Lecture 10: Changes of basis

Admin: Reading

Contents

Outline: Change of basis, orthogonal bases
Projections and Gram-Schmidt orthogonalization

Important concepts:

1. Span
2. Linear independence
3. Basis — orthogonal basis, change of basis
4. Dimension

① Span$(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r\})$

$$= \left\{ \begin{array}{c} \text{all (finite) linear combinations} \\ \text{of those vectors} \\ \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r \end{array} \right\} = \underbrace{R}_{\text{range/column space}}\left(\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_r \\ | & | & & | \end{pmatrix} \right)$$

② $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is **linearly independent** if no vector $\vec{v}_j$ lies in the span of the others.

- equivalently, if the only solution to
$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r = \vec{0}$$
is $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$

- equivalently, if
$$\underbrace{N}_{\text{nullspace}}\left(\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_r \\ | & | & & | \end{pmatrix} \right) = \{\vec{0}\}$$

③ A **basis** for a vector space V is a set of vectors that
- spans V, and
- is linearly independent

Equivalently, it is a minimal set of vectors that spans V.

④ **Dimension** (a vector space)
$= \#$ of vectors in a basis

Two ways of checking linear independence:

① Gaussian elimination preserves the rowspace

① Gaussian elimination preserves the rowspace

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 0 & 9 & 3 \end{pmatrix} \xrightarrow{2-2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 9 & 3 \end{pmatrix} \xrightarrow{-3} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

3rd row dependent on first two!

② $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is linearly independent $\iff$ nullspace of $\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$ is $\{\vec{0}\}$.

To check if a set is linearly independent, compute the nullspace.

$A\vec{x} = \vec{0}$,

ie. $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 9 \\ 3 \end{pmatrix} x_3 = \vec{0}$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 9 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow{-1,\ -1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -3 & 9 \\ 0 & -1 & 3 \end{pmatrix} \xrightarrow{-1/3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x_3$ is a **free variable**

$\Rightarrow N(A) \neq \{\vec{0}\} \Rightarrow$ linearly dependent

LINEAR TRANSFORMATIONS
$\updownarrow$
MATRICES

$$\mathcal{U} \xrightarrow{f} \mathcal{V}$$

basis $\{\vec{u}_1, \ldots, \vec{u}_n\}$ basis $\{\vec{v}_1, \ldots, \vec{v}_m\}$

Observe:

- Any $\vec{u} \in \mathcal{U}$ is a linear comb. of the $\vec{u}_j$'s

$$f\Big(\sum_j \alpha_j \vec{u}_j\Big) = \sum_j \alpha_j f(\vec{u}_j)$$

$\Rightarrow f$ is determined by $f(\vec{u}_1), \ldots, f(\vec{u}_n)$

- Each $f(\vec{u}_j) \in \mathcal{V}$ is a linear comb. of the $\vec{v}_i$'s

$$f(\vec{u}_j) = \sum_{i=1}^{m} a_{ij} \vec{v}_i$$

$$\Rightarrow A = \begin{array}{c} \begin{array}{ccc} 1 & j & n \end{array} \\ \begin{pmatrix} & | & \\ & | & \\ & | & \end{pmatrix} \end{array}$$ determines f

$$\Rightarrow A = \begin{array}{c} \\ 1 \\ i \\ m \end{array}\!\!\begin{array}{c} 1 \quad\quad j \quad\quad n \\ \begin{pmatrix} & & & \\ & & a_{ij} & \\ & & & \end{pmatrix} \end{array} \quad \text{determines } f$$

More precisely, then,

Matrix + bases	$\longrightarrow$	Linear transformation
Linear transformation + bases	$\longrightarrow$	Matrix

Remark: If you use different bases, you'll get different matrices, for the **same** linear transformation.

EXAMPLES

Example 1: $f(x,y,z) = (x,y)$

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

- using the standard basis for both

$$[f] = \begin{array}{c} \vec{e}_1 \\ \vec{e}_2 \end{array}\!\!\begin{array}{c} \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

- using $\{(1,1),\ (1,-1)\}$ as the basis for $\mathbb{R}^2$

$$[f] = \begin{array}{c} \binom{1}{1} \\ \binom{1}{-1} \end{array}\!\!\begin{array}{c} \vec{e}_1 \quad \vec{e}_2 \quad \vec{e}_3 \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \end{array}$$

since $f(\vec{e}_1) = \binom{1}{0} = \frac{1}{2}\binom{1}{1} + \frac{1}{2}\binom{1}{-1}$

$f(\vec{e}_2) = \binom{0}{1} = \frac{1}{2}\binom{1}{1} - \frac{1}{2}\binom{1}{-1}$

$f(\vec{e}_3) = 0$

Example 2: Matrix transpose

$\mathcal{U} = \mathcal{V} = \{2 \times 2 \text{ matrices}\}$

$f : \mathcal{U} \to \mathcal{V}, \quad f(A) = A^T$

In the basis $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

f corresponds to

$$\begin{array}{c|cccc} & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & 1 & 0 & 0 & 0 \\ \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & 0 & 0 & 1 & 0 \\ \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) & 0 & 1 & 0 & 0 \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) & 0 & 0 & 0 & 1 \end{array}$$

Observe: Order matters!

In the basis $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$,

f corresponds to $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

In the basis $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$,

f corresponds to $\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}$.

Remark: A basis in which a linear transformation acts diagonally is called an "eigenbasis".

CHANGING BASIS

Problems:

- Given a vector expressed in one basis, how do we express it in a different basis?
- Given a linear transformation expressed with respect to two bases, how do we express it in different bases?

Notation: $[f]_{B_{\mathcal{U}}, B_{\mathcal{V}}}$

denotes the matrix representation of $f : \mathcal{U} \to \mathcal{V}$ in the respective bases $B_{\mathcal{U}}, B_{\mathcal{V}}$.

Example: $g(x,y) = (x,y)$

the identity $\mathbb{R}^2 \to \mathbb{R}^2$

In any basis $\{\vec{u}_1, \vec{u}_2\}$,

$$[g] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But for $\vec{u}_1 = (1,1)$, $\vec{u}_2 = (1,-1)$

$\vec{e}_1 = \frac{1}{2}(\vec{u}_1 + \vec{u}_2)$, $\vec{e}_2 = \frac{1}{2}(\vec{u}_1 - \vec{u}_2)$

$$[g]_{\{e_1,e_2\} \to \{\vec{u}_1,u_2\}} = \begin{array}{c c} & \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \end{array}$$

For $f(x,y,z) = (x,y)$,

$$[f]_{\{e_1,e_2,e_3\},\{\vec{e}_1,\vec{e}_2\}} = \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$[f]_{\{e_1,e_2,e_3\},\{u_1,u_2\}} = \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \end{array}$$

$$= \begin{array}{c c} & \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \end{array} \begin{array}{c c} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$\begin{tikzcd} \mathcal{U} \text{ basis } B_\mathcal{U} \arrow[r, "f"] & V \text{ basis } B_V \arrow[r, "\text{identity}"] & V \text{ basis } B'_V \end{tikzcd}$$

$$[f]_{B_\mathcal{U}, B'_V} = [\text{identity}]_{B_V, B'_V} \, [f]_{B_\mathcal{U}, B_V}$$

You can similarly change the basis for $\mathcal{U}$ by computing $[\text{identity}]_{B'_\mathcal{U}, B_\mathcal{U}}$:

$$\begin{tikzcd} \mathcal{U} \text{ basis } B'_\mathcal{U} \arrow[r, "\text{identity}"] & \mathcal{U} \text{ basis } B_\mathcal{U} \arrow[r, "f"] & V \text{ basis } B_V \end{tikzcd}$$

$$[f]_{B'_\mathcal{U} B_V} = [f]_{B_\mathcal{U}, B_V} \, [\text{identity}]_{B'_\mathcal{U}, B_\mathcal{U}}$$

__Exercise__: Consider the linear operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$A(x,y) = (-2y, 3x+2y).$$

a) Express A as a 2×2 matrix in the standard basis.

$$[A] = \begin{pmatrix} 0 & -2 \\ 3 & 2 \end{pmatrix}$$

b) Let $\mathcal{B}'$ be the basis $\underbrace{\tfrac{1}{\sqrt{2}}(1,1)}_{\vec{v}_1}, \underbrace{\tfrac{1}{\sqrt{2}}(-1,1)}_{\vec{v}_2}$.
Give the change-of-basis matrices for $\mathcal{B}'$ to and from the standard basis, and use them to express A in the basis $\mathcal{B}'$.

$$[\mathcal{B}' \to \mathcal{B}] = \boxed{\tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}$$

$$[\mathcal{B} \to \mathcal{B}'] = [\mathcal{B}' \to \mathcal{B}]^{-1} = \boxed{\tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}$$

↑check: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \vec{v}_1 - \tfrac{1}{\sqrt{2}} \vec{v}_2$ ✓

$$\Rightarrow [A]_{\mathcal{B}'} = [\mathcal{B} \to \mathcal{B}'] [A]_{\mathcal{B}} [\mathcal{B}' \to \mathcal{B}]$$
$$= \tfrac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$= \boxed{\tfrac{1}{2} \begin{pmatrix} 3 & -3 \\ 7 & 1 \end{pmatrix}}$$

Check:

$$A \sqrt{2} \vec{v}_1 = A(1,1) = (-2, 5)$$

$$[A]_{\mathcal{B}'} \sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 3 \\ 7 \end{pmatrix} \quad \text{and} \quad \tfrac{1}{\sqrt{2}}(3\vec{v}_1 + 7\vec{v}_2) = \left(\tfrac{3}{2} - \tfrac{7}{2}, \tfrac{3}{2} + \tfrac{7}{2}\right) = (-2, 5) \ ✓$$

HARDER EXAMPLES

① Polynomial differentiation $f(p) = \frac{d}{dx} p$

$$f : \begin{Bmatrix} \text{polynomials in } x \\ \text{of degree} \le 4 \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{polynomials} \\ \text{of degree} \le 3 \end{Bmatrix}$$

$= \mathcal{U}$ 5-dim $= V$ 4-dim.

$B_{\mathcal{U}} = \{1, x, x^2, x^3, x^4\}$ $B_V = \{1, x, x^2, x^3\}$

The matrix for f is

$$\begin{array}{c|c} & \begin{array}{ccccc} 1 & x & x^2 & x^3 & x^4 \end{array} \\ \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{array}$$

← basis for $\mathcal{U}$

basis for V

since $f(1) = 0$ $f(x) = 1$ $f(x^2) = 2x$ $f(x^3) = 3x^2$ $f(x^4) = 4x^3$

Different bases ⟹ different matrix!

Hermite polynomials

From Wikipedia, the free encyclopedia

In mathematics, the **Hermite polynomials** are a classical orthogonal polynomial sequence.

The polynomials arise in:

- probability, such as the Edgeworth series;
- in combinatorics, as an example of an Appell sequence, obeying the umbral calculus;
- in numerical analysis as Gaussian quadrature;
- in finite element methods as shape functions for beams;
- in physics, where they give rise to the eigenstates of the quantum harmonic oscillator;
- in systems theory in connection with nonlinear operations on Gaussian noise.

The first eleven probabilists' Hermite polynomials are:

$$\begin{aligned}
He_0(x) &= 1 \\
He_1(x) &= x \\
He_2(x) &= x^2 - 1 \\
He_3(x) &= x^3 - 3x \\
He_4(x) &= x^4 - 6x^2 + 3 \\
He_5(x) &= x^5 - 10x^3 + 15x \\
He_6(x) &= x^6 - 15x^4 + 45x^2 - 15 \\
He_7(x) &= x^7 - 21x^5 + 105x^3 - 105x \\
He_8(x) &= x^8 - 28x^6 + 210x^4 - 420x^2 + 105 \\
He_9(x) &= x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x \\
He_{10}(x) &= x^{10} - 45x^8 + 630x^6 - 3150x^4 + 4725x^2 - 945
\end{aligned}$$

$$\begin{array}{c} \mathcal{U} \\ \text{degree} \le 4 \\ \text{polynomials in } x \end{array} \xrightarrow{f(p) = \frac{dp}{dx}} \begin{array}{c} V \\ \text{degree} \le 3 \\ \text{polynomials} \\ \text{in } x \end{array}$$

$\{1, x, x^2 - 1, x^3 - 3x, x^4 - 6x^2 + 3\}$ $\{1, x, x^2, x^3\}$

Hermite polynomials

$$\begin{array}{c|c} & \begin{array}{ccccc} 1 & x & x^2-1 & x^3-3x & x^4-6x^2+3 \end{array} \\ \begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \end{array} & \begin{pmatrix} 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 2 & 0 & -12 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{array}$$

$$
\begin{tikzcd}
\mathcal{U}\ \text{degree} \le 4 \text{ polynomials in } x \arrow[r, "f(p) = \frac{dp}{dx}"] & V\ \text{degree} \le 3 \text{ polynomials in } x
\end{tikzcd}
$$

$\{1,\ x,\ x^2-1,\ x^3-3x,\ x^4-6x^2+3\}$ $\qquad$ $\{1,\ x,\ x^2-1,\ x^3-3x\}$

$$
\begin{aligned}
f(1) &= 0 \\
f(x) &= 1 \\
f(x^2-1) &= 2x \\
f(x^3-3x) &= 3x^2-3 \\
&= 3(x^2-1) \\
f(x^4-6x^2+3) &= 4x^3-12x \\
&= 4(x^3-3x)
\end{aligned}
\qquad
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 4
\end{pmatrix}
$$

These alternative representations can also be computed using

$$
[\text{identity}]_{B_U' B_U} =
\begin{array}{c}
\\ 1 \\ x \\ x^2 \\ x^3 \\ x^4
\end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & x & x^2-1 & x^3-3x & x^4-6x^2+1 \end{array} \\
\begin{pmatrix}
1 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & -3 & 0 \\
0 & 0 & 1 & 0 & -6 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

$$
[\text{identity}]_{B_V, B_V'} =
\begin{array}{c}
\\ 1 \\ x \\ x^2-1 \\ x^3-3x
\end{array}
\begin{array}{c}
\begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

since $x^3 = 3 \cdot x + 1 \cdot (x^3-3x)$

since $x^2 = 1 + (x^2-1)$

$$
[f]_{B_U', B_V'} = [\text{identity}]_{B_V, B_V'} \cdot [f]_{B_U, B_V} \cdot [\text{identity}]_{B_U' B_U}
$$

$$
\begin{tikzcd}
\mathcal{U}_{B_U'} \arrow[r, "\text{identity}"] & \mathcal{U}_{B_U} \arrow[r, "f"] & V_{B_V} \arrow[r, "\text{identity}"] & V_{B_V'}
\end{tikzcd}
$$

Example 2: Rowspace to columnspace

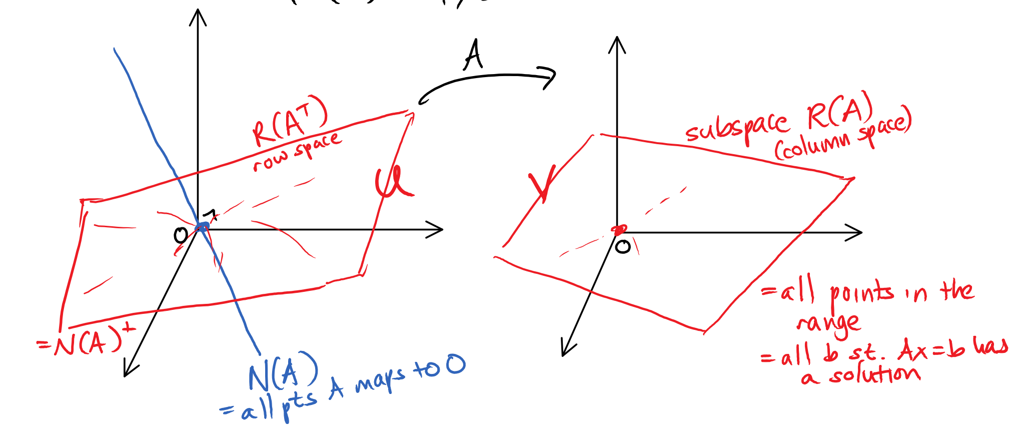
$$
\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "f(\vec{x}) = A\vec{x}"] & \mathbb{R}^3
\end{tikzcd}
\qquad
A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 4 \\ 2 & -1 & 1 \end{pmatrix}
$$

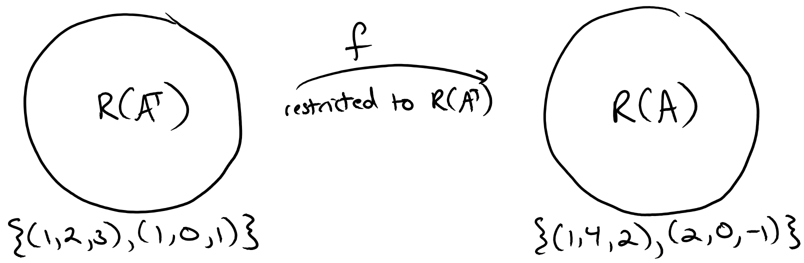
$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ $\qquad$ $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

$\Rightarrow R(A^T) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ $\qquad$ $R(A) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\}$

$\uparrow$

$$\Rightarrow R(A^T) = \text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\} \qquad R(A) = \text{Span}\left\{\begin{pmatrix}1\\4\\2\end{pmatrix}, \begin{pmatrix}2\\0\\-1\end{pmatrix}\right\}$$





Q: What is the matrix for this restricted map?

A:

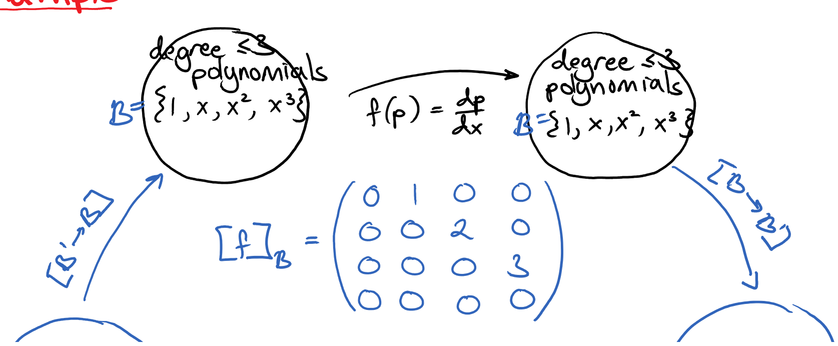
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 4 \\ 2 & -1 & 1 \end{pmatrix}$$

$$f(1,2,3) = (14, 16, 3)$$
$$= 4 \cdot (1,4,2) + 5 \cdot (2,0,-1)$$
$$f(1,0,1) = (4, 8, 3)$$
$$= 2 \cdot (1,4,2) + 1 \cdot (2,0,-1)$$

$$\Rightarrow \begin{array}{c c} & \begin{array}{cc} (1,2,3) & (1,0,1) \end{array} \\ \begin{array}{c} (1,4,2) \\ (2,0,-1) \end{array} & \begin{pmatrix} 4 & 2 \\ 5 & 1 \end{pmatrix} \end{array}$$

More Basis Change Examples

Example:



$$[f]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$[T]_B$ $\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$B' = \{1, x, x^2-1, x^3-3x\}$

B'

$$[B' \to B] = \begin{array}{c|cccc} & 1 & x & x^2-1 & x^3-3x \\ \hline 1 & 1 & 0 & -1 & 0 \\ x & 0 & 1 & 0 & -3 \\ x^2 & 0 & 0 & 1 & 0 \\ x^3 & 0 & 0 & 0 & 1 \end{array}$$

$$[B \to B'] = \begin{array}{c|cccc} & 1 & x & x^2 & x^3 \\ \hline 1 & 1 & 0 & 1 & 0 \\ x & 0 & 1 & 0 & 3 \\ x^2-1 & 0 & 0 & 1 & 0 \\ x^3-3x & 0 & 0 & 0 & 1 \end{array}$$

$$\Rightarrow [f]_{B'} = [B \to B'][f]_B[B' \to B]$$

```
>> A = [0 1 0 0 ; 0 0 2 0 ; 0 0 0 3 ; 0 0 0 0];
B = [1 0 1 0 ; 0 1 0 3 ; 0 0 1 0 ; 0 0 0 1];

inv(B) * A * B
```

= ans =

$$\begin{array}{c|cccc} & 1 & x & x^2-1 & x^3-3x \\ \hline 1 & 0 & 1 & 0 & 0 \\ x & 0 & 0 & 2 & 0 \\ x^2-1 & 0 & 0 & 0 & 3 \\ x^3-3x & 0 & 0 & 0 & 0 \end{array}$$

Note: Any nonsingular matrix A can be thought of as a basis change. A^{-1} changes back.

$$A \begin{pmatrix} \text{linear} \\ \text{trans.} \\ \text{in std. basis} \end{pmatrix} A^{-1} = \begin{pmatrix} \text{same linear} \\ \text{trans. in the} \\ \text{new basis} \end{pmatrix}$$

Example: For

$\mathcal{U}$ = degree ≤ 3 polynomials, $B_u = \{1, x, x^2, x^3\}$, $B_u' = \{1, x, x^2-1, x^3-3x\}$

g

V = degree ≤ 4 polynomials, $B_V = \{1, x, x^2, x^3, x^4\}$, $B_V' = \{1, x, x^2-1, x^3-3x, x^4-6x^2+3\}$

and $g(p) = (2+3x)\cdot p$

give the matrices $[g]_{B_V B_u}$, $[g]_{B_V' B_u}$, $[g]_{B_V B_u'}$, $[g]_{B_V', B_u'}$.

$$[B_u' \to B_u] = \begin{array}{c|cccc} & 1 & x & x^2-1 & x^3-3x \\ \hline 1 & 1 & 0 & -1 & 0 \\ x & 0 & 1 & 0 & -3 \\ x^2 & 0 & 0 & 1 & 0 \\ x^3 & 0 & 0 & 0 & 1 \end{array}$$

$$[g]_{B_u \to B_V} = \begin{array}{c|cccc} & 1 & x & x^2 & x^3 \\ \hline 1 & 2 & 0 & 0 & 0 \\ x & 3 & 2 & 0 & 0 \\ x^2 & 0 & 3 & 2 & 0 \\ x^3 & 0 & 0 & 3 & 2 \\ x^4 & 0 & 0 & 0 & 3 \end{array}$$

$$[B_V \to B_V'] = \begin{array}{c|ccccc} & 1 & x & x^2 & x^3 & x^4 \\ \hline 1 & 1 & 0 & 1 & 0 & 3 \\ x & 0 & 1 & 0 & 3 & 0 \\ x^2-1 & 0 & 0 & 1 & 0 & 6 \\ x^3-3x & 0 & 0 & 0 & 1 & 0 \\ x^4-6x^2+3 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$[g]_{B'_u \to B_v} = [g]_{B_u \to B_v} [B'_u \to B_u]$$

$$= \begin{array}{c|ccccc} & 1 & x & x^2-1 & x^3-3x \\ \hline 1 & 2 & 0 & -2 & 0 \\ x & 3 & 2 & -3 & -6 \\ x^2 & 0 & 3 & 2 & -9 \\ x^3 & 0 & 0 & 3 & 2 \\ x^4 & 0 & 0 & 0 & 3 \end{array}$$

sanity check:
$(2+3x)(x^3-3x) = 3x^4+2x^3-9x^2-6x$ ✓

$$[g]_{B_u \to B'_v} = [B_v \to B'_v] \cdot [g]_{B_u \to B_v}$$

$$= \begin{array}{c|cccc} & 1 & x & x^2 & x^3 \\ \hline 1 & 2 & 3 & 2 & 9 \\ x & 3 & 2 & 9 & 6 \\ x^2-1 & 0 & 3 & 2 & 18 \\ x^3-3x & 0 & 0 & 3 & 2 \\ x^4-6x^2+3 & 0 & 0 & 0 & 3 \end{array}$$

sanity check:
$(2+3x) \cdot x = 3x^2+2x$
$= 3(x^2-1)+2x+3$ ✓

$$[g]_{B'_u \to B'_v} = [B_v \to B'_v] \cdot [g]_{B_u \to B_v} [B'_u \to B_u]$$

$$= \begin{array}{c|cccc} & 1 & x & x^2-1 & x^3-3x \\ \hline 1 & 2 & 3 & 0 & 0 \\ x & 3 & 2 & 6 & 0 \\ x^2-1 & 0 & 3 & 2 & 9 \\ x^3-3x & 0 & 0 & 3 & 2 \\ x^4-6x^2+3 & 0 & 0 & 0 & 3 \end{array}$$

sanity check:
$(2+3x)(x^3-3x) = 3x^4+2x^3-9x^2-6x$
$= 3(x^4-6x^2+3)+2(x^3-3x)+9(x^2-1)$ ✓

Example: Differentiation can also be considered as a map

$$\underbrace{\left\{ \begin{array}{c} \text{polynomials in } x \\ \text{of degree} \le 4 \end{array} \right\}}_{= \mathcal{U}} \longrightarrow \underbrace{\left\{ \begin{array}{c} \text{polynomials in } x \\ \text{of degree} \le 4 \end{array} \right\}}_{= \mathcal{U}}$$

basis $B = \{1, x, x^2, x^3, x^4\}$

$$[f]_B = \begin{array}{c|ccccc} & 1 & x & x^2 & x^3 & x^4 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ x & 0 & 0 & 2 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 3 & 0 \\ x^3 & 0 & 0 & 0 & 0 & 4 \\ x^4 & 0 & 0 & 0 & 0 & 0 \end{array}$$

since, eg.,
$f(x^3) = 3x^2$
$= 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4$

basis $H = \{1, x, x^2-1, x^3-3x, x^4-6x^2+3\}$

basis change matrices:

$$[\text{identity}]_{H \to B} = \begin{array}{c|ccccc} & 1 & x & x^2-1 & x^3-3x & x^4-6x^2+3 \\ \hline 1 & 1 & 0 & -1 & 0 & 3 \\ x & 0 & 1 & 0 & -3 & 0 \\ x^2 & 0 & 0 & 1 & 0 & -6 \\ x^3 & 0 & 0 & 0 & 1 & 0 \\ x^4 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{c} x \\ x^3 \\ x^4 \end{array}\begin{pmatrix} & & \vdots \\ 0 & & 0 \\ & & 1 \end{pmatrix}$$

$$[\text{identity}]_{B\to H} = \begin{array}{r} \\ 1 \\ x \\ x^2-1 \\ x^3-3x \\ x^4-6x^2+3 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & x & x^2 & x^3 & x^4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ & 1 & 0 & 3 & 0 \\ & & 1 & 0 & 6 \\ & 0 & & 1 & 0 \\ & & & & 1 \end{pmatrix} \end{array}$$

eg., $x^4 = 1\cdot(x^4-6x^2+3)$
$+6\cdot(x^2-1)$
$+3\cdot 1$

Observe: $[\text{identity}]_{B\to H} = \left([\text{identity}]_{H\to B}\right)^{-1}$

```
idHtoB = (1 0 -1 0  3
          0 1  0 -3 0
          0 0  1  0 -6
          0 0  0  1 0
          0 0  0  0 1); idBtoH = (1 0 1 0 3
                                  0 1 0 3 0
                                  0 0 1 0 6
                                  0 0 0 1 0
                                  0 0 0 0 1);

idHtoB.idBtoH // MatrixForm
```

trixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{[f]_H = [id]_{B\to H}\,[f]_B\,[id]_{H\to B}} = [id]^{-1}_{H\to B}\,[f]_B\,[id]_{H\to B}$$

```
fB = (0 1 0 0 0
      0 0 2 0 0
      0 0 0 3 0
      0 0 0 0 4
      0 0 0 0 0);

idBtoH.fB.idHtoB // MatrixForm
```

trixForm=

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for example,
$\frac{d}{dx}(x^4-6x^2+3) = 4x^3-12x$
$= 4\cdot(x^3-3x)$

Note: Any nonsingular matrix A can be thought of as a basis change. A^{-1} changes back.

$$A\begin{pmatrix}\text{linear} \\ \text{trans.} \\ \text{in std. basis}\end{pmatrix}A^{-1} = \begin{pmatrix}\text{same linear} \\ \text{trans. in the} \\ \text{new basis}\end{pmatrix}$$

Haar Wavelet Bases

$\mathbb{R}^2$: $\frac{1}{\sqrt{2}}(1,1)$, $\frac{1}{\sqrt{2}}(1,-1)$

$\mathbb{R}^4$: $\frac{1}{2}(1,1,1,1)$, $\frac{1}{2}(1,1,-1,-1)$,
$\frac{1}{\sqrt{2}}(1,-1,0,0)$, $\frac{1}{\sqrt{2}}(0,0,1,-1)$

Note: These each have length 1, and are pairwise orthogonal!

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \tfrac{1}{2}(w+x+y+z)\cdot\tfrac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \tfrac{1}{2}(w+x-y-z)\cdot\tfrac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$$

$$+\tfrac{1}{\sqrt{2}}(w-x)\cdot\tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\\0\end{pmatrix} + \tfrac{1}{\sqrt{2}}(y-z)\cdot\tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\-1\end{pmatrix}$$

$\mathbb{R}^8$:

$$\tfrac{1}{\sqrt{8}}(1,1,1,1,1,1,1,1)$$
$$\tfrac{1}{\sqrt{8}}(1,1,1,1,-1,-1,-1,-1)$$
$$\tfrac{1}{2}(1,1,-1,-1,0,0,0,0)$$
$$\tfrac{1}{\sqrt{2}}(1,-1,0,0,0,0,0,0)$$
$$\tfrac{1}{\sqrt{2}}(0,0,1,-1,0,0,0,0)$$
$$\tfrac{1}{2}(0,0,0,0,1,1,-1,-1)$$
$$\tfrac{1}{\sqrt{2}}(0,0,0,0,1,-1,0,0)$$
$$\tfrac{1}{\sqrt{2}}(0,0,0,0,0,0,1,-1)$$

Intuition: It comes from recursive downsampling:

$$(\underline{a,b},\underline{c,d},\underline{e,f},\underline{g,h}) \in \mathbb{R}^8$$

$$\to (\underbrace{a+b,\ c+d,\ e+f,\ g+h}_{\text{first } \frac{8}{2}=4 \text{ coordinates come from binning coords two at a time}},\ a-b,\ c-d,\ e-f,\ g-h)$$

$$\xrightarrow{\text{downsample } 1^{st}\ 4 \text{ coords.}} \left(a+b+c+d,\ e+f+g+h,\ \begin{matrix}a+b\\-c-d\end{matrix},\ \begin{matrix}e+f\\-g-h\end{matrix},\ \ldots\right)$$

$$\xrightarrow{\text{downsample again}} \left(\begin{matrix}a+b+c+d\\+e+f+g+h\end{matrix},\ \begin{matrix}a+b+c+d\\-e-f-g-h\end{matrix},\ \ldots\right)$$

DONE! Up to scaling, this gives the above basis vectors.

Haar wavelets for arrays $\mathbb{R}^{2^k \times 2^k}$

What if your vector gives the pixel values of a 2D image?

→ We should use a 2D version of the Haar basis.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{\begin{matrix}\text{average}\\ a+b+c+d\end{matrix}} & \boxed{\begin{matrix}\text{left-right diff.}\\ a+c-b-d\end{matrix}} \\ \boxed{\begin{matrix}\text{up-down difference}\\ a+b-c-d\end{matrix}} & \boxed{\begin{matrix}\text{cross difference}\\ a-b-c+d\end{matrix}} \end{pmatrix}$$

& recurse in upper-left quadrant

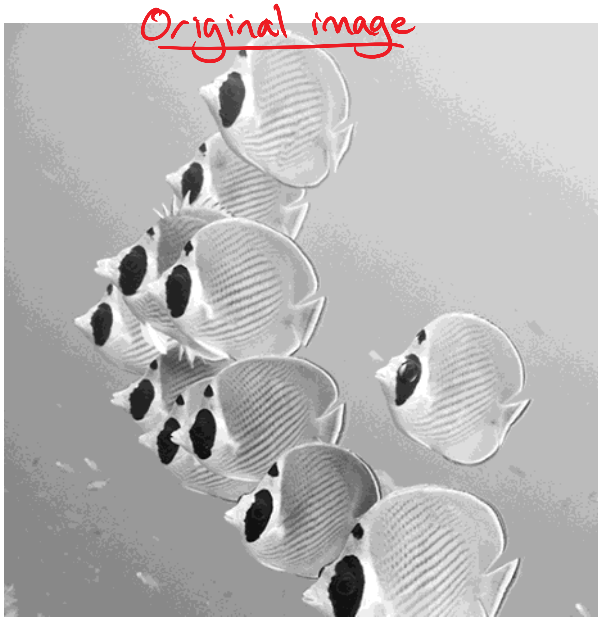
Example:

```
pkg load image;
P = double(imread('IMG_8474_g512.jpg'))/256;
```

Example:

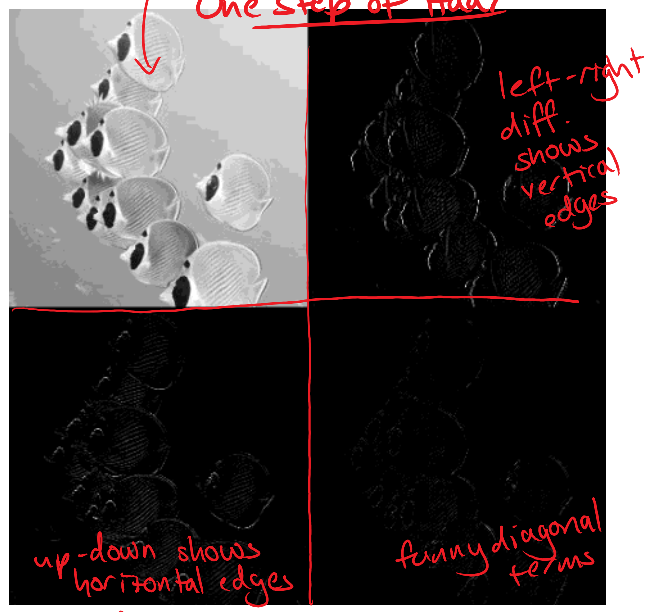
```
pkg load image;
P = double(imread('IMG_8474_g512.jpg'))/256;
PH1 = Haarstep(P, length(P));
imshow(PH1/2);
imwrite(PH1/2, 'IMG_8474_g512_Haar1.jpg');
```

Original image

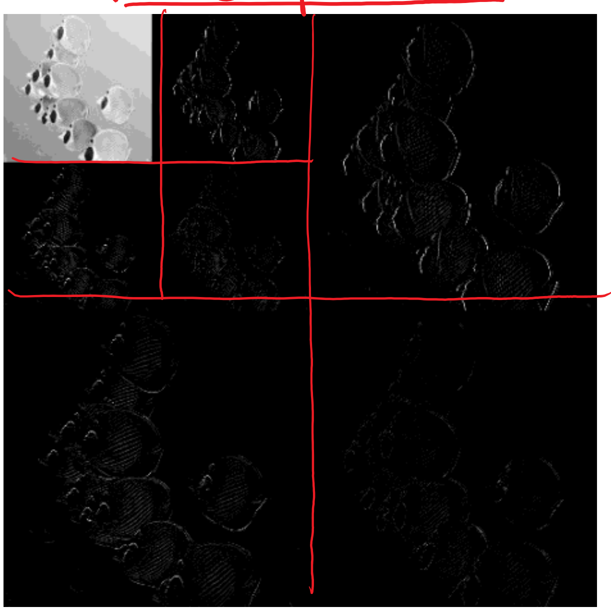


downsampled image

One step of Haar

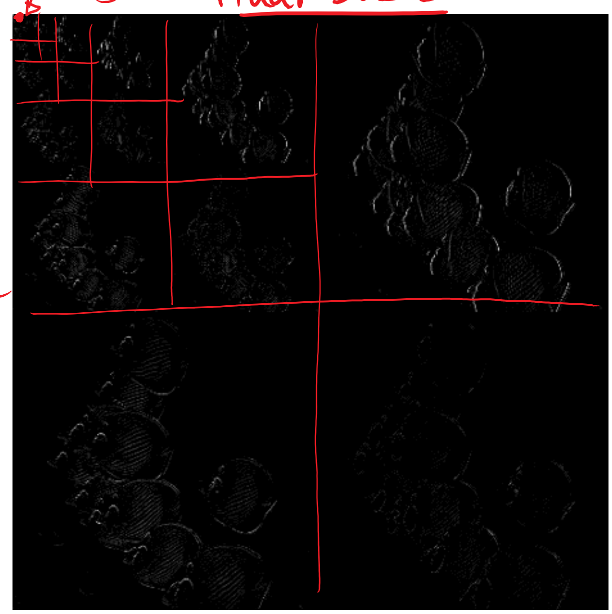


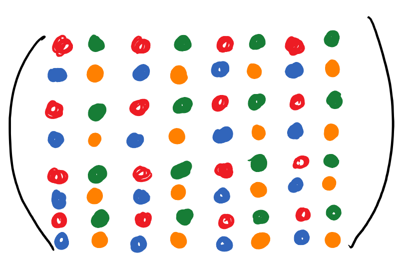
Two steps of Haar



average of everything!

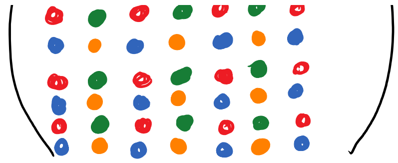
Haar basis





```
function p = Haarstep(p, n)
  oddodd   = p(1:2:n, 1:2:n);
  evenodd  = p(2:2:n, 1:2:n);
  oddeven  = p(1:2:n, 2:2:n);
  eveneven = p(2:2:n, 2:2:n);
  NW = (oddodd + evenodd + oddeven + eveneven)/2;  4
  NE = (oddodd + evenodd - oddeven - eveneven)/2;  1
  SW = (oddodd - evenodd + oddeven - eveneven)/2;  1
  SE = (oddodd - evenodd - oddeven + eveneven)/2;  1
  p(1:n/2,1:n/2) = NW;
  p(1:n/2,n/2+1:n) = NE;
```

```
function p = Haarstep(p, n)
  oddodd   = p(1:2:n, 1:2:n);
  evenodd  = p(2:2:n, 1:2:n);
  oddeven  = p(1:2:n, 2:2:n);
  eveneven = p(2:2:n, 2:2:n);
  NW = (oddodd + evenodd + oddeven + eveneven)/2; 4
  NE = (oddodd + evenodd - oddeven - eveneven)/2; 1
  SW = (oddodd - evenodd + oddeven - eveneven)/2; 1
  SE = (oddodd - evenodd - oddeven + eveneven)/2; 1
  p(1:n/2,1:n/2) = NW;
  p(1:n/2,n/2+1:n) = NE;
  p(n/2+1:n,1:n/2) = SW;
  p(n/2+1:n,n/2+1:n) = SE;
end
```

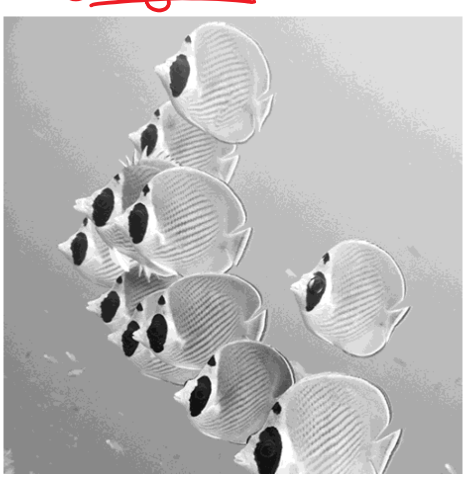


```
function p = Haar(p)
  n = length(p);
  while (n > 1)
    p = Haarstep(p, n);
    n = n/2;
  end
end
```

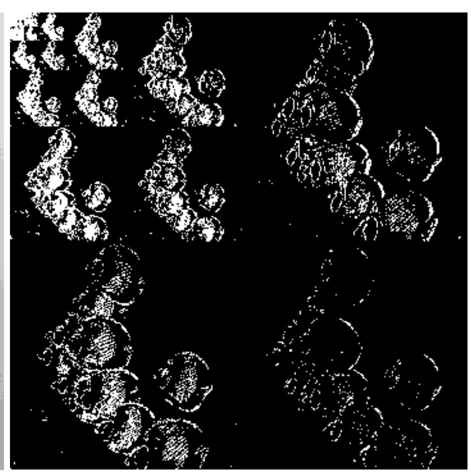
Nice property: Real-world images tend to have fairly sparse representations in the Haar basis (i.e., most Haar coordinates are close to 0).

Examples:

Original

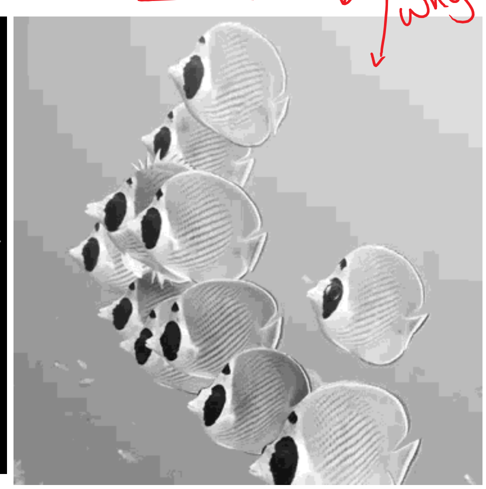


Keep 10% of coeffs



Result

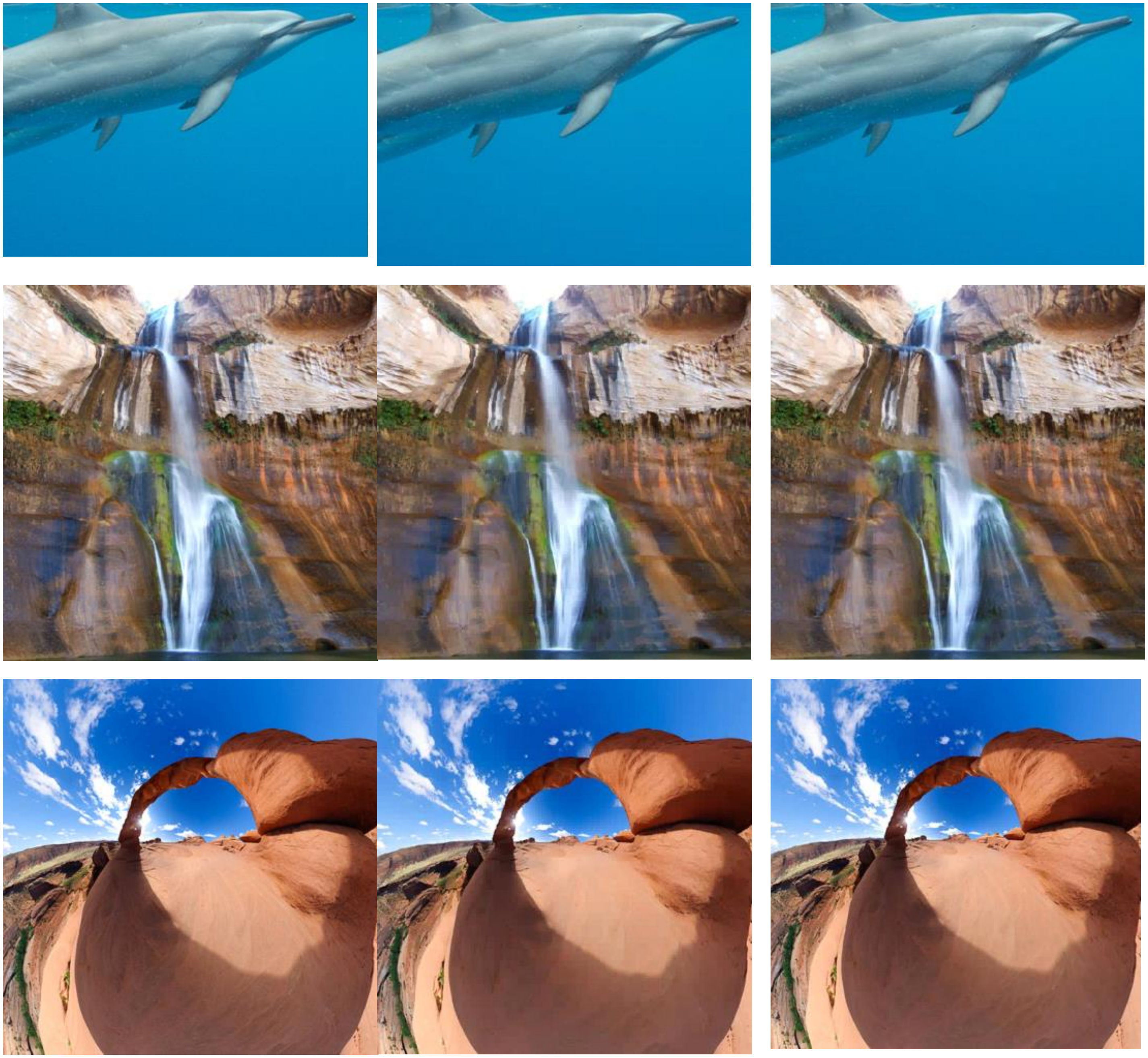
notice the blocky gradients. Why?



Original

Keeping 10%

Keeping 20%



Why are gradients so blocky?
- Other bases perform better...

Code to change back from the Haar basis:

```
function p = undoHaarstep(p, n)
  NW = p(1:n/2,1:n/2);
  NE = p(1:n/2,n/2+1:n);
  SW = p(n/2+1:n,1:n/2);
  SE = p(n/2+1:n,n/2+1:n);
  oddodd = (NW + NE + SW + SE)/2;
  evenodd = (NW + NE - SW - SE)/2;
  oddeven = (NW - NE + SW - SE)/2;
  eveneven = (NW - NE - SW + SE)/2;
  p(1:2:n, 1:2:n) = oddodd;
  p(2:2:n, 1:2:n) = evenodd;
  p(1:2:n, 2:2:n) = oddeven;
  p(2:2:n, 2:2:n) = eveneven;
```

```
function p = undoHaarstep(p, n)
  NW = p(1:n/2,1:n/2);
  NE = p(1:n/2,n/2+1:n);
  SW = p(n/2+1:n,1:n/2);
  SE = p(n/2+1:n,n/2+1:n);
  oddodd = (NW + NE + SW + SE)/2;
  evenodd = (NW + NE - SW - SE)/2;
  oddeven = (NW - NE + SW - SE)/2;
  eveneven = (NW - NE - SW + SE)/2;
  p(1:2:n, 1:2:n) = oddodd;
  p(2:2:n, 1:2:n) = evenodd;
  p(1:2:n, 2:2:n) = oddeven;
  p(2:2:n, 2:2:n) = eveneven;
end

function p = undoHaar(p)
  n = 2;
  while (n <= length(p))
    p = undoHaarstep(p, n);
    n = n*2;
  end
end
```

Moral: Real-world signals are often nearly sparse in some known basis.

Application: Image compression

- Change basis
- Throw away the small components
- Only store/transmit the large coefficients

e.g., instead of $(x_1, x_2, x_3, X_4, x_5, x_6, x_7, x_8)$,

send $4, X_4$