INNER PRODUCTS FOR MORE GENERAL VECTOR SPACES

I. Matrices

For matrices $A, B \in \mathbb{R}^{m \times n}$, define an inner product

$$\langle A, B \rangle := \underset{\text{trace}}{\mathrm{Tr}}(A^T B)$$

Example: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$\begin{aligned} \langle A, B \rangle &= \mathrm{Tr}(A^T B) \\ &= \mathrm{Tr}\begin{pmatrix} a_{11}b_{11} + a_{21}b_{21} & \text{something} \\ \text{something} & a_{12}b_{12} + a_{22}b_{22} \end{pmatrix} \\ &= a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} \end{aligned}$$

In general,

$$\begin{aligned} \mathrm{Tr}(A^T B) &= \sum_j (A^T B)_{jj} \\ &= \sum_{j,i} (A^T)_{ji} B_{ij} \\ &= \boxed{\sum_{i,j} A_{ij} B_{ij}} \end{aligned}$$

much like the inner product for vectors.

Example: 2×2 symmetric or anti-symm. matrices

Symmetric matrices $(M = M^T)$ $= \mathrm{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

$\perp$

Anti-symmetric/skew-symmetric matrices $(M = -M^T)$ $= \mathrm{Span}\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

II. Vector spaces over finite fields

Let $V = \{0, 1\}^n$ over $\mathbb{F}_2$ (addition/multiplication mod 2)

e.g., $V = \{(0,0), (0,1), (1,0), (1,1)\}$ for $n = 2$

- Lengths don't mean much

$$\|(1,1)\| \overset{?}{=} \sqrt{1^2 + 1^2} = \sqrt{2}$$

but $\frac{1}{\sqrt{2}}(1,1) \notin V$; you can't renormalize

- Angles don't mean **anything**
  there's no Euclidean geometry here

- But **orthogonality** still makes sense!
  $\vec{x} \perp \vec{y}$ if $\sum_{i=1}^{n} x_i y_i = 0 \mod 2$

  Note: $(1,1) \perp (1,1)$!!
  any even-weight vector is orthogonal to itself

  $\Rightarrow$ The Rank-Nullity Theorem still holds!

Definition: A binary linear error-correcting code is a subspace of $\{0,1\}^n$.

Example: 3-bit repetition code

$$0 \longmapsto 000$$
$$1 \longmapsto 111$$

code $= \{(0,0,0), (1,1,1)\}$ 1-dimensional subspace

$= R(A^T)$ for $A = \underbrace{(1 \;\; 1 \;\; 1)}_{\text{generator matrix}}$

$N(A) = R(A^T)^\perp$

$= R\left(\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}}_{\text{parity check matrix}}\right) = \left\{\begin{matrix} (0,0,0), (0,1,1), \\ (1,1,0), (1,0,1) \end{matrix}\right\}$

2 dimensional

$\rightsquigarrow$ a linear code can be specified by either the **generating matrix** or the **parity-check matrix**

III. Function spaces with more general inner products

A function is like a vector with ∞ly many coordinates

$\vec{v}$ has $v_1, v_2, v_3, \ldots$ $\qquad$ f has $f(1), f(2), f(\pi), \ldots$

$\left\{ \begin{matrix} \text{all functions mapping} \\ [-1,1] \longrightarrow \mathbb{R} \end{matrix} \right\}$ is a vector space

Inner product?

Inner product?

$$\langle f,g\rangle \overset{?}{=} \cancel{\sum_{x\in[-1,1]} f(x)g(x)}$$

$$= \int_{-1}^{1} dx\, f(x)g(x)$$

for square-integrable functions

Angles don't make sense, but orthogonality does!

Example: $1 \perp x$
since $\int_{-1}^{1} dx\, x = 0$

Also,
$1 \perp (x^2-\frac{1}{3})$ since $\int_{-1}^{1} (x^2-\frac{1}{3})dx = 0$
$x \perp (x^2-\frac{1}{3})$ $\int_{-1}^{1} (x^3-\frac{x}{3})dx = 0$

$\|1\| = \sqrt{\int_{-1}^{1} dx} = \sqrt{2}$, $\|x\| = \sqrt{\int_{-1}^{1} dx\, x^2} = \sqrt{\frac{2}{3}}$, $\|x^2-\frac{1}{3}\| = \sqrt{\frac{8}{45}}$

$\Rightarrow \{1, x, x^2-\frac{1}{3}\}$ is an orthogonal basis (with respect to this inner product) for polynomials of degree ≤ 2.

Legendre polynomials

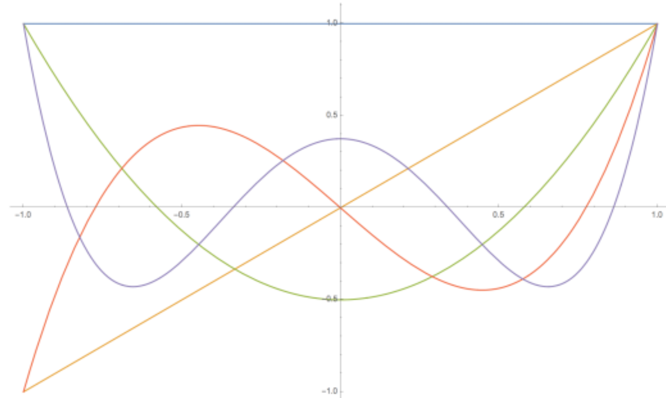
https://en.wikipedia.org/wiki/Legendre_polynomials

```
Table[{k, LegendreP[k, x]}, {k, 0, 9}] // MatrixForm
```

0	1
1	x
2	$\frac{1}{2}(-1+3x^2)$
3	$\frac{1}{2}(-3x+5x^3)$
4	$\frac{1}{8}(3-30x^2+35x^4)$
5	$\frac{1}{8}(15x-70x^3+63x^5)$
6	$\frac{1}{16}(-5+105x^2-315x^4+231x^6)$
7	$\frac{1}{16}(-35x+315x^3-693x^5+429x^7)$
8	$\frac{1}{128}(35-1260x^2+6930x^4-12012x^6+6435x^8)$
9	$\frac{1}{128}(315x-4620x^3+18018x^5-25740x^7+12155x^9)$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$$

```
Plot[Evaluate@Table[LegendreP[k, x], {k, 0, 4}], {x, -1, 1}]
```



Applications of Legendre polynomials in physics [edit]

The Legendre polynomials were first introduced in 1782 by Adrien-Marie Legendre[3] as the coefficients in the expansion of the Newtonian potential

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} = \sum_{\ell=0}^{\infty}\frac{r'^\ell}{r^{\ell+1}}P_\ell(\cos\gamma)$$

where r and r' are the lengths of the vectors $\mathbf{x}$ and $\mathbf{x}'$ respectively and γ is the angle between those two vectors. The series converges when $r > r'$. The expression gives the gravitational potential associated to a point mass or the Coulomb potential associated to a point charge. The expansion using Legendre polynomials might be useful, for instance, when integrating this expression over a continuous mass or charge distribution.

Legendre polynomials occur in the solution of Laplace's equation of the static potential, $\nabla^2\Phi(\mathbf{x}) = 0$, in a charge-free region of space, using the method of separation of variables, where the boundary conditions have axial symmetry (no dependence on an azimuthal angle). Where $\hat{\mathbf{z}}$ is the axis of symmetry and θ is the angle between the position of the observer and the $\hat{\mathbf{z}}$ axis (the zenith angle), the solution for the potential will be

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right]P_\ell(\cos\theta).$$

A_ℓ and B_ℓ are to be determined according to the boundary condition of each problem.[4]

They also appear when solving Schrödinger equation in three dimensions for a central force.

Legendre polynomials in multipole expansions [edit]

Legendre polynomials are also useful in expanding functions of the form (this is the same as before, written a little differently):

$$\frac{1}{\sqrt{1+\eta^2-2\eta x}} = \sum_{k=0}^{\infty}\eta^k P_k(x)$$

which arise naturally in multipole expansions. The left-hand side of the equation is the generating function for the Legendre polynomials.

As an example, the electric potential $\Phi(r,\theta)$ (in spherical coordinates) due to a point charge located on the z-axis at $z=a$ (Figure 2) varies like

$$\Phi(r,\theta) \propto \frac{1}{R} = \frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}}$$

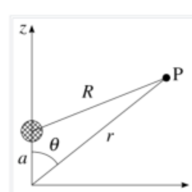


Figure 2

If the radius r of the observation point **P** is greater than a, the potential may be expanded in the Legendre polynomials

$$\Phi(r,\theta) \propto \frac{1}{r}\sum_{k=0}^{\infty}\left(\frac{a}{r}\right)^k P_k(\cos\theta)$$

where we have defined $\eta = a/r < 1$ and $x = \cos\theta$. This expansion is used to develop the normal multipole expansion.

Conversely, if the radius r of the observation point **P** is smaller than a, the potential may still be expanded in the Legendre polynomials as above, but with a and r exchanged. This expansion is the basis of interior multipole expansion.

Example: Compute $\|x+1\|^2$ for the inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)dx$.

Answer:

$$\|1+x\|^2 = \langle 1+x, 1+x\rangle$$
$$= \|1\|^2 + 2\langle 1, x\rangle + \|x\|^2$$
$$= 2 + 0 + \tfrac{2}{3}$$

You can check this by computing directly $\int_{-1}^{1}(1+x)^2\,dx$.

Other polynomial families come from different inner products.

Orthogonal polynomials

From Wikipedia, the free encyclopedia https://en.wikipedia.org/wiki/Orthogonal_polynomials

In mathematics, an **orthogonal polynomial sequence** is a family of polynomials such that any two different polynomials in the sequence are orthogonal to each other under some inner product.

The most widely used orthogonal polynomials are the classical orthogonal polynomials, consisting of the Hermite polynomials, the Laguerre polynomials, the Jacobi polynomials together with their special cases the Gegenbauer polynomials, the Chebyshev polynomials, and the Legendre polynomials.

Hermite polynomials: $\langle f,g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2/2}\,dx$

Laguerre polynomials: $\langle f,g\rangle = \int_{0}^{\infty} f(x)g(x)e^{-x}\,dx$

Chebyshev: $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\frac{1}{\sqrt{1-x^2}}\,dx$ "1st kind"

or $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\sqrt{1-x^2}\,dx$ "2nd kind"

```
Table[{k, HermiteH[k, x]}, {k, 0, 9}] // MatrixForm
```

0	1
1	$2x$
2	$-2+4x^2$
3	$-12x+8x^3$
4	$12-48x^2+16x^4$
5	$120x-160x^3+32x^5$
6	$-120+720x^2-480x^4+64x^6$
7	$-1680x+3360x^3-1344x^5+128x^7$
8	$1680-13440x^2+13440x^4-3584x^6+256x^8$
9	$30240x-80640x^3+48384x^5-9216x^7+512x^9$

```
Table[{k, LaguerreL[k, x]}, {k, 0, 9}] // MatrixForm
```

0	1
1	$1-x$
2	$\frac{1}{2}(2-4x+x^2)$
3	$\frac{1}{6}(6-18x+9x^2-x^3)$
4	$\frac{1}{24}(24-96x+72x^2-16x^3+x^4)$
5	$\frac{1}{120}(120-600x+600x^2-200x^3+25x^4-x^5)$
6	$\frac{1}{720}(720-4320x+5400x^2-2400x^3+450x^4-36x^5+x^6)$
7	$\frac{5040-35280x+52920x^2-29400x^3+7350x^4-882x^5+49x^6-x^7}{5040}$
8	$\frac{40320-322560x+564480x^2-376320x^3+117600x^4-18816x^5+1568x^6-64x^7+x^8}{40320}$
9	$\frac{362880-3265920x+6531840x^2-5080320x^3+1905120x^4-381024x^5+42336x^6-2592x^7+81x^8-x^9}{362880}$

```
Table[{k, ChebyshevT[k, x], ChebyshevU[k, x]}, {k, 0, 9}] // MatrixForm
```

0	1	1
1	x	$2x$
2	$-1+2x^2$	$-1+4x^2$
3	$-3x+4x^3$	$-4x+8x^3$
4	$1-8x^2+8x^4$	$1-12x^2+16x^4$
5	$5x-20x^3+16x^5$	$6x-32x^3+32x^5$
6	$-1+18x^2-48x^4+32x^6$	$-1+24x^2-80x^4+64x^6$
7	$-7x+56x^3-112x^5+64x^7$	$-8x+80x^3-192x^5+128x^7$
8	$1-32x^2+160x^4-256x^6+128x^8$	$1-40x^2+240x^4-448x^6+256x^8$
9	$9x-120x^3+432x^5-576x^7+256x^9$	$10x-160x^3+672x^5-1024x^7+512x^9$