- Python Setup, Basic Python, numpy, pandas and matplotlib
- Matrix Algebra
- Linear equations and transformations
- Vectors
- Vector Spaces
- Metric Spaces, Normed spaces, Inner Product Spaces
- Orthogonality
- Determinant and Trace Operator
- Matrix Decompositions (Eigen, SVD and Cholesky)
- Symmetric matrices and Quadratic Forms
- Left Inverse, Right Inverse, Pseudo Inverse

Connecting the dots 1
LINEAR REGRESSION

Connecting the dots 2
PRINCIPAL COMPONENT ANALYSIS

30

TAKE ACTION NOW!!!

COMPLETE LINEAR ALGEBRA

LINEAR ALGEBRA > Matrix Decomposition > Eigen Decomposition 1

COLLEGE COURSES

DATA SCIENCE AND MACHINE LEARNING

ENGINEERING

RESEARCH

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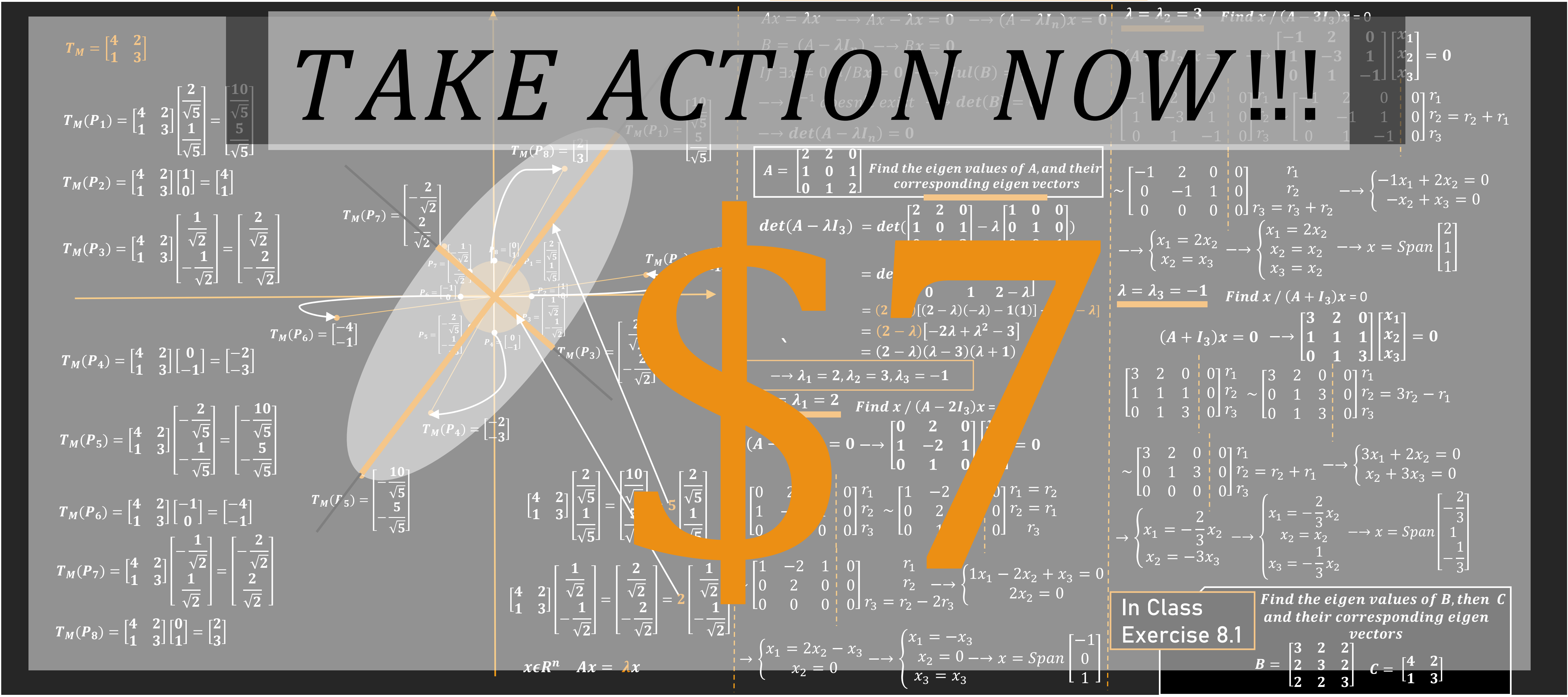
COMPLETE LINEAR ALGEBRA

LINEAR ALGEBRA > Matrix Decomposition > Eigen Decomposition 1



neuralearn.ai

In Class Exercise 8.1
SOLUTION

$$\det(B - \lambda I_3) = -(\lambda - 1)^2(\lambda - 7) \quad \boxed{\longrightarrow \lambda_1 = 1, \lambda_2 = 1. \lambda_3 = 7}$$

$$\lambda = \lambda_1 = \lambda_2 = 1 \longrightarrow x = Span\left[\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right]$$

$$\lambda = \lambda_3 = 7 \longrightarrow x = Span\left[\begin{bmatrix}1\\1\\1\end{bmatrix}\right]$$

$$\det(C - \lambda I_2) = (\lambda - 5)(\lambda - 2) \quad \boxed{\longrightarrow \lambda_1 = 5, \lambda_2 = 2}$$

$$\lambda = \lambda_1 = 5 \longrightarrow x = Span\left[\begin{bmatrix}2\\1\end{bmatrix}\right]$$

$$\lambda = \lambda_2 = 2 \longrightarrow x = Span\left[\begin{bmatrix}1\\-1\end{bmatrix}\right]$$

$$A \in R^{nxn} \quad diag_n(\lambda) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$A = Xdiag_n(\lambda)X^{-1}$$

$$X = [x_1 x_2 \ldots x_n]$$

$$T_M = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \quad \lambda = \lambda_1 = 5 \longrightarrow x = Span\left[\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right] \quad \lambda = \lambda_2 = 2 \longrightarrow x = Span\left[\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right] \longrightarrow X = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\boxed{\longrightarrow T_M = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}}$$

$$T_M = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} \quad \lambda = \lambda_1 = \lambda_2 = 1 \longrightarrow x = Span\left[\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right] \quad \lambda = \lambda_3 = 7 \longrightarrow x = Span\left[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right] \longrightarrow X = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\boxed{\rightarrow T_M = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}}$$

$$A = XDX^{-1} \quad D = diag_n(\lambda)$$

$$A^2 = (XDX^{-1})(XDX^{-1}) = (XDX^{-1}XDX^{-1}) = (XDI_nDX^{-1}) = (XDDX^{-1})$$

$$A^2 = (XD^2X^{-1})$$

$$A^3 = (XDX^{-1})(XDX^{-1})(XDX^{-1}) = (XDX^{-1}XDX^{-1}XDX^{-1}) = (XDI_nDI_nDX^{-1})$$

$$= (XDDDX^{-1})$$

$$A^3 = (XD^3X^{-1}) \qquad Let\ A^k = (XD^kX^{-1})$$

$$A^{k+1} = A^kA = (XD^kX^{-1})A = (XD^kX^{-1})(XDX^{-1}) = (XD^kX^{-1}XDX^{-1})$$

$$= (XD^kI_nDX^{-1}) = (XD^kDX^{-1})$$

$$A^{k+1} = (XD^{k+1}X^{-1}) \qquad \boxed{\longrightarrow A^k = (XD^kX^{-1})}$$

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \longrightarrow A^p = \begin{bmatrix} a_{11}^p & 0 & 0 & \cdots & 0 \\ 0 & a_{22}^p & 0 & \cdots & 0 \\ 0 & 0 & a_{33}^p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^p \end{bmatrix}$$

$$diag_n(\lambda) = D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \longrightarrow D^k = \begin{bmatrix} \lambda_1^p & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^p & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n^p \end{bmatrix}$$

$$T_M = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix} \longrightarrow T_M^{10} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}^{10} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5^{10} & 0 \\ 0 & 2^{10} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

In Class Exercise 8.2

Evaluate B^{100} $\quad B = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$

In Class Exercise 8.2
SOLUTION

$$B = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$\longrightarrow B^{100} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}^{100} = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1^{100} & 0 & 0 \\ 0 & 1^{100} & 0 \\ 0 & 0 & 7^{100} \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Cholesky Decomposition

$$A = A^T$$

$$\lambda_1, \lambda_2, \dots, \lambda_n > 0 \qquad L = \begin{bmatrix} l_{1,1} & 0 & 0 & \cdots & 0 \\ l_{2,1} & l_{2,2} & 0 & \cdots & 0 \\ l_{3,1} & l_{3,2} & l_{3,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n,1} & l_{n,2} & l_{n,3} & \cdots & l_{n,n} \end{bmatrix} \qquad L^T = \begin{bmatrix} l_{1,1} & l_{1,2} = l_{2,1} & l_{1,3} = l_{3,1} & \cdots & l_{1,n} = l_{n,1} \\ 0 & l_{2,2} & l_{2,3} = l_{3,2} & \cdots & l_{2,n} = l_{n,2} \\ 0 & 0 & l_{3,3} & \cdots & l_{3,n} = l_{n,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & l_{n,n} \end{bmatrix}$$

$$A = LL^T$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} = \begin{bmatrix} l_{1,1} & 0 & 0 \\ l_{2,1} & l_{2,2} & 0 \\ l_{3,1} & l_{3,2} & l_{3,3} \end{bmatrix} \begin{bmatrix} l_{1,1} & l_{2,1} & l_{3,1} \\ 0 & l_{2,2} & l_{3,2} \\ 0 & 0 & l_{3,3} \end{bmatrix} = \begin{bmatrix} l_{1,1}^2 & l_{2,1}l_{1,1} & l_{3,1}l_{1,1} \\ l_{2,1}l_{1,1} & l_{2,1}^2 + l_{2,2}^2 & a_{2,3} \\ l_{3,1}l_{1,1} & l_{3,1}l_{2,1} + l_{3,2}l_{2,2} & l_{3,1}^2 + l_{3,2}^2 + l_{3,3}^2 \end{bmatrix}$$

$$l_{1,1}^2 = a_{1,1} \longrightarrow l_{1,1} = \sqrt{a_{1,1}}$$

$$l_{2,1}l_{1,1} = a_{2,1} \longrightarrow l_{2,1} = \frac{a_{2,1}}{l_{1,1}}$$

$$l_{3,1}l_{1,1} = a_{3,1} \longrightarrow l_{3,1} = \frac{a_{3,1}}{l_{1,1}}$$

$$l_{2,1}^2 + l_{2,2}^2 = a_{2,2} \longrightarrow l_{2,2}^2 = a_{2,2} - l_{2,1}^2$$

$$\longrightarrow l_{2,2} = \sqrt{a_{2,2} - l_{2,1}^2}$$

$$l_{3,1}l_{2,1} + l_{3,2}l_{2,2} = a_{3,2} \longrightarrow l_{3,2}l_{2,2} = a_{3,2} - l_{3,1}l_{2,1}$$

$$\longrightarrow l_{3,2} = \frac{a_{3,2} - l_{3,1}l_{2,1}}{l_{2,2}}$$

$$l_{3,1}^2 + l_{3,2}^2 + l_{3,3}^2 = a_{3,3} \longrightarrow l_{3,3}^2 = a_{3,3} - l_{3,1}^2 - l_{3,2}^2$$

$$\longrightarrow l_{3,3} = \sqrt{a_{3,3} - l_{3,1}^2 - l_{3,2}^2}$$

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

$$l_{1,1} = \sqrt{a_{1,1}} = \sqrt{3} \qquad l_{2,1} = \frac{a_{2,1}}{l_{1,1}} = \frac{2}{\sqrt{3}}$$

$$l_{3,1} = \frac{a_{3,1}}{l_{1,1}} = \frac{2}{\sqrt{3}}$$

$$l_{2,2} = \sqrt{a_{2,2} - l_{2,1}^2} = \sqrt{3 - \left(\frac{2}{\sqrt{3}}\right)^2} = \sqrt{\frac{5}{3}}$$

$$l_{3,2} = \frac{a_{3,2} - l_{3,1}l_{2,1}}{l_{2,2}} = \frac{2 - \frac{2}{\sqrt{3}}\frac{2}{\sqrt{3}}}{\sqrt{\frac{5}{3}}} = \frac{2}{3}\sqrt{\frac{3}{5}}$$

$$l_{3,3} = \sqrt{a_{3,3} - l_{3,1}^2 - l_{3,2}^2} = \sqrt{3 - \frac{4}{3} - \frac{12}{45}} = \sqrt{\frac{7}{5}}$$

$$\longrightarrow A = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ \frac{2}{\sqrt{3}} & \sqrt{\frac{5}{3}} & 0 \\ \frac{2}{\sqrt{3}} & \frac{2}{3}\sqrt{\frac{3}{5}} & \sqrt{\frac{7}{5}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ 0 & \sqrt{\frac{5}{3}} & \frac{2}{3}\sqrt{\frac{3}{5}} \\ 0 & 0 & \sqrt{\frac{7}{5}} \end{bmatrix}$$

In Class Exercise 8.3

Find the cholesky factor of B

$$B = \begin{bmatrix} 2 & 6 & 2 \\ 2 & 8 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

In Class Exercise 8.3
SOLUTION

$Doesn't\ exist$

$$\underset{m\times n}{A} = \underset{m\times m}{U}\;\underset{m\times n}{\Sigma}\;\underset{n\times n}{V^T}$$

mxn mxm mxn nxn

$$\underset{m\times n}{A} = \underset{m\times m}{U}\;\underset{m\times n}{\Sigma}\;\underset{n\times n}{V^T}$$

mxn mxm mxn nxn

$A \in R^{mxn}$ $U \in R^{mxm}$ $V \in R^{nxn}$ $\Sigma \in R^{mxn}$

$U^{-1} = U^T$ $V^{-1} = V^T$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & \sigma_2 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \sigma_i = \sigma_r & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & \sigma_2 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \sigma_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \sigma_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \sigma_n = \sigma_r \end{pmatrix}$$

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_r$$

$A^T A \in R^{nxn}$

$$A^T A = X diag_n(\lambda) X^{-1} = X diag_n(\lambda) X^T = X \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \lambda_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix} X^T \text{---}(1)$$

$A = U\Sigma V^T$

$$\longrightarrow A^T A = \left(U\Sigma V^T\right)^T \left(U\Sigma V^T\right) = \left(\left(V^T\right)^T \Sigma^T U^T\right)\left(U\Sigma V^T\right) = \left(V\Sigma^T U^T\right)\left(U\Sigma V^T\right)$$

$$= \left(V\Sigma^T U^T U\Sigma V^T\right) = \left(V\Sigma^T I_m \Sigma V^T\right) = \left(V\Sigma^T\Sigma V^T\right)$$

$$= V \begin{pmatrix} \sigma_1^2 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & \sigma_2^2 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \sigma_i^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \sigma_{n-1}^2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \sigma_n^2 \end{pmatrix} V^T \text{---}(2) \longrightarrow \begin{cases} V = X \\ \sigma_i^2 = \lambda_i \end{cases} \longrightarrow \begin{cases} V = X \\ \sigma_i = \sqrt{\lambda_i} \end{cases}$$

$AA^T \in R^{mxm}$

$$AA^T = X diag_n(\lambda) X^{-1} = X diag_n(\lambda) X^T = X \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \lambda_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix} X^T \text{---}(1)$$

$A = U\Sigma V^T$

$$\longrightarrow AA^T = \left(U\Sigma V^T\right)\left(U\Sigma V^T\right)^T = \left(U\Sigma V^T\right)\left(\left(V^T\right)^T \Sigma^T U^T\right) = \left(U\Sigma V^T\right)\left(V\Sigma^T U^T\right)$$

$$= \left(U\Sigma V^T V\Sigma^T U^T\right) = \left(U\Sigma I_n \Sigma^T U^T\right) = \left(U\Sigma\Sigma^T U^T\right)$$

$$= U \begin{pmatrix} \sigma_1^2 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & \sigma_2^2 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \sigma_i^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \sigma_{n-1}^2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \sigma_m^2 \end{pmatrix} U^T \text{---}(2) \longrightarrow \begin{cases} U = X \\ \sigma_i^2 = \lambda_i \end{cases} \longrightarrow \begin{cases} U = X \\ \sigma_i = \sqrt{\lambda_i} \end{cases}$$

$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$ *Construct the SVD of A*

$A \in R^{3x2}$ & $A = U\Sigma V^T \longrightarrow A^TA \in R^{2x2}$ & $AA^T \in R^{3x3}$

Let $P = A^TA$ & $Q = AA^T$

$$P = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$$

$$det(P - \lambda I_2) = det\left(\begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = det\left(\begin{bmatrix} 9-\lambda & -9 \\ -9 & 9-\lambda \end{bmatrix}\right)$$

$$= (9-\lambda)(9-\lambda) - (-9\ X -9) = \lambda(\lambda - 18)$$

$$\boxed{\longrightarrow \lambda_1 = 0, \lambda_2 = 18}$$

$\lambda = \lambda_1 = 0$ *Find* $x \,/\, (P - 0I_2)x = 0$

$$(P - 0I_2)x = 0 \longrightarrow \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 9 & -9 & 0 \\ -9 & 9 & 0 \end{array}\right]\begin{array}{c} r_1 \\ r_2 \end{array} \sim \left[\begin{array}{cc|c} 9 & -9 & 0 \\ 0 & 0 & 0 \end{array}\right]\begin{array}{c} r_1 \\ r_2 = r_2 + r_1 \end{array} \rightarrow 9x_1 - 9x_2 = 0$$

$$\rightarrow \begin{cases} x_1 = x_2 \\ x_2 = x_2 \end{cases} \rightarrow x = Span\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = \lambda_2 = 18$ *Find* $x \,/\, (P - 18I_2)x = 0$

$$(P - 18I_2)x = 0 \longrightarrow \begin{bmatrix} -9 & -9 \\ -9 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -9 & -9 & 0 \\ -9 & -9 & 0 \end{array}\right]\begin{array}{c} r_1 \\ r_2 \end{array} \sim \left[\begin{array}{cc|c} -9 & -9 & 0 \\ 0 & 0 & 0 \end{array}\right]\begin{array}{c} r_1 \\ r_2 = r_2 - r_1 \end{array} \rightarrow -9x_1 - 9x_2 = 0$$

$$\longrightarrow \begin{cases} x_1 = -x_2 \\ x_2 = x_2 \end{cases} \longrightarrow x = Span\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\longrightarrow A^TA = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 18 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \longrightarrow V = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{bmatrix}$$

$$\longrightarrow Q - \lambda I_3 = \begin{bmatrix} 2-\lambda & -4 & 4 \\ -4 & 8-\lambda & -8 \\ 4 & -8 & 8-\lambda \end{bmatrix}$$

$\lambda = \lambda_1 = 0$ *Find* $x \,/\, (Q - 0I_3)x = 0$

$$(Q - 0I_2)x = 0 \longrightarrow \begin{bmatrix} 2 & -4 & 4 \\ -4 & 8 & -8 \\ 4 & -8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -4 & 4 & 0 \\ -4 & 8 & -8 & 0 \\ 4 & -8 & 8 & 0 \end{array}\right]\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{c} r_1 = 0.5 * r_1 \\ r_2 = 2r_1 + r_2 \\ r_3 = r_3 + r_2 \end{array}$$

$$\longrightarrow x_1 - 2x_2 + 2x_3 = 0 \longrightarrow \begin{cases} x_1 = 2x_2 - 2x_3 \\ x_2 = 1x_2 + 0x_3 \\ x_3 = 0x_2 + 1x_3 \end{cases}$$

$$\longrightarrow x = Span\left[\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right]$$

$\lambda = \lambda_2 = 18$ *Find* $x \,/\, (Q - 18I_3)x = 0$

$$(Q - 18I_3)x = 0 \longrightarrow \begin{bmatrix} -16 & -4 & 4 \\ -4 & -10 & -8 \\ 4 & -8 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -16 & -4 & 4 & 0 \\ -4 & -10 & -8 & 0 \\ 4 & -8 & -10 & 0 \end{array}\right]\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} \sim \left[\begin{array}{ccc|c} -4 & -1 & 1 & 0 \\ -4 & -10 & -8 & 0 \\ 0 & -18 & -18 & 0 \end{array}\right]\begin{array}{c} \frac{r_1}{4} \\ r_2 \\ r_3 + r_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -4 & -1 & 1 & 0 \\ 0 & -9 & -9 & 0 \\ 0 & -18 & -18 & 0 \end{array}\right]\begin{array}{c} r_1 \\ r_2 - r_1 \\ r_3 \end{array} \sim \left[\begin{array}{ccc|c} -4 & -1 & 1 & 0 \\ 0 & -9 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{c} r_1 \\ r_2 \\ r_3 = r_3 - 2r_2 \end{array}$$

$$\rightarrow \begin{cases} -4x_1 - x_2 + x_3 = 0 \\ -9x_2 - 9x_3 = 0 \end{cases} \rightarrow \begin{cases} x_1 = -\frac{1}{4}x_2 + \frac{1}{4}x_3 = -\frac{1}{2}x_2 \\ x_2 = x_2 \\ x_3 = -x_2 \end{cases}$$

$$\longrightarrow x = Span\left[\begin{bmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{bmatrix}\right] \longrightarrow U = \begin{bmatrix} -\frac{1}{2} & 2 & -2 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$U_1' = U_1 = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{bmatrix} \qquad U_2' = U_2 - \frac{U_2 \cdot U_1'}{U_1' \cdot U_1'}U_1' = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{bmatrix}}\begin{bmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$U_3' = U_3 - \left(\frac{U_3 \cdot U_1'}{U_1' \cdot U_1'}U_1' + \frac{U_3 \cdot U_2'}{U_2' \cdot U_2'}U_2'\right) = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{bmatrix}}\begin{bmatrix} -\frac{1}{2} \\ 1 \\ -1 \end{bmatrix} + \frac{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -\frac{2}{5} \\ \frac{4}{5} \\ 1 \end{bmatrix}$$

$$\longrightarrow U' = \begin{bmatrix} -\frac{1}{2} & 2 & -\frac{2}{5} \\ 1 & 1 & \frac{4}{5} \\ -1 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow U'' = \begin{bmatrix} \left(\frac{-\frac{1}{2}}{\frac{3}{2}}\right) & \frac{2}{\sqrt{5}} & -\frac{\frac{2}{5}}{\left(\frac{\sqrt{45}}{5}\right)} \\ \left(\frac{1}{\frac{3}{2}}\right) & \frac{1}{\sqrt{5}} & \frac{\frac{4}{5}}{\left(\frac{\sqrt{45}}{5}\right)} \\ \left(\frac{-1}{\frac{3}{2}}\right) & \frac{0}{\sqrt{5}} & \frac{1}{\left(\frac{\sqrt{45}}{5}\right)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ -\frac{2}{3} & \frac{0}{\sqrt{5}} & \frac{5}{\sqrt{45}} \end{bmatrix}$$

$$V = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \longrightarrow V' = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ -\frac{2}{3} & \frac{0}{\sqrt{5}} & \frac{5}{\sqrt{45}} \end{bmatrix}\begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

In Class Exercise 8.4

Find U from V, without having to find the eigen decomposition of AA^T

In Class Exercise 8.4 SOLUTION

$$A = U\Sigma V^T \quad \longrightarrow AV = U\Sigma V^T V \quad \longrightarrow AV = U\Sigma$$

$$\longrightarrow \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix}\begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{cases} -2 = \sqrt{18}u_{11} & 0 = 0 \\ 4 = \sqrt{18}u_{21} & 0 = 0 \\ -4 = \sqrt{18}u_{31} & 0 = 0 \end{cases}$$

$$\longrightarrow \begin{cases} u_{11} = -\frac{\sqrt{2}}{3} \\ u_{21} = \frac{2\sqrt{2}}{3} \\ u_{31} = -\frac{2\sqrt{2}}{3} \end{cases} \quad \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} \perp \begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} \perp \begin{bmatrix} u_{13} \\ u_{23} \\ u_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} \cdot \begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} = 0 \;\; \& \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} \cdot \begin{bmatrix} u_{13} \\ u_{23} \\ u_{33} \end{bmatrix} = 0$$

$$\longrightarrow \begin{cases} -\frac{\sqrt{2}}{3}u_{12} + \frac{2\sqrt{2}}{3}u_{22} - \frac{2\sqrt{2}}{3}u_{32} = 0 \\ -\frac{\sqrt{2}}{3}u_{13} + -\frac{2\sqrt{2}}{3}u_{23} - \frac{-2\sqrt{2}}{3}u_{33} = 0 \end{cases} \longrightarrow -\frac{\sqrt{2}}{3}u_{12} + \frac{2\sqrt{2}}{3}u_{22} - \frac{2\sqrt{2}}{3}u_{32} = 0 \longrightarrow u_{12} = 2u_{22} - 2u_{32}$$

$$\longrightarrow \begin{cases} u_{12} = 2u_{22} - 2u_{32} \\ u_{22} = u_{22} \\ u_{32} = u_{32} \end{cases} \longrightarrow \begin{cases} u_{12} = 2u_{22} - 2u_{32} \\ u_{22} = 1u_{22} + 0u_{32} \\ u_{32} = 0u_{22} + 1u_{32} \end{cases} \longrightarrow x = Span\left[\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right]$$

$$\longrightarrow U = \begin{bmatrix} -\frac{\sqrt{2}}{3} & 2 & -2 \\ \frac{2\sqrt{2}}{3} & 1 & 0 \\ -\frac{2\sqrt{2}}{3} & 0 & 1 \end{bmatrix}$$

$$A = [\underset{m\,x\,r}{U_1} \quad \underset{m\,x(m-r)}{U_2}] \begin{bmatrix} \underset{r\,x\,r}{\Sigma_r} & \underset{r\,x\,(n-r)}{0} \\ \underset{(m-r)\,x\,r}{0} & \underset{(m-r)x(n-r)}{0} \end{bmatrix} [\underset{nxr}{V_1} \quad \underset{nx(n-r)}{V_2}]^T \longrightarrow A = \underset{m\,xr}{U_1}\,\underset{rxr}{\Sigma_r}\,\underset{n\,xr}{V_1^T}$$

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ -\frac{2}{3} & \frac{0}{\sqrt{5}} & \frac{5}{\sqrt{45}} \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 0.48 & -0.73 & 0.48 & 0 \\ 0.47 & -0.24 & -0.84 & 0 \\ 0.23 & 0.20 & 0.07 & -0.94 \\ 0.69 & 0.60 & 0.21 & 0.31 \end{bmatrix} \begin{bmatrix} 9.54 & 0 & 0 & 0 \\ 0 & 1.22 & 0 & 0 \\ 0 & 0 & 0.60 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.05 & 0.1 & 0.44 & 0.88 \\ -0.59 & -0.79 & 0.049 & 0.098 \\ 0.80 & -0.59 & 0.0087 & 0.017 \\ 0 & 0 & 0.89 & -0.44 \end{bmatrix}^T$$

$$\begin{bmatrix} -\frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{45}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{\sqrt{45}} \\ -\frac{2}{3} & \frac{0}{\sqrt{5}} & \frac{5}{\sqrt{45}} \end{bmatrix} \begin{bmatrix} \sqrt{18} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} \frac{-\sqrt{18}}{3} & 0 \\ \frac{2\sqrt{18}}{3} & 0 \\ \frac{-2\sqrt{18}}{3} & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} [\sqrt{18}] \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} \frac{-\sqrt{18}}{3} \\ \frac{2\sqrt{18}}{3} \\ \frac{-2\sqrt{18}}{3} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{18}}{3}(-\frac{1}{\sqrt{2}}) & \frac{-\sqrt{18}}{3}(\frac{1}{\sqrt{2}}) \\ \frac{2\sqrt{18}}{3}(-\frac{1}{\sqrt{2}}) & \frac{2\sqrt{18}}{3}(\frac{1}{\sqrt{2}}) \\ \frac{-2\sqrt{18}}{3}(-\frac{1}{\sqrt{2}}) & \frac{-2\sqrt{18}}{3}(\frac{1}{\sqrt{2}}) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$$

COMPLETE LINEAR ALGEBRA

LINEAR ALGEBRA > Matrix Decomposition > Low Rank Approximation & condition number

$$A = \underset{m\,x\,k}{[U_1} \quad \underset{m\,x(m-k)}{U_2]} \begin{bmatrix} \underset{k\,x\,k}{\Sigma_k} & \underset{k\,x\,(n-k)}{0} \\ \underset{(m-k)\,x\,k}{0} & \underset{(m-k)x(n-k)}{0} \end{bmatrix} \underset{nxk}{[V_1} \quad \underset{nx(n-k)}{V_2]}^T \longrightarrow A = \underset{m\,xk}{U_1}\,\underset{kxk}{\Sigma_k}\,\underset{n\,xk}{V_1^T}$$

$k < r$

Percentage restored Information= $\frac{\sum_{i=1}^{k}\sigma_i^2}{\sum_{i=1}^{r}\sigma_i^2} * 100$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{17}{20} & 0 & 0 & -\frac{21}{40} \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{21}{40} & 0 & 0 & \frac{17}{20} \end{bmatrix} \begin{bmatrix} \frac{34}{21} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{13}{21} \end{bmatrix} \begin{bmatrix} -\frac{17}{20} & 0 & -\frac{21}{40} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ \frac{21}{40} & 0 & -\frac{17}{20} & 0 \end{bmatrix}$$

Percentage restored Information= $\frac{\sum_{i=1}^{k}\sigma_i^2}{\sum_{i=1}^{r}\sigma_i^2} = \frac{\left(\frac{34}{21}\right)^2 + 1^2 + 1^2 + \left(\frac{13}{21}\right)^2}{\left(\frac{34}{21}\right)^2 + 1^2 + 1^2 + \left(\frac{13}{21}\right)^2} * 100 = 100$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{17}{20} & 0 & 0 & -\frac{21}{40} \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{21}{40} & 0 & 0 & \frac{17}{20} \end{bmatrix} \begin{bmatrix} \frac{34}{21} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{17}{20} & 0 & -\frac{21}{40} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ \frac{21}{40} & 0 & -\frac{17}{20} & 0 \end{bmatrix}$$

Percentage restored Information= $\frac{\sum_{i=1}^{k}\sigma_i^2}{\sum_{i=1}^{r}\sigma_i^2} = \frac{\left(\frac{34}{21}\right)^2 + 1^2 + 1^2}{\left(\frac{34}{21}\right)^2 + 1^2 + 1^2 + \left(\frac{13}{21}\right)^2} * 100 = 92.3$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{17}{20} & 0 & 0 & -\frac{21}{40} \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{21}{40} & 0 & 0 & \frac{17}{20} \end{bmatrix} \begin{bmatrix} \frac{34}{21} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{17}{20} & 0 & -\frac{21}{40} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ \frac{21}{40} & 0 & -\frac{17}{20} & 0 \end{bmatrix}$$

Percentage restored Information= $\frac{\sum_{i=1}^{k}\sigma_i^2}{\sum_{i=1}^{r}\sigma_i^2} = \frac{\left(\frac{34}{21}\right)^2 + 1^2}{\left(\frac{34}{21}\right)^2 + 1^2 + 1^2 + \left(\frac{13}{21}\right)^2} * 100 = 72.3$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{17}{20} & 0 & 0 & -\frac{21}{40} \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{21}{40} & 0 & 0 & \frac{17}{20} \end{bmatrix} \begin{bmatrix} \frac{34}{21} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{17}{20} & 0 & -\frac{21}{40} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ \frac{21}{40} & 0 & -\frac{17}{20} & 0 \end{bmatrix}$$

Percentage restored Information= $\frac{\sum_{i=1}^{k}\sigma_i^2}{\sum_{i=1}^{r}\sigma_i^2} = \frac{\left(\frac{34}{21}\right)^2}{\left(\frac{34}{21}\right)^2 + 1^2 + 1^2 + \left(\frac{13}{21}\right)^2} * 100 = 52.3$

$$Condition\ Number = \frac{\sigma_{max.}}{\sigma_{min}}$$

COMPLETE LINEAR ALGEBRA

Column Space

$A = U_1\Sigma_r V_1^T \in R^{mxn}$

$b \in Col(A) \longrightarrow \exists\, x \in R^{nx1} \,/ b = Ax$

$\longrightarrow b = U_1\Sigma_r V_1^T x \longrightarrow b = U_1 x^* \longrightarrow b \in Col(U_1)$

$b \in Col(U_1) \longrightarrow \exists\, x \in R^{rx1} \,/ b = U_1 x \quad A = U_1\Sigma_r V_1^T \longrightarrow U_1 = AV_1\Sigma_r^{-1}$

$\longrightarrow b = AV_1\Sigma_r^{-1} x \longrightarrow \exists\, x^* \in R^{nx1} \,/ b = Ax^*$

$\longrightarrow b \in Col(A)$

$\boxed{\longrightarrow Col(A) = Col(U_1) = Span(U_1)}$

Row Space

$A = U_1\Sigma_r V_1^T \in R^{mxn}$

$b \in Col(A^T) \longrightarrow \exists\, x \in R^{mx1} \,/ b = A^T x \longrightarrow b = \left(U_1\Sigma_r V_1^T\right)^T x$

$\longrightarrow b = V_1\Sigma_r^T U_1^T x \longrightarrow b = V_1 x^* \longrightarrow b \in Col(V_1)$

$b \in Col(V_1) \longrightarrow \exists\, x \in R^{rx1} \,/ b = V_1 x \quad A = U_1\Sigma_r V_1^T \longrightarrow V_1 = \left(\Sigma_r^{-1}U_1^T A\right)^T$

$\longrightarrow b = A^T U_1 {\Sigma_r^{-1}}^T x \longrightarrow b = A^T x^*$

$\longrightarrow b \in Col(A^T) \longrightarrow \exists\, x^* \in R^{mx1} \,/ b = A^T x^*$

$\boxed{\longrightarrow Col(A^T) = Row(A) = Col(V_1) = Span(V_1)}$

Null Space

$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T \;\&\; A = U\Sigma V^T \in R^{mxn} \longrightarrow \Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} --- (1)$

$A = U\Sigma V^T \longrightarrow \Sigma = U^T AV \longrightarrow \Sigma = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} A \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} U_1 A \\ U_2 A \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}$

$= \begin{bmatrix} U_1AV_1 & U_1AV_2 \\ U_2AV_1 & U_2AV_2 \end{bmatrix} --- (2)$

$(1)\&(2) \longrightarrow \begin{bmatrix} U_1AV_2 \\ U_2AV_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} AV_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow AV_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow V_2 \in Null(A) \qquad n = dim\big(N(A)\big) + r$

$\longrightarrow dim\big(N(A)\big) = n - r = N^0 \text{ of Columns of } V_2$ $\boxed{\longrightarrow Null(A) = Span(V_2)}$

Left Null Space

$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T \;\&\; A = U\Sigma V^T \in R^{mxn} \longrightarrow \Sigma^T = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} --- (1)$

$A = U\Sigma V^T \longrightarrow \Sigma^T = \left(U^T AV\right)^T = V^T A^T U \longrightarrow \Sigma = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} A^T \begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} V_1 A^T \\ V_2 A^T \end{bmatrix} \begin{bmatrix} U_1 & U_2 \end{bmatrix}$

$= \begin{bmatrix} V_1A^TU_1 & V_1A^TU_2 \\ V_2A^TU_1 & V_2A^TU_2 \end{bmatrix} --- (2)$

$(1)\&(2) \longrightarrow \begin{bmatrix} V_1A^TU_2 \\ V_2A^TU_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} A^TU_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow A^TU_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\longrightarrow U_2 \in Null(A^T)$

$m = dim\left(N(A^T)\right) + r \longrightarrow dim\left(N(A^T)\right) = m - r = N^0 \text{ of Columns of } U_2$

$\boxed{\longrightarrow Null(A^T) = Span(U_2)}$

LINEAR ALGEBRA > MATRIX DECOMPOSITIONS

DECOMPOSITIONS:

If we break 30 into its factors, we obtain 30 = 2 * 3 * 5.

This decomposition of 30 helps us see certain hidden properties of 30, which may useful when doing certain calculations.

What if we are able to decompose matrices, in a way that some of their hidden properties become exposed?

Several matrix decomposition techniques exist. We shall look at the eigen decomposition, Singular value decomposition and Cholesky decomposition

EIGEN DECOMPOSITION

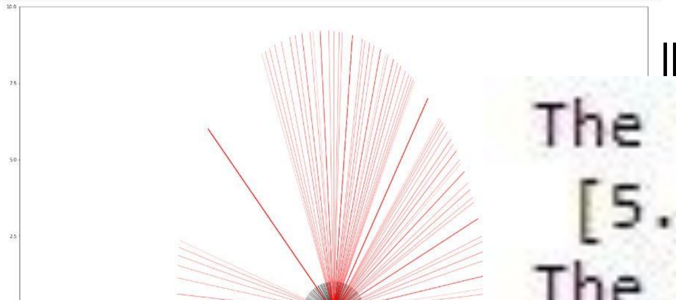
Remember, we saw that for every linear transformation, we were able to come up with a transformation matrix, which could transform a vector by simple matrix multiplication.

Nonetheless, there exist a set of vectors which when a linear transformation is applied on them, still maintain the same direction, but may different magnitude.

To be more precise, it stays on the line that it spans out, without getting out of it.

EXPL73

This vectors which are stretched or contracted by a factor



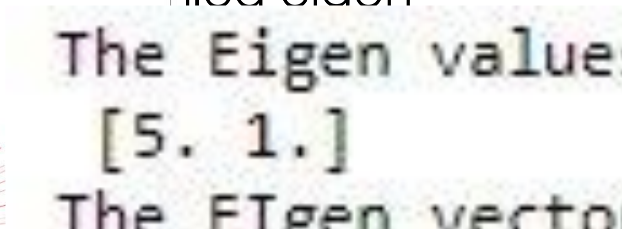
lled eigen

vectors and the corresponding factor lambda, by which t

value. Hence constructing matrices with specific eigen va

contract space in a particular direction. An eigen vector

which verifies the condition:



LINEAR ALGEBRA: MATRIX ALGEBRA

LINEAR ALGEBRA > MATRIX DECOMPOSITIONS

EXPL74

It should be noted that if p eigen vectors of the n by n matrix A have p distinct eigen values, then the eigen vectors are linearly independent.

THE CHARACTERISTIC EQUATION

With the aid of the characteristic equation, we shall see how to trix.

EXPL75

Now that we know how to find the eigen vectors and their corre t the eigen decomposition of a matrix, A.

EXPL76

DIAGONALIZATION

An n by n matrix is said to be diagonalizable if it has n linearly inc

To diagonalize a matrix means to put it in its eigen decompositic

Once we've put a matrix in this form, we can easily calculate pc

```
A1 = [[2,2,0],
      [1,0,1],
      [0,1,2]]

eigen_values_1, eigen_vectors_1 = np.linalg.eig(A1)

print("The Eigen values of A1 are  = \n", eigen_values_1)

print("The EIgen vectors of A1 are = \n", eigen_vectors_1)

print("\n")

A2 = [[2,2,0],
      [1,0,1],
      [0,1,2]]

eigen_values_2, eigen_vectors_2 = np.linalg.eig(A2)

print("The Eigen values of A2 are  = \n", eigen_values_2)

print("The EIgen vectors of A2 are = \n", eigen_vectors_2)

print("\n")
```

```
The Eigen values of A1 are  =
 [-1.  3.  2.]
The EIgen vectors of A1 are =
 [[ 5.34522484e-01  8.16496581e-01 -7.07106781e-01]
 [-8.01783726e-01  4.08248290e-01 -8.69072711e-17]
 [ 2.67261242e-01  4.08248290e-01  7.07106781e-01]]


The Eigen values of A2 are  =
 [-1.  3.  2.]
The EIgen vectors of A2 are =
 [[ 5.34522484e-01  8.16496581e-01 -7.07106781e-01]
 [-8.01783726e-01  4.08248290e-01 -8.69072711e-17]
 [ 2.67261242e-01  4.08248290e-01  7.07106781e-01]]
```

EXPL77

SYMMETRIC MATRICES

This is a matrix which is the same as its transpose

A = AT

SKEW SYMMETRIC A = -AT

Hence diagonal matrices are symmetric

POSITIVE DEFINITE MATRIX:

This is a matrix with :

EXPL78

LINEAR ALGEBRA > MATRIX DECOMPOSITIONS

A matrix is said to be orthogonally diagonalizable if there exists a diagonal matrix V, with V-1 = VT and a diagonal matrix, such that A = PDP-1 = PDPT
All symmetric matrices are orthogonally diagonalizable
CHOLESKY DECOMPOSITION
It only holds for symmetric positive definite matrices.
EXPL79
SINGULAR VALUE DECOMPOSITION (SVD)
MIT Professor, Gilbert Strang describes the Singular value decomposition as the 'fundamental theorem of linear algebra.'
This is a central decomposition method in linear algebra.
If A Is a rectangular m by n matrix of rank, r =[0:min(m,n)], the SVD of the matrix Is of the form:
EXPL80
SVD CONSTRUCTION
EXPL81
Now we are able to construct the SVD of a matrix. Let's gain more insight on how the matrices U, sigma and V are constructed.
DETAILED SVD CONSTRUCTION.
EXPL82
SVD AND THE FOUR SUBSPACES
EXPL83
THE CONDITION NUMBER

LINEAR ALGEBRA: MATRIX ALGEBRA

LINEAR ALGEBRA > MATRIX DECOMPOSITIONS

EXPL84

The smaller the condition number, the easier it is to calculate AX = b, more accurately.
If the condition number of a matrix is too large, the matrix is said to be ill-conditioned.

LOW RANK APPROXIMATION

This is an important application of the SVD. The low rank approximation of a matrix, A, given by Ak, is a matrix of same dimension as A, but of lower rank. This will mean that Ak will require less amount of data to be stored. This reduction in rank of the matrix is done in a way that not too much information is lost.

This reduction is done by setting some of the sigma i s to zero#

EXPL86

Lets see how to measure the information left after reducing the rank of a matrix:

EXPL85

So the closer L is to 1, the more information it stores