EE 510 09/25/2020

Outline:

+ Inner Product & Orthogonality
+ Rank-nullity theorem
+ Projection
+ Linear Transformation

Inner Product: is a transformation from a vector space V to a field $\mathbb{F}$ of scalars $\mathbb{F}$

$$H : V \times V \longrightarrow \mathbb{F} \quad ; \mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

$$(x, y) \longrightarrow \langle x, y \rangle$$

1°) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ $\quad (\bar{z} = a - ib ; z = a + ib)$

2°) $\langle \alpha x_1 + x_2, y \rangle = \bar{\alpha} \langle x_1, y \rangle + \langle x_2, y \rangle$

3°) $\langle x, x \rangle \geq 0$

4°) $\langle x, x \rangle = 0 \implies x = 0.$

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- $V = \mathbb{R}^n$; $\mathbb{F} = \mathbb{R}$

  $\langle x, y \rangle = x^T y$

- $V = \mathbb{C}^n$; $\mathbb{F} = \mathbb{C}$

  $\langle x, y \rangle = \bar{x}^T y = x^* y \quad (x^* = \bar{x}^T)$

- $V = \mathbb{R}^{n \times n}$; $\mathbb{F} = \mathbb{R}$

  $\langle A, B \rangle = tr(A^T B)$

- V the vectors space of real continuous functions over $[0, 1]$.

  $$\langle f, g \rangle = \int_0^1 f(x) g(x) \, dx$$

<u>Orthogonality.</u>

For a set V, the orthogonal complement of V is given as

$V^\perp = \{ y$; y is orthogonal to all the vectors of $V \}$

$= \{ y ; \langle y, x \rangle = 0 ; \forall x \in V \}$

$y_1 \in V^\perp$; $y_2 \in V^\perp$; $\alpha \in \mathbb{R}$; $x \in V$

$$\langle \alpha y_1 + y_2, x \rangle = \alpha \underbrace{\langle y_1, x \rangle}_{=0} + \underbrace{\langle y_2, x \rangle}_{=0} = 0$$

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$\Rightarrow \alpha Y_1 + Y_2 \in V^{\perp}$

$\Rightarrow V^{\perp}$ is a vector space.

- **Rank-nullity Theorem:** $A \in \mathbb{R}^{m \times n}$
  - $\dim(R(A)) = \dim(R(A^T)) = \text{rank}(A) = r$
  - $\dim N(A) = n - \dim(R(A)) = n - r$
  - $\dim N(A^T) = m - \dim(R(A^T)) = m - r$
  - $N(A) = (R(A^T))^{\perp}$; $N(A^T) = (R(A))^{\perp}$

Expl: $m = 3$; $n = 4$; $\text{rank}(A) \le \min(m, n)$

$$A = \begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & -1 & -4 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$

r	3
$\dim R(A)$	3
$\dim R(A^T)$	3
$\dim N(A)$	1
$\dim N(A^T)$	0

$$\rightarrow \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & -3 & -9 & 0 \\ 0 & 2 & 6 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 5 & 0 \\ 0 & -3 & -9 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Exercise:

Let F, G two vector ~~spa~~ Subspaces of the vector space $\mathbb{R}^n$.

Prove that

1°) $(F \cup G)^{\perp} = F^{\perp} \cap G^{\perp}$

2°) $F^{\perp} + G^{\perp} = (F \cap G)^{\perp}$ (Rq: $E+F = \text{Span}(E \cup F)$)

1_ $F \subset F \cup G$; $G \subset F \cup G$

$\Rightarrow (F \cup G)^{\perp} \subset F^{\perp}$; $(F \cup G)^{\perp} \subset G^{\perp}$ ($E \subset F$: $F^{\perp} \subset E^{\perp}$)

$\Rightarrow (F \cup G)^{\perp} \subset F^{\perp} \cap G^{\perp}$ (1)

Let $x \in F^{\perp} \cap G^{\perp}$, $\langle x, y \rangle = 0 \ \forall y \in F \cup G$

$\Rightarrow x \in (F \cup G)^{\perp}$

$\Rightarrow F^{\perp} \cap G^{\perp} \subset (F \cup G)^{\perp}$ (2)

(1) and (2) $\Rightarrow$

$(F \cup G)^{\perp} = F^{\perp} \cap G^{\perp}$.

2°) $F \cap G \subset F \Rightarrow F^{\perp} \subset (F \cap G)^{\perp}$

$F \cap G \subset G \Rightarrow G^{\perp} \subset (F \cap G)^{\perp}$

$\Rightarrow F^{\perp} + G^{\perp} \subset (F \cap G)^{\perp}$

- ~~$\dim V + \dim V^\perp =$~~ $\quad V \in \mathbb{R}^n$
- $\dim(\operatorname{span} V) + \dim V^\perp = n.$
- $\dim(V+W) + \dim(V \cap W) = \dim V + \dim W$
- $V \subset W \;;\; \dim V = \dim W.$

$\Rightarrow \quad V = W.$

$$\dim(F^\perp + G^\perp) = \dim F^\perp + \dim G^\perp - \dim(F^\perp \cap G^\perp)$$
$$= \dim F^\perp + \dim G^\perp - \dim((F \cup G)^\perp)$$
$$= \dim F^\perp + \dim G^\perp - n + \dim(\operatorname{span}(F \cup G))$$
$$= n - \dim F + \cancel{n} - \dim G - \cancel{n} + \dim(\operatorname{span}(F \cup G))$$

$$= n - \dim F - \dim G + \dim(F+G)$$
$$= n - \dim(F \cap G)$$
$$= \dim(F \cap G)^\perp$$

- $F^\perp + G^\perp \subset (F \cap G)^\perp$
- $\dim(F^\perp + G^\perp) = \dim(F \cap G)^\perp$

$\Rightarrow \quad F^\perp + G^\perp = (F \cap G)^\perp.$

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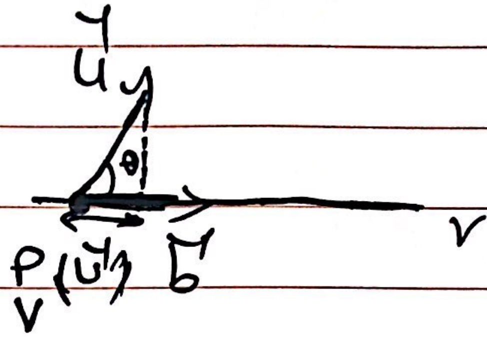
Projection:



$$P_v(\vec u) = \|\vec u\| \cos\theta \cdot \frac{\vec b}{\|\vec b\|}$$

$$= \frac{(\vec b \cdot \vec u)}{\|\vec b\|} \cdot \frac{\vec b}{\|\vec b\|} = \frac{b^T u}{\|b\|^2} \vec b = \frac{b b^T}{\|b\|^2} \vec u$$

$$P_v = \frac{b b^T}{\|b\|^2} \; ; \quad P_v u = \frac{b b^T}{\|b\|^2} u .$$

Exple: $v_1 = (3i, 2, 5)$; $V_2 = (5, 3, 3i)$

$P_{v_2}(v_1)$?

$$P_{v_2}(v_1) = \frac{v_2 \cdot v_1}{\|v_2\|^2} \cdot v_2 = \frac{(5 \;\; 3 \;\; -3i)\begin{pmatrix} 3i \\ 2 \\ 5 \end{pmatrix}}{43} v_2$$

$$= \frac{6}{43} \vec v_2 .$$

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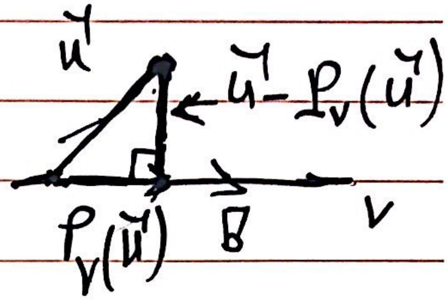
Graphical interpretation:

1D

$e = \vec{u} - P_V(\vec{u})$

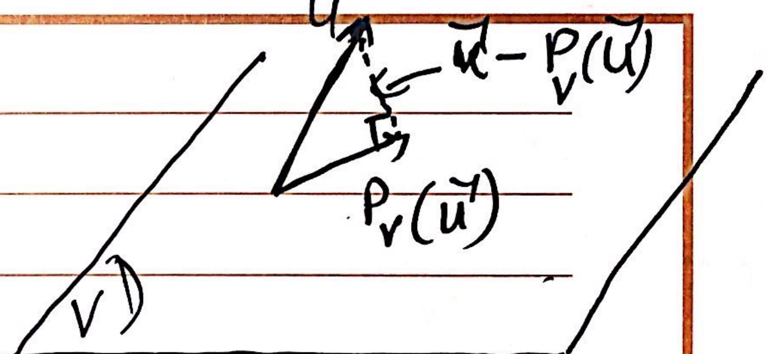


$$\min \|e\| = \min_{y \in V} \|u - y\|$$

$$\Rightarrow \underset{y \in V}{\arg\min} \|u - y\| = P_V(u).$$

$$u - P_V(u) \perp b \Rightarrow \boxed{u - P_V(u) \in V^{\perp}}$$

2D:



$$u - P_V(u) \in V^{\perp}$$

$$V = \text{span}(b_1, b_2).$$

$$A = \begin{bmatrix} | & | \\ b_1 & b_2 \\ | & | \end{bmatrix} \Rightarrow u - P_V(u) \in N(A^T).$$

$$\langle u - P_V(u), b_1 \rangle = 0 \; , \; \langle u - P_V(u), b_2 \rangle = 0$$

$$\begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \end{bmatrix} \begin{bmatrix} u - P_V(u) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow A^T(u - P_V(u)) = 0$$

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Expl:

$$b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ; b_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} , u = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} ; P_V(u) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -x \\ 3-y \\ 2-z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \Leftrightarrow A^T P_V(u) = A^T u .$$

$$A^T(u - P_V(u)) = 0$$

$$\Rightarrow x = 3/2 ; \; y = 3/2 ; \; z = 2$$

$$P_V(u) = \begin{pmatrix} 3/2 \\ 3/2 \\ 2 \end{pmatrix} \quad ; \; u - P_V(u) = \begin{pmatrix} -3/2 \\ 3/2 \\ 0 \end{pmatrix}$$

Special Case:

$$\langle b_1, b_2 \rangle = 0 \quad ; \; V = \mathrm{Span}(b_1, b_2)$$

$$P_V(u) = P_{b_1}(u) + P_{b_2}(u) = \frac{b_1 b_1^T}{\|b_1\|^2} u + \frac{b_2 b_2^T}{\|b_2\|^2} u .$$

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$\{b_1, \ldots, b_m\}$, $b_i \in R^n$; $i = 1, \ldots, m$.

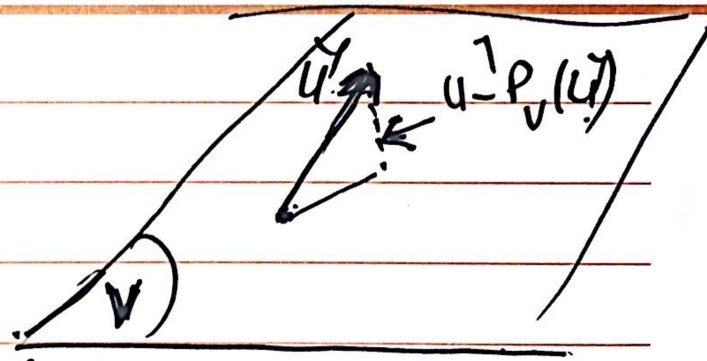
$V = \text{span}\{b_1, \ldots, b_m\}$.

$u - P_V(u) \in N(A^T)$

$$A = \begin{bmatrix} | & & | \\ b_1 & \cdots & b_m \\ | & & | \end{bmatrix}$$

$$P_V = A(AA^T)^{-1}A^T \; ; \; P_V\vec{u} = A(AA^T)^{-1}A^T u.$$

$A X = \vec{u}$



if $\vec{u} \in$ span A

AX = u has solution.

$\cancel{x^* = u}$

$\vec{u} \notin$ Span A

The closest vetor to u that is in the span of $\{b_i\}_{i=1}^{m} = V \Rightarrow P_V(u)$

Least-squares solutions.