# Homework 6 Gram-Schmidt

(1) Using the Gram-Schmidt procedure, find an orthonormal basis for the span of

$$(1,1,1,-1),\ (2,-1,-1,1) \text{ and } (-1,2,2,1).$$

(2) a) Use the Gram-Schmidt procedure to find an orthonormal basis for the rowspace, columnspace, nullspace and left nullspace of

$$A = \begin{pmatrix} 1 & -2 & -1 & 3 \\ 2 & -4 & -2 & 6 \\ 5 & -10 & -5 & 15 \end{pmatrix}$$

b) What is the projection of $(1,1,1)$ onto $N(A^T)$?

(3) Orthonormal functions

For two functions $f,g: [0,1] \to \mathbb{R}$, define their inner product to be

$$f \cdot g := \int_0^1 dx\, f(x) g(x).$$

a) Notice that the monomials

$$1, \quad x, \quad x^2, \quad x^3$$

form a basis for the set of polynomials of degree $\leq 3$ (a vector space).

A polynomial can be represented by its vector of coordinates in this basis, e.g.,

$$\underbrace{1 + 2x - x^2}_{\text{polynomial}} \longleftrightarrow \underbrace{\begin{array}{cccc} 1 & x & x^2 & x^3 \\ (1, & 2, & -1, & 0) \end{array}}_{\substack{\text{vector of coefficients} \\ \text{in basis } \{1,\, x,\, x^2,\, x^3\}}}$$

Write a function that takes

Input: Two polynomials $f$ and $g$, given by their coefficient vectors

Output: $f \cdot g$

Check your answer! For example,

$$\text{polynomialdot}\left([\overset{1}{1},\overset{x}{2},\overset{x^2}{0},\overset{x^3}{0}],\ [\overset{1}{3},\overset{x}{4},\overset{x^2}{0},\overset{x^3}{0}]\right)$$

should return 10.667, since

$$(1+2x)\cdot(3+4x) = \int_0^1 (1+2x)(3+4x)\,dx = \frac{32}{3}$$

(This is different from the usual dot product $(1,2,0,0)\cdot(3,4,0,0) = 1\cdot 3 + 2\cdot 4 = 11$.)

b) Using your function from part a, verify that the polynomials

$$v_0(x) = 1$$
$$v_1(x) = \sqrt{3}(2x-1)$$
$$v_2(x) = \sqrt{5}(6x^2-6x+1)$$
$$v_3(x) = \sqrt{7}(20x^3-30x^2+12x-1)$$

form an orthonormal basis for the set of polynomials of degree $\leq 3$. In other words,

$$v_i \cdot v_j = \int_0^1 dx\, v_i(x)\, v_j(x) = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i\neq j. \end{cases}$$

**Remark**: Here's some Mathematica code that checks orthonormality:

```
basis = {v0, v1, v2, v3} = {
    1,
    2 √3 (x - 1/2),
    √5 (6 x^2 - 6 x + 1),
    √7 (20 x^3 - 30 x^2 + 12 x - 1)
  };
Table[
  Integrate[basis[[i]] basis[[j]], {x, 0, 1}],
  {i, 1, 4}, {j, 1, 4}
 ] // MatrixForm
```

atrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

I want you to do it in Matlab or python, using the vector representation of these functions. For example,

$$v_1 = (\overset{1}{-\sqrt{3}}, \overset{x}{2\sqrt{3}}, \overset{x^2}{0}, \overset{x^3}{0})$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c) Start with the basis

$(*)\quad 1, x, x^2, x^3, x^4, \ldots, x^9, x^{10}$

for polynomials of degree $\leq 10$.

Use the Gram-Schmidt method with your code from part ⓐ to change this into an orthonormal basis.

**Deliverable:**

Please turn in your source code, as well as a printout of a matrix whose rows are your final orthonormal polynomials.

For example, when I did this working numerically in Matlab, the first four polynomials I got were the rows:

```
octave:381> Q(1:4,:)
ans =

 Columns 1 through 4:
```

| | $1$ | $x$ | $x^2$ | $x^3$ |
|---|---|---|---|---|
| $1 \rightarrow$ | 1.0000e+00 | 0.0000e+00 | 0.0000e+00 | 0.0000e+00 |
| $\sqrt{3}(2x-1) \rightarrow$ | -1.7321e+00 | 3.4641e+00 | 0.0000e+00 | 0.0000e+00 |
| $v_2 \rightarrow$ | 2.2361e+00 | -1.3416e+01 | 1.3416e+01 | 0.0000e+00 |
| $v_3 \rightarrow$ | -2.6458e+00 | 3.1749e+01 | -7.9373e+01 | 5.2915e+01 |

the other columns are 0s

These are the same four polynomials found above.

**Extra:** Does your code still work for finding an orthonormal basis for polynomials of degree $\leq 40$? Why or why not?