Lecture 7: Linear independence, bases and dimension (class)

Reading:  2.3  4.3–4.4

Today: Linear independence, bases and dimension

Example:

Definition: The matrix A is **diagonally dominant** if for all rows j,

$$|a_{jj}| \geq \sum_{i \neq j} |a_{ji}| .$$

It is strictly diagonally dominant if the inequality is strict ($>$) for all rows.

Theorem: If A is strictly diagonally dominant, then $N(A) = \{\vec{0}\}$

Proof: Let $\vec{x} \in N(A)$. Assume $\vec{x} \neq \vec{0}$.

Let k be the index so $|x_k| = \max_j |x_j|$ (*)

$$\vec{0} = A\vec{x} \qquad (A\vec{x})_k = 0 = \sum_{j=1}^{n} a_{kj} x_j = a_{kk} x_k + \sum_{j \neq k} a_{kj} x_j$$

$$|a_{kk} x_k| = \Big|\sum_{j \neq k} a_{kj} x_j\Big| \leq \underbrace{\sum_{j \neq k} |a_{kj}|}_{<\,|a_{kk}| \text{ by strict diag. dominance}} \cdot \underbrace{|x_j|}_{\leq |x_k| \ (*)} \quad (\Delta \text{ ineq.})$$

$x_k \neq 0 \Rightarrow$ can cancel $\quad |a_{kk}| < |a_{kk}|$

Contradiction! $\Rightarrow \vec{x} = \vec{0}$. $\square$

Recall:

Span (a set of vectors) = smallest subspace that contains them all

$= \left\{ \begin{matrix} \text{all (finite) linear combinations} \\ \text{of those vectors} \\ \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r \end{matrix} \right\}$

Example:

$$\mathbb{R}^3 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \Bigg| \text{ since } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbb{R}^3$$

$$= \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

$$= \text{Span}(\mathbb{R}^3)$$

any $V = \text{Span}(V)$ — vector space

Example:

Span (a finite set of vectors $v_1, \ldots, v_n$)

$$= R\left(\text{matrix} \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}\right) \quad \text{range/column space}$$

$$\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} = v_1 e_1^T + v_2 e_2^T + \cdots + v_n e_n^T$$

Goal for today: Given a subspace, find the a smallest

$$v_1 e_1^T + v_2 e_2^T + \cdots + v_n e_n^T$$

Goal for today: Given a subspace, find ~~the~~ a smallest possible spanning set for it.
(smaller → simpler, easier to work with/understand)

Outline:
- Linear independence ✓
- Basis ✓
- Dimension ✓
- Rank, Rank-nullity theorem

LINEAR INDEPENDENCE

Definitions:

- A vector $\vec{v}$ is **linearly independent of a set S** if
$$\vec{v} \notin \mathrm{Span}(S)$$

(If $v \in \mathrm{Span}(S)$, we say "v is linearly dependent on S.")

- A set of vectors is **linearly independent** if no vector can be expressed as a linear combination of the others (i.e., every vector is linearly independent of the others).

if $\vec{v}_1 \in \mathrm{Span}\{v_2, v_3\}$
$\Rightarrow \exists\, a, b$ s.t. $v_1 = a v_2 + b v_3$
$\Rightarrow \underline{1} \cdot v_1 - \underline{a} \cdot v_2 - \underline{b} \cdot v_3 = \vec{0}$
$\Rightarrow$ lin. dependent by 2nd definition

Example: $S = \{v_1, v_2, v_3\}$ is linearly independent.
$$\Updownarrow$$
$$v_1 \notin \mathrm{Span}\{v_2, v_3\}, \quad v_2 \notin \mathrm{Span}\{v_1, v_3\}, \quad v_3 \notin \mathrm{Span}\{v_1, v_2\}$$

Equivalent definition: A set $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is **linearly independent** if the only solution to
$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n = \vec{0}$$
is $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Eg.
$$2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$\alpha_1 = 2, \alpha_2 = -1$

Why are these definitions equivalent?
- If $v_1 \in \mathrm{Span}(v_2, v_3)$, say $v_1 = \alpha_2 v_2 + \alpha_3 v_3$, then
$$-v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0.$$
- If $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ with $\alpha_j \neq 0$, then
$$v_j = -\frac{1}{\alpha_j}\left(\sum_{i \neq j} \alpha_i v_i\right) \in \mathrm{Span}(\{v_i \mid i \neq j\}).$$

Observe:

$S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent $\iff$ nullspace of $\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$ is $\{\vec{0}\}$.

To check if a set is linearly independent, compute the nullspace.

Example: Are these vectors linearly independent of each other?
$$v_1 = (1, 1, 1), \quad v_2 = (2, -1, 1), \quad v_3 = (0, 9, 3) \in \mathbb{R}^3$$

Answer:

```
data = {{1, 1, 1}, {2, -1, 1}, {0, 9, 3}};
ParametricPlot3D[data*u, {u, 0, 1}, AxesLabel → {"x", "y", "z"}] /. Line → Arrow
```

$V = \mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1, v_2\}$

Answer:

```
data = {{1, 1, 1}, {2, -1, 1}, {0, 9, 3}};
ParametricPlot3D[data*u, {u, 0, 1}, AxesLabel → {"x", "y", "z"}] /. Line → Arrow
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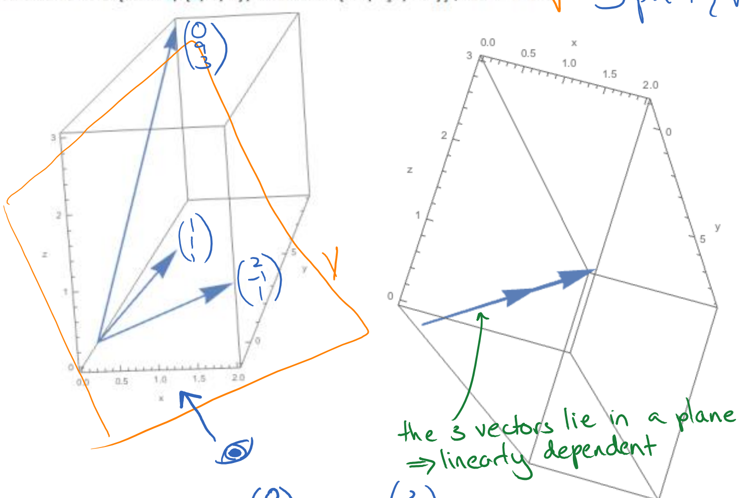


the 3 vectors lie in a plane
$\Rightarrow$ linearly dependent

$V = \text{Span}\{v_1, v_2, v_3\} = \text{Span}\{v_1, v_2\}$
$= \text{Span}\{v_2, v_3\}$
$= \text{Span}\{10v_1 + v_2, v_3\}$

all valid bases

$\dim(V) = 2$

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}0\\9\\3\end{pmatrix} + \frac{1}{2}\begin{pmatrix}2\\-1\\1\end{pmatrix} \Rightarrow \begin{pmatrix}1\\-1/6\\-1/2\end{pmatrix} \in N\left(\begin{pmatrix}1 & 0 & 2\\1 & 9 & -1\\1 & 3 & 1\end{pmatrix}\right)$$

Basis and Dimension

Definition: A basis for a vector space V is a linearly independent set that spans V.

$$\boxed{\text{dimension}(V) = \#\text{ of vectors in a basis}}$$

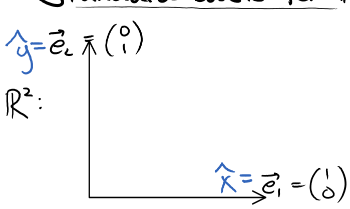
(Theorem: Two bases for the same space must have the same size.)

Example: $\mathbb{R}^n$

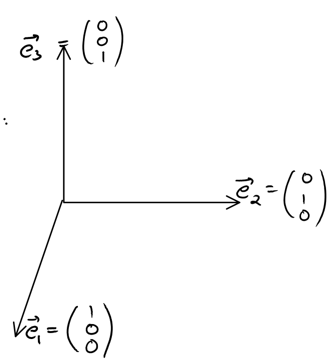
$\dim(\mathbb{R}^n) = n$

Standard basis for $\mathbb{R}^n$

$\mathbb{R}^2$:



$\mathbb{R}^3$:

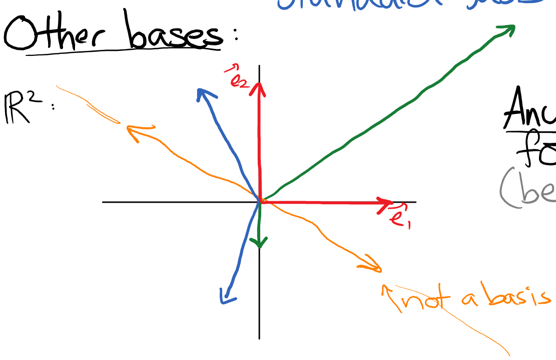


A basis for $\mathbb{R}^n$ is
$\{\vec{e}_1 = (1,0,\dots,0), \vec{e}_2 = (0,1,0,\dots,0), \dots, \vec{e}_n = (0,\dots,0,1)\}$.

"standard basis" "computational basis"

Other bases:

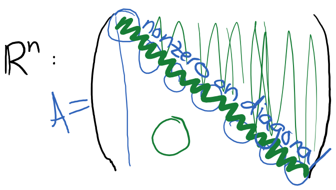
$\mathbb{R}^2$:



Any two linearly independent vectors form a basis
(because they span the plane)

linearly independent $\Leftrightarrow N(A) = \{\vec{0}\}$

not a basis

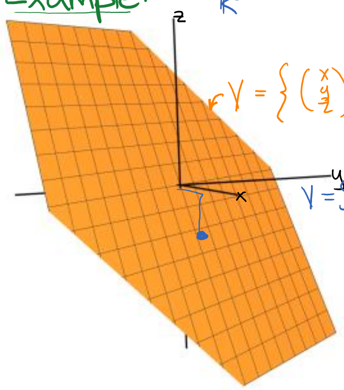


The columns are a basis

linearly independent $\Leftrightarrow N(A)=\{\vec{0}\}$ ✓
no free variables

span all $\mathbb{R}^n$ ✓

Example: $\mathbb{R}^3$



$$V=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; x+y+z=0\right\} = N\left(\begin{pmatrix}1 & 1 & 1\end{pmatrix}\right)$$

$$V=\text{Span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\} = \text{Span}\left\{\begin{pmatrix}1\\1\\-2\end{pmatrix}, \begin{pmatrix}-2\\1\\1\end{pmatrix}\right\}$$

any $\vec{x}\in V$, $v=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x\\y\\-x-y\end{pmatrix}=x\begin{pmatrix}1\\0\\-1\end{pmatrix}+y\begin{pmatrix}0\\1\\-1\end{pmatrix}$

$x+y+z=0$

$\begin{pmatrix}x\\y\\-x-y\end{pmatrix}=a\begin{pmatrix}1\\1\\-2\end{pmatrix}+b\begin{pmatrix}-2\\1\\1\end{pmatrix}$ for some a,b

Observe: Any $\vec{v}\in V$ is in $\text{Span}\left(\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}\right)$ ✓

Example: Give a basis for the span of
$(1,1,1)$, $(2,-1,1)$, $(0,9,3)$

EXAMPLES

Space	Dimension	Basis
• 2×2 matrices $\begin{pmatrix}a & b\\ c & d\end{pmatrix}=a\begin{pmatrix}1&0\\0&0\end{pmatrix}+b\begin{pmatrix}0&1\\0&0\end{pmatrix}+c\begin{pmatrix}0&0\\1&0\end{pmatrix}+d\begin{pmatrix}0&0\\0&1\end{pmatrix}$	4	$\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}$
• 2×2 symmetric matrices $A=A^T=\begin{pmatrix}a & b\\ b & c\end{pmatrix}$	3	$\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}$
• 2×2 antisymmetric matrices $A=-A^T=\begin{pmatrix}0 & a\\ -a & 0\end{pmatrix}$	1	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ or $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ or $\begin{pmatrix}0&10\\-10&0\end{pmatrix}$
• polynomials in x of degree ≤ 2 $\quad a+bx+cx^2$	3	$1, x, x^2$
• polynomials in x and y with total degree ≤ 2 $\quad a+bx+cy+dx^2+exy+fy^2$	6	$1, x, y, xy, x^2, y^2$
• polynomials in x	∞	$1, x, x^2, x^3, x^4, \ldots$
• $N\left(\begin{pmatrix}1 & 1 & 1 & \cdots & 1\end{pmatrix}\right) = \{\vec{x}\in\mathbb{R}^n \mid x_1+x_2+\cdots+x_n=0\}$	$n-1$	$(1,0,0,\ldots,0,-1)$, $(0,1,0,\ldots,0,-1)$, $(0,0,1,\ \ ,0,-1)$, $(0,0,0,\ \ ,1,-1)$

- functions $f:\mathbb{R}\to\mathbb{R}$ $\quad\infty$
- functions $f:\{0,1\}^2\to\mathbb{R}$ $\quad 4$

  $\{0,1\}^2 = \{(0,0),(0,1),(1,0),(1,1)\}$
- functions $f:\{0,1\}^2\to\mathbb{R}$ that are symmetric: $f(x,y)=f(y,x)$ $\quad 3$

  $f(0,0),\ f(0,1)=f(1,0),\ f(1,1)$

$$\begin{array}{c|c} \{0,1\}^2 & \mathbb{R} \\ \hline 00 & f(00) \\ 01 & f(01) \\ 10 & f(10) \\ 11 & f(11) \end{array} = f(00)\left[\begin{array}{c|c} \{0,1\}^2 & \mathbb{R} \\ \hline 00 & 1 \\ 01 & 0 \\ 10 & 0 \\ 11 & 0 \end{array}\right] + \cdots + f(11)\left[\begin{array}{c|c} \{0,1\}^2 & \mathbb{R} \\ \hline 00 & 0 \\ 01 & 0 \\ 10 & 0 \\ 11 & 1 \end{array}\right]$$

$$\left[\begin{array}{c|c} \{0,1\}^2 & \mathbb{R} \\ \hline 00 & 0 \\ 01 & 1 \\ 10 & 1 \\ 11 & 0 \end{array}\right]$$

EXAMPLES: Different bases for the same space

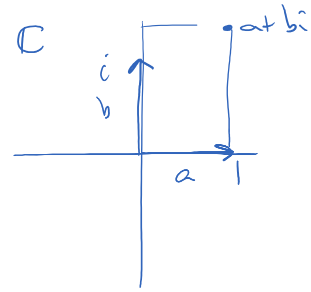
- Any linearly independent set S is a basis for $\text{Span}(S)$.

  dimension $=|S|$
- $V=\{\text{polynomials of degree} \le n\}$
  - $\{1,x,x^2,\dots,x^n\}$ form a basis
  - Hermite polynomials $\{1,\ x,\ x^2-1,\ x^3-3x,\ x^4-6x^2+3,\ \dots,\ (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})\}$ is another

  dimension $=n+1$
- $\mathbb{C}^n$ $\quad$ n dimensions over $\mathbb{C}$
  - standard basis $e_1,\dots,e_n$
  - Fourier basis $v_j=\frac{1}{\sqrt{n}}\sum_{k=0}^{n}\omega^{jk}e_k$ where $\omega=e^{2\pi i/n}$

  Exercise: Why is $\{v_j\}$ linearly independent?

$\dim_{\mathbb{R}}\mathbb{R}=1$



Example: $\mathbb{R}$

$\mathbb{C}=a+bi$ $\quad i=\sqrt{-1}$, $a,b\in\mathbb{R}$

$\mathbb{C}$ is a vector space over $\mathbb{R}$

dimension$_{\mathbb{R}}(\mathbb{C})$

basis is $1,\ i$

any $a+bi\in\mathbb{C}$ $\quad a\cdot 1+b\cdot i$

$\mathbb{C}$ is also a vector space over $\mathbb{C}$

$i=i\cdot 1$ $\quad$ i is dependent on 1

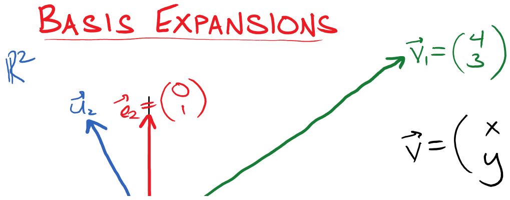
basis is 1

dimension$_{\mathbb{C}}(\mathbb{C})=1$

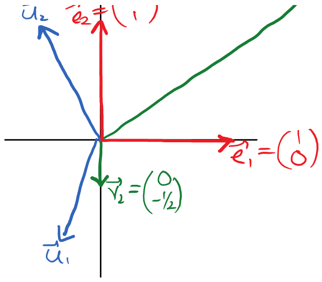
$\dim_{\mathbb{C}}\mathbb{C}^n=n$ $\quad \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\dots,\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}$

$\dim_{\mathbb{R}}\mathbb{C}^n=2n$ $\quad \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},\begin{pmatrix}i\\0\\\vdots\\0\end{pmatrix},\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix},\begin{pmatrix}0\\i\\\vdots\\0\end{pmatrix},\dots$

BASIS EXPANSIONS



$\vec{v}=\begin{pmatrix}x\\y\end{pmatrix}=x\begin{pmatrix}1\\0\end{pmatrix}+y\begin{pmatrix}0\\1\end{pmatrix}$ $\quad$ where $a=\frac{x}{4}$



$$\vec{v}=\begin{pmatrix}x\\y\end{pmatrix}=x\begin{pmatrix}1\\0\end{pmatrix}+y\begin{pmatrix}0\\1\end{pmatrix}$$
$$=a\begin{pmatrix}4\\3\end{pmatrix}+b\begin{pmatrix}0\\-1/2\end{pmatrix}$$
where $a=\frac{x}{4}$, $b=-2\left(y-\frac{3}{4}x\right)$

Def: $\begin{pmatrix}x\\y\end{pmatrix}$ are the coordinates of $\vec{v}$ in the standard basis

(a,b) are $\vec{v}$'s coordinates in the $\{\vec{v}_1,\vec{v}_2\}$ basis

Example: "Hadamard basis" for $\mathbb{R}^2$

$\mathcal{H}=\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right\}$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{x+y}{2}\\\frac{x-y}{2}\end{pmatrix}_{\mathcal{H}}=\frac{1}{2}\begin{pmatrix}x+y\\x-y\end{pmatrix}_{\mathcal{H}}$$

a) Let $\vec{x}=\begin{pmatrix}x\\y\end{pmatrix}$

What are its coordinates in the Hadamard basis?

$$\begin{pmatrix}x\\y\end{pmatrix}=\left(a=\frac{x+y}{2}\right)\begin{pmatrix}1\\1\end{pmatrix}+\left(b=\frac{x-y}{2}\right)\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$
$\Rightarrow a=\frac{x+y}{2}$
$\Rightarrow b=\frac{x-y}{2}$

b) In the Hadamard basis, let $\vec{y}=\begin{pmatrix}y_1\\y_2\end{pmatrix}$.

Express $\vec{y}$ in the standard basis.

$$\vec{y}=\begin{pmatrix}y_1\\y_2\end{pmatrix}_{\mathcal{H}}=y_1\begin{pmatrix}1\\1\end{pmatrix}+y_2\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}y_1+y_2\\y_1-y_2\end{pmatrix}\ \checkmark$$

To change from $\mathcal{H}$ to std. basis, multiply by

$$[\mathcal{H}\to\text{Std}]=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

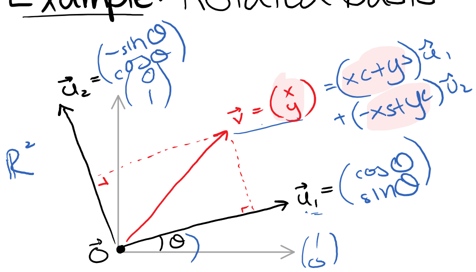
To change basis from std. to $\mathcal{H}$ basis, multiply by

$$\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}1/2&1/2\\1/2&-1/2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}1/2&1/2\\1/2&-1/2\end{pmatrix}=[\text{Std}\to\mathcal{H}]$$

$$\begin{pmatrix}1&1\\1&-1\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Example: Rotated basis



$$[\text{Std}\to\text{Rotated basis}]=\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}$$

Observe: The new coordinates are related to the old coordinates by an invertible matrix:

$$\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\cos\theta+y\sin\theta\\-x\sin\theta+y\cos\theta\end{pmatrix}$$

Why care?

① Makes all (finite-dimensional, real) vector spaces look just like $\mathbb{R}^n$

e.g. $a+bx+cx^2+dx^3\longleftrightarrow(a,b,c,d)\in\mathbb{R}^4$

for the polynomial basis $1,x,x^2,x^3$

② Specifying a linear transformation on a basis gives its values everywhere.

$\mathbb{R}^2$: $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}$

$f(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \sum_j \alpha_j f(v_j)$

③ Dimension measures the "size" of a vector space ...

$\begin{pmatrix} x \\ y \end{pmatrix}$: $x \quad 0 \quad y$

$0 \quad x \quad y-x$

Example: Polynomials in x of degree ≤ 3

one basis: $1, x, x^2, x^3$

coordinates of $a+bx+cx^2+dx^3$ are $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ in basis $1, x, x^2, x^3$

$\left[\begin{pmatrix} d \\ c \\ b \\ a \end{pmatrix}\right.$ in basis $\left.x^3, x^2, x, 1\right]$

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}_{4\times 4} : \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \longmapsto \begin{pmatrix} b \\ 2c \\ 3d \\ 0 \end{pmatrix}$$

$a+bx+cx^2+dx^3 \qquad b+2cx+3dx^2$

A implements $\frac{d}{dx}$

Theorem 1: Any two bases for the same space must have the **same # of elements**. ("Dimension makes sense")

Proof: By contradiction.

Consider bases $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m$ Assume $m<n$

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$

u-vectors are a basis $\Rightarrow$ any $\vec{v}_j = \sum_{i=1}^{m} a_{ij} \vec{u}_i$

for some coeffs. a_{ij}

Let $A = \begin{pmatrix} & a_{ij} & \end{pmatrix}_{m\times n}$ ie., column j is the expansion of $\vec{v}_j$ in the U-basis

$\xRightarrow{\text{by lemma}} N(A) \neq \{0\} \Rightarrow V$-vectors are dependent. Contradiction. □

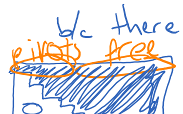
Lemma:

$\underset{m}{\boxed{\overset{n}{A}}}$ $m<n$ $\Rightarrow N(A) \neq \{\vec{0}\}$ ie., the columns are dependent

ie., it is **more**!

b/c there are at least $n-m$ free variables (in GE) ✓

GE



$\Rightarrow$ so there is a basis U with $|U|=n$

Corollary: If $\dim(V)=n$, then any linearly independent set $S \subseteq V$ with n elements is a basis for V.

Let S be a set of n lin. indep. vectors in V.

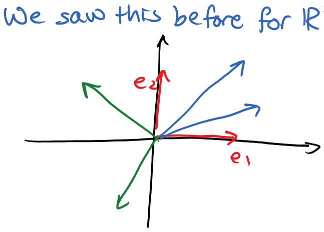
If S is not a basis, $\operatorname{Span}(S) \neq V$

there is a $\vec{v} \in V \setminus \operatorname{Span}(S)$

$\Rightarrow S \cup \{\vec{v}\}$ is linearly independent in V

Now use argument from Thm 1

We saw this before for $\mathbb{R}^2$:



$$U\begin{pmatrix} S & & \vec{v} \end{pmatrix}_{n\times(n+1)}$$

Theorem 2: If U is a subspace of V and $U \neq V$, then $\dim(U) < \dim(V)$.

Theorem 2: If $\mathcal{U}$ is a subspace of V and $\mathcal{U} \neq V$, then $\dim(\mathcal{U}) < \dim(V)$.

Proof: Assume $\dim(\mathcal{U}) = \dim(V) = n$.

$\underset{\text{by Corolary}}{\Rightarrow}$ $\mathcal{U}$ spans V, ie., $\mathcal{U} = V$. , contradiction. □

$n \times (n+1)$

$\Rightarrow S \cup \{\vec{v}\}$ is dependent, a contradiction. □

Another way to think about it:

- a basis is a minimal spanning set (if you remove anything it won't span V)
- a basis is a maximal independent set (adding anything will create a dependency)

Use the proof of Theorem 1 to show this formally.

Changing basis

Example: "Haar basis" for $\mathbb{R}^4$

$$\left\{ \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\end{pmatrix} \right\}$$

why are they linearly indep.? all perpendicular

Let $\vec{x} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4$

What are its coordinates in the Haar basis?

$$\vec{x} = a\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + b\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} + c\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} + d\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \underbrace{\begin{pmatrix}1&1&1&0\\1&1&-1&0\\1&-1&0&1\\1&-1&0&-1\end{pmatrix}}_{\substack{\| \\ A = [\text{Haar} \to \text{Std.}]}}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

$\Rightarrow a = \frac{x_1+x_2+x_3+x_4}{4}$, $c = \frac{x_1-x_2}{2}$

$b = \frac{x_1+x_2-x_3-x_4}{4}$, $d = \frac{x_3-x_4}{2}$

$$A^{-1} = \frac{1}{4}\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\2&-2&0&0\\0&0&2&-2\end{pmatrix} = [\text{Std} \to \text{Haar}]$$

Example: Two bases for $\mathbb{R}^4$

$$\mathcal{B} = \left\{ \vec{e}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \vec{e}_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \vec{e}_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \vec{e}_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$$

$$\mathcal{B}' = \left\{ \vec{v}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \vec{v}_3 = \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \vec{v}_4 = \begin{pmatrix}0\\-3\\0\\1\end{pmatrix} \right\}$$

If $\vec{\omega} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4$

$= \begin{pmatrix}1\\2\\3\\4\end{pmatrix}_{\mathcal{B}'}$ ← in the $\mathcal{B}'$ basis

$$= \begin{pmatrix} \vdots \\ 3 \\ 4 \end{pmatrix}_{\mathcal{B}'}$$ ← in the $\mathcal{B}'$ basis

$\vec{\omega} =$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -10 \\ 3 \\ 4 \end{pmatrix}_{\mathcal{B}}$$ ← in the $\mathcal{B}$ basis

since $\vec{v}_1 = \vec{e}_1$, $\vec{v}_2 = \vec{e}_2$, $\vec{v}_3 = -\vec{e}_1 + \vec{e}_3$, $\vec{v}_4 = -3\vec{e}_2 + \vec{e}_4$

Similarly,

$\vec{e}_1 = \vec{v}_1$

$\vec{e}_2 = \vec{v}_2$

$\vec{e}_3 = \vec{v}_1 + \vec{v}_3$

$\vec{e}_4 = 3\vec{v}_2 + \vec{v}_4$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ -10 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$[\mathcal{B}' \to \mathcal{B}] = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

changes from $\mathcal{B}'$ expansion to $\mathcal{B}$ basis expansion

$$[\mathcal{B} \to \mathcal{B}'] = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

changes from $\mathcal{B}$ to $\mathcal{B}'$.

Observe: These matrices are inverses of each other:

$$[\mathcal{B}' \to \mathcal{B}] \cdot [\mathcal{B} \to \mathcal{B}'] = [\mathcal{B} \to \mathcal{B}'][\mathcal{B}' \to \mathcal{B}]$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

(The most common mistake here is giving the transposes of the desired matrices.)