Reading:  2.3  4.3–4.4

Today: Linear independence, bases and dimension

Recall:

Span (a set of vectors) = smallest subspace that contains them all

$$= \left\{ \begin{array}{c} \text{all (finite) linear combinations of those vectors} \\ \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r \end{array} \right\}$$

Example:

$$\mathbb{R}^3 = \operatorname{Span}\{\underbrace{(1,0,0)}_{=\vec{e}_1}, \underbrace{(0,1,0)}_{=\vec{e}_2}, \underbrace{(0,0,1)}_{=\vec{e}_3}\}$$

$$= \operatorname{Span}\{\vec{e}_1, \underbrace{\vec{e}_1+\vec{e}_2}_{=(1,1,0)}, \underbrace{\vec{e}_1+\vec{e}_2+\vec{e}_3}_{=(1,1,1)}, \vec{e}_1-\vec{e}_2, \vec{e}_2-\vec{e}_3\}$$

$$= \operatorname{Span}(\mathbb{R}^3)$$

Example:

Span (a finite set of vectors $v_1, \ldots, v_n$)

$$= R\left(\text{matrix}\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}\right) \quad \text{range/column space}$$

$$= v_1 e_1^T + v_2 e_2^T + \cdots + v_n e_n^T$$

Goal for today: Given a subspace, find ~~the~~ a smallest possible spanning set for it.

(smaller → simpler, easier to work with/understand)

Outline:
- Linear independence
- Basis
- Dimension
- Rank, Rank-nullity theorem

LINEAR INDEPENDENCE

Definitions:

- A vector v is linearly independent of a set S if

- A vector v is linearly independent of a set S if

$$v \notin \mathrm{Span}(S)$$

(If $v \in \mathrm{Span}(S)$, we say "v is linearly dependent on S.")

- A set of vectors is linearly independent if no vector can be expressed as a linear combination of the others (i.e., every vector is linearly independent of the others).

Example: Assume $S = \{v_1, v_2, v_3\}$ is linearly independent.

$$\Rightarrow v_1 \notin \mathrm{Span}(v_2, v_3)$$
$$v_2 \notin \mathrm{Span}(v_1, v_3)$$
$$v_3 \notin \mathrm{Span}(v_1, v_2)$$

$\Rightarrow$ Knowing $f(v_2), f(v_3)$ tells us nothing about $f(v_1)$, etc.

Equivalent definition: A set $S = \{v_1, v_2, \dots, v_n\}$ is linearly independent if the only solution to

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n = 0$$

is $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Why are these definitions equivalent?

- If $v_1 \in \mathrm{Span}(v_2, v_3)$, say $v_1 = \alpha_2 v_2 + \alpha_3 v_3$, then
$$-v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0.$$
- If $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ with $\alpha_j \neq 0$, then
$$v_j = -\frac{1}{\alpha_j}\Big(\sum_{i \neq j} \alpha_i v_i\Big) \in \mathrm{Span}(\{v_i \mid i \neq j\}).$$

Observe:

$S = \{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent $\iff$ nullspace of $\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$ is $\{\vec{0}\}$.

To check if a set is linearly independent, compute the nullspace.

Example: Are these vectors linearly independent of each other?

$$(1, 1, 1), \quad (2, -1, 1), \quad (0, 9, 3)$$

Answer:

```
data = {{1, 1, 1}, {2, -1, 1}, {0, 9, 3}};
ParametricPlot3D[data*u, {u, 0, 1}, AxesLabel → {"x", "y", "z"}] /. Line → Arrow
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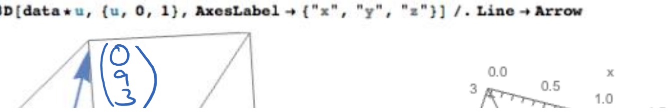

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ParametricPlot3D[data*u, {u, 0, 1}, AxesLabel → {"x", "y", "z"}] /. Line → Arrow
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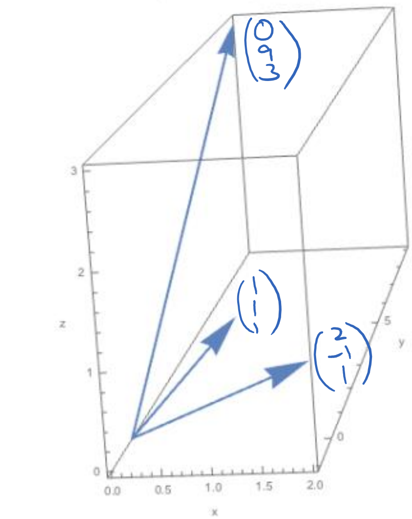
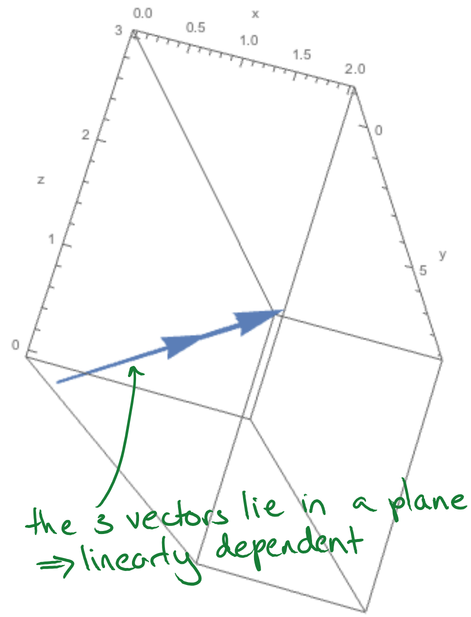




the 3 vectors lie in a plane
$\Rightarrow$ linearly dependent

$$\begin{pmatrix}0\\9\\3\end{pmatrix} = 6\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} - 3\begin{pmatrix}2\\-1\\1\end{pmatrix}$$

OR $$6\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} - 3\begin{pmatrix}2\\-1\\1\end{pmatrix} - \begin{pmatrix}0\\9\\3\end{pmatrix} = \vec{0}$$

$\therefore$ For any linear function f,

$$f(0,9,3) = 6\cdot f(1,1,1) - 3\cdot f(2,-1,1).$$

Basis and Dimension

Definition: A **basis** for a vector space V is a linearly independent set that spans V.

$$\boxed{\text{dimension}(V) = \#\text{ of vectors in a basis}}$$

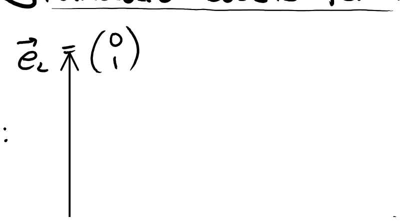
(**Theorem:** Two bases for the same space must have the same size.)

Example: $\mathbb{R}^n$

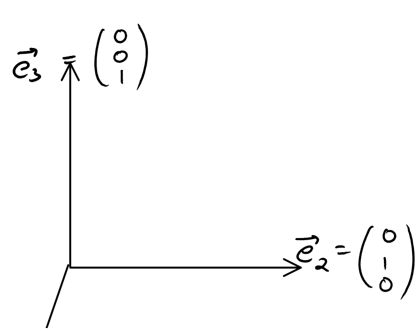
$\dim(\mathbb{R}^n) = n$

Standard basis for $\mathbb{R}^n$

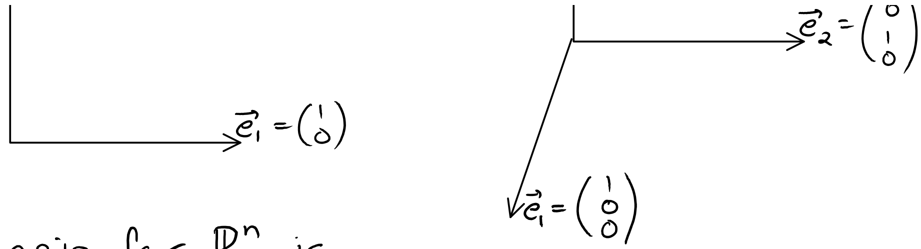
$\mathbb{R}^2$:



$\mathbb{R}^3$:



$\mathbb{R}^2$:



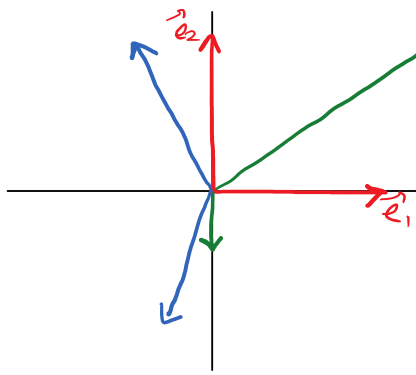
A basis for $\mathbb{R}^n$ is

$$\{\ \vec{e}_1 = (1,0,\dots,0),\ \vec{e}_2 = (0,1,0,\dots,0),\ \dots,\ \vec{e}_n = (0,\dots,0,1)\}.$$

linearly indep. ✓ span ✓

Other bases:

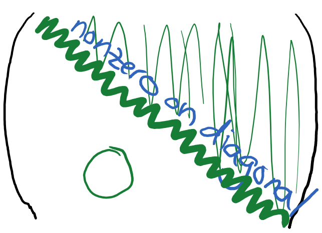
$\mathbb{R}^2$:



Any two linearly independent vectors form a basis

(because they span the plane)

$\mathbb{R}^n$:

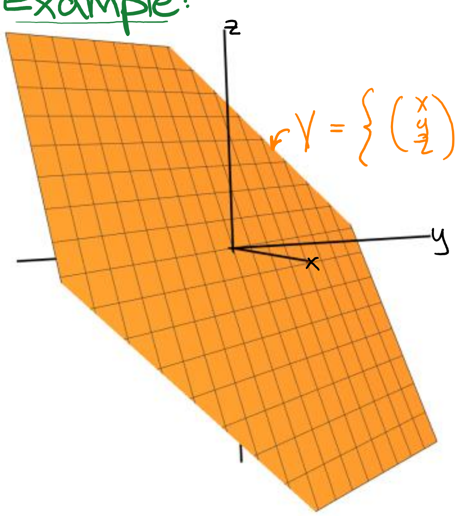


The columns are a basis

lin. indep. ✓

span $\mathbb{R}^n$ ✓

Example:



$$V = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x+y+z=0 \right\} = N\big(\begin{pmatrix}1 & 1 & 1\end{pmatrix}\big)$$

$\mathrm{Dim}(V) = 2$

many bases, eg.,

$\boxed{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}}$ or $\boxed{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}}$

Observe: Any $\vec{v} \in V$ is in $\mathrm{Span}\left(\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}\right)$

$$\begin{pmatrix}x\\y\\-x-y\end{pmatrix} = x\begin{pmatrix}1\\0\\-1\end{pmatrix} + y\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

Example: Give a basis for the span of

$(1,1,1)\ ,\quad (2,-1,1)\ ,\quad (0,9,3)$

Answer: Let $V = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

Then $V = R(A)$ for $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & -1 & 9 \\ 1 & 1 & 3 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 0 & 9 & 3 \end{pmatrix} \xrightarrow{-2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ \cancel{0} & \cancel{9} & \cancel{3} \end{pmatrix} {\scriptstyle -3}$$

$$\Rightarrow V = \text{Span}\boxed{\{\vec{v}_1, \vec{v}_2\}} = \text{Span}\boxed{\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\3\\1\end{pmatrix}\right\}}$$

different bases for V

EXAMPLES

Space	Dimension	Basis
• 2×2 matrices	4	$\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ $\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}$
• 2×2 symmetric matrices $A = A^T = \begin{pmatrix}a&b\\b&c\end{pmatrix}$	3	$\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}$
• 2×2 antisymmetric matrices $A = -A^T = \begin{pmatrix}0&a\\-a&0\end{pmatrix}$	1	$\left\{\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}$
• polynomials in x of degree ≤ 2 $a+bx+cx^2$	3	$\{1, x, x^2\}$
• polynomials in x and y with total degree ≤ 2 $a+bx+cx^2+dy+ey^2+fxy$	6	$\{1, x, x^2, y, y^2, xy\}$
• polynomials in x	∞	$\{1, x, x^2, x^3, \dots\}$
• $N((1\ 1\ 1\ \cdots\ 1))$ $= \{\vec{x}\in\mathbb{R}^n \mid x_1+x_2+\cdots+x_n = 0\}$	$n-1$	$\{\vec{e}_1-\vec{e}_2, \vec{e}_1-\vec{e}_3, \dots, \vec{e}_1-\vec{e}_n\}$
• functions $f:\mathbb{R}\to\mathbb{R}$	∞ !	
• functions $f:\{0,1\}^2 \to \mathbb{R}$	4	

- functions $f: \{0,1\}^2 \to \mathbb{R}$ that are symmetric: $f(x,y) = f(y,x)$ 3

  $f(0,0)$, $f(0,1) = f(1,0)$, $f(1,1)$

Examples: Different bases for the same space

- Any linearly independent set S is a basis for $\mathrm{Span}(S)$.

  dimension $= |S|$

- $V = \{$polynomials of degree $\leq n\}$
  - $\{1, x, x^2, \ldots, x^n\}$ form a basis
  - Hermite polynomials $\{1, x, x^2-1, x^3-3x, x^4-6x^2+3, \ldots, (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})\}$ is another

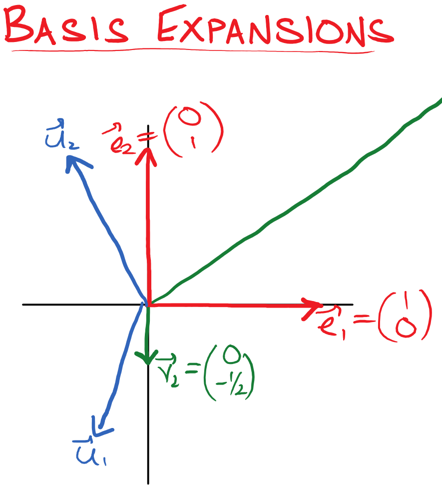
  dimension $= n+1$

- $\mathbb{C}^n$ n dimensions over $\mathbb{C}$
  - standard basis $e_1, \ldots, e_n$
  - Fourier basis $v_j = \frac{1}{\sqrt{n}} \sum_{k=0}^{n} \omega^{jk} e_k$

    where $\omega = e^{2\pi i/n}$

  Exercise: Why is $\{v_j\}$ linearly independent?

Basis Expansions



$$\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix} = x\vec{e}_1 + y\vec{e}_2$$

$$= \underbrace{\frac{x}{4}\vec{v}_1}_{\begin{pmatrix} x \\ \frac{3}{4}x \end{pmatrix}} - 2\left(y - \frac{3}{4}x\right)\vec{v}_2$$

Def: $\begin{pmatrix} x \\ y \end{pmatrix}$ are the coordinates of $\vec{v}$ in the standard basis

$\left(\frac{x}{4}, -2\left(y - \frac{3}{4}x\right)\right)$ are $\vec{v}$'s coordinates in the $\{\vec{v}_1, \vec{v}_2\}$ basis

Example: "Hadamard basis" for $\mathbb{R}^2$

$$\mathcal{B}_H = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

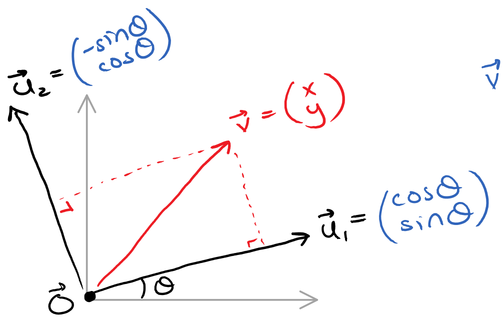
a) Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2$.
What are its coordinates in the Hadamard basis?

$$\frac{x_1 + x_2}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{x_1 - x_2}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \vec{x} \quad \checkmark$$

b) In the Hadamard basis, let $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.
Express $\vec{y}$ in the standard basis.

$$\vec{y} = y_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ y_1 - y_2 \end{pmatrix} = (y_1 + y_2)\vec{e}_1 + (y_1 - y_2)\vec{e}_2.$$

Example: Rotated basis



$$\vec{v} = \begin{pmatrix} x\cos\theta \\ +y\sin\theta \end{pmatrix} \vec{u}_1 + \begin{pmatrix} -x\sin\theta \\ +y\cos\theta \end{pmatrix} \vec{u}_2$$

Observe: The new coordinates are related to the old coordinates by an invertible matrix:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix}$$

Why care?

1. Makes all (finite-dimensional, real) vector spaces look just like $\mathbb{R}^n$

   e.g. $a + bx + cx^2 + dx^3 \longleftrightarrow (a, b, c, d) \in \mathbb{R}^4$
   for the polynomial basis $1, x, x^2, x^3$

2. Specifying a linear transformation on a basis gives its values everywhere.

$$f(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \sum_j \alpha_j f(v_j)$$

3. Dimension measures the "size" of a vector space ...

Example: Polynomials in x of degree ≤ 3
one basis: $1, x, x^2, x^3$
coordinates of $a + bx + cx^2 + dx^3$ are $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

coordinates of $a+bx+cx^2+dx^3$ are $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

In this basis, consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{4\times 4} \qquad A\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} b \\ 2c \\ 3d \\ 0 \end{pmatrix} \longleftrightarrow b+2cx+3dx^2$$

$\Rightarrow$ A corresponds to differentiation $\left(\frac{d}{dx}\right)$.

Theorem 1: Any two bases for the same space must have the same # of elements. ("Dimension makes sense")

Theorem 2: If U is a subspace of V and $U \neq V$, then

$$\dim(U) < \dim(V).$$

Proof of Theorem 1:

Let $S = \{s_1, \ldots, s_m\}$ be two bases for a vector space V.
$T = \{t_1, \ldots, t_n\}$

Assume $m < n$.

Goal: Show T is linearly dependent.

Let

$$B = \begin{pmatrix} | & | & & | \\ t_1 & t_2 & \cdots & t_n \\ | & | & & | \end{pmatrix}$$

be the matrix whose columns are the vectors t_j expanded in the basis S. (Thus $t_j = \sum_{i=1}^{m} B_{ij} s_i$.)

B is an $m \times n$ matrix.

Lemma:

$$\underset{m<n}{\overset{\quad n}{m\,\boxed{\;A\;}}} \Rightarrow N(A) \neq \{\vec{0}\}$$

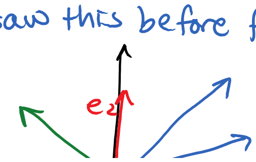
i.e., it is **more**!

$\Rightarrow N(B) \neq \{0\}$

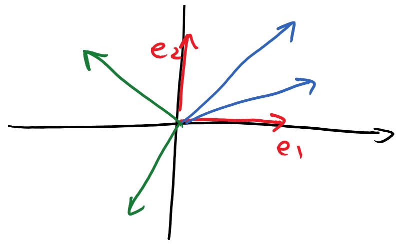
$\Rightarrow$ its columns, the vectors in T, are linearly dependent. ✓ $\square$

Corollary: If $\dim(V) = n$, then any linearly independent set $S \subseteq V$ with n elements is a basis for V.

We saw this before for $\mathbb{R}^2$:



Proof idea: If $\mathrm{Span}(S) \neq V$, then there is $v \in V$, $v \notin \mathrm{Span}(S)$.
Let $T = S \cup \{v\}$, with $n+1$ elts.



there is $v \in V$, $v \notin \mathrm{Span}(S)$.
Let $T = S \cup \{v\}$, with $n+1$ elts.
By the above argument, T is linearly dependent, a contradiction. $\square$

Theorem 2: If U is a subspace of V and $U \neq V$, then

$$\dim(U) < \dim(V).$$

Proof: Assume $\dim(U) = \dim(V) = n$.
Let S be a basis for U. By the corollary, S is a basis for V, too: $U = V$. Contradiction. $\square$

Another way to think about it:

- a basis is a minimal spanning set (if you remove anything, it won't span V)
- a basis is a maximal independent set (adding anything will create a dependency)

Use the proof of Theorem 1 to show this formally.

Changing basis

Example: "Haar basis" for R^4

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 + x_4\vec{e}_4$

What are its coordinates in the Haar basis?

$$\vec{x} = a(1,1,1,1) + b(1,1,-1,-1) + c(1,-1,0,0) + d(0,0,1,-1)$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\Rightarrow a = \frac{x_1 + x_2 + x_3 + x_4}{4}, \quad c = \frac{x_1 - x_2}{2}$$

$$b = \frac{x_1 + x_2 - x_3 - x_4}{4}, \quad d = \frac{x_3 - x_4}{2}$$

Example: Two bases for $\mathbb{R}^4$

$$\mathcal{B} = \left\{ \vec{e}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \vec{e}_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \vec{e}_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \vec{e}_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$$

$$\mathcal{B}' = \left\{ \vec{v}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \vec{v}_3 = \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \vec{v}_4 = \begin{pmatrix}0\\-3\\0\\1\end{pmatrix} \right\}$$

If

$$\vec{\omega} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4$$

$$= \begin{pmatrix}1\\2\\3\\4\end{pmatrix}_{\mathcal{B}'} \leftarrow \text{in the } \mathcal{B}' \text{ basis}$$

$$\vec{\omega} = \begin{pmatrix}1 & 0 & -1 & 0\\0 & 1 & 0 & -3\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \begin{pmatrix}-2\\-10\\3\\4\end{pmatrix}_{\mathcal{B}} \leftarrow \text{in the } \mathcal{B} \text{ basis}$$

since $\vec{v}_1 = \vec{e}_1$, $\vec{v}_2 = \vec{e}_2$, $\vec{v}_3 = -\vec{e}_1 + \vec{e}_3$, $\vec{v}_4 = -3\vec{e}_2 + \vec{e}_4$

Similarly,

$\vec{e}_1 = \vec{v}_1$

$\vec{e}_2 = \vec{v}_2$

$\vec{e}_3 = \vec{v}_1 + \vec{v}_3$

$\vec{e}_4 = 3\vec{v}_2 + \vec{v}_4$

$$\begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 3\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-2\\-10\\3\\4\end{pmatrix} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

$$[\mathcal{B}' \to \mathcal{B}] = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} & \begin{pmatrix}1 & 0 & -1 & 0\\0 & 1 & 0 & -3\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix} \end{array}$$

changes from $\mathcal{B}'$ expansion to $\mathcal{B}$ basis expansion

$$[\mathcal{B} \to \mathcal{B}'] = \begin{array}{c c} & \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 3\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix} \end{array}$$

changes from $\mathcal{B}$ to $\mathcal{B}'$.

Observe: These matrices are inverses of each other:

$$[\mathcal{B}' \to \mathcal{B}] \cdot [\mathcal{B} \to \mathcal{B}'] = [\mathcal{B} \to \mathcal{B}'][\mathcal{B}' \to \mathcal{B}]$$

$$= \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix} = I$$

(The most common mistake here is giving the <u>transposes</u> of the desired matrices.)