Homework 6 Gram-Schmidt answers

① Using the Gram-Schmidt procedure, find an orthonormal basis for the span of

$$(1,1,1,-1),\ (2,-1,-1,1) \text{ and } (-1,2,2,1).$$

Answer: $u_1 = (1,1,1,-1)$, $u_2 = (2,-1,-1,1)$, $u_3 = (-1,2,2,1)$

$$\boxed{v_1 = \frac{u_1}{\|u_1\|} = \frac{1}{2}(1,1,1,-1)}$$

$$u_2 - (v_1 \cdot u_2)v_1 = u_2 + \frac{1}{4}u_1 = \left(\frac{9}{4}, -\frac{3}{4}, -\frac{3}{4}, \frac{3}{4}\right)$$
$$= \frac{3}{4}(3,-1,-1,1)$$

$$\boxed{v_2 = \frac{1}{2\sqrt{3}}(3,-1,-1,1)}$$

$$u_3 - (v_1 \cdot u_3)v_1 - (v_2 \cdot u_3)v_2$$
$$= u_3 - \frac{1}{2}u_1 - \frac{1}{12}(-6)\cdot(3,-1,-1,1)$$
$$= (0,1,1,2)$$

$$\boxed{v_3 = \frac{1}{\sqrt{6}}(0,1,1,2)}$$

② a) Use the Gram-Schmidt procedure to find an orthonormal basis for the rowspace, columnspace, nullspace and left nullspace of

$$A = \begin{pmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & -2 & 6 \\ 5 & -10 & -5 & 15 \end{pmatrix}$$

b) What is the projection of $(1,1,1)$ onto $N(A^T)$?

Answer: The second and third rows are multiples of the first row. Hence $\operatorname{rank}(A) = 1 = \dim R(A) = \dim R(A^T)$.

$$R(A^T) = \operatorname{Span}((1,-2,-1,3))$$
$$= \operatorname{Span}\left(\frac{1}{\sqrt{15}}(1,-2,1,3)\right)$$

← o.n. basis

$$R(A) = \operatorname{Span}((1,2,5))$$
$$= \operatorname{Span}\left(\frac{1}{\sqrt{30}}(1,2,5)\right)$$

By rank-nullity, $\dim N(A) = 3$, $\dim N(A^T) = 2$.

$$N(A^T) = \operatorname{Span}\{(-2,1,0), (-5,0,1)\}$$

$$\frac{1}{\sqrt{5}}(-2,1,0),\quad (-5,0,1) - ((-5,0,1)\cdot(-2,1,0))(-2,1,0)$$

$$\tfrac{1}{\sqrt{5}}(-2,1,0), \quad (-5,0,1) - \frac{((-5,0,1)\cdot(-2,1,0))(-2,1,0)}{5}$$

$$= (-1,-2,1)$$

$$\hookrightarrow \tfrac{1}{\sqrt{6}}(-1,-2,1)$$

$$= \mathrm{Span}\left\{\tfrac{1}{\sqrt{5}}(-2,1,0), \tfrac{1}{\sqrt{6}}(-1,-2,1)\right\} \checkmark$$

$$N(A) = \mathrm{Span}\{(2,1,0,0), (1,0,1,0), (-3,0,0,1)\}$$

$$\tfrac{1}{\sqrt{5}}(2,1,0,0)$$

subtract $\frac{2}{5}(2,1,0,0)$

$$\left(\tfrac{1}{5}, -\tfrac{2}{5}, 1, 0\right)$$

normalize

$$\tfrac{1}{\sqrt{30}}(1,-2,5,0)$$

subtract $-\frac{6}{5}(2,1,0,0)$ $-\frac{3}{30}(1,-2,5,0)$

$$\left(-\tfrac{1}{2}, 1, \tfrac{1}{2}, 1\right)$$

normalize

$$\tfrac{1}{\sqrt{10}}(-1,2,1,2)$$

$$= \mathrm{Span}\left\{\tfrac{1}{\sqrt{5}}(2,1,0,0), \tfrac{1}{\sqrt{30}}(1,-2,5,0), \tfrac{1}{\sqrt{10}}(-1,2,1,2)\right\}$$

b) Let $u_1 = \frac{1}{\sqrt{5}}(-2,1,0)$, $u_2 = \frac{1}{\sqrt{6}}(-1,-2,1)$.

The projection of $v = (1,1,1)$ onto $\mathrm{Span}(u_1, u_2)$ is given by

$$(u_1 \cdot v)u_1 + (u_2 \cdot v)u_2$$

$$= -\tfrac{1}{5}(-2,1,0) - \tfrac{2}{6}(-1,-2,1)$$

$$= \left(\tfrac{11}{15}, \tfrac{7}{15}, -\tfrac{1}{3}\right).$$

③ Orthonormal functions

For two functions $f,g: [0,1] \to \mathbb{R}$, define their inner product to be

$$f \cdot g := \int_0^1 dx\, f(x) g(x).$$

a) Notice that the monomials

$$1, \quad x, \quad x^2, \quad x^3$$

form a basis for the set of polynomials of degree ≤ 3 (a vector space).

A polynomial can be represented by its vector of coordinates in this basis, e.g.,

$$\underbrace{1 + 2x - x^2}_{\text{polynomial}} \longleftrightarrow \underbrace{\overset{1}{(1}, \overset{x}{2}, \overset{x^2}{-1}, \overset{x^3}{0})}_{\substack{\text{vector of coefficients} \\ \text{in basis } \{1, x, x^2, x^3\}}}$$

Write a function that takes

Input: Two polynomials f and g,

given by their coefficient vectors

Output: $f \cdot g$

Check your answer! For example,

$$\text{polynomialdot}\left(\begin{array}{cccc} 1 & x & x^2 & x^3 \\ [1, & 2, & 0, & 0] \end{array}, \begin{array}{cccc} 1 & x & x^2 & x^3 \\ [3, & 4, & 0, & 0] \end{array}\right)$$

should return 10.667, since

$$(1+2x) \cdot (3+4x) = \int_0^1 (1+2x)(3+4x)\,dx = \frac{32}{3}$$

(This is different from the usual dot product $(1,2,0,0) \cdot (3,4,0,0) = 1 \cdot 3 + 2 \cdot 4 = 11$.)

b) Using your function from part a, verify that the polynomials

$$v_0(x) = 1$$
$$v_1(x) = \sqrt{3}(2x-1)$$
$$v_2(x) = \sqrt{5}(6x^2 - 6x + 1)$$
$$v_3(x) = \sqrt{7}(20x^3 - 30x^2 + 12x - 1)$$

form an orthonormal basis for the set of polynomials of degree ≤ 3. In other words,

$$v_i \cdot v_j = \int_0^1 dx\, v_i(x)\, v_j(x) = \begin{cases} 1 & , \text{ if } i = j \\ 0 & , \text{ if } i \neq j. \end{cases}$$

Remark: Here's some Mathematica code that checks orthonormality:

```
basis = {v0, v1, v2, v3} = {
    1,
    2 √3 (x - 1/2),
    √5 (6 x^2 - 6 x + 1),
    √7 (20 x^3 - 30 x^2 + 12 x - 1)
  };
Table[
  Integrate[basis[[i]] basis[[j]], {x, 0, 1}],
  {i, 1, 4}, {j, 1, 4}
 ] // MatrixForm
```

atrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

I want you to do it in Matlab or python, using the vector representation of these functions. For example,

$$\begin{array}{ccccc} & 1 & x & x^2 & x^3 \\ v_1 = & (-\sqrt{3}, & 2\sqrt{3}, & 0, & 0) \end{array}$$

c) Start with the basis

(*) $1, x, x^2, x^3, x^4, \ldots, x^9, x^{10}$

for polynomials of degree ≤ 10.

Use the Gram-Schmidt method with your code from part ⓐ to change this into an orthonormal basis.

Deliverable:

Please turn in your source code, as well as a printout of a matrix whose rows are your final orthonormal polynomials.

For example, when I did this working numerically in Matlab, the first four polynomials I got were the rows:

```
octave:381> Q(1:4,:)
ans =

 Columns 1 through 4:
```

	1	x	x^2	x^3
$1 \longrightarrow$	1.0000e+00	0.0000e+00	0.0000e+00	0.0000e+00
$\sqrt{3}(2x-1) \longrightarrow$	-1.7321e+00	3.4641e+00	0.0000e+00	0.0000e+00
$v_2 \longrightarrow$	2.2361e+00	-1.3416e+01	1.3416e+01	0.0000e+00
$v_3 \longrightarrow$	-2.6458e+00	3.1749e+01	-7.9373e+01	5.2915e+01

the other columns are 0s

These are the same four polynomials found above.

Extra: Does your code still work for finding an orthonormal basis for polynomials of degree ≤ 40? Why or why not?

Answer:

```
% Between two polynomials p(x) and q(x), we are using the inner product
%  p dot q = int_0^1 p(x)q(x)dx .
% If polynomials are represented as vectors, via
%  (a,b,c) <-> a+bx+cx^2,
% this function computes the dot product between two polynomials.
% For example,
%  dot([1,2], [3,4]) = 32/3 = 10.667
% since (1+2x)*(3+4x) = 3+10x+8x^2 and int_0^1 dx (3+10x+8x^2) = 3+5+8/3.
function c = dot(p,q)
  c = 0;
  for i = 1:length(p)
    for j = 1:length(q)
      c = c + p(i)*q(j)/(i+j-1);
      % this is integral_0^1 dx x^(i+j-2), for the cross-term p(i)x^{i-1} * q(j)x^{j-1}
    end
  end
end

% this is standard Gram-Schmidt code, except using the above custom dot function
%  to evaluate the inner products
n = 11;
Q = eye(n);  % start with the 11x11 identity matrix;
```

```
%  to evaluate the inner products
n = 11;
Q = eye(n);  % start with the 11x11 identity matrix;
             %  its rows represent the polynomials 1,x,x^2,x^3,...,x^10
for i = 1:n
  Q(i,:) /= sqrt(dot(Q(i,:),Q(i,:)));
  for j = i+1:n
    Q(j,:) -= dot(Q(i,:),Q(j,:)) * Q(i,:);
  end
end
```

```
octave:28> Q
Q =

   1.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00
  -1.7321e+00   3.4641e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00
   2.2361e+00  -1.3416e+01   1.3416e+01   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00
  -2.6458e+00   3.1749e+01  -7.9373e+01   5.2915e+01   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00
   3.0000e+00  -6.0000e+01   2.7000e+02  -4.2000e+02   2.1000e+02   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00
  -3.3166e+00   9.9499e+01  -6.9649e+02   1.8573e+03  -2.0895e+03   8.3579e+02   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00
   3.6056e+00  -1.5143e+02   1.5143e+03  -6.0573e+03   1.1357e+04  -9.9946e+03   3.3315e+03   0.0000e+00   0.0000e+00   0.0000e+00   0.0000e+00
  -3.8730e+00   2.1689e+02  -2.9280e+03   1.6267e+04  -4.4733e+04   6.4415e+04  -4.6522e+04   1.3292e+04   0.0000e+00   0.0000e+00   0.0000e+00
   4.1231e+00  -2.9686e+02   5.1951e+03  -3.8097e+04   1.4287e+05  -2.9716e+05   3.4669e+05  -2.1226e+05   5.3064e+04   0.0000e+00   0.0000e+00
  -4.3587e+00   3.9229e+02  -8.6304e+03   8.0551e+04  -3.9269e+05   1.0995e+06  -1.8325e+06   1.7951e+06  -9.5367e+05   2.1193e+05   0.0000e+00
   4.5756e+00  -5.0341e+02   1.3594e+04  -1.5710e+05   9.6234e+05  -3.4647e+06   7.6997e+06  -1.0686e+07   9.0165e+06  -4.2301e+06   8.4605e+05
```