EE510 09/18/2020

Outline:

+ Subspaces of a matrix
+ Linear Independence, Basis, Dimension
+ Orthogonality.

$A \in \mathbb{R}^{m \times n}$

$R(A); R(A^T), N(A); N(A^T)$

$R(A) = \{ y \in \mathbb{R}^m \text{ such that } y = Ax, x \in \mathbb{R}^n \}$

= Span of columns of A

= All vectors y such that $Ax = y$ has a solution

$R(A^T)$; Row space

$R(A^T) = \{ y \in \mathbb{R}^n$ such that $y = A^T x ; x \in \mathbb{R}^m \}$

$=$ span of rows of A.

$N(A) = \{ X \in \mathbb{R}^n$ such that $AX = 0 \}$

$N(A^T) = \{ X \in \mathbb{R}^m$ such that $A^T X = 0 \}$.

How to find $R(A)$; $R(A^T)$, $N(A)$, $N(A^T)$?

$$A = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix}$$

$R(A) = \{ y = Ax \}$.

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & 3 & 5 & y_1 \\ 0 & 0 & \boxed{2} & 2 & y_2 - 2y_1 \\ 0 & 0 & -2 & -2 & y_3 - 3y_1 \end{array}\right) \to \text{Echelon form.}$$

x, z are pivots $\qquad Ux = c$

y, t free variables

$$R(A) = \text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}3\\8\\7\end{pmatrix}\right\}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 5 & y_1 \\ 0 & 0 & 2 & 2 & y_2 - 2y_1 \\ 0 & 0 & 0 & 0 & y_2 + y_3 - 5y_1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 4y_1 - \frac{3}{2}y_2 \\ 0 & 0 & 1 & 1 & \frac{y_2}{2} - y_1 \\ 0 & 0 & 0 & 0 & y_2 + y_3 - 5y_1 \end{array}\right)$$

$\Rightarrow$ Reduced form. $Rx = d$

$N(A) = N(R)$

$$\begin{cases} x + 2y + 2\cancel{t}\,z = 0 \\ z + t = 0 \end{cases}$$

$y = 1,\ t = 0$; $z = 0,\ x = -2 \rightarrow \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$y = 0,\ t = 1$; $z = -1,\ x = -2 \rightarrow \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

$N(A) = \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$

Exercise: Let $A \in \mathbb{R}^{n \times n}$. Prove that the following statements are equivalents.

1 - $N(A) \cap R(A) = \{0_{\mathbb{R}^n}\}$

2 - $N(A^2) = N(A)$.

1/- implies 2/ : $N(A) \cap R(A) = \{0_{\mathbb{R}^n}\}$.

$N(A) \subset N(A^2)$ $\left(\begin{array}{l} X \in N(A) \Rightarrow AX = 0 \\ A^2X = 0 \Rightarrow X \in N(A^2). \end{array} \right.$

Let $X \in N(A^2) \Rightarrow A^2X = 0$

$A(AX) = 0 \Rightarrow \underline{AX \in N(A)}$ (1)

$\underbrace{AX \in R(A)}_{(2)} \Rightarrow$ ~~A~~

(1) and (2) $\Rightarrow AX \in N(A) \cap R(A) = \{0_{\mathbb{R}^n}\}$.

$\forall X \in N(A^2) \Rightarrow AX \in$ ~~$\mathbb{R}^n$~~ $N(A) \cap R(A) = \{0_{\mathbb{R}^n}\}$.

$\Rightarrow AX = 0 \Rightarrow X \in N(A)$

$N(A^2) \subset N(A)$

$\Rightarrow N(A) = N(A^2)$

2°- implies 1°- $N(A^2) = N(A)$

$\{0\} \subset N(A) \cap R(A)$ ($R(A)$ and $N(A)$ are vector spaces)

Let $X \in N(A) \cap R(A)$

$X \in N(A) \Rightarrow AX = 0$

$X \in R(A) \Rightarrow \exists Y \in \mathbb{R}^n / X = AY$

$A(AY) = 0 \Rightarrow A^2 Y = 0 \Rightarrow Y \in N(A^2) = N(A)$

$\Rightarrow Y \in N(A) \Rightarrow \underbrace{AY}_{X} = 0 \Rightarrow X = 0$

$\Rightarrow N(A) \cap R(A) \subset \{0_{\mathbb{R}^n}\}$

$\Rightarrow N(A) \cap R(A) = \{0_{\mathbb{R}^n}\}$.

Linear Independence:

- A set of vectors $\{v_1, \dots, v_n\} \subset V$ is linearly independent if $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$

$$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ | \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ | \\ 0 \end{bmatrix} \Rightarrow N(A) = 0$$

- We say $v_1, \dots, v_n$ span a vector space V, if any vector $x \in V$ can be expressed as

$$x = \sum_{i=1}^{n} \alpha_i v_i$$

- Given a vector space $(V, +, \times)$ over $\mathbb{F}$, a basis for V is the set of vectors that <u>span</u> V and <u>linearly independent</u>

$B = \{v_1, \ldots, v_n\}$ is a basis for V

$\Rightarrow \begin{cases} \{v_1, \ldots, v_n\} \text{ spans } V \\ \{v_1, \ldots, v_n\} \text{ linearly independent} \end{cases}$

+ If two sets of vectors are basis for V, then they have the same number of vectors

$\Rightarrow$ The number of vector in a basis for the vector space V is called the dimension V, and noted as $\dim(V)$. (degree of freedom)

Exercise:

$$S_1 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} ; \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} ; \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

Is S_1 a basis for $\mathbb{R}^3$?

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x = y = z = 0$

$\Rightarrow$ The set S_1 is linearly indepent

$$\cdot \left\{ \begin{array}{l} V_1 \subset V \\ \dim V_1 = \dim V \end{array} \right. \Rightarrow V_1 = V$$

$$\left\{ \begin{array}{l} \operatorname{Span}\{S_1\} \in \mathbb{R}^3 \\ \dim(\operatorname{Span}\{S_1\}) = 3 \end{array} \right. \Rightarrow \operatorname{Span} S_1 = \mathbb{R}^3$$

<u>Exercise:</u> Let V a vector space of $\mathbb{R}^4$ spanned by: $U_1 = \begin{pmatrix} 1 \\ -2 \\ 5 \\ -3 \end{pmatrix}$; $U_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ -4 \end{pmatrix}$; $U_3 = \begin{pmatrix} 3 \\ 8 \\ -3 \\ -5 \end{pmatrix}$

Find a basis and the dimension of V.

$$\begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array}\begin{pmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{pmatrix}\begin{array}{l} U_1 \\ U_2 - 2U_1 \\ U_3 - 3U_1 \end{array}$$

$$\begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{array}{l} U_1 \\ U_2 - 2U_1 \\ U_3 - 2U_2 + U_1 \end{array}$$

$U_1 + U_3 - 2U_2 = 0 \Rightarrow \{U_1, U_2, U_3\}$ is linearly dependent

$V = \mathrm{Span}\{U_1, U_2, U_3\} = \mathrm{Span}\{U_1, U_2 - 2U_1\}$

$-2U_1 + U_2 = \begin{pmatrix} 0 \\ 7 \\ -9 \\ 2 \end{pmatrix}$; $\alpha \begin{pmatrix} 1 \\ -2 \\ 5 \\ -3 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 7 \\ -9 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}$

U_1 and $U_2 - 2U_1$ are linealy independents

$\Rightarrow B = \{U_1, U_2 - 2U_1\}$ is a basis for V

$\dim V = 2$.

Exercise: 1) Prove that the set

$$S = \left\{ \begin{pmatrix} x+y & -x+3y & y \\ y & x & -2y \end{pmatrix} ; x, y \in \mathbb{R}^2 \right\}$$

is a vector sub-space of the vector space $\mathbb{R}^{2\times 3}$.

2) Find a basis B for S / dimension of S

3) Is B a basis for $\mathbb{R}^{2\times 3}$.

$$S = \left\{ \begin{pmatrix} x+y & -x+3y & y \\ y & x & -2y \end{pmatrix} ; (x, y) \in \mathbb{R}^2 \right\}$$

$$= \left\{ x \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} + y \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & -2 \end{pmatrix} ; (y, x) \in \mathbb{R}^2 \right\}$$

$= \{ x \cdot A_1 + y A_2 \; ; (x, y) \in \mathbb{R}^2 \} ; \; A_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$A_2 = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 0 & -2 \end{pmatrix}$

$= \text{Span}\{A_1, A_2\}$.

$\Rightarrow$ S is a vector subspace of $\mathbb{R}^{2\times 3}$.

2°) $x A_1 + y A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

$\Rightarrow$ A_1 and A_2 are linealy indepenbent

$\Rightarrow B = \{A_1, A_2\}$ is a basis for S.

dim S = 2.

3°) No, $\dim \mathbb{R}^{2\times 3} = 6$.

# of B = 2 $\Rightarrow$ B cannot be a basis for $\mathbb{R}^{2\times 3}$.