Homework 3 answers

Note: For full credit, show your work! You are welcome to discuss the problems with others, but write up your own solutions.

① Which of the following sets span $\mathbb{R}^3$?

ⓐ $\{(1,1,1)\}$

No, one vector spans only a line.

ⓑ $\{(1,0,0), (0,0,1), (1,0,1)\}$

No, the third vector is the sum of the first two. They span the xz-plane.

ⓒ $\{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$

Yes, the first three vectors already span $\mathbb{R}^3$.

ⓓ $\{(1,2,1), (2,0,-1), (4,4,1)\}$

No. These vectors span the row-space of the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ 4 & 4 & 1 \end{pmatrix} \xrightarrow{-2,\ -4} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -4 & -3 \\ 0 & -4 & -3 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Applying Gaussian elimination does not change the row space. Therefore it is two-dimensional, spanned by $(1,2,1)$ and $(0,4,3)$.

ⓔ $\{(1,2,1), (2,0,-1), (4,4,0)\}$

Yes. This is almost the same as part ⓓ, except now the 3^{rd} vector does not lie in the span of the first two. Therefore they span a 3D space: all $\mathbb{R}^3$.

② What is the span of the union

$\{3\times3$ symmetric matrices, i.e. $A = A^T\} \cup \{3\times3$ upper-triangular matrices$\}$?

The span is the set of $\boxed{\text{all } 3\times3 \text{ matrices}}$.

Indeed, any 3×3 matrix A can be expressed as

$$A = D + L + U,$$

where D is the diagonal portion of A, L the part of A below its diagonal, and U the part above the diagonal.

Hence

$$A = \underbrace{(D + L + L^T)}_{\text{a symmetric matrix}} + \underbrace{(U - L^T)}_{\text{an upper-triangular matrix}}$$

③ Let $S = \{s_1, \dots, s_r\}$, $T = \{s_1, \dots, s_r, s_{r+1}\}$ be two sets of vectors from the same vector space.

When is $\text{Span}(S) = \text{Span}(T)$?

Since S is contained in T, $\text{Span}(S) \subseteq \text{Span}(T)$.

- If the new vector $s_{r+1} \in \text{Span}(S)$, then $\text{Span}(S) = \text{Span}(T)$.
  (This is because any linear combination that includes a multiple of s_{r+1} can be reexpressed in terms of just $s_1, \dots, s_r$ by substituting the expansion of s_{r+1} in terms of $s_1, \dots, s_r$.)
- If $s_{r+1} \notin \text{Span}(S)$, then $\text{Span}(S) \neq \text{Span}(T)$.
  (This is because $s_{r+1} \in \text{Span}(T)$ but not in $\text{Span}(S)$.)

$\Rightarrow$ Hence, $\text{Span}(S) = \text{Span}(T)$ if and only if $s_{r+1} \in \text{Span}(S)$.

④ Find the rowspace, columnspace, nullspace and left nullspace for each of the following matrices. Express each answer as the span of a minimal set of vectors.
(The left nullspace of a matrix A is $N(A^T)$.)

	A	col. space $R(A)$	row space $R(A^T)$	nullspace $N(A)$	left nullspace $N(A^T)$
ⓐ	$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$	span of $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$	span of $(1,2,0,1)$ $(0,1,1,0)$	span of $\begin{pmatrix}2\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}$	span of $(1,0,-1)$
ⓑ	$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	span of $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$	same as above	same as above	span of $(0,0,1)$
ⓒ	$\begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{pmatrix}$	span of $\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}$	span of $(1,0,3)$ $(0,1,2)$	span of $(3,2,-1)$	span of $\begin{pmatrix}4\\-1\\-2\end{pmatrix}$
ⓓ	$\begin{pmatrix} 1 & -4 & 4 \\ 4 & -8 & 6 \\ 0 & -4 & 5 \end{pmatrix}$	same as above	span of $(1,0,-1)$ $(0,4,-5)$	span of $(4,5,4)$	same as above

⑤ Construct a 3×3 matrix A with

$$R(A) = R(A^T) = \mathrm{Span}\left(\{(1,0,1),\ (1,5,0)\}\right)$$

If $A = \boxed{\begin{pmatrix} 1 & 1 & 4/5 \\ 1 & 5 & 0 \\ 4/5 & 0 & 4/5 \end{pmatrix}}$, e.g., then $R(A) = R(A^T) = \mathrm{Span}\left(\begin{pmatrix}1\\5\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right)$ ✓

There are lots of matrices that work, and lots of ways of solving this problem.

For example:

A doesn't have to be symmetric, but it will be easier if it is. So let's assume $A = A^T$.

$$\begin{pmatrix} 1 & \blacksquare & \blacksquare \\ 0 & \blacksquare & \blacksquare \\ 1 & \blacksquare & \blacksquare \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & \blacksquare & \blacksquare \\ 1 & \blacksquare & \blacksquare \end{pmatrix}$$

$A = A^T$ fills in the first row

to get $\begin{pmatrix}1\\0\\1\end{pmatrix}$ in $R(A)$, make it a column

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & \blacksquare & 5 \\ 1 & 5 & 0 \end{pmatrix}$$

next I filled column 3 with $\begin{pmatrix}1\\5\\0\end{pmatrix}$ to get that in $R(A)$ (also row 3)

$\begin{pmatrix}1\\5\\0\end{pmatrix}$ doesn't fit in col/row 2

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & x & 5 \\ 1 & 5 & 0 \end{pmatrix}$$

What should x be?

We want column 2 $\in \mathrm{Span}\{$columns 1 and 3$\}$ (so it doesn't make $R(A)$ too big).

Writing
$$\begin{pmatrix}0\\x\\5\end{pmatrix} = a\begin{pmatrix}1\\0\\1\end{pmatrix} + b\begin{pmatrix}1\\5\\0\end{pmatrix} = \begin{pmatrix}a+b\\5b\\a\end{pmatrix},$$

we get $a = 5 \Rightarrow b = -5 \Rightarrow x = 5b = -25$.

Thus
$$A = \boxed{\begin{pmatrix} 1 & 0 & 1 \\ 0 & -25 & 5 \\ 1 & 5 & 0 \end{pmatrix}}$$
works.

⑥ Which of the row, column, null and left-null subspaces are the same for these matrices of different sizes?

ⓐ A and $\begin{bmatrix} A \\ A \end{bmatrix}$ — same row and null spaces.

(b) $\begin{bmatrix} A \\ A \end{bmatrix}$ and $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$. same column and left-null spaces

(7) Explain why a nonzero row of a matrix A cannot be in its nullspace.

Let r be the jth row of A. Then the jth coordinate of Ar is $r^T r$. If $Ar = 0$, therefore $r^T r = 0$, implying that $r = 0$. Therefore, a row of a matrix <u>can</u> be in that matrix's nullspace, but only if the row is all-zeros.

(8) Matrix multiplication and the nullspace

(a) Prove that

(*) $$N(A) \subseteq N(BA)$$

for any matrices A and B so BA is defined.

$$x \in N(A) \Rightarrow Ax = 0 \Rightarrow BAx = 0 \Rightarrow x \in N(BA)$$

Therefore $N(A) \subseteq N(BA)$.

(b) Prove that if B is invertible, then

$$N(BA) = N(A).$$

Given Eq. (*), we have to show additionally that $N(BA) \subseteq N(A)$, i.e., if $x \in N(BA)$, so $BAx = 0$ then $Ax = 0$. Indeed, multiplying both sides of the equation $BAx = 0$ by B^{-1} gives $Ax = 0$, so $x \in N(A)$. ✓

(c) Prove that

$$N(A^T A) = N(A).$$

(<u>Hint</u>: Equation (*) implies that $N(A^T A) \supseteq N(A)$, so it is enough to show the opposite inclusion, $N(A^T A) \subseteq N(A)$. Note that a vector z is zero if and only if $z^T z = \sum_i (z_i)^2 = 0$.)

Let x be a vector with $A^T A x = 0$. We want to show that $Ax = 0$. Indeed, multiplying on the left by x^T gives $(Ax)^T (Ax) = 0$, which implies that $Ax = 0$ (as in the previous problem). ✓

(d) [Optional]

Assuming that $R(B^T) \supseteq R(A)$, prove that

$N(BA) = N(A)$.

Note that this implies parts (b) and (c) as special cases.

(Hint: This is slightly harder than (c), but almost the same proof should work. Start by applying a row transformation to B.)

As a partial converse, prove that if $R(B^T) \subset R(A)$ strictly, then $N(BA) \supset N(A)$ strictly.

We have already shown that $N(BA) \supseteq N(A)$.

It remains to show $N(BA) \subseteq N(A)$.

Let $\vec{x} \in N(BA)$. We know that $BA\vec{x} = \vec{0}$, and our goal is to show that $A\vec{x} = \vec{0}$.

Let $\vec{y} = A\vec{x}$. Since $R(A) \subseteq R(B^T)$, $\vec{y} \in R(B^T)$.

Writing

$$B = \begin{pmatrix} - r_1^T - \\ - r_2^T - \\ \vdots \\ - r_m^T - \end{pmatrix}, \quad y \in \mathrm{Span}\{\vec{r}_1, \ldots, \vec{r}_m\}$$

$$\Rightarrow \exists\, c_1, \ldots, c_m : \vec{y} = \sum_i c_i \vec{r}_i$$

$B\vec{y} = \vec{0} \Rightarrow$ for all j, $\vec{r}_j \cdot \vec{y} = 0$

$$\Rightarrow \sum_j c_j (\vec{r}_j \cdot \vec{y}) = 0$$

substitute

$$= \left(\sum_j c_j \vec{r}_j\right) \cdot \left(\sum_i c_i \vec{r}_i\right) = \vec{y} \cdot \vec{y} = \|\vec{y}\|^2 = 0$$

$$\Rightarrow \vec{y} = \vec{0} \checkmark$$

Here's another proof:

By assumption, the row space of B contains the column space of A. Thus each column, say c_i, of A can be written as a linear combination of the rows b_j of B:

$$c_i = \sum_j q_{ij} b_j$$

Let $Q = (q_{ij})_{ij}$ be the matrix of coefficients q_{ij}.

Then

$$QB = A^T$$

By Eq. (*), $N(QBA) \supseteq N(BA) \supseteq N(A)$, and by part (b), $N(QBA) = N(A^T A) = N(A)$. Hence $N(BA) = N(A)$. ✓

(e) Give an example of matrices A, B where $R(B^T) \not\supseteq R(A)$ and yet $N(BA) = N(A)$.

(Hint: The smallest example is with A a 2×1 matrix and B a 1×2 matrix.)

Let $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \end{pmatrix}$, so $BA = (1)$.
Then $N(A) = N(BA) = \{\vec{0}\}$. ✓

⑨ Finite fields

Use Gaussian elimination, by hand, to solve the following system of equations, mod 2:

$$\begin{aligned} x_1 + x_2 + x_3 \phantom{{}+x_4} &= 0 \\ x_2 + x_3 + x_4 &= 1 \\ x_1 \phantom{{}+x_2} + x_3 \phantom{{}+x_4} &= 1 \\ x_1 + x_2 \phantom{{}+x_3+x_4} &= 1 \end{aligned} \qquad (\text{mod } 2)$$

Answer:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

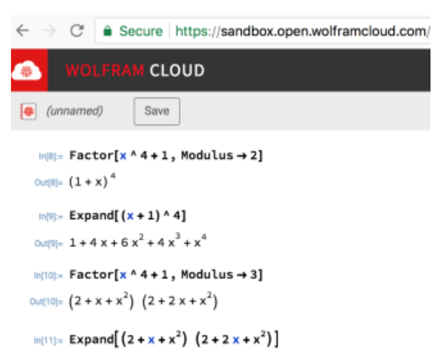
$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\Longrightarrow \boxed{\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$$

⑩ Finite fields

ⓐ Show that the polynomial x^4+1 is reducible (can be factored) mod 2, mod 3, and mod 5.
(In fact, it is reducible over any finite field.)
Hint: Feel free to use Mathematica, as in class.
If you haven't installed it, try https://sandbox.open.wolframcloud.com/



Answer:

Mod 2: $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$
$= x^4 + 1$

Mod 3: $(x^2+x+2)(x^2+2x+2)$
$= x^4 + 3x^3 + 6x^2 + 6x + 4$
$= x^4 + 1$

$= x^4+3x^3+6x^2+6x+4$

$= x^4+1$

Mod 5: $(x^2+3)(x^2+2) = x^4+5x^2+6$

$= x^4+1$

```
In[10]:= Factor[x ^ 4 + 1, Modulus → 3]
Out[10]= (2 + x + x^2) (2 + 2 x + x^2)

In[11]:= Expand[(2 + x + x^2) (2 + 2 x + x^2)]
Out[11]= 4 + 6 x + 6 x^2 + 3 x^3 + x^4

In[12]:= Factor[x ^ 4 + 1, Modulus → 5]
Out[12]= (2 + x^2) (3 + x^2)

In[13]:= Expand[(2 + x^2) (3 + x^2)]
Out[13]= 6 + 5 x^2 + x^4
```

(b) The polynomial x^3+x+1 is irreducible mod 2, so gives a field of size $2^3=8$. (The field elements are $0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1$.) Write out the multiplication table.

What are the inverses of the nonzero elements?

$(1^{-1}=?,\ x^{-1}=?,\ (x+1)^{-1}=?,\ \ldots,\ (x^2+x+1)^{-1}=?)$

Answer:

×	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
0	0	0	0	0	0	0	0	0
1		1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
x			x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
$x+1$				x^2+1	x^2+x+1	x^2	1	x
x^2					x^2+x	x	x^2+1	1
x^2+1						x^2+x+1	$x+1$	x^2+x
x^2+x							x	x^2
x^2+x+1								$x+1$

$1^{-1}=1,\ x^{-1}=x^2+1,\ (x+1)^{-1}=x^2+x,\ (x^2)^{-1}=x^2+x+1$

(c) Recall from class the field $\mathbb{F}_4$ with elements $0, 1, x, x+1$ and operations mod 2 and mod x^2+x+1.

(Here is the multiplication table:

×	0	1	x	$x+1$
0	0	0	0	0
1	0	1	x	$x+1$
x	0	x	$x+1$	1
$x+1$	0	$x+1$	1	x

)

Write out all the elements of

$$\mathrm{Span}\left(\begin{pmatrix}1\\x\\x+1\end{pmatrix}, \begin{pmatrix}x\\x\\x\end{pmatrix}\right).$$

(Your answer should have 16 vectors, a subspace of $\mathbb{F}_4^3$.)

Answer:

1\. Start by computing all the multiples of $\begin{pmatrix}1\\x\\x+1\end{pmatrix}$:

2\. Then get all multiples of $\begin{pmatrix}x\\x\\x\end{pmatrix}$:

	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\x\\x+1\end{pmatrix}$	$\begin{pmatrix}x\\x+1\\1\end{pmatrix}$	$\begin{pmatrix}x+1\\1\\x\end{pmatrix}$
$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}x\\ -1\end{pmatrix}$	$\begin{pmatrix}x\\x+1\\1\end{pmatrix}$	$\begin{pmatrix}x+1\\1\\x\end{pmatrix}$
$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$		$\begin{pmatrix}x+1\\x\\0\end{pmatrix}$	$\begin{pmatrix}x\\0\\x+1\end{pmatrix}$
$\begin{pmatrix}x\\x\\x\end{pmatrix}$	$\begin{pmatrix}x\\x\\x\end{pmatrix}$		$\begin{pmatrix}0\\1\\x+1\end{pmatrix}$	$\begin{pmatrix}1\\x+1\\0\end{pmatrix}$
$\begin{pmatrix}x+1\\x+1\\x+1\end{pmatrix}$	$\begin{pmatrix}x+1\\x+1\\x+1\end{pmatrix}$		$\begin{pmatrix}1\\0\\x\end{pmatrix}$	$\begin{pmatrix}0\\x\\1\end{pmatrix}$

3\. Finally, compute all sums $\alpha\begin{pmatrix}1\\x\\x+1\end{pmatrix}+\beta\begin{pmatrix}x\\x\\x\end{pmatrix}$:

② Over the same field $\mathbb{F}_4$, solve for the nullspace of

$$\begin{pmatrix}1 & x & x & 1 & 0\\ 0 & 1 & x & x & 1\\ 1 & 0 & 1 & x & x\\ x & 1 & 0 & 1 & x\\ x & x & 1 & 0 & 1\end{pmatrix}.$$

Answer: Use Gaussian elimination:

$$\begin{pmatrix}1 & x & x & 1 & 0\\ 0 & 1 & x & x & 1\\ 1 & 0 & 1 & x & x\\ x & 1 & 0 & 1 & x\\ x & x & 1 & 0 & 1\end{pmatrix} \longrightarrow \begin{pmatrix}1 & x & x & 1 & 0\\ 0 & 1 & x & x & 1\\ \cancel{0} & \cancel{x} & \cancel{x+1} & \cancel{x+1} & \cancel{x}\\ \cancel{0} & \cancel{x} & \cancel{x+1} & \cancel{x+1} & \cancel{x}\\ \cancel{0} & \cancel{1} & \cancel{x} & \cancel{x} & \cancel{1}\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 0 & 1 & x & x\\ 0 & 1 & x & x & 1\end{pmatrix}$$

$$\Rightarrow \quad a_1 = a_3 + x\cdot a_4 + x\cdot a_5$$
$$a_2 = x\cdot a_3 + x\cdot a_4 + a_5$$

Thus the nullspace is

$$\boxed{\operatorname{Span}\left(\underset{a_3}{\begin{pmatrix}1\\x\\1\\0\\0\end{pmatrix}},\ \underset{a_4}{\begin{pmatrix}x\\x\\0\\1\\0\end{pmatrix}},\ \underset{a_5}{\begin{pmatrix}x\\1\\0\\0\\1\end{pmatrix}}\right)}$$

⑪

a) By discretizing the interval and setting up a system of linear equations, solve numerically the differential equation

$$f''(t) - 2f'(t) = \cos(t)$$

on the interval $[0, \pi]$, with boundary conditions

$$f(0) = 0 \quad , \quad f(\pi) = 0$$

After setting up the sparse equations, solve them.
Experiment with finer discretizations.
Show your work, and plot the results

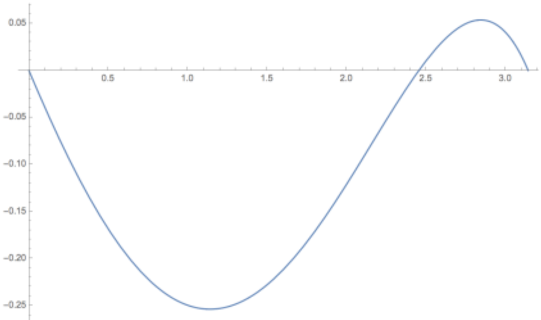
b) Do the same, except with boundary conditions

$$f(0) = 0 \quad , \quad f(\pi) = -1$$

Here are the exact solutions, according to Mathematica:

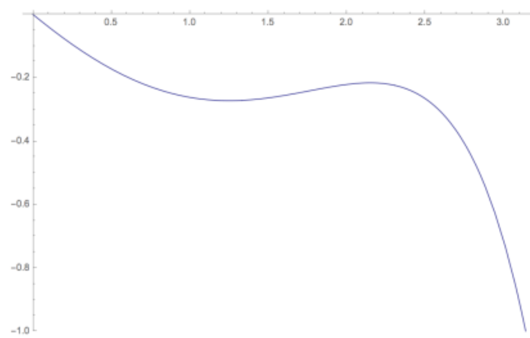
```
exactsolution = f /. DSolve[{f''[t] - 2 f'[t] == Cos[t], f[0] == 0, f[π] == 0}, f, t][[1]] //
  FullSimplify
Plot[exactsolution[t], {t, 0, π}, PlotRange → Automatic]
```

$$\text{Function}\left[\{t\}, -\frac{-1 - e^{2\pi} + 2e^{2t} - \text{Cos}[t] + e^{2\pi}\text{Cos}[t] - 2\,\text{Sin}[t] + 2e^{2\pi}\,\text{Sin}[t]}{5\left(-1 + e^{2\pi}\right)}\right]$$




```
exactsolution = f /. DSolve[{f''[t] - 2 f'[t] == Cos[t], f[0] == 0, f[π] == -1}, f, t][[1]] //
  FullSimplify
Plot[exactsolution[t], {t, 0, π}, PlotRange → {-1, 0}]
```

$$\text{Function}\left[\{t\}, -\frac{-6 - e^{2\pi} + 7e^{2t} - \text{Cos}[t] + e^{2\pi}\text{Cos}[t] - 2\,\text{Sin}[t] + 2e^{2\pi}\,\text{Sin}[t]}{5\left(-1 + e^{2\pi}\right)}\right]$$



Answer:

Discretize the interval $[0, \pi]$ as

and introduce variables $y_0, \ldots, y_n$, where we desire

$$y_j = f\left(\frac{j\pi}{n}\right), \quad j = 0, 1, 2, \ldots, n$$

(so $j \longleftrightarrow t = \frac{j\pi}{n}$).

Then

$$f'\left(\frac{j\pi}{n}\right) \approx \frac{f\left(\frac{(j+1)\pi}{n}\right) - f\left(\frac{j\pi}{n}\right)}{(\pi/n)} = \frac{n}{\pi}\left(y_{j+1} - y_j\right)$$

$$f''\left(\frac{j\pi}{n}\right) \approx \frac{f\left((j+1)\frac{\pi}{n}\right) - 2f\left(j\frac{\pi}{n}\right) + f\left((j-1)\frac{\pi}{n}\right)}{(\pi/n)^2}$$

$$= \left(\frac{n}{\pi}\right)^2\left(y_{j+1} - 2y_j + y_{j-1}\right)$$

Using the forward approximation to $f'(t)$, we get the equations

$$(*) \qquad \frac{1}{(\pi/n)^2}\left(y_{j-1} - 2y_j + y_{j+1}\right) - \frac{2}{\pi/n}\left(-y_j + y_{j+1}\right) = \cos\left(\frac{j\pi}{n}\right)$$

at all the interior points $j = 1, 2, \ldots, n-1$.

Plus, the boundary conditions give

$$y_0 = 0 \quad \text{and} \quad y_n = -1.$$

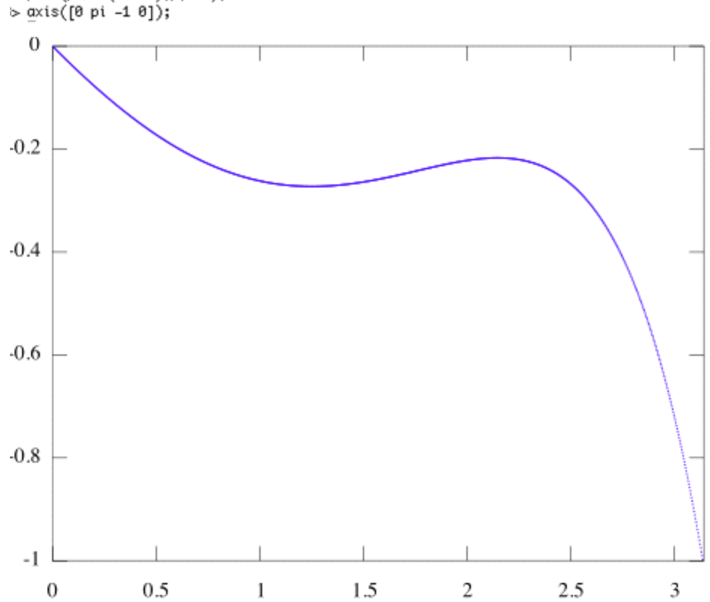
Since $y_0 = 0$, we can just drop that term from the $j=1$ equation. But the $j = n-1$ equation is trickier: substitute into it $y_n = -1$ to get

$$(**) \qquad \frac{1}{(\pi/n)^2}\left(y_{n-2} - 2y_{n-1}\right) - \frac{2}{\pi/n}\left(-y_{n-1}\right) = \cos\left(\frac{(n-1)\pi}{n}\right) + \frac{1}{(\pi/n)^2} - \frac{2}{\pi/n}$$

Now we'll use Eq. $(*)$ for $j = 1, \ldots, n-2$, and the last equation, Eq. $(**)$, to solve for $y_1, \ldots, y_{n-1}$.

Matlab commands

```
> n = 1000;
> e = ones(n-1,1);
> A = spdiags([e*(n/pi)^2,e*(-2*(n/pi)^2+2*(n/pi)),e*((n/pi)^2-2*(n/pi))], -1:1, n-1,n-1);
> b = cos((pi/n)*(1:n-1))';
> b(n-1) += (n/pi)^2 + 2*n/pi;
> y = A\b;
> plot(pi/n*(1:n-1),y,'.');
> axis([0 pi -1 0]);
```

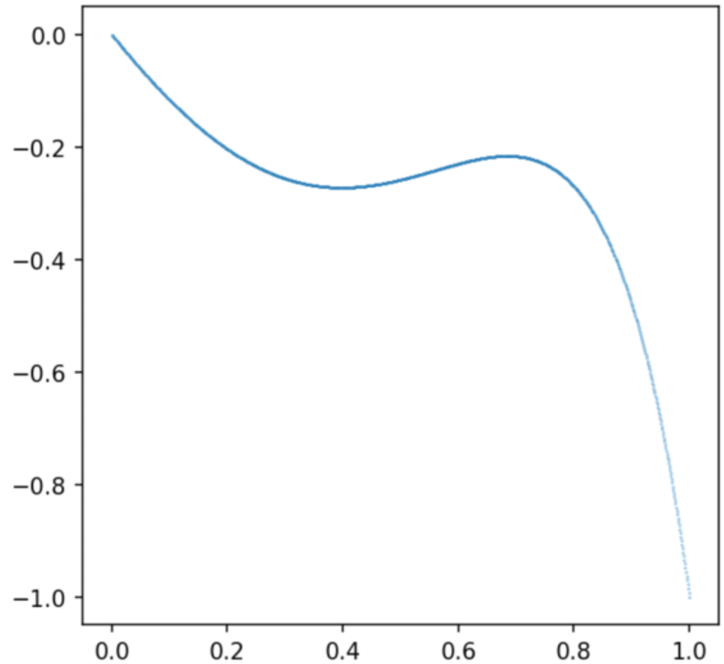


Python

Here, instead of correcting the first and last coordinates of $\vec{b}$ for the boundary conditions, I instead add new entries at the top and bottom (so $\vec{b} \in \mathbb{R}^{n+1}$ instead of $n-1$).

```python
import numpy as np
from scipy import sparse

n = 1000
dt = np.pi / n

# matrix for first derivative d/dt
DDt = 1/(2*dt) * sparse.diags((-1, 1), offsets=(-1,1),
                              shape=(n+1, n+1), format='csr')
DDt[0,:] = DDt[-1,:] = 0  # derivative d/dt only defined at internal vertices

# matrix for second derivative d^2/dt^2
D2Dt = 1/(dt ** 2) * sparse.diags((1, -2, 1), offsets=(-1,0,1),
                                  shape=(n+1, n+1), format='csr')
D2Dt[0,:] = D2Dt[-1,:] = 0

# A is the combined differential operator, with boundary conditions
A = D2Dt - 2 * DDt
# set the boundary conditions, using the (currently all-0) first and last rows
A[0,0] = A[-1,-1] = 1
print(A[:4,:4].toarray())

b = np.cos(dt * np.arange(n+1))
# set the boundary conditions
b[0], b[-1] = 0, -1

# Solve and plot the solution
from scipy.sparse.linalg import bicg
import matplotlib.pyplot as plt

y, exitCode = bicg(A, b)
```

```python
from scipy.sparse.linalg import bicg
import matplotlib.pyplot as plt

y, exitCode = bicg(A, b)

plt.figure(figsize=(5,5), dpi=150)
plt.scatter(np.arange(n+1)/n, y, marker='.', s=.2)
plt.show()
```



Mathematica code

a)

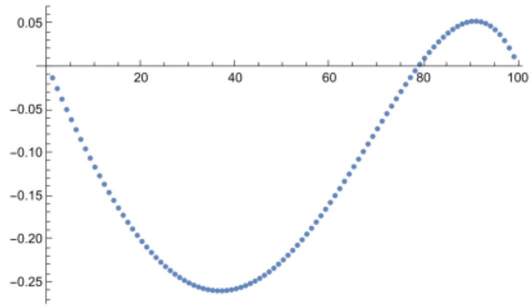
5. Discretizing the differential equation
$f''(t) - 2f'(t) = \mathrm{Cos}[t]$ with boundary conditions f(0)=0, f(π)=0.

Here I set up the equations using a dense matrix. The left-hand sides come from $f''[\frac{j\pi}{n}] - 2f'[\frac{j\pi}{n}]$.

```
In[189]:= n = 100;

"Allocate an all-zeros matrix";
A = Table[0, {n - 1}, {n - 1}];
"Then fill in each row of the matrix";
For[j = 1, j ≤ n - 1, j++,
  A[[j, j]] = (n/π)^2 (-2) - 2 (n/π (-1));
  "← coefficient of y_j in the jth equation";
  If[j < n - 1,
   A[[j, j + 1]] = (n/π)^2 1 - 2 (n/π (+1)); "← coefficient of y_(j+1)";
  ];
  If[j > 1,
   A[[j, j - 1]] = (n/π)^2 1; "← coefficient of y_(j-1)";
  ];
 ];

"Set up the right-hand sides of the equations, numerically";
b = Table[Cos[j π / n], {j, 1, n - 1}] // N;

x = LinearSolve[A, b];
ListPlot[x]
```

```
b = Table[Cos[j π / n], {j, 1, n - 1}] // N;

x = LinearSolve[A, b];
ListPlot[x]
```

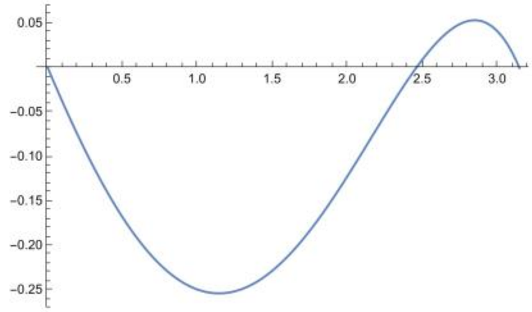
Out[197]=



Here I do it using sparse matrix routines. Note that I only specify the nonzero entries of the matrix. This saves a lot of memory! It also makes it much faster.

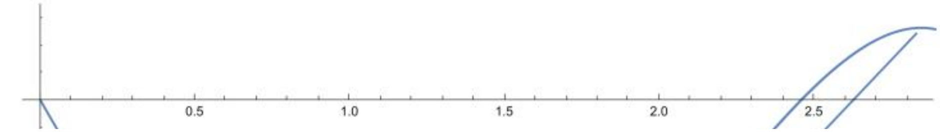
In[233]:=
```
n = 10 000;
A = SparseArray[Join[
    Table[{j, j} → -2 (n/π)^2 + 2 n/π, {j, 1, n - 1}],
    Table[{j, j - 1} → (n/π)^2, {j, 2, n - 1}],
    Table[{j, j + 1} → (n/π)^2 - 2 n/π, {j, 1, n - 2}]
  ]];
b = Table[Cos[π j/n], {j, 1, n - 1}] // N;

x = LinearSolve[A, b];

"I convert the x axis to go from 0 to π, instead of 0 to n";
x = Table[{π j/n, x〚j〛}, {j, 1, n - 1}];
plot4 = ListPlot[x, Joined → True]
```

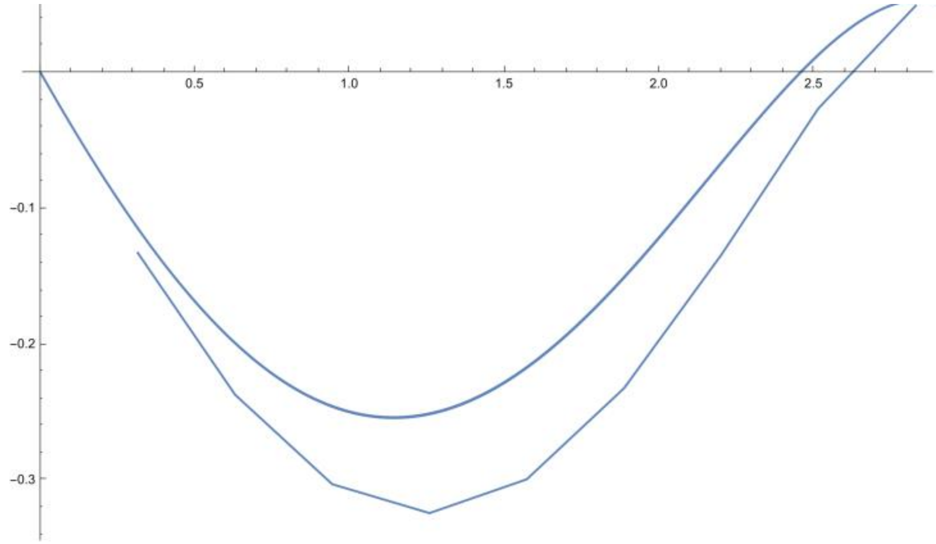
Out[239]=



Here I am combining the plots for different values of n, starting with $n = 10$. There are big errors early on, but then the plots converge nicely.

In[240]:=
```
Show[plot1, plot2, plot2, plot3, plot4]
```





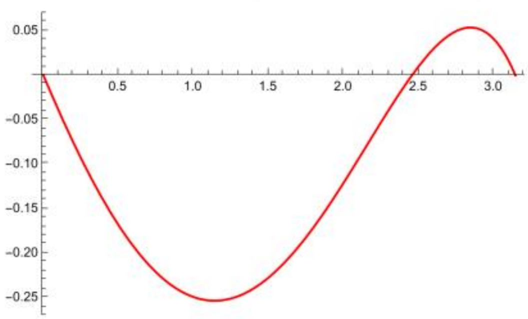
Out[240]=

This is *Mathematica*'s internal numerical differential equation solving routine. (It basically discretizes the interval the same way we have; the "InterpolatingFunction" it returns draws a straight line between adjacent points.)

In[245]:=
```
fsol = f /. NDSolve[{f''[t] - 2 f'[t] == Cos[t], f[0] == 0, f[π] == 0}, f, {t, 0, π}][[1]]
plot = Plot[fsol[t], {t, 0, π}, PlotStyle → Red]
```

Out[245]= InterpolatingFunction[Domain: {{0., 3.14}} Output: scalar]

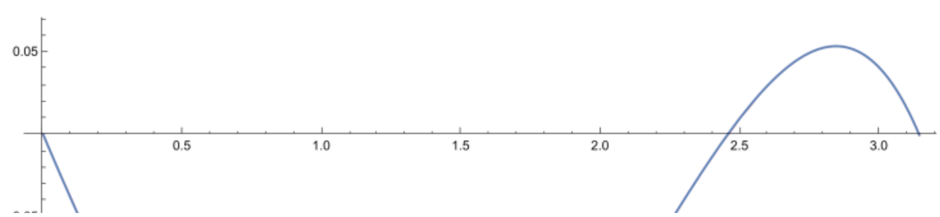
Out[246]=

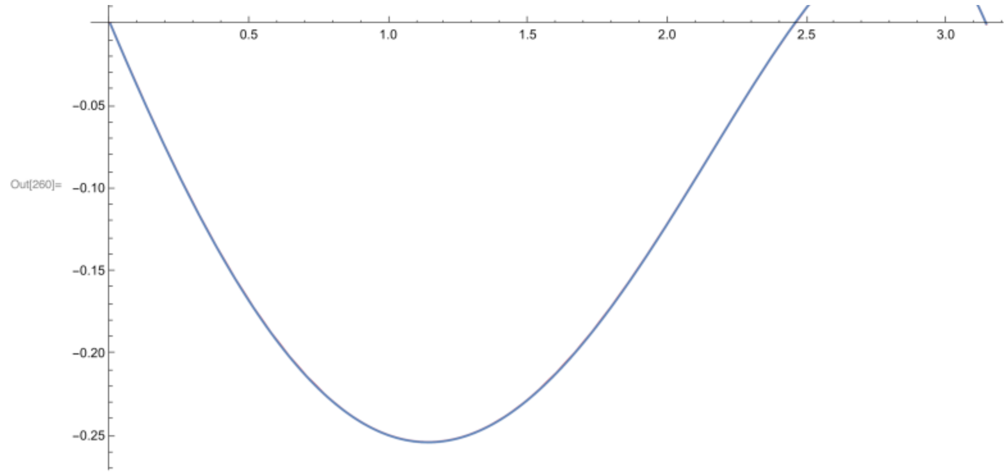


There's actually a simple exact solution to the differential equation. :)

In[258]:=
```
exactsolution =
  f /. DSolve[{f''[t] - 2 f'[t] == Cos[t], f[0] == 0, f[π] == 0}, f, t][[1]] // FullSimplify
exactplot = Plot[exactsolution[t], {t, 0, π}, PlotStyle → Red];
Show[exactplot, plot4]
```

Out[258]= Function$\left[\{t\}, -\dfrac{-1 - e^{2\pi} + 2 e^{2t} - \mathrm{Cos}[t] + e^{2\pi}\mathrm{Cos}[t] - 2\,\mathrm{Sin}[t] + 2 e^{2\pi}\mathrm{Sin}[t]}{5\left(-1 + e^{2\pi}\right)}\right]$





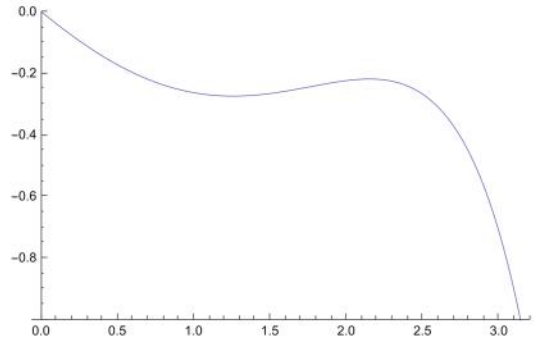
b)

Discretizing the differential equation
$f''(t) - 2f'(t) = \text{Cos}[t]$ with boundary conditions f(0)=0, f(π)=-1.


```
n = 1000;
A = SparseArray[Join[
     Table[{j, j} → -2 (n/π)^2 + 2 n/π, {j, 1, n - 1}],
     Table[{j, j - 1} → (n/π)^2, {j, 2, n - 1}],
     Table[{j, j + 1} → (n/π)^2 - 2 n/π, {j, 1, n - 2}]
    ]];
b = Table[Cos[π j/n], {j, 1, n - 1}] // N;
"Now fix the last entry in b to match the boundary condition:";
ylast = -1;
b[[n - 1]] = Cos[(n - 1)/n π] - ((n/π)^2 + 2 n/π) ylast;

x = LinearSolve[A, b];
x = Table[{π j/n, x[[j]]}, {j, 1, n - 1}];
plot4 = ListPlot[x, PlotRange → {-1, 0}, Joined → True]
```

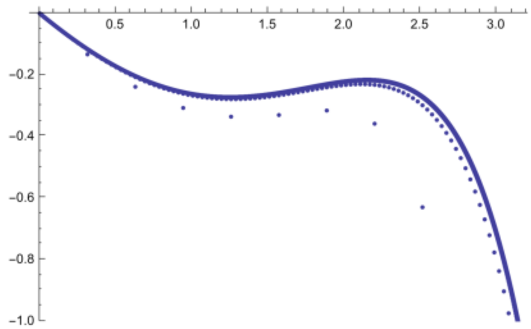
0.0

```
x = Table[{ π j / n, x[[j]]}, {j, 1, n - 1}];
plot4 = ListPlot[x, PlotRange → {-1, 0}, Joined → True]
```



Here I am combining the plots for different values of n, starting with $n = 10$. There are big errors early on, but then the plots converge nicely.

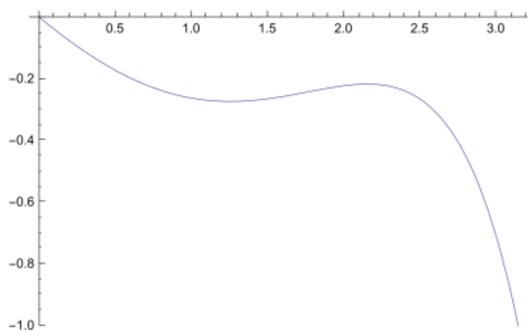
```
Show[plot, plot1, plot2, plot3, plot4, PlotRange → {-1, 0}]
```



This is *Mathematica*'s internal numerical differential equation solving routine. (It basically discretizes the interval the same way we have; the "InterpolatingFunction" it returns draws a straight line between adjacent points.)

```
fsol = f /. NDSolve[{f''[t] - 2 f'[t] == Cos[t], f[0] == 0, f[π] == -1}, f, {t, 0, π}][[1]]
plot = Plot[fsol[t], {t, 0, π}, PlotRange → {-1, 0}]
```

```
InterpolatingFunction[{{0., 3.14159}}, <>]
```



There's actually a simple exact solution to the differential equation. :)


```
exactsolution =
 f /. DSolve[{f''[t] - 2 f'[t] == Cos[t], f[0] == 0, f[π] == -1}, f, t][[1]] // FullSimplify
Plot[exactsolution[t], {t, 0, π}, PlotRange → {-1, 0}]
```

$$\text{Function}\left[\{t\},\ -\frac{-6 - e^{2\pi} + 7\,e^{2t} - \text{Cos}[t] + e^{2\pi}\,\text{Cos}[t] - 2\,\text{Sin}[t] + 2\,e^{2\pi}\,\text{Sin}[t]}{5\left(-1 + e^{2\pi}\right)}\right]$$

```
exactsolution =
 f /. DSolve[{f''[t] - 2 f'[t] == Cos[t], f[0] == 0, f[π] == -1}, f, t][[1]] // FullSimplify
Plot[exactsolution[t], {t, 0, π}, PlotRange → {-1, 0}]
```

Function$\left[\{t\}, -\frac{-6 - e^{2\pi} + 7 e^{2t} - \text{Cos}[t] + e^{2\pi}\text{Cos}[t] - 2\,\text{Sin}[t] + 2 e^{2\pi}\text{Sin}[t]}{5\left(-1 + e^{2\pi}\right)}\right]$