Lecture 20: How to find eigenvectors (class)

Reading: Meyer Ch. 6, 7.1–7.4
Strang Chs. 4–5

Recall:

$$A\vec{v} = \underbrace{\lambda}_{\text{eigenvalue}} \vec{v}$$

($\vec{v}$: eigenvector)

A is diagonalizable $\iff$ $A = U D U^{-1}$

where D is diagonal, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$, and the columns of U are eigenvectors: $A\vec{u}_i = \lambda_i \vec{u}_i$

$\iff$ there is a basis of eigenvectors

Example:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

evector — e-value

Example: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is $\underline{\underline{\text{not}}}$ diagonalizable.

$A\vec{e}_1 = 0 \cdot \vec{e}_1$ but that's it!

Note: The first step can be "justified" by noticing that

$$\lim_{k\to\infty} \frac{A^k \vec{v}}{\|A^k \vec{v}\|}$$

converges to an eigenvector (or eigenspace) with largest magnitude eigenvalue. So it makes sense to look at successive powers $A^k\vec{v}$.

DETERMINANTS (in brief)

Definition: The determinant of an $n \times n$ square matrix A is

$$\det(A) = \sum_{\substack{n\text{-element} \\ \text{permutations } \sigma}} \underbrace{\text{sign}(\sigma)}_{= (-1)^{\#\text{ of transpositions required to get to } \sigma}} \cdot \prod_{i=1}^{n} A_{i,\sigma(i)}$$

$n!$ terms $\sim \left(\frac{n}{e}\right)^n$

Examples:

- $n=1$: $A = (A_{11})$

  $\det(A) = A_{11}$

$n=2$ $\quad 2! = 2$

- $n=2$: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Permutations	Sign	
id: $1 \mapsto 1, 2 \mapsto 2$	$+1$	(0 transpositions)
$1 \leftrightarrow 2$	-1	

$$\Rightarrow \det A = +1\,(a_{11}a_{22}) - 1\,(a_{12}a_{21})$$

- $n=3$:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Permutation σ	Sign(σ)	$\prod_{i=1}^{3} a_{i,\sigma(i)}$
- identity	$+1$	$a_{11}a_{22}a_{33}$
- $2 \leftrightarrow 3$	-1	$a_{11}a_{23}a_{32}$
- $1 \leftrightarrow 2$	-1	$a_{12}a_{21}a_{33}$
- $1 \to 2 \to 3 \to 1$	$+1$	$a_{12}a_{23}a_{31}$
- $1 \to 3 \to 2 \to 1$	$+1$	$a_{13}a_{21}a_{32}$
- $1 \leftrightarrow 3$	-1	$a_{13}a_{22}a_{31}$

since $1 \to 2 \to 3 \to 1$ = $1 \leftrightarrow 2$ followed by $1 \leftrightarrow 3$ (two transpositions)

add these up to get the determinant!

Observe: $\det(A) = a_{11} \cdot \det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \cdot \det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \cdot \det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$

- General n: the same recursion works

$a_{11} \cdot \det$($n-1 \times n-1$ submatrix missing 1^{st} row, 1^{st} col)

$- a_{12} \cdot \det$($n-1 \times n-1$ submatrix missing row 1, col 2)

$+ a_{13} \cdot \cdots\cdots$

$\vdots$

$+(-1)^n a_{1n} \cdot \det$(submatrix missing row 1, col n)

[You can actually do this across any row or down any column; note that $\det(A) = \det(A^T)$.]

Problem: There are $n! \approx \left(\frac{n}{e}\right)^n$ permutations of n elements; although

$$\det(A) = \sum_{\substack{n\text{-element} \\ \text{permutations } \sigma}} \operatorname{sign}(\sigma) \cdot \prod_{i=1}^{n} A_{i,\sigma(i)}$$

defines the determinant, that's **not** how it is actually computed!

<u>Compare</u>: The permanent of a square matrix A is

$$\operatorname{perm}(A) = \sum_{\substack{n\text{-element} \\ \text{permutations } \sigma}} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

without the sign! Computing the permanent is NP-hard.

<u>**More examples**</u>:

- $\operatorname{Det}\begin{pmatrix} a & & & 0 \\ & b & & \\ & & c & \\ 0 & & & d \end{pmatrix} = abcd + 0 = abcd$

  diagonal matrix

- $\operatorname{Det}\begin{pmatrix} a & e & f & g \\ 0 & b & h & i \\ 0 & 0 & c & j \\ 0 & 0 & 0 & d \end{pmatrix} = abcd$

  upper triangular matrix

- $\operatorname{Det} A = \operatorname{Det} A^T$ ✓

true for permanent, too

false for permanent!

<u>Cool fact</u>:

$$\operatorname{Det}(AB) = \operatorname{Det}(A) \cdot \operatorname{Det}(B)$$

<u>Cor. 1</u>: This is how you can compute $\operatorname{Det} A$ is $O(n^3)$ time

Use the LU decomposition (Gaussian elimination)

$$A = P \cdot L \cdot U$$

$$\Rightarrow \operatorname{Det}(A) = \underbrace{\operatorname{Det}(P)}_{\in\{1,-1\}} \cdot \underbrace{\operatorname{Det}(L)}_{=\prod L_{jj}} \cdot \underbrace{\operatorname{Det}(U)}_{=\prod_j U_{jj}}$$

<u>Corollary 2</u>: If A is invertible,

then $\operatorname{Det}(A) = \dfrac{1}{\operatorname{Det}(A^{-1})} \neq 0$. $\left(\text{because } A \cdot A^{-1} = \underbrace{I}_{\det I = 1}\right)$

<u>Corollary 3</u>: If A is diagonalizable,

$\det I = 1$

Corollary 3 If A is diagonalizable,

$$A = U \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} U^{-1},$$

then $\det(A) = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$.

product of the eigenvalues

Proof: $\mathrm{Det}(A) = \cancel{\mathrm{Det}\, U} \cdot \mathrm{Det}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \cdot \cancel{\mathrm{Det}\, U^{-1}} = \lambda_1 \cdots \lambda_n$. ✓

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \to \mathrm{Det} = -1$

$\tau = 1.618 = \frac{1+\sqrt{5}}{2}$

$-\frac{1}{\tau}$

Example:

```
>> n = 50;
>> A = randn(n,n);
>> B = randn(n,n);
>> det(A*B)

ans =

   3.1827e+61

>> det(A) * det(B)

ans =

   3.1827e+61
```

Note:

```
>> factorial(50)

ans =

   3.0414e+64
```

— how did Matlab compute $\det(A), \det(B)$?

Answer: It takes the LU decomposition:

$$A = \underset{\text{permutation matrix}}{P} \cdot \underset{\text{lower triangular}}{\begin{pmatrix} * & & 0 \\ * & * & \\ * & * & * \end{pmatrix}} \cdot \underset{\text{upper triangular}}{\begin{pmatrix} * & * & * \\ & * & * \\ 0 & & * \end{pmatrix}}$$

$\Rightarrow \det(A) = \det(P) \cdot \det(L) \cdot \det(U)$.

More on this later...

How to compute the determinant efficiently

Gaussian elimination

Observe: The determinant function is not linear

$$\det(A) = \sum_{\text{permutations } \sigma} \mathrm{sign}(\sigma) \cdot \prod_{i=1}^{n} A_{i,\sigma(i)}$$

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}, \quad \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + \cdots$$

$$\Rightarrow \det(5A) = 5^n \cdot \det(A)$$

But it is linear in any one row:

$$\det\begin{pmatrix} 5a_{11} & 5a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 5 \cdot \det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 5a_{21} & 5a_{22} & 5a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 5 \cdot \det A$$

$$\det\begin{pmatrix} - & \vec{u} & - \\ & B & \end{pmatrix} =$$

$$\det\begin{pmatrix} - & \vec u & - \\ & B & \end{pmatrix} =$$

$$\Rightarrow \det\begin{pmatrix} - & \vec u+\vec v & - \\ & B & \end{pmatrix} = \det\begin{pmatrix} - & \vec u & - \\ & B & \end{pmatrix} + \det\begin{pmatrix} - & \vec v & - \\ & B & \end{pmatrix}$$

Observe: Switching any two rows (or columns) negates $\det(A)$.

$$\det\begin{pmatrix} - & \vec u & - \\ - & \vec v & - \\ & B & \end{pmatrix} = -\det\begin{pmatrix} - & \vec v & - \\ - & \vec u & - \\ & B & \end{pmatrix}$$

b/c it adds one transposition to every permutation

Corollary 1: If the same row occurs twice, $\det(A)=0$.

Proof:

$$\begin{pmatrix} - & \vec u & - \\ - & \vec u & - \\ & B & \end{pmatrix} \qquad \begin{pmatrix} - & \vec u & - \\ - & \vec u & - \\ & B & \end{pmatrix}$$

Corollary 2: We can use Gaussian elimination to compute det.

- $\det\begin{pmatrix} - & 5\vec u & - \end{pmatrix} = 5\det\begin{pmatrix} - & \vec u & - \end{pmatrix}$
- $\det\begin{pmatrix} - & \vec u & - \\ - & \vec u+\vec v & - \end{pmatrix} = \underbrace{\cancel{\det\begin{pmatrix} - & \vec u & - \\ - & \vec u & - \end{pmatrix}}}_{0} + \det\begin{pmatrix} - & \vec u & - \\ - & \vec v & - \end{pmatrix}$

Corollary 3:

$\det(A) \neq 0 \iff A$ is invertible

~ "nonsingular"

~ inverse exists

~ $\operatorname{rank}(A)=n$

$\det(A) = 0 \iff A$ is not invertible

~ $\operatorname{rank}(A)<n$

Proof: Apply Gaussian elim.:

$$A \xrightarrow{GE} R = \begin{pmatrix} b_{11} & & \\ & b_{22} & \\ & & \ddots \end{pmatrix}$$

Proof: Apply Gaussian elim.:

$$A \xrightarrow{GE} B = \begin{pmatrix} b_{11} & & * \\ & b_{22} & \\ 0 & & \ddots \end{pmatrix}$$

$$\operatorname{rank}(A) = \operatorname{rank}(B)$$

$$\det(A) = \begin{pmatrix}\text{something}\\ \text{nonzero}\end{pmatrix} \times \det(B) = \text{product of diagonal entries}$$

$$= 0 \iff \text{at least one } b_{jj} = 0$$

$$\iff \operatorname{rank}(B) < n \qquad \square$$

Example: Is this matrix invertible?

$$\begin{pmatrix} 1 & a & b & c+d \\ 1 & b & c & a+d \\ 1 & c & d & a+b \\ 1 & d & a & b+c \end{pmatrix}$$

Answer:

- Don't try to compute the inverse.
- Don't try to brute-force the determinant.
- Look for a pattern.

COROLLARY: HOW TO FIND EIGENVECTORS:

Observe: λ is an eigenvalue of A

$$\Updownarrow$$

for some $\vec{v} \neq \vec{0}$, $A\vec{v} = \lambda\vec{v} \iff$ for some $\vec{v} \neq \vec{0}$ $(A - \lambda I)\vec{v} = \vec{0}$

$$\iff \dim N(A - \lambda I) \geq 1$$

$$\underset{\substack{\text{LU}\\ \text{decomp.}}}{A - \lambda I = P \cdot L} \cdot \begin{pmatrix} * & * \\ 0 & \ddots \end{pmatrix} \iff \text{with } \det U = 0$$

P, L invertible; U

$$\Updownarrow$$

$$\det(A - \lambda I) = 0$$

$$\boxed{\det(A - \lambda I) = 0 \iff \lambda \text{ is an eigenvalue of } A}$$

Recipe:

1. Compute $\det(A - \lambda I)$ as a degree n polynomial in λ
2. Find all roots over $\mathbb{C}$, $\det(A - \lambda I) = 0$
3. For each λ, corresponding eigenspace is $N(A - \lambda I)$ — find it with

Gaussian elimination!

Problem: Calculate the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Answer:

$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = (3-\lambda)^2 - 1^2$$

$$= \lambda^2 - 6\lambda + 8$$

$$= (\lambda-4)(\lambda-2)$$

e-values | e-spaces

$\lambda_1 = 4$ $\quad N(A-4I) = N\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

$\lambda_2 = 2$ $\quad N(A-2I) = N\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \text{Span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$

Comments

1. $\text{Det } A = 9 - 1 = 8 = \lambda_1 \cdot \lambda_2$
2. $\text{Tr } A = A_{11} + A_{22} = 6 = \lambda_1 + \lambda_2$

In general, $\text{Tr } A = \sum_j \lambda_j$
if A is diagonalizable

Pf: $A = UDU^{-1}$

$$\text{Tr } A = \text{Tr}(UDU^{-1}) = \text{Tr}(U^{-1}UD)$$

$$= \text{Tr } D \quad \square$$

3. Claim: If $A\vec{v} = \lambda\vec{v}$,
then $(A+cI)\vec{v} = (\lambda+c)\vec{v}$.

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 3I$$

e-values of $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$: $3+1=4$, $3-1=2$

e-values of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: 1, -1

Observe: If A is an $n \times n$ matrix, $\det(A-\lambda I)$ is a polynomial of degree n in λ.

Observe: 1) $\det(A) = \lambda_1 \cdot \lambda_2 = 8$ ✓

2) $\text{Trace}(A) = $ sum of diagonal elements
$= \lambda_1 + \lambda_2 = 6$

These observations are true in general, and for 2×2 matrices these two equations suffice to solve for λ_1, λ_2.

Next, the eigenspaces:

- $N(A-\lambda_1 I) =$

- $N(A-\lambda_2 I) =$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$$

e-vectors e-values

Observe: The eigenvectors of $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ are the same as those for $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

B "

If $B\vec{v} = \lambda \vec{v}$, $(B+3I)\vec{v} = (\lambda+3)\vec{v}$

— But the eigenvalues of the sum of two matrices will not generally be the sums of the eigenvalues, unless they have the same eigenvectors.

Problem: Calculate the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} -y \\ 1 \end{pmatrix} = A\begin{pmatrix} 1 \\ y \end{pmatrix} = i\begin{pmatrix} 1 \\ y \end{pmatrix} \Rightarrow y = -i$$

Answer: $\mathrm{Det}(A-\lambda I) = \cdots$

$$\left.\begin{array}{l} \mathrm{Tr}\, A = \lambda_1 + \lambda_2 = 0 \\ \mathrm{Det}\, A = \lambda_1 \lambda_2 = 1 \end{array}\right\} \Rightarrow \lambda = -\frac{1}{\lambda} \Rightarrow \begin{array}{l} \lambda_1 = i \\ \lambda_2 = -i \end{array}$$

$$N(A-\lambda_1 I) = N\left(\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\right)$$

E-values	E-vectors
i	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$
$-i$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

Problem: What are the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \quad ?$$

Not diagonalizable

Answer:

$$\mathrm{Det}(A - \lambda I) = \mathrm{Det}\begin{pmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} = (3-\lambda)^2$$

$$= 3I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

e-value: 3 e-vector: $\vec{e}_1$

Problem: What are the eigenvalues of

$$B = \begin{pmatrix} 3 & 5 & 8 & 13 \\ & 2 & 3 & 5 \\ & & 1 & 2 \\ 0 & & & 1 \end{pmatrix} \quad ?$$

$$\mathrm{Det}(B - \lambda I) = (3-\lambda)(2-\lambda)(1-\lambda)^2$$

Moral: Triangular matrices are easy!

Question: Why does an $n \times n$ matrix have at most n distinct eigenvalues?

Answer 1:

Because $\det(A - \lambda I)$ is a degree n polynomial in λ, it always has ≥ 1 and $\leq n$ roots.

Answer 2:

Assume there are $n+1$ different eigenvalues:

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$
$$\vdots$$
$$A\vec{v}_n = \lambda_n \vec{v}_n$$
$$A\vec{v}_{n+1} = \lambda_{n+1}\vec{v}_{n+1}$$

$\Rightarrow$ e-vectors cannot be linearly independent

If, say, the first n are lin. indep., then

$$\vec{v}_{n+1} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$$

for some constants $\alpha_1, \ldots, \alpha_n$

Apply A to both sides

$$A\vec{v}_{n+1} = \lambda_{n+1}\vec{v}_{n+1}$$

$$\alpha_1\lambda_1\vec{v}_1 + \cdots + \alpha_n\lambda_n\vec{v}_n = \alpha_1\lambda_{n+1}\vec{v}_1 + \cdots + \alpha_n\lambda_{n+1}\vec{v}_n$$

$$\Rightarrow \sum_{j=1}^{n} \alpha_j(\lambda_j - \lambda_{n+1})\vec{v}_j = 0$$

$$\Rightarrow \sum_{j=1}^{n} \alpha_j \underbrace{(\lambda_j - \lambda_{n+1})}_{\text{nonzero!}} \vec{v}_j = 0$$

$\Rightarrow$ contradicts that $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are linearly independent.

Note: Dotting the i's, this argument actually proves

Lemma: Any set of eigenvectors for a matrix, corresponding to distinct eigenvalues, is linearly independent.

Summary of Eigenvalues & Eigenvectors

A $n \times n$ matrix   b/c $A\vec{v} = \lambda \vec{v} \Leftrightarrow (A - \lambda I)\vec{v} = \vec{0}$
$\Rightarrow \vec{v} \in N(A - \lambda I)$

λ an eigenvalue $\Leftrightarrow \det(A - \lambda I) = 0$

since $A\vec{v} = \lambda \vec{v}$ for some $\vec{v} \neq 0$
$\Leftrightarrow$ $A - \lambda I$ is singular

$\det(A - \lambda I)$ = degree n polynomial in λ

$$= (\lambda - \lambda_1)^{\alpha_1} (\lambda - \lambda_2)^{\alpha_2} \cdots (\lambda - \lambda_k)^{\alpha_k}$$

distinct e-values (AKA "spectrum")

multiplicities

$\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$, $k \leq n$

eigenspace for λ_j is $N(A - \lambda_j I)$

$$1 \leq \dim N(A - \lambda_j I) \leq \alpha_j$$

There are at least k independent e-vectors.

A is diagonalizable (n indep. e-vectors)

$\Leftrightarrow \dim N(A - \lambda_j I) = \alpha_j$ for all $j = 1, \ldots, k$

Admin:

Midterm 2, Nov. 12

HW 10a due Nov. 11 Wednesday

10b due Nov. 17 Tuesday

HW 11 due Nov. 24 Tues (last class day)

sample problems for midterm 2 ...

Corollary: A and A^T have the same e-values. (maybe diff. e-vectors)

(They also have the singular values)

Why is $\dim N(A - \lambda_i I) \leq \alpha_i$?

Proof: We want to show that $\operatorname{rank}(A-\lambda_i I) \geq n-\alpha_i$;
the claim $\dim N(A-\lambda_i I) \leq \alpha_i$ then follows by the Rank-Nullity theorem.

Useful claim: If U is invertible, then the eigenvalues of A are the same as those of UAU^{-1}.

Proof: If $A\vec{v} = \lambda\vec{v}$, then

$$\begin{aligned}(UAU^{-1})(U\vec{v}) &= UA\vec{v} \\ &= \lambda(U\vec{v}) \quad \checkmark\end{aligned}$$

Now apply Gaussian elimination to A; assuming no row interchanges are required, this gives

$$A = \left(L \right)\left(U \right)$$

where L has 1s along the diagonal and U has each λ_i on its diagonal α_i times. (Think about it....)

Thus we can also write

$$A = L U' L^{-1};$$

this means applying the same operations to the columns of U as were done to the rows of A in Gaussian elimination. It leaves the diagonal entries unchanged.

But then

$$\begin{aligned}\operatorname{rank}(A-\lambda_i I) &= \operatorname{rank}(U'-\lambda_i I) \\ &\geq \#\text{ of nonzero entries along the diagonal, since } U'-\lambda_i I \text{ is upper triangular} \\ &= n-\alpha_i. \checkmark \quad \square\end{aligned}$$

Note: The recipe

1. Compute $p(\lambda) = \det(A-\lambda I)$
2. Find its factors $\lambda_1, \dots, \lambda_k \Rightarrow$ eigenvalues of A
3. Compute nullspaces $N(A-\lambda_i I) \Rightarrow$ eigenspaces

works in theory, and for 2×2, 3×3 matrices.

But it quickly becomes impractical. Writing down $p(\lambda)$, and then finding its roots, is very time-consuming.

Next time we'll learn a faster way...

In fact, a common way of solving for the roots of a polynomial

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

is to compute the eigenvalues of a matrix!

For

$$A = \begin{pmatrix} 0 & & & -a_0 \\ 1 & 0 & 0 & -a_1 \\ & 1 & \ddots & \vdots \\ 0 & & 1 & -a_{n-1} \end{pmatrix},$$

$$\det(A - \lambda I) = p(\lambda).$$

$$\begin{vmatrix} \lambda & a_0 \\ -1 & \lambda + a_1 \end{vmatrix} = \lambda^2 + \lambda a_1 + a_0$$

$$\begin{vmatrix} \lambda & 0 & a_0 \\ -1 & \lambda & a_1 \\ 0 & -1 & \lambda + a_2 \end{vmatrix} = \lambda^3 + \lambda^2 a_2 + \lambda a_1 + a_0$$

$$\begin{vmatrix} \lambda & 0 & 0 & a_0 \\ -1 & \lambda & 0 & a_1 \\ 0 & -1 & \lambda & a_2 \\ 0 & 0 & -1 & \lambda + a_3 \end{vmatrix} = \lambda^4 + \lambda^3 a_3 + \lambda^2 a_2 + \lambda a_1 + a_0$$

$$\vdots$$

Corollary: A and its transpose A^T have the same eigenvalues.

Proof:

Exercises:

Problem: Calculate the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

Answer:

E-values	E-vectors
i	$\begin{pmatrix} 1 \\ -i \end{pmatrix}$
$-i$	$\begin{pmatrix} 1 \\ i \end{pmatrix}$

Related problems: What are the spectral decompositions of

$$B = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}?$$

e-vectors e-vectors

$$\begin{pmatrix} & & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

e-values	e-vectors	e-values	e-vectors
i	$(1,-i,0,0)$	i	$(0,1,-i,0)$
$-i$	$(1,i,0,0)$	$-i$	$(0,1,i,0)$
i	$(0,0,1,-i)$	i	$(1,0,0,-i)$
$-i$	$(0,0,1,i)$	$-i$	$(1,0,0,i)$

Answer: Observe that both matrices B and C break up into two copies of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence we see immediately:

B

E-values	E-vectors
i	$(1,-i,0,0)$
$-i$	$(1,i,0,0)$
i	$(0,0,1,-i)$
$-i$	$(0,0,1,i)$

C

E-values	E-vectors
i	$(1,0,0,-i)$
$-i$	$(1,0,0,i)$
i	$(0,1,-i,0)$
$-i$	$(0,1,i,0)$

Problem: What are the eigenvalues and eigenvectors of

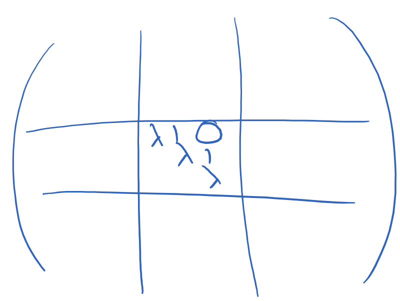
$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \ ?$$

Answer:

E-values	E-vectors
3 (multiplicity 2)	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ only!

Problem ("Jordan normal form"):
Find the spectral decomposition of

$$\left(\begin{array}{cc|ccc|cccc}
0 & 1 & & & & & & & \\
0 & 0 & & 0 & & & 0 & & \\ \hline
& & 1 & 1 & 0 & & & & \\
& 0 & & 1 & 1 & & 0 & & \\
& & 0 & & 1 & & & & \\ \hline
& & & & & 2 & 1 & 0 & 0 \\
& 0 & & 0 & & & 2 & 1 & 0 \\
& & & & & & & 2 & 1 \\
& & & & & 0 & & & 2
\end{array}\right)$$



Answer: Since the matrix is upper triangular, we can read off the e-values: 0, 1, 2.
The matrix breaks into 3 blocks

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = I + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad 2I + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix breaks into 3 blocks

$$\underset{A}{\begin{pmatrix}0&1\\0&0\end{pmatrix}}, \quad \begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix} = I + \underset{B}{\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}}, \quad 2I + \underset{C}{\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}}$$

Let's consider C (A and B are similar).

$0 \leftarrow$ the unique e-value

e-space $N(C - 0 \cdot I) = N(C) = \mathrm{Span}(\{\vec{e}_1\})$

$\uparrow$ one-dimensional!

$\mathrm{Rank}(C) = 3 \Rightarrow \dim N(C) = 4 - 3 = 1$

(not diagonalizable)

Putting everything together:

$$\begin{pmatrix} \begin{smallmatrix}0&1\\0&0\end{smallmatrix} & 0 & 0 \\ 0 & \begin{smallmatrix}1&1&0\\&1&1\\0&&1\end{smallmatrix} & 0 \\ 0 & 0 & \begin{smallmatrix}2&1&0&0\\&2&1&0\\&&2&1\\0&&&2\end{smallmatrix} \end{pmatrix}$$

E-value	E-vector
0	(1,0, | 0,0,0 | 0,0,0,0)
1	(0,0, | 1,0,0, | 0,0,0,0)
2	(0,0, | 0,0,0, | 1,0,0,0)

More examples

Eigenvalues

$$A = \begin{pmatrix} 1+2\cos\theta & -2\sin\theta \\ 2\sin\theta & 1+2\cos\theta \end{pmatrix} = I + 2\underbrace{\begin{pmatrix} c & -s \\ s & c \end{pmatrix}}_{e^{\pm i\theta}}$$

$1+2e^{i\theta}$, $1+2e^{-i\theta}$

$(1,-i)$ $\quad$ $(1,i)$

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = I + 2\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\substack{Tr=0 \\ Det=-1}} \Rightarrow \lambda = \pm 1$$

$1+2(+1)$, $1+2\cdot(-1)$

$= 3$ $\quad$ $= -1$

$(1,1)$ $\quad$ $(1,-1)$

$$C = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$3, -1, 3$

$(1,1,0)$ $\quad$ $(1,-1,0)$ $\quad$ $\vec{e}_3$

$$D = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

$3, -1, 3$

$$E = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 3 & 0 & 0 \end{pmatrix} \qquad E\begin{pmatrix}1\\1\\1\end{pmatrix} = 3\begin{pmatrix}1\\1\\1\end{pmatrix} \qquad \underline{3},\ \underline{\lambda,\ -1-\lambda}$$

$$\text{Det}\,E = 3\lambda(1-\lambda) = -9\ ?$$

$$F = \begin{pmatrix} 0 & \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} \\ \begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix} & 0 \end{pmatrix} \qquad 3,\ -1,\ -3,\ +1$$

(see HW)

$$G = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 2 & 1 & 4 & 2 \\ 2 & 4 & 1 & 2 \\ 4 & 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} B & 2B \\ 2B & B \end{pmatrix} \qquad -3,\ -3,\ 1,\ 9$$

(see HW)

$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 & \ & \ \end{matrix} \\ \begin{matrix} \\ \\ \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} O & O & O & * & * \\ O & O & O & * & * \\ O & O & O & * & * \\ * & * & * & O & O \\ * & * & * & O & O \end{pmatrix} \end{array} \qquad \begin{array}{c c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} O & O & O & * & * \\ O & O & O & * & * \\ O & O & O & X & * \\ * & * & * & O & O \\ * & * & * & O & O \end{pmatrix} \end{array}$$

$$A\vec{u}_j = \sigma_j \vec{v}_j \qquad A\vec{v} = \lambda\vec{v}$$