EE 510 10/16/2020

Outline:

+ Projections
+ Norms

- Inner Product over $\mathbb{R}^{n\times n}$:

$$\langle A, B\rangle = tr(A^T B) \ ; \ \|A\|_F^2 = tr(A^T A).$$

Exercise

Let $P = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

Let $f: \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$

$M \longmapsto PMP$

1°) Prove f is linear.

$\alpha \in \mathbb{R}, M \in \mathbb{R}^{2\times 2}, N \in \mathbb{R}^{2\times 2}$

$$f(\alpha M + N) = P(\alpha M + N)P = \alpha PMP + PNP$$
$$= \alpha f(M) + f(N)$$

$\Rightarrow f$ is linear.

2°) Calculate the matrix ~~of~~ A of f in the standard basis

$$B = \Big\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{e_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{e_2}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{e_3}, \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{e_4} \Big\}$$

$$f\begin{pmatrix}1&0\\0&0\end{pmatrix} = P\begin{pmatrix}1&0\\0&0\end{pmatrix}P = \begin{pmatrix}1&-1\\-1&1\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$$

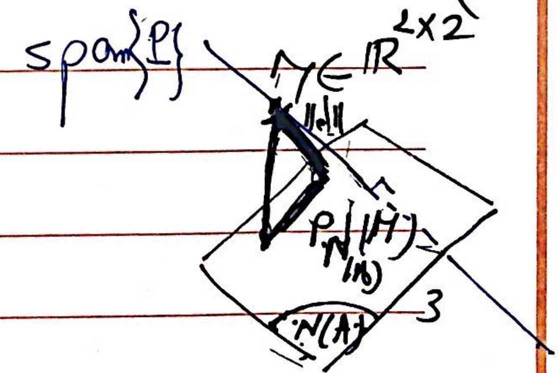
$$= \begin{pmatrix}1&-1\\-1&1\end{pmatrix}$$

$$f\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}1&-1\\-1&1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&-1\\-1&1\end{pmatrix} - \begin{pmatrix}1&-1\\-1&1\end{pmatrix}$$

$$f(e_3) = \begin{pmatrix}-1&1\\1&-1\end{pmatrix} ; f(e_4) = \begin{pmatrix}1&-1\\-1&1\end{pmatrix}$$

$$A = [f]_B = \begin{bmatrix}1&1&-1&1\\-1&-1&1&-1\\-1&-1&1&-1\\1&1&-1&1\end{bmatrix}$$

3) Prove $N(A)^{\perp} = \text{span}\{P\}$ and deduce the distance between a matrix $M \in \mathbb{R}^{2\times 2}$ and $N(A)$



$\text{rank}(A) = 1$

$\Rightarrow \dim N(A) = 3$

$N(A)^{\perp} = R(A).$

$P = f(X)$?!

$$f(P) = PPP = \begin{pmatrix}2&-2\\-2&2\end{pmatrix}\begin{pmatrix}1&-1\\-1&1\end{pmatrix} = \begin{pmatrix}4&-4\\-4&4\end{pmatrix}$$

$$= 4P$$

$$f\left(\frac{1}{4}P\right)=P \Rightarrow P \in R(f)=R(A)$$

$$\Rightarrow \operatorname{Span}(P) \subset R(A)$$

$$\Rightarrow \dim \operatorname{Span}(P) = 1 = \dim R(A)$$

$$\Rightarrow$$

$$R(A)=\operatorname{span}\{P\} \Rightarrow N(A)^{\perp}=\operatorname{Span}\{P\}$$

$$d(M, N(A)) = \min_{X\in N(A)} \|M-X\|_F$$

$$= \|M - P_{N(A)}(M)\|_F = \|P_{N(A)^{\perp}}(M)\|_F$$

$$P_V + P_{V^\perp} = I \Rightarrow P_{V^\perp} = I - P_V$$

$$P_{N(A)^\perp}(M) = \frac{\langle M, P\rangle}{\|P\|_F^2} P = \frac{1}{4}(a-b-c+d)P$$

$$\|P\|_F^2 = \langle P, P\rangle = tr\left(\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right) = tr\begin{pmatrix}2 & -2\\ -2 & 2\end{pmatrix} = 4$$

$\odot$ $M \in \begin{pmatrix}a & b\\ c & d\end{pmatrix}$; $\langle M, P\rangle = tr\left(\begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\right)$

$$= tr\begin{pmatrix}a-c & c-a\\ b-d & d-b\end{pmatrix} = a-b-c+d$$

$$d(M, N(A)) = \left\|\frac{1}{4}(a-b-c+d)P\right\|_F = \frac{|a-b-c+d|}{4}\|P\|_F$$

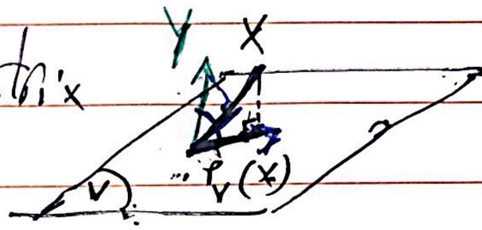
$\|\alpha X\| = |\alpha|\,\|X\|$

$$= \frac{1}{2}|a-b-c+d|$$

• $P = \frac{1}{4}\begin{bmatrix}3 & -1 & 1 & 1\\ -1 & 3 & 1 & 1\\ 1 & 1 & 3 & -1\\ 1 & 1 & -1 & 3\end{bmatrix}$ is projection matrix ?

$P^2 = P$; $P^T = P$

• P is called a projection matrix if $P = P^T$ and $P^2 = P$.

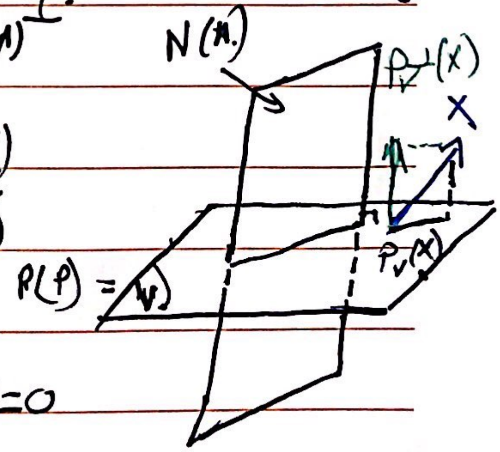


If P is a ~~matrix~~ projection matrix $\Rightarrow$ $I - P$ is projection matrix.

$(I - P)^2 = I - 2P + P^2 = I - 2P + P = I - P$

• $\langle f(X), Y \rangle = \langle X, f(Y) \rangle$: symmetrix linear tranoformation f.

$P = P_{R(P)}$; $I - P = P_{N(A) = R(A)^{\perp}}$

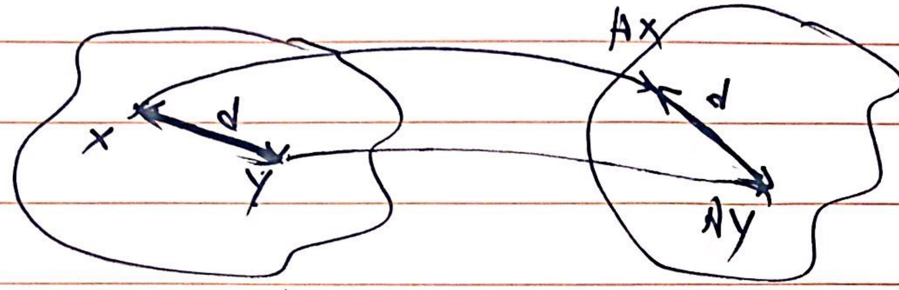


$X \in \mathbb{R}^n$; $X = \underbrace{X - P_V(X)}_{N(P)} + \underbrace{P_V(X)}_{R(P)}$

$P\left(X - P_V(X)\right) = P_V(X) - P_V^2(X)$
$= P_V(X) - P_V(X) = 0$

Isometries:

$$\|Ax\| = \|x\| \; ; \forall x$$



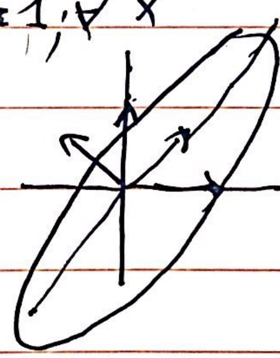
- Rotation, Reflection,

In general: $A^T A = I \Rightarrow$ orthogonal matrix

} Unitary matrix.

$A \in \mathbb{R}^{n \times n}$; $A^{-1} = A^T$

$$\|Ax\| = \|x\| \Rightarrow \frac{\|Ax\|}{\|x\|} = 1, \forall x$$



$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

$$= \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=1} \left\| A \frac{x}{\|x\|} \right\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

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$A=\begin{pmatrix}1 & -1\\ -3 & 3\end{pmatrix}$; $\|A\|=?$

$\max\limits_{\|x\|=1}\|Ax\|=?$ $\quad Ax=\begin{pmatrix}x_1-x_2\\ -3x_1+3x_2\end{pmatrix}$

$\|Ax\|^2 = x_1^2+9x_1^2+x_2^2+9x_2^2-2x_1x_2-18x_1x_2$

$= 10\left[(x_1^2+x_2^2)-2x_1x_2\right]$

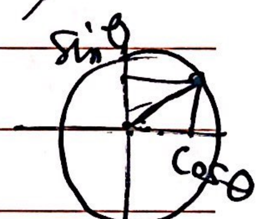
$= 10\left[1-2x_1x_2\right]$

$\max\limits_{\|x\|=1}\|Ax\|^2 = 20$; x_1 and x_2 are symmetric

$|x_1|=|x_2|$; $x_1=-x_2=\frac{1}{\sqrt{2}}$

$\|A\| = \max\limits_{\|x\|=1}\|Ax\| = 2\sqrt{5}$

$A=\begin{pmatrix}-\sqrt{10} & 4\\ \sqrt{6} & 0\end{pmatrix}$; $A\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}-\sqrt{10}\,x_1+4x_2\\ \sqrt{6}\,x_2\end{pmatrix}$



$\|Ax\|^2 = 16-8\sqrt{10}\,x_1x_2$

$= 16-8\sqrt{10}\cos\theta\sin\theta$

$= 18-4\sqrt{10}\sin 2\theta.$ $\quad (\sin 2\theta = 2\cos\theta\sin\theta)$

$\sin 2\theta = -1 \Rightarrow \theta = -\frac{\pi}{4}$

$\Rightarrow x_1=\frac{1}{\sqrt{2}}$, $x_2=-\frac{1}{\sqrt{2}}$ $\quad \|A\| = 4\sqrt{1+\sqrt{\frac{5}{8}}}$

$$A = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 7 \\ 4 & 0 & 0 \end{pmatrix} = 3\, e_1 e_2^T + 4 e_3 e_1^T + 7\, e_2 e_3^T$$

$$\|A\| = 7.$$

$$A = \begin{pmatrix} 2 & 4 & -6 \\ 1 & 2 & -3 \\ -3 & -6 & 9 \end{pmatrix}, \quad \operatorname{rank}(A) = 1$$

$$\|A\| = \sqrt{\sum_{i,j} |a_{ij}|^2} = \sqrt{196} = 14$$

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -3 & 0 & 0 & 3 \end{pmatrix} \qquad \|A\| = \max\left\{ \left\| \begin{matrix} 4 & 0 \\ 0 & 2 \end{matrix} \right\|, \left\| \begin{matrix} 1 & -1 \\ -3 & 3 \end{matrix} \right\| \right\}$$

$$= \max\{4, 2\sqrt{5}\} = 2\sqrt{5}.$$

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