EE 510 10/23/2020

Outline:

+ Norms:
+ SVD decomposition
+ Condition number

- $A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$; $\|A\| = ?$

$$\|A\| = \max_{\|x\|=1} \|Ax\| \qquad ; \; x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\|Ax\|^2 = \left\| \begin{pmatrix} 2x_1 \\ x_1 + x_2 \end{pmatrix} \right\| = 4x_1^2 + 2x_1 x_2 + 1$$

$$\max_{\|x\|=1} \; 4x_1^2 + 2x_1x_2 \; ; \quad x_1^2 + x_2^2 = 1$$

$$\mathcal{L}(x_1, x_2, \lambda) = 4x_1^2 + 2x_1x_2 + \lambda(x_1^2 + x_2^2 - 1)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = (8 + 2\lambda)x_1 + 2x_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 2x_1 + 2\lambda x_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = x_1^2 + x_2^2 - 1 = 0$$

$$\Rightarrow \begin{cases} (4+\lambda)x_1 + x_2 = 0 \\ x_1 + \lambda x_2 = 0 \\ x_1^2 + x_2^2 = 1 \end{cases}$$

$$x_1 = -\lambda x_2$$

$$x_2\left[1-\lambda(4+\lambda)\right] = 0$$

$$\lambda^2 + 4\lambda - 1 = 0 \implies \lambda_1 = -2-\sqrt{5}\ ;\ \lambda_2 = -2+\sqrt{5}$$

$$\lambda = -2-\sqrt{5}$$

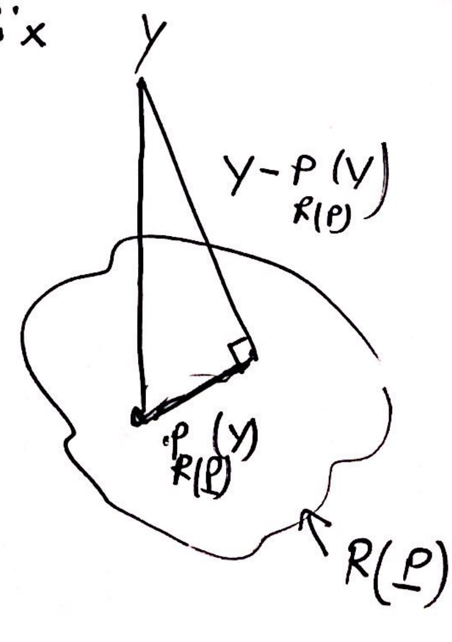
$$x_1 = (2+\sqrt{5})\,x_2 \implies x_2 = \frac{1}{\sqrt{10+4\sqrt{5}}}$$

$$x_1 = \frac{2+\sqrt{5}}{\sqrt{10+4\sqrt{5}}}$$

$$\|A\| = \sqrt{\frac{25+11\sqrt{5}}{5+2\sqrt{5}}}$$

<u>Exercise:</u> P is a projection matrix

Prove that $\|P\| \leq 1$.



$$x = P(x) + x - P(x)$$

$$\|x\|_2^2 = \|P(x) + x - P(x)\|_2^2$$

we have $\langle P(x),\ x-P(x)\rangle = 0$

$$\implies \|x\|_2^2 = \|P(x)\|_2^2 + \|x-P(x)\|_2^2$$

$$\implies \|P(x)\|_2^2 \leq \|x\|_2^2 \implies \|P(x)\|_2 \leq \|x\|_2$$

$$\implies \frac{\|P(x)\|_2}{\|x\|_2} \leq 1 \quad \forall x \implies \boxed{\|P\| \leq 1}$$

Scanned with CamScanner

SVD decomposition:

$$A \in \mathbb{R}^{m \times n} \; ; \; A = \sum_{i=1}^{r} \sigma_i v_i u_i^T \; ; \; r = \operatorname{rank}(A) \leq \min(m,n)$$

$$\|A\| = \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$$

$$m \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_m \\ | & | & & | \\ | & | & & | \end{bmatrix}}_{m} \; m \underbrace{\begin{bmatrix} \sigma_1 & & & & 0 & \cdots & 0 \\ & \sigma_2 & & & & & \\ & & \ddots & & & & \\ & 0 & & \sigma_r & & & \\ & & & & 0 & \cdots & 0 \end{bmatrix}}_{n} \begin{bmatrix} - & u_1^T & - \\ - & u_2^T & - \\ & \vdots & \\ - & u_n^T & - \end{bmatrix} n$$

$$n$$

$$A = V \Sigma U^T \; ; \quad VV^T = I$$
$$UU^T = I.$$

$$A u_j = \sum_{i=1}^{r} \sigma_i v_i u_i^T u_j = \sigma_j v_j$$

$$v_j^T A = \sum_{i=1}^{r} \sigma_i v_j^T v_i u_i^T = \sigma_j u_j$$

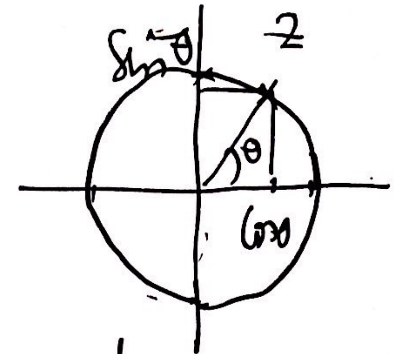
$$R(A) = \operatorname{span}\{ v_i ; \sigma_i > 0 \}$$

$$R(A^T) = \operatorname{span}\{ u_i ; \sigma_i > 0 \}$$

Scanned with CamScanner

· A ; f_A :

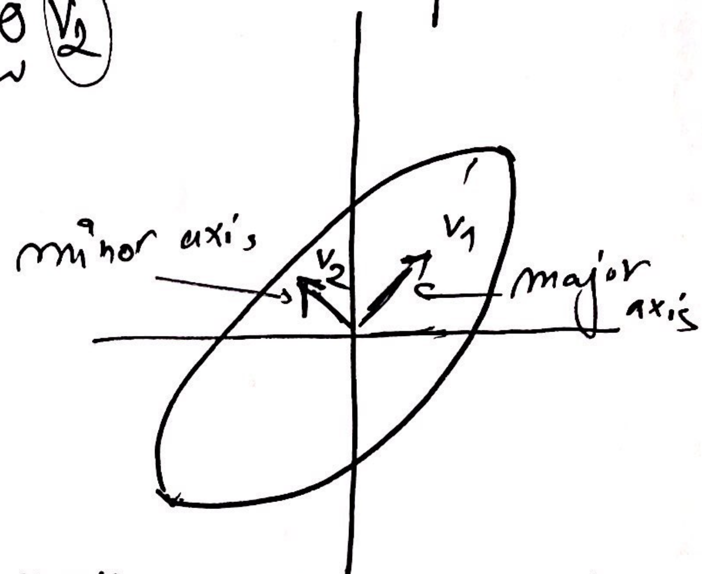
$f_A(e_1) = c_1 v_1$; $f_A(e_2) = c_2 v_2$



$z = \cos\theta\, e_1 + \sin\theta\, e_2$

$$f_A(z) = \underbrace{c_1 \cos\theta}_{x'}\, v_1 + \underbrace{c_2 \sin\theta}_{y'}\, v_2$$

$$\left(\frac{x'}{c_1}\right)^2 + \left(\frac{y'}{c_2}\right)^2 = 1$$



$\text{span}\{v_1, v_2\} = R(A)$

· Let A_k be the of matrices $\mathbb{R}^{m \times n}$ of rank at most k

$A \in \mathbb{R}^{m \times n}$; $\text{rank}(A) = r$; $k < r$

$$A_k = \underset{X \in A_k}{\text{argmin}} \|A - X\|_F \qquad \|A\|_F = \sqrt{tr(A^T A)}$$

$A_k = P_{A_k}(A)$; $A = \sum_{i=1}^{r} \sigma_i v_i u_i^T$

$$A_k = \sum_{i=1}^{k} \sigma_i v_i u_i^T = \begin{bmatrix} \\ \\ V \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_k & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} U^T$$

$$A_k = \arg\min_{X \in \mathcal{A}_k} \|A - X\|_2 \quad ; \quad \|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

$$\|A - A_k\|_2 = \sigma_{k+1}$$

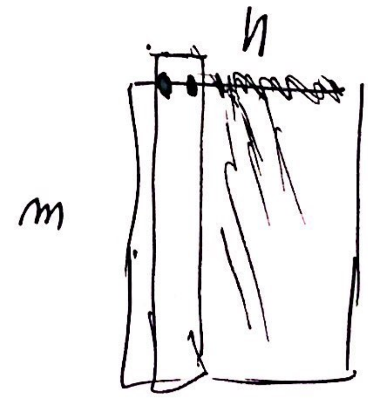
$$V \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix} U^T - V \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_k & & 0 \\ & & & 0 & \\ & & & & \ddots \; 0 \end{bmatrix} U^T$$

$$= V \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & 0 \; \sigma_{k+1} & & 0 \\ & & & \ddots \; \sigma_r & \\ 0 & & & 0 & \ddots \; 0 \end{bmatrix} U^T$$

- $A \in \mathbb{R}^{m \times n}$;

The best rank-1 approximation of A (when the error is measured in 2-norm)

$$A_1 = \|A\| \, V_1 U_1^T .$$



Pseudo-inverse:

- $$A^{+} = \sum_{\substack{i=1 \\ \sigma_i > 0}}^{r} \frac{1}{\sigma_i} u_i v_i^T \; ;$$

If A is invertible; $A \in \mathbb{R}^{n \times n}$..

$$A^{-1} = \sum_{i=1}^{n} \frac{1}{\sigma_i} u_i v_i^T$$

- $$AA^{+} = \sum_{\sigma_i > 0} v_i v_i^T = P_{R(A)}$$

- $$A^{+}A = \sum_{\sigma_i > 0} u_i u_i^T = P_{R(A^T)}.$$

If $\operatorname{rank} \boxed{A = n}$; $A \in \mathbb{R}^{m \times n}$

$\underline{n \leq m}$

$$A^T A \in \mathbb{R}^{n \times n}$$

$$\operatorname{rank}(A^T A) = \operatorname{rank}(A) = n$$

$$\Longrightarrow A^T A \text{ invertible.}$$

$$\Longrightarrow A^{+} = (A^T A)^{-1} A^T$$

If $\operatorname{rank}(A) = m$; $m < n$

$$A^{+} = A^T (AA^T)^{-1}.$$

$x \in N(A) \Rightarrow Ax = 0$

$\Rightarrow A^T A x = 0$

$\Rightarrow x \in N(A^T A).$

$N(A) \subset N(A^T A)$

$x \in N(A^T A) \Rightarrow A^T A x = 0$

$x^T A^T A x = 0$

$\Rightarrow \|Ax\|^2 = 0$

$\Rightarrow Ax = 0 \Rightarrow x \in N(A)$

$N(A^T A) \subset N(A)$

$\Rightarrow N(A) = N(A^T A)$

Condition number

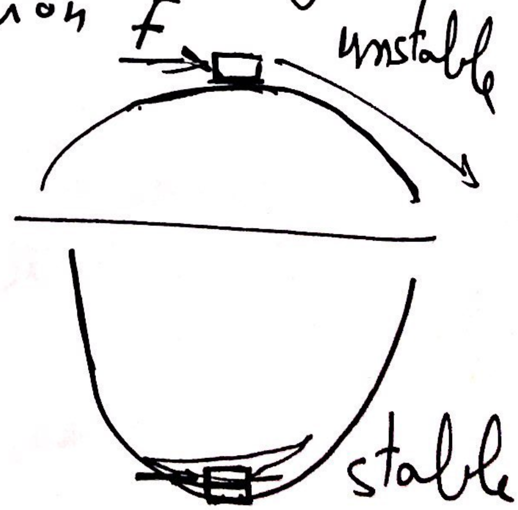
- $Ax = b$

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix}; \quad b = \begin{pmatrix} \frac{25}{12} \\ \frac{77}{60} \\ \frac{57}{80} \\ \frac{319}{420} \end{pmatrix} = \begin{pmatrix} 2{,}08 \\ 1{,}28 \\ 0{,}95 \\ 0{,}76 \end{pmatrix}$$

$$x = A^{-1} b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$b' = \begin{pmatrix} 2{,}1 \\ 1{,}3 \\ 1 \\ 0{,}8 \end{pmatrix}; \quad A x' = b' \; ; \; x' = \begin{pmatrix} 5{,}6 \\ -48 \\ 114 \\ -76 \end{pmatrix}$$

a small perturbation of the data induces a large perturbation of the solution



$Ax = b \rightarrow A$ is invertible

$b + \delta b$

$$A(x + \delta x) = b + \delta b$$

$$\frac{\|\delta x\|}{\|x\|}$$

$$Ax + A\delta x = b + \delta b$$

$$A\delta x = \delta b$$

$$\Rightarrow \delta x = A^{-1}\delta b$$

$$\Rightarrow \|\delta x\| \leq \|A^{-1}\| \|\delta b\|$$

$$b = Ax \Rightarrow \|b\| \leq \|A\| \|x\|$$

$$\frac{\|\delta x\|}{\|x\|} \leq \underbrace{\|A\| \|A^{-1}\|}_{K(A)} \frac{\|\delta b\|}{\|b\|}$$

$$K(A) = \|A\| \|A^{-1}\| \geq 1$$

$$K(I) = 1$$

$$K(A) = K(A^{-1})$$

$K(A) \simeq 1 \Rightarrow$ A is well conditionned;
Ax=b is stable

$K(A) >> 1 \Rightarrow$ A is ill conditionned
A x=b is unstable.

- $Ax = b \Rightarrow PAx = Pb$

Scanned with CamScanner