Homework 5

Linear transformations

1. a) Consider the linear transformation $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by

(*) $$f(x,y,z) = (3y,\ -x+y-2z,\ x-z).$$

Express f as a 3×3 matrix.

b) What is $f^{-1}(x,y,z)$?

2. Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that
   - First rotates the xz-plane by $\pi/3$ radians counterclockwise about the y-axis,
   - Then reflects everything about the yz-plane (i.e., switching the sign of the x coordinate),
   - Then rotates the xy-plane by $\pi/6$ radians counterclockwise about the z-axis.

   What is the 3×3 matrix representing f?
   What is the determinant of this matrix?

Changes of basis

3. Consider the 2×2 complex matrix

$$A = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Rewrite this matrix in the basis $\{(1,i),(1,-i)\}$.
Simplify your answer as much as possible.
(That is, give a 2×2 matrix that corresponds to the same linear transformation as A, but in the basis $\{(1,i),(1,-i)\}$ instead of the standard basis $\{e_1,e_2\}$.)

(4) Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 4 \\ 0 & 1 & 5 \end{pmatrix}.$$

A is a linear transformation on $\mathbb{R}^3$ (in the standard basis).

Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

$\mathcal{B}$ is another basis for $\mathbb{R}^3$.

Problem: Rewrite the matrix A in the basis $\mathcal{B}$.

(5) a) Consider the space of polynomials in x of degree at most 5. What is the dimension of this space? Give a simple basis for it.

b) Do the same for polynomials of degree at most 6.

c) Using the above bases, give the matrix that represents multiplication by $2+3x$.

d) Look up on Wikipedia Chebyshev polynomials of the first kind. Repeat parts (a), (b), (c) using these polynomials for your bases.

Check your answer with at least a simple sanity check — and show your work.

6 a) Give the dimension and a basis for

$$V = \{ x \in \mathbb{R}^n \mid x_1 + x_2 + x_3 + \cdots + x_n = 0 \}.$$

b) Consider the linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ given in the standard basis by the matrix

$$A = \begin{pmatrix} -2 & 1 & & & & & 1 \\ 1 & -2 & 1 & & & 0 & \\ & 1 & -2 & 1 & & & \\ & & & & & & \\ & 0 & & & \ddots & & \\ & & & & 1 & -2 & 1 \\ 1 & & & & & 1 & -2 \end{pmatrix}$$

(This matrix is tridiagonal except for the 1s in the upper right and lower left corners. As we saw in class, it comes from discretizing the second derivative operator on the interval where the boundaries wrap around.)

Argue that $R(A) \subseteq V$.

c) Therefore in particular $A(V) \subseteq V$. Let

$$B : V \to V$$

be the linear transformation that takes a vector in V and applies A to it (giving another vector in V).

Give the $(\dim V)$-by-$(\dim V)$ matrix for B using the basis you found in part (a).