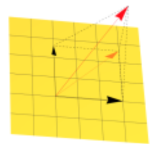
HW 7 matrix norms and SVD

More projections!



① The following code generates 10 random vectors $\vec{v}_1, \dots, \vec{v}_{10}$ and $\vec{u} \in \mathbb{R}^{10}$ (different in Matlab and Python)

```
rng(1)
d = 10;
n = 10;
v = randn(d, n);
u = randn(d, 1);
```

seed random number generator

dimension

vectors

```
np.random.seed(1)
d = 10
n = 10

v = randn(d, n)
u = randn(d, 1)
```

Using these same vectors, fill in the following table. Show your work.

k	$\lVert \vec{u} - \mathrm{Proj}_{\mathrm{span}(v_1,\dots,v_k)}(\vec{u}) \rVert$
1	
2	
$\vdots$	
n	

② Let $f(x) = \cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

For $k = 0, 1, \dots, 10$, find the degree-k polynomial $p_k(x)$ that minimizes

$$\mathbb{E}_{X \sim N(0,1)}\left[(f(X) - p_k(X))^2\right].$$

For your answer, then fill in this table:

k	$p_k(x)$	$\sqrt{\mathbb{E}_X\left[(f(X)-p_k(X))^2\right]}$
0		
1		
$\vdots$		
10		

Explain how you computed $p_k(x)$ and show at least some of your work. For example, if you use Wolfram Alpha, take at least one screenshot showing what the integrals look of your work. For example, if you use Wolfram Alpha, take at least one screenshot showing what the integrals look like.

Hints: ① Recall the Hermite polynomials

https://en.wikipedia.org/wiki/Hermite_polynomials#Definition

"probabilists' Hermite polynomials"	"physicists' Hermite polynomials"
$He_0(x) = 1,$	$H_0(x) = 1,$
$He_1(x) = x,$	$H_1(x) = 2x,$
$He_2(x) = x^2 - 1,$	$H_2(x) = 4x^2 - 2,$
$He_3(x) = x^3 - 3x,$	$H_3(x) = 8x^3 - 12x,$
$He_4(x) = x^4 - 6x^2 + 3,$	$H_4(x) = 16x^4 - 48x^2 + 12,$
$He_5(x) = x^5 - 10x^3 + 15x,$	$H_5(x) = 32x^5 - 160x^3 + 120x,$
$He_6(x) = x^6 - 15x^4 + 45x^2 - 15,$	$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120,$
$He_7(x) = x^7 - 21x^5 + 105x^3 - 105x,$	$H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x,$
$He_8(x) = x^8 - 28x^6 + 210x^4 - 420x^2 + 105,$	$H_8(x) = 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680,$
$He_9(x) = x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x,$	$H_9(x) = 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x,$
$He_{10}(x) = x^{10} - 45x^8 + 630x^6 - 3150x^4 + 4725x^2 - 945.$	$H_{10}(x) = 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240.$
↑ orthogonal with respect to $\langle f,g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2/2}\,dx$	↑ orthogonal with respect to $\langle f,g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,dx$

The Mathematica function HermiteH[j, x] returns the jth physicist poly.

② You can easily use Mathematica or Wolfram Alpha to compute any necessary integrals. For example,

https://www.wolframalpha.com/

Mathematica:

$$\int_{-\infty}^{\infty} \texttt{HermiteH[3, x]}^2\, e^{-x^2/2}\, dx$$

$528\sqrt{2\pi}$

WolframAlpha computational intelligence.

Integrate[HermiteH[3,x]^2 * e^(-x^2/2), {x, -infinity, infinity}]

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Definite integral: More digits

$$\int_{-\infty}^{\infty} H_3(x)^2\, e^{-x^2/2}\, dx = 528\sqrt{2\pi} \approx 1323.5$$

$H_n(x)$ is the n^{th} Hermite polynomial in x

③ I'll give you one for free.
$p_0(x) = \mathbb{E}[\cosh X] = \sqrt{e}$, and $\sqrt{\mathbb{E}_X[(f(X) - p_0(X))^2]} = \frac{e-1}{\sqrt{2}} \approx 1.215$

Matrix norms

③ Compute the exact norms of these matrices:

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} \cdots & \cdots \\ -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} \cdots & \cdots & \cdots \\ 4 & -2 & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & -3 \\ -3 & 0 & 2 \end{pmatrix} \qquad D = \frac{1}{\sqrt{3}}\begin{pmatrix} 3 & -1 \\ 0 & \sqrt{8} \end{pmatrix}$$

$$E = D^{-1} \qquad F = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Do the calculations by hand.
Use calculus. Do not use eigenvalues/eigenvectors.
Feel free to use Matlab/Mathematica to check your answers.

4

a) Give an example of a 2×2 matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

such that the matrix $B = \begin{pmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ has norm $\|B\| > \|A\|$. (Feel free to use Matlab.)

b) If the entries a_{ij} are all ≥ 0, prove that $\|B\| \leq \|A\|$.

5 Solve parts a,b,c,d in order.
Let $n = 50$ and let A be the $n\times n$ matrix given by

$$A_{ij} = \begin{cases} 0 & \text{if } i = j = 1 \\ r^{i+j-2} & \text{otherwise,} \end{cases}$$

where $r = 1 - \frac{1}{1000}$.

Here's some Matlab code to generate A:

```
n = 50;
r = 1 - 1/1000;
v = r .^ (0:n-1)';
A = v * v';
A(1,1) = 0;
```

Let $S = \sum_{i=0}^{n-1} r^{2i} = \frac{1-r^{2n}}{1-r^2} \approx 47.6277$.

a) Use the triangle inequality to argue that

$$S - 1 \leq \|A\| \leq S + 1$$

Can you give a simple argument that $\|A\| \leq S$?

a) Use the triangle inequality to argue that,

$$S-1 \le \|A\| \le S+1$$

Can you give a simple argument that $\|A\| \le S$?

b) Implement the following pseudocode in Matlab, and use it to lower-bound $\|A\|$:

repeat 10^6 times:
  $\vec{v} \leftarrow$ random vector
  lower bound $\leftarrow \dfrac{\|A\vec{v}\|}{\|\vec{v}\|}$
output best bound found

c) Prove that $\|A\| \geq 47.6$

Do this by finding a vector $\vec{v}$ with $\dfrac{\|A\vec{v}\|}{\|\vec{v}\|} \geq 47.6$

(6) Using the definition $\|A\| = \max\limits_{\vec{x} \neq 0} \dfrac{\|A\vec{x}\|}{\|\vec{x}\|}$,

prove that for any invertible matrix A,

$$\|A\| = \frac{1}{\min\limits_{\vec{y}:\|\vec{y}\|=1} \|A^{-1}\vec{y}\|}.$$

(7) Recall the $n \times n$ matrix

$$A_n \equiv \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & 0 & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & 0 & & \ddots & \ddots & \ddots \\ & & & & 1 & -2 & 1 \\ 1 & & & & & 1 & -2 \end{pmatrix}.$$

(This matrix arose as a discretization of the second derivative, on an interval with periodic boundary conditions.) In this problem, you will solve for $\|A_n\|$, at least when n is even.

(a) Using Matlab or similar software, determine numerically the norms of

$$A_4 = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}$$

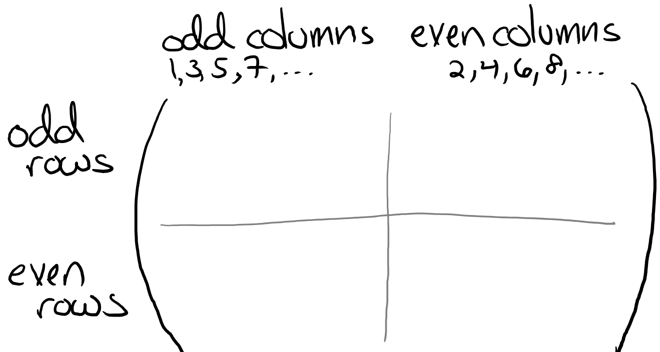
$$\begin{pmatrix} 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

(You <u>do not</u> have to prove that your answers are correct.)

Assuming that n is even:

b) Permute the rows and columns of A_n like this



This doesn't change the norm!

Find a vector $\vec{v}$ so that $\frac{\|A_n\vec{v}\|}{\|\vec{v}\|} = 4$.

Conclude that $\|A_n\| \geq 4$.

d) What is the norm of a permutation matrix, e.g.,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ? \text{ Why?}$$

e) Now prove an upper bound on $\|A_n\|$ that matches the lower bound you gave in part c.

This finishes the calculation of $\|A_n\|$ (for n even).

(<u>Hint</u>: "Break A_n into pieces." That is, write A_n as the sum of three or four permutation matrices, possibly with weights, and then use the triangle inequality $\|B+C\| \leq \|B\| + \|C\|$.)

f) [Optional] What about $\|A_n\|$ for n odd?

SVD

(8) Compute the singular-value decompositions of the following matrices. Do not use a computer! Don't use eigenvalues either.

(a) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

(b) $B = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 4 & 0 & 0 \end{pmatrix}$

(c) $C = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{pmatrix}$

(d) $D = \begin{pmatrix} 0 & 0 & 5 \\ \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 \end{pmatrix}$

(e) $E = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}$

What are the norms of the matrices?

(9) Repeat problem 6, but for the matrices

(f) $F = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

(g) $G = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

It is okay to use a computer. However, please give **exact** answers (not just numerical).