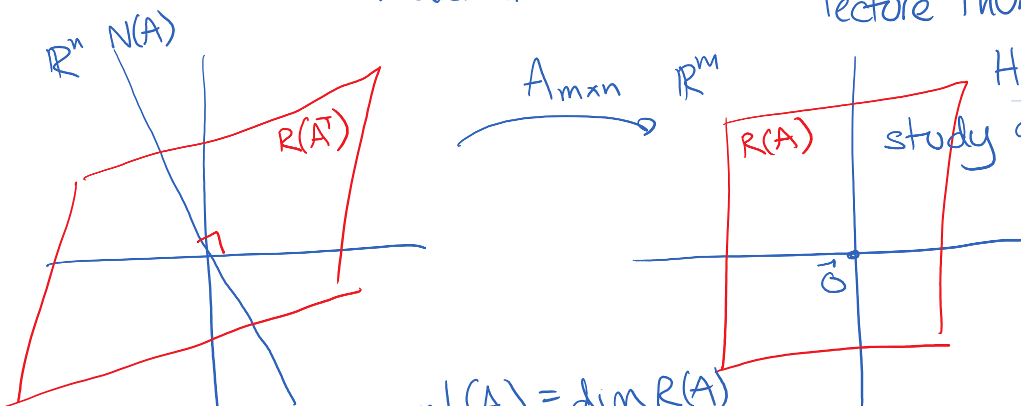
# Lecture 11: Orthogonal bases (class)

**Admin:** Midterm Thursday Oct. 8 — 11am – 11pm Pacific time
2.5 hours total (start by 8:30pm)
next Tuesday cover what material is on exam
lecture Thursday?
Homework?
study questions — probably




$\text{rank}(A) = \dim R(A)$
$= \dim R(A^T)$

## Pairwise Orthogonal Sets of Vectors

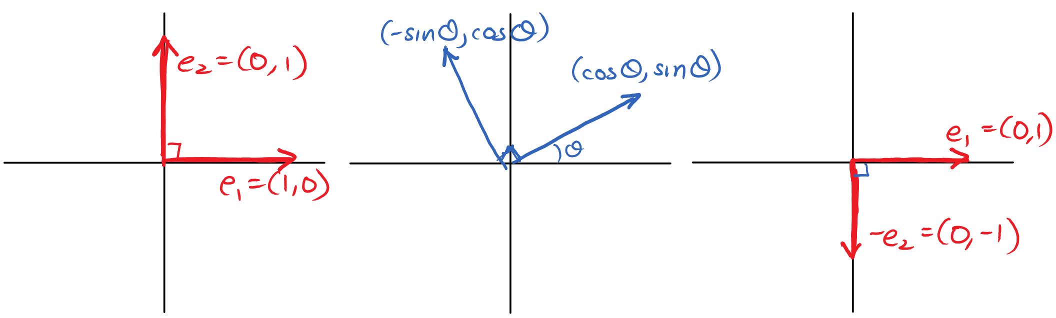
### Examples:

① Standard basis in $\mathbb{R}^n$

$\vec{e}_1 = (1, 0, \ldots, 0), \ldots, \vec{e}_n = (0, \ldots, 0, 1)$

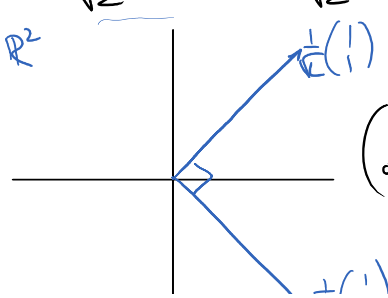
$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

② Rotations/reflections of the standard basis

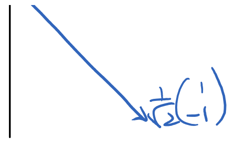



③ Hadamard basis

$\frac{1}{\sqrt{2}}(1,1) \quad , \quad \frac{1}{\sqrt{2}}(1,-1)$




= Fourier basis mod 2
(foundation for Fourier analysis of boolean functions)



change-of-basis matrix $H = \frac{1}{\sqrt{2}} \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{cc} \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} & \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \\ \multicolumn{2}{c}{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}_{2\times 2}} \end{array}$

Hadamard basis in dimension 4:

$$H_4 = \frac{1}{2}\begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}}^{H} & \boxed{\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}}^{H} \\ {}^{H}\boxed{\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}} & \boxed{\begin{matrix} -1 & -1 \\ -1 & 1 \end{matrix}}_{-H} \end{pmatrix}_{4\times 4}$$

<= * all entries either $\pm\frac{1}{\sqrt{n}}$

columns are orthogonal pairwise

In dimensions $n = 2^k$:

$$H_{2^k} = \frac{1}{\sqrt{2}}\begin{pmatrix} \boxed{H_{2^{k-1}}} & \boxed{H_{2^{k-1}}} \\ \boxed{H_{2^{k-1}}} & \boxed{-H_{2^{k-1}}} \end{pmatrix}$$

**Applications**: Coding, control theory, statistics, quantum computing, Fourier analysis of boolean functions

**Conjecture** (1936): If $n$ is a multiple of 4, then there is a Hadamard basis for $\mathbb{R}^n$.

**Open**: Find a Hadamard basis for $n = 668$.

④ Haar wavelet basis

$(1, 1, 1, 1)$
$(1, 1, -1, -1)$
$(1, -1, 0, 0)$
$(0, 0, 1, -1)$

for $\mathbb{R}^4$

$\frac{1}{\sqrt{8}}(1, 1, 1, 1, 1, 1, 1, 1)$
$\frac{1}{\sqrt{8}}(1, 1, 1, 1, -1, -1, -1, -1)$
$\frac{1}{2}(1, 1, -1, -1, 0, 0, 0, 0)$
$\frac{1}{\sqrt{2}}(1, -1, 0, 0, 0, 0, 0, 0)$
$\frac{1}{\sqrt{2}}(0, 0, 1, -1, 0, 0, 0, 0)$
$\frac{1}{2}(0, 0, 0, 0, 1, 1, -1, -1)$
$\frac{1}{\sqrt{2}}(0, 0, 0, 0, 1, -1, 0, 0)$
$\frac{1}{\sqrt{2}}(0, 0, 0, 0, 0, 0, 1, -1)$

for $\mathbb{R}^8$

-used in image compression

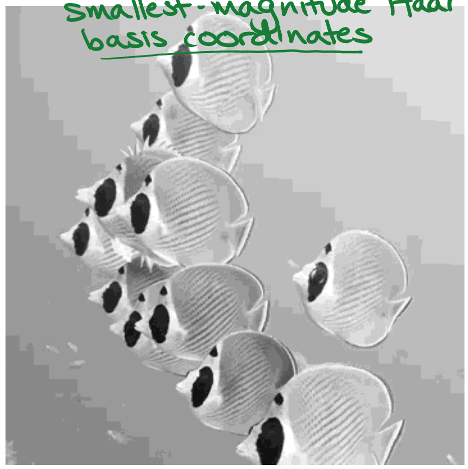
a vector in $\mathbb{R}^8$ with few jumps will have a **sparse** representation in this basis, e.g.,

$$(1, 1, 1, 0, -2, -2, -2, -2)$$
$$= -\frac{5}{8}(1, 1, 1, 1, 1, 1, 1, 1)$$
$$+\frac{11}{8}(1, 1, 1, 1, -1, -1, -1, -1)$$
$$+\frac{1}{4}(1, 1, -1, -1, 0, 0, 0, 0)$$

$+\frac{1}{2}(0,0,1,-1,0,0,0\ 0)$

Original image

... after discarding 0.9n smallest-magnitude Haar basis coordinates



(5) Fourier basis for $\mathbb{C}^n$

-used everywhere!

$(\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_{n-1})$

where $\vec{v}_j = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \exp\left(\frac{2\pi i}{n} \cdot jk\right) \vec{e}_k$

$$= \frac{1}{\sqrt{n}} \begin{array}{c} \begin{array}{ccccc} \vec{v}_0 & \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & k \end{array} \\ \left( \begin{array}{ccccc} 1 & 1 & 1 & 1 & \cdots \\ 1 & \omega & \omega^2 & \omega^3 & \\ 1 & \omega^2 & \omega^4 & \omega^6 & \\ 1 & \omega^3 & \omega^6 & \omega^9 & \\ \vdots & \vdots & \vdots & \vdots & \omega^{jk} \\ 1 & \omega^{n-1} & \omega^{n-2} & \omega^{n-3} & \end{array} \right)_{n \times n} \end{array}$$

$\to$ std basis $j$

where $\omega = \exp \frac{\pi i}{n})$

$= \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$

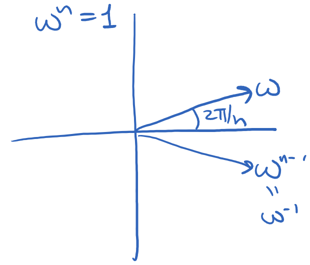
$\vec{v}_j \cdot \vec{v}_k = \frac{1}{n} \sum_{l=0}^{n-1} (v_j)^*_l (v_k)_l$

$= \frac{1}{n} \sum_l \omega^{-jl} \omega^{kl}$

$= \frac{1}{n} \sum_l \omega^{(k-j)l} = 0$ if $j \neq k$ mod n

$\omega = e^{2\pi i/n}$

$\omega^n = 1$




$\Rightarrow \vec{v}_j \cdot \vec{v}_{j'} = \frac{1}{n} \sum_{k,k'=0}^{n-1} \exp\left(\frac{2\pi i}{n}(-jk + j'k')\right) \underbrace{(\vec{e}_k \cdot \vec{e}_{k'})}_{= \delta_{kk'}}$

$= \frac{1}{n} \sum_{k=0}^{n-1} \exp\left(\frac{2\pi i}{n} k(j'-j)\right)$

case $j=j'$: $\vec{v}_j \cdot \vec{v}_j = \frac{1}{n}(1+1+\cdots+1) = 1$

case $j \neq j'$:

$\vec{v}_j \cdot \vec{v}_{j'} = \frac{1}{n} \cdot$ (geometric series $1 + \omega^{j'-j} + \omega^{2(j'-j)} + \cdots + \omega^{(n-1)(j'-j)}$)

where $\omega = e^{\frac{2\pi i}{n}} = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$

$1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$

$= 0$ since $\omega^n = 1$

⑥ Various bases for functions
e.g., Hermite, Laguerre, Chebyshev
sines and cosines, .......

There are lots of named examples because

# PAIRWISE ORTHOGONAL SETS OF VECTORS ARE NICE!!

$\{\vec{v}_1, \dots, \vec{v}_n\}$
with $\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$

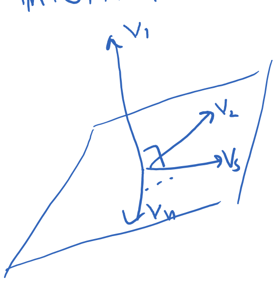
## ① Simplifies deciding linear independence

Normally, to check if $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is lin. indep.,
compute $N\left(\left(\vec{v}_1 \,|\,|\,|\, \vec{v}_n\right)\right) \stackrel{?}{=} \{\vec{0}\}$.
$\sum_j \vec{v}_j \vec{e}_j^{\,T}$

**Lemma:**

$$\boxed{\begin{array}{l} S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \\ \text{non zero, pairwise orthogonal} \\ \quad v_i \cdot v_j = 0 \text{ for } i \neq j \end{array}} \Rightarrow \boxed{S \text{ is linearly independent}}$$

Intuition:




Proof: Assume
$a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n = \vec{0}$

Goal: show $a_1 = \dots = a_n = 0$

Trick: dot both sides with $v_j$:

$a_1 \vec{v}_j \cdot \vec{v}_1 + a_2 \vec{v}_j \cdot \vec{v}_2 + \dots + a_n \vec{v}_j \cdot \vec{v}_n = \vec{v}_j \cdot \vec{0} = 0$

simplifies to $a_j \vec{v}_j \cdot \vec{v}_j = a_j \|v_j\|^2$

$\Rightarrow a_j = 0$

## ② Pairwise ⊥ity simplifies deciding if $S$ is a basis

**Corollary:** In an n-dimensional space, any set of n pairwise orthogonal vectors forms a basis.

(because any n linearly indep. vectors is a basis)

**Definition:** Orthogonal basis = basis of pairwise orthogonal vectors

Orthonormal basis = basis of orthogonal, length-one vectors

→ Orthonormal basis = basis of orthogonal, length-one vectors

$$\vec{v}_j \longrightarrow \frac{\vec{v}_j}{\|v_j\|}$$

## ③ Simplifies computing inner products & lengths

Let $\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m$

**Q**: What is $\|u\|$?

**A**: $\|\vec{u}\|^2 = \vec{u}\cdot\vec{u} = \left(\sum_j a_j\vec{v}_j\right)\cdot\left(\sum_k a_k\vec{v}_k\right)$

$$= \sum_{jk} a_j^* a_k (\vec{v}_j\cdot\vec{v}_k)$$

need to add up $m^2$ terms

If the basis $v_1,\dots,v_m$ is orthogonal

$$= \sum_{j=1}^{m} |a_j|^2\,\|\vec{v}_j\|^2 \quad \longleftarrow m \text{ terms to add}$$

If orthonormal $(\|v_j\|=1)$

$$= \sum_j |a_j|^2$$

MORAL: Orthonormal bases behave just like the standard basis.

If $\vec{u} = \sum_{j=1}^{m} a_j\vec{v}_j$ and $\vec{w} = \sum_j b_j\vec{v}_j$

for orthogonal basis: $\vec{u}\cdot\vec{w} = \sum_j a_j^* b_j \|v_j\|^2$

O.N. basis: $\vec{u}\cdot\vec{w} = \sum_j a_j^* b_j$

## ④ Simplifies basis expansions

Let $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m)$ be a basis for $V \subseteq \mathbb{R}^n$, and $\vec{u}\in V$.

What is the expansion of $\vec{u}$ in the basis $\mathcal{B}$?

In general: Solve the equations

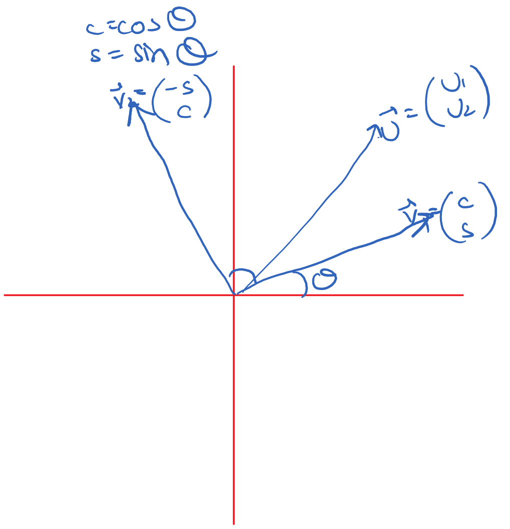
$$\vec{u} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots = \boxed{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{pmatrix}\begin{pmatrix} | \\ \vec{x} \\ | \end{pmatrix} = \vec{u}}$$

If $\mathcal{B}$ is ortho~~normal~~ goonal: use the trick to solve (Trick: dot both sides with $v_j$)

$$\vec{v}_j\cdot\vec{u} = \vec{v}_j\cdot\left(\sum_k x_k\vec{v}_k\right) = x_j\|v_j\|^2$$

$$\Rightarrow x_j = \frac{v_j \cdot u}{\|v_j\|^2}$$

$$\Rightarrow \vec{u} = \sum_{j=1}^{m} \frac{v_j \cdot u}{\|v_j\|^2} \vec{v}_j \underset{\substack{\uparrow \\ \text{if orthonormal}}}{=} \sum_{j=1}^{n} (v_j \cdot u) \vec{v}_j$$




$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_S = \begin{pmatrix} u_1 c + u_2 s \\ -u_1 s + u_2 c \end{pmatrix}_{\vec{v}_1, \vec{v}_2}$$

$$\Rightarrow \boxed{\vec{u} = \sum_{j=1}^{m} (\vec{v}_j \cdot \vec{u}) \vec{v}_j}$$

- just compute these inner products
- needn't solve any equations!

**Example:** Haar wavelet basis for $\mathbb{R}^8$

$v_1$ (1, 1, 1, 1, 1, 1, 1, 1)
$v_2$ (1, 1, 1, 1, −1, −1, −1, −1)
$v_3$ (1, 1, −1, −1, 0, 0, 0, 0)
$v_4$ (1, −1, 0, 0, 0, 0, 0, 0)
$v_5$ (0, 0, 1, −1, 0, 0, 0, 0)
$v_6$ (0, 0, 0, 0, 1, 1, −1, −1)
$v_7$ (0, 0, 0, 0, 1, −1, 0, 0)
$v_8$ (0, 0, 0, 0, 0, 0, 1, −1)

Pairwise $\perp$
$\Downarrow$
Linearly independent
$\Downarrow$
Must be a basis!

$\|u\|^2 = 19$

**Exercise:** Expand $\vec{u} = (1, 1, 1, 0, -2, -2, -2, -2)$ in the above basis.

**Answer:**

$v_1 \cdot u = -5$
$v_2 \cdot u = 11$
$v_3 \cdot u = 1$
$v_4 \cdot u = 0$
$v_5 \cdot u = 1$
$\vdots$

$$\Rightarrow \vec{u} = \sum_{j=1}^{8} \frac{v_j \cdot u}{\|v_j\|^2} \vec{v}_j$$

$$= -\frac{5}{8}\vec{v}_1 + \frac{11}{8} v_2 + \frac{1}{4} v_3 + \frac{1}{2} \vec{v}_5 + \cdots$$

$$\left(-\frac{5}{8}\right)^2 \|v_1\|^2 + \left(\frac{11}{8}\right)^2 \|v\|^2 +$$

$$\frac{5^2}{8} + \frac{11^2}{8} + \frac{1^2}{4} + \frac{1^2}{2} = 19$$

We ~~could~~ compute $v_6 \cdot u$, $v_7 \cdot u$, $v_8 \cdot u$ — and its easy — but in fact, observe

$$\|u\|^2 = 19$$

and

$$\frac{|v_1 \cdot u|^2}{\|v_1\|^2} + \underbrace{\frac{|v_2 \cdot u|^2}{\|v_2\|^2}}^{\text{squared coeff. of } v_2/\|v_2\|} + \frac{|v_3 \cdot u|^2}{\|v_3\|^2} + \frac{|v_5 \cdot u|^2}{\|v_5\|^2}$$

$$= \frac{25}{8} + \frac{121}{8} + \frac{1}{4} + \frac{1}{2}$$

$$= 19$$

$\Rightarrow$ all other coefficients **must** be 0, since all of $\vec{u}$'s length is accounted for!
(This is a common trick, to save time.)

## ④ Simplifies matrix basis expansions

$f : U \to V$

$B_U = \{\vec{u}_1, \dots, \vec{u}_n\}$ $\quad$ $B_V = \{\vec{v}_1, \dots, \vec{v}_m\}$ orthonormal basis

$$\Rightarrow [f]_{B_U \to B_V} = \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} & \vdots & \\ \cdots & v_i \cdot f(\vec{u}_j) & \\ & & \end{pmatrix}_{m \times n} \end{array}$$

(rows $1, \dots, m$)

since $f(\vec{u}_j) = \sum_{i=1}^{m} \left(\vec{v}_i \cdot f(\vec{u}_j)\right) \vec{v}_i$

## ⑤ Simplifies changing basis

**Recall:** $f : U \to V$ linear

bases $B_U, B_U'$ $\quad$ bases $B_V, B_V'$

$$[f]_{B_U' \to B_V'} = [I]_{B_V \to B_V'} [f]_{B_U \to B_V} [I]_{B_U' \to B_U}$$

$$\underset{B_V'}{V} \xleftarrow{I_V} \underset{B_V}{V} \xleftarrow{f} \underset{B_U}{U} \xleftarrow{I_U} \underset{B_U'}{U}$$

if $U = V$,

$$[f]_{B' \to B'} = [I]_{B \to B'} [f]_{B \to B} [I]_{B' \to B}$$

$$[I]_{B \to B'} = \left([I]_{B' \to B}\right)^{-1}$$

Say $S : \vec{e}_1, \dots, \vec{e}_n$
$B : \vec{v}_1, \dots, \vec{v}_n$ orthonormal

$$[B \to S] = \begin{array}{c|c} & \vec{v}_1 \;\; j \;\; v_n \\ \hline \begin{matrix} e_1 \\ i \\ e_n \end{matrix} & \begin{pmatrix} \\ e_i \cdot v_j \\ \\ \end{pmatrix} \end{array}$$

$$[S \to B] = \begin{array}{c|c} & 1 \;\; j \;\; e_n \\ \hline \begin{matrix} v_1 \\ i \\ v_n \end{matrix} & \begin{pmatrix} \\ v_i \cdot e_j \\ \\ \end{pmatrix} \end{array}$$

$$[B \to S] = [S \to B]^T$$

Example:

$\left([\;\;]_{B'\to B}\right)$

**Example:**

Consider the $2\times 2$ complex matrix $\quad S: \vec{e}_1, \vec{e}_2$

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Write this matrix in the basis $C$: $\left\{ \underbrace{\tfrac{1}{\sqrt{2}}(1,i)}_{\vec v_1}, \underbrace{\tfrac{1}{\sqrt{2}}(1,-i)}_{\vec v_2} \right\}$.

$$[A]_C = [S\to C]\; A\; [C\to S]$$

$$[S\to C] = \begin{array}{c|cc} & e_1 & e_2 \\ \hline v_1 & 1/\sqrt{2} & -i/\sqrt{2} \\ v_2 & 1/\sqrt{2} & +i/\sqrt{2} \end{array} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$[C\to S] = \begin{array}{c|cc} & v_1 & v_2 \\ \hline e_1 & \frac{1}{\sqrt{2}}1 & \frac{1}{\sqrt{2}}1 \\ e_2 & \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}(-i) \end{array} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

transpose conjugates

```
= 1/2 (1  -i) . (Cos[θ]  -Sin[θ]) . (1   1) // FullSimplify // MatrixForm
      (1   i)   (Sin[θ]   Cos[θ])   (i  -i)
```

atrixForm=

$$\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$ diagonal!

$A\vec{v}_1 = e^{-i\theta}\vec{v}_1, \quad A\vec{v}_2 = e^{i\theta}\vec{v}_2$

eigenvectors

eigenvalues

**Answer:** Initial basis $B = \{(1,0),(0,1)\}$.

New basis $C = \{\tfrac{1}{\sqrt{2}}(1,i), \tfrac{1}{\sqrt{2}}(1,-i)\}$.

**Observe:** $C$ is orthonormal!

$$\left\|\tfrac{1}{\sqrt{2}}(1,\pm i)\right\|^2 = \tfrac{1}{2}\|(1,\pm i)\|^2$$
$$= \tfrac{1}{2}\left(1^2 + |\pm i|^2\right) = 1 \checkmark$$

$$\tfrac{1}{\sqrt{2}}(1,i)\cdot\tfrac{1}{\sqrt{2}}(1,-i) = \tfrac{1}{2}\left(1 + i^*(-i)\right)$$
$$= \tfrac{1}{2}\left(1 + i^2\right) = 0 \checkmark$$

$C\to B$ basis change:

$$\begin{array}{c|cc} & \frac{1}{\sqrt{2}}(1,i) & \frac{1}{\sqrt{2}}(1,-i) \\ \hline (1,0) & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ (0,1) & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{array} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$B\to C$ basis change

$$\begin{array}{c|cc} & (1,0) & (0,1) \\ \hline \frac{1}{\sqrt{2}}(1,i) & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{array} \quad \frac{1}{\;}\begin{pmatrix} 1 & -i \end{pmatrix}$$

$B \to C$ basis change

$$\begin{array}{c|cc} & (1,0) & (0,1) \\ \hline \frac{1}{\sqrt{2}}(1,i) & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}(1,-i) & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{array} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

since, eg, $\frac{1}{\sqrt{2}}(1,i)\cdot(0,1) = \frac{-i}{\sqrt{2}}$

Check: $(1,0) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(1,i) + \frac{1}{\sqrt{2}}(1,-i)\right)$ ✓

$(0,1) = \frac{1}{\sqrt{2}}\left(\frac{-i}{\sqrt{2}}(1,i) + \frac{i}{\sqrt{2}}(1,-i)\right)$ ✓

$$\Rightarrow [A]_{C\to C} = [B\to C]\cdot[A]_{B\to B}\cdot[C\to B]$$

(read this right to left)

$$= \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

In[1]:= $\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ // FullSimplify // MatrixForm

Out[1]//MatrixForm=

$$= \boxed{\begin{pmatrix} \text{Cos}[\theta] & i\,\text{Sin}[\theta] \\ i\,\text{Sin}[\theta] & \text{Cos}[\theta] \end{pmatrix}}$$

(Mathematica)

Observe: 1. This was really easy!

2. $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

— they're inverses

3. They differ only by transpose and complex conjugate.

In general,

If $B$, $C$ are each orthonormal:

$B = \{\vec{b}_1, \dots, \vec{b}_n\}$, $C = \{\vec{c}_1, \dots, \vec{c}_n\}$.

$$[I]_{C\to B} = \begin{array}{c} \\ i \end{array}\begin{array}{c} \vec{c}_j \\ \begin{pmatrix} & \vdots & \\ \cdots & \boxed{\vec{b}_i\cdot\vec{c}_j} & \\ & \vdots & \end{pmatrix} \end{array}$$

since $\vec{c}_j = \sum_{i=1}^{n}(b_i\cdot c_j)\vec{b}_i$

$$[I]_{B\to C} = \begin{array}{c} \\ j \end{array}\begin{array}{c} \vec{b}_i \\ \begin{pmatrix} & \vdots & \\ \cdots & \boxed{\vec{c}_j\cdot\vec{b}_i} & \\ & \vdots & \end{pmatrix} \end{array}$$

since $\vec{b}_i = \sum_{j=1}^{n}(c_j\cdot b_i)\vec{c}_j$

$$\boxed{\Rightarrow [I]_{B\to C} = \left([I]_{C\to B}\right)^{\dagger}}$$

"adjoint" = transpose + complex conjugate

(same as transpose for real matrices)

(6.5) "Better for choosing random vectors

How to compute projections onto any subspace $V$

① Compute an O.N. basis for $V$

$\vec{v}_1, \dots, \vec{v}_m$

② The projection matrix is

$\sum_j \vec{v}_j \vec{v}_j^T$

in particular

$\mathrm{Proj}_V(\vec{u}) = \sum_j \vec{v}_j (v_j \cdot u)$

"std basis isn't special"

Why this works?

$\mathrm{Projection}_{\underline{xy\text{-plane}}}(\vec{u}) = \underline{(u_1, u_2, 0, \dots, 0)}$

(7) Simplifies computing projections

**Problem**: How can we project onto a higher-dimensional subspace?

**Example**: $P_{xy\text{-plane}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

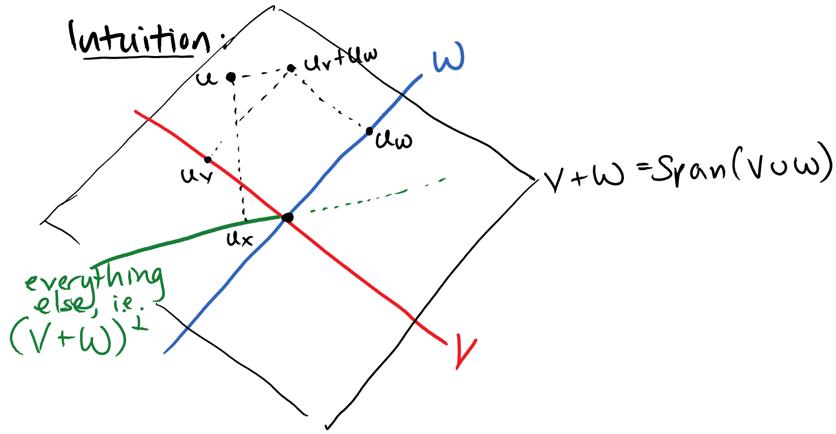
$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$P_{x\text{-axis}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}(1\ 0\ 0) \quad P_{y\text{-axis}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(0\ 1\ 0)$

**Key property**: If $V \perp W$,

$$\boxed{P_V + P_W = P_{V+W}}$$

**Intuition**:




$\Rightarrow$ To project into any space:
"Split" it into orthogonal lines,
and add up those projections

**FACT**: For a subspace $\mathcal{U} \subseteq \mathbb{R}^n$ with orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_k\}$,
orthogonal projection onto $\mathcal{U}$

$$P_{\mathcal{U}} = \boxed{\sum_{j=1}^{k} \underbrace{\vec{u}_j \vec{u}_j^T}_{n \times n \text{ matrix}}}$$

Proof: $P_{\mathcal{U}} = P_{\mathrm{span}\{u_1, \dots, u_{k-1}\}} + \underset{\displaystyle u_k u_k^T}{\underset{\shortparallel}{P_{\mathrm{span}\{u_k\}}}}$

**Exercise**: Compute the projection of $e_1 = (1, 0, 0, \dots)$ onto a random 50-dimensional subspace of $\mathbb{R}^{100}$.

Answer:

```
n = 100;
d = 50;

v = zeros(n,1);
v(1) = 1;

A = randn(n, d);    # choose d random vectors in R^n,
                    #  with normally distributed coordinates
[Q,R] = qr(A,0);    # generate Q, whose columns form an
                    #  orthonormal basis for R(A),
                    #  the span of A's columns
```

the vector $e_1$

we'll explain this function later

$Q(:,1), Q(:,2), \ldots, Q(:,d)$ form an orthonormal basis for $R(A)$

① First approach

build sum $\sum_{i=1}^{d} q_i q_i^T v$ one term at a time

```
projectedv = zeros(n,1);
for i = 1:d
    projectedv += Q(:,i) * (Q(:,i)' * v);
end for;
projectedv
```

② Second try: Note: $Q = \sum_{i=1}^{d} q_i e_i^T$ where $q_i = Q(:,i)$ ith column

$$\Rightarrow Q Q^T = \left(\sum_i q_i e_i^T\right)\left(\sum_j e_j q_j^T\right)$$

$$= \sum_{i,j} q_i \underbrace{(e_i^T e_j)}_{e_i \cdot e_j} q_j^T \qquad e_i \cdot e_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$= \sum_{i=1}^{d} q_i q_i^T$$

```
projectedv2 = Q * (Q' * v);

sum(abs(projectedv2 - projectedv))
```

Trick: To check that two vectors are the same, add up the absolute values of the coord. differences

③ Check the answer:

```
# 1. Is projectedv in R(A)?
x = A\projectedv;          # Solve for a linear comb. of the columns of A
err = A*x - projectedv     # that gives projectedv
sum(abs(err))

# 2. Is v-projectedv perpendicular to R(A)?
(v - projectedv)' * A
```

(Question: What is the expected squared length of the projection? projectedv' * projectedv)

**Key Example**: If $V = \mathbb{R}^n$, "resolution of the identity" is

$$\boxed{\sum_{j=1}^{n} \vec{v}_j \vec{v}_j^T = \underset{\text{identity matrix}}{I}}$$

This gives an easy derivation of the other facts:

Example: I forget, is

$$\vec{u} = \sum_j (\vec{v}_j \cdot \vec{u}) \vec{v}_j \quad \text{or} \quad \vec{u} = \sum_j (\vec{u} \cdot \vec{v}_j) \vec{v}_j \ ?$$

over $\mathbb{C}$, they're different!

$$\vec{u} = I u = \sum_j v_j v_j^T u = \sum_j v_j (v_j^T u) = \boxed{\sum_j (\vec{v}_j \cdot \vec{u}) \vec{v}_j}$$

$$\vec{u} = I u$$
$$= \sum_j v_j v_j^T u = \sum_j v_j (v_j^T u) = \boxed{\sum_j (\vec{v}_j \cdot \vec{u}) \vec{v}_j}$$

*Example:*

$$\|\vec{u}\|^2 = \vec{u}^T \vec{u}$$
$$= u^T I u$$
$$= u^T \sum_j v_j v_j^T u \qquad \Rightarrow \|u\|^2 = \sum_j |\vec{u} \cdot \vec{v}_j|^2$$
$$= \sum_j (u^T v_j)(v_j^T u)$$

Any basis expansion can be done by inserting
$$I = \sum_j v_j v_j^T.$$

Even changes of basis:
$$B = \{\vec{b}_1, \dots, \vec{b}_n\} \qquad C = \{\vec{c}_1, \dots, \vec{c}_n\}$$
orthonormal sets

$$I = I \cdot I$$
$$= \sum_i c_i c_i^T \sum_j b_j b_j^T$$
$$= \sum_{i,j} \vec{c}_i (c_i \cdot b_j) b_j^T$$

## 8 Orthonormal bases simplify many arguments

- If you prove something for the standard basis, it most likely holds for any orthonormal basis.
  Intuition and arguments are usually basis independent.

**Example:** If $V \perp W$,
$$P_{V+W} = P_V + P_W.$$

Why? Work in an orthonormal basis
$$\{\underbrace{\vec{b}_1, \dots, \vec{b}_p}_{\text{basis for } V}, \underbrace{\vec{b}_{p+1}, \dots, \vec{b}_{p+q}}_{\text{basis for } W}, \vec{b}_{q+p+1}, \dots, \vec{b}_n\}$$

$$P_V = \begin{array}{c} \\ p \\ q \\ n-p-q \end{array} \begin{array}{c} \begin{array}{ccc} p & q & n-p-q \end{array} \\ \left(\begin{array}{c|c|c} I_p & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \qquad P_W = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & I_q & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

$$P_V + P_W = \begin{array}{c} \\ p+q \\ n-p-q \end{array} \begin{array}{c} \begin{array}{cc} p+q & n-p-q \end{array} \\ \left(\begin{array}{cc|c} I_p & 0 & 0 \\ 0 & I_q & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{array} \quad \checkmark$$

**Example:** If $V \subseteq W$,

$$P_V P_W = P_W P_V = P_V.$$

Check this.

But how do we find orthonormal bases for spaces?

(How does Matlab's qr function work?)