Homework 10 Eigenvectors answers

Some Basic Facts

① If A is an $n \times n$ matrix, explain why for all $c \in \mathbb{C}$,

$$\det(cA) = c^n \cdot \det(A).$$

Answer: By definition,

$$\det(A) = \sum_{\text{permutations } \sigma} \text{sign}(\sigma) \cdot \prod_{i=1}^{n} A_{i,\sigma(i)}$$

If A is replaced by cA, then each $A_{i,\sigma(i)}$ term picks up a factor of c, so a factor of c^n overall.

Review

② a) Is the matrix

$$\mathcal{U} = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & \frac{-2i}{\sqrt{6}} \end{pmatrix}$$

unitary? Why or why not?

Answer:

It suffices to check that the rows of $\mathcal{U}$ are orthonormal, or that the columns are orthonormal (it doesn't matter which you do). That's easy enough to do by hand. (Just remember when to take complex conjugates.)

In Mathematica, you can do it like this:

$$\mathbf{U} = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{6}} & -\sqrt{\frac{2}{3}}\, i \end{pmatrix};$$

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U.Conjugate[Transpose[U]]
{{1, 0}, {0, 1}}
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← identity matrix ✓

$\Rightarrow \mathcal{U}^{\dagger} = \mathcal{U}^{-1}$

b) Give a 2×2 matrix that maps

$u = (1+i, i)$ to $2u$

and $v = (1+i, -2i)$ to $-3v$.

Check your answer!!

Answer: u and v are orthogonal vectors; they are proportional to the rows of the matrix from part ⓐ.

Let $\tilde{u} = \frac{u}{\|u\|} = \frac{1}{\sqrt{3}} u$, $\tilde{v} = \frac{v}{\|v\|} = \frac{1}{\sqrt{6}} v$.

Then these are orthonormal vectors, and

$$A = 2\tilde{u}\tilde{u}^{+} - 3\tilde{v}\tilde{v}^{+}$$

does what we want:

$$A\tilde{u} = 2\tilde{u}\tilde{u}^{+}\tilde{u} - 3\tilde{v}\underbrace{\cancel{\tilde{v}^{+}\tilde{u}}}_{0}$$
$$= 2\tilde{u}$$
$$A\tilde{v} = 2\tilde{u}\underbrace{\tilde{u}^{+}\tilde{v}}_{0} - 3\tilde{v}\underbrace{\tilde{v}^{+}\tilde{v}}_{1}$$
$$= -3\tilde{v} \quad ✓$$

It's easy enough to expand out A by hand, but I'll do it in Mathematica:

```
u = 1/Sqrt[3] {{1 + i}, {i}};
v = 1/Sqrt[6] {{1 + i}, {-2 i}};
A = 2 u.Conjugate[Transpose[u]] - 3 v.Conjugate[Transpose[v]];
A // ~~mf~~ Matrixform

"\nCheck the answer:"
A.{1 + i, i}
A.{1 + i, -2 i}
```

atrixForm=

$$\begin{pmatrix} \frac{1}{3} & \frac{5}{3} - \frac{5i}{3} \\ \frac{5}{3} + \frac{5i}{3} & -\frac{4}{3} \end{pmatrix}$$

Check the answer:

{2 + 2 i, 2 i} $= 2u$ ✓

{-3 - 3 i, 6 i} $= -3v$ ✓

Introduction to eigenvectors

③ Prove that if $\mathcal{U}$ is invertible, then the eigenvalues of

$$A$$

are the same as the eigenvalues of

$$\mathcal{U} A \mathcal{U}^{-1}.$$

How are the eigenvectors of A related to the eigenvectors of $\mathcal{U}A\mathcal{U}^{-1}$?

Answer:

Let $\vec{v}$ be an eigenvector of A, with eigenvalue λ.

$$\Rightarrow A\vec{v} = \lambda \vec{v}$$

$$\Rightarrow (\mathcal{U}A\mathcal{U}^{-1})(\mathcal{U}\vec{v}) = \mathcal{U} A \vec{v}$$

$$= \lambda\, \mathcal{U}\vec{v}$$

$\Rightarrow$ $\mathcal{U}\vec{v}$ is an eigenvector of $\mathcal{U}A\mathcal{U}^{-1}$, with the same eigenvalue.

④ Let

$$A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

be a diagonal matrix, with all $\lambda_j \in \mathbb{C}$.

a) Give an SVD of A; what are the singular values and corresponding right and left singular vectors?

Singular value	Right s. vector	Left s. vector
$\lvert\lambda_j\rvert$	$\vec{e}_j$	$\frac{\lambda_j}{\lvert\lambda_j\rvert}\vec{e}_j$

for $j = 1, \dots, n$

b) Give a spectral decomposition of A; what are the eigenvalues and corresponding eigenvectors?

Eigenvalue	Eigenvector
λ_j	$\vec{e}_j$

for $j = 1, \dots, n$

⑤ For a polynomial $p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_k x^k$, and a square matrix A, define $p(A)$ to be the matrix

$$p(A) = \alpha_0 I + \alpha_1 A + \cdots + \alpha_k A^k.$$

Show that if $\vec{v}$ is an eigenvector of A, with corresponding eigenvalue λ, then $\vec{v}$ is also an eigenvector of $p(A)$, with corresponding eigenvalue $p(\lambda)$.

Answer:

We are given that $A\vec{v} = \lambda\vec{v}$.

Therefore $$A^2\vec{v} = A\cdot(\lambda\vec{v}) = \lambda\cdot A\vec{v}$$

and by a simple induction argument, $A^j\vec{v} = \lambda^j\vec{v}$ for all integers $j \geq 0$. (For $j=0$, $A^j = I$, so $A^0\vec{v} = \vec{v}$.)

$$\Rightarrow p(A)\vec{v} = \sum_{j=0}^{k} \alpha_j A^j \vec{v} = \sum_j \alpha_j \lambda^j \vec{v} = p(\lambda)\cdot\vec{v}$$ ✓

⑥ If $\vec{v}_1, \ldots, \vec{v}_k$ are eigenvectors of A, all with the same eigenvalue λ, explain why every nonzero linear combination

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

is also an eigenvector of A, with eigenvalue λ. (This is why we talk of "eigenspaces".)

Answer:

$$A\vec{v} = A\cdot\sum_{j=1}^{k}\alpha_j\vec{v}_j \qquad \text{(definition of } \vec{v}\text{)}$$
$$= \sum_{j=1}^{k}\alpha_j A\vec{v}_j \qquad \text{(by linearity of } A\text{)}$$
$$= \sum_{j=1}^{k}\alpha_j \lambda \vec{v}_j \qquad \text{(since } A\vec{v}_j = \lambda\vec{v}_j\text{)}$$
$$= \lambda\vec{v}$$ ✓

Some Calculations

⑦ ⓐ By hand (not using Matlab), find the eigenvalues and corresponding eigenvectors for the matrix

$A = \begin{pmatrix} 3 & 1 \end{pmatrix}$

0 2).

Answer: In general, for an upper-triangular (or lower-triangular) matrix, the eigenvalues can be read off the diagonal.
So in this case, the eigenvalues are 3 and 2.

We can also compute the e-values by finding the roots of

$$0 = \mathrm{Det}(A - \lambda I)$$
$$= \mathrm{Det}\begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (3-\lambda)\cdot(2-\lambda)$$

$\Rightarrow \lambda = 3$ or $\lambda = 2$.

To find the corresponding eigenspaces, we compute the nullspaces of $A-3I$ and $A-2I$:

$$A-3I = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow N(A-3I) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

$$A-2I = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow N(A-2I) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

In summary,

Eigenvalues	Eigenvectors
3	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
2	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Check your answer!

$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} \checkmark$$

$$A\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \checkmark$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \checkmark$$

(b) What are the eigenvalues of $\begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}$?

What are the eigenvalues of $\begin{pmatrix} 5 & 1 \\ 0 & 4 \end{pmatrix}$?

In general, what are the eigenvalues of $\alpha I + A$? Why?

Answer: In general, if $A\vec{v} = \lambda\vec{v}$, then

$$(A+\alpha I)\vec{v} = A\vec{v} + \alpha I\vec{v}$$
$$= \lambda\vec{v} + \alpha\vec{v}$$
$$= (\lambda+\alpha)\vec{v}$$

Hence $\vec{v}$ is an eigenvector of $A+\alpha I$, with corresponding eigenvalue $\lambda+\alpha$.

Therefore, adding αI to a matrix simply **shifts** the entire spectrum by α.

(c) What are the eigenvalues and corresponding eigenvectors for

$$B = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} ?$$

What are the eigenvalues for

$$C = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} ?$$

In general, explain why knowing the eigenvalues and eigenvectors of matrices A and B also gives you the eigenvalues and eigenvectors of

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Answer:

$$B = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Eigenvalues	Eigenvectors
3	(1, 0, 0, 0)
2	(1, -1, 0, 0)
5	(0, 0, 1, 0)
4	(0, 0, 1, -1)

Eigenvalues Eigenvectors

$$C = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

Eigenvalues	Eigenvectors
3	(1, 0, 0, 0, 0, 0)
2	(1, -1, 0, 0, 0, 0)
5	(0, 0, 1, 0, 0, 0)
4	(0, 0, 1, -1, 0, 0)
7	(0, 0, 0, 0, 1, 0)
6	(0, 0, 0, 0, 1, -1)

Why?

The block-diagonal form means that we can diagonalize each block independently.

Say $C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Then if $A\vec{v} = \lambda\vec{v}$,

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} \vec{v} \\ \vec{0} \end{pmatrix} = \lambda\begin{pmatrix} \vec{v} \\ \vec{0} \end{pmatrix} \checkmark$$

and if $B\vec{\omega} = \gamma\vec{\omega}$,

$$C\begin{pmatrix} \vec{0} \\ \vec{\omega} \end{pmatrix} = \gamma\begin{pmatrix} \vec{0} \\ \vec{\omega} \end{pmatrix} \checkmark$$

② What are the eigenvalues and corresponding eigenvectors for

$$D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$E = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \text{ and}$$

$$F = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}?$$

Feel free to use Matlab.

Then explain why, given the answers to parts ⓐ and ⓒ, using Matlab is unnecessary.

Answer:

$$D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues	Eigenvectors
5	(1,0,0,0)
4	(0,1,0,0)
3	(0,0,1,0)
2	(0,0,1,−1)

$$E = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues	Eigenvectors
3	(1,0,0,0)
5	(0,1,0,0)
4	(0,0,1,0)
2	(1,0,0,−1)

$$F = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues	Eigenvectors
3	(1,0,0,0)
5	(0,1,0,0)
4	(0,1,−1,0)
2	(1,0,0,−1)

D is upper-triangular, so the e-values can be read off the diagonal. The last two e-vectors just come from those of the submatrix $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$.

E and F are also block diagonal, where the coordinates in the blocks are $\begin{array}{c|cc} & 1 & 4 \\ \hline 1 & & \\ 4 & & \end{array}$ and $\begin{array}{c|cc} & 2 & 3 \\ \hline 2 & & \\ 3 & & \end{array}$.

These blocks are highlighted above. Therefore, by part c, the e-vectors for E and F come from the e-vectors for these 2×2 blocks, but padded with zeros.

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ⓐ What are the eigenvalues and corresponding eigenspaces of the n×n matrix

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & 1 & \cdots & & & 1 \\ 1 & 1 & 2 & 1 & \cdots & & & 1 \\ \vdots & \vdots & 1 & 2 & \ddots & & 1 & \vdots \\ & & & \ddots & & \ddots & 2 & 1 \\ 1 & 1 & \cdots & & & 1 & 1 & 2 \end{pmatrix}$$

(2's on the diagonal, 1's everywhere else)?

(2's on the diagonal, 1's everywhere else)? What is its determinant?

Answer: Call the matrix A. It equals

$$A = I + J,$$

where J is the all-ones matrix. Therefore A has the same eigenspaces as J, with eigenvalues shifted up by 1.

J is a rank-one matrix, so its eigenvalues are easy to compute. They are:

Eigenspace	Eigenvalue of J	Eigenvalue of A
$\mathrm{Span}(1,1,\dots,1)$	n	$n+1$
$\{v \in \mathbb{R}^n : \sum_{i=1}^{n} v_i = 0\}$	0	1

Then $\det(A)$ = product of its eigenvalues
$= n+1$. ✓

ⓑ What is the determinant of the $n \times n$ matrix

$$\begin{pmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 1 & 1 & & 1 & 1-n \end{pmatrix}$$

(1-n along the diagonal, 1's off the diagonal)?

Hint: What are the eigenvalues?

Answer:

Just as in part ⓐ, this matrix is $J - n \cdot I$, so its eigenvalues are

0 (with multiplicity 1), and

$-n$ (with multiplicity $n-1$).

The determinant is therefore $\boxed{0}$.

9 ⓐ If $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$, find A^{100} by diagonalizing A.

(b) Diagonalize $B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ as $B = UDU^{-1}$ for some diagonal matrix D. Then use the diagonalization to prove that

$$B^k = \begin{pmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{pmatrix}.$$

(a) $\text{Det}(A - \lambda I) = \text{Det}\begin{pmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 5$

$= (\lambda - 1)(\lambda - 5)$

E-values	E-vectors
1	(1, -1)
5	(3, 1)

$$\Rightarrow A = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

$$\Rightarrow A^{100} = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5^{100} & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

$$\boxed{= \frac{1}{4}\begin{pmatrix} 3 \cdot 5^{100} + 1 & 3 \cdot (5^{100} - 1) \\ 5^{100} - 1 & 5^{100} + 3 \end{pmatrix}}$$

(b) Similarly,

$\text{Det}(B - \lambda I) = \text{Det}\begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2)$

E-values	E-vectors
3	(1, 0)
2	(1, -1)

$$\Rightarrow B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow B^k = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3^k & 0 \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3^k & 3^k \\ 0 & -2^k \end{pmatrix}$$

$$= \begin{pmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{pmatrix} \quad \checkmark$$

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The higher-order differential equation $y'' + y = 0$ can be written as a first-order system by introducing the derivative y' as another unknown:

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -y \end{pmatrix}.$$

If this is $\frac{d\vec{v}}{dt} = A\vec{v}$, what is the 2×2 matrix A?

Find its eigenvalues and eigenvectors, and compute the solution that starts from $y(0)=2$, $y'(0)=0$.

Answer:

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y \\ y'' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{= A}\begin{pmatrix} y \\ y' \end{pmatrix}$$

<u>E-values</u>	<u>E-vectors</u>
i	$(1, i)$
i –	$(1, -i)$

$$\Rightarrow A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y \\ y' \end{pmatrix}(t) = e^{At}\begin{pmatrix} y \\ y' \end{pmatrix}(t=0) = e^{At}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$e^{At} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{it}+e^{-it} & ie^{-it}-ie^{it} \\ ie^{it}-ie^{-it} & e^{it}+e^{-it} \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$\Rightarrow \boxed{y(t) = 2\cos t}$$