Lecture 6: Subspaces of a matrix

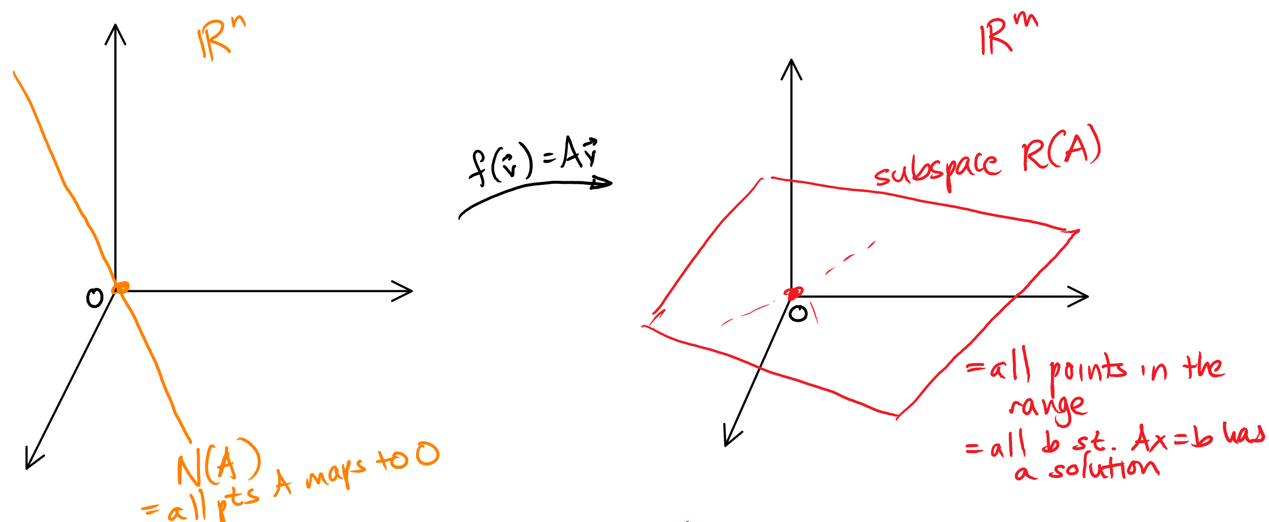
Admin: Reading: Strang §2.4, Meyer §4.2

Definition: For an $m \times n$ matrix A,

- The nullspace of A is
$$N(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0\}.$$
(everything A sends to $\vec{0}$)

- The range or column space of A is
$$R(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$
$$= \text{Span(columns of } A)$$
(everything reachable by A)

- The rowspace of A is $R(A^T)$
(span of the rows of A)



Observe: $\vec{b} \in \text{Range}(A) \iff A\vec{x} = \vec{b}$ has a solution

Example: Let $\vec{u} = \begin{pmatrix}1\\2\end{pmatrix}$, $\vec{v} = \begin{pmatrix}3\\4\end{pmatrix}$, and let
$$A = \vec{u}\,\vec{v}^T = \begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix} = \begin{pmatrix}3 & 4\\6 & 8\end{pmatrix}.$$
What are $R(A)$, $R(A^T)$, $N(A)$, $N(A^T)$?

Answer: Yes, we could apply Gaussian elimination.
But we can also just observe that for any vector $\vec{x}$,
$$A\vec{x} = \vec{u}\,\vec{v}^T\vec{x} = (\vec{v}^T\vec{x}) \cdot \vec{u}$$
$$\Rightarrow R(A) = \text{Span}(\{\vec{u}\}), \text{ a line.}$$

Similarly, $R(A^T) = \text{Span}(\{\vec{v}\})$, a line.

Also $A\vec{x} = 0 \iff v^T x = 0$

$$v^T x = 3x_1 + 4x_2$$

$$\iff x \propto (-4,3)$$

$$\Rightarrow N(A) = \text{Span}(\{(-4,3)\}),$$ the line perpendicular to u

Similarly $N(A^T) = \text{Span}(\{(-2,1)\})$. ✓

NULLSPACE $N(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$

homogeneous equations

Claim: This is a subspace.

Proof:

- closure under multiplication:

$$\vec{x} \in N(A) \Rightarrow A\vec{x} = \vec{0}$$
$$\Rightarrow \text{for any } c \in \mathbb{R},$$
$$A(c\cdot\vec{x}) = c\cdot A\vec{x} = c\cdot\vec{0} = \vec{0}$$
$$\Rightarrow c\cdot\vec{x} \in N(A) \checkmark$$

- closure under addition:

$$\vec{x},\vec{y} \in N(A) \Rightarrow A\vec{x} = \vec{0},\ A\vec{y} = \vec{0}$$
$$\Rightarrow A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$$
$$= \vec{0} + \vec{0} = \vec{0}$$
$$\Rightarrow \vec{x}+\vec{y} \in N(A) \checkmark$$

$\square$

Problem:

Let $M = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix}$

What is $N(M)$?

Answer: Solving for $N(M) \iff$ Solving $M\vec{x} = 0$

$$M \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 2 & 6 & 8 \\ 0 & 0 & 1 & 3 & 4 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{GE} \begin{pmatrix} 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow x_1 = -2x_2 + 2x_4 - x_5$$

$x_3 = -3x_4 - 4x_5$

$\Rightarrow$ General solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \boxed{N(M) = \operatorname{Span}\left(\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \end{pmatrix} \right\} \right)} \checkmark$$

Observe: Row operations don't change the nullspace.

Indeed, if U is invertible, $N(A) = N(U \cdot A)$.

Claim: Consider the equations $A\vec{x} = \vec{b}$, with $\vec{b} \neq \vec{0}$.

Let $\vec{y}$ be a solution ($A\vec{y} = \vec{b}$). Then

$$\{\vec{x} \mid A\vec{x} = \vec{b}\} = \vec{y} + \{\vec{x} \mid A\vec{x} = \vec{0}\}$$
$$= \vec{y} + N(A)$$

Thus, the solutions to a set of **non-homogeneous** equations form an **affine subspace**.

(In English: The general solution of the nonhomogeneous system is given by a particular solution ($\vec{y}$) plus the general solution of the associated homogeneous system.)

Proof: Let $V_b = \{x \mid Ax = b\}$.

To show equality, we need to verify $V_b \subseteq y + N(A)$
and $V_b \supseteq y + N(A)$.

- $V_b \supseteq y + N(A)$:

  Indeed, if $z \in N(A)$, then $A(y+z) = Ay + Az$
  $= b + 0 = b$

  $\Rightarrow y + z \in V_b$ ✓

- $V_b \subseteq y + N(A)$:

  Indeed, if $x \in V_b$, then $Ax = b$, so

$$A(y-x) = Ay - Ax = b - b = 0$$
$$\Rightarrow y - x \in N(A) \checkmark \qquad \square$$

Corollary: The equations $A\vec{x} = \vec{b}$ have <u>infinitely many solutions</u>

if and only if

$N(A) \neq \{\vec{0}\}$ and there is at least one solution.

RANGE $R(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$

Claim:
1. $\vec{b} \in R(A) \iff A\vec{x} = \vec{b}$ has a solution
2. $R(A)$ is a subspace.
3. $R(A) = \text{Span}(\text{columns of } A)$

this is why it is often called the "column space"

Proof of 2:

Closure under addition:

Let $\vec{b}, \vec{c} \in R(A) \Rightarrow$ there exist $\vec{x}, \vec{y} \in \mathbb{R}^n$ with $A\vec{x} = \vec{b}$, $A\vec{y} = \vec{c}$

$$\Rightarrow A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$$
$$= \vec{b} + \vec{c}$$
$$\Rightarrow \vec{b} + \vec{c} \in R(A) \checkmark$$

Closure under multiplication:

Let $\vec{b} \in R(A)$ and $c \in \mathbb{R} \Rightarrow \exists\, \vec{x} \in \mathbb{R}^n$ with $A\vec{x} = \vec{b}$

$$\Rightarrow A(c \cdot \vec{x}) = c \cdot A\vec{x}$$
$$= c \cdot \vec{b}$$
$$\Rightarrow c \cdot \vec{b} \in R(A) \checkmark \qquad \square$$

Proof of 3, $R(A) = \text{Span}(\text{columns of } A)$:

Let the columns be $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^m$

$$A = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$$

$$R(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} \qquad (\text{def}^{\underline{n}} \text{ of } R(A))$$
$$= \{x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \mid x_1, x_2, \dots, x_n \in \mathbb{R}\}$$
$$= \text{Span}(\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}) \qquad (\text{def. Span}) \quad \square$$

$$= \{ x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \mid x_1, x_2, \ldots, x_n \in \mathbb{R} \}$$
$$= \mathrm{Span}(\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}) \quad \text{(def. Span)} \quad \square$$

ROW SPACE $R(A^T) = \mathrm{Span}(\text{rows of } A)$

Problem:

Let $M = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix}$ (as before)

What are the row and column spaces of M?

Answer:

$$M \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 2 & 6 & 8 \\ 0 & 0 & 1 & 3 & 4 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\xrightarrow{GE} \begin{pmatrix} 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe: The row operations of Gaussian elimination do not change the row space.

(Because $\mathrm{Span}(v_1, v_2, v_3, \ldots, v_k) = \mathrm{Span}(v_1, v_2 + \alpha v_1, v_3, \ldots, v_k)$.)

$$\Rightarrow \underbrace{R(M^T)}_{\text{row space}} = \mathrm{Span}(\{(1,2,0,-2,1), (0,0,1,3,4)\}). \checkmark$$

How to compute $R(M)$?

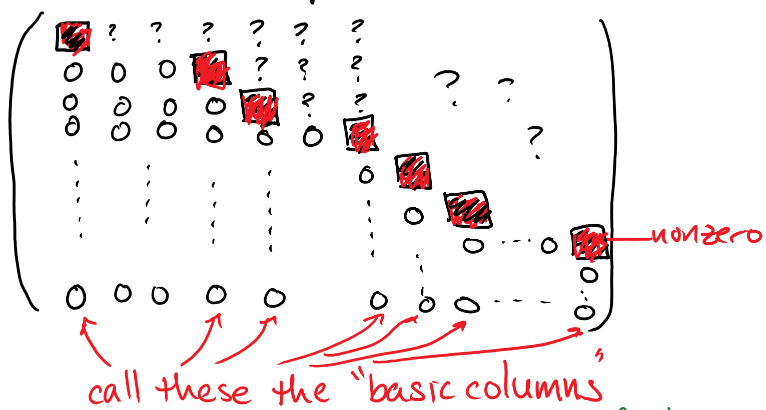
① Apply Gaussian elimination to M^T

$$M^T = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 1 & 0 & 2 \\ 1 & 4 & 4 \\ 5 & -2 & 9 \end{pmatrix} \begin{matrix} \\ -2 \\ -1 \\ -1 \\ -5 \end{matrix} \longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 6 & 3 \\ 0 & 8 & 4 \end{pmatrix} \begin{matrix} \\ \\ \\ -3 \\ -4 \end{matrix}$$
$$\longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow R(M) = \mathrm{Span}(\{(1,-2,1), (0,2,1)\})$$

② But if we have already applied G.E. to M, there is no need to apply it again to M^T:

<u>Observe</u>: The row operations of G.E. finish with a matrix like



Claim: Every column is a linear combination of the basic columns.

Proof: Take any non-basic column $\vec{c}$.

Use the last basic column to cancel $\vec{c}$'s last coordinate.

⋮

Use the first basic column to cancel $\vec{c}$'s first coord. □

$$M = \begin{pmatrix} 1 & 2 & 1 & 1 & 5 \\ -2 & -4 & 0 & 4 & -2 \\ 1 & 2 & 2 & 4 & 9 \end{pmatrix} \xrightarrow{G.E.} G = \begin{pmatrix} \boxed{1} & 2 & 1 & 1 & 5 \\ 0 & 0 & \boxed{1} & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

basic columns

e.g. $(1,3,0) = 3(1,1,0) - 2(1,0,0)$ ✓

The **same** linear combination of columns works for M:

$$\underset{\substack{\text{column } 4 \\ M\vec{e}_4}}{(1,4,4)} = 3\underset{\substack{\text{column } 3 \\ M\vec{e}_3}}{(1,0,2)} - 2\underset{\substack{\text{column } 1 \\ M\vec{e}_1}}{(1,-2,1)}$$

Why?

$M = QG$, where Q $(=PL)$ is the matrix that implements the row operations in Gaussian Elim.

$$\Rightarrow \text{If } \underset{j\text{th col}}{Ge_j} = \sum_k \alpha_k \underset{k\text{th column}}{Ge_k},$$

$$\text{then } \underset{\substack{\| \\ Me_j}}{QGe_j} = \underset{\substack{\| \\ \sum_k \alpha_k Me_k}}{Q\left(\sum_k \alpha_k Ge_k\right)}$$

$$\Rightarrow \text{Range}(M) = \text{Span}(\{(1,-2,1)\ (1,0,2)\})$$

$$\Rightarrow \text{Range}(M) = \text{Span}\left(\{(1,-2,1),(1,0,2)\}\right)$$

In general:

After Gaussian elimination, identify the basic columns. Then

$$R(M) = \text{Span}\left(\text{those same columns, in } M\right)$$

Exercise: Prove

① $R(AB) \subseteq R(A)$

(Multiplying on the right can only reduce the range.)

$R(AB) = R(A)$ if B is invertible.

② $N(BA) \supseteq N(A)$

(Multiplying on the left can only increase the nullspace.)

$N(BA) = N(A)$ if B is invertible

Easy example: Take $B = 0$, so $AB = 0$, $BA = 0$

$R(AB) = \{0\} \subseteq R(A)$, $N(BA) = \mathbb{R}^n \supseteq N(A)$ ✓

Note: These spaces are all sensitive to numeral errors and perturbations.

E.g.

	Range	Nullspace
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	x-axis $\{(x,0,0) \mid x \in \mathbb{R}\}$	yz-plane $\{(0,y,z) \mid y,z \in \mathbb{R}\}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$ $\varepsilon \neq 0$	$\mathbb{R}^3$	$\{\vec{0}\}$

(so be careful with Matlab's null and orth commands)

Example: What is the nullspace of

$$A = \begin{pmatrix} -1 & 1 & & & & & \\ 1 & -2 & 1 & & & 0 & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & \\ & 0 & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} ?$$

Answer:

Intuition: Recall that A is the matrix we got from discretizing the second derivative operator

$$f''(t) = \lim_{\delta t \to 0} \frac{1}{(\delta t)^2}\left(f(t+\delta t) - 2f(t) + f(t-\delta t)\right).$$

The functions with second derivative 0 are exactly constant functions $f(t) = \text{constant}$.

$\Rightarrow$ We expect the nullspace of A to be the set of vectors with all-equal coordinates.

Claim: $N(A) = \operatorname{Span}\left\{\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}\right\}$

$=$ set of all vectors with all-equal coordinates.

Proof:

$$A\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = \begin{pmatrix}\text{sum of } a_{ij} \text{ across row } 1\\ \vdots \\ \text{sum across row } n\end{pmatrix} = 0$$

$\Rightarrow (1, 1, \ldots, 1) \in N(A)$

$\Rightarrow \operatorname{Span}(\{(1, \ldots, 1)\}) \subseteq N(A)$.

But are there other vectors in $N(A)$?

$$A = \begin{pmatrix} -1 & 1 & & & & & \\ 1 & -2 & 1 & & & 0 & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & 0 & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} {+1} \longrightarrow \begin{pmatrix} -1 & 1 & & & & & \\ 0 & -1 & 1 & & & 0 & \\ & 1 & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & 0 & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} {+1}$$

$$\longrightarrow \begin{pmatrix} -1 & 1 & & & & & \\ 0 & -1 & 1 & & & 0 & \\ & 0 & -1 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & 0 & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -1 \end{pmatrix} {+1}$$

$$\vdots$$

$$\longrightarrow \begin{pmatrix} -1 & 1 & & & & \\ 0 & -1 & 1 & & 0 & \\ & 0 & -1 & 1 & & \\ & & 0 & -1 & 1 & \\ & & & & \ddots & \end{pmatrix}$$

$$\vec{B} = \begin{pmatrix} 0 & -1 & 1 & & & & \\ & 0 & -1 & 1 & & \bigcirc & \\ & & 0 & -1 & 1 & & \\ & & & & \ddots & & \\ & \bigcirc & & & 0 & -1 & 1 \\ & & & & & 1 & -1 \end{pmatrix}$$

$$A\vec{x} = 0 \iff B\vec{x} = 0$$

$$\iff x_1 = x_2 \quad \text{(first row)}$$

$$x_2 = x_3 \quad (2^{nd}\text{ row})$$

$$\vdots$$

$$x_{n-1} = x_n$$

$$\iff \vec{x} \in \text{Span}\{(1, 1, 1, \dots, 1)\}$$

Question: What is $R(A)$? (Note $A = A^T$.)

Example:

Definition: The matrix A is **diagonally dominant** if for all rows j,

$$|a_{jj}| \geq \sum_{i \neq j} |a_{ji}| .$$

It is **strictly diagonally dominant** if the inequality is strict ($>$) for all rows.

Theorem: If A is strictly diagonally dominant, then

$$N(A) = \{\vec{0}\}$$

Proof:

Let $\vec{x} \in N(A)$, so $A\vec{x} = \vec{0}$.

Let j be the coordinate so $|x_j|$ is largest.

$$0 = (A\vec{x})_j = a_{jj}x_j + \sum_{i \neq j} a_{ji}x_i$$

$$\Rightarrow |a_{jj}x_j| = \left|\sum_{i \neq j} a_{ji}x_i\right|$$

$$\leq \sum_{i \neq j} |a_{ji}| \cdot |x_i|$$

$$\leq \left(\sum_{i \neq j} |a_{ji}|\right) \cdot |x_j|$$

$< |a_{jj}| \cdot |x_j|$, unless $x_j = 0$.

This is a contradiction unless $x_j = 0$.

$\Rightarrow x_j = 0 \Rightarrow \vec{x} = \vec{0}$

$\Rightarrow N(A) = \{\vec{0}\}$. ✓ $\square$

Question: Can you characterize the nullspaces of diagonally dominant matrices?

Example:

$$A = \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ 1 & & & & 1 & -2 \end{pmatrix}$$

diagonally dominant, but not strictly so

has nullspace

$$N(A) = \mathrm{Span}(\{(1,1,1,\dots,1)\})$$

$= \text{set of all constant vectors.}$

Exercise: Linear codes

a) List all elements of the subspace over $\mathbb{F}_2$

$$R\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\right).$$

$$\begin{array}{ccc} \shortparallel & \shortparallel & \shortparallel \\ v_1 & v_2 & v_3 \end{array}$$

Answer: The subspace includes all elements

$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3$

with $a, b, c \in \{0, 1\}$.

0000000	0110011
1010101	1100110
0001111	0111100

1011010 1101001

b) Compute the nullspace of

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Answer: Use Gaussian elimination:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

pivots, free vars.

Hence $N(A)$ equals the columnspace of

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Observe that this is the same space we found in part ⓐ!

Remark: Subspaces over finite fields are also known as "linear error-correcting codes."

A vector in the subspace is called a "codeword."

$$\text{distance(code } V) \equiv \min_{\vec{u},\vec{v} \in V} \left(\#\text{ nonzero entries of } \vec{u}-\vec{v}\right)$$

(# nonzero entries of $\vec{u}-\vec{v}$ = # coords where $\vec{u}$ & $\vec{v}$ differ)

$$= \min_{\vec{u} \in V} \left(\#\text{ nonzero entries of } \vec{u}\right).$$

(by linearity, $\vec{u}-\vec{v} \in V$)

Claim 3: For $S \subseteq \mathbb{R}^n$, let

$$A(S) = \{ A\vec{x} \mid \vec{x} \in S \}$$

(map the set S forward by applying A)

1. If S is a subspace, so is $A(S)$.
2. If $s_1, \ldots, s_k$ span S, then $As_1, \ldots, As_k$ span $A(S)$.

Proof:

1) Closure under addition:

Let $y, z \in A(S)$. Is $y+z \in A(S)$?

Yes! $y = Ax$, $z = Ax'$

$y + z = A(x + x') \in A(S)$ ✓

Closure under multiplication is similar ✓

2) Any vector $y \in A(S)$ can be written $y = Ax$ for some $x \in S$.

$s_1, \ldots, s_k$ span S $\Rightarrow$ $\underset{\in S}{x} = \sum_{j=1}^{k} \alpha_j s_j$ for some scalars $\alpha_1, \ldots, \alpha_k$.

$\Rightarrow$ by linearity, $y = Ax = \sum_{j=1}^{k} \alpha_j (As_j)$

$\Rightarrow y \in \text{Span}(\{As_1, \ldots, As_k\})$ ✓ $\square$

Observe: ① $\vec{b} \in R(A) \iff A\vec{x} = \vec{b}$ has a solution.

② $R(A) = A(\mathbb{R}^n)$
$= \text{Span}(\text{columns of } A)$

Why?

$\mathbb{R}^n$ is spanned by $(1,0,0,\ldots,0), \ldots, (0,0,\ldots,0,1)$

By the exercise, $A(\mathbb{R}^n)$ is spanned by

$A\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = 1^{st}$ column, $\ldots$, $A\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix} = $ last column of A ✓

Thus the range is also known as the column space of A.

Observe: ③ $\underset{n \times n}{A}$ is invertible (A^{-1} exists)

$\iff$ every point $b \in \mathbb{R}^n$ has a preimage x, $Ax = b$

$\Longleftrightarrow R(A) = \mathbb{R}^{n}$ (everything)