

# 1 Model description

We have 2 interconnected neural populations, one excitatory ( $E$ ) and one inhibitory ( $I$ ), which interactions are parametrized by coupling coefficients  $c_{xy} \in \mathbb{R}^+$  with  $x, y \in \{e, i\}$ .

For a given population  $x$  we then have the following local coupling:  $(c_{xe}E - c_{xi}I)$ .

*Note: Capital letters "E, I" refer to state variables, and lower case letters "e, i" specify parameter association to neural population.*

In the case of a coupled system, the excitatory population (potentially) receives input from other excitatory populations elsewhere in the brain through a network input term  $N_{input}(t)$  that can be set to 0 if modelling a single node dynamics. Here we will consider it to be defined for a given node  $n$  in a network  $M$  as delayed coupling propagated through some connectivity matrices of coupling strength ( $A$ ) and axonal distance ( $D$ ) e.g., derived from diffusion weighted MRI (DWI) and parametrized by a global scaling factor ( $G$ ) and a conduction speed term ( $K$ ):

$$N_{input}(t) = G \sum_{m \in M} A_{nm} E_m(t - \frac{D_{nm}}{K})$$

The input to each neural population is subject to a threshold parameter  $\theta_x$  and passed through a sigmoid activation function  $S_x(X)$  with a parametrized slope  $a_x$  that operates on  $X$ , the total (thresholded) input to the population:

$$S_x(X) = \frac{1}{1 + e^{-a_x \cdot X}}, x \in \{e, i\}$$

The activation response is the scaled by a parametrized refractory term  $(k_x - r_x x)$  enabling only some subset of the neural population to respond to the input (unless set to  $k_x = 1, r_x = 0$ ). Finally, the temporal scale for the dynamics of each population is parametrized by the time constants  $\tau_x$ . This gives us the following system:

$$\begin{aligned} \tau_e \frac{dE}{dt} &= -E + (k_e - r_e E) \cdot S_e(c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e) \\ \tau_i \frac{dI}{dt} &= -I + (k_i - r_i I) \cdot S_i(c_{ie}E - c_{ii}I - \theta_i) \end{aligned}$$

## 2 Jacobian of the system

The jacobian of the system is defined as the matrix

$$J = \begin{pmatrix} \frac{d\dot{E}}{dE} & \frac{d\dot{E}}{dI} \\ \frac{d\dot{I}}{dE} & \frac{d\dot{I}}{dI} \end{pmatrix}$$

*Note:*  $\dot{Y} = \frac{dY}{dt}$

For a given population  $Y \in \{E, I\}$  we have the following dynamical equation

$$\tau_y \frac{dY}{dt} = -Y + (k_y - r_y Y) \cdot S_y(X)$$

Taking the derivative of  $Y$  with respect to itself ( $\frac{d\dot{Y}}{dY}$ ) and the other population  $Z$  ( $\frac{d\dot{Y}}{dZ}$ ) gives:

$$\begin{aligned} \frac{d\dot{Y}}{dY} &= \frac{1}{\tau_y} \frac{d}{dY} \left[ -Y + (k_y - r_y Y) \cdot S_y(X) \right] \\ &= \frac{1}{\tau_y} \left( -1 + \frac{d}{dY} \left[ (k_y - r_y Y) \right] \cdot S_y(X) + (k_y - r_y Y) \cdot \frac{d}{dY} \left[ S_y(X) \right] \right) \\ &= \frac{1}{\tau_y} \left( -1 - r_y \cdot S_y(X) + (k_y - r_y Y) \cdot \frac{d}{dY} \left[ S_y(X) \right] \right) \\ \frac{d\dot{Y}}{dZ} &= \frac{1}{\tau_y} \frac{d}{dZ} \left[ -Y + (k_y - r_y Y) \cdot S_y(X) \right] \\ &= \frac{1}{\tau_y} \left( (k_y - r_y Y) \cdot \frac{d}{dZ} \left[ S_y(X) \right] \right) \\ &= \frac{(k_y - r_y Y)}{\tau_y} \frac{d}{dZ} \left[ S_y(X) \right] \end{aligned}$$

We can derive  $\frac{d}{dY} [S_z(X)]$  for any arbitrary  $Y \in \{E, I\}, z \in \{e, i\}$  pair up to some point as:

$$\begin{aligned} \frac{d}{dY} [S_z(X)] &= \frac{d}{dY} \left[ \frac{1}{1 + e^{-a_z \cdot X}} \right] \\ &= \frac{d}{dY} \left[ (1 + e^{-a_z \cdot X})^{-1} \right] \\ &= -(1 + e^{-a_z \cdot X})^{-2} \cdot \frac{d}{dY} \left[ e^{-a_z \cdot X} \right] \\ &= -(1 + e^{-a_z \cdot X})^{-2} \cdot \left( -a_z e^{-a_z \cdot X} \cdot \frac{d}{dY} [X] \right) \\ &= \frac{a_z e^{-a_z \cdot X}}{(1 + e^{-a_z \cdot X})^2} \cdot \frac{d}{dY} [X] \end{aligned}$$

Given that  $X$  is a function of both  $Y$  and  $Z$ , the above works for the 4 possible combinations of state variable-derivative pairs. We then only need to define all 4 combinations for  $\frac{d}{dY}[X]$  separately:

$$\begin{aligned}\frac{d}{dE} \left[ c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e \right] &= c_{ee} \\ \frac{d}{dI} \left[ c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e \right] &= -c_{ei} \\ \frac{d}{dE} \left[ c_{ie}E - c_{ii}I - \theta_i \right] &= c_{ie} \\ \frac{d}{dI} \left[ c_{ie}E - c_{ii}I - \theta_i \right] &= -c_{ii}\end{aligned}$$

The entries of the jacobian can then be found to be:

$$\begin{aligned}\tau_e \frac{d\dot{E}}{dE} &= -1 - \frac{r_e}{(1 + e^{-a_e \cdot (c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e)})} + (k_e - r_e E) \cdot \frac{a_e \cdot e^{-a_e \cdot (c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e)}}{(1 + e^{-a_e \cdot (c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e)})^2} \cdot c_{ee} \\ \tau_e \frac{d\dot{E}}{dI} &= (k_e - r_e E) \cdot \frac{a_e \cdot e^{-a_e \cdot (c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e)}}{(1 + e^{-a_e \cdot (c_{ee}E - c_{ei}I + N_{input}(t) - \theta_e)})^2} \cdot -c_{ei} \\ \tau_i \frac{d\dot{I}}{dE} &= (k_i - r_i I) \cdot \frac{a_i \cdot e^{-a_i \cdot (c_{ie}E - c_{ii}I - \theta_i)}}{(1 + e^{-a_i \cdot (c_{ie}E - c_{ii}I - \theta_i)})^2} \cdot c_{ie} \\ \tau_i \frac{d\dot{I}}{dI} &= -1 - \frac{r_i}{(1 + e^{-a_i \cdot (c_{ie}E - c_{ii}I - \theta_i)})} + (k_i - r_i I) \cdot \frac{a_i \cdot e^{-a_i \cdot (c_{ie}E - c_{ii}I - \theta_i)}}{(1 + e^{-a_i \cdot (c_{ie}E - c_{ii}I - \theta_i)})^2} \cdot -c_{ii}\end{aligned}$$

### 3 Nullclines

To find the nullcline for a given population  $Y \in \{E, I\}$  we only have to set the its derivatives to  $\frac{dY}{dt} = 0$  and solve for the other population  $Z$ :

$$\begin{aligned}
\tau_y \frac{dY}{dt} &= -Y + (k_y - r_y Y) \cdot S_y(X) \\
0 &= -Y + (k_y - r_y Y) \cdot S_y(X) \\
\frac{Y}{(k_y - r_y Y)} &= S_y(X) \\
\frac{(k_y - r_y Y)}{Y} &= 1 + \exp\left(-a_y X\right) \\
\frac{(k_y - r_y Y)}{Y} - 1 &= \exp\left(-a_y X\right) \\
\ln\left[\frac{(k_y - r_y Y)}{Y} - 1\right] &= -a_y X \\
-\frac{1}{a_y} \ln\left[\frac{(k_y - r_y Y)}{Y} - 1\right] &= X
\end{aligned}$$

We can then find  $E_{null}$  and  $I_{null}$  to be

$$\begin{aligned}
c_{ee}E - c_{ei}I_{null} + N_{input}(t) - \theta_e &= -\frac{1}{a_e} \ln\left[\frac{(k_e - r_e E)}{E} - 1\right] \\
I_{null} &= -\frac{-\ln\left[\frac{(k_e - r_e E)}{E} - 1\right] + a_e \theta_e - a_e c_{ee}E - a_e N_{input}(t)}{a_e c_{ei}} \\
c_{ie}E_{null} - c_{ii}I - \theta_i &= -\frac{1}{a_i} \ln\left[\frac{(k_i - r_i I)}{I} - 1\right] \\
E_{null} &= \frac{-\ln\left[\frac{(k_i - r_i I)}{I} - 1\right] + a_i \theta_i + a_i c_{ii}I}{a_i c_{ie}}
\end{aligned}$$

## 4 Newton method

Given that the intersection of the nullclines does not have an analytical solution (at least to my knowledge), a numerical estimation is required. We can define functions  $h_E(E)$ ,  $h_I(I)$  and apply the newton root finding method on  $h_X(X)$  with  $(X, Y) \in \{(E, I), (I, E)\}$ :

$$\begin{aligned}
 h_X(X) &= X - X_{null}(Y_{null}(X)) \\
 X_{k+1} &= X_k - \frac{h_X(X_k)}{\left. \frac{dh_X}{dX} \right|_{X_k}} \\
 \text{Where, } \left. \frac{dh_X}{dX} \right|_X &= \frac{d}{dX} \left[ X - X_{null}(Y_{null}(X)) \right] \\
 &= 1 - \frac{d}{dX} \left[ X_{null}(Y_{null}(X)) \right] \\
 &= 1 - \left. \frac{dX_{null}}{dY_{null}} \right|_{Y_{null}(X)} \left. \frac{dY_{null}}{dX} \right|_X
 \end{aligned}$$

First we can obtain  $\frac{dE_{null}}{dI}$ :

$$\begin{aligned}
 \frac{dE_{null}}{dI} &= \frac{d}{dI} \left( \frac{-\ln\left(\frac{-r_i I + k_i}{I} - 1\right) + a_i c_{ii} I + a_i \theta_i}{a_i c_{ie}} \right) \\
 &= \frac{\frac{d}{dI} \left( -\ln\left(\frac{-r_i I + k_i}{I} - 1\right) + a_i c_{ii} I \right) + a_i \theta_i}{a_i c_{ie}} \\
 &= \frac{\frac{d}{dI} \left( -\ln\left(\frac{-r_i I + k_i}{I} - 1\right) \right) + a_i c_{ii} + a_i \theta_i}{a_i c_{ie}} \\
 &= \frac{-1}{a_i c_{ie}} \frac{d}{dI} \left( \ln\left(\frac{-r_i I + k_i}{I} - 1\right) \right) + \frac{a_i c_{ii} + a_i \theta_i}{a_i c_{ie}} \\
 &= \frac{-1}{a_i c_{ie}} \frac{d}{dI} \left( \frac{-r_i I + k_i}{I} - 1 \right) \frac{1}{\left(\frac{-r_i I + k_i}{I} - 1\right)} + \frac{a_i c_{ii} + a_i \theta_i}{a_i c_{ie}} \\
 &= \frac{-1}{a_i c_{ie}} \frac{d}{dI} \left( \frac{-r_i I + k_i}{I} \right) \left( \frac{I}{-r_i I + k_i - I} \right) + \frac{a_i c_{ii} + a_i \theta_i}{a_i c_{ie}} \\
 &= \frac{-1}{a_i c_{ie}} \frac{(-r_i I - (-r_i I + k_i))}{I^2} \left( \frac{I}{-r_i I + k_i - I} \right) + \frac{a_i c_{ii} + a_i \theta_i}{a_i c_{ie}} \\
 &= \frac{-(-r_i I + r_i I - k_i)}{a_i c_{ie} (-r_i I - I + k_i) I} + \frac{a_i c_{ii} + a_i \theta_i}{a_i c_{ie}} \\
 &= \frac{k_i + c_{ii} + \theta_i}{1 + a_i c_{ie}^2 k_i I - a_i c_{ie}^2 I^2 (r_i + 1)}
 \end{aligned}$$

And, by similarity, we can find  $\frac{dI_{null}}{dE}$ :

$$\frac{dI_{null}}{dE} = \frac{k_e - c_{ee} + \theta_e - S_t}{1 + a_e c_{ei}^2 k_e E - a_e c_{ei}^2 E^2 (r_e + 1)}$$