$Y_k = CX_k + Z_k$ — Y = observation, Z = measurement noise

C relates to what you choose to measure, ex:

$C = [1\ 0]$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \end{bmatrix}$

Lecture 13: State matrix 2-D

Same thing w/ more variables. Already generalized this to $n \times n$ matrices.

Lecture 14: Control variable matrix in 2D

Same thing...

Lecture 15: Trivial computations. Same w/ lecture 16, 17

Lecture 18: The covariance matrix

$$P_k = A P_{k-1} A^T + Q$$

$$K_k = \frac{P_k H^T}{H P_k H^T + R}$$

P = state covariance matrix

Q = process noise covariance matrix

R = Measurement covariance matrix (error in measurement)

K = Kalman gain

$R \to 0 \Rightarrow K \to 1$ (adjust primarily w/ measurement update)

$R \to \infty$, $K \to 0$ (adjust by predicted state)

If $P \to 0$ measurement updates mostly ignored

Lecture 19: What is covariance matrix

$$\Sigma_{ij} = \mathrm{cov}(X_i, X_j) = E[(X_i - E[X_i])(X_j - E[X_j])]$$

$$\sigma_x \sigma_y = \frac{\sum_{i=1}^{N} (\bar{x} - x_i)(\bar{y} - y_i)}{N}$$

Probability Primer

Lecture 1.1 (just talk about basic ideas, no Lebesgue)

Banach-Tarski paradox

- non measurable sets: the measure is not preserved under translation. These sets cannot be assigned a measure

1.2 Measure theory: sigma algebra

Defn: Given a set Ω, a σ algebra on Ω is a collection $\mathcal{A} \subset 2^{\Omega}$ (power set of Ω) s.t. $\mathcal{A}$ is nonempty and $\mathcal{A}$ is:

(i) closed under complements $(E \in \mathcal{A} \Rightarrow E^c \in \mathcal{A})$

(ii) closed under countable unions $(E_1, \dots, E_n \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A})$

Rmk (1) $\Omega \in \mathcal{A}$ since $E \in \mathcal{A} \xRightarrow{\text{by (i)}} E^c \in \mathcal{A} \xRightarrow{\text{by (2)}} E \cup E^c \in \mathcal{A} \Rightarrow \Omega \in \mathcal{A}$

(2) $\emptyset \in \mathcal{A}$ since $\Omega^c = \emptyset \in \mathcal{A}$

(3) $\mathcal{A}$ is closed under countable intersections

Suppose $E_1, E_2, \dots$

$$\bigcap_{i=1}^{\infty} E_i = \bigcap_{i=1} (E_i^c)^c = \left(\bigcup E_i^c\right)^c \in \mathcal{A}$$

PP 1.3 Defn: Given $\mathcal{C} \subset 2^{\Omega}$, the σ-algebra generated by $\mathcal{C}$ written $\sigma(\mathcal{C})$ = the "smallest" σ alg containing $\mathcal{C}$

– that is $\sigma(\mathcal{C}) = \bigcap_{\substack{\mathcal{A} \supset \mathcal{C} \\ \sigma\text{-alg}}} \mathcal{A}$

Rmk: $\sigma(\mathcal{C})$ always exists because:

(i) 2^{Ω} is a σ-algebra

(ii) any intersection of σ algs is a σ alg

Examples (1) $\mathcal{A} = \{\emptyset, \Omega\}$

(2) $\mathcal{A} = \{\emptyset, E, E^c, \Omega\}$

(3) If $\Omega = \mathbb{R}$, the Borel σ-alg is $\mathcal{B} = \sigma(\mathcal{T})$ where $\mathcal{T} = \{\text{open sets of } \mathbb{R}\}$. If Ω is any topological space, defined in the same way

Defn: A measure μ on Ω with σ-alg $\mathcal{A}$ is a fn $\mu: \mathcal{A} \to [0, \infty)$ s.t.

(i) $\mu(\emptyset) = 0$ (less rigorous than real variables)

(ii) $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ for any $E_1, E_2, \ldots \in \mathcal{A}$ pairwise disjoint sets

↳ countable additivity

Defn A probability measure is a measure P s.t. $P(\Omega) = 1$

PP 1.4 Kolmogorov's axioms

Ex of measure Examples:

(1) (finite set) $\Omega = \{1, \ldots, n\}$ $\mathcal{A} = 2^{\Omega}$

$P(\{k\}) = \frac{1}{n}$ $\forall k \in \Omega$ (uniform distribution)

$P(\{1, 2, 4\}) = P(\{1\} \cup \{2\} \cup \{4\}) = P(1) + P(2) + P(4)$

(2) (countably infinite) $\Omega = \{1, 2, 3, \ldots\}$ $\mathcal{A} = 2^{\Omega}$

$P(k)$ = prob it takes k coin flips to get heads

$= \alpha(1-\alpha)^{k-1} = (\frac{1}{2})(1-\frac{1}{2})^{k-1}$

(geometric distribution)

(3) (uncountable) $\Omega = [0, \infty)$, $\mathcal{A} = \mathcal{B}([0, \infty))$

$P([0, x)) = 1 - e^{-x}$ $\forall x > 0$ (exponential distribution)

Note $P(\{x\}) = 0$ $\forall x \geq 0$

PP 1.5 (4) Lebesgue measure (on $\mathbb{R}$) $\Omega = \mathbb{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$

Basic properties of measure

$\mu((a, b)) = b - a$ for any $a, b \in \mathbb{R}$, $a < b$

- not a probability measure

Theorem (basic properties of measures)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space

(i) Monotonicity: if $E, F \in \mathcal{A}$ and $E \subset F$ then $\mu(E) \leq \mu(F)$

(ii) Subadditivity: if $E_1, \ldots \in \mathcal{A}$ then $\mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$

PP 1.6 (iii) Continuity from below if $E_1, \ldots, \in \mathcal{A}$, $E_1 \subset E_2$

then $\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_{i \to \infty} \mu(E_i)$

(iv) continuity from above $E_1, \ldots \in \mathcal{A}$ and $E_1 \supset E_2 \supset \ldots$ and $\mu(E_1) < \infty$, then $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{i \to \infty} \mu(E_i)$

PP 1.7 More properties

Facts Let $(\Omega, \mathcal{A}, P)$ be a prob measure with $E, F, E_i \in \mathcal{A}$

(1) $P(E \cup F) = P(E) + P(F)$ if $E \cap F = \emptyset$

(2) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

(3) $P(E) = 1 - P(E^c)$

(4) $P(E \cap F^c) = P(E) - P(E \cap F)$

(5) (inclusion - exclusion formula)

$$P(\bigcup_{i=1}^{n} E_i) = \sum_i P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) - \ldots + (-1)^{n+1} P(E_1 \cap E_2 \ldots \cap E_n)$$

(6) $P(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} P(E_i)$ and $P(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} P(E_i)$ — all subadditivity

PP 1.8 CDFs and Borel Probability measures

$\mathbb{R}, \mathcal{B}(\mathbb{R})$ (a, b)

Defn: A Borel measure on $\mathbb{R}$ is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Defn CDF (cumulative distribution function) is a function $F: \mathbb{R} \to \mathbb{R}$ s.t.

(i) F is non decreasing ($x \leq y \Rightarrow F(x) \leq F(y)$)

(ii) F is right continuous ($\lim_{x \searrow a} F(x) = F(a)$)

(iii) $\lim_{x \to \infty} F(x) = 1$

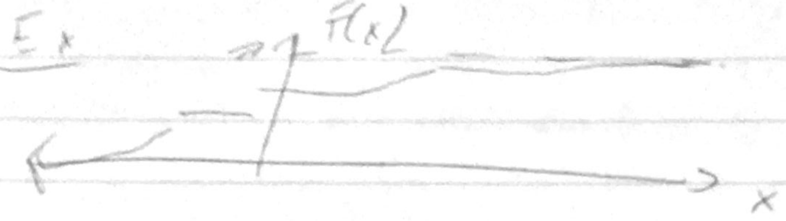
(iv) $\lim_{x \to -\infty} F(x) = 0$

Thm (i) If F is CDF then $\exists$ a unique Borel prob measure on $\mathbb{R}$ s.t. $P((-\infty, x]) = F(x)$ $\forall x \in \mathbb{R}$

(ii) If P is a Borel prob measure on $\mathbb{R}$ then there is a unique CDF F s.t. $F(x) = P((-\infty, x])$ $\forall x$

That is, there is an equivalence between CDFs and Borel prob measures

Ex



Reference

- Rudin's Principles of Math Analysis
- Jacod and Protter "Probability Essentials"
  - main ideas for first exposure
- Durrett "Prob. Theory and Examples"
  - a bit more advanced, somewhat outdated
- Folland's "Real Analysis"
- Rudin "Real and Complex Analysis" - skips intuition