

# Inverse monoids and immersions of cell complexes

Nóra Szakács

University of York

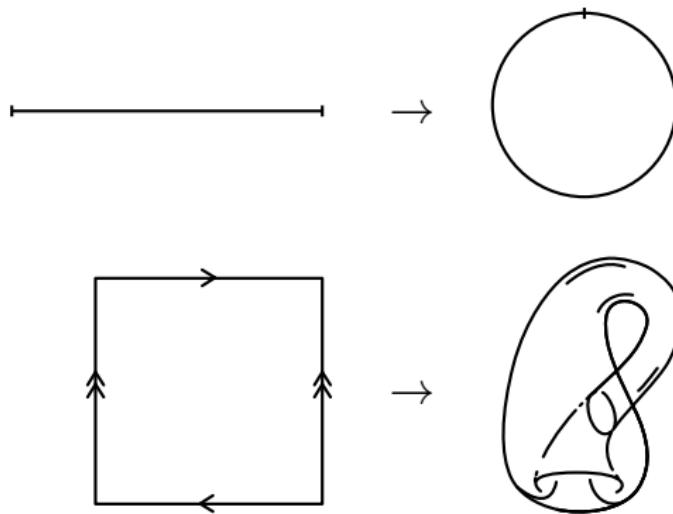
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# Immersions

## Definition

A continuous map  $f : Y \rightarrow X$  between topological spaces is called a **(topological) immersion** if every point  $y \in Y$  has a neighborhood  $U$  that is mapped homeomorphically onto  $f(U)$  by  $f$ .

## Example:



## Further examples: coverings

### Definition

A **covering** is a continuous map  $f: Y \rightarrow X$  for which there exists an open cover  $U_\alpha$  of  $X$  such that for each  $\alpha$ ,  $f^{-1}(U_\alpha)$  is a disjoint union of open sets in  $Y$ , each of which is mapped homeomorphically onto  $U_\alpha$  by  $f$ .

**Fact:** connected covers of a topological space  $\longleftrightarrow$  conjugacy classes of subgroups of its fundamental group.

**Fundamental group:** homotopy classes of closed paths around a given point, equipped with concatenation

**Why this works:** for any path  $p$  in  $X$  and any  $y$  point in  $f^{-1}(\alpha(p))$ , there exists a unique lift of  $p$  starting at  $y$ .

To characterize  $f$ , it suffices to keep track of which **closed** paths lift to **closed** paths, these correspond to a subgroup of the fundamental group.

## Immersions — the idea

If  $f: Y \rightarrow X$  is an immersion and  $p$  is a path in  $X$ , then if  $p$  lifts at some point in  $f^{-1}(\alpha(p))$ , then  $p$  is unique, however, it may be that  $p$  doesn't lift or lifts only partially.

**Idea:** find some kind of algebraic structure that enables us to distinguish when paths lift and when they don't, in addition to distinguishing when closed paths lift to closed paths.

This algebraic structure will be an **inverse monoid** of paths.

# Inverse monoids

## Definition

A monoid  $(S, \cdot)$  is called an **inverse monoid** if for all  $s \in S$  there exists an element  $s^{-1} \in S$  such that  $ss^{-1}s = s$  and  $s^{-1}ss^{-1} = s^{-1}$ , furthermore idempotents commute.

**The typical example:** the symmetric inverse monoid on a set  $X$ :  
 $X \rightarrow X$  partial injective maps under partial multiplication.

(Notation:  $\text{SIM}(X)$ )

**Natural partial order:**  $a \leq b$  iff there exists an idempotent  $e$  with  $a = be$ .

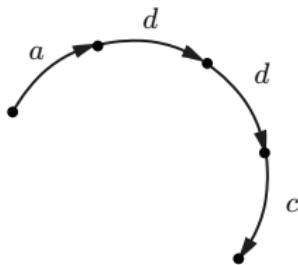
Free inverse monoids exist. (Notation:  $\text{FIM}(X)$ )

# Inverse monoid actions

## Definition

An inverse monoid  $S$  **acts** on the set  $X$  if there is a homomorphism  $S \rightarrow \text{SIM}(X)$ .

**Example:** let  $\Gamma$  be a graph edge-labeled in a deterministic and co-deterministic way over a set  $A$ , then  $\text{FIM}(A)$  acts on  $V(\Gamma)$ .



# Stabilizers

## Definition

We say  $s \in S$  stabilizes  $x \in X$  if  $x.s$  exists, and is equal to  $x$ . The stabilizer  $\text{Stab}(x)$  consists of all elements of  $S$  which stabilize  $x$ .

**Note:** the stabilizer of a set is always an inverse submonoid, and it is **closed upwards** in the natural partial order. Such inverse submonoids are called closed. (Notation:  $M \leq^\omega S$ )

Suppose  $x.s = y$ . Then

$$s^{-1} \text{Stab}(x)s \subseteq \text{Stab}(y),$$

$$s \text{Stab}(y)s^{-1} \subseteq \text{Stab}(x).$$

In this case, we say  $\text{Stab}(x)$  and  $\text{Stab}(y)$  are conjugate.

# The loop monoid of graphs

**Fact:** the homotopy equivalence on paths of a graph is induced by  $pp^{-1} \equiv \alpha(p)$  for any path  $p$ .

Let  $\approx$  denote the equivalence induced by  $pp^{-1}p \approx p$  and  $pp^{-1}qq^{-1} \approx qq^{-1}pp^{-1}$ .

## Definition (Margolis, Meakin)

The **loop monoid**  $L(\Gamma, v)$  is the inverse monoid consisting of  $\approx$ -classes of closed paths around  $v$ , with respect to concatenation.

**Note:** if  $\Gamma$  be a digraph edge-labeled over the set  $X$  in a deterministic and co-deterministic way, then Then **paths starting at  $v$  are words over  $X \cup X^{-1}$** , hence  $L(\Gamma, v) \leq \text{FIM}(X)$ , in fact  $L(\Gamma, v) = \text{Stab}(v)$  under the action of  $\text{FIM}(X)$  on  $\Gamma$ .

**Remark:**  $L(\Gamma, v)$  and  $L(\Gamma, v')$  are conjugate, but that doesn't imply isomorphic (unlike in the case of the fundamental group)

# The theorem classifying graph immersions

**An immersion between graphs:** a topological immersion that respects the graph structure.

**Theorem (Margolis, Meakin)**

*Connected immersions over a connected graph  $\Gamma \longleftrightarrow$  conjugacy classes of closed inverse submonoid of  $L(\Gamma, v)$  for any  $v \in \Gamma$ .*

**Idea of the proof:**

- ▶ if  $f: \Gamma_2 \rightarrow \Gamma_1$  is an immersion with  $f(v_2) = v_1$ , then  $L(\Gamma_2, v_2) \subseteq L(\Gamma_1, v_1)$
- ▶ for any  $M \leq^\omega L(\Gamma_1, v_1)$ , the  $\omega$ -coset graph of  $M$  immerses into  $\Gamma_1$
- ▶  $H, K \subseteq L(\Gamma_1, v_1)$  correspond to the same immersion iff they are conjugate

# Immersions in higher dimensions

**CW-complexes:** a class of topological spaces with a combinatorial structure

A CW-complex is a topological space built iteratively:

1. Start with a discrete set of points (0-cells)
2. Attach open intervals to the 0-skeleton (1-cells)
3. Attach open disks to the 1-skeleton (2-cells)
4. ...

1-dimensional CW-complexes = graphs

In a CW-complex  $\mathcal{C}$ , every cell has an attaching map  $\varphi: S^n \rightarrow \mathcal{C}$  and a characteristic map  $\sigma: B^n \rightarrow \mathcal{C}$ .

# Immersions in higher dimensions

**$\Delta$ -complexes:** CW-complexes with restricted attaching maps:

Each cell has a distinguished characteristic map  $\sigma: \Delta^n \rightarrow C$  such that the restriction to a face of  $\Delta^n$  is also a characteristic map of some cell.

Fix an ordering  $v_0, \dots, v_n$  on the vertices of  $\Delta^n$ . (Notation:  $\Delta^n = [v_0, \dots, v_n]$ .) We call the smallest vertex  $v_0$  the **root** of the simplex,  $\sigma(v_0)$  is called the **root** of the cell, denoted by  $\alpha(C)$ .

**Digraphs:** edges are 2-simplices  $[v_0, v_1]$ ;  $\alpha(e) = \sigma(v_0)$ ,  $\omega(e) = \sigma(v_1)$ .

For higher dimensional cells  $C$ , we define  $\omega(C) = \alpha(C)$ .

**Immersion between  $\Delta$ -complexes:** a topological immersion that commutes with the characteristic maps.

# The loop monoid of a $\Delta$ -complex

## The idea:

We need an algebraic structure to keep track of which paths and cells lift, and which closed paths lift to closed paths.

Let  $\mathcal{C}$  be a  $\Delta$ -complex.

A **generalized path** in  $\mathcal{C}$  is a sequence of **cells**  $s_1 \dots s_n$  such that  $\omega(s_j) = \alpha(s_{j+1})$ .

Note:

- ▶ if cells of dimension  $\geq 2$  lift, they remain "closed"
- ▶ if a cell lifts, everything in its boundary must lift as well

We will introduce equivalence relations on generalized paths (in addition to the inverse monoid relations) which reflect the above properties.

## Labeled $\Delta$ -complex

Consider a deterministic and co-deterministic labeling the  $\Delta$ -complex  $\mathcal{C}$  over a set  $X \cup P$  in way that cells of the same label have “boundaries” of the same label.

For any  $n$ -cell  $C$  ( $n \geq 2$ ), we designate the following generalized path on the boundary of  $C$ .

- ▶ if  $n = 2$ , let  $bw(C)$  be the image of the path  $(v_0, v_1, v_2, v_0)$  under  $\sigma$ ;
- ▶ if  $n > 2$ , let  $C_i = [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$ , and let  $bw(C)$  be the image of  $C_n C_{n-1} \dots C_1(v_0, v_1) C_0(v_1, v_0)$  under  $\sigma$ .

**Note:**  $bw(\rho) := \ell(bw(C))$ , where  $\ell(C) = \rho$ , is well-defined.

## The loop monoid

Take a  $\Delta$ -complex labeled over  $X \cup P$ , and consider the inverse monoid  $M_{X,P} = \langle X \cup P \rangle$ , defined by the following relations:  
for any  $\rho \in P$ ,

- ▶  $\rho^2 = \rho$ , and
- ▶  $\rho \leq \ell(bw(\rho))$ .

### Proposition

*The inverse monoid  $M_{X,P}$  acts on any complex  $\mathcal{C}$  labeled over  $X \cup P$  (consistently with boundaries).*

$L(\mathcal{C}, v) :=$  generalized paths around  $v$  wrt the above relations  
= the stabilizer of  $v$  under this action

#### Note:

- ▶  $L(\mathcal{C}, v) \leq^\omega M_{X,P}$ ;
- ▶ the greatest group homomorphic image of  $L(\mathcal{C}, v)$  is  $\pi_1(\mathcal{C})$ .

## The main theorem

### Theorem (Meakin, Sz.)

*Connected immersions over a connected  $\Delta$ -complex  $\mathcal{C} \longleftrightarrow$  conjugacy classes of closed inverse submonoid of  $L(\mathcal{C}, v)$  for any  $v \in \mathcal{C}^0$ .*

**Remark:** the above theorem was proven by Meakin and Sz. for CW-complexes in the 2-dimensional case.

Thank you for your attention!