Math 395: problem set 4

Due Mon 03/01 by 6pm, uploaded in blackboard

Instructions: Solve (with proof) and submit 2 of the following problems, arranged in order of increasing difficulty.

Notation: $[n] := \{1, \dots, n\}$, the set of the first n positive integers.

$\#\{...\}$ denotes "the number of elements in the given set. Note that in many problems the set consists of sets.

Useful facts:

Euler's totient function $\phi(n) = p_1^{\alpha_1 - 1}(p_1 - 1) \cdots p_k^{\alpha_k - 1}(p_k - 1)$ for $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ in prime decomposition.

Euler's (generalized Fermat's theorem): if $gcd(a, n) = 1$ then $a^{\phi}(n) \equiv 1 \pmod{n}$.

1. Prove that $(28)^{70} + (16)^{70}$ is divisible by 13.
2. Show that if $2^n + n^2$ is a prime number then $n \equiv 3 \pmod 6$.
3. Prove that n does not divide $2^n + 2$ for any $n > 1$ – odd.
4. Prove that $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not a prime number.
5. Prove that $x^2 \equiv -1 \pmod p$ (for p-prime) has a solution if and only if $p \equiv 1 \pmod 4$.
6. Prove that there are infinitely many prime numbers $\equiv 3 \pmod 4$.

   Harder: prove that there are infinitely many prime numbers $\equiv 1 \pmod 4$.
7. For the Putnam trainees: find (and solve, or ask me about) an interesting (and hard) problem from the Putnam (or similar) which fits the current topic. As sources you can use the Putnam Archive of Kiran Kedlaya, or IMO, or "AoPS", or any of the sources listed in the syllabus.