1 Model description

We have 2 interconnected neural populations, one excitatory (E) and one inhibitory (I), which interactions are parametrized by coupling coefficients $c_{xy} \in \mathbb{R}^+$ with $x, y \in \{e, i\}$.
For a given population x we then have the following local coupling: $\big(c_{xe}E - c_{xi}I\big)$.

Note: Capital letters "E, I" refer to state variables, and lower case letters "e, i" specify parameter association to neural population.

In the case of a coupled system, the excitatory population (potentially) receives input from other excitatory populations elsewhere in the brain through a network input term $N_{input}(t)$ that can be set to 0 if modelling a single node dynamics. Here we will consider it to be defined for a given node n in a network M as delayed coupling propagated through some connectivity matrices of coupling strength (A) and axonal distance (D) e.g., derived from diffusion weighted MRI (DWI) and parametrized by a global scaling factor (G) and a conduction speed term (K):

$$N_{input}(t) = G \sum_{m \in M} A_{nm} E_m (t - \frac{D_{nm}}{K})$$

The input to each neural population is subject to a threshold parameter θ_x and passed through a sigmoid activation function $S_x(X)$ with a parametrized slope a_x that operates on X, the total (thresholded) input to the population:

$$S_x(X) = \frac{1}{1 + e^{-a_x \cdot X}}, x \in \{e, i\}$$

The activation response is the scaled by a parametrized refractory term $(k_x - r_x x)$ enabling only some subset of the neural population to respond to the input (unless set to $k_x = 1, r_x = 0$). Finally, the temporal scale for the dynamics of each population is parametrized by the time constants τ_x. This gives us the following system:

$$\tau_e \frac{dE}{dt} = -E + (k_e - r_e E) \cdot S_e\big(c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e\big)$$
$$\tau_i \frac{dI}{dt} = -I + (k_i - r_i I) \cdot S_i\big(c_{ie}E - c_{ii}I - \theta_i\big)$$

2 Jacobian of the system

The jacobian of the system is defined as the matrix

$$J = \begin{pmatrix} \frac{d\dot{E}}{dE} & \frac{d\dot{E}}{dI} \\ \frac{d\dot{I}}{dE} & \frac{d\dot{I}}{dI} \end{pmatrix}$$

Note: $\dot{Y} = \frac{dY}{dt}$

For a given population $Y \in \{E, I\}$ we have the following dynamical equation:

$$\tau_y \frac{dY}{dt} = -Y + (k_y - r_y Y) \cdot S_y\big(X\big)$$

Taking the derivative of $\dot{Y}$ with respect to Y ($\frac{d\dot{Y}}{dY}$) and to the other population Z ($\frac{d\dot{Y}}{dZ}$) gives:

$$\begin{aligned}
\frac{d\dot{Y}}{dY} &= \frac{1}{\tau_y}\frac{d}{dY}\Bigg[-Y + (k_y - r_y Y) \cdot S_y\big(X\big)\Bigg] \\
&= \frac{1}{\tau_y}\Bigg(-1 + \frac{d}{dY}\Bigg[(k_y - r_y Y)\Bigg] \cdot S_y\big(X\big) + (k_y - r_y Y) \cdot \frac{d}{dY}\Bigg[S_y\big(X\big)\Bigg]\Bigg) \\
&= \frac{1}{\tau_y}\Bigg(-1 - r_y \cdot S_y\big(X\big) + (k_y - r_y Y) \cdot \frac{d}{dY}\Bigg[S_y\big(X\big)\Bigg]\Bigg) \\
\frac{d\dot{Y}}{dZ} &= \frac{1}{\tau_y}\frac{d}{dZ}\Bigg[-Y + (k_y - r_y Y) \cdot S_y\big(X\big)\Bigg] \\
&= \frac{1}{\tau_y}\Bigg((k_y - r_y Y) \cdot \frac{d}{dZ}\Bigg[S_y\big(X\big)\Bigg]\Bigg) \\
&= \frac{(k_y - r_y Y)}{\tau_y}\frac{d}{dZ}\Bigg[S_y\big(X\big)\Bigg]
\end{aligned}$$

We can derive $\frac{d}{dY}\big[S_z\big(X\big)\big]$ for any arbitrary $Y \in \{E, I\}$,$z \in \{e, i\}$ pair up to some point as:

$$\begin{aligned}
\frac{d}{dY}\Bigg[S_z(X)\Bigg] &= \frac{d}{dY}\Bigg[\frac{1}{1 + e^{-a_z \cdot X}}\Bigg] \\
&= \frac{d}{dY}\Bigg[\big(1 + e^{-a_z \cdot X}\big)^{-1}\Bigg] \\
&= -\big(1 + e^{-a_z \cdot X}\big)^{-2} \cdot \frac{d}{dY}\Bigg[e^{-a_z \cdot X}\Bigg] \\
&= -\big(1 + e^{-a_z \cdot X}\big)^{-2} \cdot \Bigg(-a_z e^{-a_z \cdot X} \cdot \frac{d}{dY}\Bigg[X\Bigg]\Bigg) \\
&= \frac{a_z e^{-a_z \cdot X}}{\big(1 + e^{-a_z \cdot X}\big)^2} \cdot \frac{d}{dY}\Bigg[X\Bigg]
\end{aligned}$$

Given that X is a function of both Y and Z (both populations are recurrent and interconnected), the above works for the 4 possible combinations of state variable-derivative pairs. We then only need to define all 4 combinations for $\frac{d}{dY}\Big[X\Big]$ separately:

$$\frac{d}{dE}\Bigg[c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e\Bigg] = c_{ee}$$
$$\frac{d}{dI}\Bigg[c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e\Bigg] = -c_{ei}$$
$$\frac{d}{dE}\Bigg[c_{ie}E - c_{ii}I - \theta_i\Bigg] = c_{ie}$$
$$\frac{d}{dI}\Bigg[c_{ie}E - c_{ii}I - \theta_i\Bigg] = -c_{ii}$$

The entries of the jacobian can then be found to be:

$$\tau_e \frac{d\dot{E}}{dE} = -1 - \frac{r_e}{\left(1 + e^{-a_e\cdot(c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e)}\right)} + \left(k_e - r_eE\right)\cdot\frac{a_e\cdot e^{-a_e\cdot(c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e)}}{\left(1 + e^{-a_e\cdot(c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e)}\right)^2}\cdot c_{ee}$$
$$\tau_e \frac{d\dot{E}}{dI} = \left(k_e - r_eE\right)\cdot\frac{a_e\cdot e^{-a_e\cdot(c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e)}}{\left(1 + e^{-a_e\cdot(c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e)}\right)^2}\cdot -c_{ei}$$
$$\tau_i \frac{d\dot{I}}{dE} = \left(k_i - r_iI\right)\cdot\frac{a_i\cdot e^{-a_i\cdot(c_{ie}E - c_{ii}I - \theta_i)}}{\left(1 + e^{-a_i\cdot(c_{ie}E - c_{ii}I - \theta_i)}\right)^2}\cdot c_{ie}$$
$$\tau_i \frac{d\dot{I}}{dI} = -1 - \frac{r_i}{\left(1 + e^{-a_i\cdot(c_{ie}E - c_{ii}I - \theta_i)}\right)} + \left(k_i - r_iI\right)\cdot\frac{a_i\cdot e^{-a_i\cdot(c_{ie}E - c_{ii}I - \theta_i)}}{\left(1 + e^{-a_i\cdot(c_{ie}E - c_{ii}I - \theta_i)}\right)^2}\cdot -c_{ii}$$

3 Nullclines

To find the nullcline for a given population $Y \in \{E, I\}$ we only have to set its derivatives to $\frac{dY}{dt} = 0$ and solve for the other population Z:

$$\begin{aligned}
\tau_y \frac{dY}{dt} &= -Y + (k_y - r_y Y) \cdot S_y\big(X\big) \\
0 &= -Y + (k_y - r_y Y) \cdot S_y\big(X\big) \\
\frac{Y}{(k_y - r_y Y)} &= S_y\big(X\big) \\
\frac{(k_y - r_y Y)}{Y} &= 1 + \exp\Bigg(- a_y X \Bigg) \\
\frac{(k_y - r_y Y)}{Y} - 1 &= \exp\Bigg(- a_y X \Bigg) \\
\ln\Bigg[\frac{(k_y - r_y Y)}{Y} - 1 \Bigg] &= -a_y X \\
-\frac{1}{a_y} \ln\Bigg[\frac{(k_y - r_y Y)}{Y} - 1 \Bigg] &= X
\end{aligned}$$

We can then find E_{null} and I_{null} to be:

$$\begin{aligned}
c_{ee}E - c_{ei}I_{null} + N_{input}(t) - \theta_e &= -\frac{1}{a_e} \ln\Bigg[\frac{(k_e - r_e E)}{E} - 1 \Bigg] \\
I_{null} &= -\frac{-\ln\Bigg[\frac{(k_e - r_e E)}{E} - 1 \Bigg] + a_e\theta_e - a_e c_{ee}E - a_e N_{input}(t)}{a_e c_{ei}} \\
c_{ie}E_{null} - c_{ii}I - \theta_i &= -\frac{1}{a_i} \ln\Bigg[\frac{(k_i - r_i I)}{I} - 1 \Bigg] \\
E_{null} &= \frac{-\ln\Bigg[\frac{(k_i - r_i I)}{I} - 1 \Bigg] + a_i\theta_i + a_i c_{ii} I}{a_i c_{iE}}
\end{aligned}$$

4 Newton method

Given that the intersection of the nullclines does not have an analytical solution (at least to my knowledge), a numerical estimation is required. We can define functions $h_E(E)$, $h_I(I)$ and apply the newton root finding method on $h_X(X)$ with $(X,Y) \in \{(E,I),(I,E)\}$:

$$
\begin{aligned}
h_X(X) &= X - X_{null}\big(Y_{null}(X)\big) \\
X_{k+1} &= X_k - \frac{h_X(X_k)}{\frac{dh_X}{dX}\big|_{X_k}} \\
\text{Where, } \frac{dh_X}{dX}\bigg|_X &= \frac{d}{dX}\Big[X - X_{null}\big(Y_{null}(X)\big)\Big] \\
&= 1 - \frac{d}{dX}\Big[X_{null}\big(Y_{null}(X)\big)\Big] \\
&= 1 - \frac{dX_{null}}{dY_{null}}\bigg|_{Y_{null}(X)} \frac{dY_{null}}{dX}\bigg|_X
\end{aligned}
$$

First we can obtain $\frac{dE_{null}}{dI}$:

$$
\begin{aligned}
\frac{dE_{null}}{dI} &= \frac{d}{dI}\left(\frac{-\ln\left(\frac{-r_iI+k_i}{I} - 1\right) + a_ic_{ii}I + a_i\theta_i}{a_ic_{ie}}\right) \\
&= \frac{\frac{d}{dI}\left(-\ln\left(\frac{-r_iI+k_i}{I} - 1\right) + a_ic_{ii}I\right) + a_i\theta_i}{a_ic_{ie}} \\
&= \frac{\frac{d}{dI}\left(-\ln\left(\frac{-r_iI+k_i}{I} - 1\right)\right) + a_ic_{ii} + a_i\theta_i}{a_ic_{ie}} \\
&= \frac{-1}{a_ic_{ie}}\frac{d}{dI}\left(\ln\left(\frac{-r_iI+k_i}{I} - 1\right)\right) + \frac{a_ic_{ii} + a_i\theta_i}{a_ic_{ie}} \\
&= \frac{-1}{a_ic_{ie}}\frac{d}{dI}\left(\frac{-r_iI+k_i}{I} - 1\right)\frac{1}{\left(\frac{-r_iI+k_i}{I} - 1\right)} + \frac{a_ic_{ii} + a_i\theta_i}{a_ic_{ie}} \\
&= \frac{-1}{a_ic_{ie}}\frac{d}{dI}\left(\frac{-r_iI+k_i}{I}\right)\left(\frac{I}{-r_iI+k_i-I}\right) + \frac{a_ic_{ii} + a_i\theta_i}{a_ic_{ie}} \\
&= \frac{-1}{a_ic_{ie}}\frac{(-r_iI - (-r_iI+k_i))}{I^2}\left(\frac{I}{-r_iI+k_i-I}\right) + \frac{a_ic_{ii} + a_i\theta_i}{a_ic_{ie}} \\
&= \frac{-(-r_iI + r_iI - k_i))}{a_ic_{ie}(-r_iI - I + k_i)I} + \frac{a_ic_{ii} + a_i\theta_i}{a_ic_{ie}} \\
&= \frac{k_i + c_{ii} + \theta_i}{1 + a_ic_{ie}^2k_iI - a_ic_{ie}^2I^2(r_i+1)}
\end{aligned}
$$

And, by similarity, we can find $\frac{dI_{null}}{dE}$:

$$\frac{dI_{null}}{dE} = \frac{k_e - c_{ee} + \theta_e - N_{input}(t)}{1 + a_e c_{ei}^2 k_e E - a_e c_{ei}^2 E^2 (r_e + 1)}$$

5 Fixed point classification

We can use the eigenvalues of the jacobian around the fixed points to classify the fixed points. We have:

$$J(E^*, I^*) = \begin{pmatrix} \frac{d\dot{E}}{dE}\Big|_{E^*} & \frac{d\dot{E}}{dI}\Big|_{E^*} \\ \frac{d\dot{I}}{dE}\Big|_{I^*} & \frac{d\dot{I}}{dI}\Big|_{I^*} \end{pmatrix}$$

From which we can derive the eigenvalues λ as:

$$\lambda(E^*, I^*) = \frac{(\frac{d\dot{E}}{dE}\Big|_{E^*} + \frac{d\dot{I}}{dI}\Big|_{I^*}) \pm \sqrt{(\frac{d\dot{E}}{dE}\Big|_{E^*} + \frac{d\dot{I}}{dI}\Big|_{I^*})^2 - 4\Big(\frac{d\dot{E}}{dE}\Big|_{E^*} \cdot \frac{d\dot{I}}{dI}\Big|_{I^*} - \frac{d\dot{E}}{dI}\Big|_{E^*} \cdot \frac{d\dot{I}}{dE}\Big|_{I^*}\Big)}}{2}$$