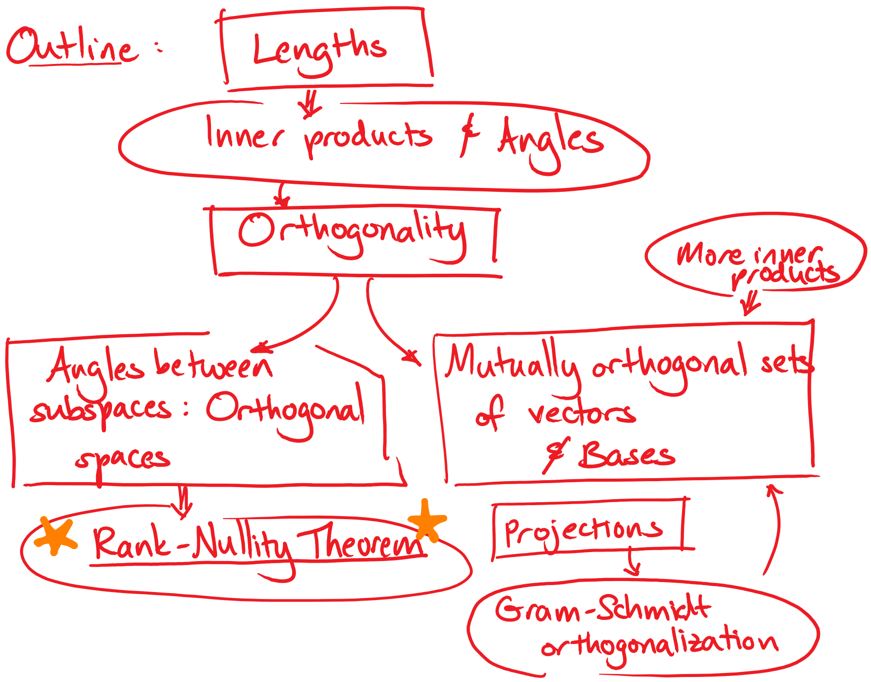
Lecture 8: Orthogonality (class)

Reading:

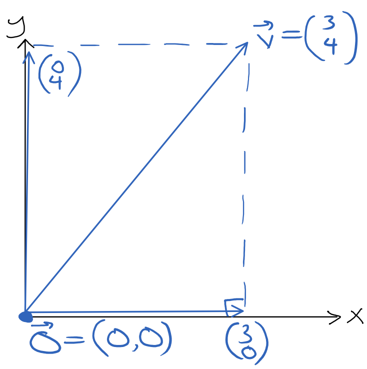
2–3

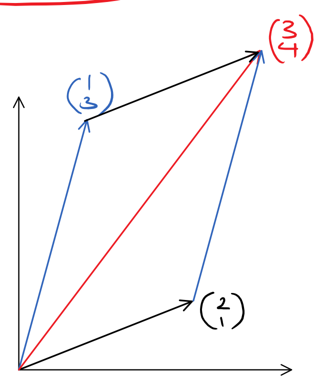
5

Outline:



VECTORS





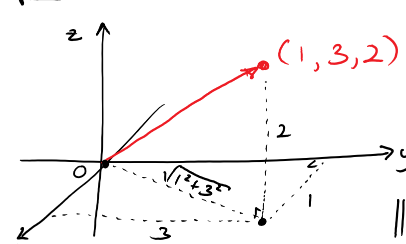
LENGTHS

- The length of a vector $\vec{v} \in \mathbb{R}^n$ (or $\mathbb{C}^n$) is

$$\|\vec{v}\| = \sqrt{|v_1|^2 + \cdots + |v_n|^2}.$$

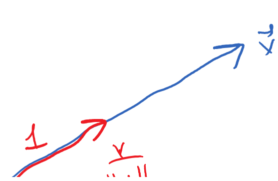
(* Note: We will define other lengths later.)

Example:



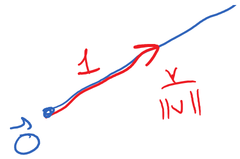
$$\|(1,3,2)\| = \sqrt{1^2+3^2+2^2} = \sqrt{14}$$

Observe: • For $v \in \mathbb{C}^n$, $\|\vec{v}\|^2 = \left(— \ \vec{v}^{*T} \ — \right) \left(\begin{matrix} | \\ \vec{v} \\ | \end{matrix} \right)$



$=\sqrt{14}$

Observe: • For $v \in \mathbb{C}^n$, $\|\vec{v}\|^2 = \underbrace{(\text{—}\ \vec{v}^{*T}\ \text{—})}_{1\times n}\underbrace{\begin{pmatrix} | \\ \vec{v} \\ | \end{pmatrix}}_{n\times 1}$

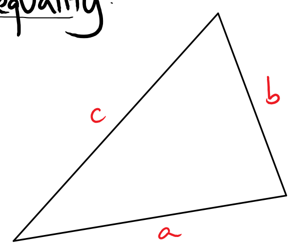


• Scaling: $\|\alpha \vec{v}\| = |\alpha| \cdot \|\vec{v}\|$

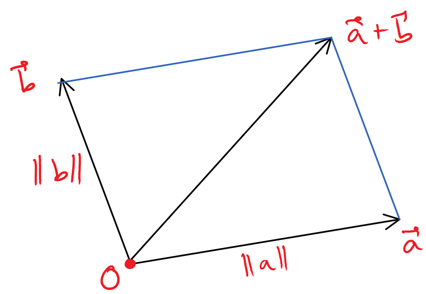
$\Rightarrow$ for any $\vec{v} \neq \vec{0}$, $\frac{\vec{v}}{\|\vec{v}\|}$ is a "unit vector"

i.e. $\left\| \frac{v}{\|v\|} \right\| = 1$

Triangle inequality:

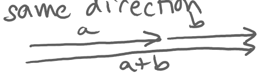


$a+b \geq c$
$a+c \geq b$
$b+c \geq a$



$$\boxed{\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|}$$

with equality if and only if a and b point in the same direction



INNER PRODUCTS & ANGLES

• The **inner product** of two vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$ is

$$\boxed{\begin{aligned}\vec{u}\cdot\vec{v} &= u^{*T} v \\ &= \sum_{i=1}^{n} u_i^* v_i\end{aligned}} \in \mathbb{R}$$

For two complex vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$,
$\vec{u}\cdot\vec{v} = \sum_i \bar{u}_i v_i \in \mathbb{C}$.
(Other notation: dot product, scalar product, $\langle u|v\rangle$, (u,v).)

Observe: * $\vec{u}\cdot\vec{v} = (\vec{v}\cdot\vec{u})^*$ $\qquad (c\vec{u})\cdot(c\vec{v}) = |c|^2(\vec{u}\cdot\vec{v})$

* $\vec{u}\cdot\vec{u} = \|\vec{u}\|^2$

* It is "bilinear":
$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
$(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot w + v\cdot w$
$\vec{u}\cdot(c\vec{v}) = c(\vec{u}\cdot\vec{v})$
$(c\vec{u})\cdot\vec{v} = c^*(\vec{u}\cdot\vec{v})$

• The **angle** between $\vec{u}, \vec{v} \neq \vec{0}$ is

$$\boxed{\cos^{-1}\left(\frac{\vec{u}}{\|u\|}\cdot\frac{\vec{v}}{\|v\|}\right)} \in [0, \pi]$$

They are **orthogonal** if $\vec{u}\cdot\vec{v} = 0$ (angle $= \pi/2$)
AKA perpendicular, $\vec{u} \perp \vec{v}$

Observe: This definition agrees with what we know from geometry:

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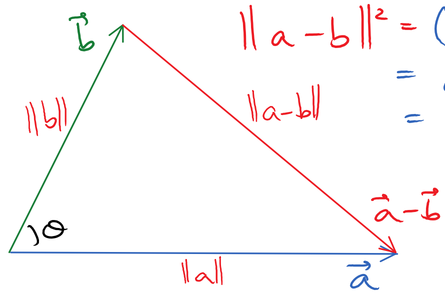


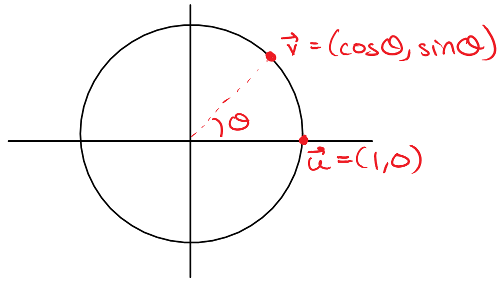
$$\|a-b\|^2 = (a-b)\cdot(a-b)$$
$$= a\cdot a + b\cdot b - 2a\cdot b$$
$$= \|a\|^2 + \|b\|^2 - 2\|a\|\,\|b\|\cos\theta$$

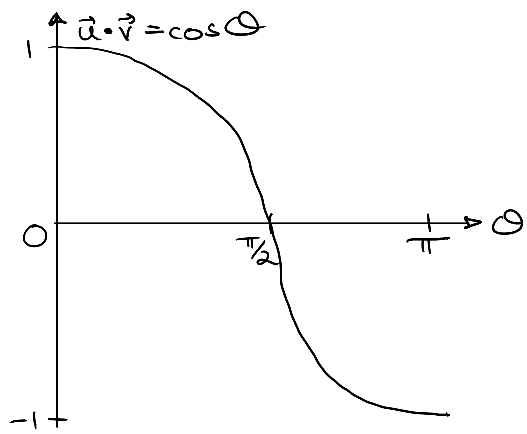
$$\|u\|\cdot\|v\|\,|\cos\theta| = |\vec{u}\cdot\vec{v}|$$

- **Cauchy-Shwarz inequality**:

$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\cdot\|\vec{v}\|$$

Example:





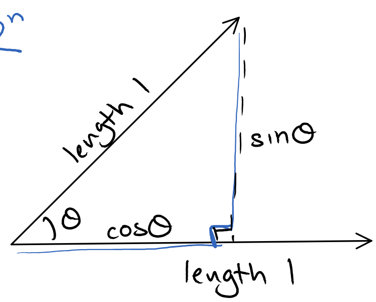
$\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|$ if they point in the same direction

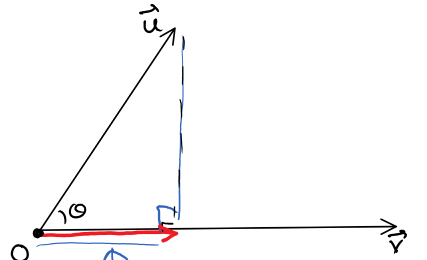
$\vec{u}\cdot\vec{v} = -\|\vec{u}\|\cdot\|\vec{v}\|$ if they point in opposite directions

otherwise, $|\vec{u}\cdot\vec{v}| < \|\vec{u}\|\cdot\|\vec{v}\|$ strictly

Observe: The magnitude $|\vec{u}\cdot\vec{v}|$ measures the "overlap" between $\vec{u}$ and $\vec{v}$:

$\mathbb{R}^n$





projection of $\vec{u}$ onto $\vec{v}$

$$(\cos\theta)\|u\|\cdot\frac{\vec{v}}{\|v\|} = \left(\frac{u}{\cancel{\|u\|}}\cdot\frac{v}{\|v\|}\right)\cancel{\|u\|}\cdot\frac{v}{\|v\|} = (u\cdot v)\frac{v}{\|v\|^2}$$

$$= \frac{v}{\|v\|^2}(v\cdot u) = \left(\frac{v\,v^T}{\|v\|^2}\right)\vec{u}$$

rank-one projection matrix

$\Rightarrow$ If $\|\vec{v}\| = 1$,

$\Rightarrow$ If $\|\vec v\| = 1$,

projection of $\vec a$ in the direction of $\vec v$
$= (\vec u \cdot \vec v)\,\vec v$

Orthogonal vectors

works over $\mathbb{R}$, $\mathbb{C}$ but also over other $\mathbb{F}$

$$\vec u \perp \vec v \iff \vec u \cdot \vec v = 0$$

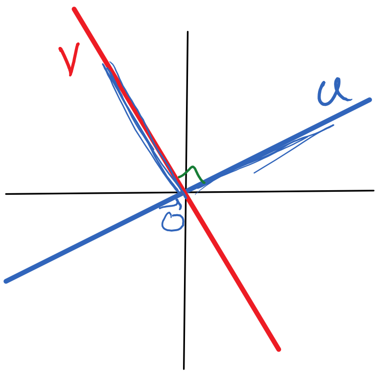
"$\vec u$ and $\vec v$ are orthogonal"

over $\mathbb{F}_2$: $\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}$
$= (1 \quad 1)\begin{pmatrix}1\\1\end{pmatrix} = 1\cdot 1 + 1\cdot 1$
$= 1+1$
$= 0 \mod 2$

$\begin{pmatrix}1\\1\end{pmatrix} \perp \begin{pmatrix}1\\1\end{pmatrix}$ over $\mathbb{F}_2$

a vector can be $\perp$ to itself!

Orthogonal Subspaces



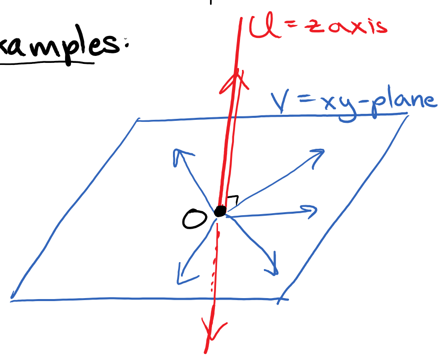
subspaces

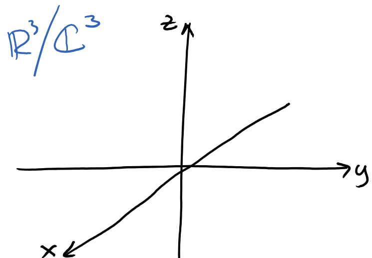
$\mathcal{U} \perp \mathcal{V}$

$\Updownarrow$

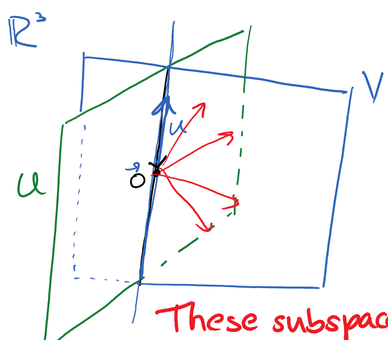
Every vector in $\mathcal{U}$ is orthogonal to every vector in $\mathcal{V}$.

Examples:





coordinate axes are pairwise orthogonal



These subspaces are NOT orthogonal!!!

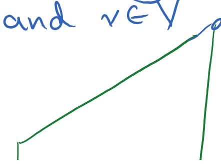
—because they intersect nontrivially

for $u \in \mathcal{U} \cap \mathcal{V}$, u makes angle 0 with itself

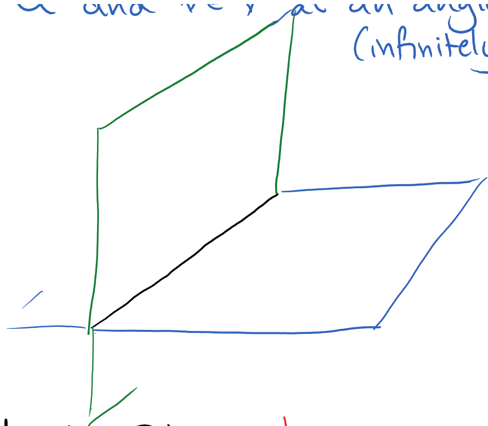
b/c for $\vec u \in \mathcal{U} \cap \mathcal{V}$, $\vec u \neq \vec 0$

$\underset{\mathcal{U}}{u} \cdot \underset{\mathcal{V}}{u} = \|u\|^2 \neq 0$

In fact, for any angle θ, there are vectors $u \in \mathcal{U}$ and $v \in \mathcal{V}$ at an angle θ to each other.
(infinitely many angles!)



(infinitely many angles!)



Example: In $\mathbb{R}^4$, wx-plane

$$\mathrm{Span}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right) \perp \mathrm{Span}\left(\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right)$$

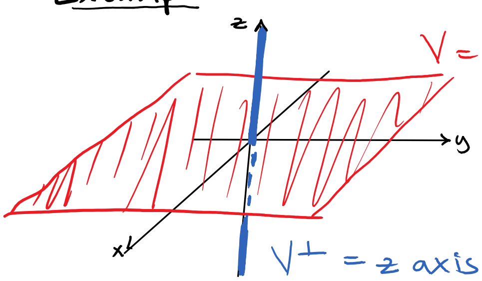
(w, x, y, z) yz-plane

Definition: For a subspace V,

$V^\perp = \{$ all vectors orthogonal to $V \}$

"orthogonal complement of V", "V perp"

Example:



$V =$ xy-plane

$V^\perp = z$ axis

$\mathbb{R}^n \subseteq \mathbb{R}^n$

$(\mathbb{R}^n)^\perp = \{\vec{0}\}$

$(\text{xy-plane})^\perp = z\text{-axis}$

$(z\text{-axis})^\perp = \text{xy-plane}$

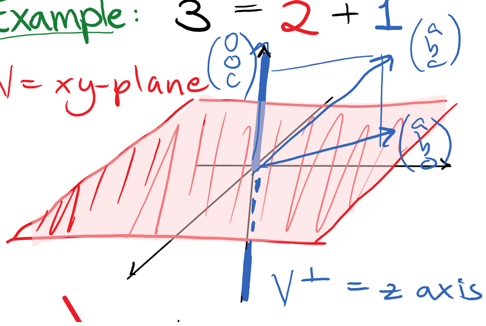
- $V \cap V^\perp = \{\vec{0}\}$ in $\mathbb{R}^n$ over $\mathbb{C}^n$ (not over finite fields)
- $(V^\perp)^\perp = V$ always
- x and z axes are orthogonal, but not complements in $\mathbb{C}^3/\mathbb{R}^3$

Theorem: (Orthogonal complementary subspaces)
Let V be a subspace of $\mathbb{R}^n$. Then,

- $\dim(V^\perp) = n - \dim(V)$ ✓ ← also true over $\mathbb{F}$
- $\mathrm{Span}(V \cup V^\perp) = \mathbb{R}^n$ ← not necessarily true over other $\mathbb{F}$

Example: $3 = 2 + 1$

$V =$ xy-plane

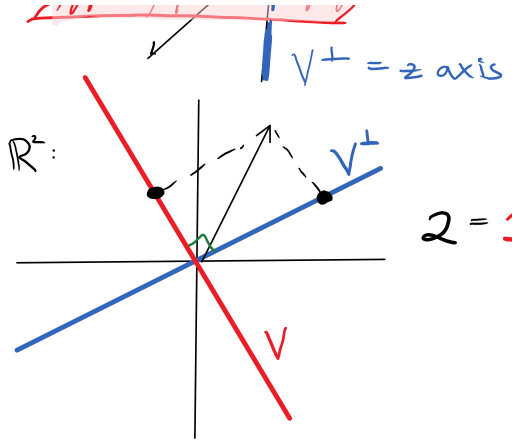


$V^\perp = z$ axis

This means that any vector can be split

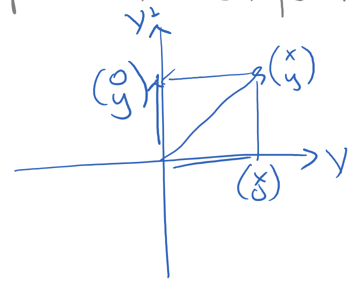
$$(a,b,c) = \underbrace{(a,b,0)}_{\in V} + \underbrace{(0,0,c)}_{\in V^\perp}$$

Exercise: Prove that this



$2 = 1 + 1$

Exercise: Prove that this decomposition is unique.



Proof of the theorem: $d = \dim V$

Let $\vec{v}_1, \dots, \vec{v}_d$ be a basis for V. Goal: show $\dim V^\perp = n - d$

Claim: $\vec{u}$ is orthogonal to all vectors in V

$\Updownarrow$

$\vec{u}$ is orthogonal to each $\vec{v}_1, \dots, \vec{v}_d$. (the basis vectors)

Proof: $\Downarrow$: obvious, as all $\vec{v}_j \in V$

$\Uparrow$: Let $\vec{v} \in V$. Goal: show $u \cdot v = 0$

$\Downarrow$

there exist $c_1, \dots, c_d$ s.t. $\vec{v} = \sum_j c_j \vec{v}_j$ (basis)

$$u \cdot v = u \cdot \left(\sum_j c_j \vec{v}_j\right) = \sum_j \vec{u} \cdot (c_j \vec{v}_j) = \sum_j c_j \overset{0}{\cancel{(u \cdot v_j)}} = 0 \quad \boxed{\checkmark}$$

$$V^\perp = \{\vec{u} \mid \vec{u} \text{ is orthogonal to all } V\}$$
$$= \{\vec{u} \mid u \cdot v_1 = u \cdot v_2 = \cdots = u \cdot v_d = 0\}$$ $\leftarrow$ d equations

$$= N\left(\begin{bmatrix} \text{---} & v_1^T & \text{---} \\ \text{---} & v_2^T & \text{---} \\ & \vdots & \\ \text{---} & v_d^T & \text{---} \end{bmatrix}\right)$$

$d \times n$ matrix

v_j's are linearly indep.

$\Rightarrow$ after GE #pivots $= d$

#free variables $= n - d$

$\dim V^\perp =$ $\square$

Next, to show $\mathrm{Span}(V \cup V^\perp) = \mathbb{R}^n$, let $\vec{u}_1, \dots, \vec{u}_{n-d}$ be a basis for $V^\perp$.

Consider the vectors

$$\vec{v}_1, \dots, \vec{v}_d, \vec{u}_1, \dots, \vec{u}_{n-d}.$$

Claim: These vectors are linearly independent.

$v_1, \dots, v_d, u_1, \dots, u_{n-d}$.

Claim: These vectors are linearly independent.

Proof: $\underbrace{\vec{v}_1, \dots, \vec{v}_d}_{\text{ind.}}, \underbrace{\vec{u}_1, \dots, \vec{u}_{n-d}}_{\text{ind.}}$

so the only way to get a dependency would be if there were

$$\underset{\substack{\cap \\ V}}{\vec{v}} + \underset{\substack{\cap \\ V^\perp}}{\vec{u}} = \vec{0} \quad \Rightarrow \underset{\substack{\cap \\ V}}{\vec{v}} = \underset{\substack{\cap \\ V^\perp}}{-\vec{u}} = \vec{0}. \qquad \square$$

form a basis for $\mathbb{R}^n$

But n linearly independent vectors in a space of dimension n must be a basis, spanning everything. $\square$

How to find $V^\perp$:

Start with a basis for V.

$\Rightarrow V^\perp = \{\text{all vectors} \perp \text{to } V\}$

$= \{\text{all vectors} \perp \text{to all basis elements}\}$

(i.e. $b_i \cdot v = 0$ for each basis elt. b_i)

Let $A = \begin{pmatrix} \text{rows a} \\ \text{basis for } V \end{pmatrix} \Rightarrow R(A^T) = V$

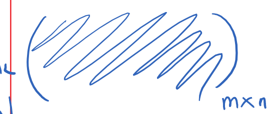
$N(A) = V^\perp$

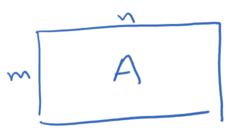
Lecture 8: Rank-Nullity Theorem (class)

RANK-NULLITY THEOREM

Definition: The rank of a matrix A is
$$\operatorname{rank}(A) = \dim R(A),$$
the dimension of the range.

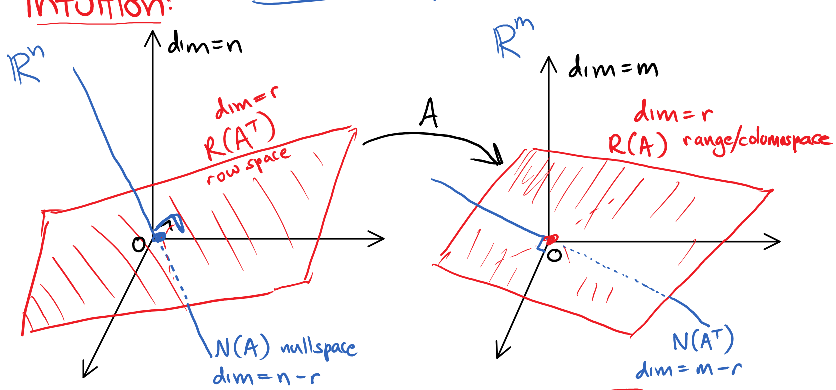
Rank-Nullity Theorem:
Let A be an $m \times n$ matrix. Then
- $\boxed{\dim R(A^T) = \dim R(A)}$
- $\dim N(A) = n - \dim R(A) = n - \text{rank}$
- $\dim N(A^T) = m - \dim R(A) = m - \text{rank}$





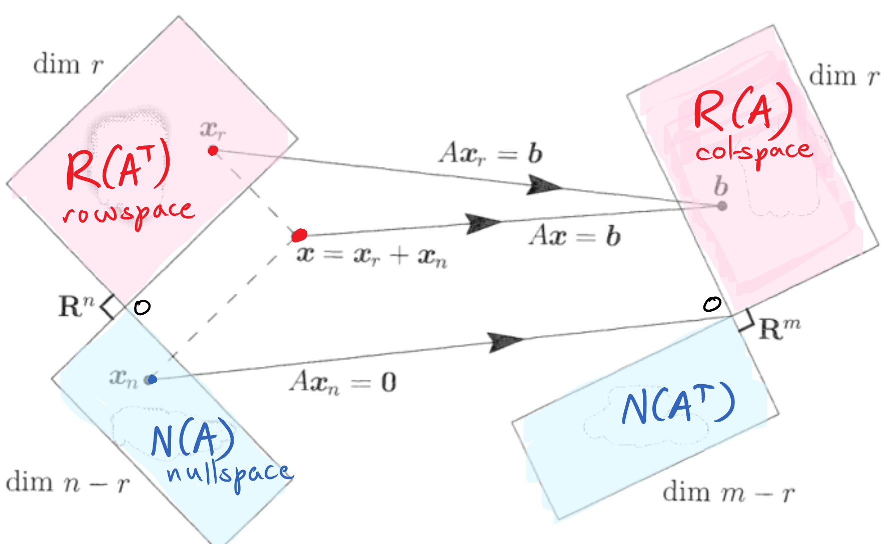
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}_{3\times 5}$$

Intuition:



$$\boxed{\dim R(A^T) = \dim R(A)}$$

$$\boxed{\dim N(A) + \dim R(A^T) = \text{total dimension } n} \qquad \boxed{\dim R(A) + \dim N(A^T) = \text{total dimension } m}$$



$$N(A) = R(A^T)^{\perp} \qquad N(A^T) = R(A)^{\perp}$$

$N(A) = R(A^T)^\perp$ $\qquad$ $N(A^T) = R(A)^\perp$

<u>Corollary</u>: A is invertible $\iff$ $m = n = \text{rank}(A)$.
"full rank"

<u>Example</u>:
<u>Problem</u>: Give a basis for $\underbrace{\{\vec{x} \in \mathbb{R}^n \mid x_1 + x_2 + \cdots + x_n = 0\}}_{\text{"}V\text{"}}$.

Answer:
$V = N(\underbrace{(1 \ 1 \ 1 \cdots 1)}_{\text{"}A\text{"}})$
$\text{rank}(A) = 1$
$\Rightarrow \dim N(A) = n - \text{rank}(A)$ $\qquad$ [Rank-Nullity]
$= n - 1$
$\Rightarrow$ <u>Any</u> linearly independent set of $n-1$ elements of V is a basis for V.
For example,
$\{\vec{e}_1 - \vec{e}_n, \vec{e}_2 - \vec{e}_n, \vec{e}_3 - \vec{e}_n, \ldots, \vec{e}_{n-1} - \vec{e}_n\}$ is a basis. ✓

Proof of the Rank-Nullity Theorem:

1) <u>Claim</u>: For <u>any</u> $m \times n$ matrix A,

$$\boxed{\begin{aligned} N(A) &= R(A^T)^\perp \\ N(A^T) &= R(A)^\perp \end{aligned}} \checkmark \qquad \left(\Rightarrow \begin{aligned} \dim N(A) &= n - \dim R(A^T) \\ \dim N(A^T) &= m - \dim R(A) \end{aligned} \right)$$

Proof: Let $\mathcal{U} = R(A^T)$.

$$A = \begin{pmatrix} \overrightarrow{\text{row } 1} \\ \overrightarrow{\text{row } 2} \\ \text{rows span } \mathcal{U} \\ \overrightarrow{\text{row } m} \end{pmatrix} \qquad N(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$$

$$A\vec{x} = \begin{pmatrix} \overrightarrow{\text{row } 1} \cdot \vec{x} \\ \overrightarrow{\text{row } 2} \cdot \vec{x} \\ \vdots \\ \overrightarrow{\text{row } m} \cdot \vec{x} \end{pmatrix}$$

$\vec{x} \perp R(A^T)$
$\Updownarrow$
$\vec{x} \cdot \overrightarrow{\text{row}_j} \ \forall j = 1, \ldots, m$

for the second statement, plug in A^T to the first statement.

2) <u>Claim</u>: $\dim R(A) = \dim R(A^T)$.

Proof #2 (works in general):
The above proof doesn't work over finite fields.
For example, if $A = (1 \ \ 1)$ over $\mathbb{F}_2$,
$N(A) = R(A^T) = \{\vec{0}, (1\ 1)\}$

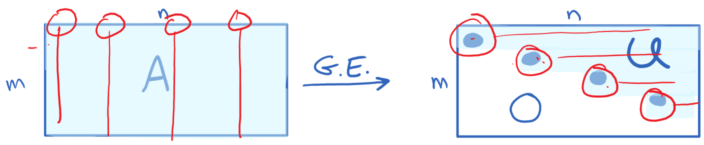
The above proof doesn't work over finite fields.

For example, if $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$ over $\mathbb{F}_2$,

$$N(A) = R(A^T) = \left\{ \vec{0}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

However $\dim R(A) = \dim R(A^T)$ is still true.

Proof using Gaussian elimination:



$\#$pivots $= \dim R(A^T)$

columns in A corresponding to pivots form a basis for $R(A)$

$\Rightarrow \#$pivots $= \dim R(A)$

Let $d = \#$ of pivot columns.

Since those rows span $R(A^T) = R(U^T)$, $d = \dim R(A^T)$.

Since the corresponding columns in A span $R(A)$ (see Lecture: Subspaces of a matrix), $d = \dim R(A)$. ✓

Also, $\#$ free variables $= n - d = \dim N(A)$. ✓ $\square$

Proof #1 (works over $\mathbb{R}$ or $\mathbb{C}$):

Let $\vec{r}_1, \dots, \vec{r}_d$ be a basis for the rowspace $R(A^T)$.

We claim that $A\vec{r}_1, \dots, A\vec{r}_d$ are linearly independent.

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 4 \\ 2 & -1 & 1 \end{pmatrix} \xrightarrow{-4,\,-2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -8 & -8 \\ \cancel{0} & \cancel{-5} & \cancel{-5} \end{pmatrix} \longrightarrow \begin{array}{ccc} & & \text{free} \\ \end{array} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R(A) = \mathrm{Span}\{(1,4,2),(2,0,-1)\} \perp N(A^T) = \mathrm{Span}((4,-5,8))$$
$$R(A^T) = \mathrm{Span}\{(1,0,1),(0,1,1)\} \perp N(A) = \mathrm{Span}((-1,-1,1))$$

$$\mathrm{rank}(A) = \dim R(A) = \dim R(A^T) = 2$$
$$\dim N(A) + \dim R(A^T) = 3$$
$$\dim N(A^T) + \dim R(A) = 3$$

Example:

$$A = \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & \ddots & & \\ & & & & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix}_{n \times n}$$

discretized 2^{nd} derivative on the circle

$$A = A^T \Rightarrow R(A^T) = R(A)$$

$$\longrightarrow \begin{pmatrix} -1 & 1 & \cdots & 1 & -1 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -1 & 1 & \cdots & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 \\ & 1 & -2 & 1 & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} -1 & 1 & \cdots & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 \\ & 0 & -1 & 1 \cdots 1 & -1 \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} -1 & & & 1 & -1 \\ & -1 & & 1 & -1 \\ & & \ddots & 1 & -1 \\ & & & & \\ 0 & 0 & 0 \cdots & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \dim R(A^T) = \mathrm{rank}(A) = n-1$$
$$\Rightarrow \dim N(A) = n - \mathrm{rank} = 1$$

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \vec{0} \qquad N(A) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\right) = N(A^T)$$

$$R(A) = R(A^T) = N(A)^{\perp} = N((1 \;\; 1 \; \cdots \; 1))$$
$$= \{ x \in \mathbb{R}^n \mid \textstyle\sum_{i=1}^{n} x_i = 0 \}$$

$$R(A) = R(A^T) = N(A) = N\big((1 \; 1 \; \cdots \; 1)\big)$$
$$= \Big\{ \vec{x} \in \mathbb{R}^n \;\Big|\; \sum_{j=1}^{n} x_j = 0 \Big\}$$

Recall: $N(A) = \mathrm{Span}((1,1,\dots,1))$

$A = A^T \Rightarrow N(A^T) = N(A)$

$\Rightarrow R(A) = N(A^T)^\perp$ $n-1$ dimensional

$= \{\text{all } \vec{b} \in \mathbb{R}^n \mid \vec{b} \cdot (1,1,\dots,1) = 0\}$

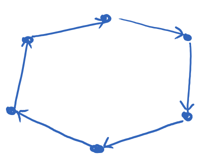
$= \{\text{all } \vec{b} \in \mathbb{R}^n \mid b_1 + b_2 + \cdots + b_n = 0\}$

Hence, $\boxed{Ax = b \text{ has a solution} \iff b_1 + \cdots + b_n = 0}$

This is much easier than using Gaussian elimination to find a basis for $R(A)$. (Essentially because $N(A^T)$ is low-dimensional.)

This example is actually very important; similar matrices arise frequently in applications.

Note that $A = -E_G E_G^T$, where G is the cycle graph and E_G is the incidence matrix.



We proved $N(E_G^T) = \mathrm{Span}((1,1,\dots,1))$, which gives an alternate proof that $N(A) = \mathrm{Span}((1,\dots,1))$.

Exercise:

Give the *dimensions* of the four fundamental subspaces (columnspace, rowspace, nullspace left nullspace) of the matrix

$$A = \begin{pmatrix} 500 & 5 & 6 & 3 & 0 \\ 200 & 2 & 8 & 4 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 500 & 5 & 0 & 0 & 0 \\ 2 & 3 & 4 & 4 & 5 \end{pmatrix}_{6 \times 5}$$

Space	Dimension
$R(A)$	4
$R(A^T)$	4
$N(A)$	$5 - \text{rank} = 1$
$N(A^T)$	$6 - \text{rank} = 2$

rank

Claim: Every step of GE leaves the rank unchanged. (true also for GE on columns)

$$\begin{pmatrix} 500 & 5 & 6 & 3 & 0 \\ 200 & 2 & 8 & 4 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 500 & 5 & 6 & 3 & 0 \\ 200 & 2 & 8 & 4 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 500 & 5 & 0 & 0 & 0 \\ 2 & 3 & 4 & 4 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 500 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 500 & 5 & 6 & 3 & 0 \\ 200 & 2 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 500 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 500 & 5 & 0 & 3 & 0 \\ 200 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 100 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 100 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 100 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{rank} = 4$$

Answer: