EE 510 10/30/2020

<u>Outline:</u>

+ Condition number
+ Least squares
+ PCA.

- Condition number: $K_A = \|A\| . \|A^{-1}\|$.

<u>Exercise:</u> $A \in \mathbb{R}^{n \times n}$ invertible:

1) Find a relation between K_{A^2} and $(K_A)^2$

$$(K_A)^2 = (\|A\| \, \|A^{-1}\|)^2$$

$$K_{A^2} = \|A^2\| \, \|(A^2)^{-1}\| \leq \|A\|^2 \|A^{-1}\|^2 \quad (\|A^2\| \leq \|A\| \|A\|)$$

$$= (K_A)^2$$

$$K_{A^2} \leq (K_A)^2.$$

2) A is symmetric, $\|A\|_2 = \max_{\|x\|=1} \|Ax\|$

Prove that $K_{A^2} = (K_A)^2$.

$$K_2(A) = \frac{\sigma_1}{\sigma_n}$$

$A = V \Sigma U^T$; $A^T = U \Sigma V^T$; $U = V$

$$\Rightarrow A = V \Sigma V^T$$

$\|A\|_2 = \sigma_1$; $\|A^{-1}\| = \frac{1}{\sigma_n}$

$$A^2 = V \Sigma \underbrace{V^T V}_{I} \Sigma V^T = V \Sigma^2 V^T \qquad \begin{pmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \ddots \ \sigma_n^2 \end{pmatrix}$$

$\|A^2\| = \sigma_1^2$

$\|(A^2)^{-1}\| = \frac{1}{\sigma_n^2}$

$$\Rightarrow K_2(A^2) = \frac{\sigma_1^2}{\sigma_n^2} = (K_2(A))^2.$$

Exercise: $A \in \mathbb{R}^{n \times n}$, invertible, B is a singular matrix (not invertible).

Prove that $K_2(A) \geq \frac{\|A\|_2}{\|A-B\|_2}$.

$\Leftrightarrow \|A\|_2 \|A^{-1}\|_2 \geq \frac{\|A\|_2}{\|A-B\|_2}$

$\Leftrightarrow \|A^{-1}\|_2 \geq \frac{1}{\|A-B\|_2} \Leftrightarrow 1 \leq \|A^{-1}\|_2 \|A-B\|_2.$

$\|I - A^{-1}B\|_2 = \|A^{-1}(A - B)\|_2 \leq \|A^{-1}\|_2 \|A-B\|_2$

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$$\|I - A^{-1}B\|_2 = \max_{x \neq 0} \frac{\|(I - A^{-1}B)x\|_2}{\|x\|_2}$$

$$\geq \frac{\|(I - A^{-1}B)y\|_2}{\|y\|} \; , \forall y \in \mathbb{R}^n.$$

Let $y \in N(B)$

$$\Rightarrow (I - A^{-1}B)y = y - A^{-1}\underbrace{By}_{=0} = y$$

$$\Rightarrow \|I - A^{-1}B\|_2 \geq \frac{\|y\|_2}{\|y\|_2} = 1$$

$$\Rightarrow 1 \leq \|I - A^{-1}B\|_2 \leq \|A^{-1}\|_2 \|A - B\|_2$$

$$\Rightarrow \frac{\|A\|_2}{\|A - B\|_2} \leq K_2(A).$$

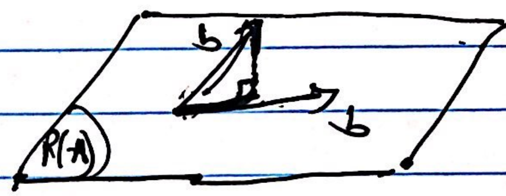
Least squares:

$AX = b \longrightarrow b \in R(A)$: at least has one solution

$\searrow b \notin R(A)$: find the best possible vector x^* such that $Ax^* - b$ is as close as possible to 0

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_2 \quad ; \quad \|v\|_2 = \sqrt{\langle v, v \rangle}$$

$A \in \mathbb{R}^{m \times n}$



• $y(t) = a + bt$

y_1 at t_1

$\vdots$

y_m at t_m

$$\text{error} = \sum_{i=1}^{m} (y_i - (a + bt_i))^2 = \overbrace{\sum_{i=1}^{m} y_i^2}^{A} + \overbrace{m}^{B} a^2 + b^2 \overbrace{\sum_{i=1}^{m} t_i^2}^{C}$$

$$+ 2ab \underbrace{\sum_{i=1}^{m} t_i}_{D} + 2a \underbrace{\left(-\sum_{i=1}^{m} y_i\right)}_{E} + 2b \underbrace{\left(-\sum_{i=1}^{m} t_i y_i\right)}_{F}$$

$$\text{error} = A + Ba^2 + b^2 C + 2abD + 2aE + 2bF.$$

$$\frac{\partial\, \text{error}}{\partial a} = 2aB + 2bD + 2E = 0$$

$$\frac{\partial\, \text{error}}{\partial b} = 2bC + 2aD + 2F = 0$$

$$a = -\frac{EC + Df}{BC - D^2} \quad ; \quad b = -\frac{Da + f}{C}$$

$$A \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ | & | \\ 1 & t_m \end{pmatrix} \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} y_1 \\ | \\ y_m \end{pmatrix}}_{b}$$

$$x = A^{+} b \quad ; \quad A^{+} = \sum_{\sigma_i > 0} \frac{1}{\sigma_i} u_i v_i^{T}$$

if $\operatorname{rank}(A^T A) = n$

$$\Rightarrow A^{+} = (A^T A)^{-1} A^T$$

$$\Rightarrow X = (A^T A)^{-1} A^T b.$$

$$AX = \underbrace{A (A^T A)^{-1} A^T}_{P_{R(A)}} b.$$

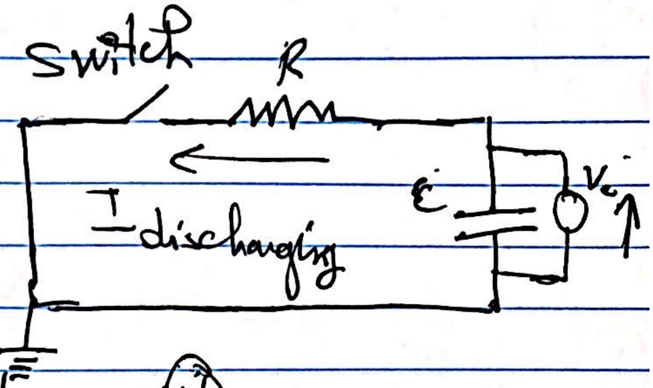
$$\cancel{AA^{+}} = A^{+} A = (A^T A)^{-1} A^T A = I$$

$$A = \sum_i \sigma_i v_i u_i^T \quad ; \quad A^T = \sum \sigma_i u_i v_i^T$$

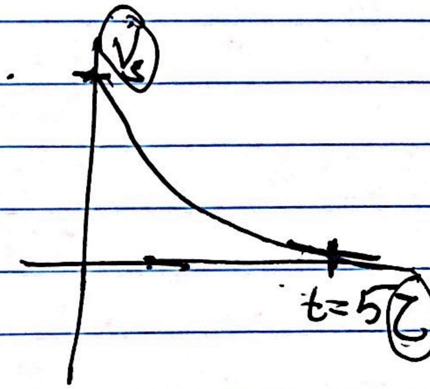
$$(A^T A)^{-1} = \left(\sum_i \sum_j \sigma_i \sigma_j u_i v_i^T v_j u_j^T \right)^{-1} = \left(\sum_i \sigma_i^2 u_i u_i^T \right)^{-1}$$

Exercise!

RC dischaging circuit



$V_c(t) = V_s\, e^{-\frac{t}{\tau}}$; $V_s > 0$, $\tau > 0$



V_c : valtage accross the capacitor.

V_s : supply valtage

t ; time.

$\tau = RC$

$v_c(t_i) \rightarrow t_i$; $v_{c,1}$; t_1

$\vdots \qquad \vdots$

$V_{c_{10}} \qquad t_{10}$

$v_1 = 5,9$	$t_1 = 0,05$
$v_2 = 3,2$	$t_2 = 0,1$
$v_3 = 2$	$t_3 = 0,15$
$v_4 = 1,6$	$t_4 = 0,2$
$v_5 = 1$	$t_5 = 0,25$
$v_6 = 0,35$	$t_6 = 0,3$
$v_7 = 0,33$	$t_7 = 0,35$
$v_8 = 0,23$	$t_8 = 0,4$
$v_9 = 0,1$	$t_9 = 0,45$
$v_{10} = 0,05$	$t_{10} = 0,5$

$$v_c(t) = v_s\, e^{-t/\tau}$$

$$\log V_c(t) = \log V_s - \frac{t}{\tau}$$

$$\underbrace{\begin{pmatrix} 1 & -t_1 \\ \vdots & -t_2 \\ \vdots & \vdots \\ 1 & -t_{10} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \log V_s \\ \frac{1}{\tau} \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} \log V_1 \\ \vdots \\ \log V_{10} \end{pmatrix}}_{b}$$

$$Ax = b$$

$$A^T A x = A^T b \quad (\text{normal equation})$$

$$\begin{pmatrix} 16 & -2,75 \\ -2,75 & 0,9625 \end{pmatrix} \begin{pmatrix} \log V_s \\ \frac{1}{\tau} \end{pmatrix} = \begin{pmatrix} -4,8252 \\ 3,4218 \end{pmatrix}$$

$$\Rightarrow \log V_s = 2,3107 \qquad\Rightarrow\qquad V_s \simeq 10$$

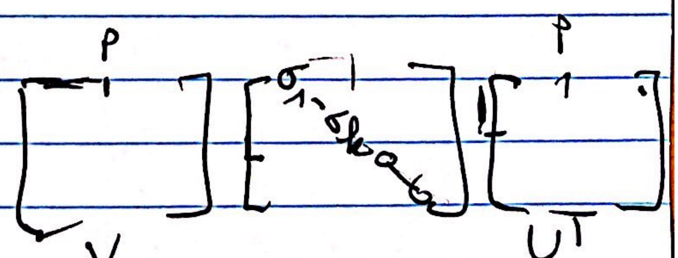
$$\frac{1}{\tau} = 10,1507 \qquad\qquad \tau = 0,0985$$

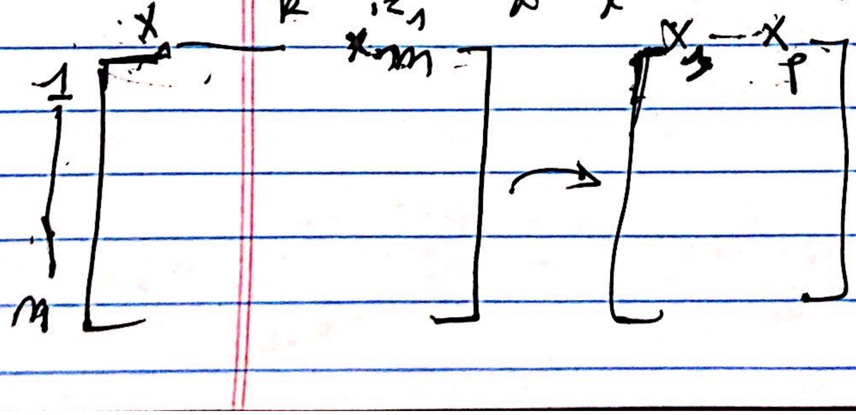
$$y(t) = F(x_1(t), x_2(t), \ldots, x_n(t))$$

$$= a_1 x_1(t) + a_2 x_2(t) \ldots + a_n x_n(t)$$

PCA: $A_k = \underset{X \in A_k}{\operatorname{argmin}} \|A - X\|_F$; $A = \sum_{i=1}^{r} \sigma_i v_i u_i^T$

$$A_k = \sum_{i=1}^{k} \sigma_i v_i u_i^T \; ;$$



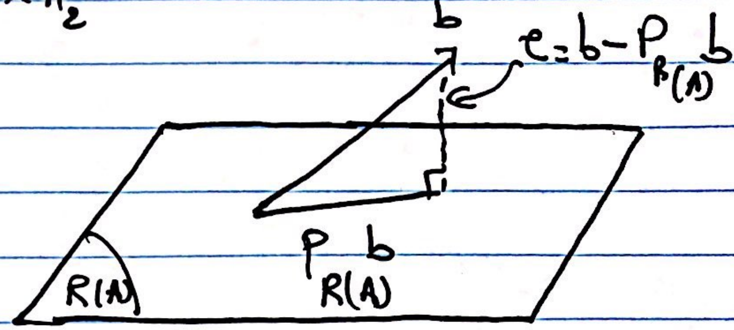


$p < m$

- **More about projection:**

$A \in \mathbb{R}^{m \times n}$, By definition:

$$P_{R(A)} b = \underset{x \in R(A)}{\arg\min} \| b - x \|_2$$



$$\Rightarrow P_{R(A)} b = A x^*$$

$e \perp R(A)$ and $e = P_{R(A)^\perp} b$

$$\Rightarrow e \in N(A^T) \Rightarrow A^T e = 0 \Rightarrow A^T(b - Ax^*) = 0$$

$$\Rightarrow A^T A x^* = A^T b.$$

The solution for $Ax = b$ is given by $x^* = A^\dagger b$

If $A = \sum_i \sigma_i v_i u_i^T \Rightarrow A^\dagger = \sum_{\sigma_i > 0} \frac{1}{\sigma_i} u_i v_i^T$

If $\operatorname{rank}(A^T A) = n \Rightarrow \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$

$$A^T A = \sum_{i,j} \sigma_i \sigma_j u_i v_i^T v_j u_j^T = \sum_i \sigma_i^2 u_i u_i^T$$

$$\Rightarrow (A^T A)^{-1} = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} u_i u_i^T$$

$$(A^T A)^{-1} A^T = \left(\sum_{i=1}^{n} \frac{1}{\sigma_i^2} u_i u_i^T \right)\left(\sum_i \sigma_i u_i v_i^T \right) = \sum_{i=1}^{n} \frac{1}{\sigma_i} u_i v_i^T$$

$$= A^\dagger$$

Then, if $\text{rank}(A^T A) = n$

$$\Longrightarrow A^+ = (A^T A)^{-1} A^T$$

$$\Rightarrow x^* = A^+ b = (A^T A)^{-1} A^T b$$

$$P_{R(A)} b = A x^* = A (A^T A)^{-1} A^T b$$

Let's $P = A (A^T A)^{-1} A^T$

$$P^2 = A \underbrace{(A^T A)^{-1} A^T A}_{I} (A^T A)^{-1} A^T = A (A^T A)^{-1} A^T = P$$

$$P^T = (A (A^T A)^{-1} A^T)^T = A (A^T A)^{-1} A^T = P$$

$\Longrightarrow P$ is a projection matrix onto the spase $R(P)$.

$$P = A A^+ = \left(\sum_i \sigma_i v_i u_i^T\right)\left(\sum_{\sigma_i > 0} \frac{1}{\sigma_i} u_i v_i^T\right)$$

$$= \sum_{i, \sigma_i > 0} v_i v_i^T$$

$\Rightarrow P$ is the projection matrix onto the space spanned $\text{Span}\{v_i ; \sigma_i > 0\}$

We know;

$$R(A) = \text{span}\{v_i ; \sigma_i > 0\}$$

$\Rightarrow$ The projection matrix onto $R(A)$ is given by $A(A^T A)^{-1} A^T$