Lecture 2: Matrices (class)

Reading:  Ch. 1,  Ch. 2

HW 1 out today
due next Thursday
@ 6pm Pacific

Outline: Matrices
- Matrix multiplication
- Examples: Diagonal, permutation, block matrices
- Matlab
- Complexity of matrix multiplication
- Matrix inverses
- Dense versus sparse matrices

MATRICES AND MATRIX MULTIPLICATION

matrix, dimensions, transpose:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

a $\underset{\#\text{rows}}{2} \times \underset{\#\text{columns}}{3}$ real matrix $\qquad$ 3×2 matrix

(i,j) entries

$$A_{1,1} = 1,\ A_{1,2} = 2,\ A_{2,3} = 6, \ldots \qquad (A^T)_{i,j} = A_{j,i}$$

sum of two matrices with the same dimensions:

$$A + \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$$

scalar multiple of a matrix:

$$2A = A\cdot 2 = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{pmatrix}, \quad 0\cdot A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Matrix multiplication:

1. Matrix-vector multiplication:

$$A\cdot\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2\cdot 2 + 3\cdot 3 \\ 4\cdot 1 + 5\cdot 2 + 6\cdot 3 \end{pmatrix} = \begin{pmatrix}14\\32\end{pmatrix} ?$$

$$A\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix} \text{ first column}$$

"standard basis vector"

$$A\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\5\end{pmatrix} \qquad A\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\6\end{pmatrix}$$

$$\begin{pmatrix}1\\2\\3\end{pmatrix} = 1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$A\begin{pmatrix}1\\2\\3\end{pmatrix} = 1\cdot A\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\cdot A\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\cdot A\begin{pmatrix}0\\0\\1\end{pmatrix} = 1\cdot\begin{pmatrix}1\\4\end{pmatrix} + 2\cdot\begin{pmatrix}2\\5\end{pmatrix} + 3\begin{pmatrix}3\\6\end{pmatrix}$$

linearity

$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

linearity

$A(\vec{u}+\vec{v}) = A\vec{u}+A\vec{v}$

Interpretation:

m × n matrix ⟷ function n-dimensional vectors to m-dim vectors

1st column of $A = A\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}_{n\times 1} \in \mathbb{R}^m$ ($\mathbb{R}^n$)

2nd column of $A = A\begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}$

$\vdots$

"linearity":

$$A\begin{pmatrix}2\\3\\0\\\vdots\\0\end{pmatrix} = 2\cdot(1^{st}\text{ column}) + 3\cdot(2^{nd}\text{ column})$$

Example:

$$A=\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix},\quad \vec{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},\quad \vec{b}=\begin{pmatrix}b_1\\b_2\end{pmatrix}$$

$$A\vec{x}=\begin{pmatrix}x_1+2x_2+3x_3\\4x_1+5x_2+6x_3\end{pmatrix}$$

$\Rightarrow A\vec{x}=\vec{b}$ is equivalent to

$$\begin{cases}x_1+2x_2+3x_3=b_1\\4x_1+5x_2+6x_3=b_2\end{cases}$$

LINEAR EQUATIONS

$$\underbrace{A}_{m\times n\text{ matrix}}\ \underbrace{\vec{x}}_{n\text{ unknowns}} = \underbrace{\vec{b}}_{m\text{-dim vector}} \iff \sum_{j=1}^{n} A_{ij}x_j = b_i \quad \text{for } i=1,\dots,n$$

2. **Matrix-matrix multiplication:** $(AB)\vec{v} = AB\vec{v} = A(B\vec{v})$

A : m×n matrix

B : n×p matrix

AB : m×p matrix — first apply B, then A (right to left)

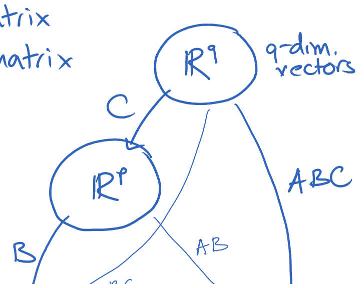
$\vec{v}\in\mathbb{R}^n$

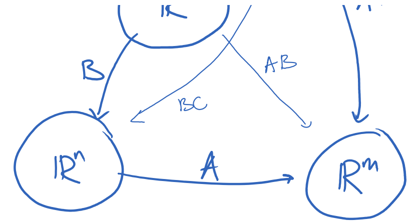
$\underset{m\times n}{A}\vec{v}_n \in \mathbb{R}^m$

$(A\vec{v})_{i} = \sum_{j=1}^{n} A_{ij}v_j$, $i=1..m$

C : p × q matrix

ABC : m × q matrix





$$\Rightarrow (AB)C = A(BC) \qquad \text{"associativity"}$$

Easy rule: across and down

$$\underset{2\times 3}{A} \cdot \underset{3\times 2}{A^T} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \left(\begin{array}{c|c} 1\cdot 1+2\cdot 2+3\cdot 3 & 1\cdot 4+2\cdot 5+3\cdot 6 \\ \hline 4\cdot 1+5\cdot 2+6\cdot 3 & 4^2+5^2+6^2 \end{array}\right)$$

$$= \begin{pmatrix} 14 & 32 \\ 32 & 77 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1\cdot 1+4\cdot 4 & 1\cdot 2+4\cdot 5 & 1\cdot 3+4\cdot 6 \\ 2\cdot 1+5\cdot 4 & 2\cdot 2+5\cdot 5 & \ddots \\ & & \end{pmatrix}$$

$$= \begin{pmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{pmatrix}$$

Formally $\boxed{(\underset{m\times n}{A}\ \underset{n\times p}{B})_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}}$

e.g. $(AB)C = A(BC) = ABC$

$$(ABC)_{il} = \sum_{j,k} A_{ij} B_{jk} C_{kl}$$

Note: Matrix-vector multiplication is a special case

$$\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}\underset{2\times 1}{\begin{pmatrix} 5 \\ 6 \end{pmatrix}} = \underset{2\times 1}{\begin{pmatrix} 17 \\ 39 \end{pmatrix}}$$

Why is this the right rule?

$$\underset{A}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\underset{B}{\begin{pmatrix} e & f \\ g & h \end{pmatrix}} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$(AB)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \text{first column of } (AB)$$

$$= A\left(B\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = A(\text{first column of } B)$$

$$= A\begin{pmatrix} e \\ g \end{pmatrix} = eA\begin{pmatrix} 1 \\ 0 \end{pmatrix} + gA\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= e\begin{pmatrix} a \\ c \end{pmatrix} + g\begin{pmatrix} b \\ d \end{pmatrix}$$

Observation 1: In general, $AB \neq BA$.
"matrix multiplication is not commutative"

$$\underset{A}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}\underset{A^T}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\neq$$

$$A^T A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Observation 2: $(AB)^T = B^T A^T$

Proof:

$$((AB)^T)_{ij} = (AB)_{ji}$$

$$= \sum_k A_{jk} B_{ki}$$

$$= \sum_k (B^T)_{ik} (A^T)_{kj}$$

$$(ABCDE)^T = E^T D^T C^T B^T A^T$$

$$= \sum_k (B^T)_{ik}(A^T)_{kj}$$
$$= (B^T A^T)_{ij} \quad ✓$$

$\square$

Corollary: AA^T is symmetric (equals its own transpose).

Proof: $(AA^T)^T = A^{T^T} A^T = AA^T$ ✓

$$AB\begin{pmatrix}1\\0\\0\end{pmatrix} = A\begin{pmatrix}4\\0\\0\end{pmatrix} = 4A\begin{pmatrix}1\\0\\0\end{pmatrix} = 4\cdot 1\begin{pmatrix}1\\0\\0\end{pmatrix}$$

Examples: Diagonal matrices, block matrices, Matlab

A. Diagonal matrices

$$\underbrace{\begin{pmatrix}1 & & 0\\ & 2 & \\ 0 & & 3\end{pmatrix}}_{=A}\underbrace{\begin{pmatrix}4 & & 0\\ & 5 & \\ 0 & & 6\end{pmatrix}}_{=B} = \begin{pmatrix}4 & & 0\\ & 10 & \\ 0 & & 18\end{pmatrix}$$

product of diagonal, square matrices is diagonal

$$BA = \begin{pmatrix}4 & & 0\\ & 10 & \\ 0 & & 18\end{pmatrix}$$

$$AD\begin{pmatrix}0\\0\\1\end{pmatrix} = 3\cdot A\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}}_{=A}\underbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}}_{=D} = \begin{pmatrix}a & 2b & 3c\\ d & 2e & 3f\\ g & 2h & 3i\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}}_{=D}\underbrace{\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}}_{=A} = \begin{pmatrix}a & b & c\\ 2d & 2e & 2f\\ 3g & 3h & 3i\end{pmatrix}$$

B. Identity matrix

$$I_1 = (1) \qquad I_3 = \begin{pmatrix}1 & & 0\\ & 1 & \\ 0 & & 1\end{pmatrix}$$

$$I = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$I\vec{v} = \vec{v}$ for any vector $\vec{v}$ of appropriate dimensions

$IA = A$ " matrix A "

$AI = A$

C. Permutation matrices

Definition: An $n \times n$ (square) matrix is a permutation matrix if every row and every column has exactly one 1 in it, and every other matrix entry is 0.

Examples:
$$\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix} \quad (1) \quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \begin{pmatrix}0 & 1 & 0\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$$

Why? Multiplying a vector by a permutation matrix permutes its entries, e.g.,

$$\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}c\\a\end{pmatrix} \qquad \Big) = a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix}$$

permutes its entries, e.g.,

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}c\\a\\b\end{pmatrix}$$

$$\begin{pmatrix}a\\b\\c\end{pmatrix}=a\begin{pmatrix}1\\0\\0\end{pmatrix}+b\begin{pmatrix}0\\1\\0\end{pmatrix}+c\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}b\\c\\a\end{pmatrix}$$

$$\underset{P}{\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}}\underset{A}{\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}}=\begin{pmatrix}g&h&i\\a&b&c\\d&e&f\end{pmatrix}$$ row permute

$$\underset{A}{\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}}\underset{P}{\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}}=\begin{pmatrix}b&c&a\\e&f&d\\h&i&g\end{pmatrix}$$ column permute

$$(AP)\begin{pmatrix}0\\0\\1\end{pmatrix}=A\begin{pmatrix}1\\0\\0\end{pmatrix} \qquad (AP)\begin{pmatrix}1\\0\\0\end{pmatrix}=A\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$(AP)\begin{pmatrix}0\\1\\0\end{pmatrix}=A\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Exercise: Find non-commuting permutation matrices P, Q.

$$\begin{pmatrix}1&&&\\&2&&\\&&3&\\&&&4\end{pmatrix}^{-1}=\begin{pmatrix}1&&&\\&\frac{1}{2}&&\\&&\frac{1}{3}&\\&&&\frac{1}{4}\end{pmatrix}$$

Example: Let

$$P=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{pmatrix}.$$

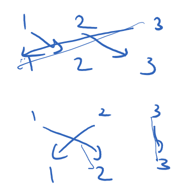
What is P^{-1}?

Answer:

P sends $1\to4$, $2\to1$, $3\to2$, $4\to3$

P^{-1} should $1\to2$, $2\to3$, $3\to4$, $4\to1$

$$P^{-1}=\begin{pmatrix}0&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}=P^T$$



P^{-1} is just the transpose P^T

(Why? Say that P has a 1 at position (i,j).

$$\begin{array}{c}\\ i\end{array}\begin{pmatrix}&&&0&&&\\&&&\vdots&&&\\0&\cdots&0&1&0&\cdots&0\\&&&\vdots&&&\\&&&0&&&\end{pmatrix}$$

It sends position j to position i. P^{-1} has to do the opposite.)

c. Block matrices

$$A=\left(\begin{array}{cc|cc}1&0&1&1\\0&1&1&1\\\hline1&1&1&0\\1&1&0&1\end{array}\right) \qquad B=\left(\begin{array}{cc|cc}1&1&1&0\\1&1&0&1\\\hline1&0&1&1\\0&1&1&1\end{array}\right)$$

What is AB?

Answer: Of course, you can multiply it out...

Or, break A and B into pieces:

$A=(I_2\ C)$ $\quad$ $B=(C\ I)$ $\quad$ $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$C \quad I_2$

and multiply in blocks (since the block dimensions match):

$$AB = \left(\begin{array}{c|c} I\cdot C + C\cdot I & I^2 + C^2 = I + 2C \\ \hline C^2 + I^2 & 2C \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \\ \hline 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \end{array}\right) \qquad C^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 = 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2C$$

D. Matlab:

```
A=[1 0 1 1;
   0 1 1 1;
   1 1 1 0;
   1 1 0 1]
A'          <- transpose
A*A'        <- matrix product
b=[1; 2; 3; 4];
b=[1 2 3 4]';
A*b
A^2         <- matrix power

I = eye(2)
C = ones(2,2)
A =[I C; C I]   <- block matrices are okay
B =[C I; I C]
A*B
```

$$\underbrace{\left(\begin{array}{c|c} B_{2\times 3} & C_{2\times 1} \\ \hline D_{3\times 3} & E_{3\times 1} \end{array}\right)}_{A} \left(\begin{array}{c} F_{3\times 100} \\ \hline G_{1\times 100} \end{array}\right) \begin{array}{l} \}\,3 \\ \}\,1 \end{array} \; 4$$

$$= \begin{pmatrix} BF + CG \\ DF + EG \end{pmatrix}$$

Asymptotic complexity of matrix multiplication

http://en.wikipedia.org/wiki/Computational_complexity_of_mathematical_operations#Matrix_algebra

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

n^2 terms to calculate; each term takes $O(n)$ time $\Rightarrow \boxed{O(n^3)}$

But Matlab is faster! (see HW 1)

$$A = \left(\begin{array}{c|c} 0_{m\times m} & B_{m\times m} \\ \hline C_{m\times m} & 0_{m\times m} \end{array}\right) \quad \Rightarrow A^{-1} = \left(\begin{array}{c|c} 0 & C^{-1} \\ \hline B^{-1} & 0 \end{array}\right)$$

Strassen's algorithm

http://en.wikipedia.org/wiki/Strassen_algorithm

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)\begin{array}{l} n/2 \\ n/2 \end{array} \qquad B = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right)$$

$$AB = \left(\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right)$$

8 $n/2 \times n/2$ matrix multiplications

time $T(n) = 8T(n/2) + O(n^2)$

$$\sim n^2 + 8\left(\tfrac{n}{2}\right)^2 + 64\left(\tfrac{n}{4}\right)^2 + \cdots \quad \log n \text{ levels}$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 4 & 0 & 5 \\ 0 & 0 & 6 & 0 \end{pmatrix}$$

time $T(n) = 8T(n/2) + O(n^2)$

$\sim n^2 + 8\left(\frac{n}{2}\right)^2 + 64\left(\frac{n}{4}\right)^2 + \cdots$ log n levels

$\sim n^2(1+2+4+8+\cdots+2^{\log_2 n})$

$= \Theta(n^3)$

$A_3 = \begin{pmatrix} 0 & 4 & 0 & \cdots \\ 0 & 0 & 6 & 0 \end{pmatrix}$

$$A^{-1} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1/2 & 0 & -1/4 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1/6 \\ 4 & -4/5 & 0 & 1/5 & 0 \end{array}$$

$\begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix}^{-1} = \frac{1}{5}\begin{pmatrix} 5 & 0 \\ -4 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 3 \\ 0 & 6 \end{pmatrix}^{-1} = \frac{1}{12}\begin{pmatrix} 6 & -3 \\ 0 & 2 \end{pmatrix}$

Try computing

$M_1 = (A_{11}+A_{22})(B_{11}+B_{22})$ $\quad M_5 = (A_{11}+A_{12})B_{22}$

$M_2 = (A_{21}+A_{22})B_{11}$ $\quad M_6 = (A_{21}-A_{11})(B_{11}+B_{12})$

$M_3 = A_{11}(B_{12}-B_{22})$ $\quad M_7 = (A_{12}-A_{22})(B_{21}+B_{22})$

$M_4 = A_{22}(B_{21}-B_{11})$

$$\Rightarrow AB = \begin{pmatrix} M_1+M_4-M_5+M_7 & M_3+M_5 \\ M_2+M_4 & M_1-M_2+M_3+M_6 \end{pmatrix}$$

$M_2 + M_4 = (A_{21}B_{11} + \cancel{A_{22}B_{11}}) + (A_{22}B_{21} - \cancel{A_{22}B_{11}})$ ✓

7 $n/2 \times n/2$ matrix multiplications

time $T(n) = 7T(n/2) + O(n^2)$

$\sim n^2\left(1 + 7\frac{1}{2^2} + 7^2\frac{1}{4^2} + \cdots + \left(\frac{7}{4}\right)^{\log_2 n}\right)$

$\sim n^{2+\log_2(7/4)}$

$= O(n^{\log_2 7 = 2.81\ldots})$

$O(n^{2.376})$ [Coppersmith-Winograd '90]

but the constant factor is impractical

Stothers 2010: 2.374

Williams 2012: 2.3728642

Le Gall 2014: 2.3728639

Question: Is the correct exponent 2?

Remark: The asymptotic complexity of solving systems of linear equations is the same as matrix multiplication.

Matrix inverses

Def.: If it exists, the *inverse* of a linear transformation A satisfies $AA^{-1} = I$, $A^{-1}A = I$.

Not all matrices are invertible!

$(0)^{-1}$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}^{-1}$ don't exist

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^{-1}$ doesn't exist

Must be square and full rank $\Updownarrow$ $N(A) = \{\vec{0}\} \Leftrightarrow \det(A) \neq 0$

$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ not invertible

1×1 matrix:

$(a)^{-1} = \left(\frac{1}{a}\right)$ if $a \neq 0$

1×1 matrix:

$$(a)^{-1} = \left(\tfrac{1}{a}\right) \quad \text{if } a \neq 0$$

2×2 matrix:

$$I^{-1} = I$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{-1} = \text{doesn't exist}$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y+2x \end{pmatrix}$$

Claim: $A^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y-2x \end{pmatrix}$

why? Then $A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} x \\ y+2x \end{pmatrix} = \begin{pmatrix} x \\ (y+2x)-2x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ ✓

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boxed{A^{-1} = \frac{1}{\underbrace{ad-bc}_{\det(A)}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}} \Rightarrow A^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

(Check: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & \cancel{ab-ba} \\ \cancel{cd-dc} & -bc+da \end{pmatrix}$) ✓

If $\det(A) = ad - bc = 0$, then A^{-1} does not exist!

Diagonal matrices

$$A = \begin{pmatrix} 1 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & 4 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} 1 & & & 0 \\ & \frac{1}{2} & & \\ & & \frac{1}{3} & \\ 0 & & & \frac{1}{4} \end{pmatrix}$$

Diagonal A is invertible $\Leftrightarrow$ diagonal entries all $\neq 0$

$$A = \begin{pmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \frac{1}{a_1} & & & 0 \\ & \frac{1}{a_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{a_n} \end{pmatrix}$$

if all $a_j \neq 0$

Permutation matrices

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$\vec{e}_1 \mapsto \vec{e}_2$
$\vec{e}_2 \mapsto \vec{e}_3$
$\vec{e}_3 \mapsto \vec{e}_4$
$\vec{e}_4 \mapsto \vec{e}_5$
$\vec{e}_5 \mapsto \vec{e}_1$

$$P^{-1} = P^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix}}_{D}\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}}_{P}$$

$$A^{-1} = P^{-1}D^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & & & 0 \\ & 1/2 & & \\ & & 1/3 & \\ 0 & & & 1/4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1/4 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \end{pmatrix}$$

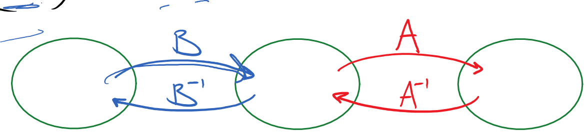
Inverse of products:

$(AB)^T = B^T A^T$

- If A and B are both invertible, then

$$(AB)^{-1} = B^{-1}A^{-1}$$

(and $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$, etc.)



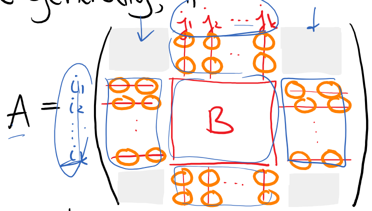
- If either A or B is not invertible, then AB is not invertible.

Block matrices

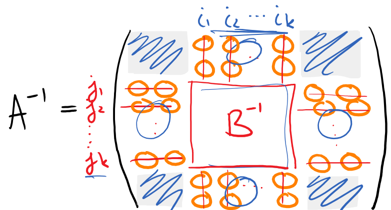
$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & a_5 \\ a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \end{pmatrix} \Rightarrow B^{-1} = \begin{pmatrix} 0 & a_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & a_2^{-1} & 0 & 0 \\ 0 & 0 & 0 & a_3^{-1} & 0 \\ 0 & 0 & 0 & 0 & a_4^{-1} \\ a_5^{-1} & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{if all } a_j \neq 0$$

$$B = P \cdot A \qquad B^{-1} = A^{-1} \cdot P^{-1}$$

More generally, if



then A^{-1} also has to have a block structure:



$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

Examples:

- $C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $C^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/5 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & -4 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

  since $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$ $\quad \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -2 & 4 \end{pmatrix}$

- True or False: The matrix $D = \begin{pmatrix} 0 & 0 & 0 & 10 & -.1 \\ 0 & 0 & 0 & .2 & .3 \\ 0 & 0 & 0 & 4 & -.7 \\ 1 & .2 & 3 & 0 & 0 \\ .8 & 2 & 1 & 0 & 0 \end{pmatrix}$ is invertible

  these blocks are not invertible

Exercise: Invert the following matrices:

$$A = \begin{pmatrix} 1 & & & & & \\ & 2 & & & 0 & \\ & & 3 & & & \\ & & & 4 & & \\ & 0 & & & 5 & \\ & & & & & 6 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 1 & & & & & \\ & 1/2 & & & 0 & \\ & & 1/3 & & & \\ & & & 1/4 & & \\ & 0 & & & 1/5 & \\ & & & & & 1/6 \end{pmatrix}$$

$$B = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ 0 & 0 & 2 & & 0 & \\ 3 & 0 & 0 & & & \\ \hline & & & 0 & 4 & 0 \\ & 0 & & 5 & 0 & 0 \\ & & & 0 & 0 & 6 \end{array}\right) \qquad B^{-1} = \left(\begin{array}{c|c} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{pmatrix}^{-1} & 0 \\ \hline 0 & \begin{pmatrix} 0 & 4 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}^{-1} \end{array}\right) = \left(\begin{array}{ccc|ccc} 0 & 0 & 1/3 & & & \\ 1 & 0 & 0 & & 0 & \\ 0 & 1/2 & 0 & & & \\ \hline & & & 0 & 1/5 & 0 \\ & 0 & & 1/4 & 0 & 0 \\ & & & 0 & 0 & 1/6 \end{array}\right)$$

$$C = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 2 \\ \hline 0 & 0 & 5 & 0 \\ 0 & 6 & 0 & 0 \end{array}\right) \qquad C^{-1} = \left(\begin{array}{c|cc|c} -2 & 0 & 0 & 1 \\ \hline 0 & 0 & 1/6 & 0 \\ 0 & 1/5 & 0 & 0 \\ \hline 3/\ldots & 0 & 0 & -1/2 \end{array}\right) \checkmark$$

$$C = \left(\begin{array}{c|cc|c} 0 & 0 & 5 & 0 \\ 0 & 6 & 0 & 0 \\ \hline 3 & 0 & 0 & 4 \end{array}\right) \qquad C^{-1} = \left(\begin{array}{c|cc|c} 0 & 0 & 1/6 & 0 \\ 0 & 1/5 & 0 & 0 \\ \hline 3/2 & 0 & 0 & -1/2 \end{array}\right) \checkmark$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

Other inverses

- Lower-triangular matrices

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & & 0 \\ 2 & 1 & \\ 3 & 4 & 1 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 0 \\ 3 & 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3/4 & 1 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} 1 & 0 \\ -3/4 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -3/8 & 1/4 \end{pmatrix}$$

- Upper-triangular matrices

$(A^T)^{-1} = (A^{-1})^T$ if A is invertible

Don't compute inverses!

- Slow
- Numerically unstable
- $(\text{sparse matrix})^{-1}$ can be dense

You can solve $A\vec{x} = \vec{b}$ without computing A^{-1}

and if you want to solve the same equations repeatedly

$$A\vec{x}_1 = \vec{b}_1,\quad A\vec{x}_2 = \vec{b}_2,\quad A\vec{x}_3 = \vec{b}_3, \ldots,$$

it is better to precompute the LU decomposition of A than to precompute A^{-1} (which might not exist)

Computing an inverse using Gaussian elimination:

Example: Computing inverse of $A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$

is equiv. to solving $AX = I$.

$$\left(\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-c} \left(\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} -a \\ -b \end{matrix}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & b & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

Observe:
If A is upper triangular, so is A^{-1}!
(A^{-1} exists $\Leftrightarrow$ all diagonal elements $\neq 0$)

Cramer's rule https://en.wikipedia.org/wiki/Cramer%27s_rule

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}^{T}$$

$\quad\quad\; \| $
$\quad\quad\; A$

This also works for $n \times n$ matrices, but is not useful.

Properties of the matrix inverse:

- $AA^{-1} = I = A^{-1}A$
  Equivalently, $(A^{-1})^{-1} = A$
- If it exists, then A^{-1} is unique.
  Proof:
  Say X and Y are both inverses of A.
  $$\begin{aligned} X = XI &= X(AY) \\ &= (XA)Y \\ &= IY \\ &= Y \quad \checkmark \end{aligned}$$

Singular-value decomposition

$$A = \sum_i \sigma_i \vec{u}_i \vec{v}_i^T = USV^T$$
$$\Rightarrow A^{-1} = \sum_i \frac{1}{\sigma_i} \vec{v}_i \vec{u}_i^T = VS^{-1}U^T$$

pseudoinverse: $A^{+} = \sum_{i:\sigma_i > 0} \frac{1}{\sigma_i} \vec{v}_i \vec{u}_i^T$
(defined for all matrices)

Spectral decomposition

If A is diagonalizable,
$$A = \sum_i \lambda_i \vec{v}_i \vec{v}_i^{\dagger}$$
$$A^{-1} = \sum_i \frac{1}{\lambda_i} \vec{v}_i \vec{v}_i^{\dagger} \quad \text{if all } \lambda_i \neq 0$$

Orthogonal and unitary

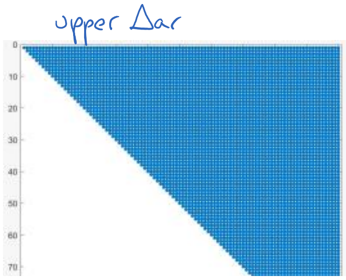
$A^{-1} = A^T \qquad\qquad A^{-1} = A^{\dagger}$

i.e., rows form orthonormal basis, as do columns

Matlab

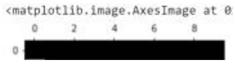
```
n = 100;
A = zeros(n);
for i = 1:n
  for j = i:n
    A(i,j) = rand();
  end;
end;
spy(inv(A));
```

upper Δar

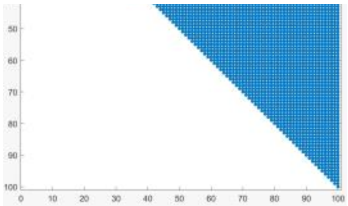


Python

```
import numpy as np

n = 10
A = np.zeros((n,n))
for i in range(n):
  for j in range(i,n):
    A[i,j] = np.random.rand()

import matplotlib.pyplot as plt
plt.spy(A)

<matplotlib.image.AxesImage at 0
```

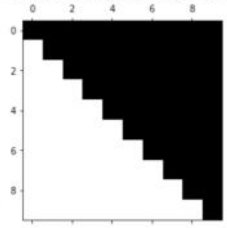


```
end;
spy(inv(A));
```

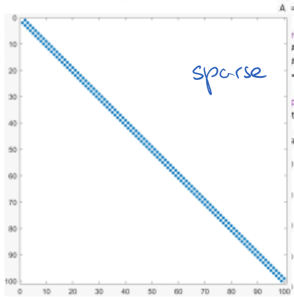


```
plt.spy(A)
```

<matplotlib.image.AxesImage at 0



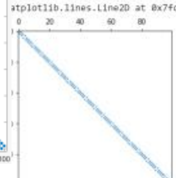
```
n = 100;
A = sparse(n,n);  % all-0 nxn sparse matrix
for i = 1:n-1
  A(i,i+1) = rand();
  A(i+1,i) = rand();
end;
spy(A);
```



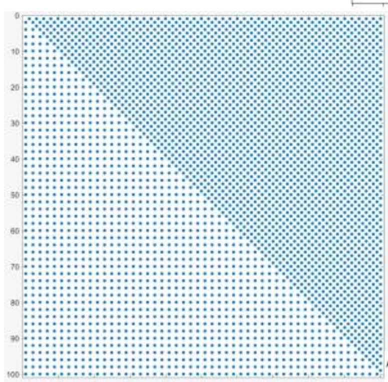
sparse

```
import numpy as np
import scipy.sparse

n = 100
A = scipy.sparse.lil_matrix((n,n))  # list-of-lists sparse format for
                                    #  incrementally building A
r i in range(n-1):
A[i,i+1] = np.random.rand()
A[i+1,i] = np.random.rand()
= A.tocsr()  # a faster sparse matrix format

port matplotlib.pyplot as plt
t.spy(A,MarkerSize=1)
```

atplotlib.lines.Line2D at 0x7fd848b4ab38>



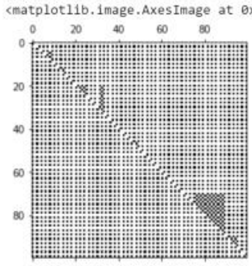
```
spy(inv(A));
```



dense ...

```
Ainv = np.linalg.inv(A.toarray())
# can also use scipy.sparse.linalg.inv(A) (but don't)
plt.spy(Ainv)
```

<matplotlib.image.AxesImage at 0x7fd848c06400>



```
Ainv[0:4,0:4]

array([[ 0.        ,  2.35909351,  0.        , -38.01358028],
       [ 2.03911397,  0.        ,  0.        ,   0.        ],
       [ 0.        ,  0.        ,  0.        ,  32.74581217],
       [ -1.84958998, -0.        ,  1.02259469, -0.        ]])
```

```
>> Ainvdense = full(inv(A));
>> Ainvdense(1:5,1:5)

ans =

         0    1.0941         0   -0.8991         0
   10.1924         0    0.0000         0   -0.0000
         0         0         0    1.0473         0
   -3.6747         0    2.3748         0    0.0000
         0         0         0         0         0
```

bipartite...