Exam sample problems

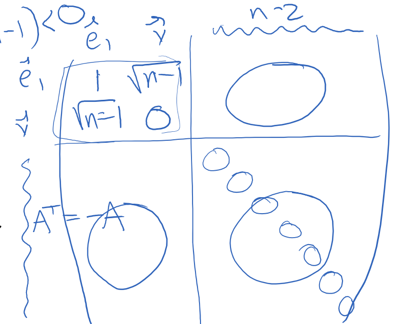
Final Thursday, Dec. 3 8am–11pm Pacific

① Let

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & & & & \\ 1 & & 0 & & \\ 1 & & & & \\ 1 & & & & \end{pmatrix}_{n\times n}$$

$\mathrm{Det}\begin{pmatrix} 1 & \sqrt{n-1} \\ \sqrt{n-1} & 0\end{pmatrix} = -(n-1) < 0$

$= \lambda_1 \lambda_2$



a) Which of the following classes does A belong to?

Symmetric, Hermitian, Normal, ~~Skew-symmetric~~

~~Orthogonal~~ ~~Unitary~~ ~~Isometry~~ ~~Invertible~~

? Positive semi-def. ~~Positive definite~~ Singular

~~Stochastic~~ Diagonalizable

Rank(A) = 2

b) What are the dimensions of

R(A) R(A^T) N(A) N(A^T)

2 2 n-2 n-2

Give ~~a~~ an orthonormal basis for each space.

$\vec e_1, \frac{1}{\sqrt{n-1}}\begin{pmatrix}0\\1\\\vdots\\1\end{pmatrix} = \vec v$

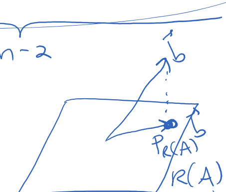
$A\vec e_1 = \vec e_1 + \sqrt{n-1}\,\vec v$

$A\vec v = \sqrt{n-1}\,\vec e_1$

$\underbrace{\frac{1}{\sqrt2}(\vec e_2 - \vec e_n), \frac{1}{\sqrt2}(\vec e_3 - \vec e_n), \ldots, \frac{1}{\sqrt2}(\vec e_{n-1} - \vec e_n)}_{n-2}$

c) Give a least-squares solution $\vec x$ to $A\vec x = \vec b = \begin{pmatrix}1\\1\\1\\3\\3\end{pmatrix} \in \mathbb{R}^5$

That is, find $\arg\min_{\vec x} \|A\vec x - \vec b\|$.



$P_{R(A)}\vec b = \vec e_1(e_1\cdot b) + \vec v(v\cdot b) = \vec e_1 + 4\vec v$

$= \begin{pmatrix}1\\2\\2\\2\\2\end{pmatrix} = A\vec x \Rightarrow \vec x = \begin{pmatrix}2\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\-1\\0\\0\end{pmatrix}$

$\begin{pmatrix}2\\-1/4\\-1/4\\-1/4\\-1/4\end{pmatrix}$

c) What are:

Rank(A) – 2

Det(A) – 0

Tr(A) – 1

$\|A\|$ =

$B = \begin{pmatrix}1 & \sqrt{n-1}\\ \sqrt{n-1} & 0\end{pmatrix}$

$\|B\|^2 = \max_x \frac{\left\|B\begin{pmatrix}1\\x\end{pmatrix}\right\|^2}{1+x^2}$

d) Without computing the spectral decomposition, argue that A has a positive e-value and a negative e-value.

since $\mathrm{Det}\begin{pmatrix}1 & \sqrt{n-1}\\ \sqrt{n-1} & 0\end{pmatrix} = -(n-1) < 0$

$= \lambda_1\lambda_2$

$\begin{pmatrix}1 & \sqrt{n-1}\\ \sqrt{n-1} & 0\end{pmatrix}\begin{pmatrix}1\\c\end{pmatrix} = \lambda\begin{pmatrix}1\\c\end{pmatrix}$

e) Give the spectral decomposition of A.

Eigenvalues **Eigenvectors**

$\lambda_1 =$

$\lambda_2 = \frac{1\pm\sqrt{4n-3}}{2}$

we can solve

$\lambda_1 + \lambda_2 = 1$

$\Rightarrow x(1-x) = b$

$-x^2 + x - b = 0$

$\Rightarrow x = \frac{\pm 1 \pm \sqrt{1 \pm 4b}}{\pm 2}$

f) Give a singular-value decomposition of A.

Sing. values	Left s.v.s	Right s.v.s
$\lvert\lambda_1\rvert$	$\frac{\lambda_1}{\lvert\lambda_1\rvert}\vec v_1$	$\vec v_1$
$\lvert\lambda_2\rvert$	$\frac{\lambda_2}{\lvert\lambda_2\rvert}\vec v_2$	$\vec v_2$

A normal $\Rightarrow$ sing. values = |eigenvalues|

$\Downarrow$

orthonormal!

$A = \sum_j \lambda_j \vec v_j \vec v_j^{\,t} \underset{\text{SVD}}{=} \sum_j \underbrace{|\lambda_j|}_{\text{singular values}} \underbrace{\left(\frac{\lambda_j}{|\lambda_j|} v_j\right)}_{\text{left s.v.s}} \underbrace{v_j^{\,t}}_{\text{rt sing. vectors}}$

g) Give a closed formula for A^n. for integer $n \geq 1$.

$\cdots$ SVD $\cdots$ left s.v.s $\cdots$ rt sing. vectors

g) Give a closed formula for A^n, for integer $n \geq 1$.
Then give a closed formula for A^p, for arbitrary $p \in \mathbb{R}$.
What is the matrix exponential e^A?

$$A = UDU^{-1}$$

$$A^n = UD^nU^{-1} = U\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}U^{-1}$$

$$e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!} = Ue^DU^{-1}$$

$$= U\begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}U^{-1}$$

$$U = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \qquad U^{-1} = U^\dagger = \begin{pmatrix} — v_1^\dagger — \\ — v_2^\dagger — \end{pmatrix}$$

Problem 2. *(12 points)* Let

$$A = \left(\begin{array}{cc|cc} 0 & 0 & 4 & -2 \\ 0 & 0 & -2 & 1 \\ \hline 4 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{array}\right).$$

$A = A^T$ symmetric/Hermitian, normal, diagonalizable

(a) Diagonalize the matrix A. That is, find its eigenvalues and corresponding eigenspaces. (**Hint:** A is a big matrix, so try to simplify the problem before attempting to compute $\mathrm{Det}(A - \lambda I)$ without a calculator.)

$$= \left(\begin{array}{c|c} 0 & B \\ \hline B & 0 \end{array}\right), \quad B = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}(2 \ -1)$$

e-values	e-vectors
5	$\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
0	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

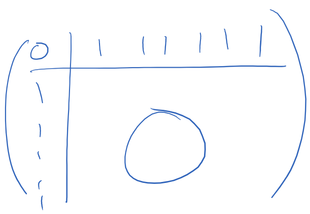
bipartite $(\lambda, -\lambda)$

Rank(A) = 2

$$B\begin{pmatrix} 2 \\ -1 \end{pmatrix} = 5\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

eigenvalue	eigenvector
0	(1, 2, 1, 2)
0	(1, 2, −1, −2)
5	(2, −1, 2, −1)
−5	(2, −1, −2, 1)

$$A\begin{pmatrix} 2 \\ -1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} = 5\begin{pmatrix} 2 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$



(b) Is A^2 positive semi-definite? Circle your answer: Yes No Why or why not?

Here are a few more questions you might ask yourself. All of these you should be able to solve without a calculator.

- What is the rank of A? Give the dimensions of the rowspace, columnspace and nullspace.
- Is A ~~invertible~~? If so, what is A^{-1}?
- Is A normal? ✓
- Is A symmetric? ✓
- ~~Is A unitary? Is it orthogonal? Is it an isometry?~~
- What is the norm, $\|A\|$? $= 5$
- What is the determinant of A? $= 0$ The trace of A? $= 0$
- What is the singular-value decomposition of A?

Example: $A = \left(\begin{array}{c|c} 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \\ \hline \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & 0 \end{array}\right)$

invertible, isometry, ~~unitary~~, $\|A\| = 1$, $\kappa = 1$, permutation, ~~symmetric~~, normal, stochastic, bipartite, order 4 $A^4 = I$

SVD

σ_i	$\vec{v}_i$	$\vec{u}_i$
1	$\vec{e}_4$	$\vec{e}_1$
1	$\vec{e}_3$	$\vec{e}_2$
1	$\vec{e}_1$	$\vec{e}_3$
1	$\vec{e}_2$	$\vec{e}_4$

Spectral decomposition

λ_i	$\vec{v}_i$
1	(1, 1, 1, 1)
−1	(1, 1, −1, −1)
i	(1, −1, i, $-i$)
$-i$	(1, −1, $-i$, i)

What about $B = \begin{pmatrix} 3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & 2 & 3 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix}$? $= 3I + 2A$

$2+3=5$
$-2+3=1$
$2i+3$
$-2i+3$

$C = \begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 3 & 2 & 0 \\ 0 & 2 & 3 & 0 \\ 2 & 0 & 0 & 3 \end{pmatrix}$?

e-values	evectors
5	(0, a, b, 0)
1	(0, c, d, 0)
5	(a, 0, 0, b)
1	(c, 0, 0, d)

$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = 3I + 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$3+2=5$
$3-2=1$

$(a,b) = (1,1)$
$(c,d) = (1,-1)$

Example: Specify the rank-one matrix B that minimizes $\|A-B\|_F = \sqrt{\sum_{ij} |a_{ij} - b_{ij}|^2}$

$$A = \begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & -1 & 7 \end{pmatrix} = \begin{pmatrix} 3 & & 3 \\ 3 & & 3 \end{pmatrix} + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & -4 \ 4 \end{array}\right)$$

Example: Specify the rank-one matrix B that minimizes $\|A-B\|_F = \sqrt{\sum_{ij} |a_{ij}-b_{ij}|^2}$

$$A = \begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & -1 & 7 \\ 3 & 3 & 7 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & & 3 \\ 3 & & & 3 \\ & 3 & 3 & \end{pmatrix} + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{smallmatrix} -4 & 4 \\ 4 & -4 \end{smallmatrix} \end{array}\right) \qquad B = 3\,\vec{1}\,\vec{1}^T$$

$\mathrm{Rank}(A) = 2$

$$= \frac{12}{4}\vec{1}\vec{1}^T - \frac{8}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}(0\ 0\ 1\ -1)$$

$$D = \left(\begin{array}{c|c} 3-0 & 4-2 \\ 0-3 & 2-4 \\ \hline 0-6 & 1-5 \\ 5-1 & 4-2 \end{array}\right) \longleftarrow \begin{pmatrix} 3 & 6 \\ 6 & 6 \end{pmatrix}$$

More Review Material [Strang]

Matrices	Eigenvalues	Eigenvectors
Symmetric: $A^T = A$	real λ's	orthogonal $x_i^T x_j = 0$
Orthogonal: $Q^T = Q^{-1}$	all $\lvert\lambda\rvert = 1$	orthogonal $\bar{x}_i^T x_j = 0$
Skew-symmetric: $A^T = -A$	imaginary λ's	orthogonal $\bar{x}_i^T x_j = 0$
Complex Hermitian: $\bar{A}^T = A$	real λ's	orthogonal $\bar{x}_i^T x_j = 0$
Positive Definite: $x^T A x > 0$	all $\lambda > 0$	orthogonal since $A^T = A$
Markov: $m_{ij} > 0, \sum_{i=1}^n m_{ij} = 1$	$\lambda_{\max} = 1$	steady state $x > 0$
Similar: $B = M^{-1}AM$	$\lambda(B) = \lambda(A)$	$x(B) = M^{-1}x(A)$
Projection: $P = P^2 = P^T$	$\lambda = 1;\ 0$	column space; nullspace
Plane Rotation	$e^{i\theta}$ and $e^{-i\theta}$	$x = (1, i)$ and $(1, -i)$
Reflection: $I - 2uu^T$	$\lambda = -1;\ 1, \ldots, 1$	u; whole plane $u^\perp$
Rank One: uv^T	$\lambda = v^T u;\ 0, \ldots, 0$	u; whole plane $v^\perp$
Inverse: A^{-1}	$1/\lambda(A)$	keep eigenvectors of A
Shift: $A + cI$	$\lambda(A) + c$	keep eigenvectors of A
Stable Powers: $A^n \to 0$	all $\lvert\lambda\rvert < 1$	any eigenvectors
Stable Exponential: $e^{At} \to 0$	all Re $\lambda < 0$	any eigenvectors
Cyclic Permutation: row 1 of I last	$\lambda_k = e^{2\pi ik/n}$	$x_k = (1, \lambda_k, \ldots, \lambda_k^{n-1})$
Tridiagonal: $-1, 2, -1$ on diagonals	$\lambda_k = 2 - 2\cos\frac{k\pi}{n+1}$	$x_k = \left(\sin\frac{k\pi}{n+1}, \sin\frac{2k\pi}{n+1}, \ldots\right)$
Diagonalizable: $A = S\Lambda S^{-1}$	diagonal of Λ	columns of S are independent
Symmetric: $A = Q\Lambda Q^T$	diagonal of Λ (real)	columns of Q are orthonormal
Schur: $A = QTQ^{-1}$	diagonal of T	columns of Q *if* $A^TA = AA^T$
Jordan: $J = M^{-1}AM$	diagonal of J	each block gives $x = (0, \ldots, 1, \ldots, 0)$
Rectangular: $A = U\Sigma V^T$	$\mathrm{rank}(A) = \mathrm{rank}(\Sigma)$	eigenvectors of A^TA, AA^T in V, U

LINEAR ALGEBRA IN A NUTSHELL

((The matrix A is n by n))

Nonsingular	**Singular**
A is invertible	A is not invertible
The columns are independent	The columns are dependent
The rows are independent	The rows are dependent
The determinant is not zero	The determinant is zero
$Ax = 0$ has one solution $x = 0$	$Ax = 0$ has infinitely many solutions
$Ax = b$ has one solution $x = A^{-1}b$	$Ax = b$ has no solution or infinitely many
A has n (nonzero) pivots	A has $r < n$ pivots
A has full rank $r = n$	A has rank $r < n$
The reduced row echelon form is $R = I$	R has at least one zero row
The column space is all of $\mathbf{R}^n$	The column space has dimension $r < n$
The row space is all of $\mathbf{R}^n$	The row space has dimension $r < n$
All eigenvalues are nonzero	Zero is an eigenvalue of A
A^TA is symmetric positive definite	A^TA is only semidefinite
A has n (positive) singular values	A has $r < n$ singular values

Fact: If $A \succeq 0$, then any submatrix using the same rows & columns

Fact: If $A \succ 0$, then any submatrix using the same rows & columns

$$\begin{array}{c|ccc} & x & 0 & y \\ \hline x & 1 & \cancel{2} & 3 \\ 0 & \cancel{2} & \cancel{4} & \cancel{5} \\ y & 3 & \cancel{5} & 6 \end{array}$$

Fact: If $A \succeq 0$, then any submatrix using the same rows & columns must also be $\succeq 0$.

Fact: If $A \succ 0$, then any submatrix using the same rows & columns must also be $\succ 0$.

$$\begin{array}{c|ccc} & & & \\ x & 1 & \cancel{2} & 3 \\ o & \cancel{2} & \cancel{4} & \cancel{5} \\ y & 3 & \cancel{5} & 6 \end{array} \qquad = \begin{array}{c|cc} & x & y \\ x & 1 & 3 \\ y & 3 & 6 \end{array}$$

Conceptual review problems [Strang]

$\forall x \quad x^t A x \geq 0 \iff A \text{ p.s.d.}$

$\forall x \neq \vec{0} \quad x^t A x > 0 \iff A \text{ p.d.}$

§1 Vectors and matrices

1.1 Which vectors are linear combinations of $v = (3, 1)$ and $w = (4, 3)$?

1.2 Compare the dot product of $v = (3, 1)$ and $w = (4, 3)$ to the product of their lengths. Which is larger? Whose inequality?

1.3 What is the cosine of the angle between v and w in Question 1.2? What is the cosine of the angle between the x-axis and v?

$A = \begin{pmatrix} I & U \\ 0 & I \end{pmatrix} \quad A\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} x + Uy \\ \vec{y} \end{pmatrix} \rightarrow A^{-1} = \begin{pmatrix} I & -U \\ 0 & I \end{pmatrix}$

§2 Solving linear equations

2.1 Multiplying a matrix A times the column vector $x = (2, -1)$ gives what combination of the columns of A? How many rows and columns in A?

2.2 If $Ax = b$ then the vector b is a linear combination of what vectors from the matrix A? In vector space language, b lies in the _____ space of A.

2.3 If A is the 2 by 2 matrix $\left[\begin{smallmatrix} 2 & 1 \\ 6 & 6 \end{smallmatrix}\right]$ what are its pivots?

2.4 If A is the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ how does elimination proceed? What permutation matrix P is involved?

2.5 If A is the matrix $\left[\begin{smallmatrix} 2 & 1 \\ 6 & 3 \end{smallmatrix}\right]$ find b and c so that $Ax = b$ has no solution and $Ax = c$ has a solution.

2.6 What 3 by 3 matrix L adds 5 times row 2 to row 3 and then adds 2 times row 1 to row 2, when it multiplies a matrix with three rows?

2.7 What 3 by 3 matrix E subtracts 2 times row 1 from row 2 and then subtracts 5 times row 2 from row 3? How is E related to L in Question 2.6?

2.8 If A is 4 by 3 and B is 3 by 7, how many *row times column* products go into AB? How many *column times row* products go into AB? How many separate small multiplications are involved (the same for both)?

2.9 Suppose $A = \left[\begin{smallmatrix} I & U \\ 0 & I \end{smallmatrix}\right]$ is a matrix with 2 by 2 blocks. What is the inverse matrix?

2.10 How can you find the inverse of A by working with $[A \;\; I]$? If you solve the n equations Ax = columns of I then the solutions x are columns of _____.

2.11 How does elimination decide whether a square matrix A is invertible?

2.12 Suppose elimination takes A to U (upper triangular) by row operations with the multipliers in L (lower triangular). Why does the last row of A agree with the last row of L times U?

2.13 What is the factorization (from elimination with possible row exchanges) of any square invertible matrix?

2.14 What is the transpose of the inverse of AB?

2.15 How do you know that the inverse of a permutation matrix is a permutation matrix? How is it related to the transpose?

§3 Vector spaces

3.1 What is the column space of an invertible n by n matrix? What is the nullspace of that matrix?

3.2 If every column of A is a multiple of the first column, what is the column space of A?

3.3 What are the two requirements for a set of vectors in $\mathbf{R}^n$ to be a subspace?

3.4 If the row reduced form R of a matrix A begins with a row of ones, how do you know that the other rows of R are zero and what is the nullspace?

3.5 Suppose the nullspace of A contains only the zero vector. What can you say about solutions to $Ax = b$?

3.6 From the row reduced form R, how would you decide the rank of A?

3.7 Suppose column 4 of A is the sum of columns 1, 2, and 3. Find a vector in the nullspace.

3.8 Describe in words the complete solution to a linear system $Ax = b$.

3.9 If $Ax = b$ has exactly one solution for every b, what can you say about A?

3.10 Give an example of vectors that span $\mathbf{R}^2$ but are not a basis for $\mathbf{R}^2$.

3.11 What is the dimension of the space of 4 by 4 symmetric matrices?

3.12 Describe the meaning of *basis* and *dimension* of a vector space.

3.13 Why is every row of A perpendicular to every vector in the nullspace?

3.14 How do you know that a column u times a row v^T (both nonzero) has rank 1?

3.15 What are the dimensions of the four fundamental subspaces, if A is 6 by 3 with rank 2?

3.16 What is the row reduced form R of a 3 by 4 matrix of all 2's?

3.17 Describe a *pivot column* of A.

3.18 True? The vectors in the left nullspace of A have the form $A^T y$.

3.19 Why do the columns of every invertible matrix yield a basis?

§4 Orthogonality and projections

4.1 What does the word *complement* mean about orthogonal subspaces?

4.2 If V is a subspace of the 7-dimensional space $\mathbf{R}^7$, the dimensions of V and its orthogonal complement add to _____.

4.3 The projection of b onto the line through a is the vector _____.

4.4 The projection matrix onto the line through a is P = _____.

4.5 The key equation to project b onto the column space of A is the *normal equation* _____.

4.6 The matrix $A^T A$ is invertible when the columns of A are _____.

4.7 The least squares solution to $Ax = b$ minimizes what error function?

4.8 What is the connection between the least squares solution of $Ax = b$ and the idea of projection onto the column space?

4.9 If you graph the best straight line to a set of 10 data points, what shape is the matrix A and where does the projection p appear in the graph?

4.10 If the columns of Q are orthonormal, why is $Q^T Q = I$?

4.11 What is the projection matrix P onto the columns of Q?

4.12 If Gram-Schmidt starts with the vectors $a = (2, 0)$ and $b = (1, 1)$, which two orthonormal vectors does it produce? If we keep $a = (2, 0)$ does Gram-Schmidt always produce the same two orthonormal vectors?

4.13 True? Every permutation matrix is an orthogonal matrix.

4.14 The inverse of the orthogonal matrix Q is _____.

§5 Determinants

$\operatorname{Det} I = 1$

5.1 What is the determinant of the matrix $-I$? $\operatorname{Det}(-I) = (-1)^n$

5.2 Explain how the determinant is a linear function of the first row.

5.3 How do you know that $\det A^{-1} = 1/\det A$?

5.4 If the pivots of A (with no row exchanges) are 2, 6, 6, what submatrices of A have known determinants?

5.5 Suppose the first row of A is 0, 0, 0, 3. What does the "big formula" for the determinant of A reduce to in this case?

5.6 Is the ordering (2, 5, 3, 4, 1) even or odd? What permutation matrix has what determinant, from your answer?

5.7 What is the cofactor C_{23} in the 3 by 3 elimination matrix E that subtracts 4 times row 1 from row 2? What entry of E^{-1} is revealed?

5.8 Explain the meaning of the cofactor formula for $\det A$ using column 1.

5.9 How does Cramer's Rule give the first component in the solution to $Ix = b$?

5.10 If I combine the entries in row 2 with the cofactors from row 1, why is $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$ automatically zero?

5.11 What is the connection between determinants and volumes?

5.12 Find the cross product of $u = (0, 0, 1)$ and $v = (0, 1, 0)$ and its direction.

5.13 If A is n by n, why is $\det(A - \lambda I)$ a polynomial in λ of degree n?

§6 Eigenvalues and eigenvectors

6.1 What equation gives the eigenvalues of A without involving the eigenvectors? How would you then find the eigenvectors?

6.2 If A is singular what does this say about its eigenvalues?

6.3 If A times A equals $4A$, what numbers can be eigenvalues of A?

6.4 Find a real matrix that has no real eigenvalues or eigenvectors.

6.5 How can you find the sum and product of the eigenvalues directly from A?

6.6 What are the eigenvalues of the rank one matrix $[1\ 2\ 1]^T[1\ 1\ 1]$?

6.7 Explain the diagonalization formula $A = S\Lambda S^{-1}$. Why is it true and when is it true?

6.8 What is the difference between the algebraic and geometric multiplicities of an eigenvalue of A? Which might be larger?

6.9 Explain why the trace of AB equals the trace of BA.

6.10 How do the eigenvectors of A help to solve $du/dt = Au$?

6.11 How do the eigenvectors of A help to solve $u_{k+1} = Au_k$?

6.12 Define the matrix exponential e^A and its inverse and its square.

6.13 If A is symmetric, what is special about its eigenvectors? Do any other matrices have eigenvectors with this property?

6.14 What is the diagonalization formula when A is symmetric?

6.15 What does it mean to say that A is *positive definite*?

6.16 When is $B = A^T A$ a positive definite matrix (A is real)?

6.17 If A is positive definite describe the surface $x^T A x = 1$ in $\mathbf{R}^n$.

6.18 What does it mean for A and B to be *similar*? What is sure to be the same for A and B?

6.19 The 3 by 3 matrix with ones for $i \geq j$ has what Jordan form?

6.20 The SVD expresses A as a product of what three types of matrices?

6.21 How is the SVD for A linked to $A^T A$?

§7 Linear transformations

7.1 Define a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$ and give one example.

7.2 If the upper middle house on the cover of the book is the original, find something nonlinear in the transformations of the other eight houses.

7.3 If a linear transformation takes every vector in the input basis into the next basis vector (and the last into zero), what is its matrix?

7.4 Suppose we change from the standard basis (the columns of I) to the basis given by the columns of A (invertible matrix). What is the change of basis matrix M?

7.5 Suppose our new basis is formed from the eigenvectors of a matrix A. What matrix represents A in this new basis?

7.6 If A and B are the matrices representing linear transformations S and T on $\mathbf{R}^n$, what matrix represents the transformation from v to $S(T(v))$?

7.7 Describe five important factorizations of a matrix A and explain when each of them succeeds (what conditions on A?).

Previously: Solve linear equations
$$A\vec{x} = \vec{b}$$

Today: Linear programs

Solve linear inequalities $A\vec{x} \leq \vec{b}$ $\quad \begin{pmatrix}1\\2\\3\end{pmatrix} \leq \begin{pmatrix}4\\5\\6\end{pmatrix}$

coordinatewise $\leq$

and optimize a linear objective over the solution set:

"primal"		"dual"
maximize$_{\vec{x}\in\mathbb{R}^n}$ $\vec{c}\cdot\vec{x}$ subject to $A\vec{x} \leq \vec{b}$, $\vec{x} \geq 0$ (coordinatewise)	$\longleftrightarrow$	minimize$_{\vec{y}\in\mathbb{R}^m}$ $\vec{y}\cdot\vec{b}$ s.t. $A^T\vec{y} \geq \vec{c}$, $\vec{y} \geq 0$ (coordinatewise)

Applications:

Linear programming can be applied to various fields of study. It is widely used in business and economics, and is also utilized for some engineering problems. Industries that use linear programming models include transportation, energy, telecommunications, and manufacturing. It has proven useful in modeling diverse types of problems in planning, routing, scheduling, assignment, and design. https://en.wikipedia.org/wiki/Linear_programming

$\sim 10^5$ variables

Next: Semidefinite programs

Optimize with linear constraints on positive semidef. matrices

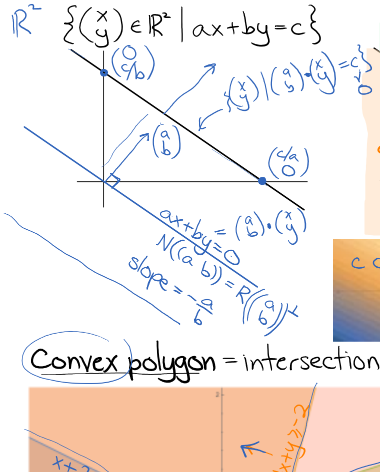
$$\begin{aligned} \text{minimize} \quad & \langle C, X\rangle \\ \text{s.t.} \quad & \langle A_i, X\rangle = b_i,\ i = 1,\dots,m \\ & X \succeq 0 \ \text{(pos. semidef.)} \end{aligned}$$

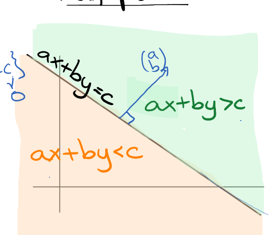
• Polytope geometry • Linear programs • Duality • Examples

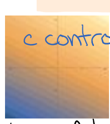
Geometry: Planes, halfspaces, polytopes

Lines: $\mathbb{R}^2$ $\left\{\begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2 \;\middle|\; ax+by=c\right\}$

Halfspaces:

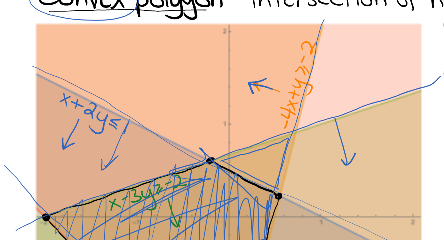


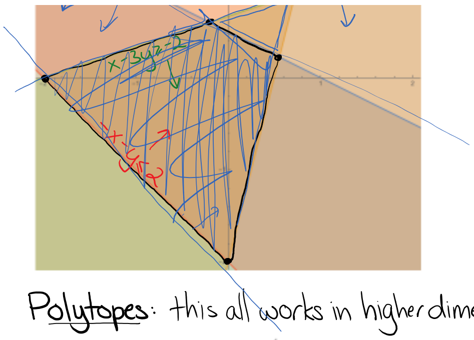




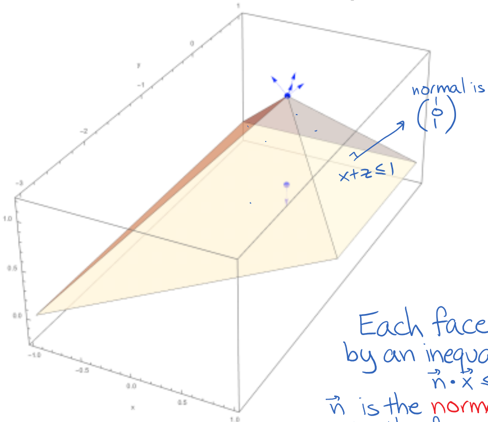
c controls translation from $\vec{0}$

Convex polygon = intersection of halfspaces





Polytopes: this all works in higher dimensions



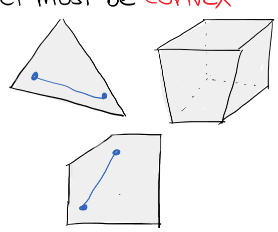
$\vec{n}\cdot\vec{x}\le b$

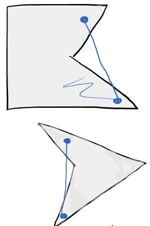
Each face is specified by an inequality $\vec{n}\cdot\vec{x}\le b$

$\vec{n}$ is the normal, perpendicular to the face

Remarks:

- Set must be convex | not convex:

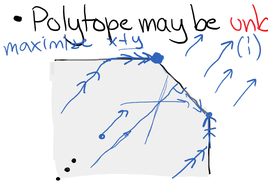


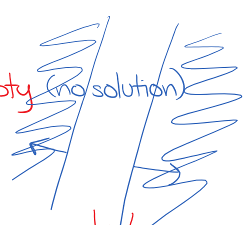


convex = between two points you stay in the set

$\vec{x},\vec{y}\in S \;\Rightarrow\; (1-p)\vec{x}+p\vec{y}\in S$ for $0\le p\le 1$

- Polytope may be unbounded or empty (no solution)

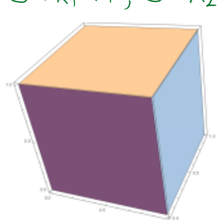




- High-dim. polytopes can be very complex!

Example: Hypercube

$0\le x_1\le 1,\; 0\le x_2\le 1,\; \dots,\; 0\le x_n\le 1$

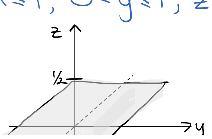


$2n$ inequalities ($2n$ faces)

but 2^n vertices!

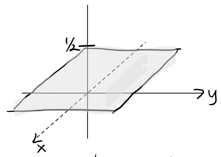
- Equalities are also okay

$0\le x\le 1,\; 0\le y\le 1,\; z=\tfrac{1}{2}$ in $\mathbb{R}^3$



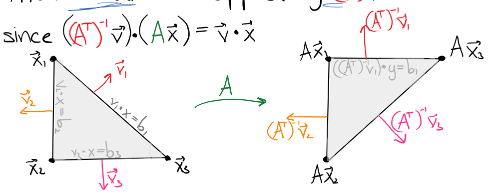
$z\ge\frac{1}{2}$

$z\le\frac{1}{2}$



$z \geq \frac{1}{2}$
$z \leq \frac{1}{2}$

- Mapping a polytope forward by A, the normals are mapped by $(A^T)^{-1}$ since $((A^T)^{-1}\vec{v})\cdot(A\vec{x}) = \vec{v}\cdot\vec{x}$



in this example, $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $(A^T)^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

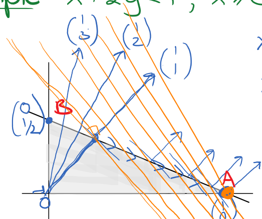
Linear programs (LPs)

A linear program maximizes (or minimizes) a linear function over a polytope (given by inequalities).

e.g., $f(x,y,z) = ax+by+cz$
$= \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

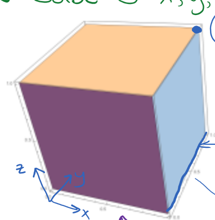
Observe: The maximum is at a vertex (or possibly all along a facet).

Example: $x+2y \leq 1$, $x \geq 0$, $y \geq 0$



$x+y$ is maximized at A
$x+3y$ maximized at B
$x+2y$ maximized along $\overline{AB}$

Example: Cube $0 \leq x,y,z \leq 1$



$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ optimizes $x+2y+3z$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

this whole edge maximize $x-z$
$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

this whole face maximizes $-x$
$\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

Examples: LPs in Microeconomics

Example:

LP: $\underset{\vec{x} \in R^4}{\text{minimize}}\ 3x_1 + x_2 + x_3 + 2x_4$

s.t. $\begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix} \vec{x} \geq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{x} \geq 0$ in each coordinate

Story:
Your firm has four production processes, that can be used to produce two outputs.
Each process uses some amount of labor, and produces one of the two outputs.
(Some processes use the other output as an intermediate input.)
You want to minimize the labor cost to produce given amounts of each output.

import cvxpy

input
output

Matlab:

```
>> c = [3 1 1 2]';
A = [1 1 -2 0; 0 -1 1 1];
b = [1 1]';

cvx_begin
  variable x(4,1)
  minimize (c' * x)
  A * x >= b
  x >= 0
```

$\Longrightarrow$

```
Status: Solved
Optimal value (cvx_optval): +5

>> x

x =

    0.5849
    0.4151
```

```
A = [1 1 -2 0; 0 -1 1 1];
b = [1 1]';

cvx_begin
  variable x(4,1)
  minimize (c' * x)
  A * x >= b
  x >= 0
cvx_end
```

$\Longrightarrow$

```
Status: Solved
Optimal value (cvx_optval): +5

>> x

x =

    0.5849
    0.4151
    0.0000
    1.4151
```

```
>> cvx_begin
  variable x(4,1)
  dual variables y1 y2
  minimize (c' * x)
  A(1,:) * x >= b(1) : y1
  A(2,:) * x >= b(2) : y2
  x >= 0
cvx_end
```

$\Longrightarrow$

```
>> [y1 y2]

ans =

    3.0000    2.0000
```

The dual variables can be interpreted as shadow costs.

Example:

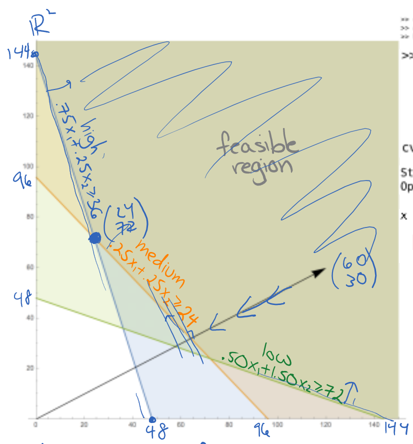
The Silverado Mining Company's contracts with uranium-processing plants require it to produce a certain amount of each grade of uranium ore every week.

Type of ore	Output (tons/hour) Mine A	Mine B	Requirements (tons/week)
High grade	.75	.25	36
Medium grade	.25	.25	24
Low grade	.50	1.50	72
Operating cost ($/hour)	$60	$30	

Variables: x_1 = # hours to operate A
x_2 = # hours for mine B

Objective: minimize $\begin{pmatrix}60\\30\end{pmatrix}\cdot\vec{x}$

Constraints $\begin{pmatrix}.75 & .25\\ .25 & .25\\ .5 & 1.5\end{pmatrix}\vec{x} \geq \begin{pmatrix}36\\24\\72\end{pmatrix}$




```
>> c = [60 30]';
>> A = [.75 .25; .25 .25; .50 1.5];
>> b = [36 24 72]';
>> cvx_begin
  variable x(2)
  dual variable y
  minimize (c' * x)
  A * x >= b : y
  x >= 0
cvx_end

Status: Solved
Optimal value (cvx_optval): +3600

x =           >> A * x       y =

   24.0000    ans =             60.0000
   72.0000                      60.0000
                 36.0000         0.0000
                 24.0000
                120.0000
```

"shadow cost"

surplus low-grade ore is produced

Interpretation of $\vec{y}$:
Requiring one extra ton of high-grade ore will increase cost by y_1.
" " medium " " & y_2.
Requiring one extra ton of low-grade ore will not change costs, since there is a surplus $(y_3 = 0)$.

Dual linear programs

"primal"

$$\max_{\vec{x}\in\mathbb{R}^n} \ \vec{c}\cdot\vec{x} \quad \text{subject to} \quad A\vec{x}\leq\vec{b},\ \vec{x}\geq 0$$

$\leq$

"dual"

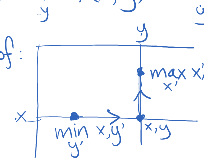
$$\min_{\vec{y}\in\mathbb{R}^m} \ \vec{y}\cdot\vec{b} \quad \text{st} \quad A^T\vec{y}\geq\vec{c},\ y\geq 0$$

$$= \max_{x\geq 0}\ \min_{y\geq 0} \left[c\cdot x + y\cdot(b - Ax)\right]$$

$$\leq \min_{y\geq 0}\ \max_{x\geq 0} \left[c\cdot x + y\cdot(b - Ax)\right]$$

For any function $f(x,y)$,
$$\max_x \min_y f(x,y) \leq \min_y \max_x f(x,y)$$

Proof:



$$\leq \min_{y\geq 0} \max_{x\geq 0} \left[c\cdot x + y\cdot(b - Ax) \right]$$

$$= \min_{y\geq 0} \max_{x\geq 0} \left[y\cdot b + \underbrace{x\cdot c - (Ax)\cdot y}_{x\cdot(c - A^T y)} \right]$$

Terminology note: The primal and dual are both LPs. Which you call "primal" and which "dual" just depends on where you started. The dual of the dual is the primal.

Weak duality theorem:

If $\vec{x}$ is primal feasible, $\vec{y}$ dual feasible, then $\quad Ax \leq b \quad x \geq 0 \quad A^T y \geq c \quad y \geq 0$

$$\vec{c}\cdot\vec{x} \leq \vec{b}\cdot\vec{y}$$

Proof: $\quad c\cdot x \underset{\substack{x\geq 0\\ c\leq A^T y}}{\leq} (A^T y)\cdot x = y\cdot(Ax) \underset{\substack{y\geq 0\\ Ax\leq b}}{\leq} y\cdot b \quad \square$

Note: Solvers can optimize primal and dual together, and stop when $b\cdot y - c\cdot x \approx 0$.

Strong duality theorem:

$$\boxed{\text{Primal value} = \text{Dual value}}$$ if either is feasible

primal $\max_{x\geq 0,\ Ax\leq b} c\cdot x \qquad$ dual $\min_{y\geq 0,\ A^T y\geq c} b\cdot y$

Four cases:
- Both feasible: $\vec{c}\cdot\vec{x}^* = \vec{b}\cdot\vec{y}^*$
- Both infeasible
- Primal $= +\infty$, Dual infeasible
- Primal infeasible, Dual $= -\infty$

For a nondegenerate program (meaning optimum achieved at intersection of exactly n halfspace),

Optimal dual variables give "prices":

Relaxing ith constraint to $b_i + \varepsilon$, objective increases by εy_i (for suff. small $\varepsilon > 0$)

Formally, letting $P(\vec{b}) = \max_{x\geq 0} \{\vec{c}\cdot\vec{x} \mid A\vec{x} \leq \vec{b}\}$,

$$\boxed{y_i^* = \frac{\partial}{\partial b_i} P(\vec{b})} \quad \text{the marginal value of resource } i$$

"**Complementary slackness**": "No leftovers"

If some input is left over $(Ax^*)_i < b_i$, then it is worthless $y_i^* = 0$.

(So if $y_i^* > 0$, then $(Ax^*)_i = b_i$.)

Example: Degenerate program

```
>> cvx_begin
>>   variable x
>>   dual variables y1 y2
>>   maximize ( x )
>>   x <= 1 : y1
>>   x <= 1 : y2
>> cvx_end
```

b_1

$\Longrightarrow$

```
>> [y1 y2]

ans =

    0.5000    0.5000
```

but in this case increasing b_1 does not increase the optimum at all

Example: LPs in Macroeconomics

n goods you can export

prices $p_1, \ldots, p_n$

inputs $v_1, \ldots, v_m$

$$A = \Big(\qquad \Big)_{m\times n}$$

inputs needed to make 1 unit of j

primal LP: $\displaystyle \max_{\vec{x}\in\mathbb{R}^n} \vec{p}\cdot\vec{x}$

s.t. $A\vec{x} \le \vec{v}$

$\vec{x} \geq 0$

$$= \max_{x\geq 0} \min_{y\geq 0} \vec{p}\cdot\vec{x} + \vec{y}\cdot(\vec{v} - A\vec{x})$$

$$= \min_{y\geq 0} \max_{x\geq 0} \vec{v}\cdot\vec{y} + \vec{x}\cdot(\vec{p} - A^T\vec{y})$$

dual LP = $\displaystyle \min_{y\in\mathbb{R}^n} \vec{v}\cdot\vec{y}$

s.t. $A^T\vec{y} \geq \vec{p}$

$\vec{y} \geq 0$

Interpretation: $\vec{y}$ is a vector of primary input prices/resource costs.
If you add ε of input i, earnings will grow by εy_i.

Dual: Find input costs to minimize production cost $(\vec{v}\cdot\vec{y})$
s.t. for every good j, cost to produce 1 unit $\geq p_j$.

Complementary slackness: if $x_j > 0$, cost $(A^T y)_j = p_j$
(0 profit made in equilibrium)

History of linear programming:

In 1939 a linear programming formulation of a problem that is equivalent to the general linear programming problem was given by the Soviet economist Leonid Kantorovich, who also proposed a method for solving it.[2] It is a way he developed, during World War II, to plan expenditures and returns in order to reduce costs of the army and to increase losses incurred to the enemy.[citation needed] Kantorovich's work was initially neglected in the USSR.[3] About the same time as Kantorovich, the Dutch-American economist T. C. Koopmans formulated classical economic problems as linear programs. Kantorovich and Koopmans later shared the 1975 Nobel prize in economics.[1] In 1941, Frank Lauren Hitchcock also formulated transportation problems as linear programs and gave a solution very similar to the later Simplex method.[2] Hitchcock had died in 1957 and the Nobel prize is not awarded posthumously.

Leonid Kantorovich

During 1946–1947, George B. Dantzig independently developed general linear programming formulation to use for planning problems in US Air Force. In 1947, Dantzig also invented the simplex method that for the first time efficiently tackled the linear programming problem in most cases. When Dantzig arranged a meeting with John von Neumann to discuss his Simplex method, Neumann immediately conjectured the theory of duality by realizing that the problem he had been working in game theory was equivalent. Dantzig provided formal proof in an unpublished report "A Theorem on Linear Inequalities" on January 5, 1948.[4] In the post-war years, many industries applied it in their daily planning.

John von Neumann

Dantzig's original example was to find the best assignment of 70 people to 70 jobs. The computing power required to test all the permutations to select the best assignment is vast; the number of possible configurations exceeds the number of particles in the observable universe. However, it takes only a moment to find the optimum solution by posing the problem as a linear program and applying the simplex algorithm. The theory behind linear programming drastically reduces the number of possible solutions that must be checked.

The linear programming problem was first shown to be solvable in polynomial time by Leonid Khachiyan in 1979, but a larger theoretical and practical breakthrough in the field came in 1984 when Narendra Karmarkar introduced a new interior-point method for solving linear-programming problems.

Spufford

Irony: In math derived for communist central planning,
prices arise naturally.

Caution!: LPs don't always look linear.

Example: $\min \|\vec{x}\|_\infty \quad = \max_i |x_i|$

s.t. $A\vec{x} = \vec{b}$

$$= \min_{t,\vec{x}} t$$

s.t. $A\vec{x} = \vec{b}$

$-t \le x_i \le t$ for $i = 1, \ldots, n$

Other norms, like $\|\vec{x}\| = \sqrt{\sum_i x_i^2}$, $\|\vec{x}\|_1 = \sum_i |x_i|$, do not give LPs.

Example: Max flows and min cuts

$$\max \sum_u f_{s\to u} - \sum_v f_{v\to s} \qquad \text{Max flow}$$

s.t. $\forall\, w \ne s,t \quad \sum_u f_{w\to u} = \sum_u f_{u\to w}$

$\forall$ edges $f_{u\to v} \le 1$

$f_{u\to v} \geq 0$

$$= \max_{f\geq 0} \min_{\substack{s_w \\ x_{u\to v}\geq 0}} \sum_u (f_{s\to u} - f_{u\to s}) + \sum_{(u,v)\in E} x_{u\to v}(1 - f_{u\to v}) + \sum_{w\ne s,t} s_w \sum_u (f_{w\to u} - f_{u\to w})$$

$$\le \min \quad \max \sum x \ldots + \sum f \ldots (-x \ldots + s \ldots - s \ldots)$$

$$\underset{\substack{s_w \\ x_{u\to v}\geq 0 \\ s_s=1,\, s_t=0}}{\quad} \quad f\geq 0$$

$$= \min_{\substack{x_{u\to v}\geq 0 \\ s_w \\ s_s=1,\, s_t=0}} \sum_{u\to v} x_{u\to v} \qquad \text{Min cut}$$

$$\text{s.t.} \quad x_{u\to v} - s_u + s_v \geq 0 \quad \forall\, u\to v$$

Proof of strong duality:

primal $\max_{x\geq 0} c\cdot x$, $Ax\leq b$ dual $\min_{y\geq 0} b\cdot y$, $A^T y\geq c$

Assume (for simplicity) that the primal optimum $\vec{x}^*\in\mathbb{R}^n$ is unique and lies at the intersection of n faces

$$(A\vec{x}^*)_i = b_i \quad \text{for } i=1,\dots,n$$
$$(A\vec{x}^*)_j < b_j \quad \text{for } j>n$$

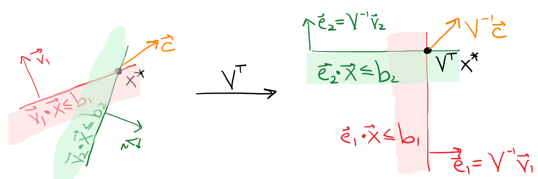
Let $V_{n\times n}$ be the matrix whose columns are the first n rows, i.e., the face normals. $A=\begin{pmatrix} V^T \\ C\end{pmatrix}$

Since the optimum is unique, the normals are lin. indep.; they form a basis.

Change coordinates into this basis.

Apply V^{-1} to every normal (row of A) and to $\vec{c}$.
Apply V^T to every point $\vec{x}$. (this preserves vector-point products)

$$\max_{\substack{x \\ A\vec{x}\leq\vec{b}}} \vec{c}\cdot\vec{x} = \max_{\substack{x \\ (AV^{T^{-1}})(V^T\vec{x})\leq\vec{b}}} (V^{-1}\vec{c})\cdot(V^T\vec{x}) = \max_{\substack{x \\ AV^{T^{-1}}\vec{x}\leq\vec{b}}} (V^{-1}\vec{c})\cdot\vec{x}$$



Observe: Taking $b_i \to b_i+\varepsilon$, optimum increases by $(V^{-1}\vec{c})_i\,\varepsilon$. ≥ 0

Now let us relate this to the dual.

Let
$$\vec{y} = \begin{pmatrix} V^{-1}\vec{c} \\ 0 \end{pmatrix}$$

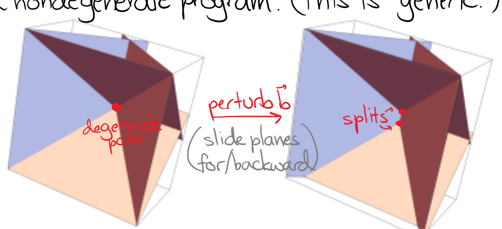
Then $\vec{y}\geq 0$.

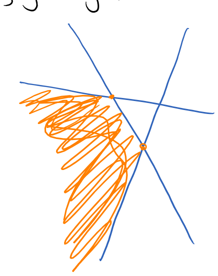
$$A^T\vec{y} = (V\ \ C^T)\begin{pmatrix} V^{-1}c \\ 0\end{pmatrix} = \vec{c}$$

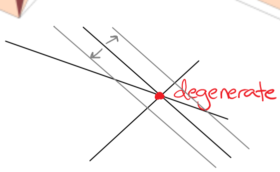
$$b\cdot y = \begin{pmatrix} b_1 \\ \vdots \\ b_n\end{pmatrix}\cdot V^{-1}\vec{c} = \left[V^{T^{-1}}\begin{pmatrix} b_1 \\ \vdots \\ b_n\end{pmatrix}\right]\cdot\vec{c} = x^*\cdot c$$

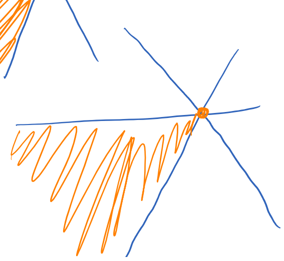
Hence $\vec{y}$ satisfies the dual constraints, and matches the primal objective. By weak duality, it's optimal. □

Remark: If you randomly perturb A and/or $\vec{b}$, you'll get a nondegenerate program. (This is "generic.")







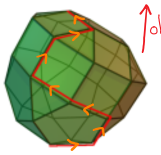


How computers solve LPs

https://en.wikipedia.org/wiki/Linear_programming#Algorithms

How computers solve LPs https://en.wikipedia.org/wiki/Linear_programming#Algorithms

Simplex method



↑ objective $\vec{c}$

walk on polytope edges to improve objective

usually good in practice,
but exponentially slow in worst case

"Interior point" methods are provably polynomial time,
and can be practical

Remarks:

* Linear programs can be generalized,
  eg, semi-definite programs (SDPs)
* But optimizing integer programs is NP-hard
  ↳ solutions must be integers

More examples:

Exercise: Compute the dual LP for

$$\min_{\vec{x}} \quad \vec{c}\cdot\vec{x}$$

$$\text{subject to} \quad A\vec{x} \leq \vec{a}, \quad B\vec{x} = \vec{b}, \quad \vec{x} \geq \vec{0}$$

Answer:

$$= \min_{\vec{x}\geq 0} \max_{\substack{\vec{y}\geq 0 \\ \vec{z}}} \vec{c}\cdot\vec{x} - \vec{y}\cdot(\vec{a}-A\vec{x}) + \vec{z}\cdot(B\vec{x}-\vec{b})$$

$$\geq \max_{\substack{\vec{y}\geq 0 \\ \vec{z}}} \min_{\vec{x}\geq 0} \left[-\vec{y}\cdot\vec{a} - \vec{z}\cdot\vec{b} + \vec{x}\cdot(\vec{c} + A^T\vec{y} + B^T\vec{z}) \right]$$

$$= \max_{\substack{\vec{y}\leq 0 \\ \vec{z}}} \vec{a}\cdot\vec{y} + \vec{b}\cdot\vec{z} \qquad \text{(switching } \vec{y} \text{ with } -\vec{y},\ \vec{z} \text{ w/ } -\vec{z}\text{)}$$

$$\text{s.t.} \quad A^T\vec{y} + B^T\vec{z} \leq \vec{c}$$

Check in Matlab:

```
>> m1 = 5; m2 = 6; n = 15;
>> A = randn(m1, n); B = randn(m2, n);
>> a = randn(m1, 1); b = randn(m2, 1); c = randn(n, 1);
>> cvx_begin
>> variables x(n, 1)
>> minimize (c' * x)
>> A * x <= a
>> B * x == b
>> x >= 0
>> cvx_end
```

Primal program:

```
Status: Solved
Optimal value (cvx_optval): -18.4292
```

```
>> cvx_begin
>> variables y(m1, 1)
>> variables z(m2, 1)
>> maximize (a' * y + b' * z)
>> A' * y + B' * z <= c
>> y <= 0
>> cvx_end
```

Dual program:

```
Status: Solved
Optimal value (cvx_optval): -18.4292
```