Lecture 13: Rotations and scaling (class)

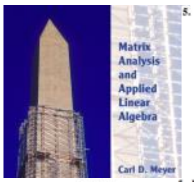
Admin: Midterm 1 20% Midterm 2 Nov. 12 → 40%

$20\% + (20\% - \text{midterm } 1)$

comprehensive

last class Nov. 24

Final also comprehensive



Concepts	Techniques	Decompositions
Vector space		
Basis	Gaussian elimination	LU decomp.
Inner products/Norm/Orthogonality		
Linear transformations		
Projections	Gram-Schmidt	QR decomp.
Rank		

Next

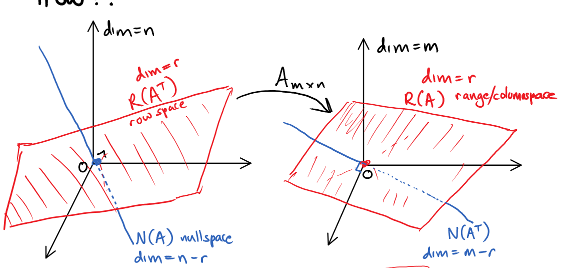
Singular values → Singular-value decomposition
Eigenvalues/vectors → Spectral decomposition

This week: **SINGULAR VALUE DECOMPOSITION**

Theoretical motivation:

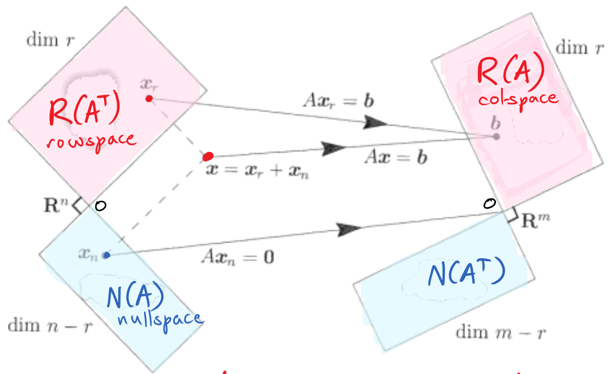
Any linear transformation A maps points in the rowspace $R(A^T)$ to distinct points in the columnspace $R(A)$. [Rank-Nullity thm.]

How??



$$\boxed{\dim R(A^T) = \dim R(A)}$$

$$\boxed{\dim N(A) + \dim R(A^T) = \text{total dimension } n} \qquad \boxed{\dim R(A) + \dim N(A^T) = \text{total dimension } m}$$



$N(A) = R(A^T)^\perp$ $\qquad$ $N(A^T) = R(A)^\perp$

Practical motivation: Many applications, including

* Solving linear equations $Ax=b$ → What is the sensitivity, e.g., to numerical errors?
  - Find the shortest solution
  - When there is no solution, find x to minimize $\|Ax-b\|$
    - Least-squares regression analysis
* Rank minimization
  - Principal Component Analysis (PCA)

  Data mining, clustering, recommendation systems, ...

SINGULAR-VALUE DECOMPOSITION (SVD)

Informally: Any linear transformation can be split into:
- a rotation, followed by
- scaling vectors in or out

Before stating the theorem formally, we'll consider these pieces.

ISOMETRIES

Definition: An isometry is a linear transformation that preserves length. (iso = same, metric = length/distance)

(That is, $\|A\vec{x}\| = \|\vec{x}\|$ for all $\vec{x}$.)

Examples:

- Identity matrix I
- Rotations, e.g., $\begin{pmatrix} \cos\theta & 0 & +\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$
- Reflections, e.g., $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- Products of isometries ~~rotations and reflections~~

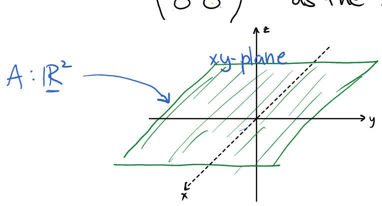
  $\forall x, \|\underline{AB x}\| = \|\underline{Bx}\| = \|x\|$

- Isometric "embeddings", e.g.

  $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ puts $\mathbb{R}^2$ into $\mathbb{R}^3$ as the xy-plane

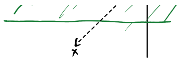
  $\mathbb{R}^2 \notin \mathbb{R}^3$ $\qquad \begin{pmatrix} x \\ y \end{pmatrix} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

  $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \\ 0 & 0 \end{pmatrix}$ ✓



$\begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ also maps $\mathbb{R}^2$ to the xy-plane of $\mathbb{R}^3$ — but does not preserve lengths

Ex: Give an isometric embedding of $\mathbb{R}^2$ into



lengths

Ex: Give an isometric embedding of $\mathbb{R}^2$ into

$$V = N\left(\begin{pmatrix}1 & 1 & 1\end{pmatrix}\right) \subseteq \mathbb{R}^3$$

Answer: $= \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x+y+z=0 \right\}$

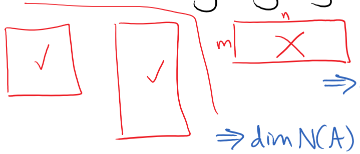
$= \text{Span}\left\{ \underset{\vec v_1}{\begin{pmatrix}1\\-1\\0\end{pmatrix}}, \underset{\vec v_2}{\begin{pmatrix}1\\0\\-1\end{pmatrix}} \right\}$

$A = \begin{pmatrix}1 & 1\\-1 & 0\\0 & -1\end{pmatrix}$ This is NOT an isometry

$\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\\0\end{pmatrix}, \frac{1}{\sqrt6}\begin{pmatrix}1\\1\\-2\end{pmatrix}$

$A = \begin{pmatrix}1/\sqrt2 & 1/\sqrt6\\-1/\sqrt2 & 1/\sqrt6\\0 & -2/\sqrt6\end{pmatrix}$ ✓

• Not an isometry: anything that reduces the dimension



with $m < n$

$\Rightarrow \text{rank}(A) = \#$ lin. indep. rows $\leq m$

$\Rightarrow \dim N(A) = n - \text{rank}(A) > 0$

$\Rightarrow$ lengths of nonzero vectors in $N(A)$ are sent to 0 — not preserved.

Claim: Preserves length $\Rightarrow$ preserves angles.

Proof: Recall the angle θ between real vectors $\vec x$ and $\vec y$ satisfies $\cos\theta = \dfrac{x \cdot y}{\|x\| \cdot \|y\|}$.

$\Rightarrow$ We have to show that dot products are preserved.

Assume A is an isometry. Goal: $\forall x, y$ $(Ax)\cdot(Ay) = x \cdot y$ ✓

Trick: Look at $x+y$

$$\|x+y\|^2 = (x+y)\cdot(x+y) = \|x\|^2 + \|y\|^2 + \underbrace{x\cdot y + y \cdot x}_{2\cdot \text{Re}(x\cdot y)}$$

$\|$

$$\|A(x+y)\|^2 = (A(x+y))\cdot(A(x+y)) = \|Ax\|^2 + \|Ay\|^2 + \underbrace{Ax\cdot Ay + Ay \cdot Ax}_{2\text{Re}((Ax)\cdot(Ay))}$$

$$= (Ax+Ay)\cdot(Ax+Ay)$$

$$\Rightarrow \text{Re}(x\cdot y) = \text{Re}(Ax \cdot Ay)$$

$i(a+bi) - i(a-bi) = -2b$

$$\|\vec x + i\vec y\|^2 = \|x\|^2 + \|y\|^2 + \underbrace{i\, x\cdot y - i\, y \cdot x}_{-2\text{Im}(x\cdot y)}$$

$\|$

$$\|A(x+iy)\|^2 = \|Ax\|^2 + \|Ay\|^2 - 2\,\text{Im}(Ax \cdot Ay) \Rightarrow \text{Im}(x\cdot y) = \text{Im}(Ax\cdot Ay)$$

$$\Rightarrow Ax \cdot Ay = x \cdot y \checkmark$$

How to tell if a matrix is an isometry?

$$A = \begin{pmatrix} | & | & & | \\ \vec v_1 & \vec v_2 & \cdots & \vec v_n \\ | & | & & | \end{pmatrix} = \sum_{j=1}^n \vec v_j \vec e_j^T$$

$$A^T A = \sum_{j,k} (v_j e_j^T)^T (v_k e_k^T) = \sum_{j,k} e_j e_k^T \cdot (\vec v_j \cdot \vec v_k)$$

Claim: A is an isometry $\Leftrightarrow$ The columns are orthonormal $\Updownarrow$ $A^TA = I$

Proof: $\Leftarrow$: Assume $v_j \cdot v_k = \delta_{jk} = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j \neq k \end{cases}$

Consider any $\vec x = \sum_j x_j \vec e_j$, $\|x\|^2 = \sum_j x_j^2$

$\|A\vec{x}\|^2 = \|\sum_j x_j \vec{v}_j\|^2 = \sum_j x_j^2 = \|x\|^2 \Rightarrow A$ is isometry ✓

$\Rightarrow$: Assume A isometry. Goal: Show columns orthonormal

Why? $\|e_j\| = 1 = \|Ae_j\| = \|v_j\|$

$j \neq k$: $e_j \cdot e_k = 0 = (Ae_j)\cdot(Ae_k) = v_j \cdot v_k$ ✓

In matrix notation:

$$\underbrace{\begin{pmatrix} — & v_1^\dagger & — \\ & \vdots & \\ — & v_n^\dagger & — \end{pmatrix}}_{A^\dagger} \underbrace{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}}_{A} = \begin{pmatrix} v_1\cdot v_1 & v_1\cdot v_2 & v_1 \cdot v_3 & \cdots \\ v_2\cdot v_1 & v_2\cdot v_2 & v_2\cdot v_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & \cdots \\ \vdots & & \ddots & \end{pmatrix} = I$$

Thus an isometry takes one orthonormal set of vectors (the standard basis) into another orthonormal set (the columns).

Exercise: Prove the converse implication:
If the columns of A are orthonormal, then A is an isometry.

A is an isometry $\Longleftrightarrow$ A preserves lengths and angles
$\Updownarrow$ $\quad\quad$ $\Updownarrow$
A's columns are orthonormal $\Longleftrightarrow$ $A^\dagger A = I$

Examples: $\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Exercise: Give an isometry from $\mathbb{R}$ to the line $L = \{(x,y,z) \mid x = y = z\} \subset \mathbb{R}^3$.

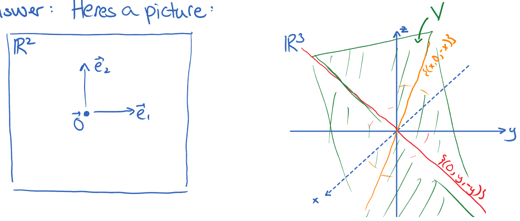
Answer: The line L consists of all multiples of the unit vector $\frac{1}{\sqrt{3}}(1,1,1)$. Therefore, the matrices

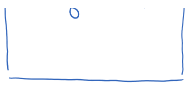
$$\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \text{and} \quad \frac{-1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

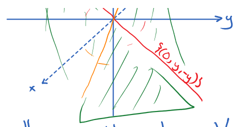
are both isometries from $\mathbb{R}$ to L.
(And these are the only such isometries.) ✓

Exercise: Give an isometry from $\mathbb{R}^2$ to the plane $V = \{(x,y,z) \mid x+y+z=0\} \subset \mathbb{R}^3$.

Answer: Here's a picture:







To map the plane $\mathbb{R}^2$ isometrically into the plane V, we just need to map $\vec{e}_1$ and $\vec{e}_2$ into two perpendicular unit vectors in V. The isometry will take

$$\vec{e}_1 \longmapsto \text{first unit vector in } V \equiv \vec{u}$$
$$\vec{e}_2 \longmapsto 2^{nd} \text{ unit vector in } V. \equiv \vec{v}$$

How to find $\vec{u}$ and $\vec{v}$?

- $\vec{u}$ can be an arbitrary unit vector
  eg., start with
  $$(1,-1,0) \in V,$$
  and normalize:
  $$\vec{u} = \tfrac{1}{\sqrt{2}}(1,-1,0).$$
- $\vec{v} = (v_1, v_2, v_3)$ has to lie in V and be perpendicular to $\vec{u}$:
  $$v_1 + v_2 + v_3 = 0 \qquad (\vec{v} \in V)$$
  $$v_1 - v_2 \quad = 0 \qquad (\vec{v} \cdot \vec{u} = 0)$$
  $$\Rightarrow \vec{v} = (1,1,-2)/\underbrace{\sqrt{6}}_{\text{normalization}} \quad \text{works}$$

What is the matrix for our isometry?

$\vec{e}_1 \mapsto \vec{u}$, $\vec{e}_2 \mapsto \vec{v}$

$$A = \begin{pmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/\sqrt{6} \end{pmatrix} \quad ✓$$

(Of course, this answer is not unique. We can also rotate or reflect the plane.)

Short answer:

An orthonormal basis for V is
$$\tfrac{1}{\sqrt{2}}(1,-1,0), \ \tfrac{1}{\sqrt{6}}(1,1,-2).$$
Therefore,
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} \text{ isometrically maps } \mathbb{R}^2 \text{ onto } V.$$
Of course, this answer is not unique. We could have rotated things around — and used any orthonormal basis for V. ✓

Orthogonal and Unitary Matrices

Definition: An "orthogonal" matrix is a square matrix isometry (ie., $n \times n$).

change of basis matrices that preserve lengths

Recall: The columns of an isometry are orthonormal,
$$A^T A = I.$$

Proposition: The rows of an orthogonal matrix are also orthonormal,
$$A A^T = I.$$

Corollary:
$$\boxed{\text{Orthogonal matrix} \quad A^T = A^{-1}}$$

rows are not orthogonal for isometric embeddings like $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Proof: Let $A = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$

A maps $\vec{e}_i$ to $\vec{v}_i$

Equivalently, in more compact notation,

$$A = \sum_{j=1}^{n} \vec{v}_j \vec{e}_j^{\,T} \qquad \begin{pmatrix} | \\ \vec{v}_j \\ | \end{pmatrix}\begin{pmatrix} — \vec{e}_j^{\,T} — \end{pmatrix}$$

Why? Check it:

$$A\vec{e}_i = \left(\sum_{j=1}^{n} \vec{v}_j \vec{e}_j^{\,T}\right)\vec{e}_i$$
$$= \sum_j \vec{v}_j (\vec{e}_j^{\,T}\vec{e}_i)$$
$$= \sum_j \vec{v}_j \underbrace{(\vec{e}_j \cdot \vec{e}_i)}_{= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}}$$
$$= \vec{v}_j \ \checkmark$$

$$\Rightarrow A^T A = \left(\sum_j v_j e_j^T\right)^T \left(\sum_k v_k e_k^T\right)$$
$$= \sum_{j,k} e_j \underbrace{v_j^T v_k}_{= v_j \cdot v_k = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{if } j \neq k \end{cases}} e_k^T$$
$$= \sum_j e_j e_j^T$$

Note: $e_1 e_1^T = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}(1\ 0\ 0 \cdots) = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \ddots \\ & \vdots & & \end{pmatrix}$

$e_2 e_2^T = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}(0\ 1\ 0 \cdots 0) = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \ddots \\ & \vdots & & \end{pmatrix}$

$\vdots$

$\Rightarrow e_1 e_1^T + e_2 e_2^T + \cdots + e_n e_n^T = I$ the identity!

Next let's compute AA^T:

$$AA^T = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}\begin{pmatrix} — & v_1^T & — \\ & \vdots & \\ — & v_n^T & — \end{pmatrix}$$
$$= \left(\sum_j v_j e_j^T\right)\left(\sum_k v_k e_k^T\right)^T$$
$$= \sum_{j,k} v_j (e_j \cdot e_k) v_k^T$$
$$= \sum_{j=1}^{n} v_j v_j^T$$

Claim: This is the identity again.

Why?

Call it M.

For any $i = 1, 2, \ldots, n$,

$$M\vec{v}_i = \sum_j v_j v_j^T v_i = \vec{v}_i \ \checkmark$$

so the vectors $\vec{v}_1, \ldots, \vec{v}_n$ are all left alone.

Any other vector can be expanded out in terms of them, like

$$\vec{u} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n$$
$$\Rightarrow M\vec{u} = \underbrace{\alpha_1 M\vec{v}_1 + \cdots + \alpha_n M\vec{v}_n}_{= \vec{u}\ \checkmark} \qquad A^T = A^{-1}\ ☑$$

Definition: An $n \times n$ complex isometry is called "unitary".

Orthogonal matrix $A^T = A^{-1}$

Unitary matrix $A^\dagger = A^{-1}$

More examples:

- Permutation matrices, e.g.,

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} : \begin{matrix} e_1 \mapsto e_2 \\ e_2 \mapsto e_3 \\ e_3 \mapsto e_4 \\ e_4 \mapsto e_1 \end{matrix}$$

- Rotations, e.g.

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ rotates the plane $\mathbb{R}^2$ counterclockwise by angle θ

$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ rotates $\mathbb{R}^3$ by θ about the z-axis

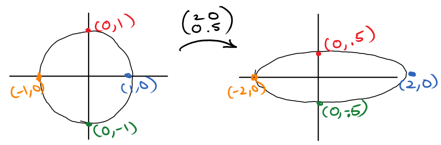
- $\frac{1}{\ }\begin{pmatrix} 1+i & 1-i \end{pmatrix}$ ✓

- $\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1+i & -1+i \end{pmatrix}$ ✓

SCALING I.

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ scales every vector up by 2

$\begin{pmatrix} 2 & 0 \\ 0 & .5 \end{pmatrix}$ scales by different amounts

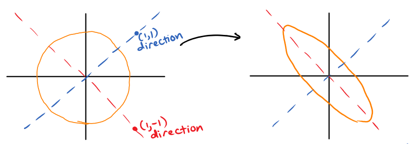


Need not be axis-aligned...

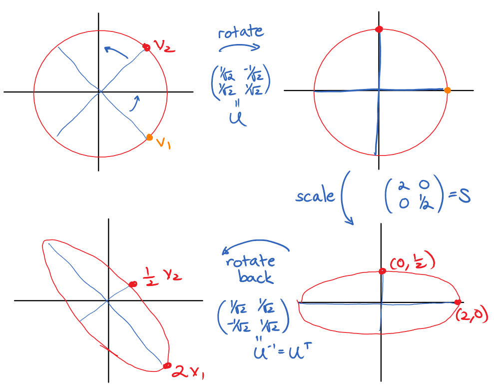
Exercise: Give a 2×2 matrix that maps

$$(1,-1) \longmapsto (2,-2) \checkmark$$
$$\perp \quad (1,1) \longmapsto \left(\tfrac{1}{2},\tfrac{1}{2}\right)$$

Answer: We want



This is the same as above, but rotated by $\pi/4$.



$\Rightarrow$ $\boxed{U^T S U}$ works

Alternative answer:

Better answer:

$\vec v_1 = \frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}$

$\vec v_2 = \frac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix}$

$$A = \boxed{\tfrac{1}{2}\, \vec v_1 \vec v_1^T + 2 \vec v_2 \vec v_2^T} = \frac{1}{2}\cdot\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + 2\cdot\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \boxed{\begin{pmatrix}5/4 & -3/4\\ -3/4 & 5/4\end{pmatrix}}$$

More generally, let $v_1, \dots, v_n$ be an orthonormal basis

let $\sigma_1, \dots, \sigma_n \in \mathbb{R}$

let $A = \sum_j \sigma_j \vec v_j \vec v_j^T$ ✓ $\quad [A]_{v_1 \cdots v_n} = \begin{pmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ & & \ddots \\ 0 & & & \sigma_n \end{pmatrix}$

Then

$$A\vec v_i = \sum_j \sigma_j v_j \underbrace{(v_j^T v_i)}_{\delta_{ij}}$$
$$= \sigma_i \vec v_i$$

SCALING II: MATRIX NORM

Definition: The spectral norm of a linear transformation A is given by

$$\|A\| = \max_{\vec x \neq 0} \frac{\|A\vec x\|}{\|\vec x\|} = \max_{\vec x,\, \|x\|=1} \|Ax\|$$

(It measures the maximum stretch of the matrix.)

(In finite dimensions, the max exists, is $<\infty$.)

Note: Often denoted $\|A\|_2$, for ℓ_2/Euclidean norm.

$$\frac{\|Ax\|}{\|x\|} = \frac{\|A\cdot c\vec x\|}{\|c\vec x\|}$$

for any $c \neq 0$

Properties I:

- For any vector $\vec x$ (of appropriate dimension), $\|A\vec x\| \le \|A\|\cdot\|\vec x\|$ Why? $\quad \frac{\|Ax\|}{\|x\|} \le \max_{z\neq 0}\frac{\|Az\|}{\|z\|} = \|A\|$ ✓
- For any real/complex number α, $\|\alpha A\| = |\alpha|\cdot\|A\|$.
- Triangle inequality: $\|A+B\| \le \|A\| + \|B\|$.

  Proof: $\|A+B\| = \max_{x:\|x\|=1} \|Ax+Bx\|$

- $\|AB\| \le \|A\|\cdot\|B\|$

  Proof: $\|A(B\vec x)\| \le \|A\|\cdot\|Bx\|$
  $\le \|A\|\cdot\|B\|\cdot\|x\|$ ✓

$\leq \max_{\|x\|=1} (\|Ax\| + \|Bx\|)$ (Δ ineq. for vectors)

$\leq (\max_{\|x\|=1} \|Ax\|) + (\max_{\|y\|=1} \|By\|)$

$= \|A\| + \|B\|$. $\square$

Examples:

- $\|I\| = 1$

  $\|\text{any isometry}\| = 1$ — If A and C are isometries $\|ABC\| \leq \|A\| \cdot \|B\| \cdot \|C\| = \|B\|$, but $\|ABC\| < \|B\|$ is possible! If C is unitary/orthogonal, then $\|ABC\| = \|B\|$.

  Exercise: Check this.

  $\|\text{any projection}\| = 1$, unless the projection is 0

- What is $\left\|\underbrace{\begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}}_{A}\right\|$? $= 1+\varepsilon$?

$A = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix} = I + \varepsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\|A\| \leq \|I\| + |\varepsilon| \cdot \left\|\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\| = 1 + |\varepsilon|$

$\|A\| \geq \frac{\|A\vec{x}\|}{\|x\|}$ for any $\vec{x} \neq \vec{0}$, $\|A\| \geq \|A\vec{e}_1\| = \left\|\begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}\right\| = \sqrt{1+\varepsilon^2} \approx 1 + \frac{1}{2}\varepsilon^2 + O(\varepsilon^4)$

Exact calculation:

$\|A\|^2 = \max_{\vec{x}, \|x\|=1} \|Ax\|^2 = \max_{\theta \in [0, 2\pi)} \left\|A\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right\|^2$

$(c+\varepsilon s)^2 + (\varepsilon c + s)^2 = \begin{pmatrix} c^2 + \varepsilon^2 s^2 \\ + 2\varepsilon cs \end{pmatrix} + \begin{pmatrix} \varepsilon^2 c^2 + s^2 \\ + 2\varepsilon cs \end{pmatrix}$

$= 1 + \varepsilon^2 + 4\varepsilon \cos\theta \cdot \sin\theta$

$\theta = \frac{\pi}{4}$

$= 1 + \varepsilon^2 + 4|\varepsilon| \cdot \frac{1}{2} = (1+|\varepsilon|)^2$

$\Rightarrow \|A\| = 1 + |\varepsilon|$ $\quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

✓

Observe: $\left\|A\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}\right\| = 1+\varepsilon > \left\|A\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\| = \left\|A\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\| = \sqrt{1+\varepsilon^2}$

Moral: Spreading out is good!

Problem: What are the operator norms of

ⓐ $\begin{pmatrix} 1 & 0 \\ 0 & 1/100 \end{pmatrix}$ ⓑ $\begin{pmatrix} 1 & 1/100 \\ 0 & 0 \end{pmatrix}$?

$$\left\|\begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & a_3 & \\ 0 & & & a_4 \end{pmatrix}\right\| = \max_j |a_j| \checkmark$$

Proof: $\|A\| \geq \|A\vec{e}_j\| = |a_j|$

$\|A\|^2 = \max_{x, \|x\|=1} \|A\vec{x}\|^2 = \max \sum_j a_j^2 x_j^2$ s.t. $\sum_j x_j^2 = 1$

let $p_j = x_j^2$

Observe:

- $\boxed{\text{In (a), you } \underline{\text{don't}} \text{ want to spread out, since there is no interaction between the two blocks of the matrix.}}$
- In general,

  $\left\|\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right\| = \max\{\|A\|, \|B\|\}$

  $\left\|\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}\right\| = \max\{\|A\|, \|B\|, \|C\|\}$, etc.

  e.g. $\left\|\begin{pmatrix} 0 & 0 & 1 & \varepsilon \\ 0 & 0 & \varepsilon & 1 \\ B & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}\right\| = \max\{\|A\|, \|B\|\}$

  $\left\|\begin{pmatrix} A & \varepsilon C \\ 0 & B \end{pmatrix}\right\| > \max\{\|A\|, \|B\|\}$

- In (b), even though $\|Ae_2\| = \varepsilon \ll \|Ae_1\| = 1$, you still want to spread between the two columns to maximize the norm.
- Also, in general,

- In (b), even though $\|Ae_2\| = \varepsilon \ll \|Ae_1\| = 1$, you still want to spread between the two columns to maximize the norm.

$\left\| \begin{pmatrix} 0 & B \end{pmatrix} \right\|$

- Also, in general,
  spectral matrix norm of a $1\times n$ matrix
  = Euclidean norm of the row vector
  (to maximize $|\vec{v}\cdot\vec{x}|$, let $\vec{x} = \vec{v}/\|\vec{v}\|$.)
  & spectral norm of an $n\times 1$ matrix
  = Euclidean norm of the column vector
  (just set $\vec{x} = (1)$.)

Claim: Let A be any rank-one matrix. Then $\|A\| = \sqrt{\sum_{ij} A_{ij}^2} = \|u\|\cdot\|v\|$

$\longrightarrow A = \begin{pmatrix} \vec{v} \end{pmatrix}$, $\|A\| = \|\vec{v}\|$

Proof: $\text{rank}(A) = 1 \Rightarrow$ there exist $\vec{u}, \vec{v}$ s.t. $A = \vec{v}\cdot\vec{u}^T$

$$\|A\| = \max_{\|x\|=1} \|Ax\| = \max_{\|x\|=1} \|\vec{v}\,(u\cdot x)\| = |u\cdot x|\cdot\|\vec{v}\| = \frac{\|u\|^2}{\|u\|}\cdot\|v\| = \|u\|\cdot\|v\| \checkmark$$

maximized by $\vec{x} = \pm\frac{\vec{u}}{\|u\|}$

$$\|u\|\cdot\|v\| = \sqrt{\sum_i u_i^2}\sqrt{\sum_j v_j^2} = \sqrt{\sum_{ij} A_{ij}^2} = \sqrt{\sum_{ij} (v_i u_j)^2} \qquad \square$$

$$\begin{pmatrix} \vec{v} & 2\vec{v} & 3\vec{v} & \tfrac{1}{2}\vec{v} \end{pmatrix} = \begin{pmatrix} \vec{v} \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & \tfrac{1}{2} \end{pmatrix}$$

Example: What is the spectral norm of the $m\times n$ all-ones matrix?

$A = m\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}_n \longrightarrow \|A\| = \sqrt{m\cdot n}$

Answer:

① Experiment numerically:

```
octave:1> m = 10;
octave:2> n = 15;
octave:3> A = ones(m,n)
A =

   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1

octave:4> norm(A)
ans =  12.247
octave:5> norm(A)^2
ans =  150.00
```

$\Rightarrow$ maybe $\|A\| = \sqrt{m\cdot n}$?

Mathematica code:

```
In[25]:= Table[
   Norm[ConstantArray[1, {m, n}]]^2,
   {m, 1, 5}, {n, 1, 5}
  ] // MatrixForm
Out[25]//MatrixForm=
```

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 3 & 6 & 9 & 12 & 15 \\ 4 & 8 & 12 & 16 & 20 \\ 5 & 10 & 15 & 20 & 25 \end{pmatrix} \qquad \|A\| - \left\|\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}\right\|^2$$

② Guess the best input:

Since the columns are all the same, it makes sense to spread out across them all, and equally:

Let $\vec{x} = \frac{1}{\sqrt{n}}(1, 1, 1, \ldots, 1) \in \mathbb{R}^n$

$\Rightarrow \|\vec{x}\| = 1$.

$$A\vec{x} = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{n}}(n, n, n, \ldots, n) \in \mathbb{R}^m$$

$\Rightarrow \|A\vec{x}\|^2 = m \cdot n$ ✓

$\Rightarrow \|A\| \geq \sqrt{m \cdot n}$

③ <u>Prove</u> that $\|A\| = \sqrt{mn}$:

One approach is to argue by symmetry that the above $\vec{x}$ is optimal.

Alternatively, notice that $\text{rank}(A) = 1$.

Since all columns are the same,

$\text{rank}(A) = \dim R(A) = \#$linearly independent columns $= 1$.

A factors as

$$A = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}_{u \in \mathbb{R}^m} \underbrace{(1\ 1\ 1\ \cdots\ 1)}_{v^T \in \mathbb{R}^n} = \vec{u}\vec{v}^T$$

$\Rightarrow A\vec{x} = (\vec{v} \cdot \vec{x})\vec{u}$

$\|Ax\| = |v \cdot x| \cdot \|u\|$,

which reaches its maximum, $\|u\| \cdot \|v\|$,

for $x = \frac{v}{\|v\|}$.

$\Rightarrow \|A\| = \|u\| \cdot \|v\| = \sqrt{m} \cdot \sqrt{n}$. ✓

<u>Observe</u>: <u>Any</u> rank-one matrix A can be factored as

$$A = \vec{u}\vec{v}^T$$

for some vectors $\vec{u}$ and $\vec{v}$. Hence $\|A\| = \|u\| \cdot \|v\|$.

Spectral norm
<u>Properties II</u>.

- $\|A\| \geq 0$, and $\|A\| = 0 \Leftrightarrow A = 0$
- $\|A\vec{x}\| \leq \underset{\text{matrix norm}}{\|A\|} \cdot \underset{\text{vector/matrix norm}}{\|\vec{x}\|}$
- $\|\alpha A\| = |\alpha| \cdot \|A\|$ for $\alpha \in \mathbb{C}$
- $\|AB\| \leq \|A\| \cdot \|B\|$

(the amount you can stretch an input by applying AB is at most the stretch from applying B times the stretch from applying A.)

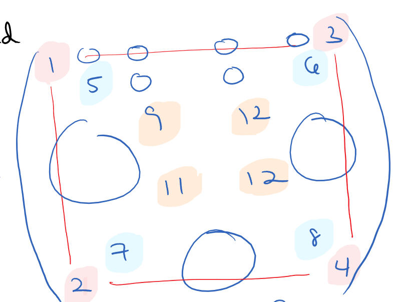
- If U and V are unitary, $\|U\| = \|V\| = 1$ and $\|UAV\| = \|A\|$

(because unitaries don't change lengths).

- $\left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\| = \max\{\|A\|, \|B\|\}$

e.g., if A is a diagonal matrix, $\|A\| = \max_i |a_{ii}|$.

- If $\text{rank}(A) = 1$, with $A = \vec{u}\vec{v}^T$, $\|A\| = \|u\| \cdot \|v\|$.

$A =$ 

A fast (crude) estimate for the spectral norm: (extra material)

<u>Claim</u>: For any $m \times n$ matrix $A = (a_{ij})$,

$$\max_{ij} |a_{ij}| \leq \|A\| \leq \sqrt{\sum_{ij} |a_{ij}|^2}$$
$$\leq \sqrt{m \cdot n} \cdot \max_{ij} |a_{ij}|.$$

<u>Observe</u>: For ~~the all-ones~~ any rank-one matrix, the upper bound ($\sqrt{mn}$) is tight, though the lower bound (1) is terrible.

Proof: Start by showing the lower bound, $\|A\| \geq \max_{ij} |a_{ij}|$.

$A = \left(\cdots\ b\ \cdots \right)$ $\|A\| \geq \|A\vec{e}_j\| = \| \cdots + b\vec{e}_i \|$

$\geq \|b\vec{e}_i\| = |b|$

Next, let us show the upper bound, $\|A\| \leq \sqrt{m \cdot n} \cdot \max_{ij} |a_{ij}|$.

$\|A\|^2 = \max_{x: \|x\|=1} \|Ax\|^2$

$\|A\| = \max\left\{ \left\| \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \right\|, \left\| \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \right\|, \left\| \begin{pmatrix} 9 & 12 \\ 11 & 12 \end{pmatrix} \right\| \right\}$

$= \left\| \begin{pmatrix} 9 & 12 \\ 11 & 12 \end{pmatrix} \right\|$

How to estimate $\|A\|$?

Δ inequality

$\|B + C\| \leq \|B\| + \|C\|$

$A = \begin{pmatrix} 9 & 12 \\ 11 & 12 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 9 & 12 \\ 9 & 12 \end{pmatrix}$

$\Rightarrow \|A\| \leq 2 + \sqrt{2} \cdot \sqrt{9^2 + 12^2}$

$= 23.213\ldots$

$\Rightarrow \|A\| \leq 12 + 12 = 24$

$A = \begin{pmatrix} 9 & 0 \\ 11 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 12 \\ 0 & 12 \end{pmatrix}$

$\Rightarrow \|A\| \leq \sqrt{11^2 + 9^2} + 12\sqrt{2}$

$\|A\| \approx 22.109$

Next, let us show the upper bound, $\|A\| \le \sqrt{m \cdot n} \cdot \max_{i,j} |a_{ij}|$.

$$\|A\|^2 = \max_{x : \|x\| = 1} \|Ax\|^2$$

Write

$$A = \begin{pmatrix} — & r_1^T & — \\ — & r_2^T & — \\ & \vdots & \\ — & r_m^T & — \end{pmatrix} = \sum_{j=1}^{m} \vec{e}_j \vec{r}_j^{\,T}$$

$$\Rightarrow \|A\vec{x}\|^2 = \Big\| \sum_j \vec{e}_j (r_j \cdot x) \Big\|^2$$

$$= \sum_{j=1}^{m} |r_j \cdot x|^2$$

$$\le \sum_j \|r_j\|^2 \|x\|^2$$

$$\Rightarrow \frac{\|Ax\|}{\|x\|} \le \sqrt{\sum_j \|r_j\|^2} = \sqrt{\sum_{i,j} |A_{ij}|^2} \quad ✓$$

$\cancel{A = \begin{pmatrix} 9 & 0 \\ & 0 \end{pmatrix} + \begin{pmatrix} 0 & \\ 0 & 12 \end{pmatrix} \Rightarrow \|A\| \le \sqrt{11^2 + 9^2} + 12\sqrt{2} = 31.18}$

Cauchy-Schwarz

$|u \cdot v| \le \|u\| \cdot \|v\|$

$$\Rightarrow \left\| \begin{pmatrix} 9 & 11 \\ 12 & 12 \end{pmatrix} \right\| \le \sqrt{9^2 + 11^2 + 12^2 + 12^2} \approx 22.136$$

$\|A\| \approx 22.101$

When is a perturbed matrix invertible?

Lemma: If $\|A\| < 1$, then $(I+A)^{-1}$ exists.

Proof:

$I + A$ is not invertible $\Leftrightarrow N(I+A) \ne \{0\}$

$\Leftrightarrow (I+A)x = 0$ for some $x \ne 0$

$\Rightarrow Ax = -x$

$\Rightarrow \|A\| \ge 1$, a contradiction. $\square$

Lemma: Let A be an invertible matrix.

If $\|B\| < 1/\|A^{-1}\|$, then $A + B$ is invertible.

Proof: $A + B = A(I + A^{-1}B)$. Now apply the previous lemma, $\square$ with $\|A^{-1}B\| \le \|A^{-1}\| \cdot \|B\|$

Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

so if $\|B\| < \frac{1}{4}$, $A + B$ is invertible.

Lemma: If $\|A\| < 1$, then

$$(I+A)^{-1} = I + A + A^2 + A^3 + A^4 + \cdots$$

Proof: Exercise.

Other Matrix Norms (extra material)

Just as we have defined multiple vector norms, like

$\|v\| = \sqrt{\sum_i |v_i|^2}$ Euclidean

$\|v\|_1 = \sum_i |v_i|$ ℓ_1 norm

$\|v\|_p \; (\sum_i |v_i|^p)^{1/p}$ ℓ_p norm,

we can define many different matrix norms.

Example:

- $\|A\|_p = \max_{x : \|x\|_p = 1} \|Ax\|_p$
- $\|A\|_{p \to q} = \max_{x : \|x\|_p = 1} \|Ax\|_q$ see, eg., arXiv: 1205.4484

Exercise: What is the matrix ℓ_1 norm, $\|A\|_1$, for

$$A = \begin{pmatrix} 5 & 9 \\ -6 & 1 \end{pmatrix} ?$$

What is it in general?

Answer:

$$\|A\|_1 = \max_{x : \|x\|_1 = 1} \|Ax\|_1$$

To evaluate this, there are two steps:

1. First, we need to find an upper bound, $\|A\|_1 \le K$.
2. Second, we need to show that this bound is achieved, ie., find x with $\|x\|_1 = 1$ so $\|Ax\|_1 = K$.

(1) $$\|A\|_1 = \max_{\substack{x_1, x_2 \\ |x_1| + |x_2| = 1}} \big(|5x_1 + 9x_2| + |-6x_1 + 1x_2| \big)$$

$$\le \max_{\substack{x_1, x_2 \\ |x_1| + |x_2| = 1}} (5+6)|x_1| + (9+1)|x_2|$$

$$\le \max\{5+6, 9+1\}$$

$$= 11$$

$\Rightarrow \|A\|_1 \le 11$

(2) The bound is achieved, $\|Ax\|_1 = 11$, for $x = (1, 0)$.

$\Rightarrow \|A\|_1 = 11$.

$\Rightarrow \|A\|_1 \le 11$

② The bound is achieved, $\|Ax\|_1 = 11$, for $x = (1, 0)$.

$\Rightarrow \|A\|_1 = 11$.

In general, $\|A\|_1 = \max_{\text{columns } j} \sum_i |a_{ij}|$

the maximum ℓ_1 norm of a column. ✓

General properties of matrix norms:

All the above norms satisfy:

- $\|A\| \geq 0$, and $\|A\| = 0 \Leftrightarrow A = 0$
- $\|\alpha A\| = |\alpha| \cdot \|A\|$ for all scalars α
- triangle inequality: $\|A + B\| \le \|A\| + \|B\|$ for same-size matrices
- sub-multiplicativity: $\|AB\| \le \|A\| \cdot \|B\|$ whenever AB is defined

Exercise: Is $f(A) = \max_{i,j} |a_{ij}|$ a matrix norm? That is, does it satisfy the above properties?

Answer: It does satisfy the first three properties. But sub-multiplicativity is harder

$A = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow AB = (2)$

$f(AB) = 2 \quad f(A) = f(B) = 1$. ✓

So NO, f is not sub-multiplicative.

Example: Frobenius norm

$\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$ ← easy to compute! $\quad \|A\| \le \|A\|_F$

Exercise: This does satisfy sub-multiplicativity and the other properties.

Observe: $\|A\|_F^2 = \text{Trace}(A^\dagger A)$ (sum of diagonal elements)

$= \sum_j (A^\dagger A)_{jj}$

$= \sum_{i,j} (A^\dagger)_{ji} (A)_{ij}$

$= \sum_{i,j} \underbrace{a^*_{ij} a_{ij}}_{|a_{ij}|^2}$ ✓

Fact: The trace is cyclic:

$$\boxed{Tr(AB) = Tr(BA)}$$

Proof: $Tr(AB) = \sum_i (AB)_{ii}$

$= \sum_{i,j} a_{ij} b_{ji}$

$= \sum_{j,i} b_{ji} a_{ij} = Tr(BA)$ ☑

Corollary: The Frobenius norm is basis-independent, i.e., $\|A\|_F = \|UAU^\dagger\|_F$ for any unitary/orthogonal matrix U.

(The Frobenius norm is the same in all orthonormal bases.)

Proof: Since U is unitary, $U^\dagger = U^{-1}$.

$\|UAU^\dagger\|_F = Tr((UAU^\dagger)^\dagger (UAU^\dagger))$

$= Tr(UA^\dagger U^\dagger U A U^\dagger)$ since $(AB)^\dagger = B^\dagger A^\dagger$

$= Tr(A^\dagger U^\dagger U A U^\dagger U)$ cyclic trace

$= Tr(A^\dagger A)$ since $U^\dagger U = I$ ✓ □

The spectral norm is also basis-independent, $\|A\|_2 = \|UAU^\dagger\|_2$ for any unitary U, since unitaries don't change lengths.

Relationships between matrix norms:

Any two matrix norms are the same up to dimension-dependent factors.

Example: For $A \in \mathbb{R}^{m \times n}$,

$\|A\|_2 \le \|A\|_F \le \min\{m, n\} \cdot \|A\|_2$

Proof:

We have already shown $\|A\|_2 \le \|A\|_F$.

$\|A\|_F \le \min\{m,n\} \cdot \|A\|_2$:

We can't prove this yet! Fortunately, it is less important.

Later, it will follow since

$\|A\|_F^2 = Tr(A^*A) = $ sum of eigenvalues of A^*A

$\|A\|_2^2 = $ largest eigenvalue of A^*A ...

See http://en.wikipedia.org/wiki/Matrix_norm#Equivalence_of_norms for more.

Examples: Compute the norms of

$A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \sqrt{2}\,U$

$= \sqrt{2} \cdot \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}}_{\text{unitary}}$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$= I + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow \|A\| \le 1 + 1 = 2$

$\|A\| \le \|A\|_F = \sqrt{1^2+1^2+1^2+1^2} = 2$

$\Rightarrow \|A\| = \sqrt{2}\,\left\|\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\right\| = \sqrt{2}$

$$B = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \\ 3 & 4 & 0 & 2 \end{array}$$

$\|B\| = \max\left\{\|(1)\|, \left\|\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\right\|\right\}$

rank-1

$= \max\left\{1, \left\|\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\|^2 = 5\right\}$

$= \boxed{5}$

$C = \begin{pmatrix} 1 & 10i \\ i & 10 \end{pmatrix}$

consider $U^{-1}C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 10i \\ i & 10 \end{pmatrix}$

$= \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 0 \\ 0 & 20 \end{pmatrix}$

$= \sqrt{2}\begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$

$\boxed{\|C\| = \|U^{-1}C\| = 10\sqrt{2}}$

$C = \underbrace{\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}}_{A}\begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$

$\Rightarrow \|C\| \le \|A\| \cdot 10 = 10\sqrt{2}$

$$D = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 10i & 0 \\ 2 & i & 0 & 10 & 0 \\ 3 & 0 & 4 & 0 & 5 \end{array}$$

$\Rightarrow \|D\| = \max\left\{\|C\|, \left\|\begin{pmatrix} 4 \\ 5 \end{pmatrix}\right\|\right\}$

$= \boxed{\|C\|}$

calculus will work...

$E = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\|E\|^2 = \max_{x : \|x\|=1} \|E\vec{x}\|^2$

$\max_{\theta \in [0,\pi)} \left\|E\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\right\|^2$

$(c+2s)^2 + (3c+4s)^2$

$\frac{d}{d\theta}(\#) = 0$

$\mathcal{L} = \|E\hat{x}\|^2 + \lambda(\|x\|^2 - 1)$

$\|E\hat{x}\|^2 = |x_1 + 2x_2|^2 + |3x_1 + 4x_2|^2$

$\frac{\partial \mathcal{L}}{\partial x_1} = 0 = \frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial \mathcal{L}}{\partial \lambda}$