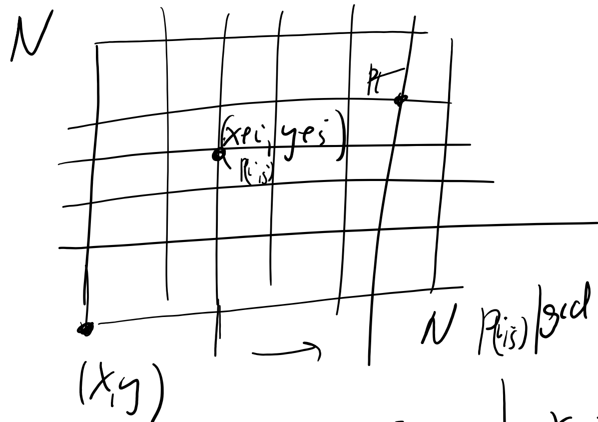


List primes: $p_1, p_2, \dots, p_{n^2}$

$p_{(i,j)}$

$p_{(i,j)} \leftrightarrow$ Primes

N $\quad p_{(i,j)} \mid \gcd(x+i, y+j)$

$i = 0 \ldots N \quad \Big| \quad x \equiv -i \pmod{p_{(i,0)} \cdots p_{(i,N)}}$

$j = 0 \ldots N \quad \Big| \quad y \equiv -j \pmod{p_{(0,j)} \cdots p_{(N,j)}}$

Polynomials: $\quad p(\cancel{x}) \in K[x]$

field: $K = \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{F}_p$

or rings: $\mathbb{Z}$.

$$p(x) = c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_n, \quad c_i \in K.$$

Division with remainder: $F(x)$ divided by $G(x)$:

$$F(x) = G(x)Q(x) + R(x), \quad \deg R(x) < \deg G(x)$$

$\underbrace{x^3}_{F}$ by $\underbrace{(x+1)^2}_{G} \to \underbrace{x^2+2x+1}_{F}\overline{)\,x^3}$

$\cancel{x}$

$-2x^2 - x$

$+2x^2 + 4x + 2$

$\dots$

$\gcd(F, G) = D(x)$: $\quad F(x) = F_1(x) D(x)$

$G(x) = G_1(x) D(x)$, $\quad \underline{\gcd(F_1(x), G_1(x)) = 1}$

$\gcd(F, G) = D(x)$: $\quad F(x) = F_1(x)\, D(x)$, $\quad G(x) = G_1(x)\, D(x)$ $\mid$ $\gcd(F_1(x), G_1(x)) = 1$

$$\Downarrow$$

$$1 = F_1(x)A(x) + G_1(x)B(x).$$

$F = x^2 - 3x + 2$

$G = (x-1)^3$

$D(x) = (x-1)$

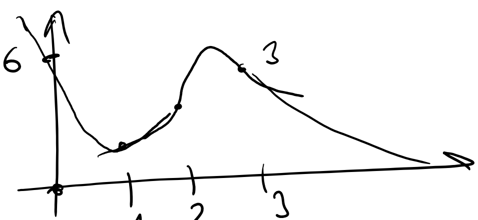
$F_1 = (x-2)$, $G_1 = (x-1)^2$

$$1 = -(x-2)\cdot\underbrace{x}_{A(x)} + (x-1)^2 \cdot \underbrace{1}_{B(x)} \quad ✓$$

Roots: $F(a) = 0 \to$ root.

$\Rightarrow F(x) = (x-a)\cdot F_1(x) \in K[x]$. (also over $\mathbb{Z}$).

Problem 1: $\deg F(x) = 3$, $F(1) = 1$, $F(2) = 2$, $F(3) = 3$, $F(0) = 6$.



$G(x) = F(x) - x$

$G(1) = 0$

$G(2) = 0$

$G(3) = 0$

$\deg G = 3$

$$\Downarrow$$

$G(x) = \overset{\text{const.}}{c}(x-1)(x-2)(x-3) \xrightarrow{x=0} c(-1)(-2)(-3) = -6c$

$G(0) = F(0) - 0 = 6$ $\quad \Rightarrow \boxed{c = -1}$

$\to G(x) = -(x-1)(x-2)(x-3) \Rightarrow F(x) = G(x) + x$

$= \boxed{-(x-1)(x-2)(x-3) + x}$.

Problem 2: $a < b < c < d \in \mathbb{Z}$

$F \in \mathbb{Z}[x]$, $F(a) = F(b) = F(c) = F(d) = 5$.

is there $k \in \mathbb{Z}$, s.t. $\boxed{F(k) = 8}$?

is there such F & k, s.t. $F(k) = 9$

$G(x) = F(x) - 5 \to$ roots a, b, c, d

$\Rightarrow G(x) = (x-a)(x-b)(x-c)(x-d)\cdot G_1(x) \in \mathbb{Z}[x]$ ...

$\Rightarrow \quad G(x) = (x-a)(x-b)(x-c)(x-d)\cdot G_1(x)$

$\underline{k}$ $\quad F(k)-5 = G(k) = (k-a)(k-b)(k-c)(k-d)\, G_1(k)$

$\boxed{3}$

at most one is ± 3, $\{\pm 1\}$ $\leftarrow$ at least 3 of $k-a$, $k-b$, ...

e.g. $k-\underline{a} = k-\underline{b} \to$ no since

$a < b < c < d$

$k-a > k-b > k-c > k-d.$

$$9-5 = 4 = \underset{+2}{(k-a)}\ \underset{+1}{(k-b)}\ \underset{-1}{(k-c)}\ \underset{-2}{(k-d)}\ \underbrace{G_1(k)}_{1}$$

$\Downarrow$

$a = b-1 = c-3 = d-4$

$\to F(x) = (x-a)(x-a+1)(x-a+3)(x-a+4)+5.$

$\underline{\text{Pr.}}$ ~~Given N~~. Given $F(x) \in K[x]$, show that $\exists\, G(x)$

$F(x) \mid G(x)$, $\quad G(x) = c_n\, x^{n\cdot N} + c_{n-1}\, x^{(n-1)N} + \dots$

$\underline{\text{ex.}}$ $F(x) = x-a \ \overset{?}{\mid}\ x^N + c$?

$$\frac{x^N - a^N}{x-a} = x^{N-1} + x^{N-2}a + x^{N-3}a^2 + \dots + a^{N-1}.$$

$x^2 - a^2 = (x-a)(x+a)$

$x^3 - a^3 = (x-a)(x^2+ax+a^2)$

$\Rightarrow \quad x-a \mid x^N - a^N \quad \Rightarrow \quad G(x) = x^N - a^N x^0$

$\boxed{\mathbb{C}}$ Fundamental theorem of algebra:

$F(x) = c(x-z_1)^{m_1}(x-z_2)^{m_2}(x-z_3)^{m_3}\dots(x-z_k)^{m_k}$, $z_i \in \mathbb{C}$.

over $\mathbb{C}$

$F(x) \mid c\,(x^N - z_1^N)^{m_1}(x^N - z_2^N)^{m_2}\dots(x^N - z_k^N)^{m_k}$

Pr. $f \in \mathbb{R}[x] \to f(x) = c_0 x^n + \dots + c_n$, $c_i \in \mathbb{R}$.

$f \to$ has all real roots $\iff f^2 \neq g^2 + h^2$, $g, h \in \mathbb{R}[x]$, $\deg g \neq \deg h$

1) $\Rightarrow$, $f = c(x-a_1)^{n_1} \dots (x-a_k)^{n_k}$ $a_i \in \mathbb{R}$, suppose $f^2 = g^2 + h^2$

$0 = f(a_i) = \underbrace{(g(a_i))^2}_{\in\mathbb{R}} + \underbrace{(h(a_i))^2}_{\in\mathbb{R}} > 0$ unless $g(a_i) = 0$, $h(a_i) = 0$.

$\Rightarrow g(x) = (x-a_1)^{r_1}(x-a_2)^{r_2} \dots (x-a_k)^{r_k} \cdot g_1(x)$ $\gcd(g_1(x), h_1(x)) = 1$.
$h(x) = (\quad)^{r_1} \dots$ $h_1(x)$

$\Rightarrow c(x-a_1)^{2(n_1-r_1)} \dots (x-a_k)^{2(n_k-r_k)} = g_1(x)^2 + h_1(x)^2$

suppose that a const $\Rightarrow$, $\exists i$: $n_i - r_i > 0$ $\Rightarrow x - a_i$ $\to 0 = g_1(a_i)^2 + h_1(a_i)^2$

$\Rightarrow g_1(a_i) = h_1(a_i) = 0 \to (x-a_i) \mid g_1(x), h_1(x)$

$\Rightarrow$ contradiction with $\gcd = 1$.

$\Rightarrow g_1, h_1 =$ constants. $\to g, h \to$ same degree $\to$ contrad.

$\Leftarrow$ f has a complex root: $f(x) = c_0 x^n + \dots + c_n$ $c_i \in \mathbb{R}$

$0 = f(z) = c_0 z^n + \dots + c_n$

$0 = \overline{f(z)} = \overline{c_0}\,\overline{z}^n + \dots + \overline{c_n} = c_0 \overline{z}^n + \dots + c_n = f(\overline{z})$

$z = a + bi \to$ root

$\Rightarrow \overline{z} = a - bi \xrightarrow{\text{also}}$ root $\Rightarrow f(x) = \underbrace{(x - a - bi)(x - a + bi)}_{} f_1(x) \dots$

$((x-a)^2 + b^2) f_1(x)$

$\Rightarrow f^2 = ((x-a)^2 + b^2)^2 (f_1(x))^2 = (\quad)^2 f_1(x)^2 + (\quad)^2 f_1(x)^2$

$\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

if $f(\bar{z}) = \overline{f(z)}$ $\forall z \in \mathbb{C}$ $\Rightarrow f \in \mathbb{R}[x]$