Lecture 10: Changes of basis (class)

Admin: Reading

Outline: Change of basis, orthogonal bases
Projections and Gram-Schmidt orthogonalization

Important concepts:

1. Span
2. Linear independence
3. Basis — orthogonal basis, change of basis
4. Dimension

① Span$(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\})$

$$= \left\{ \begin{matrix} \text{all (finite) linear combinations} \\ \text{of those vectors} \\ \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r \end{matrix} \right\} = R\left(\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_r \\ | & | & & | \end{pmatrix}\right)$$

range/column space

② $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly independent if no vector $\vec{v}_j$ lies in the span of the others.

– equivalently, if the only solution to $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r = \vec{0}$ is $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$

– equivalently, if $N\left(\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_r \\ | & | & & | \end{pmatrix}\right) = \{\vec{0}\}$

nullspace

③ A basis for a vector space V is a set of vectors that
- **spans** V, and
- is **linearly independent**

Equivalently, it is a minimal set of vectors that spans V.

④ Dimension (a vector space)
$= \#$ of vectors in a basis

Two ways of checking linear independence:

① Gaussian elimination preserves the rowspace

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 0 & 9 & 3 \end{pmatrix} \xrightarrow{2-2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 9 & 3 \end{pmatrix} \xrightarrow{-3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

3rd row dependent on first two!

② $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent $\iff$ nullspace of $\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$ is $\{\vec{0}\}$.

To check if a set is linearly independent, compute the nullspace.

$A\vec{x} = \vec{0}$,

i.e. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 9 \\ 3 \end{pmatrix} x_3 = \vec{0}$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 9 \\ 1 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 9 \\ 0 & -1 & 3 \end{pmatrix} \xrightarrow{-1/3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x_3$ is a free variable
$\Rightarrow N(A) \neq \{\vec{0}\} \Rightarrow$ linearly dependent

LINEAR TRANSFORMATIONS
$\updownarrow$
MATRICES

- $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$
- $f(c\vec{u}) = c\, f(\vec{u})$

$\Downarrow$

$f\left(\sum_j c_j \vec{u}_j\right) = \sum_j c_j f(\vec{u}_j)$

$$\mathcal{U} \xrightarrow{f} V$$

basis $\{\vec{u}_1, \ldots, \vec{u}_n\}$ basis $\{\vec{v}_1, \ldots, \vec{v}_m\}$

$\vec{u} = \sum_j c_j \vec{u}_j \xrightarrow{f} f(\vec{u}) = \sum_j c_j f(\vec{u}_j) \in V$

$f(\vec{u}_j) = \sum_i a_{ij} \vec{v}_i$

$\vec{u} = \sum_j c_j \vec{u}_j \longrightarrow f(\vec{u}) = \sum_j c_j f(\vec{u}_j) \in V$

$f(\vec{u}_j) = \sum_i a_{ij} \vec{v}_i$

$$A = \begin{array}{c} \\ i \end{array} \begin{pmatrix} & j & \\ \text{---} & a_{ij} & \end{pmatrix}_{m \times n}$$

$$f(\vec{u}) = \sum_j c_j f(\vec{u}_j) = \sum_{i,j} \vec{v}_i a_{ij} c_j = \left(A \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \right)_{\{\vec{v}_1, \dots, \vec{v}_m\}}$$

Observe:

- Any $\vec{u} \in \mathcal{U}$ is a linear comb. of the $\vec{u}_j$'s
  $f\left(\sum_j \alpha_j \vec{u}_j\right) = \sum_j \alpha_j f(\vec{u}_j)$
  $\Rightarrow f$ is determined by $f(\vec{u}_1), \dots, f(\vec{u}_n)$
- Each $f(\vec{u}_j) \in V$ is a linear comb. of the $\vec{v}_i$'s
  $f(\vec{u}_j) = \sum_{i=1}^{m} a_{ij} \vec{v}_i$

$$\Rightarrow A = \begin{array}{c} 1 \\ i \\ m \end{array} \begin{pmatrix} 1 & j & n \\ \text{---} & a_{ij} & \\ & & \end{pmatrix} \text{ determines } f$$

More precisely, then,

Matrix + bases $\longrightarrow$	Linear transformation
Linear transformation + bases $\longrightarrow$	Matrix

Remark: If you use different bases, you'll get different matrices, for the **same** linear transformation.

EXAMPLES

$$f(\vec{u}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \vec{u} = \begin{pmatrix} u_1 + 2u_2 \\ 3u_1 + 4u_2 \end{pmatrix}$$

Now let $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = [g]_{\underbrace{(1,1),(1,-1)}_{B}}$

$g(1,0) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$g(0,1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\quad [g]_{Std} = \begin{pmatrix} 5 & -1 \\ -2 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}_B$

$\downarrow g \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\frac{1}{2}\begin{pmatrix} 3 \\ 7 \end{pmatrix}_B = \frac{3}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{7}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}_B$

$\downarrow g \ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$-\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_B = -\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Example 1: $f(x, y, z) = (x, y)$

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

- using the standard basis for both

$$[f]_{Std,Std} = \begin{array}{c} \begin{array}{ccc} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{2 \times 3} \end{array}$$

$f(\vec{e}_1) = f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$A\vec{e}_1 \quad A\vec{e}_j$

- using $B = \{(1,1), (1,-1)\}$ as the basis for $\mathbb{R}^2$

$$[f]_{Std,B} = \begin{array}{c} \\ (1,1) \\ (1,-1) \end{array} \begin{array}{c} \begin{array}{ccc} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \end{pmatrix}_{2 \times 3} \end{array}$$

$f(\vec{e}_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_S = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}_B$ $\quad \left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$

$f(\vec{e}_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_S = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}_B$ $\quad \left[\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$

$f(\vec{e}_3) = \vec{0}$

Example 1.5:

$\mathcal{U} = V = \{2 \times 2 \text{ matrices}\}$

$f: \mathcal{U} \to V$ $\quad S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$f(A) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A$ $\quad$ (this is linear: $f\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $f(cA) = cf(A)$, $f(A+B) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} B$)

$$[f]_S = \begin{array}{c} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \begin{array}{cccc} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \\ \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}_{4 \times 4} \end{array}$$

$f\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$
$= 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$f\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}$

$f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$

$f\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}$

$\mathcal{B}: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$[f]_{B,S} = \begin{array}{c|cccc} & \begin{pmatrix}1&0\\0&1\end{pmatrix} & \begin{pmatrix}1&0\\0&-1\end{pmatrix} & \begin{pmatrix}0&1\\1&0\end{pmatrix} & \begin{pmatrix}0&1\\-1&0\end{pmatrix} \\ \hline \begin{pmatrix}1&0\\0&0\end{pmatrix} & 1 & & 2 & \\ \begin{pmatrix}0&1\\0&0\end{pmatrix} & 2 & ? & 1 & ? \\ \begin{pmatrix}0&0\\1&0\end{pmatrix} & 3 & & 4 & \\ \begin{pmatrix}0&0\\0&1\end{pmatrix} & 4 & & 3 & \end{array}_{4\times 4}$$

$$f(I) = \begin{pmatrix}1&2\\3&4\end{pmatrix} I = \begin{pmatrix}1&2\\3&4\end{pmatrix}$$

$$f\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = \begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}2&1\\4&3\end{pmatrix}$$

$$[f]_{B,B} = \begin{array}{c|cccc} & \begin{pmatrix}1&0\\0&1\end{pmatrix} & \begin{pmatrix}1&0\\0&-1\end{pmatrix} & \begin{pmatrix}0&1\\1&0\end{pmatrix} & \begin{pmatrix}0&1\\-1&0\end{pmatrix} \\ \hline \begin{pmatrix}1&0\\0&1\end{pmatrix} & 5/2 & & & \\ \begin{pmatrix}1&0\\0&-1\end{pmatrix} & -3/2 & & & \\ \begin{pmatrix}0&1\\1&0\end{pmatrix} & 5/2 & & & \\ \begin{pmatrix}0&1\\-1&0\end{pmatrix} & -1/2 & & & \end{array}_{4\times 4}$$

$$f(I) = \begin{pmatrix}1&2\\3&4\end{pmatrix} = \frac{5}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} - \frac{3}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} + \frac{5}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

$$[Id]_{BS} = [B\to S] = \begin{array}{c|cccc} & \begin{pmatrix}1&0\\0&1\end{pmatrix} & \begin{pmatrix}1&0\\0&-1\end{pmatrix} & \begin{pmatrix}0&1\\1&0\end{pmatrix} & \begin{pmatrix}0&1\\-1&0\end{pmatrix} \\ \hline \begin{pmatrix}1&0\\0&0\end{pmatrix} & 1 & 1 & 0 & 0 \\ \begin{pmatrix}0&1\\0&0\end{pmatrix} & 0 & 0 & 1 & 1 \\ \begin{pmatrix}0&0\\1&0\end{pmatrix} & 0 & 0 & 1 & -1 \\ \begin{pmatrix}0&0\\0&1\end{pmatrix} & 1 & -1 & 0 & 0 \end{array}_{4\times 4}$$

$$[Id]_{SB} = [S\to B] = [B\to S]^{-1}$$

$$\begin{tikzcd} B \arrow[r, "Id"] \arrow[rrr, bend left, "f"] & S \arrow[r, "f"] & S \arrow[r, "Id"] & B \end{tikzcd}$$

$$[f]_{BB} = [B\to S]^{-1}\,[f]_{SS}\,[B\to S] =$$

Example 2: Matrix transpose

$U = V = \{2\times 2 \text{ matrices}\}$

$f: U\to V$, $f(A) = A^T$

In the basis $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$

f corresponds to $\begin{array}{c|cccc} & \begin{pmatrix}1&0\\0&0\end{pmatrix} & \begin{pmatrix}0&1\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\1&0\end{pmatrix} & \begin{pmatrix}0&0\\0&1\end{pmatrix} \\ \hline \begin{pmatrix}1&0\\0&0\end{pmatrix} & 1 & 0 & 0 & 0 \\ \begin{pmatrix}0&1\\0&0\end{pmatrix} & 0 & 0 & 1 & 0 \\ \begin{pmatrix}0&0\\1&0\end{pmatrix} & 0 & 1 & 0 & 0 \\ \begin{pmatrix}0&0\\0&1\end{pmatrix} & 0 & 0 & 0 & 1 \end{array}$

Observe: Order matters!

In the basis $\left\{\begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$,

f corresponds to $\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

In the basis $\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}$,

f corresponds to $\begin{pmatrix}1&&&0\\&1&&\\&&1&\\0&&&-1\end{pmatrix}$.

Remark: A basis in which a linear transformation acts diagonally is called an "eigenbasis".

CHANGING BASIS

Problems:

- Given a vector expressed in one basis, how do we express it in a different basis?
- Given a linear transformation expressed with respect to two bases, how do we express it in different bases?

Notation: $[f]_{\mathcal{B}_U, \mathcal{B}_V}$

denotes the matrix representation of $f: U \to V$ in the respective bases $\mathcal{B}_U, \mathcal{B}_V$.

Example: $Id(x,y) = (x,y) : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \mathcal{B} = \binom{1}{1}, \binom{1}{-1}$

$$[Id]_{Std} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad [Id]_{\mathcal{B}} = \begin{array}{c|cc} & \binom{1}{1} & \binom{1}{-1} \\ \hline \binom{1}{1} & 1 & 0 \\ \binom{1}{-1} & 0 & 1 \end{array}$$

$$[Id]_{\mathcal{B},S} = \begin{array}{c|cc} & \binom{1}{1} & \binom{1}{-1} \\ \hline \binom{1}{0} & 1 & 1 \\ \binom{0}{1} & 1 & -1 \end{array}_{2\times 2} \qquad [Id]_{S,\mathcal{B}} = \begin{array}{c|cc} & \binom{1}{0} & \binom{0}{1} \\ \hline \binom{1}{1} & \boxed{1/2} & 1/2 \\ \binom{1}{-1} & \boxed{1/2} & -1/2 \end{array}$$

$$\binom{1}{0} = \tfrac{1}{2}\binom{1}{1} + \tfrac{1}{2}\binom{1}{-1} \qquad \binom{0}{1} = \binom{1/2}{-1/2}_{\mathcal{B}}$$

$$\begin{tikzcd} \mathbb{R}^2_{S} \arrow[r, "Id"] \arrow[rr, bend right, "Id"'] & \mathbb{R}^2_{\mathcal{B}} \arrow[r, "Id"] & \mathbb{R}^2_{S} \end{tikzcd}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = [Id]_{\mathcal{B}S}\,[Id]_{S\mathcal{B}}$$

In general $\boxed{[Id]_{BC} = [Id]_{CB}^{-1}}$

For $f(x,y,z) = (x,y)$,

$$[f]_{\{e_1,e_2,e_3\},\{\vec{e}_1,\vec{e}_2\}} = \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 1 & 0 & 0 \\ e_2 & 0 & 1 & 0 \end{array}$$

$$[f]_{\{e_1,e_2,e_3\},\{u_1,u_2\}} = \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline u_1 & \frac{1}{2} & \frac{1}{2} & 0 \\ u_2 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}$$

$$= \begin{array}{c|cc} & e_1 & e_2 \\ \hline u_1 & \frac{1}{2} & \frac{1}{2} \\ u_2 & \frac{1}{2} & -\frac{1}{2} \end{array} \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 1 & 0 & 0 \\ e_2 & 0 & 1 & 0 \end{array}$$

$$\begin{tikzcd} U \text{ basis } \mathcal{B}_U \arrow[r, "f"] & V \text{ basis } \mathcal{B}_V \arrow[r, "\text{identity}"] & V \text{ basis } \mathcal{B}_V' \end{tikzcd}$$

$$[f]_{\mathcal{B}_U,\mathcal{B}_V'} = [\text{identity}]_{\mathcal{B}_V,\mathcal{B}_V'}\,[f]_{\mathcal{B}_U,\mathcal{B}_V}$$

You can similarly change the basis for U by computing $[\text{identity}]_{\mathcal{B}_U',\mathcal{B}_U}$:

$$\begin{tikzcd} U \text{ basis } \mathcal{B}_U' \arrow[r, "\text{identity}"] & U \text{ basis } \mathcal{B}_U \arrow[r, "f"] & V \text{ basis } \mathcal{B}_V \end{tikzcd}$$

$$[f]_{\mathcal{B}_U' \mathcal{B}_V} = [f]_{\mathcal{B}_U,\mathcal{B}_V}\,[\text{identity}]_{\mathcal{B}_U',\mathcal{B}_U}$$

Exercise: Consider the linear operator $A: \mathbb{R}^2 \to \mathbb{R}^2$ given by
$$A(x,y) = (-2y, 3x+2y).$$

a) Express A as a 2×2 matrix in the standard basis.

$$[A] = \begin{array}{c|cc} & \binom{1}{0} & \binom{0}{1} \\ \hline \binom{1}{0} & 0 & -2 \\ \binom{0}{1} & 3 & 2 \end{array} \qquad \begin{aligned} A(1,0) &= (0,3) \\ A(0,1) &= (-2,2) \end{aligned}$$

$$[A]\binom{x}{y} = \binom{-2y}{3x+2y} \checkmark$$

b) Let $\mathcal{B}$ be the basis $(1,1)$, $(-1,1)$.
Give the change-of-basis matrices for $\mathcal{B}$ to and from the standard basis, and use them to express A in the basis $\mathcal{B}$.

$$[Id]_{S\mathcal{B}} = [S \to \mathcal{B}] = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad [Id]_{\mathcal{B}S} = [\mathcal{B} \to S] = \begin{array}{c|cc} & \binom{1}{1} & \binom{-1}{1} \\ \hline \binom{1}{0} & 1 & -1 \\ \binom{0}{1} & 1 & 1 \end{array}$$

$$A(-1,1) = (-2,-1) = -\tfrac{3}{2}\binom{1}{1} + \tfrac{1}{2}\binom{-1}{1}$$

$$[A]_{\mathcal{B}} = [S \to \mathcal{B}]\,\underbrace{[A]_{SS}}_{= \begin{pmatrix} 0 & -2 \\ 3 & 2 \end{pmatrix}}\,[\mathcal{B} \to S]$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & \boxed{-3} \\ 7 & \boxed{1} \end{pmatrix}$$

$$[A]_B = [S\to B]\,\underset{\substack{\|\\ }}{[A]_{SS}}\,[B\to S]$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & -2\\ 3 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3 & \boxed{-3}\\ 7 & \boxed{1}\end{pmatrix} \checkmark$$

$$[B' \to B] = \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}}$$

$$[B \to B'] = [B' \to B]^{-1} = \boxed{\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}}$$

$\uparrow$check: $\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\vec{v}_1 - \frac{1}{\sqrt{2}}\vec{v}_2$ ✓

$$\Rightarrow [A]_{B'} = [B\to B'][A]_B[B'\to B]$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & -2\\ 3 & 2\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

$$= \boxed{\frac{1}{2}\begin{pmatrix}3 & -3\\ 7 & 1\end{pmatrix}}$$

Check:

$A\sqrt{2}\vec{v}_1 = A(1,1) = (-2,5)$

$[A]_{B'}\sqrt{2}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}3\\7\end{pmatrix}$ and $\frac{1}{\sqrt{2}}(3\vec{v}_1 + 7\vec{v}_2) = \left(\frac{3}{2}-\frac{7}{2}, \frac{3}{2}+\frac{7}{2}\right) = (-2,5)$ ✓

HARDER EXAMPLES

① Polynomial differentiation $f(p) = \frac{d}{dx}p$

$$f: \left\{\begin{matrix}\text{polynomials in } x\\ \text{of degree} \le 4\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{polynomials}\\ \text{of degree} \le 3\end{matrix}\right\}$$

5-dim $\qquad$ 4-dim

$1, x, x^2, x^3, x^4 \qquad 1, x, x^2, x^3$

$$[f] = \begin{array}{c|ccccc} & 1 & x & x^2 & x^3 & x^4 \\ \hline 1 & & & & & \\ x & & & & & \\ x^2 & & & & & \\ x^3 & & & & & \end{array}\!\!\!\begin{pmatrix}0 & 1 & 0 & 0 & 0\\ 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 0 & 4\end{pmatrix}_{4\times 5}$$

$f(1) = \frac{d}{dx}1 = 0 \qquad f(x) = \frac{d}{dx}x = 1 \qquad f(x^2) = 2x \qquad f(x^3) = 3x^2 \qquad f(x^4) = 4x^3$

Different bases $\Rightarrow$ different matrix!

Hermite polynomials

From Wikipedia, the free encyclopedia

In mathematics, the **Hermite polynomials** are a classical orthogonal polynomial sequence.

The polynomials arise in:

- probability, such as the Edgeworth series;
- in combinatorics, as an example of an Appell sequence, obeying the umbral calculus;
- in numerical analysis as Gaussian quadrature;
- in finite element methods as shape functions for beams;
- in physics, where they give rise to the eigenstates of the quantum harmonic oscillator;
- in systems theory in connection with nonlinear operations on Gaussian noise.

The first eleven probabilists' Hermite polynomials are:

$$\begin{aligned}
He_0(x) &= 1\\
He_1(x) &= x\\
He_2(x) &= x^2 - 1\\
He_3(x) &= x^3 - 3x\\
He_4(x) &= x^4 - 6x^2 + 3\\
He_5(x) &= x^5 - 10x^3 + 15x\\
He_6(x) &= x^6 - 15x^4 + 45x^2 - 15\\
He_7(x) &= x^7 - 21x^5 + 105x^3 - 105x\\
He_8(x) &= x^8 - 28x^6 + 210x^4 - 420x^2 + 105\\
He_9(x) &= x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x\\
He_{10}(x) &= x^{10} - 45x^8 + 630x^6 - 3150x^4 + 4725x^2 - 945
\end{aligned}$$

$$\underset{\substack{\text{degree} \le 4\\ \text{polynomials in } x}}{\mathcal{U}} \xrightarrow{f(p) = \frac{dp}{dx}} \underset{\substack{\text{degree} \le 3\\ \text{polynomials}\\ \text{in } x}}{V}$$

$\{1, x, x^2-1, x^3-3x, x^4-6x^2+3\}$ $\qquad$ $\{1, x, x^2, x^3\}$

Hermite polynomials

$$\left[\frac{d}{dx}\right]_{HS} = \begin{array}{c|ccccc} & 1 & x & x^2-1 & x^3-3x & x^4-6x^2+3 \\ \hline 1 & & & & & \\ x & & & & & \\ x^2 & & & & & \\ x^3 & & & & & \end{array}\!\!\!\begin{pmatrix}0 & 1 & 0 & -3 & 0\\ 0 & 0 & 2 & 0 & -12\\ 0 & 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 0 & 4\end{pmatrix}_{4\times 5}$$

$\frac{d}{dx}(x^2-1) = 2\cdot x \qquad \frac{d}{dx}(x^3-3x) = 3x^2-3$

$$\left[\frac{d}{dx}\right]_{HS} = \underset{\substack{\|\\ \begin{pmatrix}0 & 1 & & \\ & 0 & 2 & \\ & & 0 & 3 & \\ & & & 0 & 4\end{pmatrix}}}{\left[\frac{d}{dx}\right]_S}\,[H\to S] = \begin{array}{c|ccccc} & 1 & x & x^2-1 & x^3-3x & x^4-6x^2+3 \\ \hline 1 & & & & & \\ x & & & & & \\ x^2 & & & & & \\ x^3 & & & & & \\ x^4 & & & & & \end{array}\!\!\!\begin{pmatrix}1 & 0 & -1 & 0 & 3\\ 0 & 1 & 0 & -3 & 0\\ 0 & 0 & 1 & 0 & -6\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

$$\begin{array}{c|ccccc} x^2 & 0 & 0 & 1 & 0 & -6 \\ x^3 & 0 & 0 & 0 & 1 & 0 \\ x^4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$\left[\frac{d}{dx}\right]_H = [S \to H]\left[\frac{d}{dx}\right]_S [H \to S]$$

$$\underset{\substack{\text{degree} \le 4 \\ \text{polynomials in } x}}{\mathcal{U}} \xrightarrow{f(p) = \frac{dp}{dx}} \underset{\substack{\text{degree} \le 3 \\ \text{polynomials in } x}}{V}$$

$\{1, x, x^2-1, x^3-3x, x^4-6x^2+3\}$ $\{1, x, x^2-1, x^3-3x\}$

These alternative representations can also be computed using

$$[\text{identity}]_{B'_U B_U} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x & x^2-1 & x^3-3x & x^4-6x^2+1 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$[\text{identity}]_{B_V, B'_V} = \begin{array}{c} \\ 1 \\ x \\ x^2-1 \\ x^3-3x \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

since $x^3 = 3 \cdot x + 1 \cdot (x^3 - 3x)$

since $x^2 = 1 + (x^2 - 1)$

$$[f]_{B'_U, B'_V} = [\text{identity}]_{B_V, B'_V} \cdot [f]_{B_U, B_V} \cdot [\text{identity}]_{B'_U B_U}$$

$$\mathcal{U}_{B'_U} \xrightarrow{\text{identity}} \mathcal{U}_{B_U} \xrightarrow{f} V_{B_V} \xrightarrow{\text{identity}} V_{B'_V}$$

Example: $f(p) = p(2+3x)$: polynomials of degree ≤ 3

$$[f] = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 3 & 12 & 36 \\ 0 & 0 & 9 & 54 \\ 0 & 0 & 0 & 27 \end{pmatrix} \end{array}$$

$f(1) = 1$

$f(x) = 2 + 3x$

$f(x^2) = 4 + 12x + 9x^2$

$p(x) = x^2$ $p(2+3x) = 4 + 12x + 9x^2$

$f(x^3) = (2+3x)^3 = 8 + 36x + 54x^2 + 27x^3$

Example: $f(p) = (2+3x) \cdot p$: $\text{deg} \le 2 \longrightarrow \text{deg} \le 3$

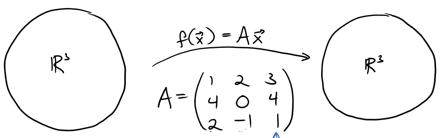
$$[f] = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{pmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}_{4 \times 3} \end{array} = 2 \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \\ & & \end{pmatrix}_{4 \times 3} + 3 \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

$f(x) = 2x + 3x^2$ $f(x^2) = 2x^2 + 3x^3$

If $f(\text{polynomial } p(x)) := p(x^2)$: $\text{deg} \le d \longrightarrow \text{deg} \le 2d$

input	output
1	1
x	x^2
x^2	x^4
x^3	x^6

$$\begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 0 & 0 \\ & & 1 & 0 \\ & & & 0 \\ & & & 1 \end{pmatrix}_{2d \times d} \end{array}$$



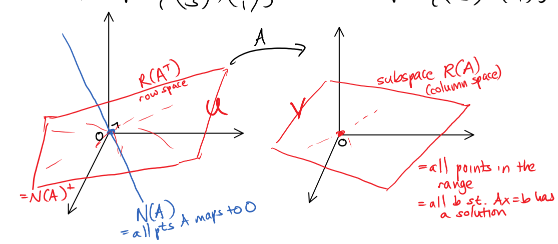
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 4 \\ 2 & -1 & 1 \end{pmatrix}$$

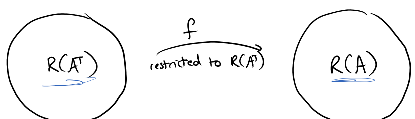
$$\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$\Rightarrow \text{Rank}(A) = 2$

$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ $\qquad$ $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

$\Rightarrow R(A^T) = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$ $\qquad$ $R(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}\right\}$





$\{(1,2,3), (1,0,1)\}$ $\qquad$ $\{(1,4,2), (2,0,-1)\}$

$$A\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 16 \\ 3 \end{pmatrix} = x\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + y\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$x = 4, \; y = 5$

$$A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 3 \end{pmatrix} = x\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + y\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$x = 2, \; y = 1$

Q: What is the matrix for this restricted map?

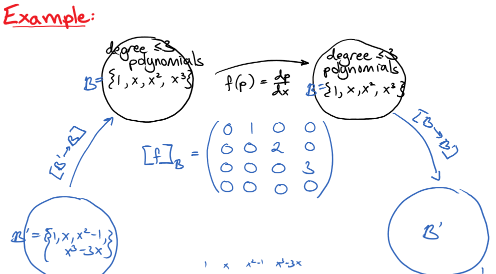
A: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 4 \\ 2 & -1 & 1 \end{pmatrix}$

$$\begin{array}{c|cc} & \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\ \hline \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} & 4 & 2 \\ \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} & 5 & 1 \end{array} \quad 2\times 2$$

$$\Rightarrow \begin{array}{c|cc} & (1,2,3) & (1,0,1) \\ \hline (1,4,2) & 4 & 2 \\ (2,0,-1) & 5 & 1 \end{array}$$

More Basis Change Examples

Example:



$$[f]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[\mathcal{B}' \to \mathcal{B}] = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2-1 & x^3-3x \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$[\mathcal{B} \to \mathcal{B}'] = \begin{array}{c} \\ 1 \\ x \\ x^2-1 \\ x^3-3x \end{array} \begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$\Rightarrow [f]_{\mathcal{B}'} = [\mathcal{B} \to \mathcal{B}'][f]_{\mathcal{B}}[\mathcal{B}' \to \mathcal{B}]$

```
>> A = [0 1 0 0 ; 0 0 2 0 ; 0 0 0 3 ; 0 0 0 0];
B = [1 0 1 0 ; 0 1 0 3 ; 0 0 1 0 ; 0 0 0 1];

inv(B) * A * B

ans =
```

$$\begin{array}{c|cccc} & 1 & x & x^2-1 & x^3-3x \\ \hline 1 & 0 & 1 & 0 & 0 \\ x & 0 & 0 & 2 & 0 \\ x^2-1 & 0 & 0 & 0 & 3 \\ x^3-3x & 0 & 0 & 0 & 0 \end{array}$$

Note: Any nonsingular matrix A can be thought of as a basis change. A^{-1} changes back.

$$A \begin{pmatrix} \text{linear} \\ \text{trans.} \\ \text{in std. basis} \end{pmatrix} A^{-1} = \begin{pmatrix} \text{same linear} \\ \text{trans. in the} \\ \text{new basis} \end{pmatrix}$$

Example: For

$$\mathcal{U} = \text{degree} \le 3 \text{ polynomials} \qquad B_{\mathcal{U}} = \{1, x, x^2, x^3\} \qquad B'_{\mathcal{U}} = \{1, x, x^2-1, x^3-3x\}$$

$$\mathcal{U} \xrightarrow{g} V$$

$$V = \text{degree} \le 4 \text{ polynomials} \qquad B_V = \{1, x, x^2, x^3, x^4\} \qquad B'_V = \{1, x, x^2-1, x^3-3x, x^4-6x^2+3\}$$

and $g(p) = (2+3x)\cdot p$

give the matrices $[g]_{B_V B_{\mathcal{U}}}, [g]_{B'_V B_{\mathcal{U}}}, [g]_{B_V B'_{\mathcal{U}}}, [g]_{B'_V, B'_{\mathcal{U}}}$.

$$[B'_{\mathcal{U}} \to B_{\mathcal{U}}] = \begin{array}{c|c} & \begin{matrix} 1 & x & x^2-1 & x^3-3x \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2 \\ x^3 \end{matrix} & \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$[g]_{B_{\mathcal{U}} \to B_V} = \begin{array}{c|c} & \begin{matrix} 1 & x & x^2 & x^3 \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{matrix} & \begin{pmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \end{array} \qquad [B_V \to B'_V] = \begin{array}{c|c} & \begin{matrix} 1 & x & x^2 & x^3 & x^4 \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2-1 \\ x^3-3x \\ x^4-6x^2+3 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$[g]_{B'_{\mathcal{U}} \to B_V} = [g]_{B_{\mathcal{U}} \to B_V} [B'_{\mathcal{U}} \to B_{\mathcal{U}}]$$

$$= \begin{array}{c|c} & \begin{matrix} 1 & x & x^2-1 & x^3-3x \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{matrix} & \begin{pmatrix} 2 & 0 & -2 & 0 \\ 3 & 2 & -3 & -6 \\ 0 & 3 & 2 & -9 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \end{array}$$

sanity check:

$(2+3x)(x^3-3x) = 3x^4+2x^3-9x^2-6x$ ✓

$$[g]_{B_{\mathcal{U}} \to B'_V} = [B_V \to B'_V] \cdot [g]_{B_{\mathcal{U}} \to B_V}$$

$$= \begin{array}{c|c} & \begin{matrix} 1 & x & x^2 & x^3 \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2-1 \\ x^3-3x \\ x^4-6x^2+3 \end{matrix} & \begin{pmatrix} 2 & 3 & 2 & 9 \\ 3 & 2 & 9 & 6 \\ 0 & 3 & 2 & 18 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \end{array}$$

sanity check:

$(2+3x)\cdot x = 3x^2+2x$

$= 3(x^2-1)+2x+3$ ✓

$$[g]_{B'_{\mathcal{U}} \to B'_V} = [B_V \to B'_V] \cdot [g]_{B_{\mathcal{U}} \to B_V} [B'_{\mathcal{U}} \to B_{\mathcal{U}}]$$

$$= \begin{array}{c|c} & \begin{matrix} 1 & x & x^2-1 & x^3-3x \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2-1 \\ x^3-3x \\ x^4-6x^2+3 \end{matrix} & \begin{pmatrix} 2 & 3 & 0 & 0 \\ 3 & 2 & 6 & 0 \\ 0 & 3 & 2 & 9 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \end{array}$$

sanity check:

$(2+3x)(x^3-3x) = 3x^4+2x^3-9x^2-6x$

$= 3(x^4-6x^2+3)+2(x^3-3x)+9(x^2-1)$ ✓

Example: Differentiation can also be considered as a map

$$\underbrace{\left\{ \begin{matrix} \text{polynomials in } x \\ \text{of degree} \le 4 \end{matrix} \right\}}_{\mathcal{U}} \longrightarrow \underbrace{\left\{ \begin{matrix} \text{polynomials in } x \\ \text{of degree} \le 4 \end{matrix} \right\}}_{\mathcal{U}}$$

basis $B = \{1, x, x^2, x^3, x^4\}$

$$[f]_B = \begin{array}{c|c} & \begin{matrix} 1 & x & x^2 & x^3 & x^4 \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{matrix} & \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 2 & & \\ & & 0 & 3 & \\ & 0 & & 0 & 4 \\ & & & & 0 \end{pmatrix} \end{array}$$

since, eg.,

$f(x^3) = 3x^2$

$= 0\cdot 1 + 0\cdot x + 3\cdot x^2$

$+ 0\cdot x^3 + 0\cdot x^4$

basis $H = \{1, x, x^2-1, x^3-3x, x^4-6x^2+3\}$

basis change matrices:

$$[\text{identity}]_{H \to B} = \begin{array}{c|c} & \begin{matrix} 1 & x & x^2-1 & x^3-3x & x^4-6x^2+3 \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{matrix} & \begin{pmatrix} 1 & 0 & -1 & 0 & 3 \\ & 1 & 0 & -3 & 0 \\ & & 1 & 0 & -6 \\ & 0 & & 1 & 0 \\ & & & & 1 \end{pmatrix} \end{array}$$

$$[\text{identity}]_{B \to H} = \begin{array}{c|c} & \begin{matrix} 1 & x & x^2 & x^3 & x^4 \end{matrix} \\ \begin{matrix} 1 \\ x \\ x^2-1 \\ x^3-3x \\ x^4-6x^2+3 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ & 1 & 0 & 3 & 0 \\ & & 1 & 0 & 6 \\ & 0 & & 1 & 0 \\ & & & & 1 \end{pmatrix} \end{array}$$

eg., $x^4 = 1\cdot(x^4-6x^2+3)$

$+6\cdot(x^2-1)$

$+3\cdot 1$

Observe: $[\text{identity}]_{B \to H} = \left([\text{identity}]_{H \to B}\right)^{-1}$

$$\text{idHtoB} = \begin{pmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \text{idBtoH} = \begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

```
idHtoB.idBtoH // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{[f]_H = [id]_{B \to H} [f]_B [id]_{H \to B}} = [id]^{-1}_{H \to B} [f]_B [id]_{H \to B}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}$$

$$fB = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

```
idBtoH.fB.idHtoB // MatrixForm
```

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for example,

$$\frac{d}{dx}(x^4-6x^2+3) = 4x^3-12x = 4\cdot(x^3-3x)$$

Note: Any nonsingular matrix A can be thought of as a basis change. A^{-1} changes back.

$$A \begin{pmatrix}\text{linear trans. in std. basis}\end{pmatrix} A^{-1} = \begin{pmatrix}\text{same linear trans. in the new basis}\end{pmatrix}$$

Haar Wavelet Bases

$\mathbb{R}^2$: $\frac{1}{\sqrt{2}}(1,1)$, $\frac{1}{\sqrt{2}}(1,-1)$

$\mathbb{R}^4$: $\frac{1}{2}(1,1,1,1)$, $\frac{1}{2}(1,1,-1,-1)$,
$\frac{1}{\sqrt{2}}(1,-1,0,0)$, $\frac{1}{\sqrt{2}}(0,0,1,-1)$

Note: These each have length 1, and are pairwise orthogonal!

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \tfrac{1}{2}(w+x+y+z)\cdot\tfrac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \tfrac{1}{2}(w+x-y-z)\cdot\tfrac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} + \tfrac{1}{\sqrt{2}}(w-x)\cdot\tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} + \tfrac{1}{\sqrt{2}}(y-z)\cdot\tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

$\mathbb{R}^8$:

$\frac{1}{\sqrt{8}}(1,1,1,1,1,1,1,1)$
$\frac{1}{\sqrt{8}}(1,1,1,1,-1,-1,-1,-1)$
$\frac{1}{2}(1,1,-1,-1,0,0,0,0)$
$\frac{1}{\sqrt{2}}(1,-1,0,0,0,0,0,0)$
$\frac{1}{\sqrt{2}}(0,0,1,-1,0,0,0,0)$
$\frac{1}{2}(0,0,0,0,1,1,-1,-1)$
$\frac{1}{\sqrt{2}}(0,0,0,0,1,-1,0,0)$
$\frac{1}{\sqrt{2}}(0,0,0,0,0,0,1,-1)$

Intuition: It comes from recursive downsampling:

$(\underline{a,b},\underline{c,d},\underline{e,f},\underline{g,h}) \in \mathbb{R}^8$

$\longrightarrow (\underbrace{a+b,\ c+d,\ e+f,\ g+h}_{\text{first } \frac{8}{2}=4 \text{ coordinates come from binning coords two at a time}},\ a-b,\ c-d,\ e-f,\ g-h)$

$\xrightarrow{\text{downsample 1st 4 coords.}} \left(a+b+c+d,\ e+f+g+h,\ \begin{matrix}a+b\\-c-d\end{matrix},\ \begin{matrix}e+f\\-g-h\end{matrix},\ \ldots\right)$

$\xrightarrow{\text{downsample again}} \left(\begin{matrix}a+b+c+d\\+e+f+g+h\end{matrix},\ \begin{matrix}a+b+c+d\\-e-f-g-h\end{matrix},\ \ldots\right)$

DONE! Up to scaling, this gives the above basis vectors.

Haar wavelets for arrays $\mathbb{R}^{2^k \times 2^k}$

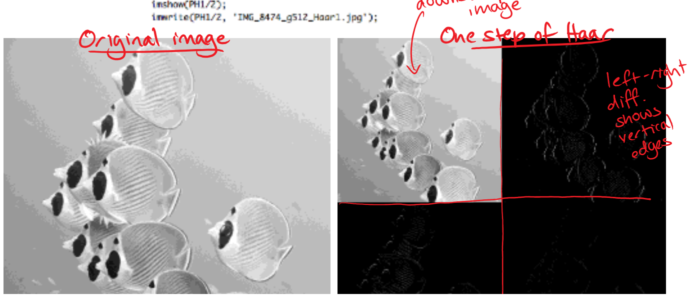
What if your vector gives the pixel values of a 2D image?
→ We should use a 2D version of the Haar basis.

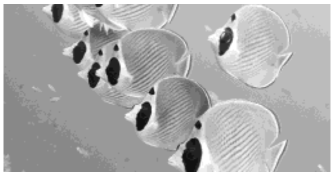
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{\begin{matrix}\text{average}\\ a+b+c+d\end{matrix}} & \boxed{\begin{matrix}\text{left-right diff.}\\ a+c-b-d\end{matrix}} \\ \boxed{\begin{matrix}\text{up-down difference}\\ a+b-c-d\end{matrix}} & \boxed{\begin{matrix}\text{cross difference}\\ a-b-c+d\end{matrix}} \end{pmatrix}$$

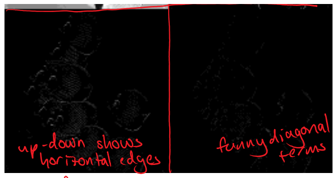
& recurse in upper-left quadrant

Example:

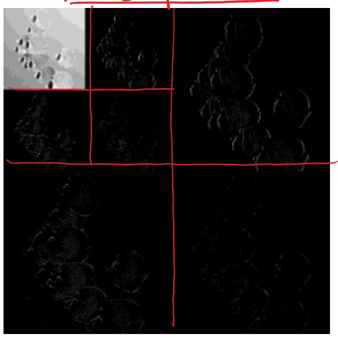
```
pkg load image;
P = double(imread('IMG_8474_g512.jpg'))/256;
PH1 = Haarstep(P, length(P));
imshow(PH1/2);
imwrite(PH1/2, 'IMG_8474_g512_Haar1.jpg');
```





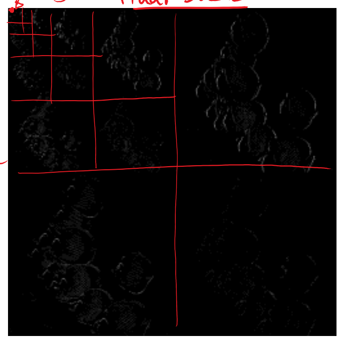


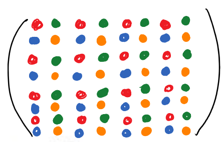
Two steps of Haar



average of everything!

Haar basis





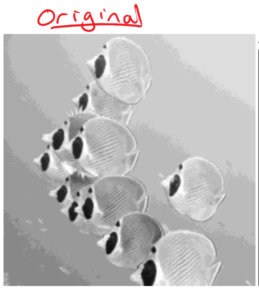
```
function p = Haarstep(p, n)
   oddodd   = p(1:2:n, 1:2:n);
   evenodd  = p(2:2:n, 1:2:n);
   oddeven  = p(1:2:n, 2:2:n);
   eveneven = p(2:2:n, 2:2:n);
   NW = (oddodd + evenodd + oddeven + eveneven)/2;   4
   NE = (oddodd + evenodd - oddeven - eveneven)/2;   1
   SW = (oddodd - evenodd + oddeven - eveneven)/2;   1
   SE = (oddodd - evenodd - oddeven + eveneven)/2;   1
   p(1:n/2,1:n/2) = NW;
   p(1:n/2,n/2+1:n) = NE;
   p(n/2+1:n,1:n/2) = SW;
   p(n/2+1:n,n/2+1:n) = SE;
end
```

```
function p = Haar(p)
  n = length(p);
  while (n > 1)
    p = Haarstep(p, n);
    n = n/2;
  end
end
```

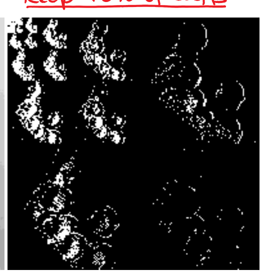
Nice property: Real-world images tend to have fairly sparse representations in the Haar basis (i.e., most Haar coordinates are close to 0).

Examples:

Original

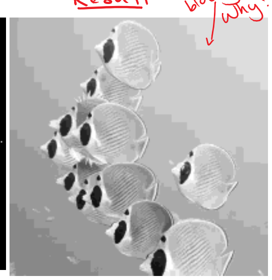


Keep 10% of coeffs



Result

notice the blocky gradients. Why?



Original

Keeping 10%

Keeping 20%

Why are gradients so blocky?

- Other bases perform better...

Code to change back from the Haar basis:

```
function p = undoHaarstep(p, n)
  NW = p(1:n/2,1:n/2);
  NE = p(1:n/2,n/2+1:n);
  SW = p(n/2+1:n,1:n/2);
  SE = p(n/2+1:n,n/2+1:n);
  oddodd = (NW + NE + SW + SE)/2;
  evenodd = (NW + NE - SW - SE)/2;
  oddeven = (NW - NE + SW - SE)/2;
  eveneven = (NW - NE - SW + SE)/2;
  p(1:2:n, 1:2:n) = oddodd;
  p(2:2:n, 1:2:n) = evenodd;
  p(1:2:n, 2:2:n) = oddeven;
  p(2:2:n, 2:2:n) = eveneven;
end

function p = undoHaar(p)
  n = 2;
  while (n <= length(p))
    p = undoHaarstep(p, n);
    n = n*2;
  end
end
```

Moral: Real-world signals are often nearly sparse in some known basis.

Application: Image compression

- Change basis
- Throw away the small components
- Only store/transmit the large coefficients

e.g., instead of $(x_1, x_2, x_3, X_4, x_5, x_6, x_7, x_8)$,

send $4, x_4$