Midterm review

Admin: Midterm

Important definitions:

Vector space
- Basis, dimension
  - Linear independence
- Span
- Inner products, angles
- Orthogonal subspaces

Linear transformation
- Range/columnspace
- Rowspace, nullspace
- Projections
- Inverses

Important theorems:

Rank-nullity theorem

LU decomposition ⟷ Gaussian elimination

Important operations:

Solve a system of linear equations
- using Gaussian elimination, LU, inverses
  - don't use inverses!
- sparse matrices
- convert equations to/from matrix equations
- homogeneous & nonhomogeneous systems

Recognize linear transformations & vector spaces

Express a linear transformation in given bases
- Change basis

Compute the span of a set of vectors

Compute the rowspace, col-space, nullspace
- compute their dimensions
- compute a basis for each

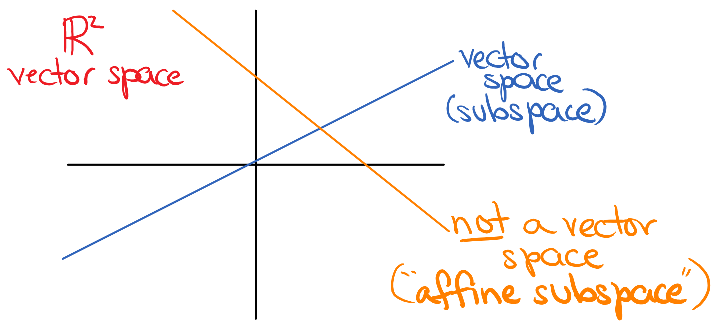
Project a vector onto a subspace

Compute (a basis for) the orthogonal complement

$V^\perp$ of a subspace V

Vector space = a set of vectors... (or polynomials, matrices, ...)
closed under addition
scalar multiplication
($\Rightarrow$ must include $\vec{0}$)

Examples:



$\{\vec{x} \mid A\vec{x} = \vec{0}\}$ vector space, nullspace $N(A)$

$\{\vec{x} \mid A\vec{x} = \underbrace{\vec{b}}_{\neq \vec{0}}\}$ not a vector space

$\{\text{functions } f: \mathbb{R} \to \mathbb{R} \text{ with } f(10) = 2\}$ no ✗

$\{\text{functions } f: \mathbb{R} \to \mathbb{R} \text{ with } f(10) = f(12)\}$ yes ✓

Linear transformation
$f: \mathcal{U} \to V$ with $f(c \cdot \vec{u}) = c \cdot f(\vec{u})$
$f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$

Linear transformations $\xleftrightarrow[\text{bases}]{\text{with}}$ Matrices

$f: \underset{\dim n}{\mathcal{U}} \to V_{\dim m}$ $\qquad [f]_{m \times n}$

Changes of basis:

$$\underset{\text{basis } B'}{U} \xrightarrow{id} \underset{\text{basis } B}{U} \xrightarrow{f} \underset{\text{basis } C}{V} \xrightarrow{id} \underset{\text{basis } C'}{V}$$

$$[f]_{B' \to C'} = [\text{identity}]_{C \to C'} [f]_{B \to C} [\text{identity}]_{B' \to B}$$

Note: $[id]_{B' \to B} = [id]^{-1}_{B \to B'}$

Span $\left(\begin{matrix}\text{a set of}\\\text{vectors}\end{matrix}\right) = \left\{\begin{matrix}\text{all linear}\\\text{combinations}\end{matrix}\right\}$ $\quad R\left(\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}\right)$

$\{\vec{v}_1, \dots, \vec{v}_n\}$ is a **Basis** for V if

1) $V = \text{Span}(\{v_1, \dots, v_n\})$
2) $\{v_1, \dots, v_n\}$ is **linearly independent**

$n =$ **dimension**

if no one lies in span(the others)

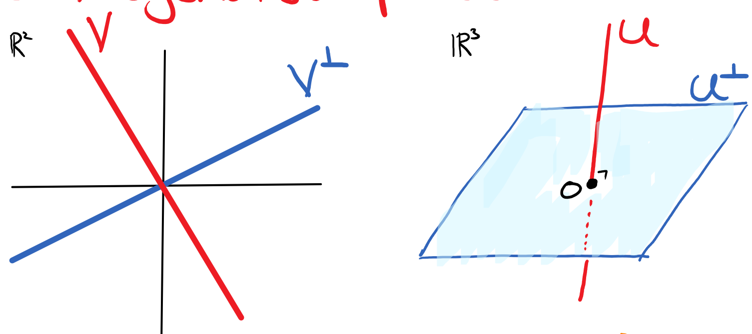
ie., $N\left(\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}\right) = \{\vec{0}\}$

Inner product $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \sum_i u_i v_i$

length $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$

angle $\cos\theta = \dfrac{\vec{u}}{\|\vec{u}\|} \cdot \dfrac{\vec{v}}{\|\vec{v}\|}$

Orthogonal subspaces



$$\dim(V) + \dim(V^\perp) = n$$

$\underbrace{R(A^T)}_{\text{rowspace}} = \underbrace{N(A)}_{\text{nullspace}}{}^\perp$

Range/columnspace

$R(A) = \text{Span(columns)} = \{\vec{b} \mid A\vec{x} = \vec{b} \text{ has solution}\}$

$\dim R(A) = \dim R(A^T) = \text{Rank}(A)$

Rowspace

$R(A^T) = \text{Span(rows)}$

Nullspace

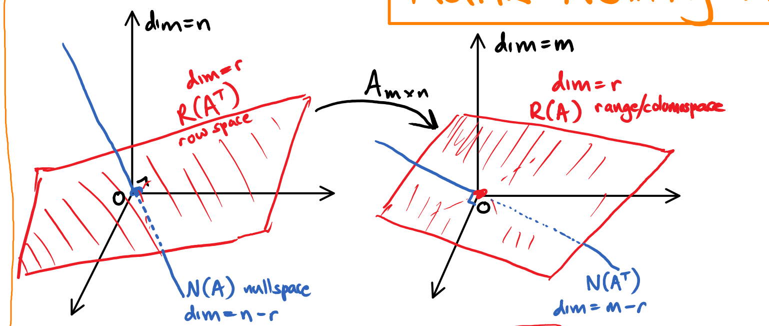
$N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$

$N(A) = R(A^T)^\perp$

(Note: Gaussian elim. leaves row & nullspace unchanged)

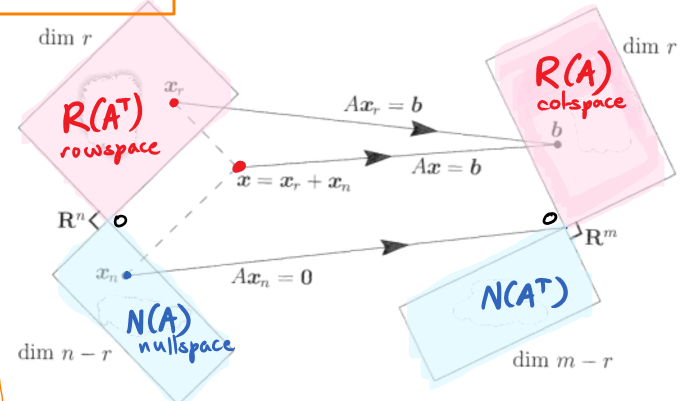
$N(A) = \{x \mid Ax = 0\}$

Rank-Nullity Theorem



$$\boxed{\dim R(A^T) = \dim R(A)}$$

$$\boxed{\begin{aligned}&\dim N(A) + \dim R(A^T)\\ &= \text{total dimension } n\end{aligned}} \qquad \boxed{\begin{aligned}&\dim R(A) + \dim N(A^T)\\ &= \text{total dimension } m\end{aligned}}$$



$N(A) = R(A^T)^{\perp}$ $\qquad$ $N(A^T) = R(A)^{\perp}$

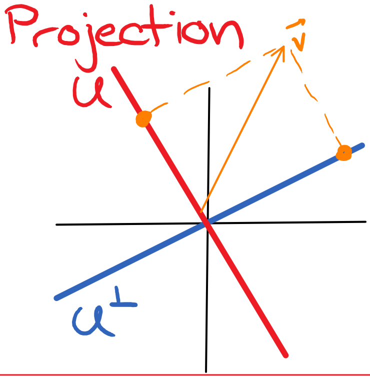
Example:

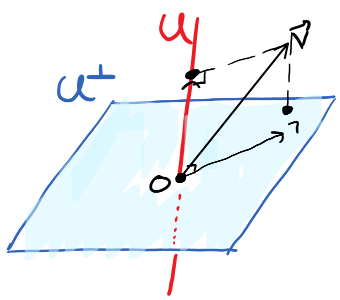
Square matrix A is invertible/nonsingular $\iff$ $N(A) = \{\vec{0}\}$

i.e., $\dim N(A) = 0$

$\text{Rank}(A) = n$

Projection





Rules:
- If $\|v\| = 1$, $\vec{v}\vec{v}^T$ projects to line $\text{Span}(\{\vec{v}\})$
- If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is orthonormal, $\sum_{j=1}^{k} \vec{v}_j \vec{v}_j^T$ projects to their span (k-dimensional)
- $I - P_u$ projects to $u^{\perp}$

Matrix multiplication:

Two ways of thinking about it:

① For $A = \begin{pmatrix} | & | & & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & & | \end{pmatrix} = \sum_j \vec{c}_j \vec{e}_j^T$

① For $A = \begin{pmatrix} | & | & & | \\ \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \\ | & | & & | \end{pmatrix} = \sum_j c_j e_j^T$

$$A\vec{x} = x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n \Rightarrow R(A) = \mathrm{Span}(\{\vec{c}_1, \ldots, \vec{c}_n\})$$

② For $A = \begin{pmatrix} \text{---} \; \vec{r}_1^T \; \text{---} \\ \text{---} \; \vec{r}_2^T \; \text{---} \\ \vdots \\ \text{---} \; \vec{r}_m^T \; \text{---} \end{pmatrix} = \sum_i \vec{e}_i \vec{r}_i^T$

$$A\vec{x} = \begin{pmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{pmatrix} \Rightarrow N(A) = R(A^T)^\perp$$

Exercise: Rank-Nullity Theorem

Compute the dimensions of $R(A)$, $R(A^T)$, $N(A)$, $N(A^T)$ for

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 3 \\ 4 & 5 & 6 & 7 & 7 \\ 8 & 9 & 0 & 1 & 1 \\ 8 & 10 & 2 & 4 & 4 \end{pmatrix}$$

Space	Dimension
$R(A)$	
$R(A^T)$	
$N(A)$	
$N(A^T)$	

Exercise: Rank-Nullity Theorem

Compute the dimensions of $R(A)$, $R(A^T)$, $N(A)$, $N(A^T)$ for
over $\mathbb{F}_2 = \{0, 1\}$

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Space	Dimension
$R(A)$	
$R(A^T)$	
$N(A)$	
$N(A^T)$	

Exercise: Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$f(\;) = \begin{pmatrix} 1 & 2 \\ & \end{pmatrix}\begin{pmatrix} x \\ \end{pmatrix} = \begin{pmatrix} x + 2y \\ \end{pmatrix}$

Exercise: Let $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$f(x,y) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$$

Let $V = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$

Let $g: V_{\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}} \longrightarrow \mathbb{R}^2_{\{e_1, e_2\}}$ be given by $g(\vec{v}) = f(\vec{v})$.

$$\underset{2\times 1}{[g]} =$$

Exercise: Let $V = \{\text{all } 2\times 2 \text{ matrices}\}$

Let $f: V \longrightarrow V$ given by

$$f(A) = A\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} A$$

Let $\mathcal{B}$ be the basis $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Express f as a 4×4 matrix in this basis.