Homework 3

Note: For full credit, show your work!
You are welcome to discuss the problems with others, but write up your own solutions.

1. Which of the following sets span R^3?
   (a) $\{(1,1,1)\}$
   (b) $\{(1,0,0), (0,0,1), (1,0,1)\}$
   (c) $\{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$
   (d) $\{(1,2,1), (2,0,-1), (4,4,1)\}$
   (e) $\{(1,2,1), (2,0,-1), (4,4,0)\}$

2. What is the span of the union
   $\{3\times 3 \text{ symmetric matrices, i.e. } A=A^T\} \cup \{3\times 3 \text{ upper-triangular matrices}\}$?

3. Let $S=\{s_1,\dots,s_r\}$, $T=\{s_1,\dots,s_r,s_{r+1}\}$ be two sets of vectors from the same vector space.
   When is $\mathrm{Span}(S)=\mathrm{Span}(T)$?

4. Find the rowspace, columnspace, nullspace and left nullspace for each of the following matrices. Express each answer as the span of a minimal set of vectors.
   (The left nullspace of a matrix A is $N(A^T)$.)

   (a) $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{pmatrix}$

   (d) (1 −4 4

$$\begin{pmatrix} \vdots \\ 1 \; 2 \; 7 \end{pmatrix}$$

(d) $\begin{pmatrix} 1 & -4 & 4 \\ 4 & -8 & 6 \\ 0 & -4 & 5 \end{pmatrix}$

(5) Construct a 3×3 matrix A with
$$R(A) = R(A^T) = \text{Span}\left(\{(1,0,1), (1,5,0)\}\right)$$

(6) Which of the row, column, null and left-null subspaces are the same for these matrices of different sizes?

(a) A and $\begin{bmatrix} A \\ A \end{bmatrix}$

(b) $\begin{bmatrix} A \\ A \end{bmatrix}$ and $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$

(7) Explain why a nonzero row of a matrix A cannot be in its nullspace.

(8) Matrix multiplication and the nullspace

(a) Prove that

(*) $$N(A) \subseteq N(BA)$$

for any matrices A and B so BA is defined.

(b) Prove that if B is invertible, then
$$N(BA) = N(A)$$

(c) Prove that
$$N(A^TA) = N(A).$$

(Hint: Equation (*) implies that $N(A^TA) \supseteq N(A)$, so it is enough to show the opposite inclusion, $N(A^TA) \subseteq N(A)$. Note that a vector z is zero if and only if $z^Tz = \sum_i (z_i)^2 = 0$.)

(d) [Optional]

Assuming that $R(B^T) \supseteq R(A)$, prove that
$$N(BA) = N(A).$$

Note that this implies parts (b) and (c) as special cases.

$N(BA) = N(A)$.

Note that this implies parts ⓑ and ⓒ as special cases.

(Hint: This is slightly harder than ⓒ, but almost the same proof should work. Start by applying a row transformation to B.)

As a partial converse, prove that if $R(B^T) \subset R(A)$ strictly, then $N(BA) \supset N(A)$ strictly.

ⓔ Give an example of matrices A, B where $R(B^T) \not\supseteq R(A)$ and yet $N(BA) = N(A)$.

(Hint: The smallest example is with A a 1×2 matrix and B a 2×1 matrix.)

⑨ Finite fields

Use Gaussian elimination, by hand, to solve the following system of equations, mod 2:

$$\begin{aligned} x_1 + x_2 + x_3 \phantom{{}+x_4} &= 0 \\ x_2 + x_3 + x_4 &= 1 \\ x_1 \phantom{{}+x_2} + x_3 \phantom{{}+x_4} &= 1 \\ x_1 + x_2 \phantom{{}+x_3+x_4} &= 1 \end{aligned} \qquad (\text{mod } 2)$$

⑩ Finite fields

ⓐ Show that the polynomial x^4+1 is reducible (can be factored) mod 2, mod 3, and mod 5.

(In fact, it is reducible over any finite field.)

Hint: Feel free to use Mathematica, as in class.

If you haven't installed it, try https://sandbox.open.wolframcloud.com/

https://www.wolframalpha.com/

ⓑ The polynomial x^3+x+1 is irreducible mod 2, so gives a field of size $2^3 = 8$. (The field elements are $0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1$.) Write out the multiplication table.

What are the inverses of the nonzero elements?

$(1^{-1} = ?,\ x^{-1} = ?,\ (x+1)^{-1} = ?,\ \ldots,\ (x^2+x+1)^{-1} = ?)$

(c) Recall from class the field $\mathbb{F}_4$ with elements $0, 1, x, x+1$ and operations mod 2 and mod x^2+x+1.

Here is the multiplication table:

$\times$	0	1	x	$x+1$
0	0	0	0	0
1	0	1	x	$x+1$
x	0	x	$x+1$	1
$x+1$	0	$x+1$	1	x

Write out all the elements of

$$\mathrm{Span}\left(\begin{pmatrix}1\\x\\x+1\end{pmatrix}, \begin{pmatrix}x\\x\\x\end{pmatrix}\right).$$

(Your answer should have 16 vectors, a subspace of $\mathbb{F}_4^3$.)

(d) Over the same field $\mathbb{F}_4$, solve for the nullspace of

$$\begin{pmatrix} 1 & x & x & 1 & 0 \\ 0 & 1 & x & x & 1 \\ 1 & 0 & 1 & x & x \\ x & 1 & 0 & 1 & x \\ x & x & 1 & 0 & 1 \end{pmatrix}$$

(11)

a) By discretizing the interval and setting up a system of linear equations, solve numerically the differential equation

$$f''(t) - 2f'(t) = \cos(t)$$

on the interval $[0, \pi]$, with boundary conditions

$$f(0) = 0 \quad , \quad f(\pi) = 0$$

After setting up the sparse equations, solve them.
Experiment with finer discretizations.
Show your work, and plot the results

b) Do the same, except with boundary conditions

$$f(0) = 0 \quad , \quad f(\pi) = -1$$