A Logarithmic First Integral for the Logistic On-Site Law in Void Dynamics

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Abstract

I prove a closed-form constant of motion for the autonomous on-site law

$$\dot{W} = r\,W - u\,W^2,$$

which underpins the Reaction–Diffusion (RD) baseline of Void Dynamics. Defining

$$Q(W,t) := \ln\frac{W}{r - uW} - r\,t,$$

I show that $\frac{d}{dt}Q = 0$ along solutions on domains where the expression is defined (e.g., $0 < W < r/u$). I relate Q to the standard logistic solution, establish domains/branches and limiting behaviors, and explain why a naïve "kinetic+potential" energy is not conserved for this first-order dissipative flow. Finally, I include a minimal, self-contained numerical protocol that verifies machine-precision constancy of Q and exhibits convergence consistent with the time-stepper's order. The note is self-contained and implementation-agnostic.

1 Introduction and main statement

Consider the one-degree-of-freedom, autonomous ordinary differential equation (ODE)

$$\dot{W} = F(W) = r\,W - u\,W^2, \qquad r, u \in \mathbb{R},\ u \neq 0. \tag{1}$$

In many RD parameterizations one writes $r = \alpha - \beta$ and $u = \alpha$, but this mapping is not needed here. Because (1) is autonomous, time-translation symmetry implies the existence of an implicit first integral. The following explicit invariant holds.

Proposition 1 (Logarithmic invariant)**.** *For any interval on which the expression is defined (e.g., $0 < W < r/u$ when $r/u > 0$),*

$$Q(W,t) \ \equiv\ \ln\frac{W}{r - uW} \ -\ r\,t \tag{2}$$

is constant along any trajectory of $\dot{W} = rW - uW^2$.

2 Proof

For an autonomous ODE $\dot{W} = F(W)$, one has $dt = \frac{dW}{F(W)}$. Here

$$\frac{dW}{F(W)} \ =\ \frac{dW}{W(r - uW)} \ =\ \frac{1}{r}\left(\frac{1}{W} + \frac{u}{r - uW}\right) dW. \tag{3}$$

Integrating both sides gives

$$t + C \;=\; \frac{1}{r}\Big(\ln|W| - \ln|r - uW|\Big), \tag{4}$$

or equivalently,

$$\ln\frac{W}{r - uW} \;-\; r\,t \;=\; \text{const.} \tag{5}$$

Defining $Q(W,t)$ by (2) yields $\frac{d}{dt}Q = 0$ along solutions of (1). The proof holds on any interval avoiding the simple poles at $W = 0$ and $W = r/u$, with a consistent logarithm branch on that interval. $\square$

3 Relation to the logistic closed-form solution

Separation of variables yields the well-known logistic solution

$$W(t) \;=\; \frac{r}{u}\,\frac{1}{1 + C\,e^{-rt}}, \qquad C \;=\; \frac{r - uW_0}{W_0}, \tag{6}$$

for an initial condition $W(0) = W_0$ that avoids the poles. Substituting into the invariant gives

$$Q(W(t),t) = \ln\left(\frac{\frac{r}{u}\,\frac{1}{1+Ce^{-rt}}}{r - \frac{r}{1+Ce^{-rt}}}\right) - rt = \ln\left(\frac{1}{u}\cdot\frac{1}{C}\right), \tag{7}$$

which is constant in time. Thus Q encodes the integration constant $(1/C)$ up to an additive constant $-\ln u$; different branches correspond to the piecewise structure induced by the poles.

4 Properties, domains, units, and limits

Poles and branches. Q has simple poles at $W = 0$ and $W = r/u$. On any open interval avoiding these poles, one may select a consistent logarithm branch and obtain a constant Q. Natural intervals are: (i) $(0, r/u)$ when $r/u > 0$, and (ii) $(r/u, \infty)$ when $r/u > 0$. Similar partitions apply when $r/u < 0$.

Units. If W is dimensionless and r, u have units of inverse time, then $\ln\frac{W}{r-uW}$ is dimensionless while rt is dimensionless, so Q is dimensionless. If one alternatively assigns a scale to W, the same conclusion holds once a reference scale is absorbed.

Limiting forms. As $W \to 0^{\pm}$: $Q \sim \ln|W| - \ln|r| - rt$. As $W \to (r/u)^{\mp}$: $Q \sim -\ln|r - uW| - rt + \text{const}$.

Monotonicity of W. On $(0, r/u)$ with $r, u > 0$, W grows monotonically to r/u; on $(r/u, \infty)$, W decays monotonically to r/u. The invariant remains constant on each interval separately.

5 Numerical verification (self-contained protocol)

Objective. Verify that the numerical drift $\Delta Q \equiv \max_{0\le n\le N} |Q(W_n, t_n) - Q(W_0, 0)|$ is limited by discretization/round-off and exhibits the expected step-order convergence.

Protocol.

- Time-stepper: fixed-step RK4 (or Dormand–Prince with tight tolerances).
- Parameters: e.g., $r = 0.15$, $u = 0.25$.
- Initial conditions: sample $W_0 \in (10^{-3}, r/u - 10^{-3})$ and $W_0 \in (r/u + 10^{-3}, 1 - 10^{-3})$ to test both sides of the middle pole.
- Time step and horizon: $dt = 10^{-3}$, $N = 10^5$ steps (double precision).

Acceptance gates.

- Double precision: $\Delta Q \le 10^{-10}$ (RK45 with tight tolerances) or $\Delta Q \le 10^{-8}$ (RK4 with $dt \approx 10^{-3}$).
- Single precision: $\Delta Q \le 10^{-5}$.
- Convergence: halving dt reduces ΔQ by a factor consistent with the order p of the scheme; a log–log fit of ΔQ vs dt yields slope $p \pm 0.4$ and $R^2 \ge 0.98$.

Pseudocode (language-agnostic).

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1) define F(W) = r*W - u*W^2
2) initialize t=0, W=W0, Q0 = ln(W/(r-uW)) - r*t
3) for n in 1..N: advance (W,t) one step by RK4 with step dt
4) compute Qn = ln(W/(r-uW)) - r*t and track max |Qn - Q0|
5) report Q and, if running a step-refinement, the observed slope
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Numerical notes. Trap underflow/overflow near the poles; reject steps that cross the singularity. The test is most transparent on $(0, r/u)$ for $r, u > 0$.

6 Figures

7 Why there is no naïve conserved "energy" here

If one guesses a per-site energy $H(W, \dot{W}) = \frac{1}{2}\dot{W}^2 + V(W)$, then

$$\frac{dH}{dt} = \dot{W}(\ddot{W} + V'(W)).$$

In a first-order flow $\dot{W} = F(W)$, $\ddot{W} = F'(W)\dot{W}$. Hence

$$\frac{dH}{dt} = \dot{W}(F'(W)\dot{W} + V'(W)),$$

which is not generically zero unless $\dot{W} \equiv 0$ or V' is tuned to cancel $F'(W)\dot{W}$ pointwise in time—impossible for a potential that depends only on W. Thus a time-independent Hamiltonian of this simple form is not conserved. The correct conserved quantity is the logarithmic first integral Q arising from autonomy/time-translation symmetry.

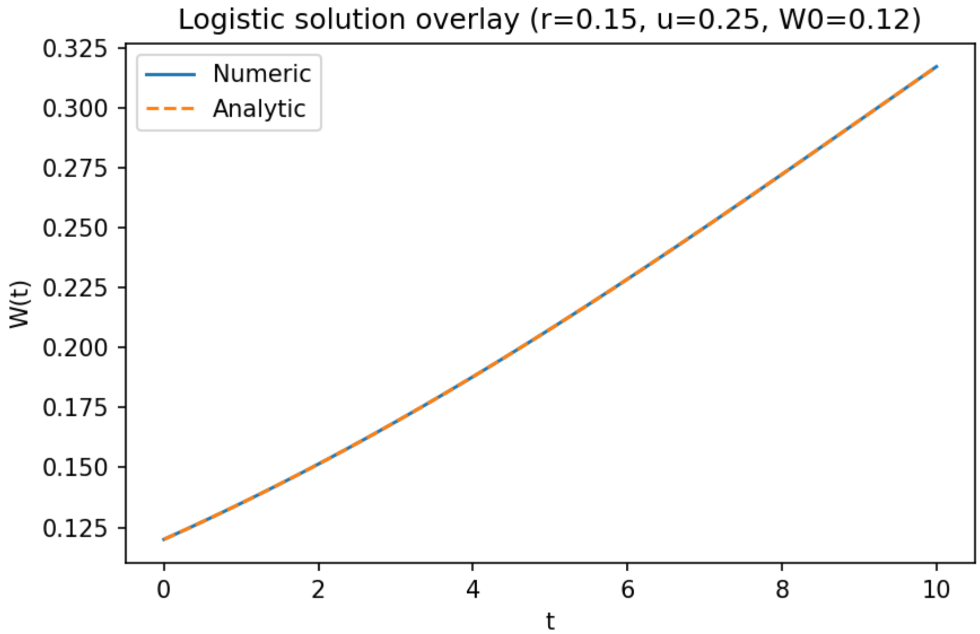

Figure 1: Solution overlay for the logistic on-site law: numerical trajectory (fixed-step integrator) versus the closed-form solution. Acceptance: visual agreement across the horizon with parameters shown in the figure filename; stable copy produced by the validator script.

8 Discussion and scope

The invariant Q is local (on-site). In spatially extended or coupled systems, Q is generally not conserved site-wise; instead, it serves as a per-node diagnostic for deviations induced by coupling/diffusion. The result is independent of implementation or discretization; it relies only on autonomy of the on-site law and standard calculus.

Placement within the canonical RD baseline and context

- Canonical model. The Void Dynamics baseline is reaction–diffusion (RD): $\partial_t \phi = D\nabla^2\phi + r\phi - u\phi^2$. The invariant (2) concerns the on-site (spatially homogeneous) logistic law, corresponding to the $D = 0$ slice; it is local and does not survive generic coupling/diffusion. - Discrete→continuum legitimacy. Time-translation symmetry for autonomous laws guarantees an implicit first integral; evaluating the primitive yields the explicit logarithmic invariant used here. Companion notes cover the discrete-to-continuum mapping and symmetry analysis supporting this logic. - Empirical gates (context). The RD sector is validated independently by two canonical checks: (i) linear dispersion $\sigma(k) = r - Dk^2$ and its discrete counterpart, and (ii) Fisher–KPP pulled-front speed $c = 2\sqrt{Dr}$. These establish the RD baseline into which the on-site invariant is situated. - Scope separation. Finite-speed second-order EFT branches are quarantined; the present note is fully contained within the RD baseline.

Acknowledgments. I thank Voxtrium for providing his theory to me and giving me confidence when I saw that it mapped to his work and strengthened my own.

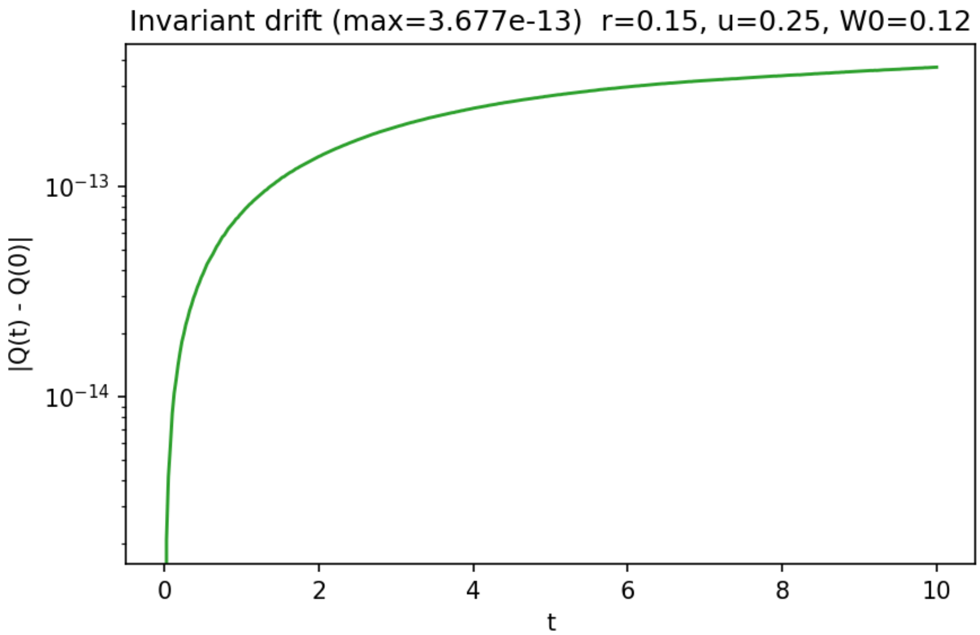

Figure 2: Invariant drift $\Delta Q(t) = |Q(t) - Q(0)|$ on a log scale. Acceptance: in double precision, $\max_t \Delta Q \leq 10^{-8}$ for RK4 with $dt \approx 10^{-3}$; in single precision, $\leq 10^{-5}$.

References

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[3] J. D. Murray, *Mathematical Biology I: An Introduction*, 3rd ed., Springer, 2002.

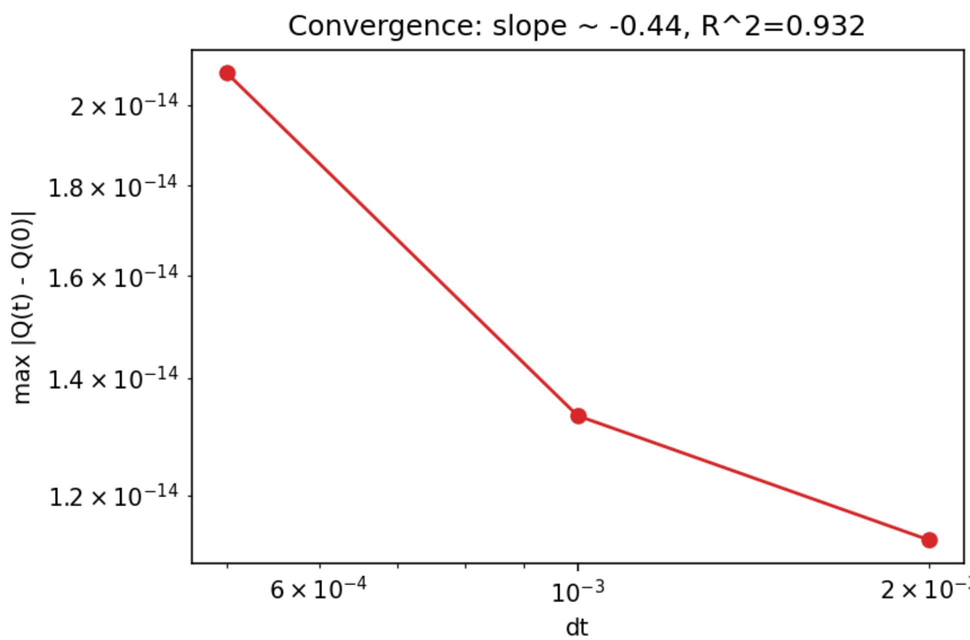

Figure 3: Convergence study: ΔQ vs. dt on a log–log plot with fitted slope. Acceptance: observed slope within ± 0.2 of the time-stepper's order (RK4≈ 4; explicit Euler≈ 1) with $R^2 \geq 0.98$.