University of Southern California

EE 510: Discussion 2

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1 Gaussian elimination

An elementary operation on the rows of a matrix A is one of the following transformations

- Permuting two different rows R and R'.
- Replacing a row R by αR where α is nonzero ($\alpha \neq 0$).
- Replacing a row R by $R + \beta R'$ where R' is another row and β can take any value.

  $\Rightarrow$ Gaussian elimination consists of applying a finite number of elementary operations on the rows.

Remark 1 *:* Note that Gaussian elimination can also be performed on columns.

2 Linear Systems: Solution using Gaussian elimination

Let $(\mathcal{S})$ be a linear system with n equations and m unknowns, and let $A|b$ its complete (augmented) matrix. In order to solve $(\mathcal{S})$, one can proceed as follows.

- **First step**: One looks for a new augmented matrix $A_0|b_0$ by performing Gaussian elimination on the augmented matrix $A|b$ (using only elementary operations on the rows).
- **Second step**: The new augmented matrix $A_0|b_0$ defines an equivalent linear system $\mathcal{S}_0$. One solve the linear system $(\mathcal{S}_0)$ by back substitution, beginning by the last row. The set of solutions of $\mathcal{S}_0$ is nothing but the set of solutions of $(\mathcal{S})$.

Exercise 1 *: Solve the following linear system*

$$(\mathcal{S}_1): \begin{cases} x + y - z = 8 \\ x + 2y - 3z = 5 \\ 3x + 2y - z = 2 \end{cases}.$$

Solution: The augmented matrix for the linear system is given as

$$A|b = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 8 \\ 1 & 2 & -3 & 5 \\ 3 & 2 & -1 & 2 \end{array}\right).$$

We use the Gauss-Jordan method applied to $A|b$.

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 8 \\ 1 & 2 & -3 & 5 \\ 3 & 2 & -1 & 2 \end{array}\right) \xrightarrow[\substack{R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - 3R_1}]{\text{Pivot : 1}} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 8 \\ 0 & \boxed{1} & -2 & -3 \\ 0 & -1 & 2 & -22 \end{array}\right)$$

$$\xrightarrow[R_3 \leftarrow R_3 + R_2]{\text{Pivot : 1}} A'|b' = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 8 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & -25 \end{array}\right).$$

The original linear system $\mathcal{S}_1$ is equivalent to the system $(\mathcal{S}_1')$:

$$(\mathcal{S}_1'): \begin{cases} x + y - z = 8 \\ y - 2z = -3 \\ 0 = -25 \end{cases}$$

This system $(\mathcal{S}_1')$ does not have solution since $(0 \neq -25)$ in the third equation; hence $(\mathcal{S}_1)$ does not have any solution.

Exercise 2 *: Solve the following linear system*

$$(\mathcal{S}_2): \begin{cases} x + 2y - 4z = -4 \\ 2x + 5y - 9z = -10 \\ 3x - 2y + 3z = 11 \end{cases}.$$

Solution:

The augmented matrix is given as $A|b = \left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 2 & 5 & -9 & -10 \\ 3 & -2 & 3 & 11 \end{array}\right)$. First, we determine an equivalent augmented matrix:

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & -4 & -4 \\ 2 & 5 & -9 & -10 \\ 3 & -2 & 3 & 11 \end{array}\right) \xrightarrow[\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1}]{\text{Pivot : 1}} \left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & \boxed{1} & -1 & -2 \\ 0 & -8 & 15 & 23 \end{array}\right)$$

$$\xrightarrow[R_3 \leftarrow R_3 + 8R_2]{\text{Pivot : 1}} \left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 7 & 7 \end{array}\right)$$

$$\xrightarrow[R_3 \leftarrow \frac{1}{7}R_3]{\text{Pivot : 1}} A'|b' = \left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

The linear system $(\mathcal{S}_2)$ is equivalent to the linear system $(\mathcal{S}_2')$. In fact, $(\mathcal{S}_2)$ is equivalent to

$$(\mathcal{S}_2') : \begin{cases} x + 2y - 4z = -4 \\ \quad\quad y - z = -2 \\ \quad\quad\quad z = 1 \end{cases}$$

Using back substitution, we determine the unique solution for this linear system given by $X = (2, -1, 1)$. Therefore the set of solutions of $(\mathcal{S}_2)$ is

$$\{(2, -1, 1)\}.$$

Exercise 3 *: Solve the following linear system*

$$(\mathcal{S}_3) \begin{cases} x + 2y - 3z + 2t = 2 \\ 2x + 5y - 8z + 6t = 5 \\ 3x + 4y - 5z + 2t = 4 \end{cases}.$$

Solution

The augmented matrix is $A|b = \left(\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 2 \\ 2 & 5 & -8 & 6 & 5 \\ 3 & 4 & -5 & 2 & 4 \end{array}\right)$. Now, we determine an equivalent augmented matrix:

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & -3 & 2 & 2 \\ 2 & 5 & -8 & 6 & 5 \\ 3 & 4 & -5 & 2 & 4 \end{array}\right) \xrightarrow[\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1}]{\text{Pivot : 1}} \left(\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 2 \\ 0 & \boxed{1} & -2 & 2 & 1 \\ 0 & -2 & 4 & -4 & -2 \end{array}\right)$$

$$\xrightarrow[R_3 \leftarrow R_3 + 2R_2]{\text{Pivot : 1}} \left(\begin{array}{cccc|c} 1 & 2 & -3 & 2 & 2 \\ 0 & \boxed{1} & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_1 \leftarrow R_1 - 2R_2]{\text{Pivot : 1}} A'|b' = \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The linear system $(\mathcal{S}_3)$ is equivalent to the linear system $(\mathcal{S}_3')$ whose augmented matrix is $A'|b'$

$$(\mathcal{S}_3') : \begin{cases} x \quad + z - 2t = 0 \\ \quad y - 2z + 2t = 1 \end{cases}$$

Using back substitution, we find the following set of solutions

$$\left\{(-z+2t,\ 2z-2t+1,\ z,\ t) \;\middle|\; (z,t) \in \mathbb{R}^2\right\}.$$

The linear system $(\mathcal{S}_3)$ has an infinite number of solutions.

Exercise 4 *Let m be a real number. Solve the following system, depending on the parameter m, using Gaussian elimination*

$$(\mathcal{S}_m) \begin{cases} mx + my - z = 0 \\ mx + y - mz = 0 \\ x + my - mz = 0 \end{cases}$$

Solution

The augmented matrix of the linear system $(\mathcal{S}_m)$ is : $A_m|b = \left(\begin{array}{ccc|c} m & m & -1 & 0 \\ m & 1 & -m & 0 \\ 1 & m & -m & 0 \end{array}\right)$. Using Gauss-Jordan method applied to $A_m|b$.

$$\left(\begin{array}{ccc|c} m & m & -1 & 0 \\ m & 1 & -m & 0 \\ 1 & m & -m & 0 \end{array}\right) \xrightarrow[R_1 \leftrightarrow R_3]{} \left(\begin{array}{ccc|c} \boxed{1} & m & -m & 0 \\ m & 1 & -m & 0 \\ m & m & -1 & 0 \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \leftarrow R_2 - mR_1 \\ R_3 \leftarrow R_3 - mR_1}]{\text{Pivot : } 1} \left(\begin{array}{ccc|c} 1 & m & -m & 0 \\ 0 & 1-m^2 & -m+m^2 & 0 \\ 0 & m-m^2 & -1+m^2 & 0 \end{array}\right)$$

$$\xrightarrow[R_2 \leftarrow R_2 - R_3]{} A'_m|b = \left(\begin{array}{ccc|c} 1 & m & -m & 0 \\ 0 & 1-m & 1-m & 0 \\ 0 & m-m^2 & -1+m^2 & 0 \end{array}\right).$$

The solution dependes on the value of $1-m$.

- 1^{st} case : $1-m=0$ i.e. $m=1$.

  The linear system $(\mathcal{S}_1)$ is equivalent to the linear system whose augmented matrix is

  $$A'_1|b = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

  This means that $(\mathcal{S}_1)$ is equivalent to the equation

  $$(\mathcal{S}'_1) : x+y-z=0.$$

  The set of solutions is given as

  $$\left\{(x,\ y,\ z) \in \mathbb{R}^3 \ \middle|\ x+y-z=0\right\} = \left\{(x,\ y,\ x+y) \ \middle|\ (x,\ y) \in \mathbb{R}^2\right\}.$$

  Hence, the linear system $(\mathcal{S}_1)$ has an infinite number of solutions.

- 2^{nd} case : $1-m \neq 0$ i.e. $m \neq 1$.

  We continue the Gauss elimination procedure

  $$A'_m|b = \left(\begin{array}{ccc|c} 1 & m & -m & 0 \\ 0 & 1-m & 1-m & 0 \\ 0 & m-m^2 & -1+m^2 & 0 \end{array}\right)$$

  $$\xrightarrow[\substack{R_2 \leftarrow \frac{1}{1-m} R_2 \\ R_3 \leftarrow \frac{1}{1-m} R_3}]{} \left(\begin{array}{ccc|c} 1 & m & -m & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & m & -1-m & 0 \end{array}\right)$$

  $$\xrightarrow[R_3 \leftarrow R_3 - mR_2]{\text{Pivot : } 1} B_m|b = \left(\begin{array}{ccc|c} 1 & m & -m & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1-2m & 0 \end{array}\right).$$

  Two cases are possible depending on the value of m in $\mathbb{R} \setminus \{1\}$.

– If $-1-2m=0$, i.e., $m=-\frac{1}{2}$, the set of solutions is given as

$$\left\{(x,\ x,\ -x)\ \middle|\ x\in\mathbb{R}\right\}.$$

– If $m\in\mathbb{R}\setminus\{1;-\frac{1}{2}\}$, there exists an unique solution given by

$$(0,\ 0,\ 0).$$

Consequently,

1. If $m=1$, the set of solutions is given by

$$\left\{(x,\ y,\ x+y)\ \middle|\ (x,\ y)\in\mathbb{R}^2\right\};$$

2. If $m=-\frac{1}{2}$, the set of solutions is given by

$$\left\{(x,\ x,\ -x)\ \middle|\ x\in\mathbb{R}\right\};$$

3. If $m\in\mathbb{R}\setminus\{1;-\frac{1}{2}\}$, the linear system has a trivial solution

$$(0,\ 0,\ 0).$$

Exercise 5 : *Prove that the following matrices are invertibles and determine A^{-1} and B^{-1}*

$$A=\begin{pmatrix}1&1&1&1\\1&0&1&0\\1&1&0&0\\-1&0&1&1\end{pmatrix};\qquad B=\begin{pmatrix}2&5&2&3\\2&3&3&4\\3&6&3&2\\4&12&0&8\end{pmatrix}.$$

Solution –

- **Matrix A:** The augmented matrix $A|I_4$ is :

$$A|I_4=\left(\begin{array}{cccc|cccc}\boxed{1}&1&1&1&1&0&0&0\\1&0&1&0&0&1&0&0\\1&1&0&0&0&0&1&0\\-1&0&1&1&0&0&0&1\end{array}\right)$$

$$\xrightarrow[\substack{R_2\leftarrow R_2-R_1\\ R_3\leftarrow R_3-R_1\\ R_4\leftarrow R_4-R_1}]{\text{Pivot : 1}}\left(\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\0&\boxed{-1}&0&-1&-1&1&0&0\\0&0&-1&-1&-1&0&1&0\\0&1&2&2&1&0&0&1\end{array}\right)$$

$$\xrightarrow[R_4 \leftarrow R_4 + R_2]{\text{Pivot : -1}} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & \boxed{-1} & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_4 \leftarrow R_4 + 2R_3]{\text{Pivot : -1}} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{-1} & -2 & 1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_3 \leftarrow R_3 - R_4 \\ R_2 \leftarrow R_2 - R_4 \\ R_1 \leftarrow R_1 + R_4}]{\text{Pivot : -1}} \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & -1 & 1 & 2 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & \boxed{-1} & 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & -1 & -2 & 1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \leftarrow R_1 + R_3]{\text{Pivot : -1}} \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{-1} & 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & -1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -2 & 1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \leftarrow R_1 + R_2]{\text{Pivot : -1}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & -1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -2 & 1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \leftarrow -R_2 \\ R_3 \leftarrow -R_3 \\ R_4 \leftarrow -R_4}]{} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 & -2 & -1 \end{array}\right).$$

Hence, A is invertible and A^{-1} is given as

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ -1 & 0 & 2 & 1 \\ -1 & 1 & 1 & 1 \\ 2 & -1 & -2 & -1 \end{pmatrix}.$$

- **Matrix B:** The augmented matrix $B|I_4$ is :

$$B|I_4 = \left(\begin{array}{cccc|cccc} 2 & 5 & 2 & 3 & 1 & 0 & 0 & 0 \\ 2 & 3 & 3 & 4 & 0 & 1 & 0 & 0 \\ 3 & 6 & 3 & 2 & 0 & 0 & 1 & 0 \\ 4 & 12 & 0 & 8 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \leftarrow \frac{1}{2}R_1]{} \left(\begin{array}{cccc|cccc} 1 & \frac{5}{2} & 1 & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 2 & 3 & 3 & 4 & 0 & 1 & 0 & 0 \\ 3 & 6 & 3 & 2 & 0 & 0 & 1 & 0 \\ 4 & 12 & 0 & 8 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \\ R_4 \leftarrow R_4 - 4R_1}]{\text{Pivot : 1}} \left(\begin{array}{cccc|cccc} \boxed{1} & \frac{5}{2} & 1 & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & -\frac{5}{2} & -\frac{3}{2} & 0 & 1 & 0 \\ 0 & 2 & -4 & 2 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \leftarrow -\frac{1}{2}R_2]{} \left(\begin{array}{cccc|cccc} 1 & \frac{5}{2} & 1 & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & -\frac{5}{2} & -\frac{3}{2} & 0 & 1 & 0 \\ 0 & 2 & -4 & 2 & -2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_1 \leftarrow R_1 - \frac{5}{2}R_2 \\ R_3 \leftarrow R_3 + \frac{3}{2}R_2 \\ R_4 \leftarrow R_4 - 2R_2}]{\text{Pivot : 1}} \left(\begin{array}{cccc|cccc} 1 & 0 & \frac{9}{4} & \frac{11}{4} & -\frac{3}{4} & \frac{5}{4} & 0 & 0 \\ 0 & \boxed{1} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{3}{4} & -\frac{13}{4} & -\frac{3}{4} & -\frac{3}{4} & 1 & 0 \\ 0 & 0 & -3 & 3 & -3 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \leftarrow -\frac{4}{3}R_3]{} \left(\begin{array}{cccc|cccc} 1 & 0 & \frac{9}{4} & \frac{11}{4} & -\frac{3}{4} & \frac{5}{4} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{13}{3} & 1 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & -3 & 3 & -3 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{R_1 \leftarrow R_1 - \frac{9}{4}R_3 \\ R_2 \leftarrow R_2 + \frac{1}{2}R_3 \\ R_4 \leftarrow R_4 + 3R_3}]{\text{Pivot : 1}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -7 & -3 & 1 & 3 & 0 \\ 0 & 1 & 0 & \frac{5}{3} & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & \boxed{1} & \frac{13}{3} & 1 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 16 & 0 & 4 & -4 & 1 \end{array}\right)$$

$$\xrightarrow[R_4 \leftarrow \frac{1}{16}R_4]{} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -7 & -3 & 1 & 3 & 0 \\ 0 & 1 & 0 & \frac{5}{3} & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{13}{3} & 1 & 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{16} \end{array}\right)$$

$$\xrightarrow[\substack{R_1 \leftarrow R_1 + 7R_4 \\ R_2 \leftarrow R_2 - \frac{5}{3}R_4 \\ R_3 \leftarrow R_3 - \frac{13}{3}R_4}]{\text{Pivot : 1}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -3 & \frac{3}{4} & \frac{5}{4} & \frac{7}{16} \\ 0 & 1 & 0 & 0 & 1 & -\frac{5}{12} & -\frac{1}{4} & -\frac{5}{48} \\ 0 & 0 & 1 & 0 & 1 & -\frac{1}{12} & -\frac{1}{4} & -\frac{13}{48} \\ 0 & 0 & 0 & \boxed{1} & 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{16} \end{array}\right)$$

Thus, the matrix B is invertible and B^{-1} is given as

$$B^{-1} = \begin{pmatrix} -3 & \frac{3}{4} & \frac{5}{4} & \frac{7}{16} \\ 1 & -\frac{5}{12} & -\frac{1}{4} & -\frac{5}{48} \\ 1 & -\frac{1}{12} & -\frac{1}{4} & -\frac{13}{48} \\ 0 & \frac{1}{4} & -\frac{1}{4} & \frac{1}{16} \end{pmatrix}.$$

Exercise 6 *: Compute the LU decomposition of the following matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{pmatrix}.$$

Solution: We define a matrix $A^{(1)}$ as

$$A^{(1)} = L_1 \times A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 4 & 6 & 8 \end{pmatrix}, \text{ where } L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then,

$$A^{(2)} = L_2 \times A^{(1)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{pmatrix}, \text{ where } L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}.$$

Finally,

$$A^{(3)} = L_3 \times A^{(2)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}, \text{ where } L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

As we can remark the matrix $A^{(3)}$ is an upper triangular matrix, so $U = A^{(3)}$, and we have

$$L_3 \times L_2 \times L_1 \times A = A^{(3)} = U$$

$$L_2 \times L_1 \times A = L_3^{-1} \times U$$

$$L_1 \times A = L_2^{-1} \times L_3^{-1} \times U$$

$$A = L_1^{-1} \times L_2^{-1} \times L_3^{-1} \times U.$$

We have

$$L_1^{-1} \times L_2^{-1} \times L_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}.$$

Consequently,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}$$

Hence,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}$$

Remark 2 *:* Note that the LU decomposition of a matrix is not unique.