Homework 5 answers

Linear transformations

① a) Consider the linear transformation $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by

(*) $$f(x,y,z) = (3y, -x+y-2z, x-z).$$

Express f as a 3×3 matrix.

b) What is $f^{-1}(x,y,z)$?

Answer:

a) $$\begin{pmatrix} 0 & 3 & 0 \\ -1 & 1 & -2 \\ 1 & 0 & -1 \end{pmatrix}$$

b)

```
>> format rat
>> A = [0 3 0; -1 1 -2; 1 0 -1];
>> A^-1

ans =

       1/9           -1/3            2/3
       1/3            0              0
       1/9           -1/3           -1/3
```

$$A^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\frac{x}{9} - \frac{y}{3} + \frac{2z}{3}, \frac{x}{3}, \frac{x}{9} - \frac{y}{3} - \frac{z}{3}\right)$$

② Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that

- First rotates the xz-plane by $\pi/3$ radians counterclockwise about the y-axis,
- Then reflects everything about the yz-plane (i.e., switching the sign of the x coordinate),
- Then rotates the xy-plane by $\pi/6$ radians counterclockwise about the z-axis.

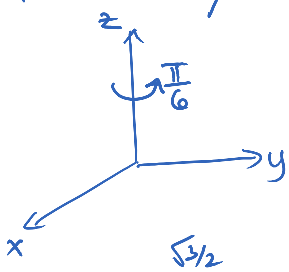
What is the 3×3 matrix representing f?
What is the determinant of this matrix?

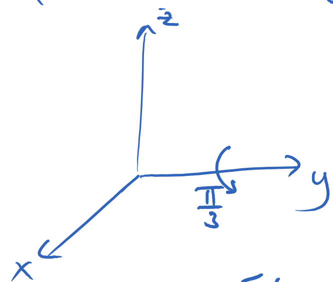
Answer: The matrix is

$$\begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\pi}{3} & 0 & \sin\frac{\pi}{3} \end{pmatrix}$$

Answer: The matrix is

$$\begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\frac{\pi}{3} & 0 & \sin\frac{\pi}{3} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{3} & 0 & \cos\frac{\pi}{3} \end{pmatrix}$$





$\sqrt{3}/2$ $\quad$ $1/2$ $\quad$ $\sqrt{3}/2$

$= \frac{1}{2}$

```
In[5]:= {{Cos[π/6], -Sin[π/6], 0}, {Sin[π/6], Cos[π/6], 0}, {0, 0, 1}} . {{-1, 0, 0}, {0, 1, 0}, {0, 0, 1}} . {{Cos[π/3], 0, Sin[π/3]}, {0, 1, 0}, {-Sin[π/3], 0, Cos[π/3]}} // Simplify // MatrixForm
```

Out[5]//MatrixForm=

$$\begin{pmatrix} -\frac{\sqrt{3}}{4} & -\frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{4} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

The determinant is the product of the determinants of the three matrices: $1 \cdot (-1) \cdot 1 = -1$. ✓

Changes of basis

(3) Consider the 2×2 complex matrix

$$A = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Rewrite this matrix in the basis $\{(1,i),(1,-i)\}$. Simplify your answer as much as possible.

(That is, give a 2×2 matrix that corresponds to the same linear transformation as A, but in the basis $\{(1,i),(1,-i)\}$ instead of the standard basis $\{e_1, e_2\}$.)

Answer:

$$B = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$ changes from the new basis to the

$B = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ changes from the new basis to the standard basis.

$\Rightarrow B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ changes back.

$\Rightarrow A$ in the new basis is given by

$$B^{-1}AB = \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} e^{i\theta} & -ie^{-i\theta} \\ e^{i\theta} & ie^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} e^{i\theta}+e^{-i\theta} & e^{i\theta}-e^{-i\theta} \\ e^{i\theta}-e^{-i\theta} & e^{i\theta}+e^{-i\theta} \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$$

where we have used the identity $e^{i\theta} = \cos\theta + i\sin\theta$

4 Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 4 \\ 0 & 1 & 5 \end{pmatrix}.$$

A is a linear transformation on $\mathbb{R}^3$ (in the standard basis).

Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

$\mathcal{B}$ is another basis for $\mathbb{R}^3$.

Problem: Rewrite the matrix A in the basis $\mathcal{B}$.

Answer 1 Direct calculation:

$$A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \\ 6 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 7\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

```
>> B = [1 1 1; 1 2 2; 1 2 3]'

B =

     1     1     1
     1     2     2
     1     2     3

>> B^-1 * [3;8;6]

ans =

    -2
     7
    -2
```

$$A\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 12 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 9\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - 1\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

```
>> A = [1 2 0; 3 1 4; 0 1 5];
>> A * [1;2;2]

ans =
```

```
>> A = [1 2 0; 3 1 4; 0 1 5];
>> A * [1;2;2]

ans =

     5
    13
    12

>> B^-1 * ans

ans =

    -3
     9
    -1
```

$$A\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}5\\17\\17\end{pmatrix}=-7\begin{pmatrix}1\\1\\1\end{pmatrix}+12\begin{pmatrix}1\\2\\2\end{pmatrix}+0\begin{pmatrix}1\\2\\3\end{pmatrix}$$

```
>> A * [1;2;3]

ans =

     5
    17
    17

>> B^-1 * ans

ans =

    -7
    12
     0
```

Therefore, we get the matrix

$$[A]_B = \begin{array}{c} \\ \begin{pmatrix}1\\1\\1\end{pmatrix} \\ \begin{pmatrix}1\\2\\2\end{pmatrix} \\ \begin{pmatrix}1\\2\\3\end{pmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} \begin{pmatrix}1\\1\\1\end{pmatrix} & \begin{pmatrix}1\\2\\2\end{pmatrix} & \begin{pmatrix}1\\2\\3\end{pmatrix} \end{array} \\ \begin{pmatrix} -2 & -3 & -7 \\ 7 & 9 & 12 \\ -2 & -1 & 0 \end{pmatrix} \end{array}$$

<u>Answer 2 Using change-of-basis matrices:</u>

Let $S=\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ the standard basis.

To change from B into S, use

$$[B\to S] = \begin{array}{c} \begin{array}{ccc} \begin{pmatrix}1\\1\\1\end{pmatrix} & \begin{pmatrix}1\\2\\2\end{pmatrix} & \begin{pmatrix}1\\2\\3\end{pmatrix} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \end{array}$$

Then

$$[A]_B = \underbrace{[S\to B]}_{=[B\to S]^{-1}}[A]_S[B\to S]$$

```
>> BtoS = [1 1 1; 1 2 2; 1 2 3]

BtoS =

     1     1     1
     1     2     2
     1     2     3

>> A = [1 2 0; 3 1 4; 0 1 5]

A =

     1     2     0
```

```
A =

     1     2     0
     3     1     4
     0     1     5

>> BtoS^-1 * A * BtoS

ans =

    -2    -3    -7
     7     9    12
    -2    -1     0
```

$$= \begin{pmatrix} -2 & -3 & -7 \\ 7 & 9 & 12 \\ -2 & -1 & 0 \end{pmatrix}$$

Check your answer!

Here I take a random test vector, and verify that I get the same result whether I apply A to it in the standard basis or in the new basis.

```
>> A = [1 2 0; 3 1 4; 0 1 5];
>> Anew = [-2 -3 -7; 7 9 12; -2 -1 0];
>> testnew = randn(3,1)

testnew =

   -2.2588
    0.8622
    0.3188

>> BtoS = [1 1 1; 1 2 2; 1 2 3];
>> A * BtoS * testnew

ans =

   -0.8718
   -1.4435
    2.2120

>> BtoS * Anew * testnew

ans =

   -0.8718
   -1.4435
    2.2120
```

$$\text{BtoS} * \text{testnew} = -2.25\begin{pmatrix}1\\1\\1\end{pmatrix} + .86\begin{pmatrix}1\\2\\2\end{pmatrix} + .32\begin{pmatrix}1\\2\\3\end{pmatrix}$$

apply A in std basis

apply A in new basis

same result!

(5) a) Consider the space of polynomials in x of degree at most 5. What is the dimension of this space? Give a simple basis for it.

basis $\{1, x, x^2, x^3, x^4, x^5\}$, dimension 6

b) Do the same for polynomials of degree at most 6.

basis $\{1, x, x^2, x^3, x^4, x^5, x^6\}$, dimension 7

c) Using the above bases, give the matrix that represents multiplication by $2+3x$.

$$
\begin{array}{c|cccccc}
 & 1 & x & x^2 & x^3 & x^4 & x^5 \\
\hline
1 & 2 & 0 & 0 & 0 & 0 & 0 \\
x & 3 & 2 & 0 & 0 & 0 & 0 \\
x^2 & 0 & 3 & 2 & 0 & 0 & 0 \\
x^3 & 0 & 0 & 3 & 2 & 0 & 0 \\
x^4 & 0 & 0 & 0 & 3 & 2 & 0 \\
x^5 & 0 & 0 & 0 & 0 & 3 & 2 \\
x^6 & 0 & 0 & 0 & 0 & 0 & 3
\end{array}
$$

d) Look up on Wikipedia Chebyshev polynomials of the first kind. Repeat parts a, b, c using these polynomials for your bases.

Check your answer with at least a simple sanity check — and show your work.

The Chebyshev polynomials of the first kind are given by

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x).$$

Thus they are:

n	$T_n(x)$
0	1
1	x
2	$2x^2-1$
3	$4x^3-3x$

2	$2x^2-1$
3	$4x^3-3x$
4	$8x^4-8x^2+1$
5	$16x^5-20x^3+5x$
6	$32x^6-48x^4+18x^2-1$

$T_0(x)$ through $T_5(x)$ form a basis for polynomials of degree ≤ 5, and $T_0(x)$ through $T_6(x)$ form a basis for polynomials of degree ≤ 6.

$$(2+3x)T_0 = 2+3x = 3T_1 + 2T_0$$
$$(2+3x)T_1 = 2x+3x^2 = \tfrac{3}{2}T_2 + 2T_1 + \tfrac{3}{2}T_0$$
$$(2+3x)T_2 = 6x^3+4x^2-3x-2$$
$$= \tfrac{3}{2}T_3 + 2T_2 + \tfrac{3}{2}T_1$$
$$(2+3x)T_3 = 12x^4+8x^3-9x^2-6x$$
$$= \tfrac{3}{2}T_4 + 2T_3 + \tfrac{3}{2}T_2$$

By now, the pattern should be clear: for $n \geq 1$,
$$(2+3x)T_n = \tfrac{3}{2}T_{n+1} + 2T_n + \tfrac{3}{2}T_{n-1}.$$

Indeed, to prove this it is enough to show that
$$3xT_n = \tfrac{3}{2}\left(T_{n+1} + T_{n-1}\right),$$
since the $2T_n$ factors on both sides cancel.
$$\iff 2xT_n = T_{n+1} + T_{n-1},$$
which follows from the defining identity.

Thus the matrix representing multiplication by $2+3x$ is given in this basis by

$$\begin{array}{c} \\ T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{array}
\begin{array}{c}
\begin{array}{cccccc} T_0 & T_1 & T_2 & T_3 & T_4 & T_5 \end{array} \\
\begin{pmatrix}
2 & 3/2 & & & & \\
3 & 2 & 3/2 & & & \\
 & 3/2 & 2 & 3/2 & & \\
 & & 3/2 & 2 & 3/2 & \\
 & & & 3/2 & 2 & 3/2 \\
 & & & & 3/2 & 2
\end{pmatrix}
\end{array}.$$

$$\begin{array}{c} T_4 \\ T_5 \\ T_6 \end{array} \begin{pmatrix} & & \frac{1}{2} & \alpha & \frac{1}{2} \\ & 0 & & \frac{3}{2} & 2 \\ & & & & \frac{3}{2} \end{pmatrix}$$

⑥ a) Give the dimension and a basis for

$$V = \{ x \in \mathbb{R}^n \mid x_1 + x_2 + x_3 + \cdots + x_n = 0 \}.$$

The dimension is $n-1$. This follows by the rank-nullity theorem since there is one constraint.

A basis is the set

$$e_2 - e_1,\ e_3 - e_2,\ e_4 - e_3,\ \ldots,\ e_n - e_{n-1},$$

where the e_j are the standard basis vectors.

Note that we do not include $e_1 - e_n$ here; it lies in the span of the above basis vectors (being minus their sum).

Why is the above set linearly independent? If $\sum_{j=1}^{n-1} \alpha_j (e_{j+1} - e_j) = 0$, then collecting like terms gives

$$(*) \quad \boxed{0 = -\alpha_1 e_1 + (\alpha_1 - \alpha_2) e_2 + (\alpha_2 - \alpha_3) e_3 + \cdots + (\alpha_{n-2} - \alpha_{n-1}) e_{n-1} + \alpha_{n-1} e_n}$$

Since $\{e_1, \ldots, e_n\}$ is linearly independent, for the right-hand side of $(*)$ to be 0, each coefficient must be 0, so $0 = \alpha_1 = \alpha_1 - \alpha_2 = \alpha_2 - \alpha_3 = \cdots$, implying that all α_j are 0.

b) Consider the linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ given in the standard basis by the matrix

$$A = \begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & 0 \\ & 1 & -2 & 1 & \\ & 0 & & \ddots & \end{pmatrix}$$

$$A = \begin{pmatrix} & & & \ddots & & \\ & O & & & & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix}$$

(This matrix is tridiagonal except for the 1s in the upper right and lower left corners. As we saw in class, it comes from discretizing the second derivative operator on the interval where the boundaries wrap around.)

Argue that $R(A) \subseteq V$.

$R(A) = \operatorname{Span}\{Ae_1, Ae_2, Ae_3, \dots, Ae_n\}$, ie., the span of the columns of A, so it is enough to show that every column of A lies in V. Indeed, the sum down any column is $1-2+1=0$. ✓

c) Therefore in particular $A(V) \subseteq V$. Let

$$B: V \to V$$

be the linear transformation that takes a vector in V and applies A to it (giving another vector in V).

Give the $(\dim V)$-by-$(\dim V)$ matrix for B using the basis you found in part ⓐ.

$Ae_j = e_{j-1} - 2e_j + e_{j+1}$ (where indices wrap around as necessary, e.g., $n+1 \to 1$)

$$\Rightarrow A(e_j - e_{j-1}) = (e_{j+1} - 2e_j + e_{j-1}) - (e_j - 2e_{j-1} + e_{j-2})$$
$$= e_{j+1} - 3e_j + 3e_{j-1} - e_{j-2}$$
$$= (e_{j+1} - e_j) - 2(e_j - e_{j-1}) + (e_{j-1} - e_{j-2}).$$

This basically allows us to read off the matrix; B should be tridiagonal with $+1$'s above and below the diagonal and -2's on the diagonal — just like

A itself!

But it is not quite that simple, since our basis is missing $e_1 - e_n$. So the first and last columns of B will have to reexpress $e_1 - e_n$ in terms of the basis vectors. We get

$$B = \begin{pmatrix}
-3 & 1 & & & & & -1\\
0 & -2 & & & 0 & & \vdots\\
-1 & 1 & 1 & & & & \\
-1 & & -2 & 1 & & & -1\\
-1 & & 1 & -2 & \ddots & & -1\\
\vdots & & & 1 & \ddots & 1 & -1\\
 & & & & \ddots & -2 & 0\\
-1 & & & & & 1 & -3
\end{pmatrix}.$$

Your answer might be more complicated (or simpler!) if you used a different basis in part a. This illustrates the principle that a good basis is one in which the linear transformations you care about are simple to express.