Homework 10B Eigenvectors

Spectral Decomposition: A Proof

1. Prove that a matrix that is upper- or lower-triangular is normal $\iff$ it is diagonal.

Spectral Decomposition: Experiments

2. a) Experiment. Generate 1×1, 2×2 and 3×3 random symmetric matrices A, and compare the eigenvalues and eigenvectors of A to those of $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$.

What pattern do you find? Explain why.

(**Notice**: $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ have different eigenvalues, in general!)

(**Hint**: It is easy to enter block matrices into Matlab, for example as follows:

```
>> m = 1;
>> n = 3;
>> A = randn(m, n)

A =

    0.8886   -0.7648   -1.4023

>> B = [zeros(m,m) A; A' zeros(n,n)]

B =

         0    0.8886   -0.7648   -1.4023
    0.8886         0         0         0
   -0.7648         0         0         0
   -1.4023         0         0         0
```
)

b) What are the eigenvalues and eigenvectors of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$?

Now, as in part ⓐ, experiment with different matrices A to try to understand how the eigenvalues and eigenvectors of A relate to those of

$$\begin{pmatrix} 3A & A \\ 0 & 2A \end{pmatrix}.$$

What pattern do you find? Why?

SPECTRAL DECOMPOSITION: POWER METHOD

3

ⓐ What are the eigenvalues and corresponding eigenvectors of

$$A = \begin{pmatrix} 0 & -3 & -2 \\ 2 & 5 & 2 \\ -2 & -3 & 0 \end{pmatrix} ?$$

What is its determinant?

Find a nonsingular matrix U such that $U^{-1}AU$ is a diagonal matrix.

Do this 3 ways:

- By hand, using $\mathrm{Det}(A-\lambda I)=0, \ldots$ Show your work, and check your answers.
- In Matlab, using the power method

  In Matlab, using the built-in functions

ⓑ What are the eigenvalues and corresponding eigenvectors for

$$B = \begin{pmatrix} 1 & -3 & -2 \\ 2 & 6 & 2 \\ -2 & -3 & 1 \end{pmatrix} ?$$

<u>Hint</u>: Use your result from part ⓐ. If this takes more than a line or two to solve, you're doing it wrong...

4 The following code creates a 100×100 <u>symmetric</u> matrix, each entry of which is uniformly random from [0, 1]:

```
rng(1)  % seed the random number generator
        % this way everyone will get the same answer!
n = 100;
A = rand(n,n)
A = diag(diag(A)) + triu(A,1) + triu(A,1)'
    % (triu(A,1) extracts the portion of A strictly above the diagonal,
    % so triu(A,1)' mirrors that below the diagonal)
```

```python
import numpy as np

np.random.seed(1)

n = 100
A = np.random.rand(n, n)
A = np.diag(np.diag(A)) + np.triu(A,1) + np.triu(A,1).T
```

Run this code in Matlab or python to initialize A.

ⓐ Use the power method to compute the eigenvector of A corresponding to the largest-magnitude eigenvalue. Verify that you have indeed computed an eigenvector. What is the eigenvalue? Show your work.
(After you are done, you can check your answer by calling eigs(A, 1), but please solve this problem, and the parts ⓑ, ⓒ, ⓓ below, without using the eig() or eigs() functions.)

ⓑ A is a nonsingular matrix. Now use the power method to compute the smallest-magnitude eigenvalue and a corresponding eigenvector. Of course, show your work.
(Hint: If λ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1} — so the smallest-magnitude eigenvalues of A correspond to the largest-magnitude eigenvalues of A^{-1}.)
Please do this without computing the inverse matrix A^{-1}. You don't need to compute the LU decomposition of A, but why might doing so speed up your calculations?

(c) Finally, use the power method to find the 2^{nd} & 3^{rd} smallest magnitude eigenvalues and corresponding eigen-vectors.

(Note: You can check your answer by calling

eigs(A, 3, 'sm')

The 'sm' option tells Matlab to look for the smallest-magnitude eigenvalues. Once again, though, don't use this in your solution. I want you to use the power method)