Lecture 11: Orthogonal bases

Admin:

Pairwise Orthogonal Sets of Vectors

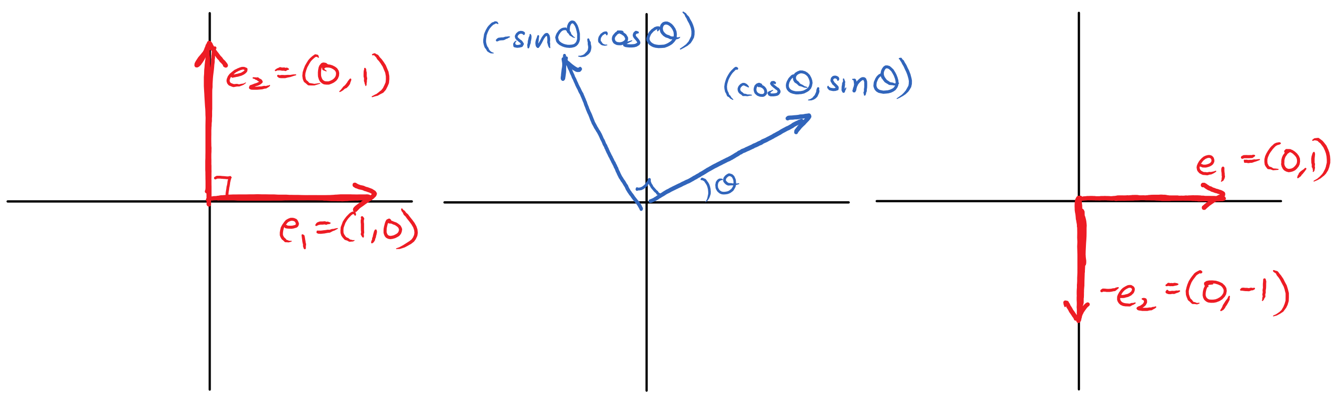
Examples:

1. Standard basis in $\mathbb{R}^n$

$$\vec{e}_1 = (1, 0, \dots, 0), \dots, \vec{e}_n = (0, \dots, 0, 1)$$

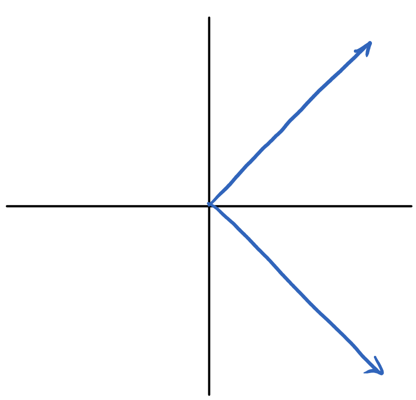
$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

2. Rotations/reflections of the standard basis



3. Hadamard basis

$$\frac{1}{\sqrt{2}}(1,1) \quad , \quad \frac{1}{\sqrt{2}}(1,-1)$$



= Fourier basis mod 2

(foundation for Fourier analysis of boolean functions)

change-of-basis matrix $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

change-of-basis matrix $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Hadamard basis in dimension 4:

$$H_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

* all entries either $\pm\frac{1}{\sqrt{n}}$

columns are orthogonal pairwise

Applications: Coding, control theory, statistics, quantum computing, Fourier analysis of boolean functions

In dimensions $n = 2^k$:

$$H_{2^k} = \frac{1}{\sqrt{2}}\begin{pmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{pmatrix}$$

Conjecture (1936): If n is a multiple of 4, then there is a Hadamard basis for $\mathbb{R}^n$.

Open: Find a Hadamard basis for $n = 668$.

④ Haar wavelet basis

$(1, 1, 1, 1)$
$(1, 1, -1, -1)$
$(1, 1, 0, 0)$
$(0, 0, 1, -1)$

for $\mathbb{R}^4$

$(1, 1, 1, 1, 1, 1, 1, 1)$
$(1, 1, 1, 1, -1, -1, -1, -1)$
$(1, 1, -1, -1, 0, 0, 0, 0)$
$(1, -1, 0, 0, 0, 0, 0, 0)$
$(0, 0, 1, -1, 0, 0, 0, 0)$
$(0, 0, 0, 0, 1, 1, -1, -1)$
$(0, 0, 0, 0, 1, -1, 0, 0)$
$(0, 0, 0, 0, 0, 0, 1, -1)$

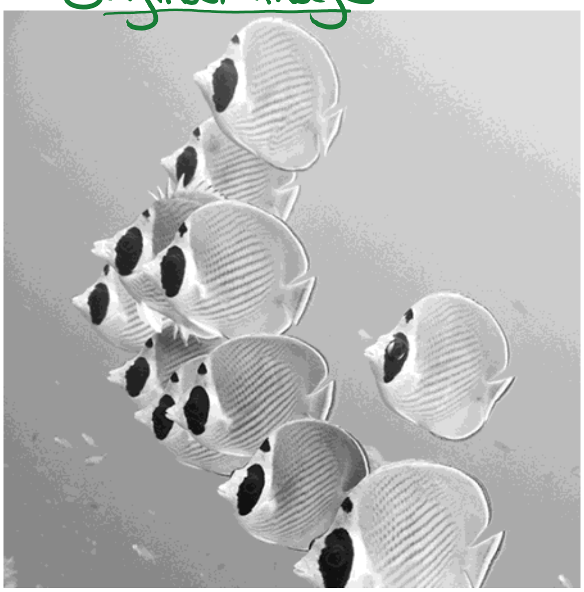
for $\mathbb{R}^8$

- used in image compression

a vector in $\mathbb{R}^8$ with few jumps will have a sparse representation in this basis, eg.,

$$(1,1,1,0,-2,-2,-2,-2)$$
$$= -\frac{5}{8}(1,1,1,1,1,1,1,1)$$
$$+\frac{11}{8}(1,1,1,1,-1,-1,-1,-1)$$
$$+\frac{1}{4}(1,1,-1,-1,0,0,0,0)$$
$$+\frac{1}{2}(0,0,1,-1,0,0,0,0)$$

Original image



... after discarding 0.9n smallest-magnitude Haar basis coordinates

⑤ Fourier basis for $\mathbb{C}^n$

-used everywhere!

$(\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{n-1})$

where $\vec{v}_j = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \exp\left(\frac{2\pi i}{n} \cdot jk\right) \vec{e}_k$

$$[I]_{\substack{\text{Fourier}\\ \to \text{std basis}}} = \frac{1}{\sqrt{n}} \begin{array}{c} \\ \vec{e}_0 \\ \vec{e}_1 \\ \\ \\ \\ \vec{e}_{n-1} \end{array} \begin{array}{ccccc} \vec{v}_0 & \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \\ \end{array}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & \omega & \omega^2 & \omega^3 & \cdots \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \omega^{3(n-1)} & \cdots \end{pmatrix}$$

where $\omega = \exp\left(\frac{2\pi i}{n}\right)$

$$= \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

$$\Rightarrow \vec{v}_j \cdot \vec{v}_{j'} = \frac{1}{n}\sum_{k,k'=0}^{n-1} \exp\left(\frac{2\pi i}{n}(-jk + j'k')\right)(\underbrace{\vec{e}_k \cdot \vec{e}_{k'}}_{\delta_{kk'}})$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} \exp\left(\frac{2\pi i}{n} k(j'-j)\right)$$

case $j = j'$: $\vec{v}_j \cdot \vec{v}_j = \frac{1}{n}(1+1+\cdots+1) = 1$

case $j \neq j'$:

$$\vec{v}_j \cdot \vec{v}_{j'} = \frac{1}{n} \cdot \left(\begin{array}{l}\text{geometric series} \\ 1 + \omega^{j'-j} + \omega^{2(j'-j)} + \cdots + \omega^{(n-1)(j'-j)}\end{array}\right)$$

where $\omega = e^{\frac{2\pi i}{n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}$$

$= 0$ since $\omega^n = 1$

⑥ Various bases for functions

e.g., Hermite, Laguerre, Chebyshev

sines and cosines, ...

There are lots of named examples because

Pairwise Orthogonal Sets of Vectors Are Nice!!

$\{\vec{v}_1, \ldots, \vec{v}_n\}$

with $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$

① Simplifies deciding linear independence

Normally, to check if $S = \{v_1, \ldots, v_n\}$ is lin. indep., compute

$$N\left(\left(v_1 \middle| \cdots \middle| v_n\right)\right) \stackrel{?}{=} \{0\}.$$

Lemma:

$$\boxed{\begin{array}{l} S = \{v_1, v_2, \dots, v_n\} \\ \text{non zero, pairwise orthogonal} \\ \qquad v_i \cdot v_j = 0 \text{ for } i \neq j \end{array}} \Rightarrow \boxed{\text{S is linearly independent}}$$

Proof: Assume $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n = 0$.

$$\Rightarrow \alpha_1 (\vec{v}_i \cdot v_1) + \alpha_2 (\vec{v}_i \cdot \vec{v}_2) + \cdots + \alpha_n (\vec{v}_i \cdot \vec{v}_n) = 0$$

$$\| \quad \alpha_i \vec{v}_i \cdot \vec{v}_i = \alpha_i \|v_i\|^2$$

$\Rightarrow \alpha_i = 0$ (since $v_i \neq 0$)

for all i □

② Pairwise $\perp$ity simplifies deciding if S is a basis

Corollary: In an n-dimensional space, any set of n pairwise orthogonal vectors forms a basis.

(because any n linearly indep. vectors is a basis)

Definition: Orthogonal basis = basis of pairwise orthogonal vectors

Orthonormal basis = basis of orthogonal, length-one vectors

Example: The standard basis $\vec{e}_1, \dots, \vec{e}_n$ is orthonormal.

Dividing by the lengths, orthogonal $\to$ orthonormal basis.

③ Simplifies computing inner products & lengths

Let $\vec{u} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_m \vec{v}_m$

Q: What is $\|u\|$?

A: $\|\vec{u}\|^2 = \vec{u}\cdot\vec{u}$

$$= \left(\sum_i \alpha_i \vec{v}_i\right)\cdot\left(\sum_j \alpha_j \vec{v}_j\right)$$

$$= \sum_{i,j=1}^{n} \alpha_i^* \alpha_j \left(\vec{v}_i \cdot \vec{v}_j\right)$$

n^2 terms to sum up!

But if $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is orthogonal,

$$\|\vec{u}\|^2 = \sum_{i=1}^{n} |\alpha_i|^2 \cdot \|\vec{v}_i\|^2$$

since $\vec{v}_i \cdot \vec{v}_j$ terms are otherwise 0,

And if $\{\vec{x}_1, \ldots, \vec{v}_n\}$ is orthonormal,

$$\boxed{\|u\|^2 = \sum_{i=1}^{n} |\alpha_i|^2}$$

the exact same expression as works in the standard basis

MORAL: Orthonormal bases behave just like the standard basis.

e.g., for $\vec{u} = \sum_j \alpha_j \vec{v}_j$, $\vec{v} = \sum_j \beta_j \vec{v}_j$,

$$\vec{u}\cdot\vec{v} = \sum_j \alpha_j^* \beta_j$$

④ Simplifies basis expansions

Let $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$ be a basis for $V \subseteq \mathbb{R}^n$, and $\vec{u} \in V$.

What is the expansion of $\vec{u}$ in the basis $\mathcal{B}$?

In general: Write

$$\left(\vec{u}\right) = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{pmatrix} \begin{pmatrix} \vec{x} \end{pmatrix}$$

and solve these equations.

If $\mathcal{B}$ is orthonormal:

$$\vec{u} = x_1\vec{v}_1 + \cdots + x_j\vec{v}_j + \cdots + x_m\vec{v}_m$$

$$\Downarrow$$

$$\vec{v}_j\cdot\vec{u} = \vec{v}_j\cdot\left(x_1\vec{v}_1 + \cdots + x_j\vec{v}_j + \cdots + x_m\vec{v}_m\right)$$

$$= 0 + 0 + \cdots + x_j + \cdots + 0$$

$$\Rightarrow \boxed{\vec{u} = \sum_{j=1}^{m} \underbrace{(\vec{v}_j \cdot \vec{u})}_{} \vec{v}_j} \quad (\vec{v}_j \cdot \vec{u}) = x_j$$

- just compute these inner products
- needn't solve any equations!

Example: Haar wavelet basis for $\mathbb{R}^8$

v_1 $(1, 1, 1, 1, 1, 1, 1, 1)$
v_2 $(1, 1, 1, 1, -1, -1, -1, -1)$
v_3 $(1, 1, -1, -1, 0, 0, 0, 0)$
v_4 $(1, -1, 0, 0, 0, 0, 0, 0)$
v_5 $(0, 0, 1, -1, 0, 0, 0, 0)$
v_6 $(0, 0, 0, 0, 1, 1, -1, -1)$
v_7 $(0, 0, 0, 0, 1, -1, 0, 0)$
v_8 $(0, 0, 0, 0, 0, 0, 1, -1)$

Pairwise $\perp$
$\Downarrow$
Linearly independent
$\Downarrow$
Must be a basis!

Exercise: Expand $\overset{u}{(1, 1, 1, 0, -2, -2, -2, -2)}$ in the above basis.

Answer:
$v_1 \cdot u = -5$
$v_2 \cdot u = 11$
$v_3 \cdot u = 1$
$v_4 \cdot u = 0$
$v_5 \cdot u = 1$
$\vdots$

$$\Rightarrow \vec{u} = \sum_j \frac{(\vec{v}_j \cdot \vec{u}) \vec{v}_j}{\|v_j\|^2}$$

$$= -\frac{5}{8}\vec{v}_1 + \frac{11}{8} v_2 + \frac{1}{4} v_3 + \frac{1}{2} v_5 + \cdots$$

We could compute $v_6 \cdot u$, $v_7 \cdot u$, $v_8 \cdot u$ — and its easy — but in fact, observe

$\|u\|^2 = 19$

and
$$\frac{|v_1 \cdot u|^2}{\|v_1\|^2} + \underbrace{\frac{|v_2 \cdot u|^2}{\|v_2\|^2}}_{\text{squared coeff. of } v_2/\|v_2\|} + \frac{|v_3 \cdot u|^2}{\|v_3\|^2} + \frac{|v_5 \cdot u|^2}{\|v_5\|^2}$$

$$= \frac{25}{8} + \frac{121}{8} + \frac{1}{4} + \frac{1}{2}$$

$\Rightarrow$ all other coefficients **must** be 0, since all of $\vec{u}$'s length is accounted for! ($= 19$)
(This is a common trick, to save time.)

④ Simplifies matrix basis expansions

$f: \mathcal{U} \to \mathcal{V}$

$B_{\mathcal{U}} = \{\vec{u}_1, \ldots, \vec{u}_n\}$

$B_{\mathcal{V}} = \{\vec{v}_1, \ldots, \vec{v}_m\}$ orthonormal basis

$$\Rightarrow [f]_{B_{\mathcal{U}} \to B_{\mathcal{V}}} = \begin{array}{c c} & j \\ i & \begin{pmatrix} & \vdots & \\ \cdots & \vec{v}_i \cdot f(\vec{u}_j) & \\ & & \end{pmatrix} \end{array}$$

since $f(\vec{u}_j) = \sum_{i=1}^{m} \left(\vec{v}_i \cdot f(\vec{u}_j)\right) \vec{v}_i$

⑤ Simplifies changing basis

Recall: $f: \mathcal{U} \to \mathcal{V}$ linear

$\mathcal{U}$: bases $B_{\mathcal{U}}, B'_{\mathcal{U}}$ $\quad$ $\mathcal{V}$: bases $B_{\mathcal{V}}, B'_{\mathcal{V}}$

$$[f]_{B'_{\mathcal{U}} \to B'_{\mathcal{V}}} = [I]_{B_{\mathcal{V}} \to B'_{\mathcal{V}}} \, [f]_{B_{\mathcal{U}} \to B_{\mathcal{V}}} \, [I]_{B'_{\mathcal{U}} \to B_{\mathcal{U}}}$$

$$\underset{B'_{\mathcal{V}}}{\mathcal{V}} \xleftarrow{I_{\mathcal{V}}} \underset{B_{\mathcal{V}}}{\mathcal{V}} \xleftarrow{f} \underset{B_{\mathcal{U}}}{\mathcal{U}} \xleftarrow{I_{\mathcal{U}}} \underset{B'_{\mathcal{U}}}{\mathcal{U}}$$

if $\mathcal{U} = \mathcal{V}$,

$$[f]_{B' \to B'} = [I]_{B \to B'} [f]_{B \to B} [I]_{B' \to B}$$

$$[I]_{B \to B'} = \left([I]_{B' \to B}\right)^{-1}$$

Example:

Consider the 2×2 complex matrix

$$A = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Write this matrix in the basis $\left\{\frac{1}{\sqrt{2}}(1,i), \frac{1}{\sqrt{2}}(1,-i)\right\}$.

Answer: Initial basis $B = \{(1,0),(0,1)\}$.

New basis $C = \left\{\frac{1}{\sqrt{2}}(1,i), \frac{1}{\sqrt{2}}(1,-i)\right\}$.

Observe: C is orthonormal!

$$\left\|\tfrac{1}{\sqrt{2}}(1,\pm i)\right\|^2 = \tfrac{1}{2}\left\|(1,\pm i)\right\|^2$$
$$= \tfrac{1}{2}\left(1^2 + |\pm i|^2\right) = 1 \checkmark$$

$$\tfrac{1}{\sqrt{2}}(1,i)\cdot\tfrac{1}{\sqrt{2}}(1,-i) = \tfrac{1}{2}\left(1 + i^*(-i)\right)$$
$$= \tfrac{1}{2}\left(1 + i^2\right) = 0 \checkmark$$

$C \to B$ basis change:

$$\begin{array}{c|cc} & \frac{1}{\sqrt{2}}(1,i) & \frac{1}{\sqrt{2}}(1,-i) \\ \hline \begin{array}{c}(1,0)\\(0,1)\end{array} & \multicolumn{2}{c}{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}} \end{array} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$B \to C$ basis change

$$\begin{array}{c|cc} & (1,0) & (0,1) \\ \hline \begin{array}{c}\frac{1}{\sqrt{2}}(1,i)\\ \frac{1}{\sqrt{2}}(1,-i)\end{array} & \multicolumn{2}{c}{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}} \end{array} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

since, eg, $\frac{1}{\sqrt{2}}(1,i)\cdot(0,1) = \frac{-i}{\sqrt{2}}$

<u>Check</u>: $(1,0) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(1,i) + \frac{1}{\sqrt{2}}(1,-i)\right) \checkmark$

$(0,1) = \frac{1}{\sqrt{2}}\left(\frac{-i}{\sqrt{2}}(1,i) + \frac{i}{\sqrt{2}}(1,-i)\right) \checkmark$

$$\Rightarrow [A]_{C\to C} = [B\to C]\cdot[A]_{B\to B}\cdot[C\to B]$$

(read this right to left)

$$= \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

In[1]:= $\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ `// FullSimplify // MatrixForm`

Out[1]//MatrixForm=

$$= \boxed{\begin{pmatrix} \text{Cos}[\theta] & i\,\text{Sin}[\theta] \\ i\,\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}}$$

(Mathematica)

Out[1]//MatrixForm=

$$= \boxed{\begin{pmatrix} \text{Cos}[\theta] & \text{i Sin}[\theta] \\ \text{i Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}}$$

(Mathematica)

Observe:
1. This was really easy!
2. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   — they're inverses
3. They differ only by transpose and complex conjugate.

In general,

If $\mathcal{B}$, C are each orthonormal:

$$\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}, \quad C = \{\vec{c}_1, \ldots, \vec{c}_n\}.$$

$$[I]_{C\to\mathcal{B}} = \begin{array}{c} \\ i \end{array}\begin{pmatrix} & \vec{c}_j & \\ \cdots & \boxed{\vec{b}_i \cdot \vec{c}_j} & \\ & & \end{pmatrix} \quad \text{since } \vec{c}_j = \sum_{i=1}^{n} (b_i \cdot c_j)\vec{b}_i$$

$$[I]_{\mathcal{B}\to C} = \begin{array}{c} \\ j \end{array}\begin{pmatrix} & \vec{b}_i & \\ \cdots & \boxed{\vec{c}_j \cdot \vec{b}_i} & \\ & & \end{pmatrix} \quad \text{since } \vec{b}_i = \sum_{j=1}^{n} (c_j \cdot b_i)\vec{c}_j$$

$$\Rightarrow \boxed{[I]_{\mathcal{B}\to C} = \left([I]_{C\to\mathcal{B}}\right)^{\dagger}}$$

"adjoint"
= transpose
+ complex conjugate
(same as transpose for real matrices)

⑦ Simplifies computing projections

Problem: How can we project onto a higher-dimensional subspace?

Example: $P_{xy\text{-plane}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Example: $P_{xy\text{-plane}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

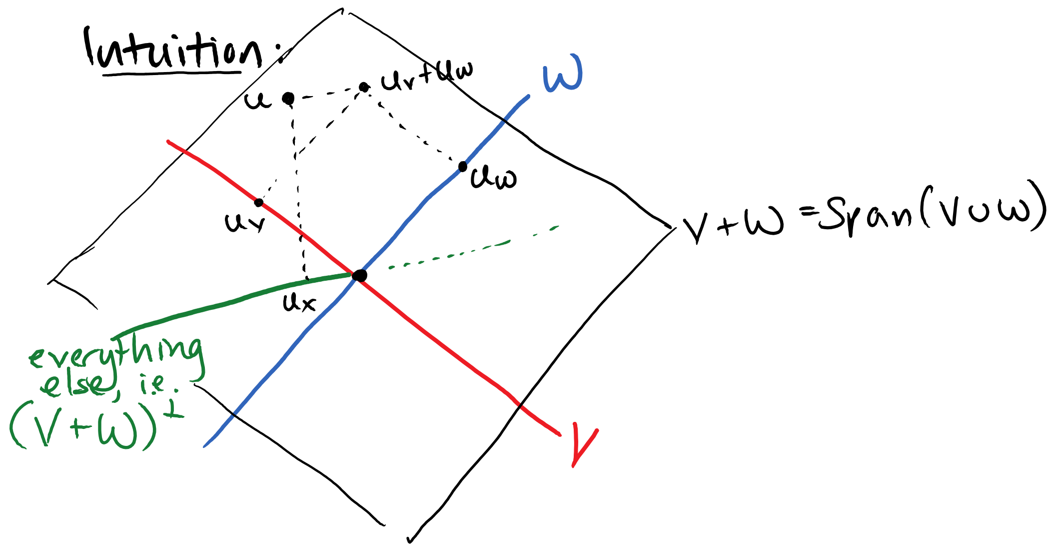
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P_{x\text{-axis}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(1\ 0\ 0) \qquad P_{y\text{-axis}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(0\ 1\ 0)$$

Key property: If $V \perp W$,

$$\boxed{P_V + P_W = P_{V+W}}$$

Intuition:



$\Rightarrow$ To project into any space:
"Split" it into orthogonal lines,
and add up those projections

FACT: For a subspace $\mathcal{U} \subseteq \mathbb{R}^n$ with orthonormal basis
$\{\vec{u}_1, \dots, \vec{u}_k\}$,
orthogonal projection onto $\mathcal{U}$

$$P_{\mathcal{U}} = \boxed{\sum_{j=1}^{k} \underbrace{\vec{u}_j\, \vec{u}_j^{\,T}}_{n \times n \text{ matrix}}}$$

Proof: $P_{\mathcal{U}} = P_{\mathrm{span}\{u_1,\ldots,u_{k-1}\}} + P_{\mathrm{span}\{u_k\}}$

$$P_{\mathrm{span}\{u_k\}} = u_k u_k^T$$

Exercise: Compute the projection of $e_1 = (1,0,0,\ldots)$ onto a random 50-dimensional subspace of $\mathbb{R}^{100}$.

Answer:

```
n = 100;
d = 50;

v = zeros(n,1);          <- the vector e1
v(1) = 1;

A = randn(n, d);     # choose d random vectors in R^n,
                         #  with normally distributed coordinates
[Q,R] = qr(A,0);     # generate Q, whose columns form an
                         #  orthonormal basis for R(A),
                         #  the span of A's columns
```

we'll explain this function later

$Q(:,1), Q(:,2), \ldots, Q(:,d)$ form an orthonormal basis for $R(A)$

① First approach

build sum $\sum_{i=1}^{d} q_i q_i^T v$ one term at a time

```
projectedv = zeros(n,1);
for i = 1:d
    projectedv += Q(:,i) * (Q(:,i)' * v);
end for;
projectedv
```

② Second try: Note: $Q = \sum_{i=1}^{d} q_i e_i^T$ where $q_i = Q(:,i)$ ith column

$$\Rightarrow QQ^T = \left(\sum_i q_i e_i^T\right)\left(\sum_j e_j q_j^T\right)$$

$$= \sum_{i,j} q_i \underbrace{(e_i^T e_j)}_{e_i \cdot e_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}} q_j^T$$

$$= \sum_{i=1}^{d} q_i q_i^T$$

```
projectedv2 = Q * (Q' * v);

sum(abs(projectedv2 - projectedv))
```

Trick: to check that two vectors

```
projectedv2 = Q * (Q' * v);

sum(abs(projectedv2 - projectedv))
```

Trick: To check that two vectors are the same, add up the absolute values of the coord. differences

(5) Check the answer:

```
# 1. Is projectedv in R(A)?
x = A\projectedv;          # Solve for a linear comb. of the columns of A
err = A*x - projectedv     #  that gives projectedv
sum(abs(err))

# 2. Is v-projectedv perpendicular to R(A)?
(v - projectedv)' * A
```

(Question: What is the expected squared length of the projection? projectedv' * projectedv)

Key Example: If $V = \mathbb{R}^n$, "resolution of the identity" is

$$\boxed{\sum_{j=1}^{n} \vec{v}_j \vec{v}_j^{\,T} = \underbrace{I}_{\text{identity matrix}}}$$

This gives an easy derivation of the other facts:

Example: I forget, is

$$\vec{u} = \sum_j (\vec{v}_j \cdot \vec{u}) \vec{v}_j \quad \text{or} \quad \vec{u} = \sum_j (\vec{u} \cdot \vec{v}_j) \vec{v}_j \ ?$$

over $\mathbb{C}$, they're different!

$$\vec{u} = I u$$
$$= \sum_j v_j v_j^T u = \sum_j v_j (v_j^T u) = \boxed{\sum_j (\vec{v}_j \cdot \vec{u}) \vec{v}_j}$$

Example:

$$\|\vec{u}\|^2 = \vec{u}^T \vec{u}$$
$$= u^T I u$$
$$= u^T \sum_j v_j v_j^T u$$
$$= \sum_j (u^T v_j)(v_j^T u)$$

$$\Rightarrow \|u\|^2 = \sum_j |\vec{u} \cdot \vec{v}_j|^2$$

Any basis expansion can be done by inserting

$$I = \sum_j v_j v_j^T .$$

Even changes of basis:

$$B = \{\vec{b}_1, \dots, \vec{b}_n\} \qquad C = \{\vec{c}_1, \dots, \vec{c}_n\}$$

orthonormal sets

$$I = I \cdot I$$

$$= \sum_i c_i c_i^T \sum_j b_j b_j^T$$

$$= \sum_{i,j} \vec{c}_i (c_i \cdot b_j) b_j^T$$

8 Orthonormal bases simplify many arguments

- If you prove something for the standard basis, it most likely holds for any orthonormal basis.
  Intuition and arguments are usually basis independent.

Example: If $V \perp W$,

$$P_{V+W} = P_V + P_W .$$

Why? Work in an orthonormal basis

$$\{\underbrace{\vec{b}_1, \dots, \vec{b}_p}_{\text{basis for } V}, \underbrace{\vec{b}_{p+1}, \dots, \vec{b}_{p+q}}_{\text{basis for } W}, \vec{b}_{q+p+1}, \dots, \vec{b}_n\}$$

$$P_V = \begin{array}{c|c} & \begin{array}{ccc} p & q & n-p-q \end{array} \\ \begin{array}{c} p \\ q \\ n-p-q \end{array} & \left(\begin{array}{c|c|c} I_p & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \qquad P_W = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & I_q & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

$$P_V + P_W = \begin{array}{c|c} & \begin{array}{cc} p+q & n-p-q \end{array} \\ \begin{array}{c} p+q \\ n-p-q \end{array} & \left(\begin{array}{cc|c} I_p & 0 & 0 \\ 0 & I_q & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \quad \checkmark$$

Example: If $V \subseteq W$,

$$P_V P_W = P_W P_V = P_V .$$

Check this.

But how do we find orthonormal bases for spaces?

(How does Matlab's qr function work?)