HW7, pr. 4:

Part a: $f(x) = (1+x)^p - 1 - x^p$ ← poly of deg ≤ p-1

↓ $\mathbb{F}_p[x]$ mod p.

Know: $\boxed{a^p \equiv a \pmod p}$ for every $a = 0, 1, \ldots, p-1$

$\Rightarrow f(a) = (1+a)^p - 1 - a^p \equiv 1+a-1-a \equiv 0 \pmod p$

$\Rightarrow a = 0, 1, \ldots, p-1$ → roots of $f(x)$ in $\mathbb{F}_p[x]$

p roots, deg ≤ p-1 $\Rightarrow$ $f(x) \equiv 0$ in $\mathbb{F}_p[x]$

$\Rightarrow (1+x)^p - 1 - x^p = \sum_{k=1}^{p-1} \binom{p}{k} x^k \equiv 0 \pmod p$ iff

$\Longleftrightarrow \binom{p}{k} \equiv 0 \pmod p$.

$(1+x)^{2p} = \left((1+x)^p\right)^2 = (1+x^p)^2 = 1 + 2x^p + x^{2p}$

in $\mathbb{F}_p[x]$, $(1+x)^p = 1+x^p$

$\binom{2p}{k} \equiv \begin{cases} 2 & k = p \\ 1 & k = 0, 2p \\ 0 & \text{o.w.} \end{cases}$

$\binom{ap}{bp} \overset{?}{\equiv} ? \pmod p$

$x(x+2)(x^2-1) - 24$

$(x-1)x(x+1)(x+2) = 1 \cdot 2 \cdot 3 \cdot 4$.

$x=2$: 1 2 3 4

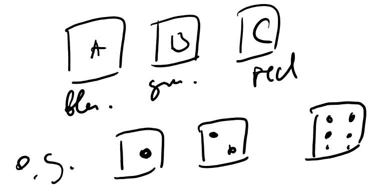
$x=-3$: -4 -3 -2 -1

Probability

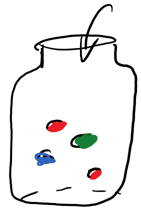
Side problems:

... red. What is the prob. we get

① 3 dice blue, green, red. What is the prob. we get $A < B < C$?



$1 < 2 < 6$ ✓

② 

10 red

20 blue

30 green

What is the prob to eat all the red ones before blue and green ?

③

H

1

toss ... e.g. HHGT we get H H H T H H H H H H in a row. $\mathbb{E}$[# of tosses until HHH].

toss until THH $\mathbb{E}$[# of tosses until THH]..

Coin toss game: Player 1 picks a sequence of 3 outcomes. e.g. THT

Then 2 picks another sequence e.g. HTH.

Start tossing until one sequence shows up

H T T H T ... → Player 1 wins.

Q: What is a winning strategy and should you be player 1 or 2 ?

④ ① ② ... ⓝ

Each turn → pick two balls at random ⓘ ⓙ → ⓘ ⓘ ↘ put them back

Stop when all have same color

$\mathbb{E}$[# of turns] = ?

$$\mathbb{P}(A) = \frac{\#\text{ outcomes in } A}{\text{total } \#\text{ outcomes}}$$ / if each is equally likely

T / H → 1/2

$$\mathbb{P}(\text{see HHH}) = \frac{1(HHHH) + 1(THHH) + 1(HHHT)}{2 \cdot 2 \cdot 2 \cdot 2} = \frac{3}{16}$$

H	T	H	T
1	2	3	4

1 2 3 4

biased coin: $H \to p$, $T \to 1-p$

$P(\text{see } HHH) = \Pr(THHH) + \Pr(HHHH) + \Pr(HHHT)$

1 2 3 4

$(1-p)\cdot p\cdot p\cdot p + p\cdot p\cdot p\cdot p + p\cdot p\cdot p\cdot(1-p)$

$= p^3(2-p)$

Conditional probability: $P(A|B) := \dfrac{P(A\cap B)}{P(B)}$

"event A given event B"

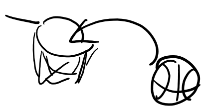
$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots + P(A|B_k)P(B_k)$

$B_1 \cup B_2 \cup \dots \cup B_k \to$ all possible outcomes

Bayes rule $= \dfrac{P(B|A)\cdot P(A)}{P(B)}$

Putnam: 2002, B-1

Shaquille O'Neal



1st: hit: H

2nd: miss: M

throws: 1 2 ... n

$\Pr(\text{throw } n+1 \text{ is a hit}) = \dfrac{\#\text{ hits first } n \text{ throws}}{n}$

$\Pr(\text{3rd throw is a hit}) = 1/2$

HHH 2/3, 1/2

$\Pr(\text{4th throw is a hit}) = \Pr(\text{4th throw is a hit} \,/\, \text{3rd throw is a hit})\Pr(\text{3rd throw is a hit})$

$+ \Pr(\text{4th throw is a hit} \,/\, \text{3rd throw is a miss})\Pr(\text{3rd is a miss})$

HMM 1/2

1/3

$= \tfrac{2}{3}\cdot\tfrac{1}{2} + \tfrac{1}{3}\cdot\tfrac{1}{2} = \tfrac{1}{2}$

Problem: Prob(she has exactly 50 hits after 100 throws) = ?

$P_n(i) = \text{Prob}(i \text{ hits from } n \text{ total shots})$

$P_{n+1}(i) = \Pr\left(\begin{matrix}n+1 \text{ 1st shot is } H \\ i \text{ hits total}\end{matrix}\right) + \Pr\left(\begin{matrix}n+1 \text{ 1st shot is } M \\ i \text{ hits total}\end{matrix}\right)$

$P_n(i-1)$

$$p_{n+1}(i) = Pr\left(\frac{n+1 \text{ 'st shot is } H}{i \text{ hits total}}\right) \cdot \qquad \underbrace{\quad i \text{ hits total}}_{p_n(i-1)}$$

$$Pr(\underbrace{\text{last shot is } H}_{A} \mid \underbrace{i-1 \text{ hits in } n \text{ shots}}_{B}) \, Pr(i-1 \text{ hits in } n \text{ shots})$$

$$+ \; Pr(\text{last shot is } M \mid i \text{ hits in } n \text{ shot}) \, \underbrace{Pr(i \text{ hits in } n \text{ shots})}_{p_n(i)}$$

$$\frac{i-1}{n} \qquad \left(1-\frac{i}{n}\right)$$

$$p_{n+1}(i) = \frac{i-1}{n}\, p_n(i-1) + \frac{n-i}{n} \cdot p_n(i) \quad \text{for every } i \in \{1, 2, \dots, n\}$$

$$p_3(i) = \frac{i-1}{2} \cdot \underbrace{p_2(i-1)} + \dots \to 1/2$$

$$p_3(1) = 1/2, \quad p_3(2) = 1/2$$

$$p_4(1) = \frac{3-1}{3} \cdot p_3(1) = 1/3, \quad p_4(2) = 1/3, \; p_4(3) = 1/3.$$

Guess: $p_n(i) = \frac{1}{n-1}$ ✓

Induction:

$$p_{n+1}(i) = \frac{i-1}{n} \cdot \frac{1}{n-1} + \frac{n-i}{n} \cdot \frac{1}{n-1} =$$

$$= \frac{1}{\cancel{n-1}} \cdot \left(\frac{\cancel{n-1}}{n}\right) = 1/n.$$

Answer: $\to Pr(\underbrace{50}_{i} \text{ hits in } \underbrace{100}_{n} \text{ shots}) = \frac{1}{99}$.

1000 coins $\quad$ 1 coin is $\underline{H/H}$, rest $\to H/T$

You pick a coin $\to$ toss 10 times $\Rightarrow \underbrace{HHH\dots H}_{10}$.

$Pr(\text{you picked the HH coin}) = ?$

Event: $A \to$ chosen coin is the HH coin. $\qquad P(A \mid B) = ?$

$Pr\left(\text{fair coin} \to HH\dots H\right)$

$\to \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} = \frac{1}{2^{10}}$

event: $A \to$ chosen coin is the ~~HH~~ HH coin. $P(A|B) = ?$

event: $B \to$ 10 tosses $\to$ HH...H

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\overbrace{P(B|A)}^{=1}\,\overbrace{P(A)}^{1/1000}}{P(B)}$$

$$P(B) = \underbrace{P(B|A)}_{1}\,\underbrace{P(A)}_{\frac{1}{1000}} + \underbrace{P(B|\bar{A})}_{\frac{1}{1024}}\,\underbrace{P(\bar{A})}_{\left(1-\frac{1}{1000}\right)} = \frac{1024+999}{1024 \cdot 1000}$$

$$\to P(A|B) = \frac{\frac{1}{\cancel{1000}}}{1024+999} \cdot 1024 \cdot \cancel{1000} = \frac{1024}{2023}$$