Lecture 20: How to find eigenvectors

Reading: Meyer Ch. 6, 7.1–7.4
Strang Chs. 4–5

Recall:

$$\underbrace{A\vec{v}}_{\text{eigenvector}} = \overbrace{\lambda}^{\text{eigenvalue}}\vec{v}$$

A is **diagonalizable** $\iff$ $A = U D U^{-1}$

diagonal: $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}$

U: columns are eigenvectors $A\vec{u}_i = \lambda_i \vec{u}_i$

$\iff$ there is a basis of eigenvectors

Example:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

e-vector e-value

Example: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is <u>not</u> diagonalizable.

$A\vec{e}_1 = 0 \cdot \vec{e}_1$ but that's it!

WHAT MATRICES HAVE AN EIGENVECTOR?

Theorem: Every (real or complex) square matrix has at least one eigenvector (over $\mathbb{C}$!).

Proof: Let A be an $n \times n$ matrix.

Step 1. Let $\vec{v}$ be any nonzero vector.

Consider the vectors

$$\vec{v}, \ A\vec{v}, \ A^2\vec{v}, \ \ldots, \ A^n\vec{v}$$

Since there are $n+1$ vectors in an n-dimensional space, they must be linearly dependent.

(<u>Note</u>: This argument only works in finite dimensions.)

Say

$$\alpha_0 v + \alpha_1 A v + \alpha_2 A^2 v + \cdots + \alpha_n A^n v = 0$$

with not all α_j's 0.

Step 2. Thus

$$\left(\sum_{j=0}^{n} \alpha_j A^j\right)\vec{v} = 0$$

Factor the polynomial

$$p(x) = \sum_j \alpha_j x^j = \prod_j (x - \lambda_j)$$

$$\Rightarrow \left[\prod_j (A - \lambda_j I)\right]\vec{v} = 0$$

Step 3.

$\Rightarrow$ at least one of the matrix terms $A - \lambda_j I$ must be singular

(the product of two nonsingular matrices is nonsingular!)

✓ Done. $\square$

Note: The first step can be "justified" by noticing that

$$\lim_{k\to\infty} \frac{A^k \vec{v}}{\|A^k \vec{v}\|}$$

converges to an eigenvector (or eigenspace) with largest magnitude eigenvalue. So it makes sense to look at successive powers $A^k \vec{v}$.

DETERMINANTS (in brief)

Definition: The determinant of an $n \times n$ square matrix A is

$$\sum_{\substack{n\text{-element} \\ \text{permutations } \sigma}} \underbrace{\text{sign}(\sigma)}_{= (-1)^{\#\text{ of transpositions required to get to } \sigma}} \cdot \prod_{i=1}^{n} A_{i,\sigma(i)}$$

Examples:

- $n=1$: $A = (A_{11})$

  $\det(A) = A_{11}$

- $n=2$: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

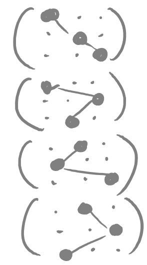
Permutations	Sign
$1 \mapsto 1, 2 \mapsto 2$	$+1$ (0 transpositions)

$$\overset{1\leftrightarrow 2}{\Rightarrow} \det A = \big((+1)\cdot a_{11}a_{22}\big) + \big((-1)\cdot a_{12}a_{21}\big)$$
$$= a_{11}a_{22} - a_{12}a_{21}$$

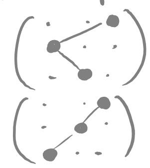
- $n=3$:

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix}$$

Permutation σ	Sign(σ)	$\prod_{i=1}^{3} a_{i,\sigma(i)}$
identity	$+1$	$a_{11}a_{22}a_{33}$
$2\leftrightarrow 3$	-1	$a_{11}a_{23}a_{32}$
$1\leftrightarrow 2$	-1	$a_{12}a_{21}a_{33}$
$1\to 2\to 3\to 1$	$+1$	$a_{12}a_{23}a_{31}$
$1\to 3\to 2\to 1$	$+1$	$a_{13}a_{21}a_{32}$
$1\leftrightarrow 3$	-1	$a_{13}a_{22}a_{31}$



since $(1\to 2\to 3\to 1)$ = $1\leftrightarrow 2$ followed by $1\leftrightarrow 3$ (two transpositions)



add these up to get the determinant!

Observe: $\det(A) = a_{11}\cdot\det\begin{pmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{pmatrix} - a_{12}\cdot\det\begin{pmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{pmatrix} + a_{13}\cdot\det\begin{pmatrix} a_{21} & a_{22}\\ a_{31} & a_{32}\end{pmatrix}$

- General n: the same recursion works

$a_{11}\cdot\det(n-1\times n-1$ submatrix missing 1st row, 1st col$)$
$-a_{12}\cdot\det(n-1\times n-1$ submatrix missing row 1, col 2$)$
$+a_{13}\cdot \cdots$
$\vdots$
$+(-1)^n a_{1n}\cdot\det($submatrix missing row 1, col $n)$

[You can actually do this across any row or down any column; note that $\det(A)=\det(A^T)$.]

Problem: There are $n! \approx \left(\frac{n}{e}\right)^n$ permutations of n elements; although

$$\det(A) = \sum_{\sigma} \operatorname{sign}(\sigma)\cdot\prod_{i=1}^{n} A_{i,\sigma(i)}$$

n-element permutations σ

defines the determinant, that's not how it is actually computed!

Compare: The permanent of a square matrix A is

$$\mathrm{perm}(A) = \sum_{\substack{n\text{-element} \\ \text{permutations } \sigma}} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

without the sign! Computing the permanent is NP-hard.

More examples:

- $\mathrm{Det}\begin{pmatrix} a & & & 0 \\ & b & & \\ & & c & \\ 0 & & & d \end{pmatrix} = abcd$ product of diagonal entries

  diagonal matrix

- $\mathrm{Det}\begin{pmatrix} a & e & f & g \\ 0 & b & h & \vdots \\ 0 & 0 & c & g \\ 0 & 0 & 0 & d \end{pmatrix} = abcd$ still!

  upper triangular matrix

true for permanent, too

false for permanent!

Cool fact (that I won't prove or use):

$$\mathrm{Det}(AB) = \mathrm{Det}(A) \cdot \mathrm{Det}(B)$$

Corollary: If A is diagonalizable,

$$A = U \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} U^{-1},$$

then $\det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$.

product of the eigenvalues

Proof:
$$\begin{aligned} \det A &= \det(U D U^{-1}) \\ &= \det(U) \cdot \det(D) \cdot \det(U^{-1}) \\ &= \det(D) \cdot \underbrace{\det(U U^{-1})}_{\det(I) = 1} \end{aligned}$$
$\square$

Example:

```
>> n = 50;
>> A = randn(n,n);
>> B = randn(n,n);
```

Note:

```
>> factorial(50)

ans =
```

— how did Matlab compute det(A), det(B)?

```
>> n = 50;
>> A = randn(n,n);
>> B = randn(n,n);
>> det(A*B)

ans =

   3.1827e+61

>> det(A) * det(B)

ans =

   3.1827e+61
```

Note:

```
>> factorial(50)

ans =

   3.0414e+64
```

— how did Matlab compute det(A), det(B)?

Answer: It takes the LU decomposition:

$$A = \underset{\substack{\text{permutation}\\ \text{matrix}}}{P} \cdot \underset{\text{lower triangular}}{\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}} \cdot \underset{\text{upper triangular}}{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}$$

$$\Rightarrow \det(A) = \det(P) \cdot \det(L) \cdot \det(U).$$

More on this later...

How to compute the determinant efficiently

Gaussian elimination

Observe: The determinant function is not linear

$$\det(A) = \sum_{\text{permutations } \sigma} \operatorname{sign}(\sigma) \cdot \prod_{i=1}^{n} A_{i,\sigma(i)}$$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}, \quad \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{matrix} a_{11}a_{22}a_{33} \\ -a_{11}a_{23}a_{32} \\ -a_{12}a_{21}a_{33} + \cdots \end{matrix}$$

$$\Rightarrow \det(5A) = 5^n \cdot \det(A)$$

But it is linear in any one row:

$$\det\begin{pmatrix} 5a_{11} & 5a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 5 \cdot \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 5a_{21} & 5a_{22} & 5a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 5 \cdot \det A$$

$$\det\begin{pmatrix} \text{—}\ \vec{u}\ \text{—} \\ B \end{pmatrix} = (\quad)\, u_1 + (\quad)\, u_2 + \cdots + (\quad) \cdot u_n .$$

$$\Rightarrow \det\begin{pmatrix} \text{—}\ \vec{u}+\vec{v}\ \text{—} \\ B \end{pmatrix} = \det\begin{pmatrix} \text{—}\ \vec{u}\ \text{—} \\ B \end{pmatrix} + \det\begin{pmatrix} \text{—}\ \vec{v}\ \text{—} \\ B \end{pmatrix}$$

Observe: Switching any two rows (or columns) negates det(A).

$$\det\begin{pmatrix} \text{—}\ \vec{u}\ \text{—} \\ \text{—}\ \vec{v}\ \text{—} \\ B \end{pmatrix} = -\det\begin{pmatrix} \text{—}\ \vec{v}\ \text{—} \\ \text{—}\ \vec{u}\ \text{—} \\ B \end{pmatrix}$$

$$\det\begin{pmatrix} — \vec{v} — \\ B \end{pmatrix} = -\det\begin{pmatrix} — \vec{u} — \\ B \end{pmatrix}$$

b/c it adds one transposition to every permutation

Corollary 1: If the same row occurs twice, $\det(A)=0$.

Proof:

$$\det\begin{pmatrix} — \vec{u} — \\ — \vec{u} — \\ B \end{pmatrix} = -\det\begin{pmatrix} — \vec{u} — \\ — \vec{u} — \\ B \end{pmatrix}$$

$\Rightarrow$ it must be 0 $\square$

Corollary 2: We can use Gaussian elimination to compute det.

- $\det\begin{pmatrix} \\ — 5\vec{u} — \\ \\ \end{pmatrix} = 5\det\begin{pmatrix} \\ — \vec{u} — \\ \\ \end{pmatrix}$

- $\det\begin{pmatrix} — \vec{u} — \\ \\ — \vec{u}+\vec{v} — \\ \\ \end{pmatrix} = \underbrace{\cancel{\det\begin{pmatrix} — \vec{u} — \\ \\ — \vec{u} — \\ \\ \end{pmatrix}}}_{0} + \det\begin{pmatrix} — \vec{u} — \\ \\ — \vec{v} — \\ \\ \end{pmatrix}$

Corollary 3:

$\det(A) \neq 0 \iff A$ is invertible

~ "nonsingular"

~ inverse exists

~ $\text{rank}(A)=n$

$\det(A) = 0 \iff A$ is not invertible

~ $\text{rank}(A)<n$

Proof: Apply Gaussian elim.:

$$A \xrightarrow{GE} B = \begin{pmatrix} b_{11} & & & \\ & b_{22} & & \\ & & \ddots & \\ & 0 & & \end{pmatrix}$$

$\text{rank}(A) = \text{rank}(B)$

$\det(A) = \begin{pmatrix}\text{something} \\ \text{nonzero}\end{pmatrix} \cdot \det(B) = \text{product of diagonal entries}$

$= 0 \iff$ at least one $b_{ii}=0$

$\iff \operatorname{rank}(B) < n$ $\square$

Example: Is this matrix invertible?

$$\begin{pmatrix} 1 & a & b & c+d \\ 1 & b & c & a+d \\ 1 & c & d & a+b \\ 1 & d & a & b+c \end{pmatrix}$$

Answer:

- Don't try to compute the inverse.
- Don't try to brute-force the determinant.
- Look for a pattern.

COROLLARY: HOW TO FIND EIGENVECTORS:

Observe: λ is an eigenvalue of A

$$\Updownarrow$$

$A\vec{v} = \lambda\vec{v}$ for a nonzero vector $\vec{v}$

$$\Updownarrow$$

$(A-\lambda I)\vec{v} = 0$ for $\vec{v} \neq 0$

$$\Updownarrow$$

$N(A-\lambda I) \neq \{0\}$.

$$\Updownarrow$$

$A-\lambda I$ is singular!

$$\Updownarrow$$

$\det(A-\lambda I) = 0$

$$\boxed{\det(A-\lambda I) = 0 \iff \lambda \text{ is an eigenvalue of } A}$$

Recipe:

1. Compute $\det(A-\lambda I)$
2. Find all roots, $\det(A-\lambda I) = 0$
3. For each λ, corresponding eigenspace is $N(A-\lambda I)$ — find it with Gaussian elimination!

Problem: Calculate the eigenvalues and eigenvectors of

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Answer: A "

$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$

$$= (3-\lambda)(3-\lambda) - 1$$

$$= \lambda^2 - 6\lambda + 8$$

$$= (\lambda-4)(\lambda-2) = 0$$

$\Rightarrow$ Two eigenvalues, $\lambda_1 = 4$ and $\lambda_2 = 2$

Observe: If A is an $n \times n$ matrix, $\det(A-\lambda I)$ is a polynomial of degree n in λ.

Observe: 1) $\det(A) = \lambda_1 \cdot \lambda_2 = 8$ ✓

2) $\text{Trace}(A) =$ sum of diagonal elements $= \lambda_1 + \lambda_2 = 6$

These observations are true in general, and for 2×2 matrices these two equations suffice to solve for λ_1, λ_2.

Next, the eigenspaces:

- $N(A-\lambda_1 I) = N\left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right)$

$$= \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} ✓$$

- $N(A-\lambda_2 I) = N\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)$

$$= \text{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

$$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2\cdot\begin{pmatrix} 1 \\ -1 \end{pmatrix} ✓$$

Done: $$A = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{\text{e-vectors}}\underbrace{\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}}_{\text{e-values}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

Observe: The eigenvectors of $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ are the same as those for $\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{B\text{ "}}$.

as those for $\underset{B}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$.

If $B\vec{v} = \lambda\vec{v}$, $(B+3I)\vec{v} = (\lambda+3)\vec{v}$

— But the eigenvalues of the sum of two matrices will not generally be the sums of the eigenvalues, unless they have the same eigenvectors.

Problem: Calculate the eigenvalues and eigenvectors of

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Answer: First, give it a name (always a good idea!).

Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Now we solve for λ in

$$0 = \det(A - \lambda I)$$
$$= \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$$
$$= \lambda^2 + 1$$
$$\Rightarrow \lambda \in \{i, -i\}$$

They add to 0 since Trace$(A) = 0$.
Their product is $1 = \det(A)$. ✓

To calculate the respective eigenvectors, we could solve for $N(A \pm iI)$. But it is easier just to go directly:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ y \end{pmatrix} = i\begin{pmatrix} 1 \\ y \end{pmatrix}$$
$$\begin{pmatrix} -y \\ 1 \end{pmatrix} \quad \Rightarrow y = -i \checkmark$$

Similarly,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = -i\begin{pmatrix} 1 \\ i \end{pmatrix} \checkmark$$

Problem: What are the eigenvalues and eigenvectors of

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} ?$$

Answer: A

$A - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix}$ is upper triangular

$\Rightarrow$ determinant is product of diagonal entries

$\Rightarrow$ eigenvalue 3 (with multiplicity 2)

Eigenvectors: Solve

$$0 = (A - 3I)x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\Rightarrow x_2 = 0$

and we may scale so $x_1 = 1$

$\Rightarrow$ eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

is the <u>only</u> eigenvector!

(The matrix is not diagonalizable)

Problem: What are the eigenvalues of

$$\begin{pmatrix} 3 & 5 & 8 & 13 \\ & 2 & 3 & 5 \\ & & 1 & 2 \\ 0 & & & 1 \end{pmatrix} ?$$

Answer: Just as above, since the matrix is upper triangular, its eigenvalues are the diagonal elements: 3, 2 and 1 (with multiplicity 2). ✓

Moral: Triangular matrices are easy!

Question: Why does an $n \times n$ matrix have at most n distinct eigenvalues?

Answer 1:

Because $\det(A - \lambda I)$ is a degree n polynomial in λ, it always has ≥ 1 and $\leq n$ roots.

Answer 2:

Assume there are $n+1$ <u>different</u> eigenvalues:

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$
$$\vdots$$
$$A\vec{v}_n = \lambda_n \vec{v}_n$$
$$A\vec{v}_{n+1} = \lambda_{n+1} \vec{v}_{n+1}$$

$\Rightarrow$ e-vectors cannot be linearly independent

If, say, the first n are lin. indep., then

$$\vec{v}_{n+1} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n$$

for some constants $\alpha_1, \dots, \alpha_n$

Apply A to both sides

$$A\vec{v}_{n+1} = \lambda_{n+1}\vec{v}_{n+1}$$

$$\alpha_1\lambda_1\vec{v}_1 + \cdots + \alpha_n\lambda_n\vec{v}_n = \alpha_1\lambda_{n+1}\vec{v}_1 + \cdots + \alpha_n\lambda_{n+1}\vec{v}_n$$

$$\Rightarrow \sum_{j=1}^{n} \alpha_j \underbrace{(\lambda_j - \lambda_{n+1})}_{\text{nonzero!}} \vec{v}_j = 0$$

$\Rightarrow$ contradicts that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are linearly independent.

<u>Note</u>: Dotting the i's, this argument actually proves

<u>Lemma</u>: Any set of eigenvectors for a matrix, corresponding to <u>distinct</u> eigenvalues, is linearly independent.

SUMMARY OF EIGENVALUES & EIGENVECTORS

A $n \times n$ matrix

λ an eigenvalue $\Leftrightarrow \det(A - \lambda I) = 0$

since $A\vec{v} = \lambda\vec{v}$ for some $\vec{v} \neq 0$

$\Leftrightarrow A - \lambda I$ is singular

$\det(A - \lambda I)$ = degree n polynomial in λ

$$= (\lambda - \lambda_1)^{\alpha_1}(\lambda - \lambda_2)^{\alpha_2} \cdots (\lambda - \lambda_k)^{\alpha_k}$$

distinct e-values (AKA "spectrum")

multiplicities

$\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$, $k \leq n$

eigenspace for λ_j is $N(A - \lambda_j I)$

$$1 \leq \dim N(A - \lambda_j I) \leq \alpha_j$$

There are at least k independent e-vectors.

A is diagonalizable (n indep. e-vectors)

$$\Leftrightarrow \dim N(A - \lambda_i I) = \alpha_i \text{ for all } i = 1, \ldots, k$$

A is diagonalizable (n indep. e-vectors)

$$\iff \dim N(A-\lambda_j I) = \alpha_j \text{ for all } j=1,\dots,k$$

Why is $\dim N(A-\lambda_i I) \le \alpha_i$?

Proof: We want to show that $\operatorname{rank}(A-\lambda_i I) \geq n-\alpha_i$; the claim $\dim N(A-\lambda_i I) \leq \alpha_i$ then follows by the Rank-Nullity theorem.

Useful claim: If $\mathcal{U}$ is invertible, then the eigenvalues of A are the same as those of $\mathcal{U} A \mathcal{U}^{-1}$.

Proof: If $A\vec{v} = \lambda \vec{v}$, then

$$(\mathcal{U}A\mathcal{U}^{-1})(\mathcal{U}\vec{v}) = \mathcal{U}A\vec{v} = \lambda(\mathcal{U}\vec{v}) \quad \checkmark$$

Now apply Gaussian elimination to A; assuming no row interchanges are required, this gives

$$A = (L)(\mathcal{U})$$

where L has 1s along the diagonal and $\mathcal{U}$ has each λ_i on its diagonal α_i times. (Think about it....)

Thus we can also write

$$A = L\,\mathcal{U}'\,L^{-1};$$

this means applying the same operations to the columns of $\mathcal{U}$ as were done to the rows of A in Gaussian elimination. It leaves the diagonal entries unchanged.

But then

$$\begin{aligned}\operatorname{rank}(A-\lambda_i I) &= \operatorname{rank}(\mathcal{U}'-\lambda_i I) \\ &\geq \#\text{ of nonzero entries along the diagonal, since } \mathcal{U}'-\lambda_i I \text{ is upper triangular} \\ &= n-\alpha_i. \checkmark \quad \square\end{aligned}$$

Note: The recipe

1. Compute $p(\lambda) = \det(A-\lambda I)$
2. Find its factors $\lambda_1,\dots,\lambda_k \Rightarrow$ eigenvalues of A
3. Compute nullspaces $N(A-\lambda_i I) \Rightarrow$ eigenspaces

② Find its factors $\lambda_1, \ldots, \lambda_k$ $\Rightarrow$ eigenvalues of A
③ Compute nullspaces $N(A-\lambda_i I)$ $\Rightarrow$ eigenspaces

works in theory, and for 2×2, 3×3 matrices.

But it quickly becomes impractical. Writing down $p(\lambda)$, and then finding its roots, is very time-consuming.

Next time we'll learn a faster way...

In fact, a common way of solving for the roots of a polynomial

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

is to compute the eigenvalues of a matrix!

For

$$A = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & 0 & & 0 & -a_1 \\ & 1 & \ddots & & \vdots \\ & 0 & & & \\ & & & 1 & -a_{n-1} \end{pmatrix},$$

$$\det(A-\lambda I) = p(\lambda).$$

$$\begin{vmatrix} \lambda & a_0 \\ -1 & \lambda+a_1 \end{vmatrix} = \lambda^2 + \lambda a_1 + a_0$$

$$\begin{vmatrix} \lambda & 0 & a_0 \\ -1 & \lambda & a_1 \\ 0 & -1 & \lambda+a_2 \end{vmatrix} = \lambda^3 + \lambda^2 a_2 + \lambda a_1 + a_0$$

$$\begin{vmatrix} \lambda & & 0 & a_0 \\ -1 & \lambda & & a_1 \\ & -1 & \lambda & a_2 \\ 0 & & -1 & \lambda+a_3 \end{vmatrix} = \lambda^4 + \lambda^3 a_3 + \lambda^2 a_2 + \lambda a_1 + a_0$$

$\vdots$

Corollary: A and its transpose A^T have the same eigenvalues.

Proof: $\mathrm{Det}(A-\lambda I) = \mathrm{Det}(A^T - \lambda I)$ since $\mathrm{Det}\, M = \mathrm{Det}\, M^T$.
$\Rightarrow$ same roots (eigenvalues) ✓ □

But the eigenvectors can be different!

Exercises:

Problem: Calculate the eigenvalues and eigenvectors of

Problem: Calculate the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

Answer:

E-values	E-vectors
i	$\begin{pmatrix} 1 \\ -i \end{pmatrix}$
$-i$	$\begin{pmatrix} 1 \\ i \end{pmatrix}$

Related problems: What are the spectral decompositions of

$$B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} ?$$

Answer: Observe that both matrices B and C break up into two copies of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence we see immediately:

B

E-values	E-vectors
i	$(1, -i, 0, 0)$
$-i$	$(1, i, 0, 0)$
i	$(0, 0, 1, -i)$
$-i$	$(0, 0, 1, i)$

C

E-values	E-vectors
i	$(1, 0, 0, -i)$
$-i$	$(1, 0, 0, i)$
i	$(0, 1, -i, 0)$
$-i$	$(0, 1, i, 0)$

Problem: What are the eigenvalues and eigenvectors of

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} ?$$

Answer:

E-values	E-vectors
3 (multiplicity 2)	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ only!

Problem ("Jordan normal form"):

Find the spectral decomposition of

$$\left(\begin{array}{cc|ccc|cccc}
0 & 1 & & & & & & & \\
0 & 0 & & 0 & & & 0 & & \\
\hline
& & 1 & 1 & 0 & & & & \\
& 0 & 0 & 1 & 1 & & 0 & & \\
& & & & 1 & & & & \\
\hline
& & & & & 2 & 1 & 0 & 0 \\
& 0 & & 0 & & & 2 & 1 & 0 \\
& & & & & 0 & & 2 & 1 \\
& & & & & & & & 2
\end{array}\right)$$

Answer: Since the matrix is upper triangular, we can read off the e-values: $0, 1, 2$.

The matrix breaks into 3 blocks

$$\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{A}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = I + \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{B}, \quad 2I + \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{C}$$

Let's consider C (A and B are similar).

$0 \leftarrow$ the unique e-value

e-space $N(C - 0 \cdot I) = N(C) = \mathrm{Span}(\{\vec{e}_1\})$ ← one-dimensional!

$\mathrm{Rank}(C) = 3 \Rightarrow \dim N(C) = 4 - 3 = 1$

(not diagonalizable)

Putting everything together:

$$\left(\begin{array}{c|c|c} \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} & 0 & 0 \\ \hline 0 & \begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix} & 0 \\ \hline 0 & 0 & \begin{smallmatrix} 2 & 1 & 0 & 0 \\ & 2 & 1 & 0 \\ & & 2 & 1 \\ 0 & & & 2 \end{smallmatrix} \end{array}\right)$$

E-value	E-vector
0	$(1,0,\vert 0,0,0\vert, 0,0,0,0)$
1	$(0,0,\vert 1,0,0,\vert 0,0,0,0)$
2	$(0,0,\vert 0,0,0,\vert 1,0,0,0)$