EE 510 10/02/2020

Outline: Linear Transformation

+ Review questions.

$f : E \longrightarrow F$

$$f(\alpha x + y) = \alpha f(x) + f(y).$$

$E = \mathbb{R}$; $F = \mathbb{R}$; $f(x) = a x$.

$E = \mathbb{C}$; $F = \mathbb{C}$;

$f(z) = z + i\bar{z}$; $\bar{z}$ = Conjugate.

$$f(i(1+i)) = f(-1+i) = -1 + i + i\overline{(-1+i)}$$
$$= -1 + i + 1 - i$$
$$= 0$$

$$i f(1+i) = i(1+i+1+i) = -2+2i$$

$$f(i(1+i)) \neq i f(1+i)$$

f is not a linear transformation

$$f(x, y, z) = (x+y,\ 2x-y+3z,\ z-y+1)$$

$$f(0,0,0) \neq (0,0,0)$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

$$f(0) = 0.$$

$$f(\alpha X + Y) = \alpha f(X) + f(Y)$$

$$f(-x+x) = -f(x) + f(x)$$

$$f(0) = 0.$$

f is not a linear transformation.

$f : V \longrightarrow W$: linear transformation

$B = \{b_1, \dots, b_m\}$ a basis for V

$C = \{c_1, \dots, c_n\}$ a basis for W.

$$[f]_{B \to C} = \begin{array}{c} \\ c_1 \\ \vdots \\ c_n \end{array} \begin{array}{c} f(b_1) \quad \cdots \quad f(b_m) \\ \begin{bmatrix} \alpha_{11} & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \vdots & \vdots & \vdots \end{bmatrix} \end{array}$$

$$f(b_1)=\sum_{i=1}^{n}\alpha_{i1}c_i$$

Exple: find the matrix of f in the standard basis:

$$f(x,y,z)=(x+y,\ 2x-y+3z;\ z-y).$$

$$f(1,0,0)=(1,2,0);\ f(0,1,0)=(1,-1,-1)$$

$$f(0,0,1)=(0,3,1)$$

$$[f]_S=\begin{bmatrix}1&1&0\\2&-1&3\\0&-1&1\end{bmatrix}$$

$$[f(x)]_S=\begin{bmatrix}1&1&0\\2&-1&3\\0&-1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x+y\\2x-y+3z\\z-y\end{bmatrix}$$

Exercise 1

$$f : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \begin{pmatrix} a-b & d-c \\ c-d & b-a \end{pmatrix}$$

1°) Show that f is a linear transformation.

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $M' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$; $\alpha \in \mathbb{R}$

$$f(\alpha M + M') = \begin{pmatrix} \alpha a + a' - \alpha d - d' & \alpha d + d' - \alpha c - c' \\ \alpha c + c' - \alpha d - d' & \alpha b + b' - \alpha a - a' \end{pmatrix}$$

$$= \alpha \begin{pmatrix} a-b & d-c \\ c-d & b-c \end{pmatrix} + \begin{pmatrix} a'-b' & d'-c' \\ c'-d' & b'-c' \end{pmatrix}$$

$$= \alpha f(M) + f(M')$$

2°) Find the matrix of the linear transformation f in the standard basis:

$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $E_{21} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

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$$f(E_{11}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = E_{11} - E_{22}$$

$$f(E_{12}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -E_{11} + E_{22}.$$

$$f(E_{21}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -E_{12} + E_{21}$$

$$f(E_{22}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = E_{12} - E_{21}$$

$$[f]_S = \begin{array}{c} \\ E_{11} \\ E_{12} \\ E_{21} \\ E_{22} \end{array} \begin{array}{c} f(E_{11}) \qquad\qquad\quad \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ -1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

3) Find the null space of f

$$N(f) = N([f]_S)$$

$$= \left\{ M \in \mathbb{R}^{2\times 2} \;/\; f(M) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;/\; \begin{pmatrix} a-b & d-c \\ c-d & b-a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} a & a \\ c & c \end{pmatrix} ;\; a, c \in \mathbb{R} \right\} = \left\{ a\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}.$$

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$\mathrm{Span}\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} \subset$

$\dim N(f) = 2.$

ii) Find $R(f)$.

From the rank-nullity theorem.

$\dim N(f) + \dim R(f) = 4$

$\Rightarrow \dim R(f) = 4 - 2 = 2$

$\{ f(E_{12}), f(E_{21}) \}$ is a basis for $R(f)$

<u>Change of basis:</u>

V a vector space of dimension n: $\mathbb{R}^n$.

$S = (e_1 \text{---}, e_n)$ $\quad$ f is a linear transformation

$B = (b_1 \text{---}, b_n)$

$$[f]_S = \begin{array}{c|c} & f(e_1) \quad \cdots \quad f(e_n) \\ \begin{matrix} e_1 \\ \vdots \\ e_n \end{matrix} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array} \quad ; \quad [f]_B = \begin{array}{c|c} & f(b_1) \quad \cdots \quad f(b_n) \\ \begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix} & \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \end{array}$$

$$[f]_S \longleftrightarrow [f]_B$$

$$\begin{tikzcd}
{[X]_B} \arrow[r, "f"] \arrow[d] & {[f(x)]_B} \\
{[X]_S} \arrow[r, "f"] & {[f(x)]_S} \arrow[u]
\end{tikzcd}$$

$$[f(x)]_B = [f]_B [X]_B$$

$$[X]_S = I[B \to S] [X]_B$$

$$[f(x)]_S = [f]_S [X]_S$$

$$[f(x)]_B = \cancel{[f(x)]_S\, I[S \to}$$

$$= I[S \to B] [f(x)]_S.$$

$$[f(x)]_B = I[S \to B] [f]_S\, I[B \to S] [X]_B$$

$$[f]_B [X]_B = I[S \to B] [f]_S\, I[B \to S] [X]_B$$

$$[f]_B = I[S \to B] [f]_S\, I[B \to S]$$

$$I[B \to S] = \left(I[S \to B]\right)^{-1}$$

Exple

$\mathbb{R}_2[X]$: set of polynomial of degree less of equal to 2.

$S=\{1, X, X^2\}$; $B=\{X+1, X-1, 2X^2\}$.

$$I[B\to S] = \begin{array}{c|c} & \begin{array}{ccc} X+1 & X-1 & 2X^2 \end{array} \\ \begin{array}{c} 1 \\ X \\ X^2 \end{array} & \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{array}$$

$$I[S\to B] = \begin{array}{c|c} & \begin{array}{ccc} 1 & X & X^2 \end{array} \\ \begin{array}{c} X+1 \\ X-1 \\ 2X^2 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

$P(X) = a + bX + cX^2$

$$[P]_S = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[P]_B = I[S\to B]\,[P]_S$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{a+b}{2} \\ \frac{b-a}{2} \\ \frac{c}{2} \end{bmatrix}$$

$$P(X) = \left(\frac{a+b}{2}\right)(X+1) + \left(\frac{b-a}{2}\right)(X-1) + \cancel{2}(2X^2)$$

Review questions:

1) $(1,2,0)$; $(3,0,1)$; $(2,-2,1)$
is basis for $\mathbb{R}^3$?
True / False

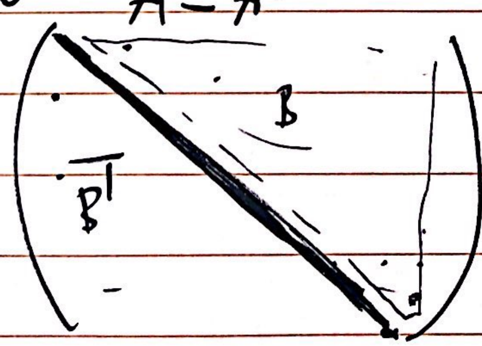
$V_1 + V_2 = V_3$

2) $A \in \mathbb{R}^{m\times n}$; $\operatorname{rank}(A) = m < n$.
$AX = b$ has ~~an~~ unique solution.
True / False

$\dim N(A) = n - m > 0$

3) Dimension of the set $n\times n$ matrices that are symmetric $= 1+2+3+\dots+n = \dfrac{n(n+1)}{2}$

$A = A^T$

$A =$ 

4) Dimension of the set of $n \times n$ matrices that are antisymmetric $= 1 + 2 + \dots + n-1 = \frac{n(n-1)}{2}$.

$$A = -A^T \Rightarrow A + A^T = 0$$

5) $f(1,2,0) = (1,3,1)$

$f(3,0,1) = (3,3,3)$

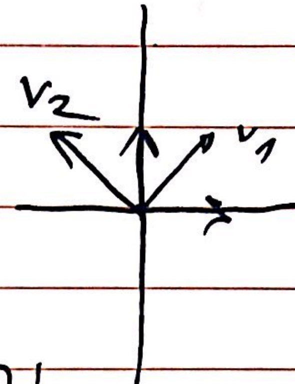
$f(2,-2,1) = ?$

$(2,-2,1) = (3,0,1) - (1,2,0)$

$$f(2,-2,1) = f(3,0,1) - f(1,2,0) = (2,0,2)$$

6')

$f(-1,1) = (0,1)$



$f(1,1) = (2,1)$

$f(x,y)\ ;\ \forall x,y \in \mathbb{R}$?!

$V_1 = \frac{1}{\sqrt{2}}(1,1)\ ;\ V_2 = \frac{1}{\sqrt{2}}(-1,1)$

$B = \{V_1, V_2\}$ is a basis for $\mathbb{R}^2$

$$[f]_{B\to S} = \begin{array}{c} \\ e_1 \\ e_2 \end{array}\begin{array}{c} \begin{array}{cc} f(V_1) & f(V_2) \end{array} \\ \begin{bmatrix} \frac{2}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \end{array}$$

$$I[B \to S] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\ ;\ I[S \to B] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$[f]_S = [f]_{B\to S}\, I[S \to B] = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$f(e_1) = e_1 \quad ; \quad f(e_2) = e_1 + e_2$

$[f]_S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix}$

$f(x,y) = (x+y, y) \; ; \; \forall x, y \in \mathbb{R}.$

7)

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \\ \boxed{2} & 0 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{6} & 0 \end{pmatrix}$$

$R(A) = \mathbb{R}^4.$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & \boxed{3} & \boxed{1} & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & \boxed{4} & \boxed{1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{-1} & 0 & \boxed{1} & 0 & 0 \\ 0 & \boxed{4} & 0 & \boxed{-3} & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ 4 & -3 \end{pmatrix}$$

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