COMPLETE LINEAR ALGEBRA

- Python Setup, Basic Python, numpy, pandas and matplotlib
- Matrix Algebra
- Linear equations and transformations
- Vectors
- Vector Spaces
- Metric Spaces, Normed spaces, Inner Product Spaces
- Orthogonality
- Determinant and Trace Operator
- Matrix Decompositions (Eigen, SVD and Cholesky)
- Symmetric matrices and Quadratic Forms
- Left Inverse, Right Inverse, Pseudo Inverse

Connecting the dots 1
LINEAR REGRESSION

Connecting the dots 2
PRINCIPAL COMPONENT ANALYSIS

$$\begin{aligned} det: R^{nxn} &\longrightarrow R \\ A &\longmapsto |A| \end{aligned} \qquad det(A) = |A| = \begin{cases} ad - bc, if\ n = 2\ and\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \sum_{j=1}^{n} (-1)^{1+j} a_{1j} det(A_{1j}) \end{cases}$$

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 4 & 9 \\ 4 & 1 & 9 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$det(A) = (-1)^{1+1} a_{11} \boxed{det(A_{11})} + (-1)^{1+2} a_{12} \boxed{det(A_{12})} + (-1)^{1+3} a_{13} \boxed{det(A_{13})}$$

$$= (1) * \begin{vmatrix} 4 & 9 \\ 1 & 9 \end{vmatrix} + (-4) * det\left(\begin{bmatrix} 1 & 9 \\ 4 & 9 \end{bmatrix}\right) + (1) * \begin{vmatrix} 1 & 4 \\ 4 & 1 \end{vmatrix}$$

$$= (1) * [(4 * 9) - (1 * 9)] + (-4) * [(1 * 9) - (4 * 9)] + (1) * [(1 * 1) - (4 * 4)]$$

$$= 120$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 7 & 9 \\ 4 & 0 & 9 \end{bmatrix}$$

$$|B| = -(0) * \begin{vmatrix} 1 & 9 \\ 4 & 9 \end{vmatrix} + (7) * \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} - (0) * \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix}$$

$$= (7) * [(1 * 9) - (4 * 1)] \quad = 35$$

$$A = \begin{bmatrix} 47 & 4 & 10 & 11 \\ 1 & 0 & 4 & 1 \\ 1 & 0 & 4 & 9 \\ 4 & 0 & 1 & 9 \end{bmatrix}$$

$$det(A) = -(4) * det\left(\begin{bmatrix} 1 & 4 & 1 \\ 1 & 4 & 9 \\ 4 & 1 & 9 \end{bmatrix}\right) + (0) * det\left(\begin{bmatrix} 47 & 10 & 11 \\ 1 & 4 & 9 \\ 4 & 1 & 9 \end{bmatrix}\right)$$

$$-(0) * det\left(\begin{bmatrix} 47 & 10 & 11 \\ 1 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}\right) + (0) * det\left(\begin{bmatrix} 47 & 10 & 11 \\ 1 & 4 & 1 \\ 1 & 4 & 9 \end{bmatrix}\right)$$

$$= -4 * 120$$

$$= -480$$

In Class Exercise 7.1

$$C = \begin{bmatrix} 1 & 3 & 5 & 9 \\ 1 & 3 & 1 & 7 \\ 4 & 3 & 9 & 7 \\ 5 & 2 & 0 & 9 \end{bmatrix}$$

In Class Exercise 7.1 SOLUTION

$$det(C) = -376$$

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad A_2 = \begin{bmatrix} a_{11} + p\,a_{21} + qa_{31} & a_{12} + pa_{22} + qa_{32} & a_{13} + pa_{23} + qa_{33} \\ a_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, p, q \in R$$

$$\boxed{det(A_1) = det(A_2)}$$

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad A_2 = \begin{bmatrix} ka_{11} + p\,a_{21} & ka_{12} + pa_{22} & ka_{13} + pa_{23} \\ a_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, k, p \in R$$

$$\boxed{det(A_1) = \frac{1}{k} det(A_2)}$$

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad A_2 = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\boxed{det(A_1) = -det(A_2)}$$

$$det\left(\prod A_i\right) = \prod det(A_i)$$

$$\longrightarrow det(A_1A_2 \dots A_n) = det(A_1)det(A_2)\dots det(A_n)$$

$$det(A) = det(A^T)$$

$$det(A^{-1}) = \frac{1}{det(A)}$$

$$det(kA) = k^n det(A)$$

$$det(A) = \begin{cases} (-1)^r \prod Pivots \\ 0, otherwise \end{cases}$$

$$r = Number\ of\ Operations$$

$$A = \begin{bmatrix} -4 & 4 & 7 \\ 1 & 1 & 9 \\ 0 & -1 & 8 \end{bmatrix} \qquad \boxed{r = 4}$$

$$|A| = det\begin{bmatrix} -4 & 4 & 7 \\ 1 & 1 & 9 \\ 0 & -1 & 8 \end{bmatrix} = -det\begin{bmatrix} 1 & 1 & 9 \\ -4 & 4 & 7 \\ 0 & -1 & 8 \end{bmatrix}$$

$$= -det\begin{bmatrix} 1 & 1 & 9 \\ -4 + 4(1) & 4 + 4(1) & 7 + 4(9) \\ 0 & -1 & 8 \end{bmatrix}$$

$$= -\frac{1}{8} * det\begin{bmatrix} 1 & 1 & 9 \\ 0 & 8 & 43 \\ 8 * 0 & 8 * -1 & 8 * 8 \end{bmatrix} = -\frac{1}{8} det\begin{bmatrix} 1 & 1 & 9 \\ 0 & 8 & 43 \\ 0 & -8 & 64 \end{bmatrix}$$

$$= -\frac{1}{8} * det\begin{bmatrix} 1 & 1 & 9 \\ 0 & 8 & 43 \\ 0 + 0 & -8 + 8 & 64 + 43 \end{bmatrix} = -\frac{1}{8} det\begin{bmatrix} 1 & 1 & 9 \\ 0 & 8 & 43 \\ 0 & 0 & 107 \end{bmatrix}$$

$$= (-1)^4\left[-\frac{1}{8} * (1 * 8 * 107)\right] = -107$$

In Class Exercise 7.2

$$C = \begin{bmatrix} 1 & 3 & 5 & 9 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 9 & 7 \\ 0 & 0 & 9 & 9 \end{bmatrix}$$

$$D = \begin{bmatrix} 8 & 5 & -1 \\ 8 & 1 & 3 \\ 4 & 0 & 2 \end{bmatrix} \text{(Use the echelon form)}$$

In Class Exercise 7.2
SOLUTION

$$det(C) = -1 * (1 * 3 * 9 * 2)$$

$$det(D) = 0$$

COMPLETE LINEAR ALGEBRA

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n \end{cases} \rightarrow \begin{bmatrix} a_{1,1} & \cdots & a_{1,i} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,i} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(n-1),1} & \cdots & a_{(n-1),i} & \cdots & a_{(n-1),n} \\ a_{n,1} & \cdots & a_{n,i} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} \leftrightarrow AX = b$$

$$x_i = \frac{det(A_i)}{det(A)} \qquad A_i = \begin{bmatrix} a_{1,1} & \cdots & b_1 & \cdots & a_{1,n} \\ a_{2,1} & \cdots & b_2 & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(n-1),1} & \cdots & b_{n-1} & \cdots & a_{(n-1),n} \\ a_{n,1} & \cdots & b_n & \cdots & a_{n,n} \end{bmatrix}$$

$$\begin{cases} x + 2z = 9 ---(1) \\ 2y + z = 8 ---(2) \\ 4x - 3y = -2 ---(3) \end{cases} \dashrightarrow \begin{cases} 1x + 0y + 2z = 9 ---(1) \\ 0x + 2y + 1z = 8 ---(2) \\ 4x - 3y + 0z = -2 ---(3) \end{cases} \dashrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 4 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ -2 \end{bmatrix}$$

$$x_1 = \frac{det(A_1)}{det(A)} = \frac{\begin{vmatrix} 9 & 0 & 2 \\ 8 & 2 & 1 \\ -2 & -3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 4 & -3 & 0 \end{vmatrix}} = \frac{9[(0*2) - (-3*1)] + 2[(8*-3) - (-2*2)]}{1[(0*2) - (-3*1)] + 2[(0*-3) - (2*4)]} = \frac{-13}{-13} = 1$$

$$x_2 = \frac{det(A_2)}{det(A)} = \frac{\begin{vmatrix} 1 & 9 & 2 \\ 0 & 8 & 1 \\ 4 & -2 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 4 & -3 & 0 \end{vmatrix}} = \frac{1[(8*0) - (-2*1)] + 4[(1*9) - (2*8)]}{-13} = \frac{-26}{-13} = 2$$

$$x_3 = \frac{det(A_3)}{det(A)} = \frac{\begin{vmatrix} 1 & 0 & 9 \\ 0 & 2 & 8 \\ 4 & -3 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 4 & -3 & 0 \end{vmatrix}} = \frac{1[(2*-2) - (-3*8)] + 9[(0*-3) - (4*2)]}{-13} = \frac{-52}{-13} = 4$$

In Class Exercise 7.2

Solve the following equation using Cramer's rule

$$\begin{cases} w + x - 3y + z = 2 ---(1) \\ -5w + 3x - 4y + z = 0 ---(2) \\ w + 2y - z = 1 ---(3) \\ w + 2x = 12 ---(4) \end{cases}$$

In Class Exercise 7.3
SOLUTION

$$w = \frac{22}{17}, x = \frac{91}{17}, y = \frac{84}{17}, z = \frac{173}{17}$$

$$A^{-1} = \frac{1}{det(A)} Adj(A)$$

$$Adj(A) = \left(Cof(A)\right)^T$$

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,i} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,i} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(n-1),1} & \cdots & a_{(n-1),i} & \cdots & a_{(n-1),n} \\ a_{n,1} & \cdots & a_{n,i} & \cdots & a_{n,n} \end{bmatrix}$$

$$Cof(A) = \begin{bmatrix} -1^{(1+1)}\det(a_{1,1}) & \cdots & -1^{(1+i)}\det(a_{1,i}) & \cdots & -1^{(1+n)}\det(a_{1n}) \\ -1^{(2+1)}\det(a_{2,1}) & \cdots & -1^{(2+i)}\det(a_{2,i}) & \cdots & -1^{(2+n)}\det(a_{2n}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1^{((n-1)+1)}\det(a_{n-1,1}) & \cdots & -1^{((n-1)+i)}\det(a_{n-1,i}) & \cdots & -1^{((n-1)+n)}\det(a_{n-1,n}) \\ -1^{(n+1)}\det(a_{n,1}) & \cdots & -1^{(n+i)}\det(a_{n,i}) & \cdots & -1^{(n+n)}\det(a_{n,n}) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 6 & 2 \end{bmatrix}$$

$$\dashrightarrow Cof(A) = \begin{bmatrix} +\begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 4 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 0 & 6 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 6 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$\dashrightarrow Cof(A) = \begin{bmatrix} -22 & 0 & 0 \\ 14 & 2 & -6 \\ 5 & -4 & 1 \end{bmatrix}$$

$$\dashrightarrow Adj(A) = \begin{bmatrix} -22 & 14 & 5 \\ 0 & 2 & -4 \\ 0 & -6 & 1 \end{bmatrix}$$

$$det(A) = 1\begin{vmatrix} 1 & 4 \\ 6 & 2 \end{vmatrix} = -22$$

$$\dashrightarrow A^{-1} = \frac{1}{-22}\begin{bmatrix} -22 & 14 & 5 \\ 0 & 2 & -4 \\ 0 & -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{7}{11} & -\frac{5}{22} \\ 0 & -\frac{1}{11} & \frac{2}{11} \\ 0 & \frac{3}{11} & -\frac{1}{22} \end{bmatrix}$$

In Class Exercise 7.4

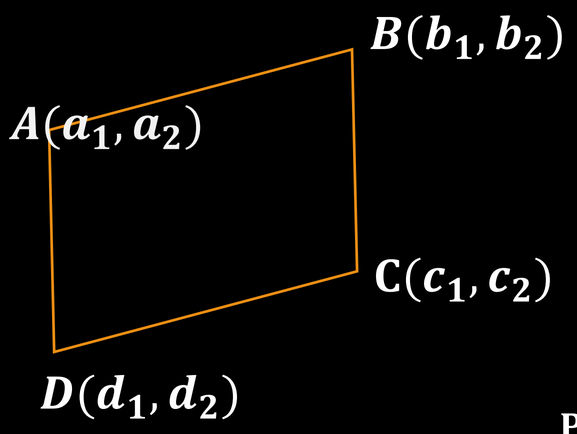
Find the inverse of the following matrix

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

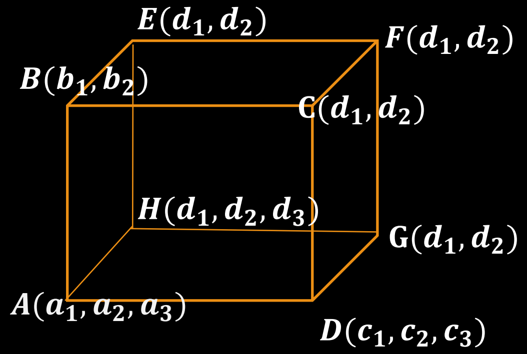
In Class Exercise 7.4
SOLUTION

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

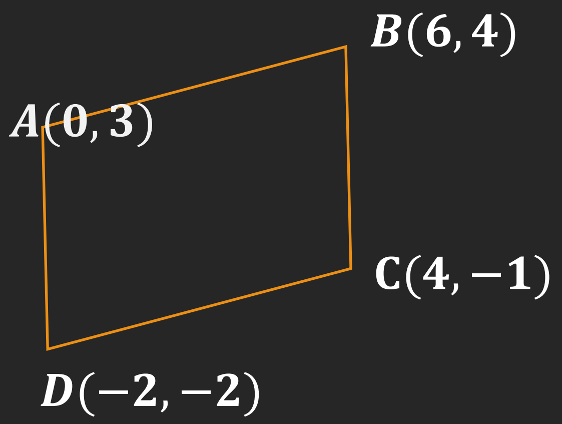
COMPLETE LINEAR ALGEBRA



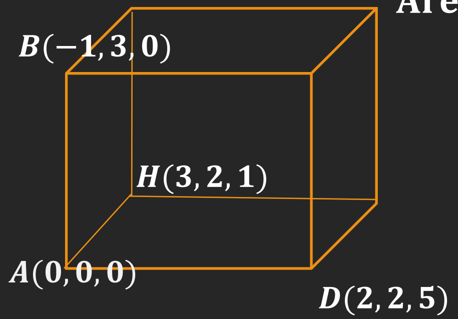
$$\text{Area} = \left|\det\left(\begin{bmatrix} DA_1 & BA_1 \\ DA_2 & BA_2 \end{bmatrix}\right)\right|$$
$$= \left|\det\left(\begin{bmatrix} DC_1 & BC_1 \\ DC_2 & BC_2 \end{bmatrix}\right)\right|$$
$$= \left|\det\left(\begin{bmatrix} AB_1 & CB_1 \\ AB_2 & CB_2 \end{bmatrix}\right)\right|$$
$$= \left|\det\left(\begin{bmatrix} AD_1 & CD_1 \\ AD_2 & CD_2 \end{bmatrix}\right)\right|$$
$$PQ = \begin{bmatrix} PQ_1 \\ PQ_2 \end{bmatrix} = \begin{bmatrix} OQ_1 - OP_1 \\ OQ_2 - OP_2 \end{bmatrix}$$



$$\text{Area} = \left|\det\left(\begin{bmatrix} DA_1 & BA_1 & HA_1 \\ DA_2 & BA_2 & HA_2 \\ DA_3 & BA_3 & HA_3 \end{bmatrix}\right)\right|$$
$$= \left|\det\left(\begin{bmatrix} CF_1 & EF_1 & GF_1 \\ CF_2 & EF_2 & GF_2 \\ CF_3 & EF_3 & GF_3 \end{bmatrix}\right)\right|$$
$$PQ = \begin{bmatrix} PQ_1 \\ PQ_2 \\ PQ_3 \end{bmatrix} = \begin{bmatrix} OQ_1 - OP_1 \\ OQ_2 - OP_2 \\ OQ_3 - OP_3 \end{bmatrix}$$



$$\text{Area} = \left|\det\left(\begin{bmatrix} DA_1 & BA_1 \\ DA_2 & BA_2 \end{bmatrix}\right)\right|$$
$$= \left|\det\left(\begin{bmatrix} 0-(-2) & 0-6 \\ 3-(-2) & 3-4 \end{bmatrix}\right)\right|$$
$$= \left|\det\left(\begin{bmatrix} 2 & -6 \\ 5 & -1 \end{bmatrix}\right)\right|$$
$$= 28 \text{ square units}$$



$$\text{Area} = \left|\det\left(\begin{bmatrix} DA_1 & BA_1 & HA_1 \\ DA_2 & BA_2 & HA_2 \\ DA_3 & BA_3 & HA_3 \end{bmatrix}\right)\right|$$
$$= \left|\det\left(\begin{bmatrix} -2 & 1 & -3 \\ -2 & -3 & -2 \\ -5 & 0 & -1 \end{bmatrix}\right)\right|$$
$$= \left| -5 * \det\left(\begin{bmatrix} 1 & -3 \\ -3 & -2 \end{bmatrix}\right) - 1 * \det\left(\begin{bmatrix} -2 & 1 \\ -2 & -3 \end{bmatrix}\right)\right|$$
$$= 47 \textit{ cubic units}$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & a_{(n-1),n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{bmatrix}$$

$$\begin{aligned} tr: R^{mxn} &\longrightarrow R \\ A &\longmapsto tr(A) = \textstyle\sum_i a_{i,i} \end{aligned}$$

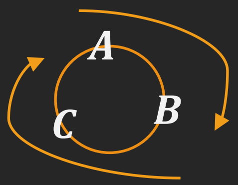
$tr(A) = tr(A^T)$

$tr(ABC) = tr(CAB) = tr(BCA)$

$tr(ABC) \neq tr(BAC)$

$tr(ABC) \neq tr(CBA)$

$tr(ABC) \neq tr(ACB)$



$tr(AB) = tr(BA)$ $\quad A \in R^{mxn}, B \in R^{nxk}$

$tr(a) = a$ $\quad$ a$\in R$

$tr(I_n) = n$

$tr(aA + B) = atr(A) + tr(B)$ $\quad A \in R^{mxn}, B \in R^{mxn}$

$\|A\|_F = \sqrt{tr(AA^T)}$

LINEAR ALGEBRA > DETERMINANT AND TRACE OPERATOR

This is a very important concept in linear algebra, as it is used to solve problems like finding the inverse of a matrix (I know you found the inverse of a matrix previously, without having to know about the determinant), solving linear equations, capturing how linear transformations change area or volume,…

Only square matrices have a determinant.

The determinant of a matrix, is denoted |A| or det(A) and is defined as:

EXPL65

PROPERTIES OF THE DETERMINANT OF AN n * n MATRIX

- The determinant is invariant to exchanging a row with the sum of that row and another row.
- The determinant is multiplied by a factor of k, if a row is multiplied by a factor of k, during an elementary operation
- The determinant changes sign, when there is an interchange of two rows
- The determinant of a matrix, A is = determinant of the transpose of the matrix A
- The determinant of the product of matrices = product of their determinants
- The determinant of A inverse = 1/ Determinant of A
- The determinant of kA = k^n(det(A))

EXPL66

MATRIX INVERTIBILITY

Since a matrix with less than n pivots has a determinant of 0, we can conclude that: A matrix, A is invertible iff det(A) != 0

LINEAR ALGEBRA: MATRIX ALGEBRA

LINEAR ALGEBRA > DETERMINANT AND TRACE OPERATOR

APPLICATIONS OF THE DETERMINANT OF A MATRIX:

SOLVING THE EQUATION AX = b (Cramer’s rule)
EXPL67
FINDING THE INVERSE OF A MATRIX
In previous sections, we have seen how to get the inverse of a matrix, but other methods exist, and the method we are about to explore, makes use of the determinant AND Adjugate of the matrix.
To obtain the adjugate, we get the transpose of the cofactor matrix.
The cofactor matrix is by using the following procedure.
EXPL68
FINDING THE INVERSE OF A 2 BY 2 MATRIX
We proceed using the formula:
EXPL69

Nonetheless, because the cramer’s rule and the formula method for getting the inverse are highly inneficient, both methods aren’t used in practice

CALCULATING AREAS

AREA OF PARALLELOGRAM SPANNED BY TWO VECTORS a and b is given by:
EXPL70

LINEAR ALGEBRA > **DETERMINANT AND TRACE OPERATOR**

CALCULATING VOLUMES

VOLUME OF PARALLELOPIPED SPANNED BY THREE VECTORS a, b and c is given by:

EXPL71

TRACE OPERATOR

This operator is very useful in manipulating certain expressions.

The trace of a matrix A, can be defined as:

EXPL72