Homework 4 answers

Systems of linear equations

1. Consider a particular species of wildflower in which each plant has several stems, leaves and flowers, and for each plant let

S = the average stem length (in cm)
L = the average leaf width (in cm)
F = the number of flowers.

Four plants are examined, and the information is tabulated in the following matrix:

$$A = \begin{array}{c} \\ \#1 \\ \#2 \\ \#3 \\ \#4 \end{array} \begin{array}{c} \begin{array}{ccc} S & L & F \end{array} \\ \begin{pmatrix} 2 & 2 & 10 \\ 4 & 2 & 12 \\ 4 & 4 & 15 \\ 6 & 4 & 17 \end{pmatrix} \end{array}.$$

For these four plants, determine whether or not there exists a linear relationship between S, L and F.
In other words, do there exist constants $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ such that $\alpha_0 + \alpha_1 S + \alpha_2 L + \alpha_3 F = 0$?

Answer:

This is equivalent to finding a nonzero vector in the nullspace of

$$\begin{pmatrix} 1 & 2 & 2 & 10 \\ 1 & 4 & 2 & 12 \\ 1 & 4 & 4 & 15 \\ 1 & 6 & 4 & 17 \end{pmatrix}.$$

You can solve it with Gaussian elimination, or in Matlab:

```
>> A = [
1 1 1 1;
2 4 4 6;
2 2 4 4;
10 12 15 17
]';
>> alpha = null(A)

alpha =

   -0.9245
   -0.1849
   -0.2774
    0.1849

>> alpha / alpha(1)
```
← divide by the 1st component to try to make it nicer

```
ans =

    1.0000
    0.2000
    0.3000
   -0.2000
```

$$\boxed{\Rightarrow \alpha_0 = 1,\ \alpha_1 = \tfrac{1}{5},\ \alpha_2 = \tfrac{3}{10},\ \alpha_3 = -\tfrac{1}{5} \text{ works}}$$

② Determine which of the following sets of vectors are linearly independent. For those sets that are linearly dependent, write one of the vectors as a linear combination of the others.

a) $\{(1,2,3),\ (2,1,0),\ (1,5,9)\}$

✗ Dependent: $\begin{pmatrix}1\\5\\9\end{pmatrix} = 3\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}2\\1\\0\end{pmatrix}$

b) $\{(1,2,3),\ (0,4,5),\ (0,0,6),\ (1,1,1)\}$

✗ Dependent (four vectors in $\mathbb{R}^3$ must be linearly dep.)

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \frac{1}{4}\begin{pmatrix}0\\4\\5\end{pmatrix} - \frac{1}{8}\begin{pmatrix}0\\0\\6\end{pmatrix}$$

c) $\{(3,2,1),\ (1,0,0),\ (2,1,0)\}$

✓ Independent

d) $\{(2,2,2,2),\ (2,2,0,2),\ (2,0,2,2)\}$

✓ Independent

e) $\left\{\begin{pmatrix}1\\2\\0\\4\\0\\3\end{pmatrix}, \begin{pmatrix}0\\2\\0\\4\\1\\3\end{pmatrix}, \begin{pmatrix}0\\2\\1\\4\\0\\3\end{pmatrix}, \begin{pmatrix}0\\2\\0\\4\\0\\3\end{pmatrix}\right\}$

$$\left\{ \begin{pmatrix} \vdots \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} \vdots \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} \vdots \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} \vdots \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\}$$

✓ Independent

Notice that each of the vectors has a coordinate that is nonzero where all 3 other vectors are 0. Therefore none of them can be expressed as a linear combination of the others.

Vector spaces

3) Which of the following are vector spaces? In each case, explain why/why not. And if it is a vector space, give the dimension and a basis.

a) the set $\mathbb{R}$ of real numbers

$\boxed{\text{Yes}}$ closed under addition ✓ & scalar multiplication ✓

$\boxed{\begin{array}{l} \text{dimension } 1 \\ \text{basis } \{1\} \end{array}}$

In fact, for any $a \neq 0$, $\{a\}$ is a basis for $\mathbb{R}$

b) the set of solutions (x_1, x_2) to the equations

$$5x_1 + 2x_2 = 0$$
$$3x_1 - 2x_2 = 2$$

$\boxed{\text{No}}$, $(0,0)$ is not a solution

c) the set of solutions (x_1, x_2) to the equation

$$x_1 x_2 = 0$$

$\boxed{\text{No}}$, the solutions are not closed under addition. For example,

$$(1,0) \quad \text{and} \quad (0,1)$$

are both solutions, but

$$(1,0) + (0,1) = (1,1)$$

is not.

d) the span of the vectors

$$(2,3,4),\quad (-1,-1,-4) \text{ and } (0,1,-4)$$
$$\qquad \vec{v}_1 \qquad\qquad\qquad \vec{v}_2 \qquad\qquad\qquad \vec{v}_3$$

Yes, the span of a set of vectors is **always** a vector space.

Since

$$\vec{v}_3 = \vec{v}_1 + 2\vec{v}_2,$$

they are not linearly independent.

$$\boxed{\begin{array}{l}\text{dimension } 2\\ \text{basis } \{\vec{v}_1, \vec{v}_2\}\end{array}}$$

e) the set of **anti-symmetric** 4×4 matrices (recall: a matrix A is antisymmetric if $A^T = -A$)

Yes A 4×4 antisymmetric matrix must have the form

$$\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \quad \text{for } a,b,c,d,e,f \in \mathbb{R}$$

Such matrices are closed under addition and scalar multiplication.

dimension 6

basis
$$\left\{ \begin{pmatrix} 0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0 \end{pmatrix} \right\}$$

Bases

④ Determine the dimension of the space spanned by the set

$$S = \left\{ \begin{pmatrix}1\\2\\-1\\3\end{pmatrix}, \begin{pmatrix}1\\0\\0\\2\end{pmatrix}, \begin{pmatrix}2\\8\\-4\\8\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}3\\3\\0\\6\end{pmatrix} \right\}$$

Answer: We'll apply Gaussian elimination to find the dimension of the rowspace of

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 1 & 0 & 0 & 2 \\ \dots \end{pmatrix} \begin{matrix} +2 \\ \\ -4 \end{matrix} \qquad \begin{pmatrix} 1 & 2 & -1 & 3 \\ 1 & 0 & 0 & 2 \\ \dots \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & -1 \\ \dots \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 3 \\ 1 & 0 & 0 & 2 \\ 2 & 8 & -4 & 8 \\ 1 & 1 & 1 & 1 \\ 3 & 3 & 0 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & -1 & 3 \\ 1 & 0 & 0 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \dim(\mathrm{Span}(S)) = 3$$

⑤ Determine whether or not the set

$$B = \{(2,3,2),\ (1,1,-1)\}$$

is a basis for the space spanned by the set

$$A = \{(1,2,3),\ (5,8,7),\ (3,4,1)\}.$$

Answer:

Since $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 7 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$,

A spans a 2-dimensional plane.

This span is the same as Span(B) if it contains both vectors in B.

Let's check this with Matlab:

```
>> A = [1 2 3; 5 8 7]'

A =

     1     5
     2     8
     3     7

>> x = A \ [2;3;2]

x =

   -0.5000
    0.5000

>> A*x

ans =

    2.0000
    3.0000
    2.0000
```

$\Rightarrow \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \in \mathrm{Span}(A)$

```
>> x = A \ [1;1;-1]

x =

   -1.5000
    0.5000

>> A*x

ans =

    1.0000
    1.0000
   -1.0000
```

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \in \mathrm{Span}(A)$.

Since the vectors in B are linearly independent, and both lie in the span of A, $\boxed{\text{yes,}}$ they form a basis for the span of A.

⑥ How many different bases are there for the 3-dimensional space $\mathbb{F}_2^3$?

Answer: There are $8-1=7$ nonzero vectors in $\mathbb{F}_2^3$

space #2 :

Answer: There are $8-1=7$ nonzero vectors in $\mathbb{F}_2^3$.
After choosing the first vector $\vec{v}_1$, there are $7-1=6$ choices for the second basis vector $\vec{v}_2$.
$\vec{v}_1$ and $\vec{v}_2$ span the 2D subspace $\{\vec{0}, \vec{v}_1, \vec{v}_2, \vec{v}_1+\vec{v}_2\}$.
The third basis vector can be anything outside this space, so there are 4 choices for $\vec{v}_3$.

$$\Rightarrow \quad 7 \cdot 6 \cdot 4 = \boxed{168} \text{ different bases}$$

If we don't care about the order, then dividing by $3!$ gives 28.
Here's another argument that also works:

There are $8-1=7$ nonzero vectors in $\mathbb{F}_2^3$.

$\binom{7}{3} = \frac{7!}{3!4!} = 35$ ways of choosing 3 different nonzero vectors

But not all of these span $\mathbb{F}_2^3$. Some will only span a 2D subspace.
There are 7 2D subspaces, e.g., $\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$.
(Each is determined by a nonzero orthogonal vector.)

$\Rightarrow$ There are $35-7=28$ different bases for $\mathbb{F}_2^3$, not counting order, or

$28 \times 3! = \boxed{168}$ ordered bases

Here's some Mathematica code that enumerates them all:

```
bases = {};
For[k = 1, k < 2^3, k++,
  kd = IntegerDigits[k, 2, 3];
  For[l = k + 1, l < 2^3, l++,
   ld = IntegerDigits[l, 2, 3];
   For[m = l + 1, m < 2^3, m++,
    md = IntegerDigits[m, 2, 3];
    If[Mod[kd + ld + md, 2] == {0, 0, 0}, Continue[];];
    AppendTo[bases, {kd, ld, md}];
    ];
   ];
  ];
MatrixForm /@ bases
bases // Length
```

$$\left\{\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&1&1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&1\\1&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&1\\1&1&0\end{pmatrix},\right.$$
$$\begin{pmatrix}0&0&1\\0&1&1\\1&1&1\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\1&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\1&1&1\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&1\\1&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&1\\1&1&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&1&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&1&1\\1&0&1\end{pmatrix},$$
$$\begin{pmatrix}0&1&0\\0&1&1\\1&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&1&1\\1&1&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\1&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\1&1&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&1\\1&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&1&0\\1&1&1\end{pmatrix}, \begin{pmatrix}0&1&1\\1&0&0\\1&0&1\end{pmatrix},$$
$$\left.\begin{pmatrix}0&1&1\\1&0&0\\1&1&0\end{pmatrix}, \begin{pmatrix}0&1&1\\1&0&1\\1&1&1\end{pmatrix}, \begin{pmatrix}0&1&1\\1&1&0\\1&1&1\end{pmatrix}, \begin{pmatrix}1&0&0\\1&0&1\\1&1&0\end{pmatrix}, \begin{pmatrix}1&0&0\\1&0&1\\1&1&1\end{pmatrix}, \begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}, \begin{pmatrix}1&0&1\\1&1&0\\1&1&1\end{pmatrix}\right\}$$

28

⑦ Let

$$A = \begin{pmatrix}2&2&5&0&1\\3&4&8&1&2\\1&6&5&5&3\end{pmatrix}, \quad \vec{v} = \begin{pmatrix}3\\1\\0\\3\\-8\end{pmatrix}.$$

Verify that $\vec{v} \in N(A)$.

Verify that $\vec{v} \in N(A)$.
Then extend $\{\vec{v}\}$ to a basis for $N(A)$.
(That is, find a basis for $N(A)$ that includes $\vec{v}$.)

Answer: Since $A\vec{v} = 0$, $\vec{v} \in N(A)$.

Let us now find $N(A)$. Applying row operations,

$$\begin{pmatrix} 2 & 2 & 5 & 0 & 1 \\ 3 & 4 & 8 & 1 & 2 \\ 1 & 6 & 5 & 5 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 6 & 5 & 5 & 3 \\ 0 & -10 & -5 & -10 & -5 \\ 0 & -14 & -7 & -14 & -7 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 6 & 5 & 5 & 3 \\ 0 & 2 & 1 & 2 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 2 & 1 & 2 & 1 \end{pmatrix}$$

Hence $\mathrm{Rank}(A) = 2$, so

$$\dim N(A) = 5 - \mathrm{Rank}(A) = 3.$$

($A\vec{x} = \vec{0}$ has two independent equations on 5 variables)

The free variables are x_2, x_4 and x_5.

Back-substituting gives

$$x_1 = -2x_3 + x_4$$
$$x_2 = -\tfrac{1}{2}(x_3 + 2x_4 + x_5)$$

Hence the general solution to $A\vec{x} = \vec{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = a \begin{pmatrix} -2 \\ -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ -1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

where a, b, c can be anything.

$$\Rightarrow N(A) = \mathrm{Span}\{(4,1,-2,0,0), (1,-1,0,1,0), (0,1,0,0,-2)\}$$

Since it is easy to see that $\vec{v}$ does not lie in the span of the first two of these ~~vectors~~,

$$N(A) = \mathrm{Span}\boxed{\{\vec{v}, (4,1,-2,0,0), (1,-1,0,1,0)\}}$$

↑ our desired basis
(of course it is not unique)

Check the answer:

```
>> A = [2 2 5 0 1; 3 4 8 1 2; 1 6 5 5 3];
>> V = [3 1 0 3 -8; 4 1 -2 0 0; 1 -1 0 1 0];
>> A = [2 2 5 0 1; 3 4 8 1 2; 1 6 5 5 3];
>> V = [3 1 0 3 -8; 4 1 -2 0 0; 1 -1 0 1 0]';
```
← (notice the transpose!)

```
>> rank(V)

ans =

     3
```

$\Rightarrow \{v, v_1, v_3\}$ is a lin. indep. set ✓

```
>> A * V

ans =

     0     0     0
     0     0     0
     0     0     0
```

$\Rightarrow A\vec{v} = A\vec{v}_1 = A\vec{v}_3 = \vec{0}$ ✓

Inner products and orthogonality

8 a) Find two different unit vectors (i.e., length-one vectors) that are orthogonal to $\vec{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \in \mathbb{R}^2$.

b) Give a basis for the orthogonal complement

$$\operatorname{Span}\left(\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}\right)^{\perp} \subseteq \mathbb{R}^3$$

a) $\frac{1}{\sqrt{13}}(2,3)$ and $\frac{-1}{\sqrt{13}}(2,3)$

b) $\{(2,3,0), (0,0,1)\}$ works.

9 a) Prove the "parallelogram law":

$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2(\|\vec{x}\|^2 + \|\vec{y}\|^2).$$

b) Show the implications:

$\vec{x}$ is orthogonal to $\vec{y}$ $\Leftrightarrow$ $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$

$\vec{x}$ is orthogonal to $\vec{y}$ $\Leftrightarrow$ $\vec{x}+\vec{y}$ and $\vec{x}-\vec{y}$ have the same length

$\vec{x}+\vec{y}$ and $\vec{x}-\vec{y}$ have the same length $\Leftrightarrow$ $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$

Answer:

a) $\|x+y\|^2 + \|x-y\|^2 = (x+y)\cdot(x+y) + (x-y)\cdot(x-y)$

$= x\cdot x + \cancel{x\cdot y} + \cancel{y\cdot x} + y\cdot y$

$+ x\cdot x - \cancel{x\cdot y} - \cancel{y\cdot x} + y\cdot y$

$$= 2\|x\|^2 + 2\|y\|^2 \quad ✓$$

b) $$\|x+y\|^2 = (x+y)\cdot(x+y)$$
$$= \|x\|^2 + \|y\|^2 + \underbrace{x\cdot y + y\cdot x}_{=2x\cdot y \text{ for real vectors}}$$

Thus, for real vectors,
$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 \iff x\cdot y = 0,$$
ie, $\vec{x}$ is orthogonal to $\vec{y}$

(**Note**: This is <u>false</u> for complex vectors.
E.g., $\vec{x} = (1)$, $\vec{y} = (i) \in \mathbb{R}^1$
$$\|x+y\|^2 = \|1+i\|^2 = 2 = \|x\|^2 + \|y\|^2.)$$

Next, $\|x+y\|^2 - \|x-y\|^2 = 4x\cdot y$ for real vectors
This is 0 iff $x\cdot y = 0$. ✓

10 What is the projection of $b = (2,1,2)$ onto the plane spanned by $(0,1,0)$ and $(0,1,1)$?

<u>Answer</u>: Notice that
$$\mathrm{Span}((0,1,0),(0,1,1)) = \mathrm{Span}((0,1,0),(0,0,1))$$
$$= \text{the } yz \text{ plane!}$$

The projection of $(2,1,2)$ onto the yz plane is just $\boxed{(0,1,2)}$.