EE 510 11/13/2020

Outlines

+ Eigenvalues & Eigenvectors
+ Determinants

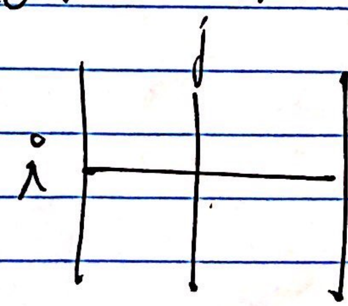
. $A \in \mathbb{R}^{n\times n}$; $P_A(\lambda) = \det(A - \lambda I)$ (Charactheristic Polynomial)

. $E_\lambda = N(A - \lambda I)$; eigenspace.

$\det(M) = ?$; $M = m_{ij}$

$$\det(M) = \sum_{i=1}^{n} m_{ij} \cdot C_{ij} \; ; \forall j \quad \Big(\underbrace{C_{ij}}_{\text{Cofactor}} = (-1)^{i+j} \underbrace{A_{ij}}_{\text{Minor}} \Big)$$

$$= \sum_{j=1}^{n} m_{ij} C_{ij} \, , \forall i$$



$$M^{-1} = \frac{1}{\det M} C^T \; ; \; C = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad ; \quad A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$M^{-1} = \frac{1}{\det(M)} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\det M = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

Cramer Rule:

$Ax = b$, A invertible : $A = \begin{pmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{pmatrix}$

$$x_i = \frac{\det(A_1, A_2, \ldots, \overset{i\text{th}}{b}, A_{i+1}, \ldots A_n)}{\det A}$$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 2 & 0 & 6 \end{pmatrix} ; \quad b = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} ; \quad \text{Solve. } Ax = b.$$

$$\det A = 1(-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = 2 \neq 0$$

$$x_1 = \frac{\begin{vmatrix} 1 & 0 & 2 \\ 5 & 1 & 2 \\ 3 & 0 & 6 \end{vmatrix}}{2} = \frac{\begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}}{2} = 0$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 5 & 6 \end{vmatrix}}{2} = \frac{8}{2} = 4$$

$$x_3 = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 5 \\ 2 & 0 & 3 \end{vmatrix}}{2} = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}}{2} = \frac{1}{2}$$

$$x^* = A^{-1} b = \begin{pmatrix} 0 \\ 4 \\ \frac{1}{2} \end{pmatrix}$$

• Exercise:

$A \in \mathbb{R}^{n \times n}$

λ is an e-value for A

① $B = A + cI$,

$$Ax = \lambda x \Rightarrow Bx = (A + cI)x$$
$$= Ax + cx = \lambda x + cx$$
$$= (\lambda + c)x$$

$\Rightarrow \lambda + c$ is an eigenvalue for B.

② A^{-1}: $\quad Ax = \lambda x$

$$\underbrace{A^{-1}A}_{I} x = \lambda A^{-1} x$$

$$\Rightarrow A^{-1}x = \frac{1}{\lambda} x$$

$\frac{1}{\lambda}$ is an e-value for A^{-1}.

③ Projection matrix P. $\left(P^T = P, \; P^2 = P\right)$.

$$Pe = \lambda e \Rightarrow P^2 e = \lambda Pe$$

$$\Rightarrow \lambda = \begin{cases} 0 \; ; N(P) \\ 1 \; ; R(P). \end{cases}$$

Exercise

$$A = UV^T \; ; \; U, V \in \mathbb{R}^n$$

1°) what is the rank of A

2°) what are the e-values and e-vectors of A

1- rank $A = 1$; $\sigma_1 = \|U\| \, \|V\|$

2°) $\dim N(A) = n-1 \Rightarrow P_A(\lambda) = \lambda^{n-1}(\lambda - \lambda_1)$

$E_0 = N(A)$

$Ax = \lambda_1 x \Rightarrow U\underbrace{V^T x} = \lambda_1 x$

$\Rightarrow V^T x \; U = \lambda_1 x$

$x = U \; , \; \lambda_1 = V^T U.$

$\boxed{tr(A) = \sum_{i=1}^{n} \lambda_i}$. $tr(A) = 0 + \lambda_1$

$\Rightarrow tr(A) = tr(UV^T) = tr(V^T U) = V^T U.$

$\boxed{\lambda_1 = V^T U}$

Exercises

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_n & 0 & \cdots & 0 \end{pmatrix}$$

1) rank(A) ~~?~~ rank(A) = 2, (two linearly independent)

~~2) Prove that the~~

$\dim N(A) = n-2$; $P_A(\lambda) = \lambda^{n-2}(\lambda-\lambda_1)(\lambda-\lambda_2)$

2) Prove that both eigenvalues have different sign.
(opposite)

$A' = BAB^{-1}$

$AX = \lambda V$; $A' B V = BAB^{-1}BV = BAV = B\lambda V$
$= \lambda BV$

$\Rightarrow$ BV is an e-vector for A' corresponding to the same e-value.

$$S = \left\{ \underset{v_1}{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \underbrace{v_3, \cdots, v_n}_{\text{basis for } N(A)} \right\}$$

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$$A' = \begin{array}{c c} & \begin{matrix} v_1 & v_2 & \cdots & v_n \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{matrix} & \begin{pmatrix} \begin{matrix} a_1 & 1 \\ \sum_{i=2}^{n} a_i^2 & 0 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$

AV_1 ? AV_2 ?

$$AV_1 = \begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & & \vdots \\ a_n & \cdots & \cdots \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} a_i^2 \\ a_1 a_2 \\ \vdots \\ a_1 a_n \end{pmatrix}$$

$$= a_1 \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} \sum_{i=2}^{n} a_i^2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = a_1 V_1 + \sum_{i=2}^{n} a_i^2 \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{V_2}$$

$$AV_2 = \begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & & \vdots \\ a_n & \cdots & \cdots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = V_1$$

$$A' = \begin{array}{c c} & \begin{matrix} v_1 & \cdots & v_n \end{matrix} \\ \begin{matrix} v_1 \\ \\ \vdots \\ v_n \end{matrix} & \begin{pmatrix} \begin{matrix} a_1 & 1 \\ \sum a_i^2 & 0 \end{matrix} & \cdots \\ \vdots & \vdots \end{pmatrix} \end{array} \Rightarrow \det \begin{vmatrix} a_1 & 1 \\ \sum_{i=2}^{n} a_i^2 & 0 \end{vmatrix} = \lambda_1 \lambda_2 = -\sum_{i=2}^{n} a_i^2 < 0$$

$\Rightarrow \lambda_1 \lambda_2 < 0 \Rightarrow \lambda_1$ and λ_2 have opposite sign.

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Exercise :

$$A = \begin{pmatrix} 1 & a & b & a \\ 1 & b & 1 & b \\ 1 & 1 & 1 & a \\ 1 & 1 & a & b \end{pmatrix}$$

$\det(A) = ?$

$$\det A = \begin{vmatrix} 1 & a & b & a \\ 0 & b-a & 1-b & b-a \\ 0 & 1-a & 1-b & 0 \\ 0 & 1-a & a-b & b-a \end{vmatrix}$$

$$= 1 \begin{vmatrix} b-a & 1-b & b-a \\ 1-a & 1-b & 0 \\ 1-a & a-b & b-a \end{vmatrix}$$

$$= (b-a) \begin{vmatrix} b-a & 1-b & 1 \\ 1-a & 1-b & 0 \\ 1-a & a-b & 1 \end{vmatrix}$$

$$= (b-a) \begin{vmatrix} b-a & 1-b & 1 \\ 1-a & 1-b & 0 \\ 1-b & a-1 & 0 \end{vmatrix} = (b-a) \times 1 \times (-1)^{1+3} \begin{vmatrix} 1-a & 1-b \\ 1-b & a-1 \end{vmatrix}$$

$$= (b-a)\left(-(a-1)^2 - (b-1)^2\right) = (a-b)\left((a-1)^2 + (b-1)^2\right)$$

Exercise

Let a_n, b_n and c_n sequences defined as follows.

$$\begin{cases} a_n = 2a_{n-1} + b_{n-1} - c_{n-1} \\ b_n = -a_{n-1} + 2b_{n-1} + c_{n-1} \\ c_n = -a_{n-1} + b_{n-1} + 2c_{n-1} \end{cases} \quad (a_0, b_0, c_0 \text{ given}).$$

Find a_n, b_n and c_n as a function of a_0, b_0, c_0.

$$X_n = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} ; \quad X_n = \underbrace{\begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{pmatrix}}_{X_{n-1}}$$

$$X_n = A X_{n-1}$$

By induction $X_n = A^n X_0$; $X_0 = \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}$

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}}_{U^{-1}}$$

$$\Rightarrow A^n = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} U^{-1}$$

$$X_n = A^n X_0 \Rightarrow \begin{cases} a_n = 2^n a_0 + (2^n - 1) b_0 + (1-2^n) c_0 \\ b_n = (2^n - 3^n) a_0 + 2^n b_0 + (3^n - 2^n) c_0 \\ c_n = (2^n - 3^n) a_0 + (2^n - 1) b_0 + (3^n - 2^n + 1) c_0 . \end{cases}$$

Exercise 1

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -2 & 0 & -2 \\ 3 & 3 & 5 \end{pmatrix}$$

1°) find $P_A(\lambda)$.

2) Prove that A is diagonalizable and find basis of eigenvectors

3) Check that $P_A(A) = 0$ and deduce A^{-1}.

1) $$P_A(\lambda) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ -2 & -\lambda & -2 \\ 3 & 3 & 5-\lambda \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 3-\lambda \\ -2 & -\lambda & -2 \\ 5-\lambda & 3 & 3 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 0 & 0 \\ -2 & 2-\lambda & 4-2\lambda \\ 5-\lambda & \lambda-2 & 3-(5-\lambda)(3-\lambda) \end{vmatrix} = -(\lambda-2) \begin{vmatrix} 1 & 0 & 0 \\ -2 & -1 & 4-2\lambda \\ 5-\lambda & 1 & 3-(5-\lambda)(3-\lambda) \end{vmatrix}$$

$$= (\lambda - 2)(-\lambda^2 + 6\lambda - 8) = -(\lambda-2)^2(\lambda-4)$$

The eivalues are 2 and 4

2) $1 \leq \dim E_\lambda \leq m_\lambda$ (multiplicity of λ)

$\dim E_4 = 1$ ✓

$\dim E_{\lambda_i} = m_{\lambda_i}; \forall \lambda_i \iff$ A is diagonalizable

$E_2 = N(A - 2I)$

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -2 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x + y + z = 0$

$E_2 = \{ (x, y, z) \in \mathbb{R}^n / x + y + z = 0 \}$.

$\Rightarrow \dim E_2 = 2$.

$\Rightarrow$ A is diagonalizable.

$(A - 2I) X = 0 \Rightarrow \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$(A - 4I) X = 0 \Rightarrow \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

3) $P_A(\lambda) = -(\lambda-2)^2(\lambda-4)$

$P_A(A) = -(A-2I)^2(A-4I) = 0$

A^{-1} ?!

$P_A(A) = -A^3 + 8A^2 - 20A + 16I = 0$

$\Rightarrow A^3 - 8A^2 + 20A = 16I$

$\Rightarrow A(A^2 - 8A + 20I) = 16I.$

$$A^{-1} = \frac{1}{16}(A^2 - 8A + 20I).$$

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