Homework 3

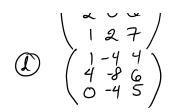
Note: For full credit, show your work! You are welcome to discuss the problems with others, but write up your own solutions.

- o Which of the following sets span R3?
 - @ {(1,1,1)}
 - (1,0,1), (1,0,1) (1,0,1) (1)
 - © {(1,0,0), (0,1,0), (0,0,1), (1,1,1)}
 - @ {(1,2,1), (2,0,-1), (4,4,1)}
 - @ {(1,2,1), (2,0,-1), (4,4,0)}
- 2 What is the span of the union {3×3 symmetric matrices,} U {3×3 upper-triangular}?

 i.e. A=AT
- 3 Let $S = \{s_1, ..., s_r\}$, $T = \{s_1, ..., s_r, s_{r+1}\}$ be two sets of vectors from the same vector space. When is Span(S) = Span(T)?
- 4) Find the rowspace, columnspace, nullspace and left nullspace for each of the following matrices. Express each answer as the span of a minimal set of vectors.

 (The left nullspace of a matrix A is N(AT).)
 - $\bigcirc
 \left(\begin{array}{cccc}
 (1 & 2 & 0 & 1 \\
 0 & 1 & 1 & 0 \\
 1 & 2 & 0 & 1
 \end{array}\right)$

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- (3) Construct a 3×3 matrix A with $R(A) = R(A^T) = Span(\{(1,0,1), (1,5,0)\})$

 $\begin{array}{c}
 \begin{bmatrix} A \\ A \end{bmatrix} \text{ and } \begin{bmatrix} A & A \\ A & A \end{bmatrix}
\end{array}$

- 3 Explain why a row of a motrix A cannot be in its nullspace.
- @ Matrix multiplication and the nullspace
- Prove that

 (*) N(A) SN(BA)

for any matrices A and B so BA is defined.

- (b) Prove that if B is invertible, then N(BA) = N(A)
- Prove that $N(A^TA) = N(A)$.

 (Lint: Equation (*) implies that $N(A^TA) \ge N(A)$, so it is enough to show the opposite inclusion, $N(A^TA) \subseteq N(A)$. Note that a vector \mathbf{z} is zero if and only if $\mathbf{z}^{\dagger}\mathbf{z} = \sum_{i} (\mathbf{z}_{i})^{2} = 0$.
- (a) [Optional] Assuming that $R(BT) \supseteq R(A)$, prove that N(BA) = N(A). Note that this implies parts (b) and (c) as special cases.

Note that this implies parts (and as special cases.

(Itint: This is slightly harder than (), but almost the same proof should work. Start by applying a row transformation to B.

As a partial converse, prove that if R(BT)CR(A) strictly, then N(BA) > N(A) strictly.

② Give an example of matrices A, B where

R(BT) ≠ R(A) and yet N(BA)=N(A).

(1tint: The smallest example is with A a 1×2)

matrix and B ~ 2×1 matrix.

9 Finite fields Use Gaussian elimination, by hand, to salve the following system of equations, mod 2:

$$x_1 + x_2 + x_3 = 0$$

 $x_2 + x_3 + x_4 = 1$
 $x_1 + x_3 = 1$
 $x_1 + x_2 = 1$
 $x_1 + x_2 = 1$

(a) Finite fields
(a) Show that the polynomial x4+1 is reducible (can be factored)

mod 2, mod 3, and mod 5.

(In fact, it is reducible over any finite field.)

Hint: Feel free to use Mothematica, as in class.

If you haven't installed it, try <a href="https://sandbox.open.wolframcloud.com/https://www.wolframalpha.com/https://www.wolframalpha.com/https://www.wolframalpha.com/

6) The polynomial x^3+x+1 is irreducible mod 2, so gives a field of size 2^2-8 . (The field elements are 0, 1, x, x+1, x^2 , x^2+1 , x^2+x , x^2+x+1 .) Write out the multiplication table. What are the inverses of the nonzero elements? $(\Gamma'=?, x^{-1}=?, (x+1)^{-1}=?, ..., (x^2+x+1)^{-1}=?)$

@ Recall from class the field #4 with elements 0,1,x,x+1 and operations mad 2 and mod x2+x+1. Here is the multiplication table: x 0' | x x + | 0 0 0 0 0 | 1 0 | x x + | x 0 x x + | | x | x + | 0 x + | | x | Write out all the elements of $Span\left(\begin{pmatrix} 1\\ \times\\ \times+1 \end{pmatrix},\begin{pmatrix} \times\\ \times\\ \times \end{pmatrix}\right)$ (Your answer should have 16 vectors, a subspace of TF3.) @ Over the same field F4, solve for the nullspace of

a) By discretizing the interval and setting up a system of livear equations, solve numerically the differential equation $f''(t) - 2f'(t) = \cos(t)$

on the interval [0,TT], with boundary conditions f(0)=0, f(T)=0

After setting up the sparse equations, solve them. Experiment with finer discretizations.

Show your work, and plot the results

b) Do the same, except with boundary conditions f(0) = 0, $f(\pi) = -1$