EE 510 10/09/2020

Outline:

+ Orthonormal bases
+ Gram-Schmidt procedure.
+ QR decomposition.

A set $V = \{v_1, \ldots, v_n\}$ is an orthormal basis:

- $\langle v_i, v_j \rangle = 0 , \forall i \neq j$
- $\| v_i \| = 1 , i = 1, \ldots, n$

$$x \in \mathrm{Span}(V) \Rightarrow x = \sum_{i=1}^{n} \alpha_i v_i$$

$$\langle v_j, x \rangle = \left\langle v_j, \sum_{i=1}^{n} \alpha_i v_i \right\rangle$$

$$= \langle v_j, \alpha_j v_j \rangle = \alpha_j \| v_j \|^2 = \alpha_j$$

$$\begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} X \end{pmatrix}$$

$B = \{b_1, \dots, b_n\}$, $X = \sum_{i=1}^{n} \beta_i b_i$

$V = \{v_1, \dots, v_n\}$ a set of vectors, span the vector space E

$E = \mathrm{Span}(V)$.

$V_1 = \{v_1, \dots, v_r\}$ linearly indipendent

$\Rightarrow \dim E = r$

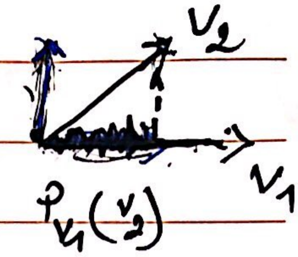
Find an orthormal basis for E from the set V_1.

$U_1 = V_1$

$U_2 \in V_1^{\perp} \Rightarrow U_2 = P_{V_1^{\perp}}(U_1) = V_2 - \langle U_1, V_2 \rangle U_1$

Expl

$\mathbb{R}^2 = \operatorname{span}(v_1, v_2)$



$U_1 = V_1$

$U_2 \in P_{V_1^\perp}(V_2) = (I - P_{V_1})(V_2)$

$= V_2 - P_{V_1}(V_2)$

$\boxed{P_V + P_{V^\perp} = I}$

$= V_2 - \langle U_1, V_2 \rangle V_1$

$U_3 \in (\operatorname{span}(U_1, U_2))^\perp$

$U_3 = P_{\operatorname{span}(U_1,U_2)^\perp}(V_3) = V_3 - P_{\operatorname{span}\{U_1,U_2\}}(V_3) = V_3 - \langle U_1, V_3 \rangle V_1 - \langle U_2, V_3 \rangle V_2$

$P_{\operatorname{span}(U_1,U_2)^\perp}(V_3) = P_{U_1^\perp}(V_3) + P_{U_2^\perp}(V_3)$

$\boxed{V \perp W \Rightarrow P_{\operatorname{span}(U,W)} = P_U + P_W}$

$\vdots$

$U_i = V_i - \sum_{j=1}^{i-1} \langle U_j, V_i \rangle U_j$

$$U_r = V_r - \langle U_1, V_r\rangle U_1 - \langle U_2, V_r\rangle U_2 - \cdots - \langle U_{r-1}, V_r\rangle U_{r-1}$$

$$b_i = \frac{U_i}{\|U_i\|} \; ; \; i = 1, \ldots, r.$$

$$\|b_i\| = 1 \; ; \; \langle b_i, b_j\rangle = \left\langle \frac{U_i}{\|U_i\|}, \frac{U_j}{\|U_j\|} \right\rangle = 0$$

<u>Exercise:</u> $x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $y = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$; $z = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Find an orthonormal basis by applying G.S. to $\{x, y, z\}$.

$$U_1 = x \Rightarrow V_1 = \frac{U_1}{\|U_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$U_2 = y - \langle y, U_1\rangle U_1 = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \Rightarrow V_2 = \frac{U_2}{\|U_2\|} = \frac{1}{\sqrt{6}}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$U_3 = z - \langle z, U_1\rangle U_1 - \langle z, U_2\rangle U_2 = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$V_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$\{V_1, V_2, V_3\}$ is an orthonormal basis.

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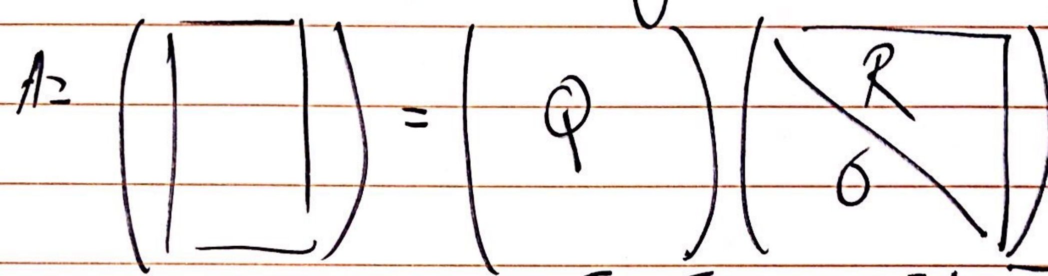
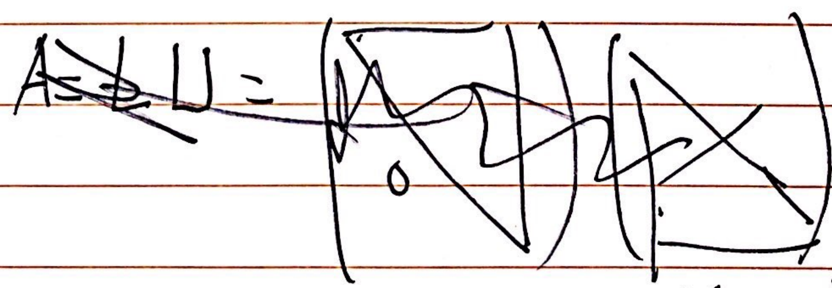
QR decomposition:

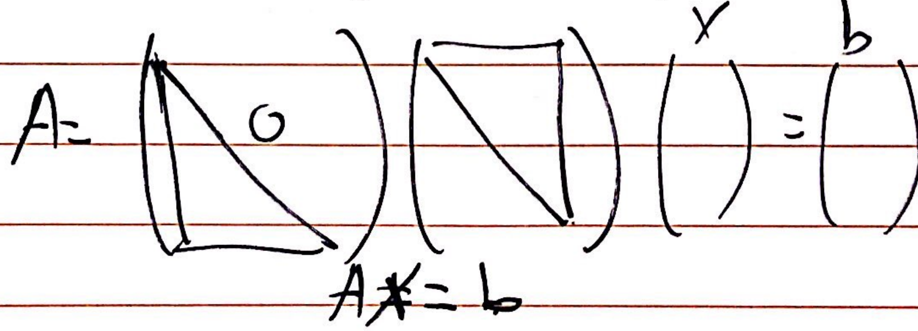
- $A \in \mathbb{R}^{n \times n} \Rightarrow A = Q.R$:
  - Q satisfies: $QQ^T = Q^TQ = I$
  - R is an upper triangular matrix.



$QQ^T = Q^TQ = I$, $Q^{-1} = Q^T$

This decomposition is unique iff A is invertible!





$Ax = b$

$$Ax = b \Longrightarrow QRx = b$$

$$\Rightarrow Rx = Q^T b$$

$$\begin{pmatrix} \diagdown\, R \\ 0 \end{pmatrix} \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} y \end{pmatrix}$$

Expl: Find QR decomposition of A and use that to solve $Ax = b$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \; ; \; b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$

$$v_1 = \frac{c_1}{\|c_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \; ; \; u_2 = c_2 - \langle v_1, c_2 \rangle v_1 = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$$

$$v_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}$$

$$u_3 = c_3 - \langle v_1, c_3 \rangle v_1 - \langle v_2, c_3 \rangle v_2 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$$

$$v_3 = \frac{u_3}{\|u_3\|} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$$

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$\langle v_i, v_j\rangle = 0\ ;\ i \neq j\ ;\ i,j = 1,2,3.$

$\langle v_i, v_i\rangle = 1$

$$Q = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}\ ;\ QQ^T = Q^TQ = \begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \end{pmatrix}\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} \langle v_1, c_1\rangle & \langle v_2, c_1\rangle & \langle v_3, c_1\rangle \\ 0 & \langle v_2, c_2\rangle & \langle v_3, c_2\rangle \\ 0 & 0 & \langle v_3, c_3\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{3} & \frac{2}{\sqrt{3}} & 0 \\ 0 & \frac{2}{\sqrt{6}} & \frac{3}{\sqrt{6}} \\ 0 & 0 & \frac{3}{\sqrt{6}} \end{pmatrix}$$

$$Ax = b \Rightarrow QRx = b \Rightarrow Rx = Q^Tb = \begin{pmatrix} 0 \\ \frac{9}{\sqrt{6}} \\ \frac{7}{\sqrt{6}} \end{pmatrix}$$

$$\Rightarrow x = \begin{pmatrix} -\frac{2}{3} \\ 1 \\ \frac{7}{3} \end{pmatrix}.$$

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Exercise:

V: The space of polynomials in x of degree at most 1

1°/ Find a basis of V. ? Deduce the dimension of V.

$$\mathbb{R}_n[x] = \{1, x, x^2, \dots, x^n\} \Rightarrow \dim \mathbb{R}_n[x] = n+1$$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0.$$

$n = 1$; $\mathbb{R}_1[x] = \text{span}\{1, x\}$.

$$\dim \mathbb{R}_1[x] = 2$$

2°) Find the projection of the function x^3 defined over the interval $[-1, 1]$ on V.

(the inner product: $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$

$P_V(x^3)$? ;

$V = \text{Span}\{1, x\}$

$\langle x, 1\rangle = \int_{-1}^{1} x \, dx = 0$

$x \perp 1 \;\Rightarrow P_V = P_x + P_1$

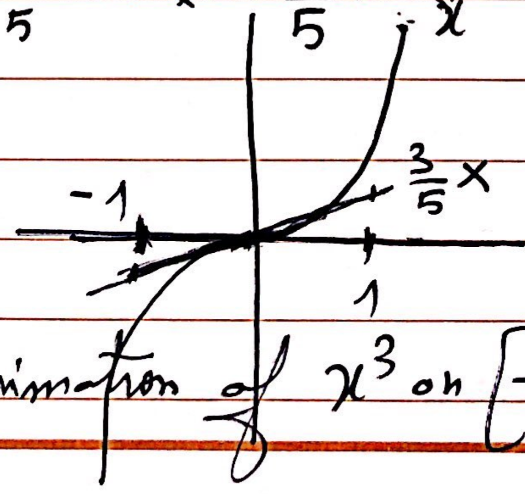
$P_V(x^3) = P_x(x^3) + P_1(x^3)$

$P_x(x^3) = \frac{\langle x; x^3\rangle}{\langle x, x\rangle} x \quad ; P_1(x^3) = \frac{\langle 1, x^3\rangle}{\langle 1, 1\rangle} 1$

$\langle 1, x^3\rangle = \int_{-1}^{1} x^3 dx = 0$

$\langle x, x^3\rangle = \int_{-1}^{1} x^4 dx = \frac{2}{5} \quad ; \langle x, x\rangle = \int_{-1}^{1} x^2 dx = \frac{2}{3}$

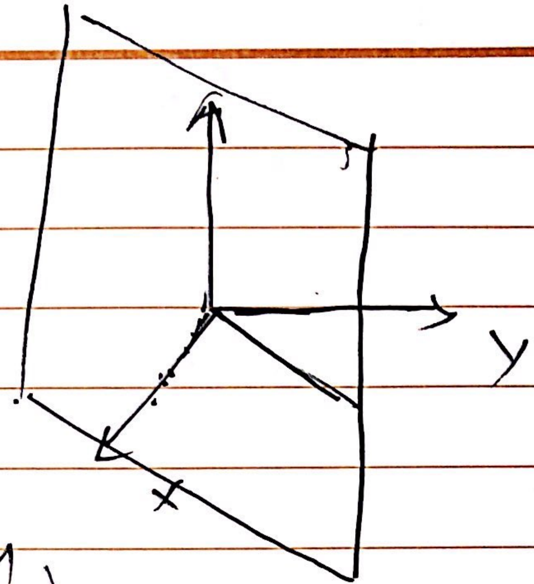
$P_V(x^3) = \frac{3}{5} x + 0 \times 1 = \frac{3}{5} x$



Linear approximation of x^3 on $[-1, 1]$.

- Reflection:

$x = y$



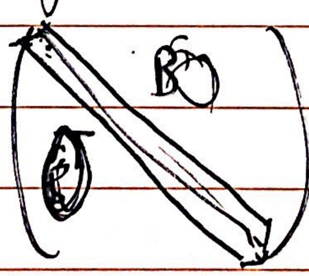
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x = z \Rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$y = z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- Dimension:
  - ~~$A = A^T$ and upper~~

    dimension of the vector space of matrices, of both symmetric and upper triangular:

$A =$  ; dim $= n$

dim $= 0$ (Anti-symmetric and upper triangular)

- $\mathbb{R}$: $\{1\}$; $\dim \mathbb{R} = 1$
- $\mathbb{C}$ over $\mathbb{R}$

$\{1, i\}$ $\dim \mathbb{C} = 2$

- $\mathbb{C}$ over $\mathbb{C}$: $\{1\}$; $\dim \mathbb{C} = 1$

- $C = \begin{pmatrix} 1 & 7 & 8 & 0 \\ 5 & 2 & 6 & 9 \\ 9 & 6 & 3 & 5 \\ 0 & 8 & 7 & 4 \end{pmatrix}$ is invertible

orthonormal basis for $R(C)$.

$R(C) = \mathbb{R}^4 \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$