# Lecture 9: Linear transformations (class)

**Reading:**  2.3  4.3–4.4

**Today:** Linear transformations $\mathbb{R}^n \to \mathbb{R}^m$

Goal: Understand the geometry of linear maps.

**Recall:**

**Definition:** For an $m \times n$ matrix $A$,

- The nullspace of $A$ is

$$N(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0\}.$$

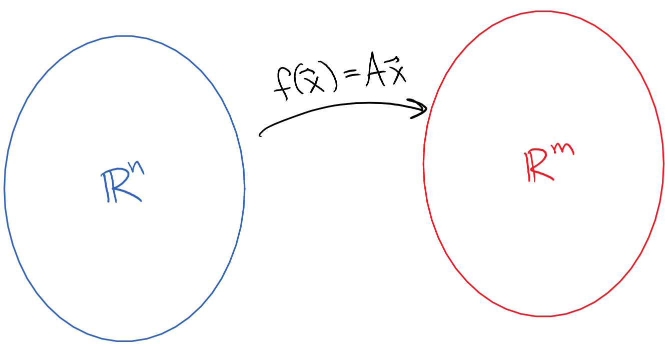
- The range (or column space) of $A$ is

$$R(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}.$$
$$= \text{Span}(\text{columns of } A)$$

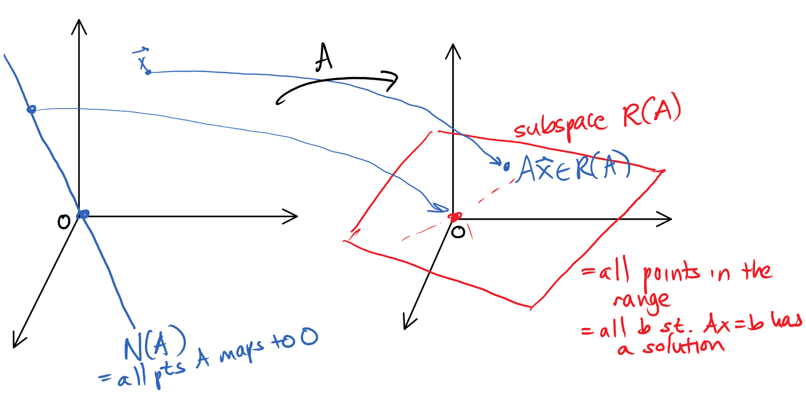
- The rowspace of $A$ is $R(A^T) = N(A)^\perp$

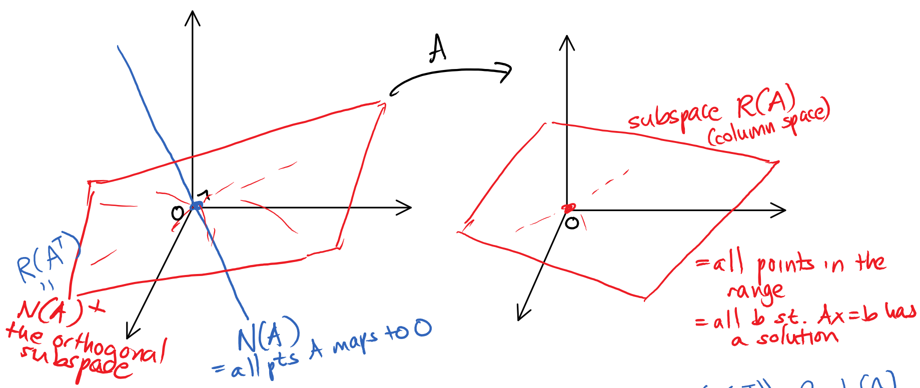
## The geometry of matrices

Let $A$ be an $m \times n$ real matrix.




Let's refine this picture...



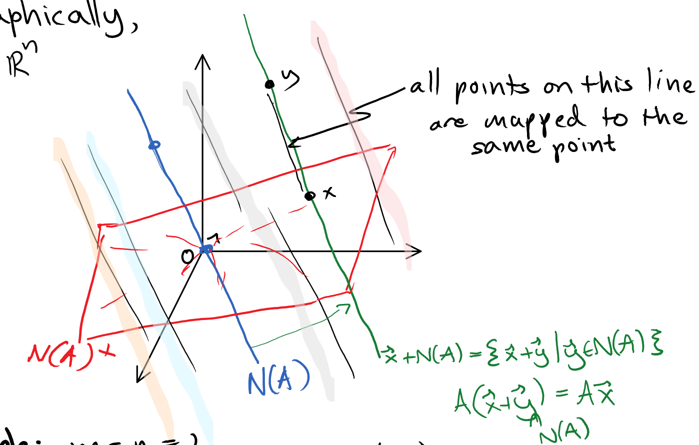



$\dim R(A) = \dim(R(A^T)) = \mathrm{Rank}(A)$

## Observations

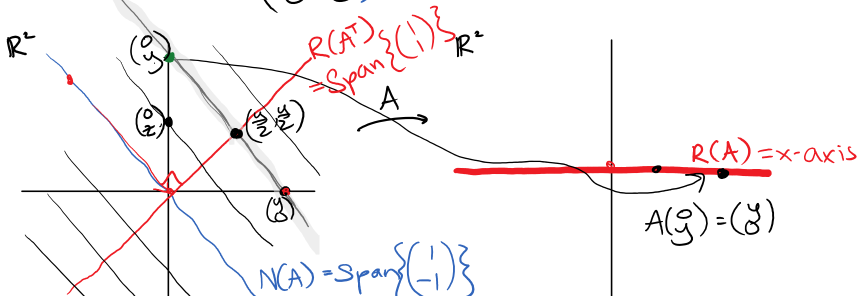
① If $\vec{x} - \vec{y} \in N(A)$, then $A\vec{x} = A\vec{y}$.

Graphically,




**Example:** $m = n = 2$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 0 \end{pmatrix}$ $\quad \mathrm{rank}(A) = 1$




The y-coordinate (direction parallel to N(A)) is irrelevant.
All that matters is the x-coordinate (perpendicular to N(A)).
That's also true in general!

In our heads, we can thus break A into two steps:

1) First map y to x,
i.e., take a point and move it parallel to N(A) to get to $N(A)^\perp$.
This is a **projection**; it flattens the space to $N(A)^\perp$.

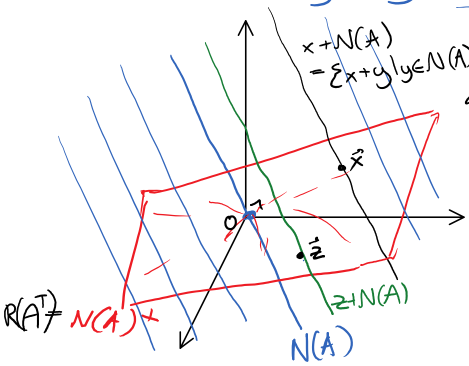
2) Map $\vec{x}$ to $A\vec{x}$.

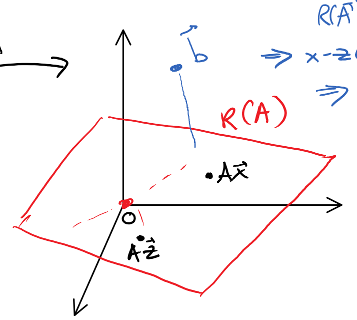
② If $\vec{x}, \vec{z}$ are distinct points in $N(A)^\perp = R(A^T)$, then $A\vec{x} \neq A\vec{z}$.

b/c the only way $A\vec{x} = A\vec{z} \Rightarrow A(x - z) = 0 \Rightarrow \underset{R(A^T)}{x} - \underset{R(A^T)}{z} \in N(A)$

b/c the only way $A\vec{x} = A\vec{z} \Rightarrow A(x-z) = 0 \Rightarrow x - z \in N(A)$

$\overset{\cap}{R(A^T)}\ \overset{\cap}{R(A^T)}$




$\Rightarrow x - z \in R(A^T) \cap N(A)$

$\Rightarrow x - z = 0$

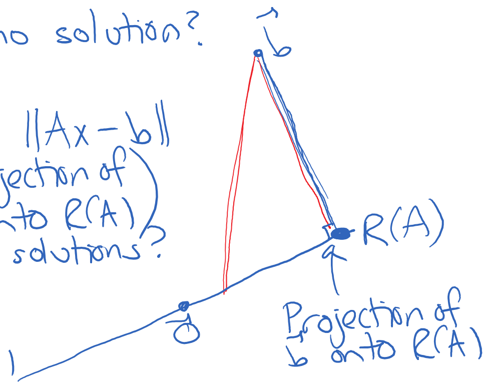
ie $x = z$

Thus A is a 1-to-1 map $N(A)^\perp \longrightarrow R(A)$. It is invertible between these spaces.

Applications: ① What if $A\vec{x} = \vec{b}$ has no solution? ($b \notin R(A)$)

Find $\vec{x}$ to minimize $\|Ax - b\|$

How? Solve $A\vec{x} =$ (Projection of $\vec{b}$ onto R(A))

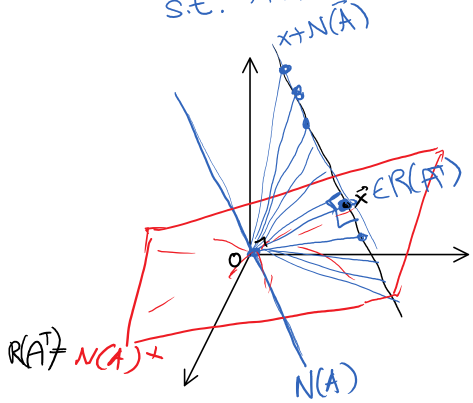


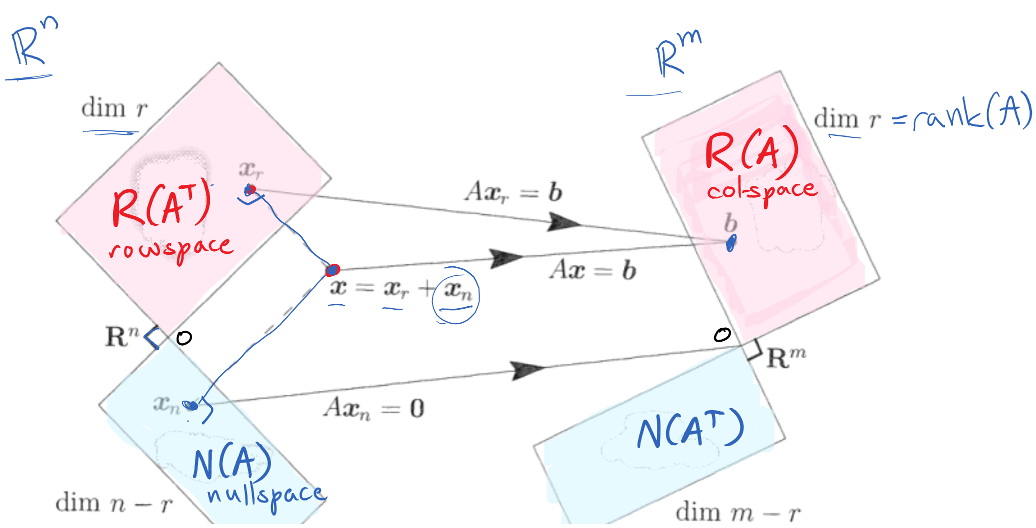

② What if $A\vec{x} = \vec{b}$ has ∞ (infinitely many) solutions?

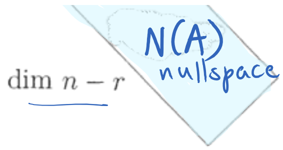
dimN(A) ≥ 1

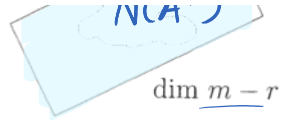
Problem: Find the shortest solution, ie..

$$\arg\min_{\vec{x}} \|\vec{x}\|^2 \quad \text{s.t. } A\vec{x} = \vec{b}$$






$N(A) = R(A^T)^\perp$ $\qquad$ $N(A^T) = R(A)^\perp$

By the Rank-Nullity theorem,

$$N(A)^\perp = R(A^T) \leftarrow \text{rowspace of } A$$

and $\dim R(A^T) = \dim R(A) \leftarrow$ called the **rank** of $A$.

> **SINGULAR-VALUE DECOMPOSITION (SVD)**
>
> Informally: Any linear transformation can be split into:
> - a rotation, followed by
> - scaling vectors in or out

# LINEAR TRANSFORMATIONS

Why matrices??? $\qquad$ why matrix multiplication? Why matrix inversion?

> **Definition**: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if
>
> $f(\alpha \vec{u}) = \alpha \cdot f(\vec{u})$ for all vectors $\vec{u} \in \mathbb{R}^n$ and scalars $\alpha \in \mathbb{R}$
>
> $f(\vec{u}+\vec{v}) = f(\vec{u}) + f(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$

$f(0 \cdot \vec{u}) = 0 \cdot f(\vec{u}) = \vec{0}$

$\shortparallel$

$f(\vec{0})$

$f(\vec{x}) = \vec{0}$ is linear

$f(\vec{x}) = \vec{y} \neq \vec{0}$ is not linear

**Examples**: For $n=m=2$,

- $f(x,y) = (0,0)$ ✓
- $f$ a rotation by $\theta$ : $f(x,y) = \begin{pmatrix} (\cos\theta)x - (\sin\theta)y, \\ (\sin\theta)x + (\cos\theta)y \end{pmatrix}$ ✓
- $f(x,y) = (x^2, \sin y)$ ✗ not linear
- $f(x,y) = (1+x, y)$ ✗ not linear

**One more example**: For a polynomial $p$,

eg., $5x^2+3x+2$

let

$$f(p) = (2+3x) \cdot p.$$

$f : \text{polyn. of deg} \le 2 \longrightarrow \text{polyn. of deg} \le 3$

$f(p(x)) = (2+3x) \cdot p(x)$

$f$ maps polynomials in $x$ to polynomials in $x$

$\uparrow$ a vector space

and it is a linear transformation!

Claim: This is linear.

① $f(c \cdot p) = (2+3x) \cdot (c \cdot p) = c \cdot (2+3x) \cdot p = c \cdot f(p)$ ✓

② $f(p+q) = (2+3x)p + (2+3x)q = f(p) + f(q)$ ✓ ✓

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Example: Recall that {all 2×2 matrices} is a 4D vector space

$f(A) = A^T$

Claim: This is linear

⓪ $f\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

① $f(cA) = (cA)^T = cA^T = c \cdot f(A)$ ✓

② $f(A+B) = A^T + B^T$ ✓

We'll see lots more examples later (eg., differentiation, ...).

$$\boxed{\begin{array}{c}\text{LINEAR TRANSFORMATIONS} \\ \Updownarrow \quad \Downarrow \\ \text{MATRICES}\end{array}}$$

We'll see the correspondence today for maps $\mathbb{R}^n \to \mathbb{R}^m$, and generalize it to arbitrary vector spaces next ~~week~~ lecture

**Theorem 1**: Let $A$ be an $m \times n$ matrix.
Define $f: \mathbb{R}^n \to \mathbb{R}^m$ by

$$f(\vec{v}) = A\vec{v}.$$

Then $f$ is a linear transformation.

Proof: 0. $f(\vec{0}) = \vec{0}$

1. $f(c\vec{v}) = A(c\vec{v}) = cA\vec{v} = c \cdot f(\vec{v})$ ✓
2. $f(\vec{v}+\vec{w}) = A(v+w) = Av + Aw = f(v) + f(w)$ ✓

**Theorem 2**: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be any linear transformation.
Then there exists an $m \times n$ matrix $A$ such that

$$f(\vec{v}) = A\vec{v}.$$

Proof:

$$A = \begin{array}{c} \begin{array}{ccc} \vec{e}_1 & \vec{e}_j & \vec{e}_n \end{array} \\ \left(\begin{array}{ccc} | & | & | \\ f(\vec{e}_1) & f(\vec{e}_j) & f(\vec{e}_n) \\ | & | & | \end{array}\right) \end{array}$$

$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

Claim: For any $\vec{v} \in \mathbb{R}^n$,
$f(\vec{v}) = A\vec{v}$.

Proof: Write $v = (v_1, \ldots, v_n)$

$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$

$$\begin{aligned} f(\vec{v}) &= f(v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n) \\ &= \sum_j f(v_j\vec{e}_j) \\ &= \sum_j v_j f(\vec{e}_j) \quad \square \end{aligned}$$

$A\vec{v} = \sum_j v_j f(e_j)$

Digression: Pictures are bad!

$$A = \begin{array}{c} \quad j \\ i \left(\begin{array}{c} a_{ij} \end{array}\right) \end{array} \iff A = \sum_{i,j=1}^{n} a_{ij}\, \vec{e}_i \vec{e}_j^T$$

Observe: $\vec{e}_i \vec{e}_j^T = i\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1} \begin{array}{c}(0 \cdots 1 \cdots 0)_{1\times n} \\ j\end{array} = i\begin{pmatrix} 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 \end{pmatrix}_{n\times n}$ (with $j$ marking the column)

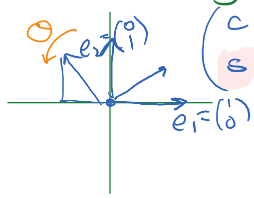
$$A = \sum_{j=1}^{n} f(\vec{e}_j)\, \vec{e}_j^T \quad (0 \;\; 1 \;\; 0)$$

$$\rightarrow A = \sum_{j=1}^{n} f(\vec{e}_j)\cdot \vec{e}_j^{\,T}$$

**Note**: This proof is important! It constructs the matrix $A$.

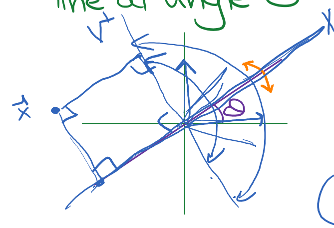
**Examples**: Give the matrices for these linear transformations:

a) Rotation of $\mathbb{R}^2$ by $\theta$:




$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

b) Reflection of $\mathbb{R}^2$ about the line at angle $\theta$:



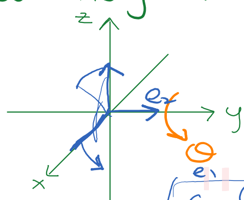

What is a reflection about a subspace $V$?
1. $x \in V$ should be fixed
2. $y \in V^\perp$ should be negated

$$(A + I)\vec{x} = 2\left(\text{Projection of } x \text{ onto } \vec{v}\right) = \frac{x \cdot v^T}{\|v\|^2} x$$

$$\Rightarrow A = 2\frac{vv^T}{\|v\|^2} - I = 2\binom{c}{s}(c \ s) - I$$

$$v = \binom{\cos\theta}{\sin\theta} \qquad = \begin{pmatrix} 2c^2 - 1 & 2cs \\ 2cs & 2s^2 - 1 \end{pmatrix} \checkmark$$

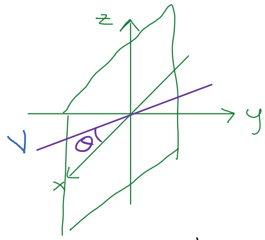
c) Rotation in $\mathbb{R}^3$ by $\theta$ about the $y$-axis:




$$\begin{array}{ccc} e_1 & e_2 & e_3 \end{array}$$
$$\begin{pmatrix} c & 0 & s \\ 0 & 1 & 0 \\ -s & 0 & c \end{pmatrix}$$

d) Reflection of $\mathbb{R}^3$ about the line in the $xz$-plane at angle $\theta$ above the $x$-axis:

$$V = \text{Span}\begin{pmatrix} c \\ 0 \\ s \end{pmatrix} = v$$

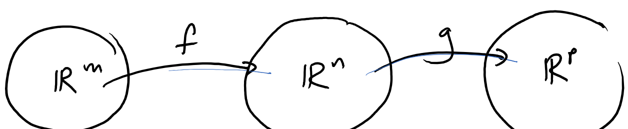



$$A = 2vv^T - I = \begin{pmatrix} 2c^2 - 1 & 0 & 2cs \\ 0 & -1 & 0 \\ 2cs & 0 & 2s^2 - 1 \end{pmatrix}$$

These theorems are why matrix-vector multiplication is defined the way it is.

What about matrix-matrix multiplication?

$$\boxed{\text{MATRIX MULTIPLICATION} \longleftrightarrow \text{LINEAR FUNCTION COMPOSITION}}$$

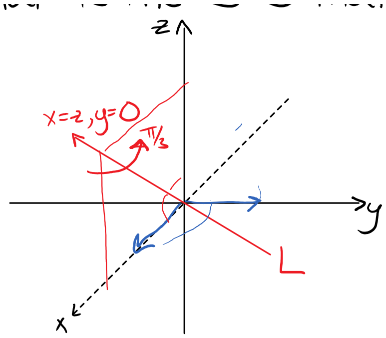



$f(u) = Au \qquad g(v) = Bv$

$$g(f(u)) = B \cdot A \cdot u$$

**Exercise**: Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that rotates by $\pi/3$ radians about the line $\{x = z, y = 0\}$. What is the $3\times3$ matrix representing $f$?


Answer: In three steps:



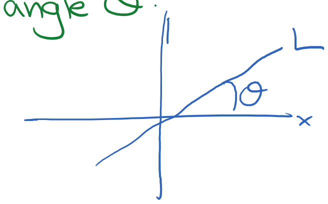

Answer: In three steps:
1. Rotate L to the x-axis
2. Rotate by $\pi/3$ about x-axis
3. Go back.

$c = \cos\frac{\pi}{3} = \frac{1}{2}$
$s = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\begin{pmatrix} | & | & | \\ f(\vec{e}_1) & f(e_2) & f(e_3) \\ | & | & | \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \overset{R_{x,\theta}}{\overset{\shortparallel}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix}}} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}}_{\text{answer!}}$$

Reflection of $\mathbb{R}^2$ about the line at angle $\theta$:




Answer: In three steps:
1. Rotate L to the x-axis
2. Reflect about x-axis
3. Go back.

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix}}$$
$$= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} c & s \\ s & -c \end{pmatrix} = \begin{pmatrix} c^2-s^2 & 2cs \\ 2cs & s^2-c^2 \end{pmatrix}$$
$$= \begin{pmatrix} 2c^2-1 & 2cs \\ 2cs & 2s^2-1 \end{pmatrix} \checkmark$$

**Exercise**: Constructing linear transformations.

a) Give a matrix mapping $(1,2,3) \mapsto (4,5,6)$.

Answer:

There are many possible answers, eg.,

$$\begin{pmatrix} 4 & 0 & 0 \\ 5 & 0 & 0 \\ 6 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

We can find all solutions: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

$$a_{11} + 2a_{12} + 3a_{13} = 4$$
$$a_{21} + 2a_{22} + 3a_{23} = 5$$
$$a_{31} + \underbrace{2a_{32} + 3a_{33}}_{\text{these are all free variables.}} = 6$$

b) Give a matrix mapping

$\vec{a} = (1,2,3) \longmapsto (4,5,6) = f(a)$
$\vec{b} = (1,1,1) \longmapsto (1,0,0) = f(b)$
$\vec{c} = (2,3,4) \longmapsto (0,1,0) = f(c)$

$\vec{c} = \vec{a} + \vec{b} \underset{\text{linearity}}{\Longrightarrow} f(c) = f(a) + f(b)$

NOT true $\Rightarrow$ no linear function (no matrix) works

$\vec{a} = (1,2,3) \longmapsto (4,5,6)$
$\vec{b} = (1,1,1) \longmapsto (1,0,0)$
$\vec{c} = (11,21,31) \longmapsto (0,1,0)$

$c = 10\vec{a} + \vec{b}$ $\quad f(c) \neq 10f(a) + f(b)$

$\Rightarrow$ no linear transformation works ✓

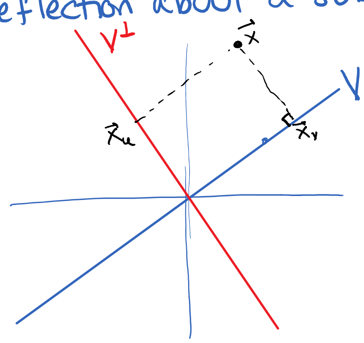
c) Give a matrix mapping
$(1,2,3) \longmapsto (4,5,6)$
$(1,1,1) \longmapsto (1,0,0)$
$(1,3,2) \longmapsto (1,2,3)$

$$(1\ 2\ 3) \longrightarrow e_1 \longrightarrow (4\ 5\ 6)$$
$$(1\ 1\ 1) \longrightarrow e_2 \longrightarrow (1\ 0\ 0)$$
$$(1\ 3\ 2) \longrightarrow e_3 \longrightarrow (1\ 2\ 3)$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix}^{-1} \qquad \begin{pmatrix} 4 & 1 & 1 \\ 5 & 0 & 2 \\ 6 & 0 & 3 \end{pmatrix}$$

Answer: $\begin{pmatrix} 4 & 1 & 1 \\ 5 & 0 & 2 \\ 6 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix}^{-1}$

Reflection about a subspace $V$:




$R_V$

- $\vec{v} \in V \Rightarrow R\vec{v} = \vec{v}$
- $\vec{u} \in V^\perp \Rightarrow R\vec{u} = -\vec{u}$

$R_V + I$

- $\vec{v} \in V \Rightarrow (R+I)\vec{v} = 2\vec{v}$
- $u \in V^\perp \Rightarrow (R+I)\vec{u} = \vec{0}$

$\Downarrow$

$\forall \vec{x},\ (R_V + I)\vec{x} = 2\vec{x}_v$
$= 2\,\mathrm{Proj}_V(\vec{x})$

$R_V = 2P_V - I$

We'll construct $P_V$ for arbitrary $V$ next week

if $\dim V = 1$, $V = \mathrm{span}\{\vec{v}\}$
then
$P_V = \dfrac{\vec{v}\,\vec{v}^T}{\|v\|^2}$

Answer 1:
This can be solved like in ⓐ, but now with

9 equations for the 9 unknowns.

Answer 2:

Let
$$B = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 1 & 1 \\ 5 & 0 & 2 \\ 6 & 0 & 3 \end{pmatrix}.$$

Then $B$ maps
$$\begin{pmatrix}1\\0\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \longmapsto \begin{pmatrix}1\\3\\2\end{pmatrix}$$

$C$ maps
$$\begin{pmatrix}1\\0\\0\end{pmatrix} \longmapsto \begin{pmatrix}4\\5\\6\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\0\\1\end{pmatrix} \longmapsto \begin{pmatrix}1\\2\\3\end{pmatrix}$$

So $C \cdot B^{-1}$ does what we want:

```
>> B = [1 1 1; 2 1 3; 3 1 2]; C = [4 1 1; 5 0 2; 6 0 3];
>> A = C * inv(B)

A =

    0.0000   -1.0000    2.0000
   -2.3333   -0.3333    2.6667
   -3.0000   -0.0000    3.0000
```

```
>> A * [1 2 3]'

ans =

     4
     5
     6

>> A * [1 1 1]'

ans =

    1.0000
    0.0000
    0.0000

>> A * [1 3 2]'

ans =

     1
     2
     3
```

Example:

- $f(p) = (2+3x)p$

degree $\le 1$ poly $\longrightarrow$ degree $\le 2$ poly

$\underline{1, x}$ $\qquad$ $\underline{1, x, x^2}$

$$[f] = \begin{array}{c} \begin{array}{cc} 1 & x \end{array} \\ \begin{pmatrix} 2 & 0 \\ 3 & 2 \\ 0 & 3 \end{pmatrix} \end{array}$$

$f(1) = 2 + 3x$
$= 2 \cdot 1 + 3 \cdot x + 0 \cdot x^2$
$\to \begin{pmatrix}2\\3\\0\end{pmatrix}$

$f(x) = 2x + 3x^2$

- $f(A) = A^T$

$f : \left\{ \begin{matrix} 2\times 2 \\ \text{matrices} \end{matrix} \right\} \circlearrowleft$

$\overset{\shortparallel}{V} = \operatorname{Span}\left\{ \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix} \right\}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}_{\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}}$$

$$[f] = \begin{array}{c|cccc} & \begin{pmatrix}1&0\\0&0\end{pmatrix} & \begin{pmatrix}0&1\\0&0\end{pmatrix} & \begin{pmatrix}0&0\\1&0\end{pmatrix} & \\ \hline \begin{pmatrix}1&0\\0&0\end{pmatrix} & 1 & 0 & 0 & 0 \\ \begin{pmatrix}0&1\\0&0\end{pmatrix} & 0 & 0 & 1 & 0 \\ \begin{pmatrix}0&0\\1&0\end{pmatrix} & 0 & 1 & 0 & 0 \\ \begin{pmatrix}0&0\\0&1\end{pmatrix} & 0 & 0 & 0 & 1 \end{array}$$

"matrix for $f$"

$$f\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\right) = \begin{pmatrix}1&0\\0&0\end{pmatrix} = 1 \cdot \begin{pmatrix}1&0\\0&0\end{pmatrix} + 0 \left(\text{the other basis vectors}\right)$$

$$f\left(\begin{pmatrix}0&1\\0&0\end{pmatrix}\right) = \begin{pmatrix}0&0\\1&0\end{pmatrix} = 0 \cdot \begin{pmatrix}1&0\\0&0\end{pmatrix} + 0 \cdot \begin{pmatrix}0&1\\0&0\end{pmatrix} + 1 \cdot \begin{pmatrix}0&0\\1&0\end{pmatrix} + 0 \cdot \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

- $f(p(x)) = p(2+3x)$

BAD example $\quad$ not linear

Exercise: Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation that

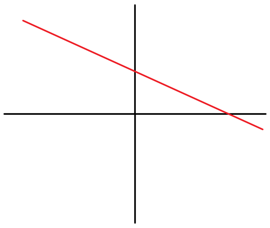
- First rotates the $xz$-plane by $\pi/3$ radians counterclockwise about the $y$-axis,
- Then reflects everything about the $yz$-plane (i.e., switching the sign of the $x$ coordinate),
- Then rotates the $xy$-plane by $\pi/6$ radians counterclockwise about the $z$-axis.

What is the 3×3 matrix representing $f$?
What is the determinant of this matrix?

Trivia:

## How to Work With Affine Spaces & Affine Transformations

**Q**: Is this a subspace?



**No**. Subspaces have to go through $\vec{0}$, to be closed under multiplication (by 0).

**Q**: Is this a linear transformation?

$$(x,y,z) \longmapsto (x+1,y,z)$$

**No**. A linear transformation has to take $\vec{0}$ to $\vec{0}$, in order to satisfy linearity under multiplication:

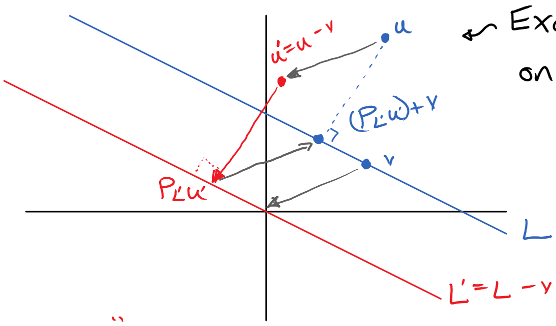
$$f(\alpha\vec{v}) = \alpha \cdot f(\vec{v})$$

$$\Rightarrow f(\vec{0}) = \vec{0}, \text{ setting } \alpha = 0.$$

Thus translations are not linear.

Of course, affine subspaces and affine transformations, that shift the origin. How can we manipulate them?

1) Shift coordinates so the space goes through $\vec{0}$,
work there,
then shift/translate back.




← Example: How to project u onto the affine subspace L.

2) Use "homogeneous coordinates"

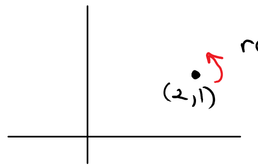
Add a new coordinate, $w = 1$. Then,

$$T: (x,y,z) \longmapsto (x+1,y,z)$$

corresponds to

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : (x,y,z,w) \mapsto (x+w,y,z,w)$$

**Example**: Using homogeneous coordinates, give a 3×3 matrix for the 2D affine transformation



rotate 30 degrees about the point $(2,1)$.

Answer: Shift to the origin $(0,0)$, rotate there, and shift back.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} x & y & w \end{array} \\ \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} x & y & w \end{array} \\ \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$c = \cos 30^\circ = \frac{\sqrt{3}}{2}$

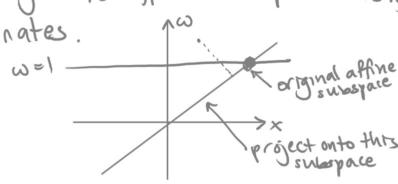
$s = \sin 30^\circ = \frac{1}{2}$

$$= \begin{array}{c} \begin{array}{ccc} x & y & w \end{array} \\ \begin{pmatrix} c & -s & 2-2c+s \\ s & c & 1-c-2s \\ 0 & 0 & 1 \end{pmatrix} \end{array} = M \quad \checkmark$$

Sanity check: $M\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix} \checkmark \quad M\begin{pmatrix}3\\1\\1\end{pmatrix}$

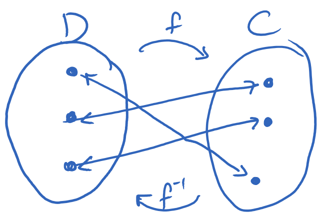
$$M\begin{pmatrix}3\\1\\1\end{pmatrix} = M\begin{pmatrix}2\\1\\1\end{pmatrix} + M\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2+c\\1+s\\1\end{pmatrix} \checkmark$$

**Note**: You can also project to affine subspaces using homogeneous coordinates.



# Matrix and function inverses

**Definition:** A function $f : \underset{\text{domain}}{D} \to \underset{\text{codomain}}{C}$ is **invertible** if every point in $C$ is the image of exactly one point in $D$.

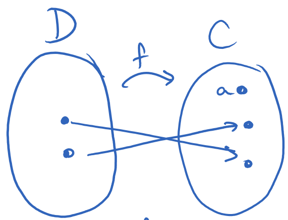



The inverse function $f^{-1} : C \to D$ takes each point in $C$ to its unique preimage.

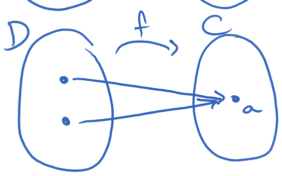
Equivalently, $f^{-1}$ satisfies

$f^{-1} \circ f$ = identity on $D$

$f \circ f^{-1}$ = identity on $C$




a isn't the image of anything
$\Rightarrow f$ not invertible




a has two preimages
$\Rightarrow f$ not invertible

**Exercise:** Prove that the inverse of a linear function, if it exists, is also linear.

**Definition:** The **inverse** of a matrix $A$ is a matrix $B$ that satisfies

$BA$ = the identity matrix and $AB$ = the identity matrix.

**Observe:** • Not every matrix is invertible, e.g.,

$A = (\,0\,)$ is not invertible.

A matrix that is not invertible is called **singular**.

• If an inverse exists, then it is unique.

Proof: Assume $B$ and $C$ are both inverses of $A$. Consider $BAC$.

$$C = (BA)C = BAC = B(AC) = B \quad \Rightarrow B = C \quad \square$$

**Example:** The $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible/nonsingular if and only if $ad - bc \neq 0$. The inverse is given by

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(Check this!)

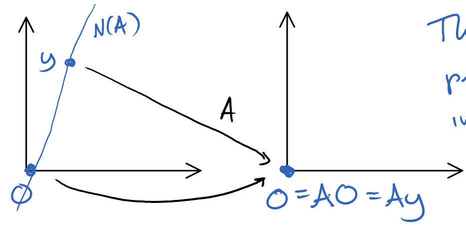
**How to compute the inverse of a matrix?**

Gaussian elimination, of course ...

## When is a matrix invertible?

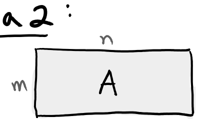
**Lemma 1:** If $N(A) \neq \{\vec{0}\}$, then $A$ is not invertible.

Proof: Take $y \in N(A)$, $y \neq 0$.




Then the point 0 has two preimages, so $A$ is not invertible. □

**Lemma 2:**




$m < n$ $\Rightarrow$ $N(A) \neq \{\vec{0}\}$

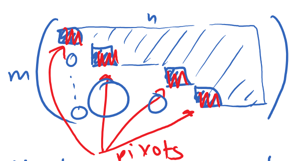
i.e., it is **more**!

Intuition:

$A$ "squashes" $n$ dimensions down to $m < n$ dimensions.

$\therefore$ Some directions must be collapsed to $\vec{0}$.

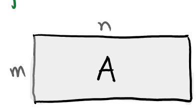
Proof: To find $N(A)$, apply Gaussian elimination:




The point is that there are at most $m$ pivots. Since $m < n$, there are necessarily at least $n - m \geq 1$ free variables so the nullspace is infinite. ✓ □

**Corollary:** No non-square matrix can be inverted.

Proof: Either

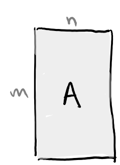



$m < n$

$\Rightarrow N(A) \neq \{\vec{0}\}$ (Lemma 2)

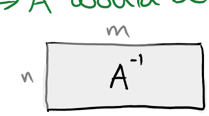
$\Rightarrow A$ not invertible (Lemma 1) ✓

or



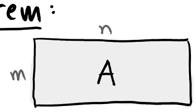

$m > n$

$\Rightarrow A^{-1}$ would be $n \times m$




and this can't be inverted ✓ □

**Theorem:**




is invertible $\Leftrightarrow$ $m = n$, $N(A) = \{\vec{0}\}$

**Remark:** This doesn't work in infinite dimensions, e.g.,

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ \vdots & 0 & 1 & \ddots \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ \vdots & 0 & 1 & \ddots \\ & \vdots & 0 & \ddots \\ & & \vdots & \end{pmatrix}$$

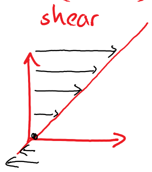
$N(A) = \{\vec{0}\}$, but $A$ is not invertible.

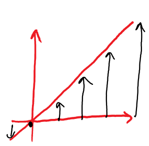
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ & & \vdots & & \ddots \end{pmatrix}$$

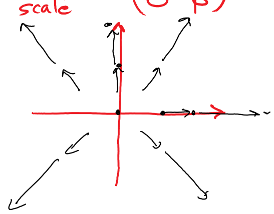
$N(A) = \mathrm{Span}(\vec{e}_1)$, but $R(A)$ = everything.

# I. 2×2 matrices as linear transformations

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ shear $\qquad A^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad S(\alpha,\beta) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ scale







**Claim**: Any 2×2 matrix can be expressed as a product of shearing and scaling matrices, eg.,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} bc & 0 \\ 0 & ad-bc \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{bc} & 0 \\ 0 & \frac{1}{ac} \end{pmatrix}$$

(if $abc \neq 0$)

Proof: Starting with any 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, G.E. row operations change it to a matrix of the form either

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$