Variance of a standardized mean effect

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14/2/2018

Contents

1 Variance of standardized mean effect

Consider $\delta = \frac{\mu}{\sigma}$ and its estimator:

$$d = \frac{\overline{Y}}{S},$$

where S is the square root of the unbiased sample variance. We are focussing on one study with sample size N. We estimate the standardized mean effect and its variance.

We know that:

$$d = \frac{1}{\sqrt{n}}\left[\frac{\mathcal{Z} + \sqrt{n}\delta}{\sqrt{\frac{V}{\nu}}}\right],$$

where $\mathcal{Z} \sim N(0,1)$, V is a Chi-squared random distributed variable and $\nu = N - 1$.

From the expression above, we see that d equals $\frac{1}{\sqrt{n}}$ times a non-central t-distributed random variable with $n-1$ degrees of freedom and $\sqrt{n}\delta$ as the non-centrality parameter.

We shall now calculate the variance of d, using the second moment of a non-central t-distribution. Before doing so, we first define some functions.

1.1 Second moment of a non-central t-distribution

The second moment of a random distributed T-variable is defined as:

$$\mathrm{Var}(T) = \frac{\nu(1+\theta^2)}{\nu-2} - \frac{\theta^2\nu}{2}\left(\frac{\Gamma((\nu-1)/2)}{\Gamma(\nu/2)}\right)^2,$$

if $\nu > 2$. Note that we use θ for the non-centrality parameter.

1.2 Parameters

Let us define N and δ:

```
N <- 20
delta <- 0.8
```

2 Functions

Function to use in second moment of non-central t-distribution:

$$FU = \frac{\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}$$

```
ratioGamma <- function(N){
  num <- gamma((N - 2)/2)
  denom <- gamma((N - 1) / 2)
  return(num/denom)
}
```

Function to calculate true correction factor (denoted as h):

$$h = \frac{\Gamma\left(\frac{N-1}{2}\right)}{\sqrt{\frac{N-1}{2}}\Gamma\left(\frac{N-2}{2}\right)}$$

```
TrueCorr <- function(N){
  num <- gamma((N - 1)/2)
  denom <- (sqrt((N - 1)/2) * gamma((N - 2)/2))
  return(num/denom)
}
```

Compare with J:

$$J = \left(1 - \frac{3}{4(n-1)-1}\right)$$

which is implemented in the **R** package `NeuRRoStat`:

```
NeuRRoStat::corrJ
```

```
function(N){
  1-(3/((4*(N-1))-1))
}
<environment: namespace:NeuRRoStat>
```

2.1 Difference between h and J

```
NeuRRoStat::corrJ(10) - TrueCorr(10)
[1] 0.0004108225
NeuRRoStat::corrJ(N) - TrueCorr(N)
[1] 8.964708e-05
```

3 Second moment of non-central t: plug in

To start with, let us take the second moment of the non-central t-distribution and plug in $\nu = N - 1$ and $\theta = \sqrt{n}\delta$.

```
varA <- ((N-1) * (1 + N*delta**2))/(N - 3)
varB <- ((N*delta**2*(N - 1))/2) * ratioGamma(N)**2
1/N*(varA - varB)
```

```
[1] 0.07660231
```

4 Version written in paper

Compare the value from previous section with the formula given in the paper:

$$\mathrm{Var}(d) = \frac{1}{n}\left[\frac{n^2\delta^2 - n\delta^2 + n - 1}{n-3}\right] - \frac{\delta^2}{h^2},$$

with h defined earlier.

```
PaperVarA <- 1/N*((N**2*delta**2 - N*delta**2+N-1)/(N-3))
PaperVarB <- (delta**2)/(TrueCorr(N)**2)
PaperVarA - PaperVarB
```

```
[1] 0.07660231
```

Same value

5 Hedges 1981: biased effect size

Now let us compare with the formula given by Hedges in 1981 for a two sample scenario, in which N corresponds to the sample size of the control group (essentially the approach of Glass).

$$\text{Var}(d) = \frac{N-1}{(N-1-2)N}\left[1 + N\delta^2\right] - \frac{\delta^2}{h^2}.$$

```
VarBiasedG <- (N - 1)/((N - 3)*N)*(1 + (delta**2*N)) - (delta**2/TrueCorr(N)**2)
VarBiasedG
```

```
[1] 0.07660231
```

Same value

6 Variance of unbiased estimator

To calculate the unbiased estimator, we need to multiply d with J (for more details, see paper). Hence we have:

$$g = d \times J$$

and

$$\begin{aligned}\text{Var}(g) &= \text{Var}(d \times J)\\ &= \text{Var}(d) \times J^2\end{aligned}$$

```
VarBiasedG * (TrueCorr(N)**2)
```

```
[1] 0.0705835
```

7 Radua

Radua uses the following formula for the variance of g:

$$\begin{aligned}\text{Var}(g) &= \frac{1}{N} + \left[1 - \left(\frac{\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}\right)^2 \times \frac{N-3}{2}\right] \times d^2\\ &= \frac{(1+d^2N)}{N} - \frac{d^2(N-3)}{2} \times \left(\frac{\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}\right)^2\end{aligned}$$

```
((1 + delta**2*N)/N) - (((delta**2*(N - 3))/2) * ratioGamma(N)**2)
```

```
[1] 0.06853891
```

Different value

8 Conclusion

- Computationally, there is no need anymore to rely on an approximation J. Might as well use h.
- Radua uses a slightly different approach to obtain Var(g). I do not know which one. At the moment, I suggest using Var(g) as given by the second moment of the non-central t-distribution.