Approximating convergence time of Stackelberg-Nash based ratio via Markov Chain Absorption Times

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Abstract

In Game Theory, more precisely in Security Games, Stackelberg Games are used to represent problems involving 2 or more actors under a hierarchical relationship between them. This setting is useful to, in particular, model the problem (from the perspective of an authority) of evasion in public transport systems [Escalona et al., 2024] [Brotcorne et al., 2021].

In the aforementioned papers it is discussed thoroughly the necessity of the operational implementation of the probabilities of inspection in public transportation systems, obtained by solving a multi-level optimization problem. This problem solves simultaneously the minimization of the evasion and the minimization of risk of the opportunistic passengers in being inspected.

In particular, it is of interest to know how much time (days) it will take to the evasion rate to converge to the one obtained through the steady-state probabilities obtained during optimization. Monte Carlo simulations were done in order to estimate this time in the papers cited previously. This approach has a latent problem. There is no *a priori* knowledge of how much time the simulation has to run in order to converge. So, in not-so-rare cases it could take more than one year time in instances, making this evasion rate both useless (as this needs to be implemented in mid term) and expensive, because these simulations are costly.

This work aims to establish a second method to obtain the time it should take the evasion rate converge to the theoretical steady-state one via Markov Chains. Specifically, this work seeks to construct a Markov Chain that permits to deduce, or approximate, this convergence time.

Keywords Markov Chain, Stackelberg games, Nash Equilibrium

1 Introduction

In Game Theory, more precisely in Security Games, Stackelberg Games are used to represent problems involving 2 or more actors under a hierarchical relationship between them. For example, but not limited to, it let us model the problem of evasion in public transport systems from the perspective of the transit authority [Escalona et al., 2024] [Brotcorne et al., 2021]. Another example is the work of [Carrasco et al., 2025], where a Stackelberg-Nash problem, which model the problem of street vending in the city of Valparaiso, Chile, is addressed. In any case, the operational implementation of the solution of the optimization problem rests on the construction of daily inspection schedules, which, in the long run, decreases the evasion rate (ER) or the Illegal Occupation Rate (IOR), respectively. Both the ER and the IOR represents the fraction of actors under an undesirable behavior (Percentage of persons evading paying the public transport tariff or of persons selling in the streets). Since there is limited resources to inspect, this daily schedules have to be implemented randomly daily for some time before it induces change in behavior. In that sense, it matters how fast this convergence occurs. To estimate that, Monte Carlo simulations were carried out, measuring the mean time it takes to have a daily ER or IOR less than some threshold.

The present work aims to estimate the convergence time to the equilibrium IOR via Markov Chain absorption times. Here, a Markov chain is built around the Nash-Equilibrium density distribution of actors $\mathbf{p}^*$, obtained through the optimal solution of the Stackelberg-Nash Problem. The chain replicates the behavior near the Nash Equilibrium, as the like of an harmonic oscillator.

This work is organized as follows. A brief resume on how the Nash Equilibrium is built is in section 2. section 3 formulates a Markov Chain that approximates the dynamic of the Stackelberg-Nash game. section 4 presents some actual computations based on the theory built in section 3. Finally, in section 5 concludes this work presenting some conclusions and advice for future work.

2 The Stackelberg-Nash model

In this section will be discussed briefly where does the probabilities, that are used as parameters for the model in this work, came from. The model Stackelberg-Nash model discussed in the next section is a one-to-one replica of the one from [Carrasco et al., 2025], where the setting is the management of street vending in the city of Valparaíso, Chile. The understanding of the variables, that will be parameters later, involved in the Nash Equilibrium Problem is important for this work.

From that, let J be the set of sites where an action could take place (e.g. street vending). It is defined a subset $L \subsetneq J$ of authorized[1] sites where the action is acceptable to occur, and $I \subsetneq J$ the subset of unauthorized sites where it is not acceptable. Obviously, $I \cap L = \emptyset$ and $I \cup L = L$. The attractiveness of a site $j \in J$ is defined by the parameter B_j, such that $\min_{i\in I} B_i > \max_{l\in L} B_l$. That is, the worst attractive unauthorized site is more attractive than the best authorized site. In terms of street vending it makes sense, because usually unauthorized places are cheaper because of lack of rental of place, flexibility, etc.

Let m be the number of actors acting over J. The distribution of the actors over the J sites is represented by the vector $\mathbf{p} = (p_j)_{j\in J} \in \mathbb{R}^{|J|}$, such that $\mathbb{1}_{|J|}^\top \mathbf{p} = 1$ and $\mathbf{p} \geq \mathbb{0}_{|J|}$.

The objective of an authority is to thwart the actions of the subjects over unauthorized sites I. To do so, the authority counts with a small set of n teams that induce aversion in the total attractiveness of a site. Let $\mathbb{P}_j$ be the probability of an intervesion ocurring in site $j \in J$, with a aversion of $\overline{W} \in \mathbb{R}$. In the same line, a total-atractiveness function $U(p_j|B_j, \mathbb{P}_j)$ is defined for every site $j \in J$. This function is strictly increasing in B_j; non-increasing in p_j, measuring the saturation on a spatial sense; and non-increasing in aversion, as higher probabilities of being inspectioned implies lower expected utility.

The Nash Equilibrium Problem that finds the distribution $\mathbf{p}$ of actors, given the aversion probabilities $\mathbb{P} \in \mathbb{R}^{|J|}$, such that this is configuration is a Nash Equilibrium, is defined by ([Carrasco et al., 2025]):

$$
\begin{aligned}
NE(\mathbb{P}): \quad p_j(\gamma - B_j U(p_j|B_j,\mathbb{P}_j)(1-\mathbb{P}_j) - \overline{W}\mathbb{P}_j) &= 0 & \forall j \in J \\
B_j U(p_j|B_j,\mathbb{P}_j)(1-\mathbb{P}_j) - \overline{W}\mathbb{P}_j &\leq \gamma & \forall j \in J \\
\mathbb{1}_{|J|}^\top \mathbb{P} &= 1 & \\
\mathbf{p} &\geq \mathbb{0}_{|J|} &
\end{aligned}
$$

The problem of finding optimal aversion probabilities $\mathbb{P}^*$ is discussed thoroughly in the aforementioned work of street vending. That discusion is out of the scope of this work, because both $\mathbb{P}^*$ and $\mathbf{p}^*$ will be treated as parameters.

3 The Markov Model

After doing the corresponding optimizations, it stays open the practical implementation of intervention teams given the aversion probabilities obtained. In [Carrasco et al., 2025], they interpreted probabilities of intervention as frequency, that is, the fraction of the days from a given time horizon that a given place must be inspected during those days. In addition, given the optimal Illegal Occupation Rate (IOR) $Ior^* := \sum_{i\in I} p_i^*$, they asked how much time, considering a day-by-day implementation of the aversion probabilities, it will take to the actors drop the daily IOR $Ior(t)$ to the optimal.

The only approach in said work was to do Monte-Carlo Simulation. The setting considered counting the fraction of time, until that day, that a given site has been intervened as the aversion probabilities. The probability of selecting a site

[1]Legal in the context of street vending, thus the capital L.

j is given by the optimal $\mathbb{P}_j^*$. By the Law of Large Numbers, this setting guarantees that $Ior(t) \to Ior^*$ as $t \to \infty$. So, day by day, they solved the Nash Equilibrium Problem $NE(t)$ with the aversion probabilities obtained by the Montecarlo simulation. After that, they computed the actual IOR, $Ior(t)$, and obtained the convergence time as the first arrival under the optimal IOR Ior^*. That is, $\tau = \min\{t \mid Ior(t) < Ior^*\}$ is the time they considered has to pass until a day-by-day implementation of the aversion probabilities makes the actors converge to the Nash Equilibrium, in the IOR sense.

In this section, an alternative method will be described to obtain the convergence time of the IOR, τ.

Let $\{X(j)_t\}_{t=\in\mathbb{N}, j\in J}$ be a Markov chain that represents the proportion of actors for every site j at time t. So, $m \cdot X(j)_t \in \{0, 1 \dots m\}$ is the quantity of actors in j at t. It is assumed that for fixed j, $\{X(j)_t\}_{t=0}^m$ is a Markov Chain on its own. For any two $j \neq j^{'} \in J$, the Markov Chains defined are independently one to another.

3.1 Transition Matrix

In this section, the construction of the transition matrix is addressed. Let $P \in \mathbb{R}^{(m+1+|J|)^2}$ be the transition matrix of the Markov Chain $\{X(j)_t\}_{t\in\mathbb{N}, j\in J}$. As the Markov Chain measures quantity (and does not model individually every actor), it is assumed that P is a block-diagonal matrix, having $|J|$ diagonal blocks $P(j) \in \mathbb{R}^{(m+1)^2}$:

$$P = \left(\begin{array}{c|c|c|c} P(j_1) & \mathbb{0} & \dots & \mathbb{0} \\ \hline \mathbb{0} & P(j_2) & \dots & \mathbb{0} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathbb{0} & \mathbb{0} & \dots & P(j_{|J|}) \end{array}\right).$$

Now, let $j \in J$ be fixed. To replicate the behavior, from the perspective of Game Theory, of the Nash Equilibrium, the probabilities will be distributed according to the distance of the actual proportion to the optimal p_j^*. That is, if the distribution of proportions is distant from the optimal, then the former carries momentum towards the optimal distribution. For sake of simplicity, the relationship between a given actual proportion and the optimal is considered to be linear and symmetrical.

Let $a, b \in \{0, \frac{1}{m} \dots 1\}$, $\Delta_a := |p_j^* - a|$ and $[p_j^*] := \lceil m \cdot p_j^* \rfloor / m$, where $\lceil \cdot \rfloor$ is the nearest-integer function. So, we have that the transition probabilities from a to b is:

$$P(j)_{ab} := \begin{cases} C_{a,1}(C_{a,2} - |p_j^* - b|), & \text{if } a \neq [p_j^*] \text{ and } |p_j^* - b| < \Delta_a, \\ 1 & \text{if } a = b = [p_j^*], \\ 0 & \text{in other case,} \end{cases}$$

where $C_{a,1}$ and $C_{a,2}$ are constants ensuring 2 properties: that this is, in fact, a discrete distribution, and that nulls out for any b such that $|p_j^* - b| \geq \Delta_A$. Following the definition, for $a \neq [p_j^*]$, then $b = a$ is the nearest point to p_j^* outside the support of $P(j)_{a\cdot}$. So:

$$P(j)_{aa} = C_{a,1}(C_{a,2} - \Delta_a) = 0 \implies C_{a,2} = \Delta_a.$$

On the other hand, for every a, $P(j)_{a\cdot}$ is a probability distribution, so it has to sum exactly to 1 over all the values of $b \in \{0, \frac{1}{m} \dots 1\}$. If we define $s_{a,j}^- := \lceil m(p_j^* - \Delta_a) \rceil$ and $s_{a,j}^+ := \lfloor m(p_j^* + \Delta_a) \rfloor$, then the condition $|p_j^* - b| < \Delta_a$ is equivalent to $m \cdot b \in \left\{s_{a,j}^-, \dots, s_{a,j}^+\right\} \subset \mathbb{N}$. So, we have that:

$$\sum_{k=0}^{m} P(j)_{a\frac{k}{m}} = 1.$$

$$\iff \sum_{k=s_{a,j}^-}^{s_{a,j}^+} C_{a,1}(\Delta_a - |p_j^* - k/m|) = 1.$$

$$\iff C_{a,1} = \left(\sum_{k=s_{a,j}^-}^{s_{a,j}^+} (\Delta_a - |p_j^* - k/m|) \right)^{-1}.$$

For example, for some $j_1 \in J$, $m = 10$, $p^*_{j_1} = 5.3$, the j_1-th transition matrix $P(j_1)$ is:

$$\begin{pmatrix}
0 & 0.036 & 0.071 & 0.107 & 0.143 & 0.179 & 0.164 & 0.129 & 0.093 & 0.057 & 0.021\\
0 & 0 & 0.054 & 0.109 & 0.163 & 0.217 & 0.196 & 0.141 & 0.087 & 0.033 & 0\\
0 & 0 & 0 & 0.093 & 0.185 & 0.278 & 0.241 & 0.148 & 0.056 & 0 & 0\\
0 & 0 & 0 & 0 & 0.192 & 0.385 & 0.308 & 0.115 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0.625 & 0.375 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0.143 & 0.5 & 0.357 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0.056 & 0.194 & 0.333 & 0.278 & 0.139 & 0 & 0 & 0\\
0 & 0 & 0.029 & 0.103 & 0.176 & 0.25 & 0.221 & 0.147 & 0.074 & 0 & 0\\
0 & 0.018 & 0.064 & 0.109 & 0.155 & 0.2 & 0.182 & 0.136 & 0.091 & 0.045 & 0
\end{pmatrix}.$$

Likewise, for $j_2 \in J$, with $p^*_{j_2} = 7.8$, the matrix $P(j_2)$ is:

$$\begin{pmatrix}
0 & 0.021 & 0.042 & 0.063 & 0.084 & 0.105 & 0.126 & 0.146 & 0.159 & 0.138 & 0.117\\
0 & 0 & 0.026 & 0.053 & 0.079 & 0.106 & 0.132 & 0.159 & 0.175 & 0.148 & 0.122\\
0 & 0 & 0 & 0.035 & 0.069 & 0.104 & 0.139 & 0.174 & 0.194 & 0.160 & 0.125\\
0 & 0 & 0 & 0 & 0.048 & 0.096 & 0.144 & 0.192 & 0.221 & 0.173 & 0.125\\
0 & 0 & 0 & 0 & 0 & 0.072 & 0.145 & 0.217 & 0.261 & 0.188 & 0.116\\
0 & 0 & 0 & 0 & 0 & 0 & 0.128 & 0.256 & 0.333 & 0.205 & 0.077\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.313 & 0.500 & 0.187 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.286 & 0.714 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0.083 & 0.292 & 0.417 & 0.208 & 0
\end{pmatrix}$$

The graph of $P(j_1)_{ab}$ is shown as a function of b in figure 1.

Absorption times and convergence

It is easy to see that every transition matrix in the diagonal of P, $P(j)$, has an absorbing class of at most 2 elements. The absorbing class usually consists of the nearest element in the lattice $\{0, \frac{1}{m}, \ldots, 1\}$ to p^*_j, or the 2 nearest elements, if it turns out to be the case (because of roundings, numerical errors, etc.).

So, with respect to P, it is easy to see that after reordering its entries according to the canonical form the matrix Q of P is given in terms of the matrices $Q(j)$ of each j-th Markov Chain in the diagonal of P:

$$Q = \left(\begin{array}{c|c|c|c}
Q(j_1) & \mathbb{0} & \ldots & \mathbb{0}\\
\hline
\mathbb{0} & Q(j_2) & \ldots & \mathbb{0}\\
\hline
\vdots & \vdots & \ddots & \vdots\\
\hline
\mathbb{0} & \mathbb{0} & \ldots & Q(j_{|J|})
\end{array}\right).$$

Once more, due to the diagonal structure of P, the fundamental matrix of P, F, is obtained as the diagonal matrix containing in its j-th diagonal block the matrix $F(j)$, where

$$F(j) := (Id - Q(j))^{-1},$$

that is

$$F = \left(\begin{array}{c|c|c|c}
F(j_1) & \mathbb{0} & \ldots & \mathbb{0}\\
\hline
\mathbb{0} & F(j_2) & \ldots & \mathbb{0}\\
\hline
\vdots & \vdots & \ddots & \vdots\\
\hline
\mathbb{0} & \mathbb{0} & \ldots & F(j_{|J|})
\end{array}\right)$$

It is known that the absorption time, starting from a transient state a is given by the formula

$$\mathbf{e}_{ma}^{\top} F \mathbb{1}_l$$

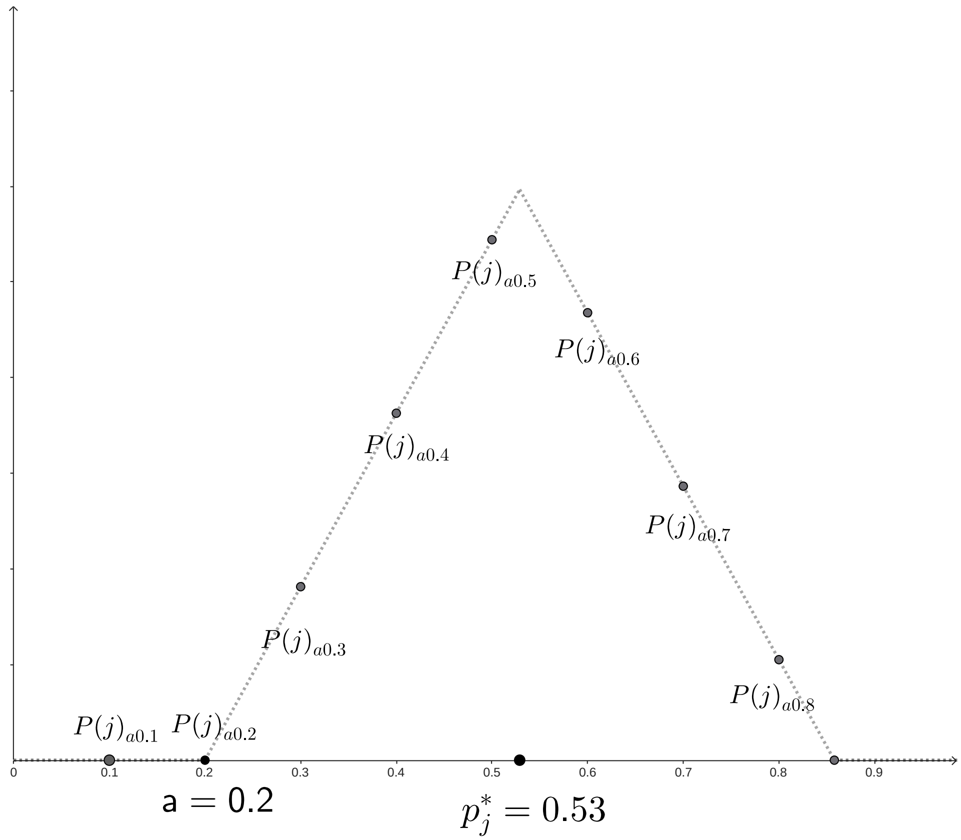

Figure 1: Example of the conditional probabilities $P(j)_{ab}$, for fixed j, a, p_j^* and for $b \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$. The dotted lines shows the continuous interpolator of the conditional probabilities.

where $\mathbf{e}_{ma}$ is the ma-th element of the canonical base of $\mathbb{R}^l$, for a addecuate constant $l \in \mathbb{N}$ (remember that $ma \in \{0, 1, \ldots, m\}$). Moreover, the expected absorption time, as the mean over all posible initial states, is given by:

$$\tau := \frac{\mathbb{1}_l^\top F \mathbb{1}_l}{l}.$$

Again, by the diagonality of P, this expected time reduces to:

$$\tau = \frac{\sum_{j\in J} \mathbb{1}_{l(j)}^\top F(j) \mathbb{1}_{l(j)}}{l},$$

where $l(j)$ is equal to the quantity of transient states of the j-th Markov Chain, that is, $m + 1 - Abs(j)$. So, $l = \sum_{j\in J} l(j) = (m + 1)|J| - Abs$, where $Abs = \sum_{j\in J} Abs(j)$ is the total quantity of absorbing states over P.

3.2 Illegal Occupation rate

Now everything is available to compute the IOR with respect to the absorbing time τ. Assuming that the initial distribution of actors across the $|J|$ sites is uniformly distributed, observe that:

$$
\begin{aligned}
Ior(\tau) &= \mathbb{E}\left(\sum_{i\in I} X(i)_{\lceil\tau\rfloor}\right),\\
&= \sum_{i\in I} \mathbb{E}(X(i)_{\lceil\tau\rfloor}),\\
&= \sum_{i\in I}\sum_{b} \mathbb{P}(X(i)_{\lceil\tau\rfloor} = b)\cdot b,\\
&= \sum_{i\in I}\sum_{b}\sum_{a} \underbrace{\mathbb{P}(X(i)_{\lceil\tau\rfloor} = b \mid \ X(i)_0 = a)}_{=P_{ab}^{\lceil\tau\rfloor}(i)} \cdot \mathbb{P}(X(i)_0 = a)\cdot b,\\
&= \sum_{i\in I}\sum_{a}\sum_{b} \frac{1}{m+1} P_{ab}^{\lceil\tau\rfloor}(i)\cdot b,\\
&= \sum_{i\in I} \frac{\mathbb{1}_{|I|}^{\top} P^{\lceil\tau\rfloor}(i)\mathbf{v}}{m+1},
\end{aligned}
$$

where $\mathbf{v} = (0, 1, \ldots, m)/m \in \mathbb{R}^{m+1}$.

4 Numerical Experiments

The data set used in this work is the same as in [Carrasco et al., 2025]. That is the common parameters $m = 974$, $|I| = 102$ and $|L| = 40$ and the varying parameter $n \in \{1, 2 \ldots 15\}$ and the optimal probabilities $\mathbf{p}^*(n)$.

Following the author's analysis, lets fix $n = 10$. One can compute the equilibrium IOR, obtaining that $Ior^* \approx 0.47681$. Then, both τ and $Ior(\tau)$ are computed, obtaining $\tau \approx 4.67541$ and $Ior(\tau) \approx 0.60012$. Considering $\tau + 1$ instead, the IOR improves dramatically, to $Ior(\tau + 1) \approx= 0.50752$. This may seem artificial, but serves as a compensation for the oversimplification of considering P block-diagonal, which holds accurate only near the equilibrium.

Further experiments show that for $t < \tau$, one may obtain unrealistic values for IOR, such as $Ior(1) \approx 16$, or $Ior(2) \approx 5$. As a ratio, this number should always be not greater than 1, but in the early stages of the iterations this could easily not be the case because some high-density cases for every site j might occur at the same time.

It is worth mentioning that for $t \gtrapprox 2\tau$, $Ior(t) \approx Ior^*$. So the transition matrix, at least in a limiting case, works as intended (the Markov IOR converges to the equilibrium one).

Now, the same experiments were carried out for every $n\{1, 2 \ldots 15\}$. The results are shown in 2 and 3.

From here, the analysis does not change essentially as n changes, i.e., the Markov Chain converges rapidly towards the Nash Equilibrium.

As reported in Carrasco et al. [2025], for $n = 10$ it is expected that after 58 days the IOR hits the threshold of 52% for the first time. This value is vastly greater than the theoretical $\approx 48\%$. This value is the best that the authorities expect to reduce the IOR, so it stays as the threshold for which any solution that achieves this value, or lesser, is considered a good solution. For other values of n there has not been an extensive analysis, so it has been left out in this work aswell.

5 Conclusions

It is clear that the Markov Model proposed has a very rapid convergence compared to the Montecarlo simulations. This may be due to the fact that the transition matrix P is built very naively. Intuitively, the graph of a set of states approaching a Nash Equilibrium should look like an spring, i.e., said states should look like they are revolving around the equilibrium. That may not be the case for the way the transition matrix was built in this work, because it puts major weight towards the equilibrium, but does not address the fact that the momentum of a given state before absorption may induce higher probabilities to not converge immediately, if it is not near the equilibrium. Coming back to the spring

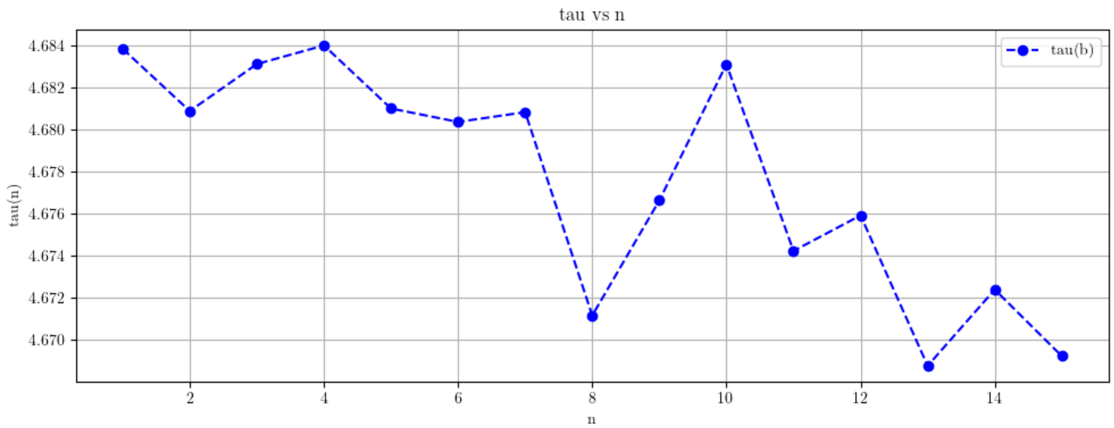

Figure 2: Plot of $\tau(n)$, for $n \in \{1, 2 \ldots 15\}$.

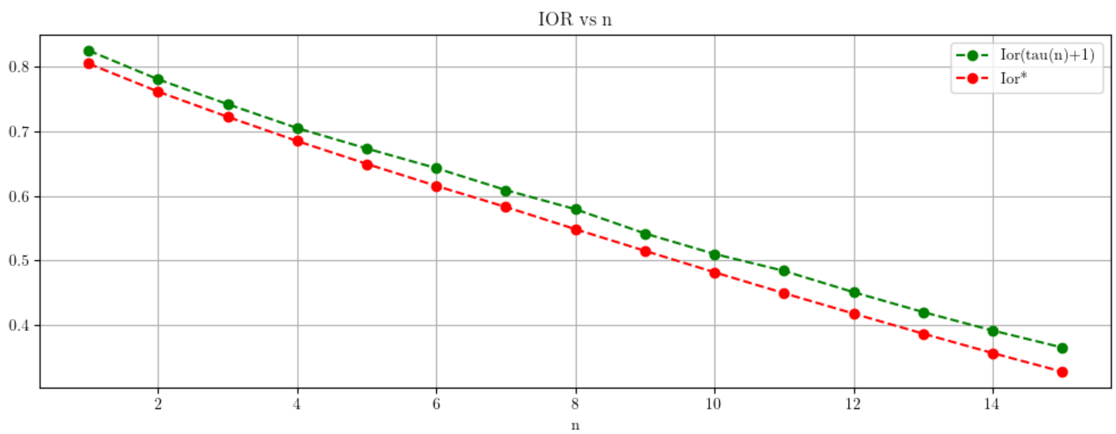

Figure 3: Plot of $Ior(\tau(n) + 1)$, for $n \in \{1, 2 \ldots 15\}$(green) and $Ior^*(n)$ (red), for reference.

comparison, an spring that is set off farther from its equilibrium will oscillate further. On other side, the analysis is still reasonable, as the IOR converges accordingly.

Improvements would come from a better modeling of the transition matrix, as described in the last paragraph.

Code

The Python programmes used in the computations of this work are up in this github repository.

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