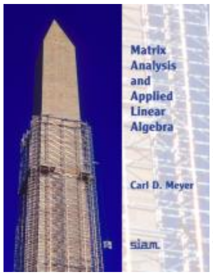
Lecture 5: Vector spaces (class)

Admin: Reading:



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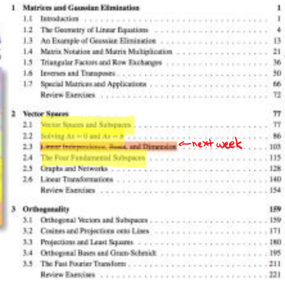


VECTOR SPACES

Why? They're everywhere!

$\mathbb{R}^4$	(a, b, c, d)
cubic polynomials	$ax^3 + bx^2 + cx + d$
2×2 matrices	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
multilinear polynomials in x and y	$axy + bx + cy + d$
⋮	

So it makes sense to abstract their properties and study them together.

And, the best way to understand **matrices** is as linear transformations on vector spaces.

Main properties: "LINEARITY"

Addition: $(a,b,c,d) + (e,f,g,h) = (a+e, b+f, c+g, d+h)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

$$(ax^3 + bx^2 + cx + d) + (ex^3 + fx^2 + gx + h) = (a+e)x^3 + (b+f)x^2 + \cdots$$

Scalar multiplication: $\mathbb{R}, \mathbb{C}, \mathbb{F}$

$$5(a,b,c,d) = (5a, 5b, 5c, 5d)$$

$$5\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 5a & 5b \\ 5c & 5d \end{pmatrix}$$

$$5(ax^3 + bx^2 + cx + d) = \cdots$$

These all behave the same way!

Note: Only under scalar multiplication, not arbitrary multiplication.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$(ax^3 + bx^2 + cx + d)(ex^3 + fx^2 + gx + h)$$
$$ae\,x^6 + (af + be)x^5 + \cdots$$

$(a,b,c,d)(e,f,g,h)$ is not defined!
(there's just the dot product $ae + bf + cg + dh$)

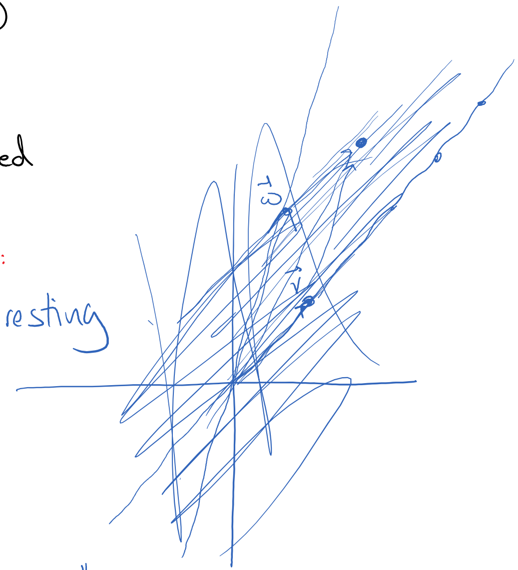
(there's just the dot product $ae+bf+cg+dh$)

Definition: A vector space consists of
- a set of "vectors" V
- a field $\mathbb{F}$ (often the reals $\mathbb{R}$ or complex #s $\mathbb{C}$)
- operations of
  - vector addition $V \times V \to V$, denoted $\vec{x}+\vec{y}$
  - scalar multiplication $\mathbb{F} \times V \to V$, denoted $\alpha\vec{x}$

that satisfy:
- closure under addition & scalar multiplication:
  for all $\alpha \in \mathbb{F}$, $\vec{x},\vec{y} \in V$ $\quad \boxed{\vec{x}+\vec{y} \in V \quad \alpha\vec{x} \in V}$ ← interesting
- existence of $\vec{0} \in V$
  $\vec{0}+\vec{x}=\vec{x}$ for all $\vec{x}$
- additive inverses
  for all $\vec{x} \in V$, there exists $\vec{y} \in V$ s.t. $\vec{x}+\vec{y}=\vec{0}$

  for all $\alpha,\beta \in \mathbb{F}$, $\vec{x},\vec{y},\vec{z} \in V$:
- commutativity $\quad \vec{x}+\vec{y}=\vec{y}+\vec{x}$
- associativity $\quad \vec{x}+(\vec{y}+\vec{z})=(\vec{x}+\vec{y})+\vec{z}$, $\quad \alpha(\beta\vec{x})=(\alpha\beta)\vec{x}$
- distributivity $\quad (\alpha+\beta)\vec{x}=\alpha\vec{x}+\beta\vec{x}$, $\quad \alpha(\vec{x}+\vec{y})=\alpha\vec{x}+\alpha\vec{y}$
- $1\vec{x}=\vec{x}$ (identity for multiplication)

usually automatically



Note: The most important properties to check are closure under addition and scalar multiplication. The other properties are usually automatic.

$\mathbb{R}^2$

Examples: Vector spaces are everywhere!

1. $\mathbb{R}^n$: real vectors $(x_1, x_2, \dots, x_n)$
   coordinate-wise addition & multiplication
2. $\mathbb{C}^n$
   matrices $\mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$
3. the single-point sets $\{0\}$ or $\{(0,0,\dots,0)\}$
   (trivially closed under addition & multiplication)

   But these are NOT vector spaces:
   $\{1\}$, $\{(1,0,0)\}$
   $\{(0,0),(1,0)\}$
   the interval $[0,1]$
4. function spaces, eg.,
   all functions $\mathbb{R}\to\mathbb{R}$
   all functions $[0,1]\to\mathbb{R}$

   addition $(f+g)(x)=f(x)+g(x)$
   multiplication $(cf)(x)=c\cdot f(x)$

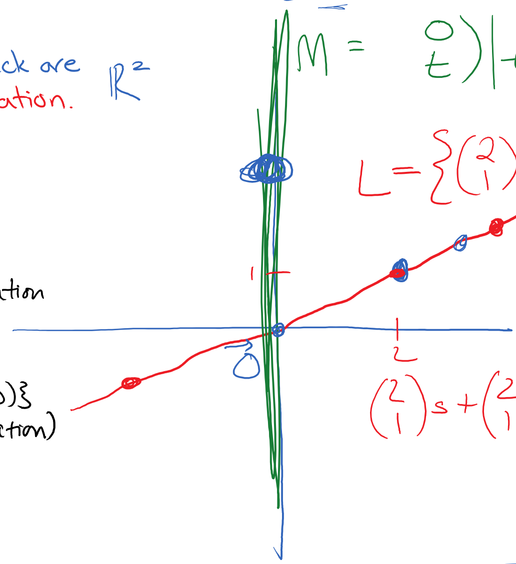
$g=5f$ means $g(x)=5f(x)$

$(f+g)(x) \equiv f(x)+g(x)$

$V=\{\vec{0}\}$ is a vector space

$M = \{\binom{0}{t} \mid t\in\mathbb{R}\}$ = y-axis

$L=\{\binom{2}{1}s \mid s\in\mathbb{R}\}$



$\binom{2}{1}s+\binom{2}{1}t=\binom{2}{1}(s+t) \in L$

$L \cup M$ is NOT a vector space!

closed under scalar multiplication ✓

NOT closed under addition ✗

Subspaces!

5. $\{(x, 2x) : x\in\mathbb{R}\}$
   includes $(0,0)$, closed under $+$, $\times$ ✓

$\mathbb{R}^2$

in $\mathbb{R}^3$:
$\{\vec{0}=\begin{pmatrix}0\\0\\0\end{pmatrix}\}$, $\mathbb{R}^3$
all lines thru $\vec{0}$

(2) $\{(x, 2x) : x \in \mathbb{R}\}$

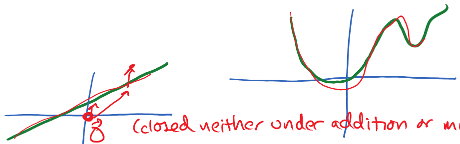
includes $(0,0)$, closed under $+$, $\times$ ✓

ALL subspaces of $\mathbb{R}^2$:

$\{\vec{0}\}$, lines through 0 $\{(x,y) : ax+by=0\}$, $\mathbb{R}^2$ itself

NOT subspaces:

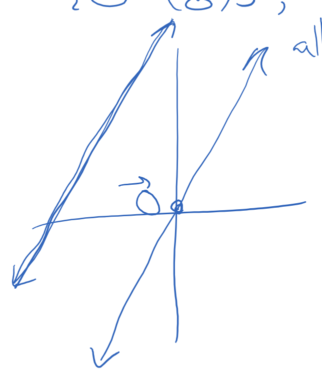
other lines: curves:



(closed neither under addition or multiplication!)

Important: Lines/planes/hyperplanes that don't go through the origin (0) are NOT subspaces!!

$\{\vec{0} = \begin{pmatrix}0\\0\\0\end{pmatrix}\}$, $\mathbb{R}^3$, all lines thru $\vec{0}$



(3) The SPAN of any (finite or infinite) set of points S

$\boxed{\text{Span}(S) \text{ is defined to be the set of all finite linear combinations of elements from } S}$

i.e., all sums $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_r \vec{v}_r = \sum_{i=1}^{r} \alpha_i \vec{v}_i$

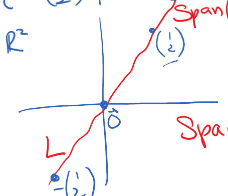
for scalars α_j and $\vec{v}_j \in S$

By definition, this is closed under $+$, $\times$, and hence is a vector space.

$$\underbrace{c\left(\sum_{j=1}^{r} \alpha_j \vec{v}_j\right)}_{\text{Span}} = \underbrace{\sum_{j=1}^{r} (c\alpha_j)\vec{v}_j}_{\text{Span}}$$

Examples: What are:

- $\text{Span}\underbrace{\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}}_{S} = \left\{c\begin{pmatrix}1\\2\end{pmatrix} \middle| c \in \mathbb{R}\right\}$



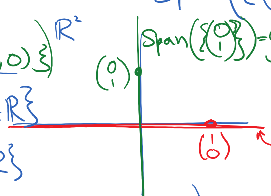
$\text{Span}(\{\vec{0}\}) = \{\vec{0}\}$

- $\text{Span}\{(1,2), (-1,-2)\} = \text{Span}\left(\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}\right) = \text{Span}\left(\left\{\begin{pmatrix}-1\\-2\end{pmatrix}\right\}\right)$
  $= \text{Span}\left(\left\{\begin{pmatrix}2\\4\end{pmatrix}\right\}\right) = L = \left\{\begin{pmatrix}x\\y\end{pmatrix} \middle| y = 2x\right\}$

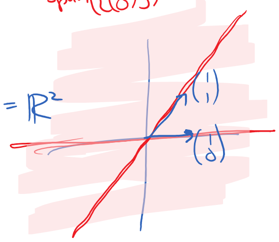
$\mathbb{R}^2 = \text{Span}(\{(0,1), (1,0)\})$

$= \left\{c\begin{pmatrix}0\\1\end{pmatrix} + d\begin{pmatrix}1\\0\end{pmatrix} \middle| c,d \in \mathbb{R}\right\}$

$= \left\{\begin{pmatrix}d\\c\end{pmatrix} \middle| c,d \in \mathbb{R}\right\}$



$\text{Span}\begin{pmatrix}\text{any two nonzero}\\ \text{vectors in } \mathbb{R}^2 \text{ that}\\ \text{are not multiples of each other}\end{pmatrix} = \mathbb{R}^2$



- $\text{Span}\{1, x, x^2, x^3, \ldots\}$
  $= \left\{\sum_{j=0}^{K} c_j x^j \middle| c_j \in \mathbb{R}, K < \infty\right\}$

- $\text{Span}\left\{\begin{matrix}(1,0,0),\\(0,0,1),\\(1,0,1)\end{matrix}\right\} = \text{Span}\left(\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}\right)$
  $= \left\{\begin{pmatrix}x\\0\\z\end{pmatrix} \middle| x,z \in \mathbb{R}\right\}$
  $= xz\text{-plane} \subseteq \mathbb{R}^3$

- S $\{(1,2,1,1,5), = \vec{v}$

$= xz\text{-plane} \subseteq \mathbb{R}^3$

- $\text{Span}\left\{\begin{matrix}(1,2,1,1,5) = \vec{v} \\ (-2,-4,0,4,-2) = \vec{w} \\ (1,2,2,4,9) = \vec{x}\end{matrix}\right\}$

- this is a subspace of $\mathbb{R}^5$ — it must be either a line, a plane, or a 3D hyperplane

Two approaches to find out:

① Add vectors one at a time

$\vec{v} = \begin{pmatrix}1\\2\\1\\1\\5\end{pmatrix}$: $\text{Span}\left\{\begin{pmatrix}1\\2\\1\\1\\5\end{pmatrix}\right\}$ = a line ...

$\vec{w}$: $\text{Span}\left(\{\vec{v},\vec{w}\}\right)$ = a plane ... $= \{c\vec{v}+d\vec{w} \mid c,d\in\mathbb{R}\}$

$\vec{w} \neq c\vec{v}$ for any $c\in\mathbb{R}$

$\vec{x}$: $\text{Span}\left(\{\vec{v},\vec{w},\vec{x}\}\right)$

?: Is $\vec{x} \in \text{Span}\{\vec{v},\vec{w}\}$?

$\Updownarrow$

do there exit $c,d\in\mathbb{R}$ s.t. $\vec{x} = c\vec{v}+d\vec{w}$?

$\Updownarrow$

is there any solution to

$\left\{\begin{matrix}(1,2,1,1,5) = \vec{v} \\ (-2,-4,0,4,-2) = \vec{w} \\ (1,2,2,4,9) = \vec{x}\end{matrix}\right.$

$$\underbrace{\begin{pmatrix}1 & -2\\2 & -4\\1 & 0\\1 & 4\\5 & -2\end{pmatrix}}_{\vec{v}\ \ \ \vec{w}}\begin{pmatrix}c\\d\end{pmatrix} = \underbrace{\begin{pmatrix}1\\2\\2\\4\\9\end{pmatrix}}_{\vec{x}} \ ?$$

$\boxed{\begin{matrix}c=2\\d=1/2\end{matrix}}$

$\vec{x} = 2\vec{v} + \frac{1}{2}\vec{w} \in \text{Span}\{\vec{v},\vec{w}\} = \text{Span}\{\vec{v},\vec{w},\vec{x}\}$

$$\text{Span}\{\vec{u}_1,\dots,\vec{u}_k\} = \left\{\vec{x} \;\middle|\; \text{there is a solution to } \begin{pmatrix}| & | & & |\\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_k\\ | & | & & |\end{pmatrix}\vec{y} = \vec{x}\right\}$$

② Start with all the vectors, and try to simplify them

This is what Gaussian elimination does:

Let $M = \begin{pmatrix}1 & 2 & 1 & 1 & 5\\-2 & -4 & 0 & 4 & -2\\1 & 2 & 2 & 4 & 9\end{pmatrix}$ $\begin{matrix}2\\-1\end{matrix}$

$M \xrightarrow{GE} \begin{pmatrix}1 & 2 & 1 & 1 & 5\\0 & 0 & 2 & 6 & 8\\0 & 0 & 1 & 3 & 4\end{pmatrix} \begin{matrix}\\-1/2\end{matrix} \xrightarrow{GE} \begin{pmatrix}1 & 2 & 1 & 1 & 5\\0 & 0 & 1 & 3 & 4\\0 & 0 & 0 & 0 & 0\end{pmatrix} -1$

$\xrightarrow{GE} \begin{pmatrix}1 & 2 & 0 & -2 & 1\\0 & 0 & 1 & 3 & 4\\0 & 0 & 0 & 0 & 0\end{pmatrix}$

$\Rightarrow \text{Span}\{\vec{v},\vec{w},\vec{x}\} = \text{Span}\left\{\begin{pmatrix}1\\2\\0\\-2\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\3\\4\end{pmatrix},\cancel{\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix}}\right\}$

$$\begin{pmatrix}\text{---} & \vec{u}_1 & \text{---}\\ \text{---} & \vec{u}_2 & \text{---}\\ & \vdots & \\ \text{---} & \vec{u}_k & \text{---}\end{pmatrix} \xrightarrow{GE} \begin{pmatrix}\text{---} & \vec{v}_1 & \text{---}\\ \text{---} & \vec{v}_2 & \text{---}\\ & \vdots & \\ \text{---} & \vec{v}_k & \text{---}\end{pmatrix}$$

$\text{Span}\left(\{\vec{u}_j\}\right) = \text{Span}\left(\{\vec{v}_j\}\right)$

GE does not change the span of the rows

GE does not change the span of the rows

Why?

① $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\} = \text{Span}\{c\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ if $c \neq 0$

$\sum_{j=1}^{k} a_j \vec{v}_j$ $\qquad$ $\frac{a_1}{c}(c\vec{v}_1) + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$

② $\text{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} = \text{Span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$

$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$ $\qquad$ $a_1(\vec{v}_1 + \vec{v}_2) + (a_2 - a_1)\vec{v}_2 + a_3\vec{v}_3 + \cdots + a_k\vec{v}_k$

Note: Adding a multiple of one row to another does not change the span of the rows.

$\text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots\}$

$= \text{Span}\{\vec{v}_1, \underbrace{\vec{v}_2 + \beta\vec{v}_1}_{\vec{v}_2'}, \vec{v}_3, \ldots\}$

since anything you can reach with a linear combination $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \cdots$ can also be reached with a linear combination of the new vectors

$(\alpha_1 - \beta\alpha_2)\vec{v}_1 + \alpha_2(\vec{v}_2 + \beta\vec{v}_1) + \alpha_3\vec{v}_3 + \cdots,$

and vice versa.

$\Rightarrow$ The nonzero rows left over after Gaussian elimination (are a minimal set of vectors) that span the same set as the original rows.

Claim: Span(S) is the smallest vector space that contains all the points in S.

Proof:

- Let T be a vector space containing all of S.

  Goal: Show $\text{Span}(S) \subseteq T$.

- Let $\vec{v} \in \text{Span}(S)$. Goal: Show $\vec{v} \in T$.

  $\Rightarrow \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r$, with all $\vec{v}_j \in S$, $c_j \in \mathbb{R}$

  $\Rightarrow$ all $\vec{v}_j \in T$ $\Rightarrow$ $c_j\vec{v}_j \in T$ by closure under multiplication

  $\Rightarrow c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \in T$ by closure under addition

  $\Rightarrow \vec{v} \in T$ ☒

⑦ Polynomials $= \text{Span}\{1, x, x^2, x^3, \ldots\}$

Continuous functions ✓

Differentiable functions ✓

Functions f with $f(1) = 0$ ✓

Definition: Affine subspace = translated subspace

i.e., a set $\vec{u} + V$ for a vector $\vec{u} \neq \vec{0}$ and subspace V.

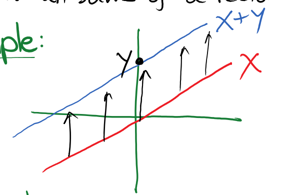
⑧ The **sum** of two subspaces is a subspace.

Definition: For subsets X and Y of a vector space,

$$X + Y \doteq \{x + y \mid x \in X, y \in Y\}.$$

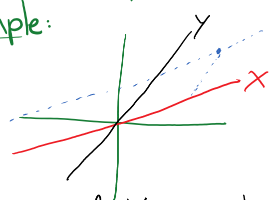
(In English: all sums of a vector in X and a vector in Y.)

Example:



not a subspace! (sometimes called an "affine subspace")

Example:



What is $X+Y$?
Answer: The whole plane!
For any other point, draw this parallelogram ✓

Claim 1: If X and Y are subspaces, then $X+Y$ is a subspace.

Proof: The key properties to check are closure under + and ×.

Closure under addition:

Want to show (WTS) if $a, b \in X+Y$, then $a+b \in X+Y$.

$a \in X+Y \Rightarrow a = x+y$ for some $x \in X, y \in Y$

$b \in X+Y \Rightarrow b = x'+y'$ " x' " y' "

$$a+b = (x+y)+(x'+y') = \underbrace{(x+x')}_{X} + \underbrace{(y+y')}_{Y} \in X+Y \checkmark$$

Closure under multiplication:

WTS: If $a \in X+Y$, then for all scalars α, $\alpha a \in X+Y$:

$a = x+y$

$$\alpha a = \alpha(x+y) = \underbrace{(\alpha x)}_{X} + \underbrace{(\alpha y)}_{Y} \checkmark$$

$\square$

Claim 2: For subsets S and T of a vector space V,

$$\mathrm{Span}(S) + \mathrm{Span}(T) = \mathrm{Span}(S \cup T)$$

Proof: $\mathrm{Span}(S) = \{$finite linear combinations of elts of $S\}$

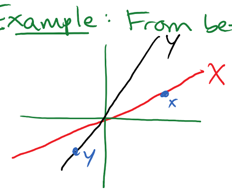
$\mathrm{Span}(T) = \{$ " " " " $T\}$

$\therefore x \in \mathrm{Span}(S) + \mathrm{Span}(T)$

$$\Leftrightarrow x = \sum_{j=1}^{k} \alpha_j \underbrace{s_j}_{S} + \sum_{j=1}^{\ell} \beta_j \underbrace{t_j}_{T}$$

$\Leftrightarrow x$ is a finite linear combination of elements of $S \cup T$. ✓ $\square$

Example: From before



$X = \mathrm{Span}(\{x\})$

$Y = \mathrm{Span}(\{y\})$

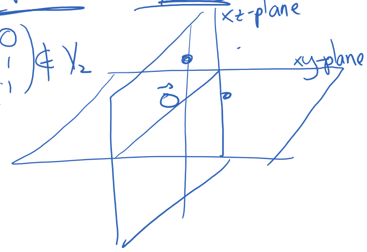
$\mathrm{Span}(\{x, y\}) = \mathbb{R}^2 = X+Y$

More examples of vector spaces

Space	Closed under addition?	Closed under multiplication?	Vector space?
$\mathbb{R}^3$ [cut off]	✓	✓	✓

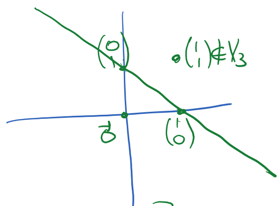
Space	Closed under addition?	Closed under multiplication?	Vector space?
$V_1 = \left\{ \begin{array}{l} (b_1, b_2, b_3) \in \mathbb{R}^3 \\ \text{s.t. } b_1 - 2b_2 + 3b_3 = 0 \end{array} \right\}$	Yes	Yes	✓
$V_2 = \left\{ \begin{array}{l} (b_1, b_2, b_3) \in \mathbb{R}^3 \\ \text{s.t. } b_2 b_3 = 0 \end{array} \right\}$	No	Yes	No

$\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \in V_2$ $\quad \begin{pmatrix}0\\1\\1\end{pmatrix} \notin V_2$



	addition	multiplication	Vector space?
$V_3 = \{\vec{b} \in \mathbb{R}^2 \mid b_1 + b_2 = 1\}$	No	No	No

does not include $\vec{0}$



$V_{\text{homogeneous}} = \{\vec{b} \in \mathbb{R}^n \mid \underset{m \times n}{A}\vec{b} = \vec{0}\}$ Yes Yes Yes

solutions to these homogeneous equations

closure under addition:

$\vec{b}, \vec{c} \in V_h \overset{\text{WTS}}{\Longrightarrow} \vec{b} + \vec{c} \in V_h$ ✓

want to show

$\Downarrow$

$A\vec{b} = \vec{0} = A\vec{c}$

$\Rightarrow A(\vec{b} + \vec{c}) = \vec{0}$

closure under multiplication

$c \in \mathbb{R}, \ \vec{b} \in V_h \overset{\text{WTS}}{\Longrightarrow} c\vec{b} \in V_h$ ✓

$\Downarrow$

$A\vec{b} = \vec{0} \Rightarrow A(c\vec{b}) = cA\vec{b}$
$= c \cdot \vec{0} = \vec{0}$

$V = \{\vec{b} \in \mathbb{R}^n \mid A\vec{b} = \vec{c}\}$ $\quad \boxed{\text{No}}$

where $A_{m \times n}$ and $\vec{c} \neq \vec{0}$ are given

$\vec{0} \notin V$ because $A\vec{0} = \vec{0} \neq \vec{c}$

	addition	multiplication	Vector space?
$V_4 = \text{Span}(\{(1,1,0), (2,0,1)\})$	✓	✓	✓
$V_5 = \{\text{upper-triangular } m \times m \text{ matrices}\}$	✓	✓	✓
$V_6 = \{\text{Diagonal } n \times n \text{ matrices}\}$	✓	✓	✓
$V_7 = \{10 \times 10 \text{ matrices } A \ \text{ s.t. } \text{Trace}(A) = \sum^{10} \ldots\}$	✓	✓	✓

$$V_7 = \left\{ \begin{array}{l} 10\times 10 \text{ matrices } A \\ \text{with } \boxed{\mathrm{Trace}(A) \doteq \sum_{j=1}^{10} a_{jj}} = 0 \end{array} \right\}$$ ✓ ✓ ✓

$$V_8 = \left\{ \begin{array}{l} 10\times 10 \text{ matrices } A \\ \text{with } \mathrm{Tr}(A) = 1 \end{array} \right\}$$ $\begin{pmatrix} 0 \end{pmatrix} \notin V_8$ ✗

$$V_9 = \left\{ \begin{array}{l} 3\times 3 \text{ matrices } A \\ \text{with } A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \end{array} \right\}$$ ✓ ✓ ✓

$$a_{11} + a_{12} + a_{13} = 0$$
$$a_{21} + a_{22} + a_{13} = 0$$
$$a_{31} + a_{32} + a_{33} = 0$$

$$V_{10} = \left\{ \begin{array}{l} \text{symmetric } n\times n \\ \text{matrices (ie } A = A^T) \end{array} \right\}$$ ✓

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

$$\forall i \neq j \quad a_{ij} - a_{ji} = 0$$

$$\left\{ \begin{array}{l} \text{arithmetic progressions,} \\ \text{i.e., sequences } (x_1, x_2, x_3, \dots) \\ \text{with } x_j - x_{j-1} \text{ constant} \end{array} \right\}$$ ✓ ✓ ✓

(e.g., $(0, 2, 4, 6, 8, 10, \dots)$ $+ (1, 2, 3, 4, 5, 6, \dots)$
$= (1, 4, 7, 10, 13, 16, \dots)$

$$\left\{ \begin{array}{l} \text{differentiable functions} \\ f: \mathbb{R} \to \mathbb{R} \text{ with } f'(2) = 3 \end{array} \right\}$$ ✗

$$c(0, 2, 4, 6, 8, \dots)$$
$$= (c\cdot 0, c2, 4c, 6c, \dots)$$

$$\left\{ \begin{array}{l} (0,0), (2,1), (4,2), (6,3), \\ \quad \dots, (94,47), (96,48) \end{array} \right\}$$

$$\subset (\mathbb{Z}/97\mathbb{Z}) \times (\mathbb{Z}/97\mathbb{Z})$$
(97 is prime)

$$\left\{ \begin{array}{l} \text{all matrices that commute} \\ \text{with a given matrix } A \end{array} \right\}$$

Proof. To prove (4.1.1), demonstrate that the two closure properties **(A1)** and **(M1)** hold for $\mathcal{S} = \mathcal{X} + \mathcal{Y}$. To show **(A1)** is valid, observe that if $\mathbf{u}, \mathbf{v} \in \mathcal{S}$, then $\mathbf{u} = \mathbf{x}_1 + \mathbf{y}_1$ and $\mathbf{v} = \mathbf{x}_2 + \mathbf{y}_2$, where $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ and $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{Y}$. Because $\mathcal{X}$ and $\mathcal{Y}$ are closed with respect to addition, it follows that $\mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{X}$ and $\mathbf{y}_1 + \mathbf{y}_2 \in \mathcal{Y}$, and therefore $\mathbf{u} + \mathbf{v} = (\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{y}_1 + \mathbf{y}_2) \in \mathcal{S}$. To verify **(M1)**, observe that $\mathcal{X}$ and $\mathcal{Y}$ are both closed with respect to scalar multiplication so that $\alpha\mathbf{x}_1 \in \mathcal{X}$ and $\alpha\mathbf{y}_1 \in \mathcal{Y}$ for all α, and consequently $\alpha\mathbf{u} = \alpha\mathbf{x}_1 + \alpha\mathbf{y}_1 \in \mathcal{S}$ for all α. To prove (4.1.2), suppose $\mathcal{S}_X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r\}$ and $\mathcal{S}_Y = \{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_t\}$, and write

$$\mathbf{z} \in span(\mathcal{S}_X \cup \mathcal{S}_Y) \Longleftrightarrow \mathbf{z} = \sum_{i=1}^{r} \alpha_i \mathbf{x}_i + \sum_{i=1}^{t} \beta_i \mathbf{y}_i = \mathbf{x} + \mathbf{y} \text{ with } \mathbf{x} \in \mathcal{X},\ \mathbf{y} \in \mathcal{Y}$$
$$\Longleftrightarrow \mathbf{z} \in \mathcal{X} + \mathcal{Y}. \quad \blacksquare$$

Example 4.1.8

If $\mathcal{X} \subseteq \Re^2$ and $\mathcal{Y} \subseteq \Re^2$ are subspaces defined by two different lines through the origin, then $\mathcal{X} + \mathcal{Y} = \Re^2$. This follows from the parallelogram law—sketch a picture for yourself.

Exercises for section 4.1

4.1.1. Determine which of the following subsets of $\Re^n$ are in fact subspaces of $\Re^n$ $(n > 2)$.

(a) $\{\mathbf{x} \mid x_i \geq 0\}$, (b) $\{\mathbf{x} \mid x_1 = 0\}$, (c) $\{\mathbf{x} \mid x_1 x_2 = 0\}$,

(d) $\left\{\mathbf{x} \,\middle|\, \sum_{j=1}^{n} x_j = 0\right\}$, (e) $\left\{\mathbf{x} \,\middle|\, \sum_{j=1}^{n} x_j = 1\right\}$,

(f) $\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \text{where}\ \mathbf{A}_{m\times n} \neq \mathbf{0}\ \text{and}\ \mathbf{b}_{m\times 1} \neq \mathbf{0}\}$.

4.1.2. Determine which of the following subsets of $\Re^{n\times n}$ are in fact subspaces of $\Re^{n\times n}$.

(a) The symmetric matrices. (b) The diagonal matrices.
(c) ~~The nonsingular matrices~~. (d) ~~The singular matrices~~.
? (e) The triangular matrices. (f) The upper-triangular matrices.
(g) All matrices that commute with a given matrix $\mathbf{A}$.
(h) All matrices such that $\mathbf{A}^2 = \mathbf{A}$.
(i) All matrices such that $trace(\mathbf{A}) = 0$.

4.1.3. If $\mathcal{X}$ is a plane passing through the origin in $\Re^3$ and $\mathcal{Y}$ is the line through the origin that is perpendicular to $\mathcal{X}$, what is $\mathcal{X} + \mathcal{Y}$?

$\mathbb{R}^3$

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4.1.4. Why must a real or complex nonzero vector space contain an infinite number of vectors?

4.1.5. Sketch a picture in $\Re^3$ of the subspace spanned by each of the following.

(a) $\left\{\begin{pmatrix}1\\3\\2\end{pmatrix}, \begin{pmatrix}2\\6\\4\end{pmatrix}, \begin{pmatrix}-3\\-9\\-6\end{pmatrix}\right\}$, line (b) $\left\{\begin{pmatrix}-4\\0\\0\end{pmatrix}, \begin{pmatrix}0\\5\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$, xy-plane

(c) $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$. $\mathbb{R}^3$

4.1.6. Which of the following are spanning sets for $\Re^3$?

(a) ✗ $\{(1\ 1\ 1)\}$ (b) ✗ $\{(1\ 0\ 0), (0\ 0\ 1)\}$,

(c) ✓ $\{(1\ 0\ 0), (0\ 1\ 0), (0\ 0\ 1), (1\ 1\ 1)\}$,

✗ (d) $\{(1\ 2\ 1), (2\ 0\ -1), (4\ 4\ 1)\}$. = 2 first + second

✓ (e) $\{(1\ 2\ 1), (2\ 0\ -1), (4\ 4\ 0)\}$.

4.1.7. For a vector space $\mathcal{V}$, and for $\mathcal{M}, \mathcal{N} \subseteq \mathcal{V}$, explain why $span(\mathcal{M} \cup \mathcal{N}) = span(\mathcal{M}) + span(\mathcal{N})$.

skip

4.1.8. Let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces of a vector space $\mathcal{V}$.

(a) Prove that the intersection $\mathcal{X} \cap \mathcal{Y}$ is also a subspace of $\mathcal{V}$. ✓

(b) Show that the union $\mathcal{X} \cup \mathcal{Y}$ need not be a subspace of $\mathcal{V}$. ✓

4.1.9. For $\mathbf{A} \in \Re^{m\times n}$ and $\mathcal{S} \subseteq \Re^{n\times 1}$, the set $\mathbf{A}(\mathcal{S}) = \{\mathbf{Ax} \mid \mathbf{x} \in \mathcal{S}\}$ contains all possible products of $\mathbf{A}$ with vectors from $\mathcal{S}$. We refer to $\mathbf{A}(\mathcal{S})$ as the set of ***images*** of $\mathcal{S}$ under $\mathbf{A}$.

(a) If $\mathcal{S}$ is a subspace of $\Re^n$, prove $\mathbf{A}(\mathcal{S})$ is a subspace of $\Re^m$. ✓

(b) If $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k$ spans $\mathcal{S}$, show $\mathbf{As}_1, \mathbf{As}_2, \ldots, \mathbf{As}_k$ spans $\mathbf{A}(\mathcal{S})$. ✓

4.1.10. With the usual addition and multiplication, determine whether or not the following sets are vector spaces over the real numbers.

(a) $\Re$, ✓ (b) $\mathcal{C}$, ✓ (c) The rational numbers. ✗

4.1.11. Let $\mathcal{M} = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r\}$ and $\mathcal{N} = \{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_r, \mathbf{v}\}$ be two sets of vectors from the same vector space. Prove that $span(\mathcal{M}) = span(\mathcal{N})$ if and only if $\mathbf{v} \in span(\mathcal{M})$. ✓

4.1.12. For a set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, prove that $span(\mathcal{S})$ is the intersection of all subspaces that contain $\mathcal{S}$. **Hint:** For $\mathcal{M} = \bigcap_{\mathcal{S}\subseteq\mathcal{V}} \mathcal{V}$, prove that $span(\mathcal{S}) \subseteq \mathcal{M}$ and $\mathcal{M} \subseteq span(\mathcal{S})$.

6 Examples over other fields

$\mathbb{F}_p$ = field of numbers mod p (for a prime p)

$\mathbb{F}_2 = \{0, 1\}$

Bit strings of length n form a vector space:

$n=3$: $(0,0,0), (0,0,1), (0,1,0), (0,1,1)$

$(1,0,0), (1,0,1), (1,1,0), (1,1,1)$

addition is coordinate-wise, mod 2

$(0,0,1) + (0,1,1) = (0,1,0)$

subspace, eg.,

$\text{Span}(\{(0,0,1), (1,0,1)\})$

$= \{(0,0,0), (0,0,1), (1,0,1), (1,0,0)\}$

Problems:

1. How many subspaces are there of $\mathbb{R}^2$?

   Answer: Infinitely many! (lines through the origin)

2. How many subspaces are there of $\mathbb{R}$?

   Answer: Two! $\{0\}$ and $\mathbb{R}$ itself.

3. How many subspaces are there of $\{0,1\}^2$?

   $\{(0,0)\}$, everything $\{0,1\}^2$

   $\{(0,0),(0,1)\}$, $\{(0,0),(1,0)\}$

   $\{(0,0),(1,0)\}$, $\{(0,0),(1,1)\}$

   and that's it!

$\{(0,0),(0,1)\}$, $\{(0,0),(1,0)\}$
$\{(0,0),(1,0)\}$, $\{(0,0),(1,1)\}$
and that's it!
(if a subspace contains two of the nonzero points, then it also includes their sum, which is the last nonzero point $\therefore$ $(1,0)+(0,1)+(1,1)=(0,0)$ means that any two sum to the third)
$1+4+1=6$

4. How many subspaces are there of $\{0,1\}^n$?
We'll answer this later! Definitely $<\infty$ though

vector $(1,1,-1)$ and automatically contains any multiple $(c,c,-c)$:

$$\textbf{Nullspace is a line} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} c & c & -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The nullspace of B is the line of all points $x=c$, $y=c$, $z=-c$. (The line goes through the origin, as any subspace must.) We want to be able, for any system $Ax=b$, to find $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A)$: all attainable right-hand sides b and all solutions to $Ax=0$.

The vectors b are in the column space and the vectors x are in the nullspace. We shall compute the dimensions of those subspaces and a convenient set of vectors to generate them. We hope to end up by understanding all *four* of the subspaces that are intimately related to each other and to A—the column space of A, the nullspace of A, and their two perpendicular spaces.

Problem Set 2.1

1. Construct a subset of the x-y plane $\mathbf{R}^2$ that is

(a) closed under vector addition and subtraction, but not scalar multiplication.

(b) closed under scalar multiplication but not under vector addition.

Hint: Starting with u and v, add and subtract for (a). Try cu and cv for (b).

2. Which of the following subsets of $\mathbf{R}^3$ are actually subspaces?

(a) The plane of vectors (b_1,b_2,b_3) with first component $b_1=0$.

(b) The plane of vectors b with $b_1=1$.

(c) The vectors b with $b_2b_3=0$ (this is the union of two subspaces, the plane $b_2=0$ and the plane $b_3=0$).

(d) All combinations of two given vectors $(1,1,0)$ and $(2,0,1)$.

(e) The plane of vectors (b_1,b_2,b_3) that satisfy $b_3-b_2+3b_1=0$.

3. Describe the column space and the nullspace of the matrices

$$A=\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad C=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. What is the smallest subspace of 3 by 3 matrices that contains all symmetric matrices *and* all lower triangular matrices? What is the largest subspace that is contained in both of those subspaces?

5. Addition and scalar multiplication are required to satisfy these eight rules:

1. $x+y=y+x$.
2. $x+(y+z)=(x+y)+z$.
3. There is a unique "zero vector" such that $x+0=x$ for all x.
4. For each x there is a unique vector $-x$ such that $x+(-x)=0$.
5. $1x=x$.
6. $(c_1c_2)x=c_1(c_2x)$.
7. $c(x+y)=cx+cy$.
8. $(c_1+c_2)x=c_1x+c_2x$.

(a) Suppose addition in $\mathbf{R}^2$ adds an extra 1 to each component, so that $(3,1)+(5,0)$ equals $(9,2)$ instead of $(8,1)$. With scalar multiplication unchanged, which rules are broken?

(b) Show that the set of all positive real numbers, with $x+y$ and cx redefined to equal the usual xy and x^c, is a vector space. What is the "zero vector"?

(c) Suppose $(x_1,x_2)+(y_1,y_2)$ is defined to be (x_1+y_2, x_2+y_1). With the usual $cx=(cx_1,cx_2)$, which of the eight conditions are not satisfied?

1. $x+y=y+x$.
2. $x+(y+z)=(x+y)+z$.
3. There is a unique "zero vector" such that $x+0=x$ for all x.
4. For each x there is a unique vector $-x$ such that $x+(-x)=0$.
5. $1x=x$.
6. $(c_1c_2)x=c_1(c_2x)$.
7. $c(x+y)=cx+cy$.
8. $(c_1+c_2)x=c_1x+c_2x$.

(a) Suppose addition in $\mathbf{R}^2$ adds an extra 1 to each component, so that $(3,1)+(5,0)$ equals $(9,2)$ instead of $(8,1)$. With scalar multiplication unchanged, which rules are broken?

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6. Let $\mathbf{P}$ be the plane in 3-space with equation $x+2y+z=6$. What is the equation of the plane $\mathbf{P}_0$ through the origin parallel to $\mathbf{P}$? Are $\mathbf{P}$ and $\mathbf{P}_0$ subspaces of $\mathbf{R}^3$?

7. Which of the following are subspaces of $\mathbf{R}^\infty$?

(a) All sequences like $(1,0,1,0,\ldots)$ that include infinitely many zeros.

(b) All sequences $(x1,x2,\ldots)$ with $x_j=0$ from some point onward.

(c) All decreasing sequences: $x_{j+1}\le x_j$ for each j.

(d) All convergent sequences: the x_j have a limit as $j\to\infty$.

(e) All arithmetic progressions: $x_{j+1}-x_j$ is the same for all j.

(f) All geometric progressions $(x_1, kx_1, k^2x_1, \ldots)$ allowing all k and x_1.

8. Which of the following descriptions are correct? The solutions x of

$$Ax=\begin{bmatrix}1&1&1\\1&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

form

(a) a plane.

(b) a line.

(c) a point.

(d) a subspace.

(e) the nullspace of A.

(f) the column space of A.

9. Show that the set of nonsingular 2 by 2 matrices is not a vector space. Show also that the set of *singular* 2 by 2 matrices is not a vector space.

10. The matrix $A=\begin{bmatrix}2&-2\\2&-2\end{bmatrix}$ is a "vector" in the space $\mathbf{M}$ of all 2 by 2 matrices. Write the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

11. (a) Describe a subspace of $\mathbf{M}$ that contains $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ but not $B=\begin{bmatrix}0&0\\0&-1\end{bmatrix}$.

(b) If a subspace of $\mathbf{M}$ contains A and B, must it contain I?

(c) Describe a subspace of $\mathbf{M}$ that contains no nonzero diagonal matrices.

12. The functions $f(x)=x^2$ and $g(x)=5x$ are "vectors" in the vector space $\mathbf{F}$ of all real functions. The combination $3f(x)-4g(x)$ is the function $h(x)=$ ____. Which rule is broken if multiplying $f(x)$ by c gives the function $f(cx)$?

13. If the sum of the "vectors" $f(x)$ and $g(x)$ in $\mathbf{F}$ is defined to be $f(g(x))$, then the "zero vector" is $g(x)=x$. Keep the usual scalar multiplication $cf(x)$, and find two rules that are broken.

14. Describe the smallest subspace of the 2 by 2 matrix space $\mathbf{M}$ that contains

(a) $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $\begin{bmatrix}0&1\\0&0\end{bmatrix}$. (b) $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $\begin{bmatrix}1&0\\0&1\end{bmatrix}$.

(c) $\begin{bmatrix}1&1\\0&0\end{bmatrix}$. (d) $\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $\begin{bmatrix}0&1\\0&1\end{bmatrix}$.

15. Let $\mathbf{P}$ be the plane in $\mathbf{R}^3$ with equation $x+y-2z=4$. The origin $(0,0,0)$ is not in $\mathbf{P}$! Find two vectors in $\mathbf{P}$ and check that their sum is not in $\mathbf{P}$.

16. $\mathbf{P}_0$ is the plane through $(0,0,0)$ parallel to the plane $\mathbf{P}$ in Problem 15. What is the equation for $\mathbf{P}_0$? Find two vectors in $\mathbf{P}_0$ and check that their sum is in $\mathbf{P}_0$.

17. The four types of subspaces of $\mathbf{R}^3$ are planes, lines, R^3 itself, or $\mathbf{Z}$ containing only $(0,0,0)$.

(a) Describe the three types of subspaces of $\mathbf{R}^2$.

(b) Describe the five types of subspaces of $\mathbf{R}^4$.

18. (a) The intersection of two planes through $(0,0,0)$ is probably a ____ but it could be a ____. It can't be the zero vector $\mathbf{Z}$!

(b) The intersection of a plane through $(0,0,0)$ with a line through $(0,0,0)$ is probably a ____ but it could be a ____.

18. (a) The intersection of two planes through $(0,0,0)$ is probably a ____ but it could be a ____. It can't be the zero vector $\mathbf{Z}$!

(b) The intersection of a plane through $(0,0,0)$ with a line through $(0,0,0)$ is probably a ____ but it could be a ____.

(c) If $\mathbf{S}$ and $\mathbf{T}$ are subspaces of $\mathbf{R}^5$, their intersection $\mathbf{S}\cap\mathbf{T}$ (vectors in both subspaces) is a subspace of $\mathbf{R}^5$. *Check the requirements on $x+y$ and cx.*

19. Suppose $\mathbf{P}$ is a plane through $(0,0,0)$ and $\mathbf{L}$ is a line through $(0,0,0)$. The smallest vector space containing both $\mathbf{P}$ and $\mathbf{L}$ is either ____ or ____.

20. True or false for $\mathbf{M}$ = all 3 by 3 matrices (check addition using an example)?

(a) The skew-symmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} = -A$) form a subspace.

(b) The unsymmetric matrices in $\mathbf{M}$ (with $A^{\mathrm{T}} \neq A$) form a subspace.

(c) The matrices that have $(1,1,1)$ in their nullspace form a subspace.

Problems 21–30 are about column spaces $C(A)$ and the equation $Ax = b$.

21. Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

22. For which right-hand sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$\text{(a)} \quad \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \qquad \text{(b)} \quad \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

23. Adding row 1 of A to row 2 produces B. Adding column 1 to column 2 produces C. A combination of the columns of ____ is also a combination of the columns of A. Which two matrices have the same column ____?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

24. For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

25. (Recommended) If we add an extra column b to a matrix A, then the column space gets larger unless ____. Give an example in which the column space gets larger and an example in which it doesn't. Why is $Ax = b$ solvable exactly when the column space *doesn't* get larger by including b?

26. The columns of AB are combinations of the columns of A. This means: *The column space of AB is contained in* (possibly equal to) *the column space of A.* Give an example where the column spaces of A and AB are not equal.

27. If A is any 8 by 8 invertible matrix, then its column space is ____. Why?

28. True or false (with a counterexample if false)?

(a) The vectors b that are not in the column space $\boldsymbol{C}(A)$ form a subspace.

(b) If $\boldsymbol{C}(A)$ contains only the zero vector, then A is the zero matrix.

(c) The column space of $2A$ equals the column space of A.

(d) The column space of $A - I$ equals the column space of A.

29. Construct a 3 by 3 matrix whose column space contains $(1,1,0)$ and $(1,0,1)$ but not $(1,1,1)$. Construct a 3 by 3 matrix whose column space is only a line.

30. If the 9 by 12 system $Ax = b$ is solvable for every b, then $\boldsymbol{C}(A)$ = ____.

31. Why isn't $\mathbf{R}^2$ a subspace of $\mathbf{R}^3$?

2.2 Solving $Ax = 0$ and $Ax = b$

Chapter 1 concentrated on square invertible matrices. There was one solution to $Ax = b$ and it was $x = -A^{-1}b$. That solution was found by elimination (not by computing A^{-1}). A rectangular matrix brings new possibilities—U may not have a full set of pivots. This section goes onward from U to a reduced form R—**the simplest matrix that elimination can give**. R reveals all solutions immediately.

For an invertible matrix, the nullspace contains only $x = 0$ (multiply $Ax = 0$ by A^{-1}). The column space is the whole space ($Ax = b$ has a solution for every b). The new questions appear when the nullspace contains *more than the zero vector* and/or the column space contains *less than all vectors*:

1. Any vector x_n in the nullspace can be added to a particular solution x_p. The solutions to all linear equations have this form, $x = x_p + x_n$:

$$\textbf{Complete solution} \qquad Ax_p = b \quad \textbf{and} \quad Ax_n = 0 \qquad \textbf{produce} \quad A(x_p + x_n) = b.$$

space contains *less than all vectors*:

1. Any vector x_n in the nullspace can be added to a particular solution x_p. The solutions to all linear equations have this form, $x = x_p + x_n$:

   Complete solution $\quad Ax_p = b$ **and** $Ax_n = 0$ $\quad$ **produce** $\quad A(x_p + x_n) = b$.

2. When the column space doesn't contain every b in $\mathbf{R}^m$, we need the conditions on b that make $Ax = b$ solvable.

A 3 by 4 example will be a good size. We will write down all solutions to $Ax = 0$. We will find the conditions for b to lie in the column space (so that $Ax = b$ is solvable). The 1 by 1 system $0x = b$, one equation and one unknown, shows two possibilities:

$0x = b$ has *no solution* unless $b = 0$. The column space of the 1 by 1 zero matrix contains only $b = 0$.

$0x = 0$ has *infinitely many solutions*. The nullspace contains *all* x. A particular solution is $x_p = 0$, and the complete solution is $x = x_p + x_n = 0 + (\text{any } x)$.

More important examples:

SUBSPACES OF A MATRIX

Let $A \in \mathbb{R}^{m\times n}$ be an $m\times n$ real-valued matrix.

- Range(A) $= \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$

  $= \text{Span}(\text{columns of } A)$

  AKA "column space" of A

  Observe: $\vec{b} \in \text{Range}(A) \iff A\vec{x} = \vec{b}$ has a solution

- Range(A^T) $= \text{Span}(\text{rows of } A)$

  "row space"

- Kernel(A) $= \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$ (solutions to the homogeneous equations)

  AKA "null space" of A

Successive Fourier approximations to the step function

$$\text{Table}\left[\frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{kmax}\frac{(-1)^k}{2k+1}\text{Cos}[(2k+1)\pi x], \{kmax, -1, 3\}\right] \text{ // Expand // TableForm}$$

$\frac{1}{2}$

$\frac{1}{2} - \frac{2\,\text{Cos}[\pi x]}{\pi}$

$\frac{1}{2} - \frac{2\,\text{Cos}[\pi x]}{\pi} + \frac{2\,\text{Cos}[3\pi x]}{3\pi}$

$\frac{1}{2} - \frac{2\,\text{Cos}[\pi x]}{\pi} + \frac{2\,\text{Cos}[3\pi x]}{3\pi} - \frac{2\,\text{Cos}[5\pi x]}{5\pi}$

$\frac{1}{2} - \frac{2\,\text{Cos}[\pi x]}{\pi} + \frac{2\,\text{Cos}[3\pi x]}{3\pi} - \frac{2\,\text{Cos}[5\pi x]}{5\pi} + \frac{2\,\text{Cos}[7\pi x]}{7\pi}$

$$\text{step} = \text{Plot}\left[\text{If}\left[x < \frac{1}{2}, 0, 1\right], \{x, 0, 1\}, \text{PlotStyle} \to \{\text{Black}, \text{Thickness}[.01]\}\right];$$

$$\text{Show}\left[\text{step}, \text{Plot}\left[\text{Evaluate@Table}\left[\frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{kmax}\frac{(-1)^k}{2k+1}\text{Cos}[(2k+1)\pi x], \{kmax, \{0, 1, 10, 100, 1000\}\}\right], \{x, 0, 1\}\right]\right]$$