Homework 11 answers

Singular value and spectral decompositions

1

In class, we saw that the nonzero eigenvalues of AA^T are the same as the nonzero eigenvalues of A^TA — just the squares of the nonzero singular values of A. In this problem, you'll relate the singular values of A to the eigenvalues of $\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$.

a) Diagonalize the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

b) Diagonalize the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

c) Let A be an arbitrary $m \times n$ real matrix, with $m \leq n$.

Let

$$B = \begin{array}{c c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \left(\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array}\right) \end{array}.$$

B is an $(m+n) \times (m+n)$ symmetric matrix. (Therefore it is unitarily diagonalizable.)

In terms of the singular values and left- and right-singular vectors of A, specify the eigenvalues and eigenvectors of B.

Note: B has $m+n$ eigenvalues, so don't forget any! Parts a and b should be helpful special cases, but feel free to experiment more with Matlab until you see the pattern. Commands like these might be helpful:

```
m = 2;
n = 3;
A = randn(m, n);
[U, S, V] = svd(A)  % <-- returns left singular vectors, singular values, right singular vectors

B = [zeros(m,m) A; A' zeros(n,n)];
[W, D] = eig(B) % <-- returns eigenvectors, eigenvalues
```

Answers:

ⓐ

Eigenvalues	Eigenvectors
1	$(1,1)$
-1	$(1,-1)$

ⓑ

Eigenvalues	Eigenvectors
$\sqrt{2}$	$(\sqrt{2},1,1)$
$-\sqrt{2}$	$(\sqrt{2},-1,-1)$
0	$(0,1,-1)$

ⓒ Let an SVD of A be

$$A = \sum_{i=1}^{m} \sigma_i \vec{u}_i \vec{v}_i^T.$$

Then for each i,

B has eigenvalues $+\sigma_i$ and $-\sigma_i$,

with corresponding e-vectors (u_i, v_i) and $(u_i, -v_i)$.

Indeed, $B\begin{pmatrix} u_i \\ \pm v_i \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}\begin{pmatrix} u_i \\ \pm v_i \end{pmatrix}$

$$= \begin{pmatrix} \pm A v_i \\ A^T u_i \end{pmatrix}$$

$$= \sigma_i \begin{pmatrix} \pm u_i \\ v_i \end{pmatrix}$$

$$= \pm \sigma_i \begin{pmatrix} u_i \\ \pm v_i \end{pmatrix} \checkmark$$

These eigenvectors are all perpendicular to each other, so are independent.

This accounts for $2m$ of the eigenvalues of B, so $n-m$ eigenvalues remain. These remaining eigenvalues are all 0. Let $\vec{\omega}_1, \ldots, \vec{\omega}_{n-m}$ extend $\vec{v}_1, \ldots, \vec{v}_m$ to an orthonormal basis for $\mathbb{R}^n$. The $\vec{\omega}_j$'s all lie in the nullspace of A, so

$$\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \omega_j \end{pmatrix} = 0.$$

Hence the vectors $(\vec{0}, \vec{\omega}_j)$ are eigenvectors of B with eigenvalue 0. Since the $\vec{\omega}_j$ vectors are perpendicular both to each other and to the $\vec{v}_i$ vectors, these e-vectors are perpendicular to those already found, completing the set.

Positive semi-definite matrices

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Positive semi-definite matrices

A real symmetric matrix is "positive definite" if its eigenvalues are all >0, "positive semi-definite" if its eigenvalues are all ≥ 0, and "indefinite" otherwise.

② The quadratic form $f(x,y)=x^2+4xy+2y^2$ has a saddle point at the origin, despite the fact that its coefficients are positive. Write f as a difference of two squares.

$$f(x,y) = (x,y)\begin{pmatrix}1 & 2\\ 2 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

```
{{λ1, v1}, {λ2, v2}} = Eigensystem[(1 2
                                    2 2)] // Transpose;
v1 /= Norm[v1]; v2 /= Norm[v2];
{{λ1, v1}, {λ2, v2}} // Simplify // MatrixForm

λ1 Transpose[{v1}].{v1} // Simplify // MatrixForm
λ2 Transpose[{v2}].{v2} // Simplify // MatrixForm
%% + % // MatrixForm
```

xForm=

$$\begin{pmatrix} \frac{1}{2}\left(3+\sqrt{17}\right) & \left\{\frac{-1+\sqrt{17}}{\sqrt{34-2\sqrt{17}}},\ \sqrt{\frac{1}{34}\left(17+\sqrt{17}\right)}\right\} \\ \frac{1}{2}\left(3-\sqrt{17}\right) & \left\{-\frac{1+\sqrt{17}}{\sqrt{2\left(17+\sqrt{17}\right)}},\ 2\sqrt{\frac{2}{17+\sqrt{17}}}\right\} \end{pmatrix}$$

xForm=

$$\begin{pmatrix} \frac{1}{2}+\frac{7}{2\sqrt{17}} & 1+\frac{3}{\sqrt{17}} \\ 1+\frac{3}{\sqrt{17}} & 1+\frac{5}{\sqrt{17}} \end{pmatrix} = \lambda_1 \vec{v}_1 \vec{v}_1^T$$

xForm=

$$\begin{pmatrix} \frac{1}{2}-\frac{7}{2\sqrt{17}} & 1-\frac{3}{\sqrt{17}} \\ 1-\frac{3}{\sqrt{17}} & 1-\frac{5}{\sqrt{17}} \end{pmatrix} = \lambda_2 \vec{v}_2 \vec{v}_2^T$$

xForm=

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\Rightarrow f(x,y) = \underbrace{\lambda_1\left(\vec{v}_1\cdot\begin{pmatrix}x\\y\end{pmatrix}\right)^2}_{\frac{1}{2}\left(1+\frac{7}{\sqrt{17}}\right)x^2+\left(1+\frac{5}{\sqrt{17}}\right)y^2+2\left(1+\frac{3}{\sqrt{17}}\right)xy} + \underbrace{\lambda_2\left(\vec{v}_2\cdot\begin{pmatrix}x\\y\end{pmatrix}\right)^2}_{\frac{1}{2}\left(1-\frac{7}{\sqrt{17}}\right)x^2+\left(1-\frac{5}{\sqrt{17}}\right)y^2+2\left(1-\frac{3}{\sqrt{17}}\right)xy}$$

$$= \boxed{\left(\sqrt{\tfrac{1}{2}\left(1+\tfrac{7}{\sqrt{17}}\right)}\,x+\sqrt{1+\tfrac{5}{\sqrt{17}}}\,y\right)^2-\left(\sqrt{\tfrac{1}{2}\left(\tfrac{7}{\sqrt{17}}-1\right)}\,x+\sqrt{\tfrac{5}{\sqrt{17}}-1}\,y\right)^2}$$

③

③

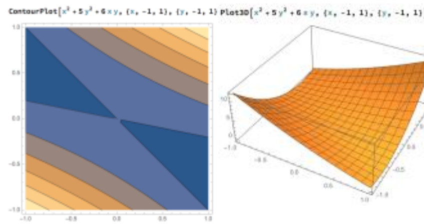
Decide for or against the positive definiteness of these matrices, and write out the corresponding quadratic form $f(\vec{x}) = \vec{x}^T A \vec{x}$:

a) $\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}$ b) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ d) $\begin{pmatrix} -1 & 2 \\ 2 & -8 \end{pmatrix}$

The determinant in (b) is 0; along what line is $f(x,y)=0$?

a) $\boxed{\text{Not}}$ p.s.d. $(\text{Det} A = -4 < 0)$

$f(x,y) = x^2 + 5y^2 + 6xy$



b) $\text{Tr} B = \underset{\lambda_1+\lambda_2}{\overset{\shortparallel}{2}}$, $\text{Det} B = \underset{\lambda_1 \cdot \lambda_2}{\overset{\shortparallel}{0}}$

$\Rightarrow$ e-values are $0, 2$ $\Rightarrow \boxed{\text{p.s.d.}}$ (but not positive definite)

$f(x,y) = x^2 + y^2 - 2xy = (x-y)^2$

c) $\text{Tr} C = 7$, $\text{Det} C = 1$ $\Rightarrow \boxed{C > 0}$

$f(x,y) = 2x^2 + 5y^2 + 6xy$

d) $(1\ 0) D \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1 < 0 \Rightarrow \boxed{\text{not}}$ p.s.d. (it is actually negative definite)

$f(x,y) = -x^2 - 8y^2 + 4xy$

④

For what range of numbers a and b are the matrices A and B positive definite?

$$A = \begin{pmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{pmatrix}$$

Answer:

$$A = 2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(1\ 1\ 1) + (a-2)I$$

$\Rightarrow$ Its e-values are

$2 \cdot 3 + (a-2) = 4 + a$
$2 \cdot 0 + (a-2) = a - 2$
$2 \cdot 0 + (a-2) = a - 2$

$A > 0 \iff \boxed{a > 2}$

B is never positive definite. The submatrix $\begin{pmatrix} 1 & 4 \\ 4 & 7 \end{pmatrix}$

B is never positive definite. The submatrix $\begin{pmatrix} 1 & 7 \\ 4 & 7 \end{pmatrix}$ has negative determinant, so must have a negative e-value.

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Positive definite or not?

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \qquad C = \underbrace{\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}}_{D}{}^{2}$$

Answer:

$A = 3I - \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ has e-values $3, 3, 0$ $\Rightarrow A \succeq 0$ but is not definite

$B - I = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \succeq 0$ has rank one $\Rightarrow$ B's e-values are 1, 1, and 1 + positive number $\Rightarrow \boxed{B \succ 0}$

$C = D^2$, D has full rank $\Rightarrow$ all e-values $\neq 0$ $\Rightarrow$ C's e-values > 0 $\boxed{C \succ 0}$

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Give a quick reason why each of these statements is true:

a) Every positive definite matrix is invertible.

b) The only pos. def. projection matrix is $P = I$.

c) A diagonal matrix with positive diagonal entries is pos. def.

d) A symmetric matrix with a positive determinant might not be positive definite.

a) all e-values > 0 $\Rightarrow$ full rank $\Rightarrow$ invertible

b) Every projection is pos. semidefinite, with e-values 0 or 1. The only projection P with trivial nullspace, $N(P) = \{\vec{0}\}$, is I.

c) The e-values are the diagonal elements, all > 0.

d) An even # of negative eigenvalues $\Rightarrow$ minus signs cancel, can give positive determinant

Eg., $-I \prec 0$ but $\mathrm{Det}(-I) = (-1)^{\text{dimension}}$.

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Classify each of the following matrices as positive definite, positive semidefinite, or indefinite. Try to do it by hand.

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(1\ 1\ 1) + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}(1\ 0\ 1)$$

As the sum of two positive semi-definite matrices, it is p.s.d.

Since its rank is only 2, one e-value is 0, so it is not positive definite.

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{pmatrix}$$

Observe that the submatrix $\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$ has determinant 0, so one of its e-values is 0. An e-vector is $(3,-1)$. This suggests that we should perturb $(3,0,-1)$ to try to to find a vector $\vec{x}$ with $\vec{x}^T B \vec{x} < 0$. Indeed,

$$(3\ \ \varepsilon\ \ -1)\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{pmatrix}\begin{pmatrix} 3 \\ \varepsilon \\ -1 \end{pmatrix} = 5\varepsilon^2 + 4\varepsilon$$

For any $\varepsilon \in (-\frac{4}{5}, 0)$, this is < 0. Hence B is indefinite.

$$C = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{pmatrix} = \left(\begin{array}{cc|c} 1 & 2 & \\ 2 & 4 & 0 \\ \hline 0 & & 0 \end{array}\right) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \left(\begin{array}{c|cc} 0 & 0 & \\ \hline 0 & 3 & -2 \\ & -2 & 3 \end{array}\right)$$

Since C is the sum of three p.s.d. matrices, it is p.s.d.

Now apply Gaussian elimination

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 5/3 \end{pmatrix}$$

Hence C has full rank (rank 4), so cannot have any 0 eigenvalues. It is positive definite.

(From this upper-triangular form, we can also see that $\mathrm{Det}(C) = 1 \cdot 2 \cdot 3 \cdot 5/3 = 10$.)

$D = B^{-1}$. B has both positive and negative eigenvalues, (since, e.g., $e_1^T B e_1 = 1 > 0$, while $(3\ \ \varepsilon\ \ -1)B\begin{pmatrix} 3 \\ \varepsilon \\ -1 \end{pmatrix} < 0$ for some values of ε). D's eigenvalues are the inverses of B's eigenvalues, so will also have both signs. D is indefinite.

$E = C^{-1}$ Positive definite, since C is positive definite.

$F = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$ Indefinite. By question 1, F's e-values are $\pm$ A's e-values.

8. - If A is positive semidefinite and $\alpha \geq 0$, prove that αA is positive semi-definite.

- If A and B are positive semi-definite, prove that $A+B$ is positive semidefinite.
(Hint: You'll probably want to use a theorem from class...)
- Conclude that if A and B are positive semi-definite, then so are the matrices $pA+(1-p)B$, for all $p\in[0,1]$.
(Thus the set of positive semi-definite matrices of a given dimension is **convex**. This is extremely important in optimization theory: namely, in semi-definite programming.)

We showed in class that $A \succeq 0 \iff$ for all $\vec{x}$, $\vec{x}^T A \vec{x} \geq 0$.

- If $\alpha > 0$ and for all x, $x^T A x \geq 0$, then also for all x,
$x^T(\alpha A)x = \alpha(x^T A x) \geq 0 \Rightarrow \alpha A \succeq 0$.
- If for all x, $x^T A x \geq 0$ and $x^T B x \geq 0$, then for all x,
$x^T(A+B)x = x^T A x + x^T B x \geq 0$. ✓
Hence $A+B \succeq 0$.
- $A \succeq 0$ and $B \succeq 0 \Rightarrow pA \succeq 0, (1-p)B \succeq 0 \Rightarrow pA+(1-p)B \succeq 0$. ✓

9

a) Prove that if $A \succeq 0$ then $A^2 \succeq 0$.

$A = A^T \Rightarrow (A^2)^T = (A^T)^2 = A^2$

A's e-values $\lambda_i \geq 0 \Rightarrow A^2$'s e-values $\lambda_i^2 \geq 0$

The definition of a positive semi-definite matrix can be used to define a partial order on symmetric matrices.

Definition: For symmetric matrices A and B of the same dimensions, define
"$A \succeq B$"
if $A-B$ is positive semi-definite.

b) Give an example of two symmetric matrices A and B such that neither $A \succeq B$ nor $B \succeq A$. These matrices are incomparable; that's why it is called a partial order.

For example,
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

c) Give an example of symmetric A and B so that

$A \succeq B$ is true,

but $A^2 \succeq B^2$ is false!

(That is, $A-B$ is positive semidefinite, but A^2-B^2 is not.)

(Hint: You can use 2×2 matrices. Play around until you find an example, and then try to simplify it to understand how it works.)

Answer: I took two random symmetric matrices in Matlab.

```
>> A = randn(2,2); A = A + A';
>> B = randn(2,2); B = B + B';
>> eig(A-B)

ans =

   -4.4474
    2.4300

>> A = A + 4.45 * eye(2,2);
>> eig(A-B)

ans =

    0.0026
    6.8800

>> eig(A^2-B^2)

ans =

   -0.6515
   75.2584
```

A is not $\succeq B$,

so I added just enough times the identity $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ to make A barely $\succeq B$.

now $A^2 \not\succeq B^2$, so we're done!

That gives a working example, but it would be nice to find a cleaner example. Let's start by choosing $A-B$.

Set
$$A-B=\begin{pmatrix}1&1\\1&1\end{pmatrix}.$$

This matrix is $\succeq 0$, but one of its e-values is 0, so it is "barely" $\succeq 0$.

Now the simplest thing to choose for B is just

$$B=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$\Rightarrow A=(A-B)+B=\begin{pmatrix}2&1\\1&1\end{pmatrix}$$

$$A^2=\begin{pmatrix}5&3\\3&2\end{pmatrix}$$

$$\Rightarrow A^2-B^2=\begin{pmatrix}4&3\\3&2\end{pmatrix}$$ has determinant $8-9=-1$, so it must have a negative e-value ✓

$\Rightarrow A^2 - B^2 = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$ has determinant of -7,
so it must have a negative e-value ✓

(We could also have started with $A - B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.
Lots of matrices work!)