EE510 11/06/2020

Outline:

+ PCA
+ Eigenvalues - Eigenvectors
+ Determinant.

PCA:

$$A_k = \underset{X \in A_k}{\operatorname{argmin}} \|A - X\|_F$$

$$\Rightarrow A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T \; ; \quad \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{\nu} \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_k & & \\ & & & 0 & \\ 0 & & & & 0 \end{bmatrix} \underbrace{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}_{U^T}$$

$$\begin{array}{c} \\ 1 \\ \vdots \\ n \end{array} \begin{array}{c} x_1 \quad \text{———} \quad x_p \\ \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} \end{array}$$

data matrix

	Living Area	# of Rooms	furnished or not	distance from the beach
Apt1				
⋮				
Aptn				

Price an apartment (Prediction)

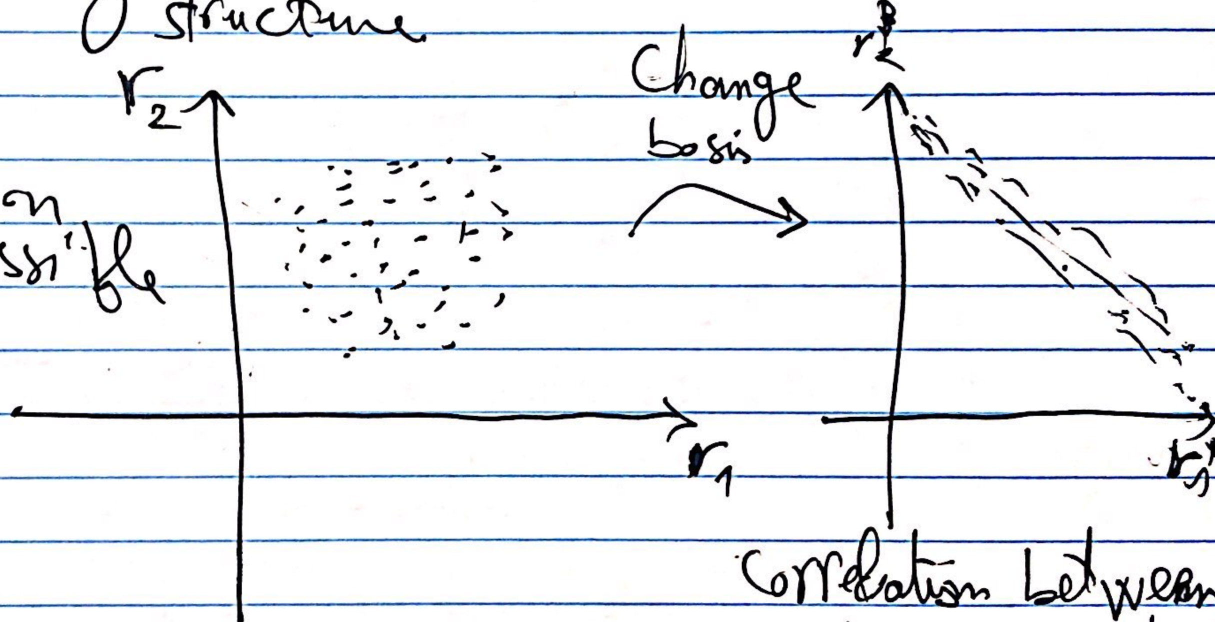
⇒ • Dimensionality reduction

- Remove redundacy
- Remove irrelevant features
- Extract hidden lower-structure from high dimensional structure

speed	distance	time

$$speed = \frac{distance}{time}$$

prediction is not possible



Correlation between r_1 and r_2 ⇒ Prediction is possible

2D subspace

$$\begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} \sigma_1 & \\ & \sigma_2 \end{pmatrix} \begin{pmatrix} - v_1^T - \\ - v_2^T - \end{pmatrix}$$

$$\begin{pmatrix} | & | \\ \sigma_1 u_1 & \sigma_2 u_2 \\ | & | \end{pmatrix}$$

+ Eigenvalue/Eigenvectors:

Def: $A \in \mathbb{R}^{n\times n}$, a nonzero vector $v \in \mathbb{R}^n$ is an eigenvector of A, if there exists $\lambda \in \mathbb{R}$

such that, $Av = \lambda v \quad (v \neq 0)$

v: eigenvector ; λ eigenvalue

• if $\lambda = 0 \Rightarrow \exists x \in \mathbb{R}^n \,/\, Ax = 0 \Rightarrow N(A) \neq \{0\}$

$\dim N(A) \geq 1$

• If A has n linearly independent eigenvectors $v_1, \ldots, v_n$

$$\Rightarrow \begin{cases} Av_1 = \lambda_1 v_1 \\ \vdots \\ Av_n = \lambda_n v_n \end{cases} \Rightarrow \begin{pmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \lambda_1 v_1 & \cdots & \lambda_n v_n \\ | & & | \end{pmatrix}$$

$$\Rightarrow A \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \underbrace{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}_{D}$$

$$\Rightarrow A = V D V^{-1}$$

$\Rightarrow$ A is diagonalizable.

$$\Rightarrow [f]_B = I[C \to B]\,[f]_C\, I[B \to C]$$

$$I[B \to C] = I[C \to B]^{-1}$$

$$A = V D V^{-1}$$

$$A^n = \underbrace{(VDV^{-1})(VDV^{-1}) \cdots (VDV^{-1})}_{n}$$

$$= V D^n V^{-1}$$

If A is diagonalizable $\Rightarrow A^n = V D^n V^{-1}$

$\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$; $\lambda_1 = 3$ $\quad v_1 = e_1$

$\lambda_2 = 4$ $\quad v_2 = e_2$

$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$; $\lambda_1 = 4$ $\quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 0$ $\quad V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}^n = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4^n & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4^n & 4^n \\ 4^n & 4^n \end{pmatrix}$$

$$\begin{pmatrix} \boxed{3} & 0 & 0 & \boxed{0} \\ 0 & \boxed{2} & \boxed{2} & 0 \\ 0 & \boxed{2} & \boxed{2} & 0 \\ \boxed{0} & 0 & 0 & \boxed{4} \end{pmatrix} \qquad \lambda_1 = 3 \qquad \lambda_2 = 4 \qquad \lambda_3 = 0$$

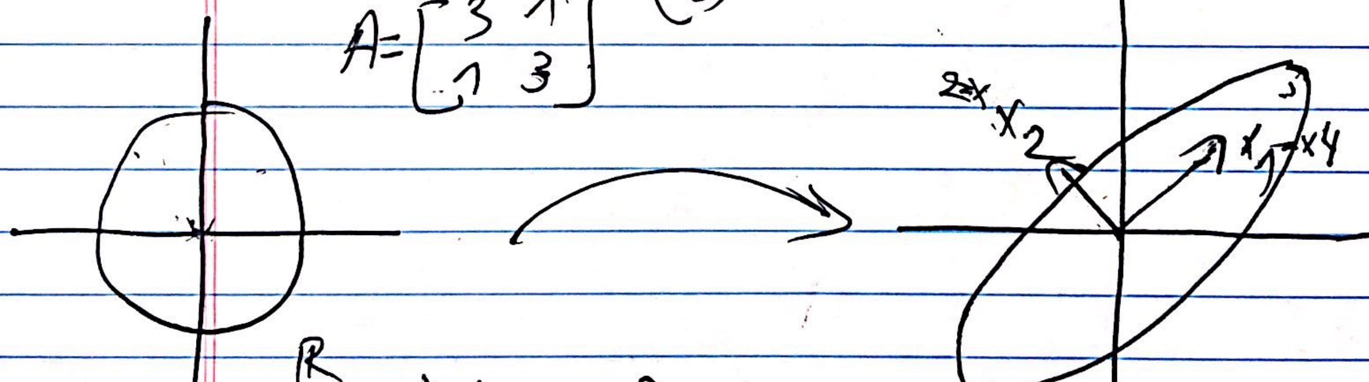
$v_1 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $\cancel{v_2 = e}$; $v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = e_4$; $v_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

$$\lambda_1 = 3 \Rightarrow v_1 = e_1$$

$$\lambda_2 = 4 \Rightarrow v_2 = e_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} ; \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

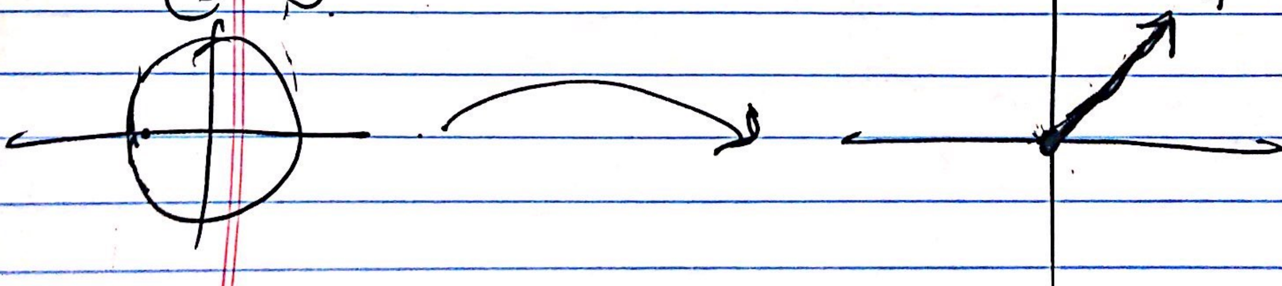
$$\lambda_3 = 0 \Rightarrow v_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$



$\lambda_1 x_1$, $\lambda_2 x_2$

$$A = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$$



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Finding eigenvalues

$$Ax = \lambda_i x \Rightarrow (A-\lambda_i I)x = 0 \Rightarrow x \in N(A-\lambda_i I)$$

$$\Rightarrow \dim N(A-\lambda_i I) \geq 1$$

$\Rightarrow A-\lambda_i I$ is a singular matrix $\Rightarrow$ ~~dim~~

$$\det(A-\lambda_i I) = 0$$

$$P(\lambda) = \det(A-\lambda I)$$

The roots of P ($P(\lambda_i)=0$) are the eigenvalues for the matrix A.

$$\det(A) = \sum_{\sigma} \operatorname{sign}\sigma \prod_{i=1}^{n} A_{i\sigma(i)}$$

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}, \quad B = \begin{bmatrix} | & | & & | \\ cA_1 & A_2 & \cdots & A_n \\ | & | & & | \end{bmatrix}; \quad D = \begin{bmatrix} | & | & | & \\ cA_1 & dA_2 & A_3 & \cdots \\ | & | & | & \end{bmatrix}$$

$$\det(B) = c\det(A)$$

$$\det(D) = cd\det(A)$$

$\Rightarrow \det(\cdot)$ is a multilinear.

- $\det(AB) = \det(A)\det(B)$
- $\det(A^{-1}) = \dfrac{1}{\det(A)}$
- $N = V \cdot M V^{-1} \Rightarrow \det(N) = \det(M)$
- $\det(A^T) = \det(A)$
- $\det(c_1, \dots, a_k + b_k, \dots c_n) = \det(c_1, \dots, a_k, \dots c_n) + \det(c_1, \dots, b_k, \dots c_n)$

$$\det(A+B) \neq \det(A) + \det(B)$$

Exercice 1

$A^T = -A \ , \ A \in \mathbb{R}^{n \times n} \ ; \ n = 2p+1$

$$\det(A^T) = \det(-A) = (-1)^{2p+1} \det(A) = -\det(A)$$

$$= \det(A) \Rightarrow \det(A) = 0 .$$

$$\begin{array}{ccccc} + & + & + & - & - \ \ - \\ a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \dots$$

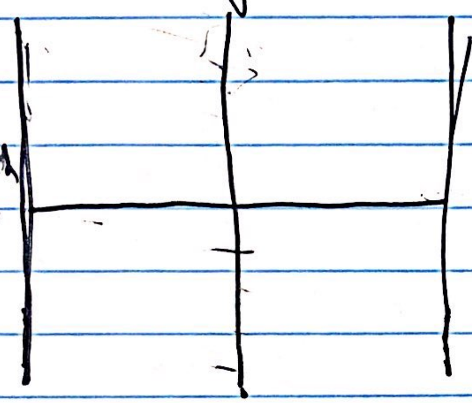
Cofactors:

$\det\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}$

$$C_{ij} = (-1)^{i+j} A_{ij}$$

Cofactor   minor

j^{th}

$A_{ij} =$ i^{th} 

$$C = \begin{pmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & & \vdots \\ C_{n1} & \cdots & C_{nn} \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A} C^T$$

$$\det(M) = \sum_{i=1}^{n} m_{ij} (-1)^{i+j} A_{ij} = \sum_{i=1}^{n} m_{ij} C_{ij} \quad \forall j$$

$$\det(M) = \sum_{j=1}^{n} m_{ij} (-1)^{i+1} A_{ij} = \sum_{j=1} m_{ij} C_{ij} \quad \forall i$$

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$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 0 & \boxed{1} & 1 \end{pmatrix} = 1 \times (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} + 1 \times (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$$

$$= -\underbrace{\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix}}_{-11} + \underbrace{\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}}_{-6} = 5.$$