Homework 10 Eigenvectors

Some Basic Facts

① If A is an $n \times n$ matrix, explain why for all $c \in \mathbb{C}$,

$$\det(cA) = c^n \cdot \det(A).$$

Review

② a) Is the matrix

$$\mathcal{U} = \begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{i}{\sqrt{3}} & \frac{-2i}{\sqrt{6}} \end{pmatrix}$$

unitary? Why or why not?

b) Give a 2×2 matrix that maps

$u = (1+i, i)$ to $2u$

and $v = (1+i, -2i)$ to $-3v$.

Check your answer!!

Introduction to eigenvectors

③ Prove that if $\mathcal{U}$ is invertible, then the eigenvalues of

A

are the same as the eigenvalues of

$\mathcal{U}A\mathcal{U}^{-1}$.

How are the eigenvectors of A related to the eigenvectors of $\mathcal{U}A\mathcal{U}^{-1}$?

④ Let

$$A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

be a diagonal matrix, with all $\lambda_i \in \mathbb{C}$.

be a diagonal matrix, with all $\lambda_i \in \mathbb{C}$.

a) Give an SVD of A; what are the singular values and corresponding right and left singular vectors?

b) Give a spectral decomposition of A; what are the eigenvalues and corresponding eigenvectors?

5. For a polynomial $p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_k x^k$, and a square matrix A, define $p(A)$ to be the matrix

$$p(A) = \alpha_0 I + \alpha_1 A + \cdots + \alpha_k A^k.$$

Show that if $\vec{v}$ is an eigenvector of A, with corresponding eigenvalue λ, then $\vec{v}$ is also an eigenvector of $p(A)$, with corresponding eigenvalue $p(\lambda)$.

6. If $\vec{v}_1, \dots, \vec{v}_k$ are eigenvectors of A, all with the same eigenvalue λ, explain why every nonzero linear combination

$$\vec{v} = c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k$$

is also an eigenvector of A, with eigenvalue λ. (This is why we talk of "eigenspaces".)

Some Calculations

7. a) By hand (not using Matlab), find the eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}.$$

b) What are the eigenvalues of $\begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}$?

What are the eigenvalues of $\begin{pmatrix} 5 & 1 \\ 0 & 4 \end{pmatrix}$?

In general, what are the eigenvalues of $\alpha I + A$? Why?

ⓒ What are the eigenvalues and corresponding eigenvectors for

$$B = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} ?$$

What are the eigenvalues for

$$C = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix} ?$$

In general, explain why knowing the eigenvalues and eigenvectors of matrices A and B also gives you the eigenvalues and eigenvectors of

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

ⓓ What are the eigenvalues and corresponding eigenvectors for

$$D = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$E = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \text{ and}$$

$$\begin{pmatrix} 0 & 0 & 0 & 2 \end{pmatrix}$$

$$F = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} ?$$

Feel free to use Matlab.
Then explain why, given the answers to parts ⓐ and ⓒ, using Matlab is unnecessary.

⑧
ⓐ What are the eigenvalues and corresponding eigenspaces of the $n \times n$ matrix

$$\begin{pmatrix} 2 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 2 & 1 & \cdots & 1 \\ \vdots & \vdots & & \ddots & & \vdots \\ 1 & 1 & \cdots & 1 & 2 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 2 \end{pmatrix}$$

(2's on the diagonal, 1's everywhere else)?
What is its determinant?

ⓑ What is the determinant of the $n \times n$ matrix

$$\begin{pmatrix} 1-n & 1 & 1 & \cdots & 1 \\ 1 & 1-n & 1 & \cdots & 1 \\ 1 & 1 & 1-n & & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 1-n \end{pmatrix}$$

(1-n along the diagonal, 1's off the diagonal)?
Hint: What are the eigenvalues?

⑨
ⓐ If $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$, find A^{100} by diagonalizing A.

ⓑ Diagonalize $B = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ as $B = UDU^{-1}$ for some diagonal matrix D. Then use the diagonalization to prove that

$$B^k = \begin{pmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{pmatrix}.$$

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The higher-order differential equation $y'' + y = 0$ can be written as a first-order system by introducing the derivative y' as another unknown:

$$\frac{d}{dt}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -y \end{pmatrix}.$$

If this is $\frac{d\vec{v}}{dt} = A\vec{v}$, what is the 2×2 matrix A?

Find its eigenvalues and eigenvectors, and compute the solution that starts from $y(0) = 2$, $y'(0) = 0$.