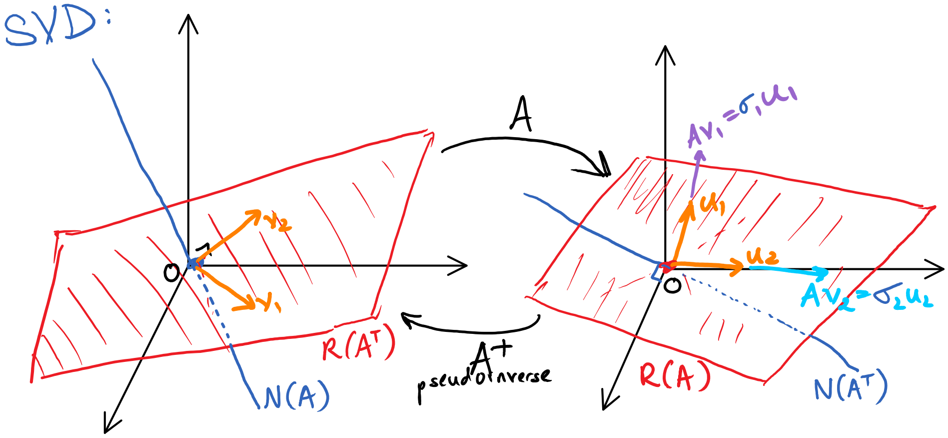
Lecture 19: Introduction to eigenvectors (class)

Admin:

INTRODUCTION TO EIGENVECTORS

SVD:



SVD $\Rightarrow$ linear transformations are simple

- map rowspace to columnspace

$\vec{v}_j \longmapsto \sigma_j \vec{u}_j$

orthonormal basis to orthogonal basis

But some matrices are even simpler!

Example: Diagonal matrices

$$A = \begin{pmatrix} 12 & & & 0 \\ & 10 & & \\ & & -1 & \\ 0 & & & 0 \end{pmatrix}$$

$Ae_1 = 12e_1$

$Ae_3 = -e_3$

$\sigma_1 = \|A\| = 12, \ \sigma_2 = 10, \ \sigma_3 = 1, \ \sigma_4 = 0$

$\text{Rank}(A) = 3$

$$A^n = \begin{pmatrix} 12^n & & & 0 \\ & 10^n & & \\ & & (-1)^n & \\ 0 & & & 0^n \end{pmatrix}$$

If $f(x) = \sum_{j=0}^{\infty} c_j x^j$,

$$f(A) := \sum_{j=0}^{\infty} c_j A^j = \begin{pmatrix} f(12) & & & 0 \\ & f(10) & & \\ & & f(-1) & \\ 0 & & & f(0) \end{pmatrix}$$

$$f(A) := \sum_{j=0} c_j A^j = \begin{pmatrix} f(10) & & \\ & f(-1) & \\ 0 & & f(0) \end{pmatrix}$$

$f(0) = c_0$

Example: Diagonalizable matrices $\leftrightarrow$ a matrix with a basis of e-vectors

$$A = \mathcal{U} \underbrace{\begin{pmatrix} 12 & & & 0 \\ & 10 & & \\ & & -1 & \\ 0 & & & 0 \end{pmatrix}}_{= D} \mathcal{U}^{-1}$$

where $\mathcal{U} = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{pmatrix} = \sum_{j=1}^{n} u_j e_j^T$

$U e_j = u_j$

$\Rightarrow U^{-1} u_j = e_j$

$$\Rightarrow A\vec{u}_j = U D U^{-1} \vec{u}_j$$
$$= U D e_j$$
$$= D_{jj} U e_j$$
$$= D_{jj} \vec{u}_j$$

Def.: $\vec{u}$ is an eigenvector of A if $A\vec{u} = \lambda \vec{u}$, $\lambda \in \mathbb{C}$ eigenvalue

Note: Directions of other vectors definitely do change,

$$A^n(\vec{u}_1 + \vec{u}_2) = 12^n \vec{u}_1 + 10^n \vec{u}_2 \neq \lambda(\vec{u}_1 + \vec{u}_2)$$

$$A(c\vec{u}_1) = 12(c\vec{u}_1) \qquad \text{"eigenspaces"}$$

If the $\{\vec{u}_j\}$ are orthonormal,

$$\|U A U^{-1}\| = \|A\| = 12$$

unitary

but they don't have to be!

$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ (λ_1, λ_2)

$$A = U D U^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} D \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$\|A\| = \sqrt{2} = \sigma_1$

$\sigma_2 = 0$

More examples of eigenvectors & of diagonalizable matrices

$A = (0) \qquad A\vec{v} = 0 \cdot \vec{v}$ for any $\vec{v}$

More examples of eigenvectors & of diagonalizable matrices

$\sigma_2 = 0$

- $A = (0)$ $\quad A\vec{v} = 0 \cdot \vec{v}$ for any $\vec{v}$
- If $\dim N(A) \geq 1 \Rightarrow N(A)$ is an eigenspace with eigenvalue 0.
- $A = I$ $\quad I\vec{v} = \vec{v}$

  all vectors are e-vectors with e-value $\lambda = 1$

- $A = cI$ $\quad$ a vectors are e-vectors with e-value $\lambda = c$ $\quad c I\vec{v} = c\vec{v}$

- $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad v_1 = e_1 \ \sigma_1 = 1 \quad u_1 = e_2$
  $\quad v_2 = e_2 \ \sigma_2 = 1 \quad u_2 = e_1$

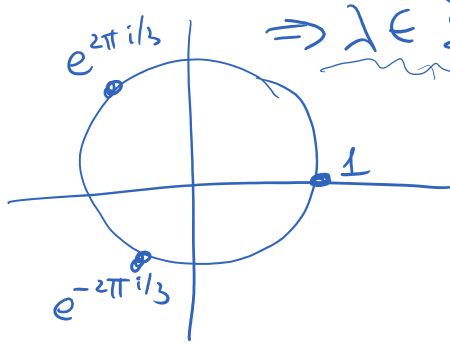
	e-vectors	e-value
$A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	1
$A\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = -1 \cdot \begin{pmatrix}1\\-1\end{pmatrix}$	$\begin{pmatrix}1\\-1\end{pmatrix}$	-1

$\rightarrow \lambda^2 = 1$

$A^2 = I$ $\quad$ If $A\vec{v} = \lambda\vec{v} \Rightarrow A^n\vec{v} = \lambda^n\vec{v}$

- $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow A^3 = I \Rightarrow \lambda^3 = 1$

  $\Rightarrow \lambda \in \{1, e^{2\pi i/3}, e^{-2\pi i/3}\}$



Generalization 1:

For any permutation matrix P,

$$P\begin{pmatrix}1\\ \vdots \\ 1\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 1 \\ \vdots\\ 1\end{pmatrix}$$

all-ones vector is an e-vector with e-value 1

$$+\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \begin{pmatrix}1\\1\\1\end{pmatrix}$$ is an e-vector with e-value 1

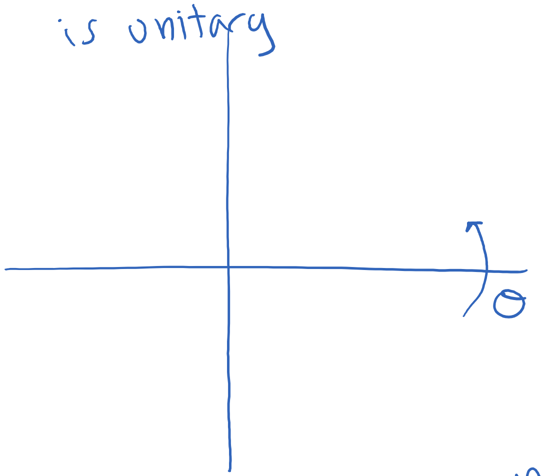
Generalization 2:

For any square matrix whose rows all sum to 1 ("row-stochastic")

$$\begin{pmatrix} .9 & .05 & .05 \\ .8 & .1 & .1 \\ .7 & .2 & .1 \end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$\Rightarrow$ all-ones vector is an e-vector with e-value 1

- $A = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$ is unitary



$$\begin{pmatrix}c & -s \\ s & c\end{pmatrix}\begin{pmatrix}1\\y\end{pmatrix} = \lambda\begin{pmatrix}1\\y\end{pmatrix}$$

$$c - sy = \lambda$$
$$s + cy = \lambda y$$

$$s + \cancel{cy} = (c - sy)y$$
$$= \cancel{cy} - sy^2$$

$$sy^2 + s = 0$$

If $\sin\theta \neq 0$, then $y^2 + 1 = 0$
$(y+i)(y-i)$

Eigenvalues	Eigenvectors
$c - si = e^{-i\theta}$	$\begin{pmatrix}1\\i\end{pmatrix}$
$c - s(-i) = e^{i\theta}$	$\begin{pmatrix}1\\-i\end{pmatrix}$

$$A = \underbrace{\begin{pmatrix}1 & 1 \\ i & -i\end{pmatrix}}_{=U}\begin{pmatrix}e^{-i\theta} & 0 \\ 0 & e^{i\theta}\end{pmatrix}U^{-1}$$

$$U^{-1} = \tfrac{1}{2}U^{\dagger} = \tfrac{1}{2}\begin{pmatrix}1 & -i \\ 1 & i\end{pmatrix}$$

Moral: Real matrices can have complex e-values & e-vectors.

In contrast, the SVD of a real-valued matrix works with real singular vectors.
Singular values are always (nonnegative) reals.
Eg., A's singular values are 1, 1.

- $A = \vec{u} \cdot \vec{v}^T$ for vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$

$$N(A) = \mathrm{Span}(\vec{v})^\perp$$

e-value 0 e-space

Moral: Every vector in the nullspace is an eigenvalue-zero eigenvector.
(Nullspace (if nonzero) is an eigenvalue-zero eigenspace)

Example:

$$A = \frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \text{all-ones} & & & 1 \\ 1 & \text{matrix} & & & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}_{n\times n} = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} = P_{\mathrm{Span}\left\{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\right\}}$$

Moral: For the projection P_V onto a subspace V,

e-values	e-space
1	V
0	$V^\perp$

- A slightly more complicated example:
  Let $A = \sum_j \sigma_j v_j u_j^T$ (an SVD)

$$A^T A = \sum_j \sigma_j^2 u_j u_j^T \qquad\qquad A A^T = \sum_j \sigma_j^2 v_j v_j^T$$

$\shortparallel$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

$$A^*A = \sum_j \sigma_j v_j v_j^*$$

$$\Downarrow$$

$\vec{v}_j$ is an e-vector with e-value σ_j^2

$$\Downarrow$$

$\vec{x}_j$ is an e-vector with e-value σ_j^2

- $\boxed{\text{A matrix is diagonalizable } \underset{n\times n}{A} = U \underset{\text{diagonal}}{D} U^{-1} \iff \text{there is a basis of eigenvectors for } \mathbb{C}^n \text{ (the columns of } U)}$

From the examples so far, it seems that lots of matrices are diagonalizable.

But not all!!

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$$

good matrix to use in examples

$$\underset{\begin{pmatrix} y \\ 0 \end{pmatrix}}{\underset{\|}{A\begin{pmatrix} x \\ y \end{pmatrix}}} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} y = \lambda x \\ 0 = \lambda y \end{matrix}$$

$\Rightarrow$ either $\underline{y = 0}$ or $\underline{\lambda = 0}$

$\Rightarrow \lambda x = 0$

$\Rightarrow \lambda = 0$

<u>e-value</u>	<u>e-space</u>
0	$N(A) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

Alternative proof:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{If } A\vec{v} = \lambda \vec{v}$$

$$\vec{0} = A^2 \vec{v} = \lambda^2 \vec{x} \Rightarrow \lambda = 0$$

"SPECTRAL THEORY"

= THEORY OF EIGENVALUES & EIGENVECTORS

Spectral theory asks the following questions:

Answers

① What matrices have an eigenvector? — $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ldots$ all square matrices — today ✓

② What matrices can be diagonalized? (That is, they have a full basis of eigenvectors.) — ? — diagonalizable matrices

③ What matrices can be diagonalized with an **orthogonal** basis of eigenvectors? — "normal" matrices — proof soon

Another natural question is when are the eigenvalues real? And, of course, how do you **find** eigenvalues & eigenvectors? We'll be studying these questions next.

But first, some applications:

Application: Solving linear recursions

Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, ...

$$f_n = f_{n-1} + f_{n-2}$$
$$f_1 = f_2 = 1$$

$$\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{= A} \begin{pmatrix} f_{n-1} \\ f_{n-2} \end{pmatrix} = A^2 \begin{pmatrix} f_{n-2} \\ f_{n-3} \end{pmatrix} = \cdots = A^{n-2} \begin{pmatrix} 1 = f_2 \\ 1 = f_1 \end{pmatrix}$$

we'll diagonalize $A = UDU^{-1} \Rightarrow A^{n-2} = UD^{n-2}U^{-1}$

Finding the e-vectors of A:

$$A\vec{x} = \lambda \vec{x}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

if $\vec{x}$ is an e-vector with $x_1 \neq 0$ then $\begin{pmatrix} 1 \\ x_2/x_1 \end{pmatrix}$ is also an e-vector w/ the same e-value.

$$\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\lambda\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

w/ the same e-value.

① $x_1=0$: $\cancel{\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}\neq\lambda\begin{pmatrix}0\\1\end{pmatrix}}$

~~not an e-vector~~

② $x_1\neq 0$: $\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}1\\x\end{pmatrix}=\lambda\begin{pmatrix}1\\x\end{pmatrix}$

$$\overset{\parallel}{\begin{pmatrix}1+x\\1\end{pmatrix}} \Rightarrow \begin{array}{l}\lambda=1+x\\ \lambda x=1\end{array}$$

$$x^2+x-1=0$$

$$\Rightarrow \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-1\pm\sqrt{5}}{2}, \quad \lambda=1+x$$

"golden ratio" **Eigenvalues**

$$\lambda_1=1+\frac{-1+\sqrt{5}}{2}=\frac{1}{2}+\frac{\sqrt{5}}{2}=1.618\ldots$$

$$\lambda_2=1+\frac{-1-\sqrt{5}}{2}=\frac{1}{2}-\frac{\sqrt{5}}{2}=\underset{1-\lambda_1}{\underset{\parallel}{-.618\ldots}}$$

Eigenvectors

$$\begin{pmatrix}1\\\frac{-1+\sqrt{5}}{2}\end{pmatrix}=\vec{v}_1$$

$\perp$ $\quad v_1\cdot v_2=0$

$$\begin{pmatrix}1\\\frac{-1-\sqrt{5}}{2}\end{pmatrix}=\vec{v}_2$$

$$A^{n-2}=\underset{U}{\underset{\parallel}{\begin{pmatrix}1&1\\\frac{-1+\sqrt{5}}{2}&\frac{-1-\sqrt{5}}{2}\end{pmatrix}}}\begin{pmatrix}\lambda_1^{n-2}&0\\0&\lambda_2^{n-2}\end{pmatrix}U^{-1}$$

$$\underset{\text{for } n \text{ large}}{\approx} U\begin{pmatrix}\lambda_1^{n-2}&0\\0&0\end{pmatrix}U^{-1}=\begin{pmatrix}1\\\frac{-1+\sqrt{5}}{2}\end{pmatrix}\begin{pmatrix}\lambda_1^{n-2}&0\end{pmatrix}U^{-1}$$

Application: Systems of differential equations

Warmup:

$$\frac{d}{dt}x(t)=x(t) \longrightarrow x(t)=\underset{x(t=0)}{\underset{\parallel}{C}}e^{t}$$

What about

$$\dot{x}=4x-5y$$
$$\dot{y}=2x-3y$$

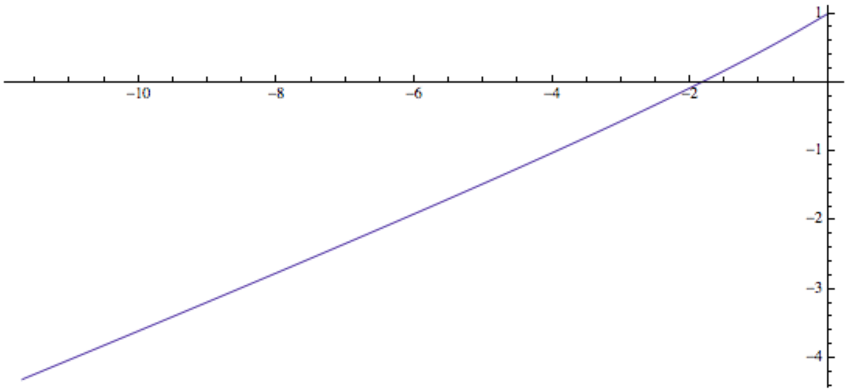
Numerical solution (Mathematica):

```
In[2]:= sol = NDSolve[{
          D[x[t], t] == 4 x[t] - 5 y[t],
          D[y[t], t] == 2 x[t] - 3 y[t],
          x[0] == 0,
          y[0] == 1
        }, {x, y}, {t, 0, 1}];
      ParametricPlot[{x[t], y[t]} /. sol, {t, 0, 1}]
```

```
In[1]:= ? NDSolve
```

NDSolve[*eqns*, *u*, {*t*, t_{min}, t_{max}}] finds a numerical solution to the ordinary differential equations *eqns* for the function *u* with the independent variable *t* in the range t_{min} to t_{max}.
NDSolve[*eqns*, *u*, {*t*, t_{min}, t_{max}}, {*x*, x_{min}, x_{max}}] finds a numerical solution to the partial differential equations *eqns*.
NDSolve[*eqns*, {u_1, u_2, …}, {*t*, t_{min}, t_{max}}] finds numerical solutions for the functions u_i. »

Out[3]=



Exact solution:

"Rule": When you see multiple equations, try to vectorize them!

$$\dot{x} = 4x - 5y$$
$$\dot{y} = 2x - 3y$$

Problem: Solve

$$\frac{d}{dt}\vec{v}(t) = A\,\vec{v}(t)$$

subject to $\vec{v}(t=0) = \vec{v}_0$ initial conditions

Answer:

① In terms of eigenvalues/eigenvectors:

- Say $A\vec{u}_1 = \lambda_1\vec{u}_1$
  and $\vec{v}_0 = \vec{u}_1$

- Say $A\vec{u}_1 = \lambda_1 u_1$

$$A\vec{u}_2 = \lambda_2 \vec{u}_2$$
and $\vec{v}_0 = \vec{u}_1 - 2\vec{u}_2$

- If A is diagonalizable, then its eigenvectors form a basis,

If A can be diagonalized,

$$A = UDU^{-1}$$

$$\Rightarrow \frac{d}{dt} v = UDU^{-1} v$$

The eigenvalues of A, λ_1 and λ_2, determine the exponents (growth rates).

We can actually simplify even further:

$$v(t) = U \cdot \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} u(0)$$

$$v(t) = \mathcal{U} \cdot \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} u(0)$$

$$= \mathcal{U} \cdot \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \mathcal{U}^{-1} v(0)$$

② In terms of matrices:

$$v(t) = e^{At} \vec{v}_0$$

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} + \cdots$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} (At)^j$$

Proof:

Functions of diagonalizable matrices:

Definition: If A is a diagonalizable matrix

$$A = \mathcal{U} \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \mathcal{U}^{-1}$$

and $f: \mathbb{C} \to \mathbb{C}$ any function, define

$$f(A) = \mathcal{U} \begin{pmatrix} f(\lambda_1) & & & 0 \\ & f(\lambda_2) & & \\ & & \ddots & \\ 0 & & & f(\lambda_n) \end{pmatrix} \mathcal{U}^{-1}.$$

Example: $f(x) = x^2$

$\Rightarrow f(A) = \mathcal{U} D^2 \mathcal{U}^{-1} = A^2$, as you'd expect ✓

Example: $f(x) = e^{tx}$ exponential

$$\Rightarrow f(A) = \mathcal{U} \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix} \mathcal{U}^{-1}$$

So,

$$v(t) = \underbrace{\mathcal{U} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \mathcal{U}^{-1}}_{e^{At}} v(0)$$

$\Rightarrow$ The solution to

$$\frac{d}{dt} v(t) = A\, v(t)$$

is $v(t) = e^{At} v(0)$.

This looks just like the single-variable case.

You can also show this using Taylor series:

$$e^{xt} = 1 + xt + \frac{(xt)^2}{2} + \cdots$$

$$= \sum_{j=0}^{\infty} \frac{(xt)^j}{j!}$$

and $e^{At} = \sum_{j=0}^{\infty} \frac{(At)^j}{j!}$ where $A^0 = \underset{\text{identity matrix}}{I}$.

Extension: Higher-order differential equations with constant coefficients.

To solve a 2nd-order diff. eq. like

$$\ddot{x} = -\dot{y} + 2x$$
$$\ddot{y} = \dot{x} + \dot{y} - 3y$$

More applications we'll consider later:

- Spectral graph partitioning
- Markov chains
- Google Pagerank

⋮

WHAT MATRICES HAVE AN EIGENVECTOR?

Theorem: Every (real or complex) square matrix has at least one eigenvector (over $\mathbb{C}$!).

Proof: Let A be an $n \times n$ matrix, real or complex.

Goal: Prove that there exists a vector $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \lambda \vec{x}$ for some λ.

Fact 1: If $a_1 a_2 \cdots a_k = 0$, then at least one $a_j = 0$.

Fact 2: (Fund. Thm. of Algebra) Any polynomial $p(x)$ can be factored over $\mathbb{C}$,

$$p(x) = c \cdot (x - r_1)(x - r_2) \cdots (x - r_k)$$

Lemma: For matrices $A_1, A_2, \ldots, A_k$, if $\dim N(A_1 A_2 \cdots A_k) \geq 1$, then at least one A_j has $\dim N(A_j) \geq 1$.

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Proof of lemma:

$\dim N(A_1 \cdots A_k) \geq 1 \Rightarrow \exists\, \vec{v} \neq \vec{0}$ s.t.

$$A_1 A_2 \cdots A_k \vec{v} = \vec{0}$$

If $A_k \vec{v} = \vec{0} \Rightarrow \vec{v} \in N(A_k) \Rightarrow \dim N(A_k) \geq 1$.

otherwise $\underbrace{A_k \vec{v}}_{\vec{v}'} \neq \vec{0} \Rightarrow \underset{\neq \vec{0}}{\vec{v}'} \in N(A_1 \cdots A_{k-1}) \Rightarrow \dim N(A_1 \cdots A_{k-1}) \geq 1$

$\square$

Proof of Theorem: Let $\vec{v} \neq \vec{0}$ be any vector.

Consider

$$\vec{v}_0 = \vec{v}, \quad \vec{v}_1 = A\vec{v}, \quad \vec{v}_2 = A^2 \vec{v}, \quad \vec{v}_n = A^n \vec{v}$$

where A is an $n \times n$ matrix. $\underbrace{\qquad}_{n+1 \text{ vectors in } \mathbb{C}^n}$

$\Rightarrow$ linearly dependent

$\Rightarrow \exists$ constants $c_0, \dots, c_n$ not all 0 s.t.

$$\sum_{j=0}^{n} c_j \vec{v}_j = \vec{0}$$

$$\Downarrow$$

$$\text{"}p(A)\vec{v}\text{"} = \sum_{j=0}^{n} c_j A^j \vec{v} = \vec{0}$$

Let $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$

have factorization

$p(x) = c_n (x - r_1) \cdots (x - r_n)$

<u>Claim</u>: $p(A)\vec{v} = c_n (A - r_1 I)(A - r_2 I) \cdots (A - r_n I)\vec{v}$

b/c $(A - r_1 I)(A - r_2 I) = A^2 - (r_1 + r_2)A + r_1 r_2 I$

$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2$ $\square$

$$p(A)\vec{v} = c_n (A - r_1 I)(A - r_2 I) \cdots (A - r_n I)\vec{v} = \vec{0}$$

$$\vec{v} \in N\left(\prod_{j=1}^{n} (A - r_j I)\right)$$

$\Rightarrow \exists$ some j with $\dim N(A - r_j I) \geq 1$.

$\Rightarrow$ some $\vec{w} \neq \vec{0}$, $A\vec{w} = r_j \vec{w}$

$\vec{w}$ is an eigenvector of A. $\square$

Recall: Factorizations

① If $abc = 0$, then $a = 0$ or $b = 0$ or $c = 0$

For matrices: If $ABC\vec{v} = \vec{0}$, $\vec{v} \neq \vec{0}$,

then $\vec{v} \in N(C)$

or $C\vec{v} \in N(B)$ and $C\vec{v} \neq \vec{0}$

or $BC\vec{v} \in N(A)$ and $BC\vec{v} \neq \vec{0}$

$\Rightarrow$ Either A, B or C is singular

(the product of invertible matrices is invertible).

② "Fundamental theorem of algebra":

Any polynomial $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$,

with real or complex coefficients, can be factored as

$$p(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)$$

for some complex roots $\lambda_j \in \mathbb{C}$, $p(\lambda_j) = 0$

Step 1. Let $\vec{v}$ be any nonzero vector.

Consider the vectors

$$\vec{v},\ A\vec{v},\ A^2\vec{v},\ \ldots,\ A^n\vec{v}$$

Since there are $n+1$ vectors in an n-dimensional space, they must be linearly dependent.

(<u>Note</u>: This argument only works in finite dimensions.)

Say

$$\alpha_0 v + \alpha_1 A v + \alpha_2 A^2 v + \cdots + \alpha_n A^n v = 0$$

with not all α_j's 0.

Step 2. Thus

$$\left(\sum_{j=0}^{n} \alpha_j A^j\right)\vec{v} = 0$$

Factor the polynomial

$$p(x) = \sum_j \alpha_j x^j = \prod_j (x - \lambda_j)$$

$$\Rightarrow \left[\prod_j (A - \lambda_j I)\right]\vec{v} = 0$$

Step 3.

$\Rightarrow$ at least one of the matrix terms $A - \lambda_j I$

must be singular
(the product of two nonsingular matrices is nonsingular!)

✓ Done. $\square$

Note: The first step can be "justified" by noticing that $\lim_{k\to\infty} \frac{A^k v}{\|A^k v\|}$ converges to an eigenvector with largest magnitude eigenvalue (though I don't want to prove this). So it makes sense to look at successive powers $A^k v$.

Note: There's a simple proof that $\mathbb{C}$ is algebraically closed in the appendix of Lang's "Linear Algebra."

More examples

Eigenvalues

$$\begin{pmatrix} 1+2\cos\theta & -2\sin\theta \\ 2\sin\theta & 1+2\cos\theta \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 3 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \begin{matrix}1&2\\2&1\end{matrix} \\ \begin{matrix}1&2\\2&1\end{matrix} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 1 & 4 & 2 \\ 2 & 4 & 1 & 2 \\ 4 & 2 & 2 & 1 \end{pmatrix}$$

Outline:

Motivation