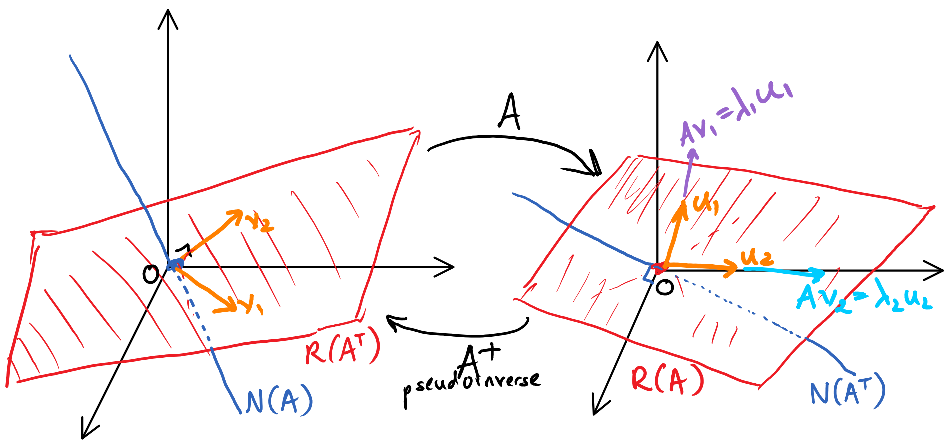
Lecture 19: Introduction to eigenvectors

Admin:

INTRODUCTION TO EIGENVECTORS



SVD $\Rightarrow$ linear transformations are simple

- map rowspace to columnspace

$$\vec{v}_j \longmapsto \sigma_j \vec{u}_j$$

orthonormal basis to orthogonal basis

But some matrices are even simpler!

Example: Diagonal matrices

$$A = \begin{pmatrix} 12 & & & 0 \\ & 10 & & \\ & & -1 & \\ 0 & & & 0 \end{pmatrix}$$

$e_1 \mapsto 12 e_1$
$e_2 \mapsto 10 e_2$
$e_3 \mapsto -e_3$
$e_4 \mapsto 0 e_4$

just scales basis vectors without a rotation!

Why this is cool?

rank = 3, norm = 12, s.v.'s = 12, 10, 1, 0

$$A^{500} = \begin{pmatrix} 12^{500} & & & 0 \\ & 10^{500} & & \\ & & (-1)^{500} & \\ 0 & & & 0 \end{pmatrix}$$

easy to take powers!

Example: Diagonalizable matrices

$$A = U \underbrace{\begin{pmatrix} 12 & & & 0 \\ & 10 & & \\ & & -1 & \\ 0 & & & 0 \end{pmatrix}}_{=D} U^{-1}$$

$$\begin{pmatrix} & & & \\ O & & -1 & \\ & & & 0 \end{pmatrix} = D$$

where $\mathcal{U} = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{pmatrix} = \sum_j u_j e_j^T$

$\mathcal{U}\vec{e}_1 = \vec{u}_1$

$\Rightarrow \mathcal{U}^{-1}\vec{u}_1 = \vec{e}_1$, so $A\vec{u}_1 = 12\vec{u}_1$

A just scales each $\vec{u}_j$, without changing directions!

$$A^{500} = \mathcal{U}D\underbrace{\mathcal{U}^{-1}\cdot\mathcal{U}}_{I}D\underbrace{\mathcal{U}^{-1}\cdot\mathcal{U}}_{I}D\mathcal{U}^{-1}\cdot \cdots$$

$$= \mathcal{U}\cdot D^{500}\cdot\mathcal{U}^{-1}$$

$$= \mathcal{U}\begin{pmatrix} 12^{500} & & & O \\ & 10^{500} & & \\ & & (-1)^{500} & \\ O & & & 0 \end{pmatrix}\mathcal{U}^{-1} \quad \text{easy!}$$

Note: Directions of other vectors definitely do change,

e.g.,
$$\underbrace{A(\vec{u}_1+\vec{u}_2)}_{= \,\mathcal{U}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}} = \underbrace{12\vec{u}_1 + 10\vec{u}_2}_{= \,\mathcal{U}\begin{pmatrix}12\\10\\0\\0\end{pmatrix}}$$

Only the special vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$, and their **multiples**, have directions unchanged. These are called **eigenvectors**.

Again, $\mathrm{rank}(A) = 3$.

If the $\{u_j\}$ are orthonormal,

then $\|A\| = 12$ and singular values are $12, 10, 1, 0$.

If the $\{u_j\}$ are not orthonormal, all bets are off!

Eg., $\mathcal{U} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathcal{U}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$$\mathcal{U}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathcal{U}^{-1}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{norm} = \sqrt{2}$$

More examples of eigenvectors & of diagonalizable matrices

- $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $A\vec{v}_1 = \vec{v}_1$ eigenvector with eigenvalue 1

$\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $A\vec{v}_2 = -\vec{v}_2$ eigenvalue -1

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

Generalization 1:

For any permutation matrix P, the all-ones vector is a $+1$ eigenvalue e-vector. (since permuting the coordinates leaves the all-ones vector alone).

Generalization 2:

For any square matrix whose rows all sum to 1 ("row-stochastic"), the all-ones vector is $+1$ ev ev.

$$A = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \qquad A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 + 1/2 \\ 1/3 + 1/3 + 1/3 \\ 1/4 + 1/4 + 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$A\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos\theta - i\sin\theta \\ \sin\theta + i\cos\theta \end{pmatrix} = \begin{pmatrix} e^{-i\theta} \\ ie^{-i\theta} \end{pmatrix} = e^{-i\theta}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

using $e^{i\theta} = \cos\theta + i\sin\theta$

$$A\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} c + is \\ s - ic \end{pmatrix} = \begin{pmatrix} c + is \\ -i(c + is) \end{pmatrix} = e^{i\theta}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\Rightarrow A = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

since $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ is unitary (col's are orthonormal)

so $U^{-1} = U^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$

Moral: Real matrices can have complex e-values & e-vectors.
In contrast, the SVD of a real-valued matrix works with real singular vectors.
Singular values are always (nonnegative) reals.
Eg., A's singular values are 1, 1.

- $A = \vec{u} \cdot \vec{v}^T$ for vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$

What is the rank of A?

1, provided $\vec{u}, \vec{v} \neq \vec{0}$

What is the nullspace N(A)? $\dim N(A) = n-1$

$$A\vec{x} = \vec{u}\vec{v}^T\vec{x} = \vec{u}(\vec{v} \cdot \vec{x})$$

$$A\vec{x} = 0 \iff \vec{v} \cdot \vec{x} = 0$$

ie. $x \perp v$

$$\vec{x} \in N(A) \Rightarrow A\vec{x} = \vec{0} = 0 \cdot \vec{x}$$

Moral: Every vector in the nullspace is an eigenvalue-zero eigenvector.

(Nullspace (if nonzero) is an eigenvalue-zero eigenspace)

Example:

$$A = \frac{1}{n}\begin{pmatrix} & & & \\ & \text{all-ones} & & \\ & \text{matrix} & & \\ & & & \end{pmatrix}_{n \times n} = \vec{u} \cdot \vec{u}^T \quad \text{for } \vec{u} = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\Rightarrow A\vec{u} = u u^T u$$
$$= \vec{u}(u \cdot u)$$
$$= \|u\|^2 \vec{u}$$
$$= \vec{u}$$

Moral: For the projection P_V onto a subspace V,

- V is a +1 eigenspace (for $x \in V$, $P_V x = x$)
- $V^\perp = N(P_V)$ is a 0 eigenspace (for $x \in V^\perp$, $P_V x = 0$)

- A slightly more complicated example:

Let $A = \sum_i \lambda_i u_i v_i^T$ (an SVD)

$\Rightarrow A^T A = \sum_i \lambda_i^2 v_i v_i^T$

it just scales each $\vec{v}_i$ by λ_i^2, leaves direction unchanged

$$\Rightarrow A^T A = \begin{pmatrix} | & | & | & | \\ | & v_i & | & | \\ | & | & | & | \end{pmatrix} \begin{pmatrix} \lambda_1^2 & & 0 \\ & \lambda_2^2 & \\ 0 & & \ddots \end{pmatrix} \begin{pmatrix} \text{———} \\ \text{———} \\ v_i^T \\ \text{———} \\ \text{———} \end{pmatrix}$$

• A matrix is diagonalizable $\Longleftrightarrow$ there is a basis of eigenvectors

$A = U \underset{\text{diagonal}}{D} U^{-1}$ (the columns of U)

From the examples so far, it seems that lots of matrices are diagonalizable.

But not all!!

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ SVD $A = e_1 \cdot e_2^T$

singular values $\lambda_1 = 1, \lambda_2 = 0$

rank 1, norm 1

good matrix to use in examples

Eigenvector $A\vec{e}_1 = \vec{0} = 0 \cdot \vec{e}_1$

eigenvalue 0

But there are no other (independent) eigenvectors!

$$A\vec{x} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \lambda x_2 = 0$$
$$\lambda x_1 = x_2$$

If $\lambda = 0$, then $x_2 = 0 \Rightarrow \vec{x} = x_1 \vec{e}_1$, which we already found

If $\lambda \neq 0$, again $x_2 = 0 \Rightarrow x_1 = 0 \Rightarrow \vec{x} = \vec{0}$ not an eigenvector

$\Rightarrow A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not diagonalizable!

It cannot be written $A = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^{-1}$

Alternative proof: Assume $A = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^{-1}$

Alternative proof: Assume $A = U\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}U^{-1}$

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}^2 = A^2 = (UDU^{-1})(UDU^{-1}) = UD^2U^{-1}$$

$$\begin{pmatrix}0&1\\0&0\end{pmatrix}^2 = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$\Rightarrow D^2 = U^{-1}\cdot 0 \cdot U = 0$$

$$\Rightarrow A = 0$$, a contradiction ✗

"SPECTRAL THEORY"
= THEORY OF EIGENVALUES & EIGENVECTORS

Spectral theory asks the following questions:

	Answers	
① What matrices have an eigenvector?	$\begin{pmatrix}0&1\\0&0\end{pmatrix}, \ldots$	all square matrices
② What matrices can be diagonalized? (That is, they have a full basis of eigenvectors.)	?	diagonalizable matrices
③ What matrices can be diagonalized with an **orthogonal** basis of eigenvectors?	?	"normal" matrices

Another natural question is when are the eigenvalues real? And, of course, how do you **find** eigenvalues & eigenvectors? We'll be studying these questions next.

But first, some applications:

Application: Solving linear recursions

Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, ...

$$f_n = f_{n-1} + f_{n-2}$$

$$f_1 = f_2 = 1$$

$$\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} f_{n-1} \\ f_{n-2} \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} f_3 \\ f_2 \end{pmatrix}$$

$$A^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} f_4 \\ f_3 \end{pmatrix}$$

$$\vdots$$

$$A^{500} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} f_{502} \\ f_{501} \end{pmatrix}$$

Let's find A's eigenvectors:

$$A\vec{x} = \lambda \vec{x} \iff \begin{aligned} x_1 + x_2 &= \lambda x_1 \\ x_1 &= \lambda x_2 \end{aligned}$$

(Note: We'll soon learn a better way...)

$$\Rightarrow (\lambda + 1) x_2 = \lambda^2 x_2$$

If $x_2 = 0$, then $\lambda = 1 \Rightarrow x_1 = 0$ no good!

$\Rightarrow$ $x_2 \neq 0$, so we can divide it out:

$$\Rightarrow \lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Let $\tau \doteq \frac{1+\sqrt{5}}{2}$ the "golden ratio", $\tau^{-1} = -\frac{1-\sqrt{5}}{2}$

$\Rightarrow$ Eigenvalues — Eigenvectors (from $x_1 = \lambda x_2$)

τ — $(\tau, 1)$

$-\tau^{-1}$ — $(-\tau^{-1}, 1)$

these are perpendicular!

$$A = \begin{pmatrix} \tau & -\tau^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & -\tau^{-1} \end{pmatrix} \begin{pmatrix} \tau & -\tau^{-1} \\ 1 & 1 \end{pmatrix}^{-1}$$

$$\Rightarrow A^{500} = \begin{pmatrix} \tau & -\tau^{-1} \\ \end{pmatrix} \begin{pmatrix} \tau^{500} & 0 \\ \end{pmatrix} \begin{pmatrix} \tau & -\tau^{-1} \\ \end{pmatrix}^{-1}$$

Multiply by $\begin{pmatrix}1\\1\end{pmatrix}$ to get out $\begin{pmatrix}f_{502}\\f_{501}\end{pmatrix}$!

Notice: $\tau \approx 1.61$

$-\tau^{-1} = 1-\tau \approx -.61$

$\Rightarrow \tau^{500} \gg |1-\tau|^{500}$

$\Rightarrow$ Asymptotically, the Fibonacci sequence grows like τ^n.

Application: Systems of differential equations

Warmup:

$$\frac{d}{dt}x(t) = x(t) \longrightarrow x(t) = \underset{\substack{\parallel \\ x(t=0)}}{C}e^t$$

What about

$$\dot{x} = 4x - 5y$$
$$\dot{y} = 2x - 3y$$

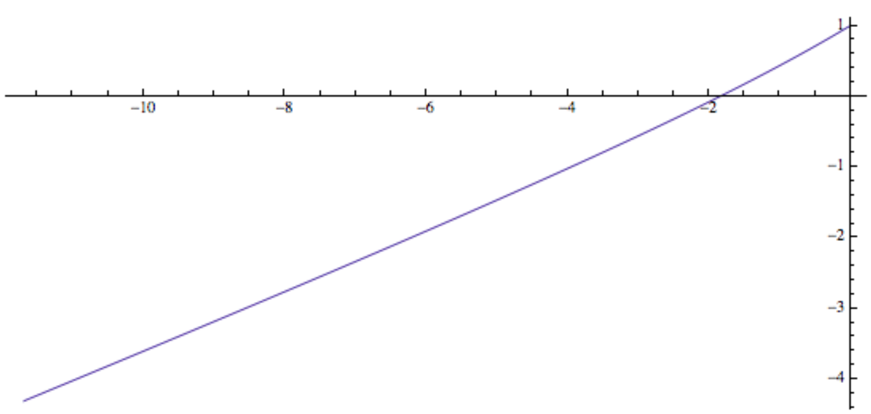
Numerical solution (Mathematica):

```
In[2]:= sol = NDSolve[{
          D[x[t], t] == 4 x[t] - 5 y[t],
          D[y[t], t] == 2 x[t] - 3 y[t],
          x[0] == 0,
          y[0] == 1
        }, {x, y}, {t, 0, 1}];
        ParametricPlot[{x[t], y[t]} /. sol, {t, 0, 1}]
```

```
In[1]:= ? NDSolve
```

NDSolve[*eqns*, *u*, {*t*, t_{min}, t_{max}}] finds a numerical solution to the ordinary differential equations *eqns* for the function *u* with the independent variable *t* in the range t_{min} to t_{max}.
NDSolve[*eqns*, *u*, {*t*, t_{min}, t_{max}}, {*x*, x_{min}, x_{max}}] finds a numerical solution to the partial differential equations *eqns*.
NDSolve[*eqns*, {u_1, u_2, ...}, {*t*, t_{min}, t_{max}}] finds numerical solutions for the functions u_i. >>

Out[3]=



Exact solution:

"Rule": When you see multiple equations, try to vectorize them!

vectorize them!

$$\dot{x} = 4x - 5y \qquad\qquad \dot{y} = 2x - 3y \longrightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Problem: Solve

$$\frac{d}{dt}\vec{v}(t) = A\,\vec{v}(t)$$

subject to $\vec{v}(t=0) = \vec{v}_0$ initial conditions

Answer:

① In terms of eigenvalues/eigenvectors:

- Say $A\vec{u}_1 = \lambda_1\vec{u}_1$
  and $\vec{v}_0 = \vec{u}_1$

  $\Rightarrow \vec{v}(t)$ is always proportional to $\vec{u}_1$
  $\vec{v}(t) = e^{\lambda t}\vec{u}_1$

  (so the time derivative is $e^{\lambda t}\cdot\lambda\vec{u}_1$
  $= A\cdot\vec{v}(t)$ ✓)

- Say $A\vec{u}_1 = \lambda_1\vec{u}_1$
  $A\vec{u}_2 = \lambda_2\vec{u}_2$
  and $\vec{v}_0 = \vec{u}_1 - 2\vec{u}_2$

  $\Rightarrow \vec{v}(t) = e^{\lambda_1 t}\vec{u}_1 - e^{\lambda_2 t}\cdot 2\vec{u}_2$

  (by linearity:
  if $\frac{d}{dt}\vec{\omega}(t) = A\vec{\omega}(t)$
  and $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$,
  then $\frac{d}{dt}\left(\vec{\omega}(t) + \vec{x}(t)\right) = A\left(\vec{\omega}(t) + \vec{x}(t)\right)$)

- If A is diagonalizable, then its eigenvectors form a basis, so any $\vec{v}_0$ can be expressed as a linear combination
  $\alpha_1\vec{u}_1 + \alpha_2\vec{u}_2 + \cdots + \alpha_n\vec{u}_n$
  of eigenvectors — and then they evolve independently
  $\vec{v}(t) = \alpha_1 e^{\lambda_1 t}\vec{u}_1 + \cdots + \alpha_n e^{\lambda_n t}\vec{u}_n$

If A can be diagonalized,

$$A = U D U^{-1}$$

$$\Rightarrow \frac{d}{dt} v = U D U^{-1} v$$

$$\Rightarrow U^{-1} \cdot \frac{d}{dt} v = \frac{d}{dt}(U^{-1} v) = D(U^{-1} v)$$

Change variables: $u = U^{-1} v$, $v = U u$,

$$\Rightarrow \frac{d}{dt} u = D u$$

$$\begin{pmatrix} \frac{d}{dt} u_1 \\ \frac{d}{dt} u_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\Rightarrow u_1 = u_1(0) e^{\lambda_1 t}$$

$$u_2 = u_2(0) e^{\lambda_2 (t)}$$

$$\Rightarrow v = U \cdot \begin{pmatrix} u_1(0)\, e^{\lambda_1 t} \\ u_2(0)\, e^{\lambda_2 t} \end{pmatrix}$$

The eigenvalues of A, λ_1 and λ_2, determine the exponents (growth rates).

We can actually simplify even further:

$$v(t) = U \cdot \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} u(0)$$

$$= U \cdot \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} U^{-1} v(0)$$

② In terms of matrices:

$$v(t) = e^{At} \vec{v}_0$$

$$\parallel$$

$$I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} (At)^j$$

Proof: $v(0) = e^{A \cdot 0} \vec{v}_0$

$$= I \vec{v}_0$$

$$= \vec{v}_0 \quad \checkmark$$

$$\bullet \; \frac{d}{dt}\left(e^{At}\vec{v}_0\right) = \left(\frac{d}{dt} e^{At}\right)\vec{v}_0$$
$$= \left(\sum_{j=0}^{\infty} \frac{1}{j!} \cdot \frac{d}{dt}\left(A^j t^j\right)\right)\vec{v}_0$$
$$= \left(\sum_{j=1}^{\infty} \frac{1}{(j-1)!} A^j t^{j-1}\right)\vec{v}_0$$
$$= A\left(\sum_{j=0}^{\infty} \frac{1}{j!}(At)^j\right)\vec{v}_0$$
$$= A\vec{v}(t) \quad \checkmark$$

Functions of diagonalizable matrices:

Definition: If A is a diagonalizable matrix

$$A = U \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} U^{-1}$$

and $f: \mathbb{C} \to \mathbb{C}$ any function, define

$$f(A) = U \begin{pmatrix} f(\lambda_1) & & & 0 \\ & f(\lambda_2) & & \\ & & \ddots & \\ 0 & & & f(\lambda_n) \end{pmatrix} U^{-1}.$$

Example: $f(x) = x^2$

$\Rightarrow f(A) = U D^2 U^{-1} = A^2$, as you'd expect ✓

Example: $f(x) = e^{tx}$ exponential

$$\Rightarrow f(A) = U \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} U^{-1}$$

So, $$v(t) = \underbrace{U \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} U^{-1}}_{e^{At}} v(0)$$

$\Rightarrow$ The solution to

$$\frac{d}{dt} v(t) = A\, v(t)$$

is $v(t) = e^{At} v(0)$.

This looks just like the single-variable case.

You can also show this using Taylor series:

$$e^{xt} = 1 + xt + \frac{(xt)^2}{2} + \cdots$$
$$= \sum_{j=0}^{\infty} \frac{(xt)^j}{j!}$$

and $$e^{At} = \sum_{j=0}^{\infty} \frac{(At)^j}{j!}$$ where $A^0 = \underset{\text{identity matrix}}{I}$.

Extension: Higher-order differential equations with constant coefficients.

To solve a 2^{nd}-order diff. eq. like

$$\ddot{x} = -\dot{y} + 2x$$
$$\ddot{y} = \dot{x} + \dot{y} - 3y$$

use the same trick we used for solving the Fibonacci sequence recursion:

$$\frac{d}{dt}\begin{pmatrix}\dot{x}\\ \dot{y}\\ x\\ y\end{pmatrix} = \begin{pmatrix} 0 & -1 & 2 & 0 \\ 1 & 1 & 0 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ x\\ y\end{pmatrix}$$

Then continue as before.

More applications we'll consider later:

- Spectral graph partitioning
- Markov chains
- Google Pagerank

⋮

WHAT MATRICES HAVE AN EIGENVECTOR?

Theorem: Every (real or complex) square matrix has at least one eigenvector (over $\mathbb{C}$!).

Proof: Let A be an $n \times n$ matrix, real or complex.

Goal: Prove that there exists a vector $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \lambda\vec{x}$ for some λ.

Recall: Factorizations

① If $abc = 0$, then $a=0$ or $b=0$ or $c=0$

Recall: Factorizations

① If $abc = 0$, then $a = 0$ or $b = 0$ or $c = 0$

For matrices: If $ABC\vec{v} = \vec{0}$, $\vec{v} \neq \vec{0}$,
then $\vec{v} \in N(C)$
or $C\vec{v} \in N(B)$ and $C\vec{v} \neq \vec{0}$
or $BC\vec{v} \in N(A)$ and $BC\vec{v} \neq \vec{0}$

$\Rightarrow$ Either A, B or C is singular
(the product of invertible matrices is invertible).

② "Fundamental theorem of algebra":
Any polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$,
with real or complex coefficients, can be factored as

$$p(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$$

for some complex roots $\lambda_j \in \mathbb{C}$, $p(\lambda_j) = 0$

Step 1. Let $\vec{v}$ be any nonzero vector.
Consider the vectors

$$\vec{v},\ A\vec{v},\ A^2\vec{v},\ \ldots,\ A^n\vec{v}$$

Since there are $n+1$ vectors in an n-dimensional space,
they must be linearly dependent.
(Note: This argument only works in finite dimensions.)
Say

$$\alpha_0 v + \alpha_1 A v + \alpha_2 A^2 v + \cdots + \alpha_n A^n v = 0$$

with not all α_j's 0.

Step 2. Thus

$$\left(\sum_{j=0}^{n} \alpha_j A^j\right)\vec{v} = 0$$

Factor the polynomial

$$p(x) = \sum_j \alpha_j x^j = \prod_j (x-\lambda_j)$$

$$\Rightarrow \left[\prod_j (A-\lambda_j I)\right]\vec{v} = 0$$

Step 3.
$\Rightarrow$ at least one of the matrix terms $A-\lambda_j I$
must be singular

(the product of two nonsingular matrices is nonsingular!)

✓ Done. $\square$

Note: The first step can be "justified" by noticing that $\lim_{k\to\infty} \frac{A^k v}{\|A^k v\|}$ converges to an eigenvector with largest magnitude eigenvalue (though I don't want to prove this). So it makes sense to look at successive powers $A^k v$.

Note: There's a simple proof that $\mathbb{C}$ is algebraically closed in the appendix of Lang's "Linear Algebra."

More examples

Eigenvalues

$$\begin{pmatrix} 1+2\cos\theta & -2\sin\theta \\ 2\sin\theta & 1+2\cos\theta \end{pmatrix} = I + 2\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \qquad 1+2e^{i\theta},\ 1+2e^{-i\theta}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = I + 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad 1+2\cdot 1 = 3,\ 1+2\cdot(-1) = -1$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 0 \\ \hline 0 & 0 & 3 \end{array}\right) \qquad \underset{\substack{\uparrow \\ \text{dim. 2} \\ \text{e-space}}}{3},\ -1$$

$$\left(\begin{array}{c|c|c} 1 & 0 & 2 \\ \hline 0 & 3 & 0 \\ \hline 2 & 0 & 1 \end{array}\right) \qquad \underset{(\text{multiplicity } 2)}{3},\ -1$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 3 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ is e-vector with e-value } 3$$

but the other e-vectors are hard to find! (even though the SVD is easy)

$$\begin{pmatrix} \bigcirc & \begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix} \\ \begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix} & \bigcirc \end{pmatrix}$$

try these in Matlab,

$$\begin{pmatrix} & 2\;1 \\ 1\;2 & \\ 2\;1 & O \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 1 & 4 & 2 \\ 2 & 4 & 1 & 2 \\ 4 & 2 & 2 & 1 \end{pmatrix}$$

try these in Matlab, see if you can find the pattern...

Outline:

Motivation