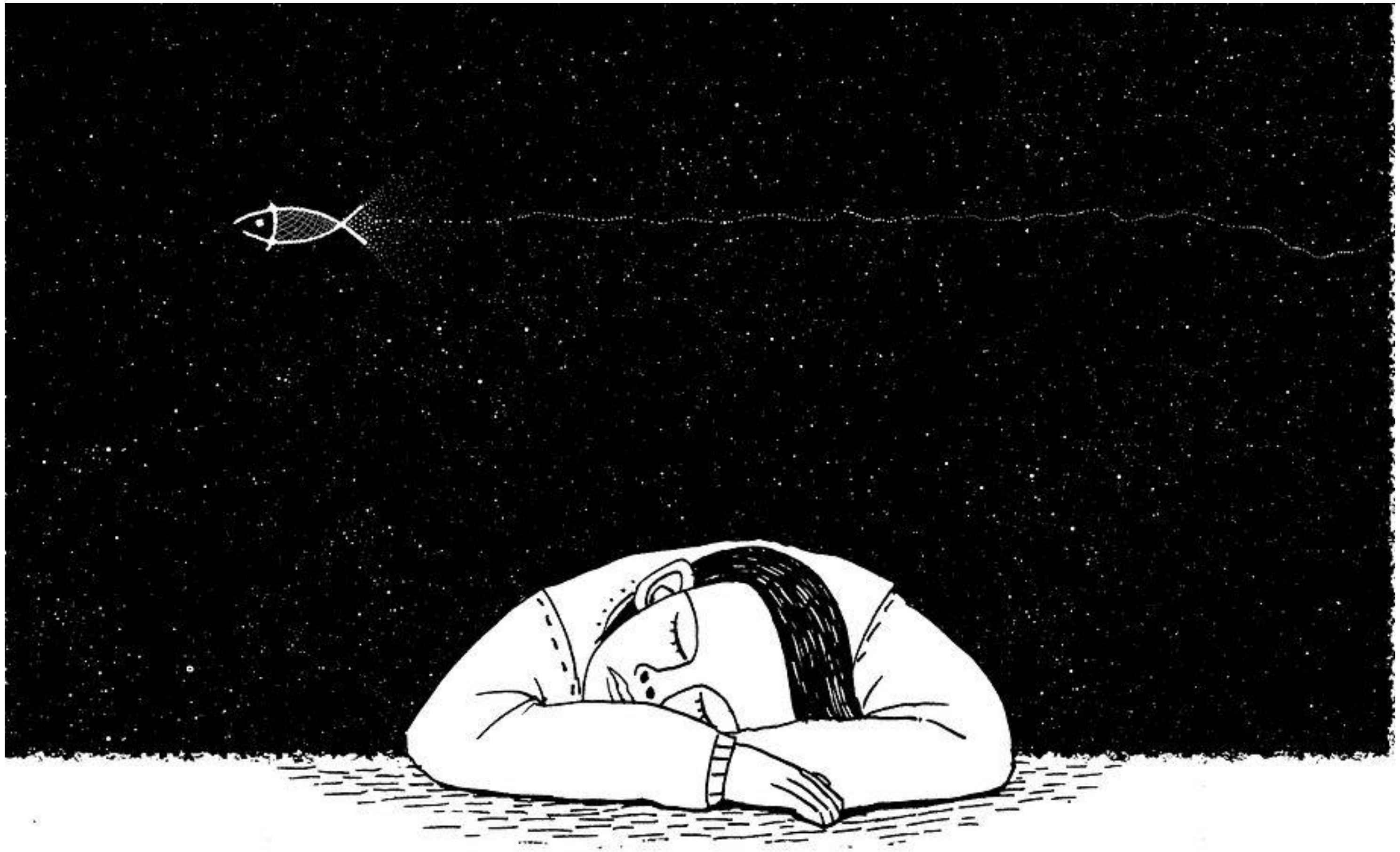
$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & -5 & 1 \\ 1 & 4 & 7 \end{bmatrix}$$

In Class Exercise 4.9 SOLUTION

- $Col(A) = Span\left\{\begin{bmatrix}3\\2\\1\end{bmatrix}, \begin{bmatrix}0\\-5\\4\end{bmatrix}, \begin{bmatrix}2\\1\\7\end{bmatrix}\right\}$
- $Row(A) = Span\left\{\begin{bmatrix}3\\0\\2\end{bmatrix}, \begin{bmatrix}2\\-5\\1\end{bmatrix}, \begin{bmatrix}1\\4\\7\end{bmatrix}\right\}$
- $Nul(A) = \{\}$
- $Nul(A^T) = \{\}$

LINEAR ALGEBRA > **VECTOR SPACES**

A vector space is a location on which vectors reside.



To be more precise, a vector space, denoted V, is a set of vectors on which two operations are defined and the following properties verified:

EXPL27

LINEAR ALGEBRA > **VECTOR SPACES**

Let's look at some examples of vector spaces.

EXPL28

The subspace H, of a vector space V is a vector space which verifies the conditions:

EXPL29

LINEAR COMBINATION OF A SET OF VECTORS

This is a very useful operation when it comes to vectors. Considering you have a set of vectors from v1 to vn , the linear combination of these vectors is obtained by multiplying each vector, vi by a scalar ai

EXPL30

The SPAN OF A SET OF VECTORS

SPAN{V1,…,VN} is the set of all vectors that can be written as linear combinations of {v1,…vn}

EXPL31

SUBSPACE GENERATED BY A SET OF VECTORS FOUND IN A VECTOR SPACE

EXPL32

NULL SPACES, COLUMN SPACES AND ROW SPACES (THE FOUR SUBSPACES)

NULL SPACE OF A MATRIX, A

This is the set of solutions of the equation AX = 0, denoted NulA

Defined as:

EXPL33

COLUMN SPACE OF A MATRIX, A

This is the set of all linear combinations of the columns of A

EXPLS34

LINEAR ALGEBRA > VECTOR SPACES

ROW SPACE OF A MATRIX, A

This is the set of all linear combinations of the rows of A and can be be denoted as Col Atranspose

EXPL35

LINEAR INDEPENDENCE

This notion comes into play when some vectors can be written as a linear combination of one or more vectors. A set of vectors are said to be linearly independent if none of these vectors can be written as a linear combination of the others (hence the term linear independence). We shall look at two examples (set of vectors) and decide from the definition earlier stated, which is linear independent or dependent.

EXPL36

So, we can say that a set of vectors are linearly independent if:

EXPL37

But this method of analyzing whether a set of vectors are linearly independent by just looking at the vectors and trying out different combinations isn't scalable.

We shall look at a systematic method of checking whether a set of vectors are linearly independent or not.

- Row reduce the matrix made up of the vectors.
- If all the columns are pivot columns, then the set of vectors are linearly independent, if not, then they aren't and the free columns can be written as a linear combinations of all the pivot columns to its left

With this we can come to conclusion that if a square matrix is linearly independent, then its inverse exists, that is its invertible

LINEAR ALGEBRA > VECTOR SPACES

BASES

A set of vectors, B in a vector space, V are a basis or minimal generating set of of a subspace H of V if :

- All the vectors which make up B are linearly independent
- H = span(B), i.e. with a linear combination of the different columns of B, we can generate every vector in H

Some examples of basis are:

EXPL39

FINDING THE BASE FROM A SET OF VECTORS WHICH SPAN A SUBSPACE

- Rewrite the set of vectors in matrix form
- Get the row echelon form of this matrix
- The base is composed of all the pivot columns (after all , if any free columns are found, they are a linear combination of pivot columns to their left)

EXPL40

COORDINATE SYSTEMS AND CHANGE OF BASIS

With the aid of coordinate systems, we are able to locate points in space. In this section, we see how to locate a point with respect to a base. Every point in the vector space Rn is differentiated by a unique set of real number a1,…,an and represented with respect to a base, with vectors {b1,…, bn} as :

EXPL41

Notice that the canonical base, doesn't practically change the form of the initial real numbers which characterize a vector in Rn

So if we are given the coordinates (c1 ,…, cn)of a point in the canonical base and we are interested in getting its coordinates in another base B={b1,…,bn}, it will suffice to get all the real numbers a1,…,an which verify:

EXPL42

We've seen how to leave from the canonical base to a base in H(subspace of Rn), now lets generalize our method and leave from a base B to a base C (B or C not necessarily the canonical base).

To do this, we write out the vectors that make up B in base C, thereby forming a matrix which can then be used to transform any vector in base B to base C.

EXPL43

DIMENSION OF A VECTOR SPACE and RANK

This is the number of vectors in the set which spans the vector space.

If this number is finite, then dim V = k

Else dim V = infinity

On the other hand, the rank of a matrix gotten from the extracting the basis of a vector space is equals the dimension of that vector space

Note:

An Rn*n Matrix with full rank is one with n pivot positions

If we find the row echelon matrix of a set of vectors which span a vector space, the dimension of this vector space will be the number of pivot columns and the remaining free columns will be the dimension of the Nul Space of A.

Hence for a vector space spanned by n vectors, n = rank A + dim NulA