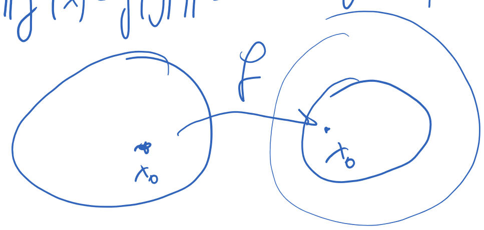
Monday, April 12, 2021 6:21 PM

Q: Classic problem 5 $a_{n+1} = a_n + e^{-a_n}$, does $a_n - \ln(n)$ converge?

Fixed points:

$f: X \to X$, $X \subset \mathbb{R}^n$ contraction map

$\|f(x) - f(y)\| \le c\|x-y\|$, $c<1$. Then f has a unique fixed point.



If $f(x_0) = x_0$, $f(y_0) = y_0$

$\|f(x_0) - f(y_0)\| \le c\|x_0 - y_0\| < \|x_0 - y_0\|$

$\| = \|x_0 - y_0\|$ contradiction.

x_1, $x_2 = f(x_1)$, $x_{n+1} = f(x_n)$

$\|x_{n+1} - x_n\| = \|f(x_n) - f(x_{n-1})\| \le c\|x_n - x_{n-1}\| \le c^2\|x_{n-1} - x_{n-2}\| \le \dots$

$\le c^{n-1}\|x_2 - x_1\|$

$\|x_{n+p} - x_n\| \le \|x_{n+p} - x_{n+p-1}\| + \|x_{n+p-1} - x_{n+p-2}\| + \dots + \|x_{n+1} - x_n\| \le$

for every p $\le c^{n+p-2}\|x_2 - x_1\| + \dots + c^{n-1}\|x_2 - x_1\| =$

$= c^{n-1}\left(\underbrace{1 + \dots + c^{p-1}}_{\frac{1-c^p}{1-c}}\right)\|x_2 - x_1\| = \dfrac{c^{n-1}(1-c^p)\|x_2-x_1\|}{1-c} \to 0$, <1

Let $\varepsilon > 0$, let $n(\varepsilon)$: $c^{n(\varepsilon)-1} < \dfrac{(1-c)}{\|x_2 - x_1\|}\cdot\varepsilon$

$\Rightarrow \|x_{n+p} - x_n\| < \varepsilon$ for every $n > n(\varepsilon)$, $\forall p$.

$\Rightarrow$ Cauchy's criterion $\to$ $x_n \to x_0$ as $n \to \infty$

Cauchy.
$\|x_m - x_n\| < \varepsilon$
for every $m, n > n(\varepsilon)$
$\Rightarrow x_n \to$ converges.

$\Rightarrow$ $x_{n+1} = f(x_n)$ f continuous

$\downarrow \quad \downarrow \quad n \to \infty$

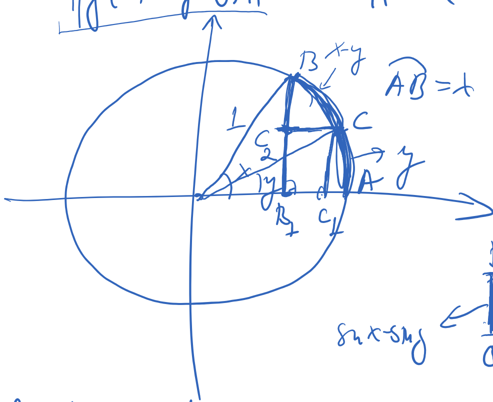
$x_0 = f(x_0)$

$x \in \mathbb{R}$

Example: let $t \in \mathbb{R}$, $c \in \mathbb{R}$: solve $x - c\cdot\sin(x) = t$ $\to$ has a unique solution in x.

$|c|<1$ $\quad x = \underbrace{c\cdot\sin(x)+t}_{f(x)} \Leftarrow$ f is continuous, $f:\mathbb{R}\to\mathbb{R}$

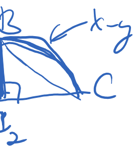
$$\|f(x)-f(y)\| = \|c(\sin(x)-\sin(y))\| = c\|\underbrace{\sin x-\sin y}_{\le |x-y|}\|$$



$\overset{\frown}{AB} = x$

$BB_1 = \sin x$, $CC_1 = \sin y$.

$BC_2 = \sin x - \sin y$



$BC_2 < BC < \text{arc } BC = x-y$

$\Rightarrow$ apply the fixed pt theorem $\to$ unique $x_0: f(x_0) = x_0$

$x_0 = c\sin(x_0)+t$

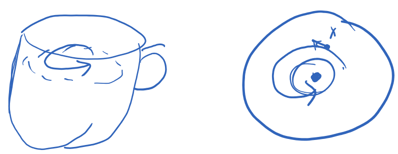
Brouwer fixed point theorem:

$f: K\to K$, for any $x\neq y$: $|f(x)-f(y)| < c|x-y|$ $\quad \boxed{c=1}$

closed compact subset of $\mathbb{R}^n$

$\Rightarrow$ f has a unique fixed point

$f(x) = \overrightarrow{x+\delta}$



Functions (cont. or not)

Ex. $\lim\limits_{x\to 0}\left(\dfrac{a_1^x+\dots+a_n^x}{n}\right)^{1/x} = \sqrt[n]{a_1 a_2\dots a_n}$ ← geometric mean

n - fixed, $a_1,\dots,a_n\in\mathbb{R}_{>0}$

try: $\left(\dfrac{a^x+1}{2}\right)^{1/x} \Rightarrow \sqrt{a}$

$\lim\limits_{x\to0}\dfrac{a^x-1}{x} \xrightarrow{\text{L'Hospital's rule}} \lim\dfrac{(a^x-1)'}{(x)'} = \dfrac{e^{\ln(a)x}\cdot\ln(a)}{1} = \ln(a)$ $\qquad a^x = e^{\ln(a)\cdot x}$

$a^x = 1+t$, $x = \dfrac{\ln(1+t)}{\ln(a)}$

$\to$ $\dfrac{t}{\ln(1+t)}\cdot\ln(a)$ $\qquad \dfrac{1}{\lim\limits_{t\to0}\ln\left(1+t\right)^{1/t}} \to 1$ $\qquad \lim\limits_{n\to\infty}\left(1+1/n\right)^n \Rightarrow e$

$f(x)x$ $\quad 1/x$

$\left(\ln(1+t)\right) \to 1$, $t\to 0$

$$\left(\frac{a_1^x - 1 + a_2^x - 1 \ldots + a_n^x - 1}{n} + 1\right)^{1/x}$$

$f(x)x$

$\ln(a_1)\cdot x$

$$\frac{\ln(a_1)\cdot x + \ldots + \ln(a_n)x}{n} + 1 = \left(\left(\frac{\ln(a_1 \ldots a_n)}{n}\cdot x + 1\right)^{1/(xA)}\right)^A$$

A

$$\left(\left(Ax+1\right)^{\frac{1}{xA}}\right)^A = e^A$$

$\downarrow e$

$$= e^{\frac{\ln(a_1 \ldots a_n)}{n}} = (a_1 \ldots a_n)^{1/n}$$

$$\left(f(x)\cdot x + 1\right)^{1/x} \to e^A \qquad f(x) \to \frac{\ln(a_1 \ldots a_n)}{n} = A \text{ as } x\to 0$$

for every $\varepsilon > 0$

$\left|\left(f(x)\cdot x+1\right)^{1/x} - e^A\right| < \varepsilon$ for all $x < \delta$

$x < \delta$? $\to$ $|f(x) - A| < \delta_1$

$$\underbrace{\left(f(x)\cdot x+1\right)^{1/x} - (Ax+1)^{1/x}}_{\wedge ?} + \underbrace{(Ax+1)^{1/x} - e^A}_{\wedge}$$

Ex. $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} \frac{f(x) - f(ax)}{x} = 0 \Rightarrow$ Prove that $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

$a \in (0,1) \to$ fixed.

Continuity g: for every $\varepsilon > 0$, $\exists\, \delta$: if $x \in (-\delta, \delta) \Rightarrow |g(x) - g(0)| < \varepsilon$

Let $g(x) = \frac{f(x) - f(ax)}{x} \Rightarrow \boxed{|f(y) - f(ay)| < \varepsilon \cdot y}$ $\forall y \in (-\delta, \delta)$

εa $\quad y = ax \in (-\delta, \delta)$

$$|f(x) - f(a^n x)| < |f(x) - f(ax)| + |f(ax) - f(a^2x)| + \ldots + |f(a^{n-1}x) - f(a^n x)|$$

$\wedge\ \varepsilon x$ $\quad \wedge\ \varepsilon \cdot a x$ $\quad \wedge\ \varepsilon a^{n-1} x =$

$$= \varepsilon x \cdot \frac{(1 + \cdots + a^{n-1})}{\frac{1-a^n}{1-a}}$$

x fixed. $\quad |f(x) - f(a^n x)| < \varepsilon x \frac{1-a^n}{1-a}$ $\quad a^n \to 0$, $f(a^n x) \to 0$

let $n \to \infty$ $\quad \to 0 \quad \Rightarrow \quad |f(x)| \le \frac{\varepsilon x}{1-a}$

$$\Rightarrow \left|\frac{f(x)}{x}\right| \le \frac{\varepsilon}{1-a} \quad \text{for } x \in (-\delta, \delta)$$

$$\Rightarrow \left|\frac{f(x)}{x}\right| \to 0 \quad \text{as } x \to 0$$

Ex. $\quad f(f(x)) = \frac{x^2}{(x^2+1)(x^4+1)}$, $\quad$ show that it has a unique fixed point.

$f: \mathbb{R} \to \mathbb{R}$

$$f(x_0) = x_0 \Rightarrow f(f(x_0)) = f(x_0) = x_0$$

$$\frac{x_0^2}{(x_0^2+1)(x_0^4+1)} = x_0 \iff x_0^2 = x_0(x_0^2+1)(x_0^4+1)$$

$$\Rightarrow 0 = x_0(\underbrace{x_0^4 + x_0^2 + 1}_{>0}) \Rightarrow \boxed{x_0 = 0}$$

Show that $x = 0$ is a fixed point.

$$f(0) = b \Rightarrow f(\underbrace{f(0)}_{b}) = \frac{0^2}{\cdots} = 0 \Rightarrow f(b) = 0$$

$$f(\underbrace{f(b)}_{0}) = \frac{b^2}{(b^2+1)(b^4+1)}$$

$$\underline{b = f(0)} \quad \Rightarrow b = \frac{b^2}{(b^2+1)(b^4+1)} \Rightarrow \boxed{b = 0}$$

same.