Homework 8

① Compute the singular-value decompositions of the following matrices. Do not use a computer!
Don't use eigenvalues either.

$$A = \begin{pmatrix} 0 & 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 3 & 2 & 1 & -2 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & \sqrt{8} & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Pseudoinverses

② By hand, compute the pseudoinverses of the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \end{pmatrix}_{1\times n} \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}_{n\times 1}$$

$$C = \begin{pmatrix} 1 & 2 & 3 & 0 \\ -4 & 2 & 0 & 2 \\ 1 & 1 & -1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{pmatrix}$$

③ Is $(AB)^+ = B^+ A^+$ always true for pseudoinverses?
No! Give a counterexample.

SVD and condition number

④

ⓐ <u>Without computing the norm exactly</u>, argue why

$$100 < \left\| \begin{pmatrix} 1 & -100 \end{pmatrix} \right\| < 101$$

ⓐ Without computing the norm exactly, argue why

$$100 \le \left\| \begin{pmatrix} 1 & -100 \\ 0 & 1 \end{pmatrix} \right\| \le 101.$$

ⓑ What is the inverse of $A = \begin{pmatrix} 1 & -100 \\ 0 & 1 \end{pmatrix}$?

What is the condition number of A?

What is the condition number of A^{-1}?

ⓒ Find vectors $\vec{b} \in \mathbb{R}^2$ and $\vec{\delta} \in \mathbb{R}^2$ such that $\|\delta\|$ is "small" compared to $\|b\|$, and yet

$$\| A^{-1}(b+\delta) - A^{-1} b \|$$

is "large" compared to $\|A^{-1} b\|$. (Use your own judgment for what should count as small or large.)

(Hint: Compute the SVD and experiment a bit, using Matlab.)

⑤

ⓐ Let $H_3 = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$.

Compute $b = H_3 x$ for $x = (1,1,1)$ and $x = (0, 6, -3.6)$.

A small change Δb produces a large change Δx.

ⓑ Compute numerically the largest and smallest singular values of the 7×7 "Hilbert matrix" H_7, a matrix whose (i,j) entry is $\frac{1}{i+j-1}$.

(Hint: Google the Matlab "hilb" command.)

ⓒ If $H_7 x = b$ with $\|b\| = 1$, how large can $\|x\|$ be?

If b has roundoff error less than 10^{-16} in norm, how large an error can this cause in x

⑥ Let A be an $m \times n$ real matrix, with singular-value decomposition

$$A = \sum_i \sigma_i \vec{v}_i \vec{u}_i^T.$$

Assume that the singular values are sorted $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$.

$A = \sum_i \sigma_i v_i u_i'$.

Assume that the singular values are sorted, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq 0$.

a) Give the SVD for the matrix AA^T.

What is its condition number?

Why is $R(AA^T) = R(A)$?

Why is $N(A^TA) = N(A)$?

b) Give the SVD for the $(m+n) \times (m+n)$ matrix

$$B = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}.$$

What is its condition number?

In terms of A^+, what is the pseudoinverse B^+?

(Hint: What is B applied to $(\vec{0}, \vec{u}_i) \in \mathbb{R}^{m+n}$?

How about B applied to $(\vec{v}_i, \vec{0}) \in \mathbb{R}^{m+n}$?)

Least-squares regression

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a) Find the projection of b onto the column space of A:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}.$$

Split b into $p + q$, with p in the column space and q perpendicular to that space. Which of the four subspaces contains q?

$R(A)$, $R(A^T)$, $N(A)$ or $N(A^T)$

b) Using your answer from part a, find the $\vec{x} \in \mathbb{R}^2$ that minimizes $\|A\vec{x} - \vec{b}\|$.

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For $\mathbf{A} \in \Re^{m \times n}$ and $\mathbf{b} \in \Re^m$, prove that $\mathbf{x}_2$ is a least squares solution for $\mathbf{Ax} = \mathbf{b}$ if and only if $\mathbf{x}_2$ is part of a solution to the larger system

$$\begin{pmatrix} \mathbf{I}_{m \times m} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0}_{n \times n} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix}.$$