Math 395: problem set 6

Due Mon 03/08 by 6pm, uploaded in blackboard

Instructions: Solve (with proof) and submit 2 of the following problems, arranged in order of increasing difficulty.

Notation: $[n] := \{1, \ldots, n\}$, the set of the first n positive integers.

$\#\{...\}$ denotes "the number of elements in the given set. Note that in many problems the set consists of sets.

Useful facts:

Euler's totient function $\phi(n) = p_1^{\alpha_1 - 1}(p_1 - 1) \cdots p_k^{\alpha_k - 1}(p_k - 1)$ for $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ in prime decomposition.

Euler's (generalized Fermat's theorem): if $gcd(a, n) = 1$ then $a^{\phi}(n) \equiv 1 \pmod{n}$.

Chinese remainder theorem:

Let $n_1, n_2, \ldots, n_k$ be integers, such that $gcd(n_i, n_j) = 1$ if $i \neq j$, and let $b_1, \ldots, b_k$ be arbitrary other integers. Then there exists an integer x, such that

$$x \equiv b_i \pmod{n_i}, i = 1, \ldots, k.$$

General hint: not all of the problems below will involve these theorems, and some are quite hard (towards the end).

1. Given some integer n, let $p_1, \ldots, p_n$ be the first n prime numbers. Do there exist n consecutive numbers $m, m+1, \ldots, m+n-1$, such that $m + i - 1$ is NOT divisible by p_i for every $i = 1, \ldots, n$.

2. Does there exist a positive integer k, such that $k2^n + 1$ is not a prime number for every $n = 1, 2, \ldots$.

3. A point $(x, y) \in \mathbb{Z}^2$ is *visible* iff $gcd(x, y) = 1$ (see our lecture for context). Let $n \in \mathbb{N}$. Prove that there exists a $2n + 1 \times 2n + 1$ square in $\mathbb{Z}^2$, such that no integer point inside it is visible.

4. Find all integers x, y, n, m, such that $gcd(n, m) = 1$ and
$$(x^2 + y^2)^n = (xy)^m.$$

5. Let p be a prime number, and $0 < a \leq b \leq c \leq d \leq p - 2$ be integers, such that
$$pa - (p-1)2^a + pd - (p-1)2^d \equiv pb - (p-1)2^b + pc - (p-1)2^c \pmod{p(p-1)}$$
Show that $a = b$ and $c = d$.

6. Let $a, b \in \mathbb{Z}$ be two positive integers, such that $ab + 1 | a^2 + b^2$. Show that
$$\frac{a^2 + b^2}{ab + 1} = c^2$$
for some $c \in \mathbb{Z}$.

7. For the Putnam trainees: find (and solve, or ask me about) an interesting (and hard) problem from the Putnam (or similar) which fits the current topic. As sources you can use the Putnam Archive of Kiran Kedlaya, or IMO, or "AoPS", or any of the sources listed in the syllabus.