EE510 11/20/2020

Outline:

Review problems:

For each question, choose True or False.

1°_ Distinct eigenvectors are linearly independent. False

v is an ~~evct~~ eigenvector, $\alpha . v$ is also an eigenvector $(\alpha \neq 0)$

2°_ If λ is an eigenvalue for A, then the eigenvectors associated to the eigenvalue λ are scalar multiple of each other. False

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

3°_ If a matrix A commutes with a matrix B, then A^T commutes with $(B^T)^4$. True

$AB = BA$; $A^T B^T = B^T A^T$;

$$A^T (B^T)^4 = A^T B^T (B^T)^3 = B^T A^T (B^T)^3 = (B^T)^4 A^T$$

4°- $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$ False

$\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$.

5°- Symmetric matrices are diagonalizables True

Theorem is lecture notes

6°- Symmetric matrices are always invertibles False

0

7°- If a matrix A commutes with a matrix B and the matrix B commutes with a matrix C, then the matrix A commutes with the matrix C. False

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ; B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} ; C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$AB = BA = BC = CB = AC = 0$$

$$CA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

8° _ There exists a linear transformation $f: \mathbb{R}^2 \to \mathbb{R}^2$ whose range space is the same as its null space. True

$[f]_S = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$; $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

9° _ If $A, B \in \mathbb{R}^{n \times n}$ and A, B are invertibles

then $(A+B)^{-1} = A^{-1} + B^{-1}$ False

$A = 2$; $B = 3$; $(A+B)^{-1} = \frac{1}{5} \neq \frac{1}{2} + \frac{1}{3}$

10° for Any symmetric matrix A ($A \in \mathbb{R}^{n \times n}$) that satisfies $A^2 = A$, then $\text{rank}(A) = \text{trace}(A)$. True

A is a projection $\Rightarrow$ e-values 0, or 1

$N(A)$ $R(A)$

$$[A]_B = \begin{bmatrix} I_{r \times r} & 0_{r \times n-r} \\ 0_{n-r, r} & 0_{n-r, n-r} \end{bmatrix}$$

$r = \text{rank}(A)$

$= \dim R(A)$.

the trace does not depend on the basis

$\text{trace}(A) = \text{trace}([A]_B) = r$

11° - Any matrix $A \in \mathbb{R}^{n \times n}$ has the same eigenvalues as A^T True $\Rightarrow \det(A - \lambda I) = \det(A^T - \lambda I)$

$\det(A) = \det(A^T)$

12° - If $A \in \mathbb{R}^{n \times n}$ such that $\|Ax\| = \|x\|$; $\forall x \in \mathbb{R}^n$, then $\exists b \in \mathbb{R}^n$ such that the equation $AX = b$ has infinetly many solutions False

A is an orthogonal matrix $\Rightarrow$ A is invertible

$AX = b$ has an unique solution $\forall b \in \mathbb{R}^n$.

13° - Eigenvectors of a symmetric matrix from different eigenspaces must be orthogonal. True

$A = UDU^T$; $UU^T = I$.

14°- Every 2x2 matrix is diagonalizable False

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

15°- If A is a 2x2 matrix whose characteristic polynomial does not have real roots, the matrix A is diagonalizable True

$P(\lambda) = 0,\ \lambda \in \mathbb{C},\ P(\lambda^*) = 0.$

$P_A(\lambda) \Rightarrow$ has 2 distinct roots $\Rightarrow$

$\dim E_\lambda = 1,\ \dim E_{\lambda^*} = 1$

16°- A is a nxn matrix with fewer than n distinct eigenvalues, then A is not diagonalizable. False

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 1 e-value

$1 < 3$

17° - Let A be a 2×2 matrix. If there exists some basis B such that $[A]_B$ is not diagonal, then A is not diagonalizable False

18° - If 0 is an eigenvalue for a matrix $A \in \mathbb{R}^{n\times n}$, then, $\text{rank}(A) < n$ True

$E_0 = N(A) \neq 0 \Rightarrow \dim N(A) \geq 1 \Rightarrow \text{rank}(A) < n.$

19° - There exists an invertible matrix A that satisfies $AA^T = A^2$ and A is not symmetric. False

$AA^T = A^2 \Rightarrow A^T = A.$

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20 - There exists a 3×3 real matrix A such that $A^4 = -I$. False.

$$\det(A^4) = (\det(A))^4 = -1 \Rightarrow \det(A) = \frac{1+i}{\sqrt{2}}$$

contradicts A is a real matrix.

21 - There exists a non-zero symmetric matrix A such that $A^5 = 0$ False.

$$A = UDU^T \Rightarrow A^5 = UD^5U^T \Rightarrow A^5 = 0 \Rightarrow UD^5U^T = 0$$
$$\Rightarrow D^5 = 0 \Rightarrow D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

22 - Let $A \in \mathbb{R}^{2\times 2}$

$A\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ and $A\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$, then for any vector $X = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$; $X^T A X = 2(x^2+y^2)$. True | $X^T(2I)X = 2\|X\|^2$

$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ e-vector. $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ e-vector. e-value 2 $\begin{pmatrix} 1 \\ -2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ are linearly independent $A = U\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}U^{-1} = 2I$

23 - If $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, $\forall X = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, then $N(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$. False

$$N(A) = \{X,\ AX = 0\} = \left\{X,\ \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} = \{X,\ x = 0\} = \left[X;\ \begin{bmatrix} 0 \\ y \end{bmatrix}\right]$$
$$= \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}.$$

Exercise: Let A be a 2×2 matrix.

1°- Show that

$$A^2 - tr(A)\,A + det(A)\,I_2 = 0$$

2°- Find all matrices $X \in \mathbb{R}^{2\times 2}$ such that and $det(X+I_2) \neq 0$

$$X^2 + X = A \;; \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

1°-

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \quad det(A - \lambda I) = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

$$P_A(\lambda) = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

$$= \lambda^2 - tr(A)\lambda + det(A).$$

$$P_A(\lambda) = \lambda^2 - tr(A)\lambda + det(A)$$

$$\left(A \in \mathbb{R}^{n\times n} ;\; P_A(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} tr(A)\,\lambda^{n-1} + \cdots + det(A) \right.$$

$$\boxed{P_A(A) = 0} \Rightarrow A^2 - tr(A)\,A + det(A)\,I_2 = 0$$

$2°-\quad X^2+X=A\ ,\quad A=\begin{pmatrix}1&1\\1&1\end{pmatrix}\ ;\ X\in\mathbb{R}^{2\times 2}.$

$X(X+I_2)=A \quad \left(X = A(X+I)^{-1}\right)$

$\Longrightarrow \det(X(X+I))=\det(A)=0$

$\cancel{\Rightarrow \det(X)}$

$\det(X)\,\underbrace{\det(X+I_2)}_{\neq 0}=0$

$\Longrightarrow \det(X)=0$

$X^2-tr(X)\,X+\underset{0}{\underset{\shortparallel}{\det(X)}}\,I_2=0$

$\Longrightarrow X^2-tr(X)\,X=0 \Longrightarrow X^2=tr(X)\,X$

$tr(X)=\alpha\ ;\quad X^2=\alpha X$

$\alpha X+X=A \Rightarrow (1+\alpha)X=A$

$\Longrightarrow X=\dfrac{1}{1+\alpha}A.$

$A^2=\begin{pmatrix}2&2\\2&2\end{pmatrix}$

$\left(\dfrac{1}{1+\alpha}\right)^2A^2+\left(\dfrac{1}{1+\alpha}\right)A=A \Longrightarrow 2\left(\dfrac{1}{1+\alpha}\right)^2+\left(\dfrac{1}{1+\alpha}\right)=1$

$$2\beta^2 + \beta - 1 = 0$$

$$\beta = -1 \; ; \; \beta = \frac{1}{2}$$

$$\Rightarrow \frac{1}{1+\alpha} = -1 \text{ or } \frac{1}{1+\alpha} = \frac{1}{2}$$

$$X = -A \text{ or } X = \frac{A}{2}$$