Homework 4

Systems of linear equations

① Consider a particular species of wildflower in which each plant has several stems, leaves and flowers, and for each plant let

S = the average stem length (in cm)

L = the average leaf width (in cm)

F = the number of flowers.

Four plants are examined, and the information is tabulated in the following matrix:

$$A = \begin{array}{c} \\ \#1 \\ \#2 \\ \#3 \\ \#4 \end{array} \begin{array}{c} \begin{array}{ccc} S & L & F \end{array} \\ \begin{pmatrix} 2 & 2 & 10 \\ 4 & 2 & 12 \\ 4 & 4 & 15 \\ 6 & 4 & 17 \end{pmatrix} \end{array}.$$

For these four plants, determine whether or not there exists a linear relationship between S, L and F.

In other words, do there exist constants $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ such that $\alpha_0 + \alpha_1 S + \alpha_2 L + \alpha_3 F = 0$?

② Determine which of the following sets of vectors are linearly independent. For those sets that are linearly dependent, write one of the vectors as a linear combination of the others.

a) $\{(1,2,3),\ (2,1,0),\ (1,5,9)\}$

b) $\{(1,2,3),\ (0,4,5),\ (0,0,6),\ (1,1,1)\}$

c) $\{(3,2,1),\ (1,0,0),\ (2,1,0)\}$

d) $\{(2,2,2,2),\ (2,2,0,2),\ (2,0,2,2)\}$

e) $\left\{ \begin{pmatrix} 1 \\ 2 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ \vdots \end{pmatrix} \right.$

e) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 4 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 4 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 4 \\ 0 \\ 3 \\ 1 \end{pmatrix} \right\}$

Vector spaces

3. Which of the following are vector spaces?
In each case, explain why/why not.
And if it is a vector space, give the dimension and a basis.

a) the set $\mathbb{R}$ of real numbers

b) the set of solutions (x_1, x_2) to the equations

$$5x_1 + 2x_2 = 0$$
$$3x_1 - 2x_2 = 2$$

c) the set of solutions (x_1, x_2) to the equation

$$x_1 x_2 = 0$$

d) the span of the vectors
$(2, 3, 4)$, $(-1, -1\ -4)$ and $(0, 1, -4)$

e) the set of **anti-symmetric** 4×4 matrices
(recall: a matrix A is antisymmetric if $A^T = -A$)

Bases

4. Determine the dimension of the space spanned by the set

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 8 \\ -4 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \\ 6 \end{pmatrix} \right\}$$

5. Determine whether or not the set

$$B = \{(2, 3, 2),\ (1, 1, -1)\}$$

is a basis for the space spanned by the set
$$A = \{(1,2,3), (5,8,7), (3,4,1)\}.$$

(6) How many different bases are there for the 3-dimensional space $\mathbb{F}_2^3$?

(7) Let
$$A = \begin{pmatrix} 2 & 2 & 5 & 0 & 1 \\ 3 & 4 & 8 & 1 & 2 \\ 1 & 6 & 5 & 5 & 3 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 3 \\ -8 \end{pmatrix}.$$
Verify that $\vec{v} \in N(A)$.
Then extend $\{\vec{v}\}$ to a basis for $N(A)$.
(That is, find a basis for $N(A)$ that includes $\vec{v}$.)

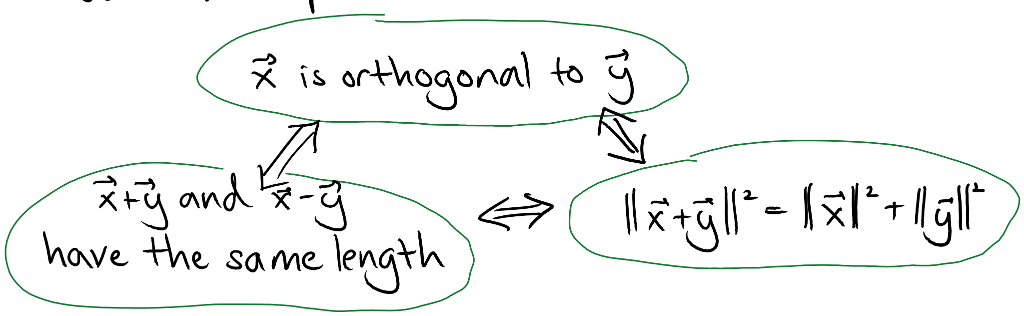
Inner products and orthogonality

(8) a) Find two different unit vectors (i.e., length-one vectors) that are orthogonal to $\vec{u} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \in \mathbb{R}^2$.

b) Give a basis for the orthogonal complement
$$\mathrm{Span}\left(\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}\right)^{\perp} \subseteq \mathbb{R}^3$$

(9) a) Prove the "parallelogram law":
$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2(\|\vec{x}\|^2 + \|\vec{y}\|^2).$$

b) Show the implications:



(10) What is the projection of $b = (2,1,2)$ onto the plane spanned by $(0,1,0)$ and $(0,1,1)$?