Homework 11

Singular value and spectral decompositions

1

In class, we saw that the nonzero eigenvalues of AA^T are the same as the nonzero eigenvalues of A^TA — just the squares of the nonzero singular values of A. In this problem, you'll relate the singular values of A to the eigenvalues of $\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$.

a) Diagonalize the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

b) Diagonalize the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

c) Let A be an arbitrary $m \times n$ real matrix, with $m \leq n$.

Let

$$B = \begin{array}{c} \\ m \\ n \end{array} \begin{array}{c} \begin{array}{cc} m & n \end{array} \\ \left(\begin{array}{c|c} 0 & A \\ \hline A^T & 0 \end{array}\right) \end{array}.$$

B is an $(m+n) \times (m+n)$ symmetric matrix. (Therefore it is unitarily diagonalizable.)

In terms of the singular values and left- and right-singular vectors of A, specify the eigenvalues and eigenvectors of B.

Note: B has $m+n$ eigenvalues, so don't forget any! Parts a) and b) should be helpful special cases, but feel free to experiment more with Matlab until you see the pattern. Commands like these might be helpful:

```
m = 2;
n = 3;
A = randn(m, n);
[U, S, V] = svd(A)  % <-- returns left singular vectors, singular values, right singular vectors

B = [zeros(m,m) A; A' zeros(n,n)];
[W, D] = eig(B) % <-- returns eigenvectors, eigenvalues
```

Positive semi-definite matrices

A real symmetric matrix is "positive definite" if its eigenvalues are all >0, "positive semi-definite" if its eigenvalues are all ≥ 0, and "indefinite" otherwise.

2

The quadratic form $f(x,y) = x^2 + 4xy + 2y^2$ has a saddle point at the origin, despite the fact that its coefficients are positive. Write f as a difference of two squares.

3

Decide for or against the positive definiteness of these matrices, and write out the corresponding quadratic form $f(\vec{x}) = \vec{x}^T A \vec{x}$:

a) $\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}$ b) $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$ d) $\begin{pmatrix} -1 & 2 \\ 2 & -8 \end{pmatrix}$

The determinant in (b) is 0; along what line is $f(x,y) = 0$?

4

For what range of numbers a and b are the matrices A and B positive definite?

$$A = \begin{pmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & b & 8 \\ 4 & 8 & 7 \end{pmatrix}$$

5

Positive definite or not?

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}^2$$

6

Give a quick reason why each of these statements is true:

a) Every positive definite matrix is invertible.

b) The only pos. def. projection matrix is $P=I$.

c) A diagonal matrix with positive diagonal entries is pos. def.

d) A symmetric matrix with a positive determinant might not be positive definite.

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Classify each of the following matrices as positive definite, positive semidefinite, or indefinite. Try to do it by hand.

$$A=\begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$B=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{pmatrix}$$

$$C=\begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 3 \end{pmatrix}$$

$D=B^{-1}$

$E=C^{-1}$

$$F=\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$$

8

- If A is positive semidefinite and $\alpha \geq 0$, prove that αA is positive semi-definite.
- If A and B are positive semi-definite, prove that $A+B$ is positive semidefinite.
  (<u>Hint</u>: You'll probably want to use a theorem from class...)
- Conclude that if A and B are positive semi-definite, then so are the matrices $pA+(1-p)B$, for all $p \in [0,1]$.

(Thus the set of positive semi-definite matrices of a given dimension is **convex**. This is extremely important in optimization theory: namely, in semi-definite programming.)

9

a) Prove that if $A \succeq 0$ then $A^2 \succeq 0$.

The definition of a positive semi-definite matrix can be used to define a partial order on symmetric matrices.

Definition: For symmetric matrices A and B of the same dimensions, define

$$\text{"}A \succeq B\text{"}$$

if $A-B$ is positive semi-definite.

b) Give an example of two symmetric matrices A and B such that <u>neither</u> $A \succeq B$ <u>nor</u> $B \succeq A$. These matrices are incomparable; that's why it is called a partial order.

c) Give an example of symmetric A and B so that

$$A \succeq B \quad \text{is true,}$$

$$\text{but} \quad A^2 \succeq B^2 \quad \text{is false!}$$

(That is, $A-B$ is positive semidefinite, but A^2-B^2 is not.)

(**Hint:** You can use 2×2 matrices. Play around until you find an example, and then try to simplify it to understand how it works.)