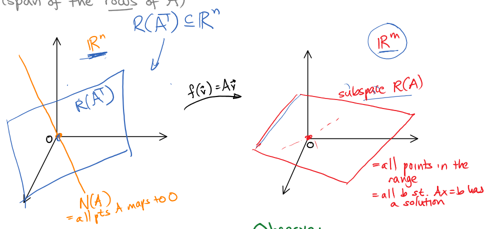
Lecture 6: Subspaces of a matrix (class)

Admin: **Reading**: Strang §2.4, Meyer §4.2

Definition: For an $m\times n$ matrix A,

- The **nullspace of A** is
  $N(A) = \{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=\vec{0}\}$
  (everything A sends to $\vec{0}$)
- The **range** or **column space** of A is
  $R(A) = \{A\vec{x} \mid \vec{x}\in\mathbb{R}^n\}$
  $= \mathrm{Span}(\text{columns of } A)$
  (everything reachable by A)
- The **rowspace** of A is $R(A^T)$
  (span of the rows of A)



Observe:
$\vec{b}\in \mathrm{Range}(A) \iff A\vec{x}=\vec{b}$ has a solution

$R(A) = \{\vec{b} \mid A\vec{x}=\vec{b} \text{ has a solution}\}$

Example: Let $\vec{u} = \binom{1}{2}$, $\vec{v} = \binom{3}{4}$, and let

$$A = \vec{u}\,\vec{v}^T = \begin{pmatrix}1\\2\end{pmatrix}(3\ \ 4) = \begin{pmatrix}3 & 4\\ 6 & 8\end{pmatrix} \xrightarrow{GE} \begin{pmatrix}3 & 4\\ 0 & 0\end{pmatrix}$$

What are $R(A)$, $R(A^T)$, $N(A)$, $N(A^T)$?
← "left nullspace"

$$N(A) = \{\vec{x} \mid A\vec{x}=\vec{0}\} = \mathrm{Span}\left(\left\{\begin{pmatrix}-4\\3\end{pmatrix}\right\}\right)$$

$$A\vec{x} = \begin{pmatrix}1\\2\end{pmatrix}(3\ \ 4)\vec{x} = \vec{u}(\vec{v}^T\vec{x}) = \vec{u}\cdot(3x_1+4x_2)$$
$$= 0 \iff 3x_1+4x_2=0$$

$$R(A) = \mathrm{Span}\left(\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}\right) \qquad R(A^T) = \mathrm{Span}\left(\left\{\begin{pmatrix}3\\4\end{pmatrix}\right\}\right)$$

$$A\vec{x} = \vec{u}(\vec{v}^T\vec{x})$$

$$N(A^T) = \mathrm{Span}\left(\left\{\begin{pmatrix}-2\\1\end{pmatrix}\right\}\right)$$

NULLSPACE $N(A) = \{\vec{x}\in\mathbb{R}^n \mid A\vec{x}=\vec{0}\}$
← homogeneous equations

Claim: This is a subspace.

Proof.

Closed under addition:
$\vec{x},\vec{y}\in N(A) \Rightarrow A\vec{x}=\vec{0},\ A\vec{y}=\vec{0}$
$\Rightarrow A(x+y) = Ax+Ay = 0+0 = 0$
$\Rightarrow x+y\in N(A)$ ✓

Closed under scalar multiplication:
$c\in\mathbb{R},\ \vec{x}\in N(A) \Rightarrow A\vec{x}=\vec{0}$
$\Rightarrow A(c\vec{x}) = c\cdot A\vec{x} = c\cdot\vec{0} = \vec{0} \Rightarrow c\cdot\vec{x}\in N(A)$ ✓ □

Problem:
Let $M = \begin{pmatrix}1 & 2 & 1 & 1 & 5\\ -2 & -4 & 0 & 4 & -2\\ 1 & 2 & 2 & 4 & 9\end{pmatrix}_{3\times 5}$

What is $N(M)$? $\subseteq \mathbb{R}^5$

Answer: Solving for $N(M) \iff$ Solving $M\vec{x}=\vec{0}$

$$M \xrightarrow{GE} \begin{pmatrix}1 & 2 & 1 & 1 & 5\\ 0 & 0 & 2 & 6 & 8\\ 0 & 0 & 1 & 3 & 4\end{pmatrix} \xrightarrow{GE} \begin{pmatrix}1 & 2 & 1 & 1 & 5\\ 0 & 0 & 1 & 3 & 4\\ 0 & 0 & 0 & 0 & 0\end{pmatrix} \xrightarrow{\text{backsub}} \begin{pmatrix}1 & 2 & 0 & -2 & 1\\ 0 & 0 & 1 & 3 & 4\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

pivots; free: x_2, x_4, x_5

$x_1 = -2x_2 + 2x_4 - x_5$
$x_3 = \quad\quad -3x_4 - 4x_5$

$$N(M) = \left\{\begin{pmatrix}-2x_2+2x_4-x_5\\ x_2\\ -3x_4-4x_5\\ x_4\\ x_5\end{pmatrix} \,\middle|\, x_2,x_4,x_5\in\mathbb{R}\right\}$$

$$\boxed{= \mathrm{Span}\left(\left\{\begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\-3\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\-4\\0\\1\end{pmatrix}\right\}\right)}$$

$\uparrow x_2 \quad \uparrow x_4 \quad \uparrow x_5$

Observe: Row operations don't change the nullspace.
Indeed, if U is invertible, $N(A) = N(U\cdot A)$.

Claim: Consider the equations $A\vec{x}=\vec{b}$, with $\vec{b}\neq\vec{0}$.
Let $\vec{y}$ be a solution ($A\vec{y}=\vec{b}$). Then

$$\{\vec{x} \mid A\vec{x}=\vec{b}\} = \vec{y} + \{\vec{x} \mid A\vec{x}=\vec{0}\}$$
$$= \vec{y} + N(A)$$

Thus, the solutions to a set of **non-homogeneous** equations form an **affine subspace**.

(In English: The general solution of the nonhomogeneous system is given by a particular solution ($\vec{y}$) plus the general solution of the associated homogeneous system.)

Proof: Let $V_b = \{x \mid Ax = b\}$.

The rest of the proof is obvious ✓

<u>Corollary</u>: The equations $A\vec{x} = \vec{b}$ have <u>infinitely many solutions</u>

if and only if

$N(A) \neq \{\vec{0}\}$ and there is at least one solution.

RANGE $R(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$

<u>Claim</u>: ① $\vec{b} \in R(A) \iff A\vec{x} = \vec{b}$ has a solution ✓

② $R(A)$ is a subspace.

③ $R(A) = \text{Span}(\text{columns of } A)$

this is why it is often called the "column space"

Proof of ②:

Closure under addition:

Let $\vec{b}, \vec{c} \in R(A) \Rightarrow$ there exist $(\exists)$ x, y such that (s.t.)

$A\vec{x} = b, \ Ay = c$

$\Rightarrow A(x+y) = Ax + Ay = b + c$

$\Rightarrow b + c \in R(A)$ ✓

Closure under scalar multiplication:

Let $\vec{b} \in R(A)$. Let $a \in \mathbb{R}$ $\Rightarrow \exists\, x : Ax = b$

$\Rightarrow A(a\vec{x}) = a \cdot A\vec{x} = a\vec{b}$

$\Rightarrow a \cdot \vec{b} \in R(A)$ ✓ □

Proof of ③, $R(A) = \text{Span}(\text{columns of } A)$:

Let the columns be $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^m$

$$A = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}_{m \times n} \qquad R(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

$$= \{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \mid x_1, \dots, x_n \in \mathbb{R}\}$$

$$= \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$$

✓

Example: Compute $R(A)$ for

$$A = \begin{pmatrix} 1 & 5 & 6 \\ 2 & 6 & 8 \\ 3 & 7 & 10 \\ 4 & 8 & 12 \end{pmatrix}_{4 \times 3} \Rightarrow R(A) \subseteq \mathbb{R}^4$$

-1

$$R(A) = \text{Span}\left(\left\{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}5\\6\\7\\8\end{pmatrix}\right\}\right) = \text{Span}\left(\left\{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}0\\1\\2\\3\end{pmatrix}\right\}\right)$$

↑ or could have

ROW SPACE $R(A^T) = \text{Span}(\text{rows of } A)$

<u>Problem</u>:

Let $M = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix}_{3\times5}$ (as before)

What are the row and column spaces of M?

<u>Answer</u>:

$$M \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 2 & 6 & 8 \\ 0 & 0 & 1 & 3 & 4 \end{pmatrix}_{3\times5} \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

<u>Observe</u>: The row operations of Gaussian elimination do not change the row space.

(Because $\text{Span}(v_1, v_2, v_3, \dots, v_k) = \text{Span}(v_1, v_2 + \alpha v_1, v_3, \dots, v_k)$.)

$$\Rightarrow \boxed{R(M^T) = \text{Span}\{(1,2,0,-2,1), (0,0,1,3,4)\}.} ✓ \subseteq \mathbb{R}^5$$

row space

$$\text{Span}\left\{\begin{pmatrix}1\\2\\0\\-2\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\3\\4\end{pmatrix}\right\}$$

<u>How to compute $R(M)$?</u>

① Apply Gaussian elimination to M^T

$$M^T = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 1 & 0 & 2 \\ 1 & 4 & 4 \\ 5 & -2 & 9 \end{pmatrix} \begin{matrix} \\ \leftarrow -2 \\ \leftarrow -1 \\ \leftarrow -1 \\ \leftarrow -5 \end{matrix} \longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 6 & 3 \\ 0 & 8 & 4 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow -3 \\ \leftarrow -4 \end{matrix} \longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

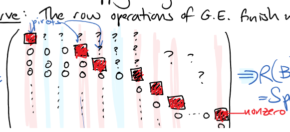
$$\Rightarrow R(M) = \text{Span}\left(\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}\right\}\right)$$

rowspace

$$R\left(\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}^T\right) = \text{Span}\left(\left\{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right\}\right)$$

② But if we have already applied G.E. to M, there is no need to apply it again to M^T:

<u>Observe</u>: The row operations of G.E. finish with a matrix like

$A \xrightarrow{G.E.} B =$ 

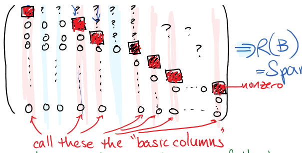
$\Rightarrow R(B) = \text{Span}\left\{\begin{matrix}\text{basic}\\\text{columns}\\\text{of } B\end{matrix}\right\}$

Big picture reason

$R(AU) = R(A)$

if U^{-1} exists

$R((UA)^T) = R(A^T)$

$A \xrightarrow{G.E.} B =$ 

$\Rightarrow R(B) = \mathrm{Span}\left\{\begin{matrix}\text{basic}\\ \text{columns}\\ \text{of } B\end{matrix}\right\}$

Claim: Every column is a linear combination of the basic columns.

Proof: Take any non-basic column $\vec{c}$.

Use the last basic column to cancel $\vec{c}$'s last coordinate.

⋮

Use the first basic column to cancel $\vec{c}$'s first coord. □

$$M = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix} \xrightarrow{G.E.} G = \begin{pmatrix} \boxed{1} & 2 & \boxed{1} & 1 & 5 \\ 0 & 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

basic columns

e.g. $(1,3,0) = 3(1,1,0) - 2(1,0,0)$ ✓

The **same** linear combination of columns works for M:

$$\underset{\substack{\text{column 4}\\ M\vec{e}_4}}{(1,4,4)} = 3\underset{\substack{\text{column 3}\\ M\vec{e}_3}}{(1,0,2)} - 2\underset{\substack{\text{column 1}\\ M\vec{e}_1}}{(1,-2,1)}$$

Why?

$M = QG$, where $Q \overset{=PL}{}$ is the matrix that implements the row operations in Gaussian Elim.

$\Rightarrow$ If $\underset{j\text{th col}}{Ge_j} = \sum_k \alpha_k \underset{k\text{th column}}{Ge_k}$,

then $QGe_j = Q\left(\sum_k \alpha_k Ge_k\right)$

$\quad\quad\; \| \quad\quad\quad\quad \|$

$\quad\quad Me_j \quad\quad \sum_k \alpha_k Me_k$

$\Rightarrow \mathrm{Range}(M) = \mathrm{Span}(\{(1,-2,1),(1,0,2)\})$

$R(AU) = R(A)$ if U^{-1} exists

$R((UA)^T) = R(A^T)$ if U^{-1} exists

when you do G.E. on the rows

$A \xrightarrow{GE} B$

$R(A^T) = R(B^T)$ rowspace

$R(A) \neq R(B)$ colomnspace in general

$A = GB$ G is from G.E. (LU decomposition) $G = P \cdot L$

if column j of B $= \sum_{i=1}^{j-1} c_i$(column i of B)

$$\Rightarrow B\begin{pmatrix} c_1 \\ \vdots \\ c_{j-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{0} \underset{\substack{\text{mult.}\\ \text{both sides}\\ \text{by } G}}{\Longrightarrow} A\begin{pmatrix} c_1 \\ \vdots \\ c_{j-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{0}$$

In general:

After Gaussian elimination, identify the basic columns. Then

$$R(A) = \mathrm{Span}(\text{those same columns, in } A)$$

Exercise: Prove

① $R(AB) \subseteq R(A)$

(Multiplying on the right can only reduce the range.)

$R(AB) = R(A)$ if B is invertible.

② $N(BA) \supseteq N(A)$

(Multiplying on the left can only increase the nullspace.)

$N(BA) = N(A)$ if B is invertible

homework?

Easy example: Take $B = 0$, so $AB = 0$, $BA = 0$

$R(AB) = \{0\} \subseteq R(A)$, $N(BA) = \mathbb{R}^n \supseteq N(A)$ ✓

Note: These spaces are all sensitive to numeral errors and perturbations

E.g.

	Range	Nullspace
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	x-axis	yz-plane
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$, $\varepsilon \neq 0$	$\mathbb{R}^3$	$\{\vec{0}\}$

(so be careful with Matlab's null and orth commands)

Example: What is the nullspace of

$$A = \begin{pmatrix} -1 & 1 & & & & & \\ 1 & -2 & 1 & & & 0 & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & 0 & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} ?$$

Answer:

Intuition: Recall that A is the matrix we got from discretizing the second derivative operator

$$f''(t) = \lim_{\delta t \to 0} \frac{1}{(\delta t)^2}\left(f(t+\delta t) - 2f(t) + f(t-\delta t)\right).$$

The functions with second derivative 0 are exactly constant functions $f(t) = $ constant.

$\Rightarrow$ We expect the nullspace of A to be the set of vectors with all-equal coordinates.

Claim: $N(A) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\right\}$

$=$ set of all vectors with all-equal coordinates.

Proof:

$$A\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \text{sum of } a_{ij} \text{ across row } 1 \\ \vdots \\ \text{sum across row } n \end{pmatrix} = 0$$

$\Rightarrow (1, 1, \ldots, 1) \in N(A)$

$\Rightarrow \mathrm{Span}(\{(1, \ldots, 1)\}) \subseteq N(A)$.

But are there other vectors in $N(A)$?

$$A = \begin{pmatrix} -1 & 1 & & & & & \\ 1 & -2 & 1 & & & 0 & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & 0 & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} \begin{matrix} \\ \curvearrowleft +1 \end{matrix} \longrightarrow \begin{pmatrix} -1 & 1 & & & & & \\ 0 & -1 & 1 & & & 0 & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & 0 & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} \begin{matrix} \\ \curvearrowleft 1 \end{matrix}$$

$$\longrightarrow \begin{pmatrix} -1 & 1 & & & & \\ 0 & -1 & 1 & & 0 & \\ & 0 & -1 & 1 & & \\ & & 1 & -2 & 1 & \\ & & & 1 & -2 & 1 \end{pmatrix} \begin{matrix} \\ \\ \curvearrowleft +1 \end{matrix}$$

$$\rightarrow \begin{pmatrix} -1 & 1 & & & & & \\ 0 & -1 & 1 & & & \bigcirc & \\ & 0 & -1 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & \bigcirc & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} \; +1$$

$$\vdots$$

$$\rightarrow B = \begin{pmatrix} -1 & 1 & & & & & \\ 0 & -1 & 1 & & & \bigcirc & \\ & 0 & -1 & 1 & & & \\ & & 0 & -1 & 1 & & \\ & & & & \ddots & & \\ & \bigcirc & & & 0 & -1 & 1 \\ & & & & & 1 & -1 \end{pmatrix}$$

$A\vec{x} = 0 \iff B\vec{x} = 0$

$\iff x_1 = x_2$ (first row)

$x_2 = x_3$ (2^{nd} row)

$\vdots$

$x_{n-1} = x_n$

$\iff \vec{x} \in \text{Span}\{(1, 1, 1, \dots, 1)\}$

Question: What is $R(A)$? (Note $A = A^T$.)

Example:

Definition: The matrix A is **diagonally dominant** if for all rows j,

$$|a_{jj}| \geq \sum_{i \neq j} |a_{ji}| .$$

It is strictly diagonally dominant if the inequality is strict ($>$) for all rows.

Theorem: If A is strictly diagonally dominant, then

$$N(A) = \{\vec{0}\}$$

Proof:

Let $\vec{x} \in N(A)$, so $A\vec{x} = \vec{0}$.

Let j be the coordinate so $|x_j|$ is largest.

$$0 = (A\vec{x})_j = a_{jj}x_j + \sum_{i \neq j} a_{ji}x_i$$

$$\Rightarrow |a_{jj}x_j| = \left|\sum_{i \neq j} a_{ji}x_i\right|$$

$$\leq \sum_{i \neq j} |a_{ji}| \cdot |x_i|$$

$$\leq \left(\sum_{i \neq j} |a_{ji}|\right) \cdot |x_j|$$

$$< |a_{jj}| \cdot |x_j| , \text{ unless } x_j = 0.$$

This is a contradiction unless $x_j = 0$.

$\Rightarrow x_j = 0 \Rightarrow \vec{x} = \vec{0}$

$\Rightarrow N(A) = \{\vec{0}\}$. ✓ □

Question: Can you characterize the nullspaces of diagonally dominant matrices?

Example:

$$A = \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & - & 1 \\ 1 & & & & 1 & -2 \end{pmatrix}$$

← diagonally dominant, but not strictly so

has nullspace

$$N(A) = \text{Span}(\{(1, 1, 1, \dots, 1)\})$$

= set of all constant vectors.

Exercise: Linear codes

"generator matrix"

a) List all elements of the subspace over $\mathbb{F}_2$

$$R\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}_{7\times 3}\right) = \text{Span}\left(\left\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \right\}\right) = \left\{ \begin{matrix} c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \\ c_1, c_2, c_3 \in \mathbb{F}_2 \end{matrix} \right\}$$

$$\begin{matrix} \| & \| & \| \\ v_1 & v_2 & v_3 \end{matrix}$$

$c_1 = c_2 = c_3 = 0$: 0000000,

$c_2 = 1$: 0110011, 0111100,

$c_1 = 1$: 0001111,

$c_3 = 1$: 1010101, 1011010, 1100110, 1101001

col-space (gen matrix) = code

$c_1 = 1$ 0001111 , 0111100 ,

$= \text{code}$

$c_3 = 1$ 1010101 1100110

1011010 1101001

Answer: The subspace includes all elements $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$ with $a, b, c \in \{0, 1\}$.

v_3 0000000 0110011

1010101 1100110 v_1

v_2

0001111 0111100

1011010 1101001

b) Compute the nullspace over $\mathbb{F}_2$ of

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}_{4\times 7}$$

parity-check matrix P

$N(P) = \text{code}$

$$\to \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

free variables

$$\to \begin{array}{c} \begin{array}{ccccccc} & & & & x_5 & x_6 & x_7 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

free

$$\to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\Longrightarrow \boxed{N(A) = \operatorname{Span}\left(\left\{ \underset{x_5}{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{x_6}{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}, \underset{x_7}{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \right\} \right)}$$

Answer: Use Gaussian elimination:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

pivots free vars.

Hence $N(A)$ equals the columnspace of

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Observe that this is the same space we found in part ⓐ!

Remark: Subspaces over finite fields are also known as "linear error-correcting codes."

A vector in the subspace is called a "codeword."

$$\text{distance}(\text{code } V) \equiv \min_{\vec{u},\vec{v} \in V} \Big(\underset{\# \text{ coords where } \vec{u} \neq \vec{v} \text{ differ}}{\# \text{nonzero entries of } \vec{u} - \vec{v}} \Big)$$

$$= \min_{\vec{u} \in V} \left(\# \text{ nonzero entries of } \vec{u} \right).$$

(by linearity, $\vec{u} - \vec{v} \in V$)

Claim 3: For $S \subseteq \mathbb{R}^n$, let

$$A(S) = \{ A\vec{x} \mid \vec{x} \in S \}$$

(map the set S forward by applying A)

1. If S is a subspace, so is $A(S)$.
2. If $s_1, \ldots, s_k$ span S, then $As_1, \ldots, As_k$ span $A(S)$.

Proof:

1) Closure under addition:
   Let $y, z \in A(S)$. Is $y+z \in A(S)$?
   Yes! $y = Ax$, $z = Ax'$
   $y+z = A(x+x') \in A(S)$ ✓

   Closure under multiplication is similar ✓

2) Any vector $y \in A(S)$ can be written $y = Ax$ for some $x \in S$.
   $s_1, \ldots, s_k$ span S $\Rightarrow$ $\underset{\substack{\cap \\ S}}{x} = \sum_{j=1}^{k} \alpha_j s_j$ for some scalars $\alpha_1, \ldots, \alpha_k$.
   $\Rightarrow$ by linearity, $y = Ax = \sum_{j=1}^{k} \alpha_j (As_j)$
   $\Rightarrow$ $y \in \mathrm{Span}(\{As_1, \ldots, As_k\})$ ✓ $\square$

Observe: ① $\vec{b} \in R(A) \Longleftrightarrow A\vec{x} = \vec{b}$ has a solution.
② $R(A) = A(\mathbb{R}^n)$
$= \mathrm{Span}(\text{columns of } A)$

Why?
$\mathbb{R}^n$ is spanned by $(1,0,0,\ldots,0), \ldots, (0,0,\ldots,0,1)$
By the exercise, $A(\mathbb{R}^n)$ is spanned by
$A\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1^{st}$ column, $\ldots$, $A\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} =$ last column of A ✓

Thus the range is also known as the **column space** of A.

Observe: ③ $A_{n \times n}$ is invertible (A^{-1} exists)
$\Longleftrightarrow$ every point $b \in \mathbb{R}^n$ has a preimage x, $Ax = b$
$\Longleftrightarrow$ $R(A) = \mathbb{R}^n$ (everything)

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$