Lecture 2: Matrices

Reading:  Ch. 1,  Ch. 2

Outline: Matrices
- Matrix multiplication
- Examples: Diagonal, permutation, block matrices
- Matlab
- Complexity of matrix multiplication
- Matrix inverses
- Dense versus sparse matrices

Matrices and Matrix Multiplication

matrix, dimensions, transpose:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

a $\underset{\#\text{rows}}{\underset{\shortparallel}{2}} \times \underset{\#\text{columns}}{\underset{\shortparallel}{3}}$ real matrix 3×2 matrix

(i, j) entries

$$A_{1,1} = 1,\ A_{1,2} = 2,\ A_{2,3} = 6, \ldots \qquad (A^T)_{i,j} = A_{j,i}$$

sum of two matrices with the same dimensions:

$$A + \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$$

scalar multiple of a matrix:

$$2A = A \cdot 2 = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}, \quad 0 \cdot A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Matrix multiplication:

1. Matrix-vector multiplication:

Matrix multiplication

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$$A\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} = \underset{2\times 3}{\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}}\underset{3\times 1}{\begin{pmatrix}1\\2\\3\end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix}1\cdot 1+2\cdot 2+3\cdot 3\\4\cdot 1+5\cdot 2+6\cdot 3\end{pmatrix}}$$

$$=\begin{pmatrix}14\\32\end{pmatrix}$$

$$A\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\cdot 1+2\cdot 0+3\cdot 0\\4\cdot 1+5\cdot 0+6\cdot 0\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix} \quad \text{first column}$$

$$A\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix} \qquad A\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\6\end{pmatrix} \quad 3^{rd}\text{ column}$$

picks out 2^{nd} column

$\Rightarrow$ By linearity,

$$A\begin{pmatrix}1\\2\\3\end{pmatrix} = A\begin{pmatrix}1\\0\\0\end{pmatrix} + A\begin{pmatrix}0\\2\\0\end{pmatrix} + A\begin{pmatrix}0\\0\\3\end{pmatrix}$$

$$= \begin{pmatrix}1\\4\end{pmatrix} + 2\cdot\begin{pmatrix}2\\5\end{pmatrix} + 3\cdot\begin{pmatrix}3\\6\end{pmatrix}$$

$$= \begin{pmatrix}14\\32\end{pmatrix} \checkmark$$

Interpretation:

m x n matrix $\longleftrightarrow$ function n-dimensional vectors to m-dim vectors

$$1^{st}\text{ column of } A = A\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}$$

$$2^{nd}\text{ column of } A = A\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}$$

$\vdots$

"linearity":

$$A\begin{pmatrix}2\\3\\0\\\vdots\\0\end{pmatrix} = 2\cdot(1^{st}\text{column}) + 3\cdot(2^{nd}\text{ column})$$

Example:

$$A = \begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}, \quad \vec{x} = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}b_1\\b_2\end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix}x_1+2x_2+3x_3\\4x_1+5x_2+6x_3\end{pmatrix}$$

$\Rightarrow A\vec{x}=\vec{b}$ is equivalent to

$$\begin{cases} x_1 + 2x_2 + 3x_3 = b_1 \\ 4x_1 + 5x_2 + 6x_3 = b_2 \end{cases}$$

LINEAR EQUATIONS

$$\underbrace{A}_{m\times n \text{ matrix}} \underbrace{\vec{x}}_{n \text{ unknowns}} = \underbrace{\vec{b}}_{m\text{-dim vector}} \iff \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{for } i = 1, \dots, n$$

2. Matrix-matrix multiplication:

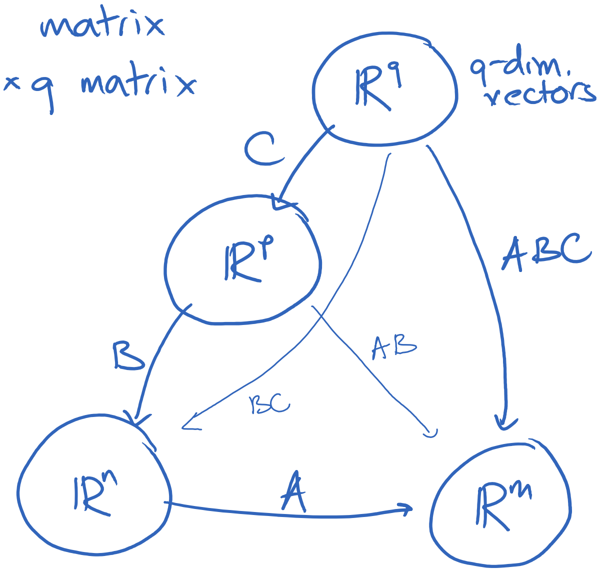
A : $m \times n$ matrix

B : $n \times p$ matrix

AB : $m \times p$ matrix — first apply B, then A (right to left)

C : $p \times q$ matrix

ABC : $m \times q$ matrix



$\Rightarrow (AB)C = A(BC)$ "associativity"

Easy rule: across and down

$$\underset{2\times 3}{A} \cdot \underset{3\times 2}{A^T} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot 3 & 1\cdot 4 + 2\cdot 5 + 3\cdot 6 \\ 4\cdot 1 + 5\cdot 2 + 6\cdot 3 & 4\cdot 4 + 5\cdot 5 + 6\cdot 6 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 4\cdot 4 & 1\cdot 2 + 4\cdot 5 & 1\cdot 3 + 4\cdot 6 \\ 2\cdot 1 + 5\cdot 4 & 2\cdot 2 + 5\cdot 5 & \ddots \\ & & \end{pmatrix}$$

$$= \begin{pmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{pmatrix}$$

Formally $$\boxed{(\underset{m\times n}{A}\ \underset{n\times p}{B})_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{k,j}}$$

Note: Matrix-vector multiplication is a special case

$$\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} 5 \\ 6 \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} 17 \\ 39 \end{pmatrix}}$$

Why is this the right rule?

$$\underset{\underset{A}{\shortparallel}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \underset{\underset{B}{\shortparallel}}{\begin{pmatrix} e & f \\ g & h \end{pmatrix}} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$(AB)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ = first column of AB

= A(first column of B)

$= A \begin{pmatrix} e \\ g \end{pmatrix}$

= (first col. of A) $\cdot e$ + (2^{nd} col. of A) $\cdot g$ ✓

Observation 1: In general, $AB \neq BA$.
"matrix multiplication is not commutative"

Observation 2: $(AB)^T = B^T A^T$

Proof:

$$\left((AB)^T\right)_{i,j} = (AB)_{j,i}$$

$$= \sum_k A_{jk} B_{ki}$$

$$= \sum_k (B^T)_{ik} (A^T)_{kj}$$
$$= (B^T A^T)_{ij} \quad \checkmark \qquad \square$$

Corollary: AA^T is symmetric
(equals its own transpose).

$$(AA^T)^T = A^{T^T} A^T = AA^T \checkmark$$

Examples: Diagonal matrices, block matrices, Matlab

A. **Diagonal matrices**

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 18 \end{pmatrix}$$

product of diagonal, square matrices is diagonal

$$= \begin{pmatrix} 4 & & \\ & 5 & \\ & & 6 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$$ —and they commute!

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} a & 2b & 3c \\ d & 2e & 3f \\ g & 2h & 3i \end{pmatrix}$$

scaling the columns (since the diagonal matrix acts first)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ 2d & 2e & 2f \\ 3g & 3h & 3i \end{pmatrix}$$

scaling the rows (since the diagonal matrix here acts second)

B. **Identity matrix**

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$I\vec{v} = \vec{v}$ for any vector $\vec{v}$ of appropriate dimensions

$IA = A$ " matrix A "

$AI = A$

c. Permutation matrices

Definition: An $n \times n$ (square) matrix is a **permutation matrix** if every row and every column has exactly one 1 in it, and every other matrix entry is 0.

Examples:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \cancel{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Why? Multiplying a vector by a permutation matrix permutes its entries, e.g.,

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ a \\ b \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix} \checkmark$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} b & c & a \\ e & f & d \\ h & i & g \end{pmatrix} \checkmark$$

easy proof: follow $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Example: Let

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

What is P^{-1}?

Answer:

P sends $1 \to 4$, $2 \to 1$, $3 \to 2$, $4 \to 3$. So P^{-1} sends $1 \to 2$, $2 \to 3$, $3 \to 4$, $4 \to 1$.

$$P^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \checkmark$$

$$P^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \checkmark$$

P^{-1} is just the transpose P^T

(Why? Say that P has a 1 at position (i, j).

$$\begin{array}{cc} & j \\ i & \begin{pmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ 0 \cdots 0 & 1 & 0 \cdots 0 \\ & 0 & \\ & \vdots & \\ & 0 & \end{pmatrix} \end{array}$$

It sends position j to position i. P^{-1} has to do the opposite.)

C. Block matrices

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

What is AB?

Answer: Of course, you can multiply it out...
Or, break A and B into pieces:

$$A = \begin{pmatrix} I & C \\ C & I \end{pmatrix} \quad B = \begin{pmatrix} C & I \\ I & C \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and multiply in blocks (since the block dimensions match):

$$AB = \begin{pmatrix} IC + CI & I^2 + C^2 \\ C^2 + I^2 & CI + IC \end{pmatrix}$$

$$= \begin{pmatrix} 2C & I + C^2 \\ I + C^2 & 2C \end{pmatrix} \qquad C^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2C$$

$$= \begin{pmatrix} 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \\ 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \end{pmatrix}_{4 \times 4} \quad \checkmark$$

D. Matlab:

```
A=[1 0 1 1;
   0 1 1 1;
```

```
0 1 1 1 ;
1 1 1 0 ;
1 1 0 1 ]
A'          <- transpose
A*A'        <- matrix product
b=[1; 2; 3; 4];
b=[1 2 3 4]';
A*b
A^2         <- matrix power

I = eye(2)
C = ones(2,2)
A = [I C; C I]     <- block matrices are okay
B = [C I; I C]
A*B
```

Asymptotic complexity of matrix multiplication

http://en.wikipedia.org/wiki/Computational_complexity_of_mathematical_operations#Matrix_algebra

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

n^2 terms to calculate

each term takes $O(n)$ time

$\Rightarrow \boxed{O(n^3)}$

But Matlab is faster! (see HW 1)

Strassen's algorithm

http://en.wikipedia.org/wiki/Strassen_algorithm

$$A = \begin{array}{cc} \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) & \begin{array}{c} n/2 \\ n/2 \end{array} \\ \begin{array}{cc} n/2 & n/2 \end{array} & \end{array} \qquad B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$$

$$AB = \left(\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right)$$

8 $n/2 \times n/2$ matrix multiplications

time $T(n) = 8T(n/2) + O(n^2)$

$$\sim n^2 + 8\left(\frac{n}{2}\right)^2 + 64\left(\frac{n}{4}\right)^2 + \cdots$$

log n levels

$$\sim n^2\left(1+2+4+8+\cdots+2^{\log_2 n}\right)$$

$$= \Theta(n^3)$$

Try computing

$$M_1 = (A_{11}+A_{22})(B_{11}+B_{22}) \qquad M_5 = (A_{11}+A_{12})B_{22}$$

$$M_2 = (A_{21}+A_{22})B_{11} \qquad M_6 = (A_{21}-A_{11})(B_{11}+B_{12})$$

$$M_3 = A_{11}(B_{12}-B_{22}) \qquad M_7 = (A_{12}-A_{22})(B_{21}+B_{22})$$

$$M_4 = A_{22}(B_{21}-B_{11})$$

$$\Rightarrow AB = \begin{pmatrix} M_1+M_4-M_5+M_7 & M_3+M_5 \\ M_2+M_4 & M_1-M_2+M_3+M_6 \end{pmatrix}$$

$$M_2 + M_4 = (A_{21}B_{11} + \cancel{A_{22}B_{11}}) + (A_{22}B_{21} - \cancel{A_{22}B_{11}}) \checkmark$$

7 $n/2 \times n/2$ matrix multiplications

time $T(n) = 7T(n/2) + O(n^2)$

$$\sim n^2\left(1 + 7\frac{1}{2^2} + 7^2\frac{1}{4^2} + \cdots + \left(\frac{7}{4}\right)^{\log_2 n}\right)$$

$$\sim n^{2+\log_2(7/4)}$$

$$= O(n^{\log_2 7 = 2.81\ldots})$$

$O(n^{2.376})$ [Coppersmith-Winograd '90]

but the constant factor is impractical

Stothers 2010: 2.374

Williams 2012: 2.3728642

Le Gall 2014: 2.3728639

Question: Is the correct exponent 2?

Remark: The asymptotic complexity of solving systems of linear equations is the same as matrix multiplication.

Matrix inverses

Matrix inverses

Def.: If it exists, the inverse of a linear transformation A satisfies
$$A A^{-1} = I, \quad A^{-1} A = I.$$

Not all matrices are invertible!

$$\left.\begin{matrix} (0)^{-1} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^{-1} \\ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^{-1} \end{matrix}\right\} \text{do not exist!}$$

Must be square and full rank
$$\Updownarrow$$
$$N(A) = \{\vec{0}\} \Leftrightarrow \det(A) \neq 0$$

1×1 matrix:
$$(a)^{-1} = \left(\frac{1}{a}\right) \quad \text{if } a \neq 0$$

2×2 matrix:
$$I^{-1} = I$$
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{-1} = \text{does not exist!}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow A^{-1} = \frac{1}{\underbrace{ad-bc}_{\det(A)}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Check: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & \cancel{ab-ba} \\ \cancel{cd-dc} & -bc+da \end{pmatrix}$ ✓

If $\det(A) = ad - bc = 0$, then A^{-1} does not exist!

Diagonal matrices
$$A = \begin{pmatrix} 1 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & 4 \end{pmatrix} \quad \Rightarrow A^{-1} = \begin{pmatrix} 1 & & & 0 \\ & 1/2 & & \\ & & 1/3 & \\ 0 & & & 1/4 \end{pmatrix}$$

Diagonal A is invertible $\Leftrightarrow$ diagonal entries all $\neq 0$

$$A = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{1}{a_1} & & & 0 \\ & \frac{1}{a_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{a_n} \end{pmatrix}$$

if all $a_j \neq 0$

Permutation matrices

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$\vec{e}_1 \mapsto \vec{e}_2$
$\vec{e}_2 \mapsto \vec{e}_3$
$\vec{e}_3 \mapsto \vec{e}_4$
$\vec{e}_4 \mapsto \vec{e}_5$
$\vec{e}_5 \mapsto \vec{e}_1$

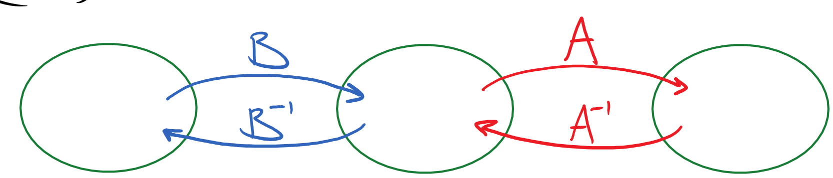
$$P^{-1} = P^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Inverse of products:

- If A and B are both invertible, then

$$(AB)^{-1} = B^{-1} A^{-1}$$

(and $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$, etc.)



- If either A or B is not invertible, then AB is not invertible.

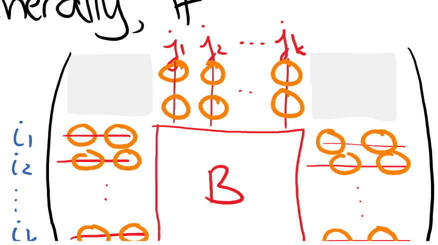
Block matrices

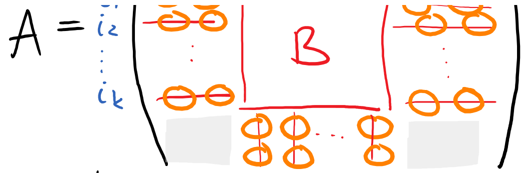
$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & a_5 \\ a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \end{pmatrix} \Rightarrow B^{-1} = \begin{pmatrix} 0 & a_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & a_2^{-1} & 0 & 0 \\ 0 & 0 & 0 & a_3^{-1} & 0 \\ 0 & 0 & 0 & 0 & a_4^{-1} \\ a_5^{-1} & 0 & 0 & 0 & 0 \end{pmatrix}$$

if all $a_j \neq 0$

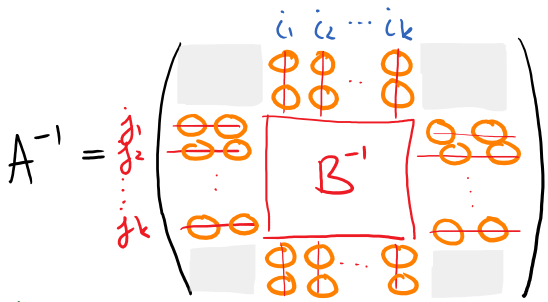
$= P \cdot A$ $\qquad$ $= A^{-1} \cdot P^{-1}$

More generally, if

$A =$ 



then A^{-1} also has to have a block structure:



Examples:

- $$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & -4 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

  since $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$ $\quad \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -2 & 4 \end{pmatrix}$

- True or False: The matrix $D = \begin{pmatrix} 0 & 0 & 0 & 10 & -.1 \\ 0 & 0 & 0 & .2 & .3 \\ 0 & 0 & 0 & 4 & -.7 \\ 1 & .2 & 3 & 0 & 0 \\ .8 & 2 & 1 & 0 & 0 \end{pmatrix}$ is invertible.

  these blocks are not invertible

Exercise: Invert the following matrices:

$$A = \begin{pmatrix} 1 & & & & & \\ & 2 & & & & \\ & & 3 & & & \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 6 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} \; \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 2 & & & \\ 3 & 0 & 0 & & & \\ & & & 0 & 4 & 0 \\ & & & 5 & 0 & 0 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} \; \end{pmatrix}$$

$$\begin{pmatrix} 3 \ 0 \ 0 & \\ & \begin{matrix} 0 & 4 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 6 \end{matrix} \\ 0 & \end{pmatrix} \qquad \begin{pmatrix} \\ \\ \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 5 & 0 \\ 0 & 6 & 0 & 0 \\ 3 & 0 & 0 & 4 \end{pmatrix} \qquad C^{-1} = \begin{pmatrix} & \\ & \end{pmatrix}$$

Other inverses

- Lower-triangular matrices

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & & 0 \\ 2 & 1 & \\ 3 & 4 & 1 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 1 & & 0 \\ -2 & 1 & \\ 5 & -4 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3/4 & 1 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 1 & 0 \\ -3/4 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -3/8 & 1/4 \end{pmatrix}$$

- Upper-triangular matrices

$(A^T)^{-1} = (A^{-1})^T$ if A is invertible

Don't compute inverses!

Slow

Numerically unstable

(sparse matrix)$^{-1}$ can be dense

You can solve $A\vec{x} = \vec{b}$ without computing A^{-1}

and if you want to solve the same equations repeatedly

You can solve $A\vec{x}=\vec{b}$ without computing A^{-1}
and if you want to solve the same equations repeatedly
$$A\vec{x}_1=\vec{b}_1,\quad A\vec{x}_2=\vec{b}_2,\quad A\vec{x}_3=\vec{b}_3,\ \dots,$$
it is better to precompute the LU decomposition of A than to precompute A^{-1} (which might not exist)

Computing an inverse using Gaussian elimination:

Example: Computing inverse of $A=\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}$
is equiv. to solving $AX=I$.

$$\left(\begin{array}{ccc|ccc}1 & a & b & 1 & 0 & 0\\ 0 & 1 & c & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right)\begin{matrix}\\ \\ \big)-c\end{matrix} \longrightarrow \left(\begin{array}{ccc|ccc}1 & a & b & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & -c\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right)\begin{matrix}\big)-a\\ \\ \big)-b\end{matrix}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc}1 & 0 & b & 1 & -a & ac-b\\ 0 & 1 & 0 & 0 & 1 & -c\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right)$$

$$\Rightarrow A^{-1}=\begin{pmatrix}1 & -a & ac-b\\ 0 & 1 & -c\\ 0 & 0 & 1\end{pmatrix}$$

Observe:
If A is upper triangular, so is A^{-1}!
(A^{-1} exists $\Leftrightarrow$ all diagonal elements $\neq 0$)

Cramer's rule https://en.wikipedia.org/wiki/Cramer%27s_rule

$$\underbrace{\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}}_{=A}^{-1}=\frac{1}{\det(A)}\begin{pmatrix}\begin{vmatrix}e & f\\ h & i\end{vmatrix} & -\begin{vmatrix}d & f\\ g & i\end{vmatrix} & \begin{vmatrix}d & e\\ g & h\end{vmatrix}\\ -\begin{vmatrix}b & c\\ h & i\end{vmatrix} & \begin{vmatrix}a & c\\ g & i\end{vmatrix} & -\begin{vmatrix}a & b\\ g & h\end{vmatrix}\\ \begin{vmatrix}b & c\\ e & f\end{vmatrix} & -\begin{vmatrix}a & c\\ d & f\end{vmatrix} & \begin{vmatrix}a & b\\ d & e\end{vmatrix}\end{pmatrix}^{T}$$

This also works for $n\times n$ matrices, but is not useful.

Properties of the matrix inverse:

- $AA^{-1}=I=A^{-1}A$

Equivalently, $(A^{-1})^{-1} = A$

- If it exists, then A^{-1} is unique.

Proof:

Say X and Y are both inverses of A.

$$
\begin{aligned}
X = XI &= X(AY) \\
&= (XA)Y \\
&= IY \\
&= Y \quad \checkmark
\end{aligned}
$$

Singular-value decomposition

$$A = \sum_i \sigma_i \vec{u}_i \vec{v}_i^T = USV^T$$

$$\Rightarrow A^{-1} = \sum_i \frac{1}{\sigma_i} \vec{v}_i \vec{u}_i^T = VS^{-1}U^T$$

pseudoinverse: $A^+ = \sum_{i : \sigma_i > 0} \frac{1}{\sigma_i} \vec{v}_i \vec{u}_i^T$

(defined for all matrices)

Spectral decomposition

If A is diagonalizable,

$$A = \sum_i \lambda_i \vec{v}_i \vec{v}_i^\dagger$$

$$A^{-1} = \sum_i \frac{1}{\lambda_i} \vec{v}_i \vec{v}_i^\dagger \quad \text{if all } \lambda_i \neq 0$$

Orthogonal and unitary

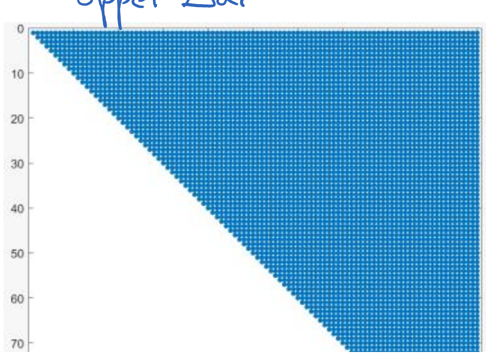
$A^{-1} = A^T \qquad\qquad A^{-1} = A^\dagger$

i.e., rows form orthonormal basis, as do columns

Matlab

```
n = 100;
A = zeros(n);
for i = 1:n
  for j = i:n
    A(i,j) = rand();
  end;
end;
spy(inv(A));
```

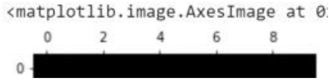
upper Δar



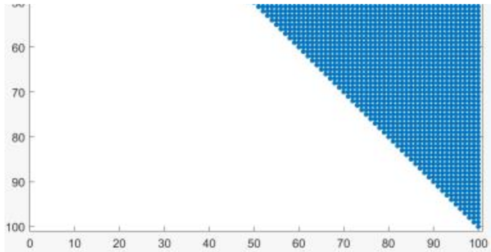
Python

```
import numpy as np

n = 10
A = np.zeros((n,n))
for i in range(n):
  for j in range(i,n):
    A[i,j] = np.random.rand()

import matplotlib.pyplot as plt
plt.spy(A)
```

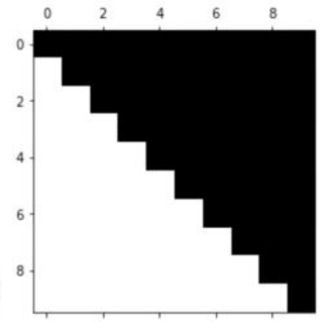
```
<matplotlib.image.AxesImage at 0:
```



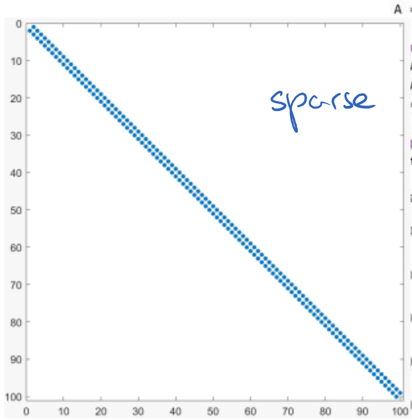
```
spy(inv(A));
```



```
<matplotlib.image.AxesImage at 0:
```



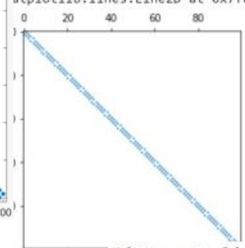
```
n = 100;
A = sparse(n,n);  % all-0 nxn sparse matrix
for i = 1:n-1
  A(i,i+1) = rand();
  A(i+1,i) = rand();
end;
spy(A);
```



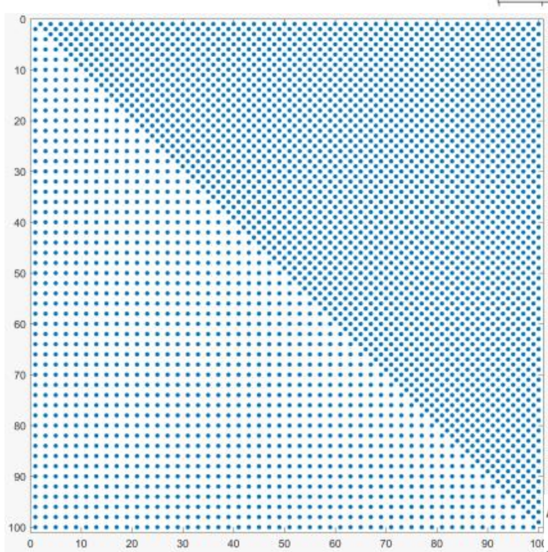
sparse

```
import numpy as np
import scipy.sparse

n = 100
A = scipy.sparse.lil_matrix((n,n))  # list-of-lists sparse format for
                                    #  incrementally building A
r i in range(n-1):
A[i,i+1] = np.random.rand()
A[i+1,i] = np.random.rand()
= A.tocsr()  # a faster sparse matrix format

port matplotlib.pyplot as plt
t.spy(A,MarkerSize=1)
```

```
atplotlib.lines.Line2D at 0x7fd848b4ab38>
```



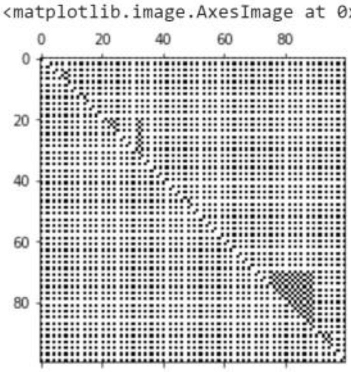
```
spy(inv(A));
```



dense ...

```
Ainv = np.linalg.inv(A.toarray())
# can also use scipy.sparse.linalg.inv(A) (but don't)
plt.spy(Ainv)
```

```
<matplotlib.image.AxesImage at 0x7fd848c06400>
```



```
Ainv[0:4,0:4]

array([[ 0.        ,  2.35909351,  0.        , -38.01358028],
       [ 2.03911397,  0.        ,  0.        ,   0.        ],
       [ 0.        ,  0.        ,  0.        ,  32.74581217],
       [ -1.84958998, -0.        ,  1.02259469,  -0.        ]])
```

```
>> Ainvdense = full(inv(A));
>> Ainvdense(1:5,1:5)

ans =

         0    1.0941         0   -0.8991         0
   10.1924         0    0.0000         0   -0.0000
         0         0         0    1.0473         0
   -3.6747         0    2.3748         0    0.0000
         0         0         0         0         0
```

bipartite...