How to solve problems?

Pr. How many subsets $S \subset \underbrace{\{1, 2, \dots, n\}}_{[n]}$ are there?

Extra:
① $\#\{S \subset [n], \text{ no } x \neq y \in S, \text{ s.t. } x+y = n+1\}$ | $3^{n/2}$ never ...

② $\#\{S \subset [n], \text{ s.t. } \#S - \text{even}\} \to 2^{n-1}$

$\#\{\dots\}$: number of objects; $\#S$: number of elements of S

③* $\#\{S \subset [n], 3 \mid \#S\}$

④ $\#\{S \subset [n], \text{ s.t. no } x, y \in S, \text{ s.t. } x - y = 1\}$

$S \subset [n] \to 2^n$

small examples:

$n=1$	$n=2$	$n=3$
$\emptyset, \{1\}$,	$\emptyset, 1, 2, 12$	$\emptyset, 1, 2, 3, 12, 13, 23, 123$
2	4	8

pattern → powers of 2 → $\boxed{2^n}$ guess

1) Bisection: $S = \{2, 3, 5\} \subset [6] \leftrightarrow$

encoding:

$$\begin{array}{cccccc} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} \leftrightarrow (a_1, a_2, \dots, a_n), \ a_i \in \{0, 1\}$$

$$a_i = \begin{cases} 0 & \text{if } i \notin S \\ 1 & \text{if } i \in S \end{cases}$$

$$\#\{(\underset{\text{2 choices}}{a_1}, \underset{\text{2 choices}}{a_2}, \dots, \underset{\text{2 choices}}{a_n}), a_i \in \{0, 1\}\} = 2 \times 2 \times 2 \dots \times 2 = 2^n.$$

2) Induction: (Recursion)

Thm: (Math Induction): If we have a statement "$P(n) = \text{true}$", s.t.:

1) $P(1) = \text{true}$ (base case)

Ind. step

1) $P(1) = \text{true}$ (base case)

2) If "$P(n) = \text{true}$ $\Rightarrow$ $P(n+1) = \text{true}$" for every $\underline{n}$.

$\Rightarrow P(n) = \text{true}$ for every n.

$$f(n) = \#\{S \subset [n]\}$$

$$P(n) = \text{"}f(n) = 2^n\text{"}$$

base cases: $P(1) = \text{"}f(1) = 2\text{"}$ ✓, $\quad S = \emptyset, \{1\}$; $P(2) = \text{"}f(2) = 4\text{"}$ ✓, $\quad S = \emptyset, 1, 2, 12$

Ind. step: Recursion: $f(n+1) = \text{function of } (f(n), f(n-1), \ldots f(1), f(0))$

$$f(n+1) = \#\{S \subset [n+1]\}$$

$n+1 \in S$: $S = S' \cup \{n+1\}$, $S' \subset [n]$

\# of such cases $\Rightarrow f(n)$

$n+1 \notin S$: $S = S' \subset [n]$ $\quad f(n)$

$$\Rightarrow \boxed{f(n+1)} = f(n) + f(n) = \boxed{2f(n)} \quad (*)$$

Proof that $f(n) = 2^n$:

1) base cases ✓

2) $P(n+1) = \text{"}f(n+1) = 2^{n+1}\text{"}$

assume $P(n) = \text{"}f(n) = 2^n\text{"}$

If $f(n) = 2^n \Rightarrow f(n+1) = 2f(n) = 2 \cdot 2^n = 2^{n+1}$ ✓ $\quad (*)$

$\square$

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③ Using Formulas

$$\#\{S \subset [n], \#S = k\} = \binom{n}{k} \quad \text{"n choose k"}$$

binomial coefficient $\quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$\boxed{\#\{S \subset [n]\}} = \sum_{k=0}^{n} \underbrace{\#\{S \subset [n], \#S = k\}}_{\binom{n}{k}} = \boxed{\sum_{k=0}^{n} \binom{n}{k}}$$

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}} \quad \text{(Newton's binomial formula)}$$

$$x = y = 1 \Rightarrow (1+1)^n = 2^n = \sum^{n} \binom{n}{k} 1^k 1^{n-k} = \sum \binom{n}{k} = \#\{S \subset [n]\}.$$

$$x=y=1 \Rightarrow (1+1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n}\binom{n}{k} = \#\{S\subset[n]\}.$$

(1) $g(n) := \#\{S \subset [n],\ \text{no } x\neq y,\ \underline{x+y=n+1}\}$

$n=1 \to \emptyset, 1 \to g(1)=2 \to 2\cdot 3^0$

$n=2 \to \emptyset, 1, 2, \cancel{12}$ ($1+2=3$) $\to g(2)=\underline{3} \to 3^1$

$n=3 \to \emptyset, 1, 2, 3, 12$ ($1+2=3\neq 4$ ✓), $\cancel{13}$, 23, $\cancel{123}$ ($x+y=4$) $\Rightarrow g(3)=6 \to 2\cdot 3^1$

$n=4 \Rightarrow \emptyset, 1, 2, 3, 4, 12, 13, \cancel{23}, \cancel{14}, 24, 34, \cancel{123}, \ldots \Rightarrow g(4)=\underline{9} \to 3^2$

no $x+y=5$ $(=n+1)$

Guess: $g(n) = \begin{cases} 3^{n/2} & \text{if } n \to \text{even} \\ 2\cdot 3^{\frac{n-1}{2}} & \text{if } n \to \text{odd} \end{cases}$

Prove? $\underline{S\subset[n]}$ $\downarrow$ $\underline{\text{bijection}}$

$S=\{1,4\}\subset[5]$ $(1+4=5\neq 5+1)$ ✓

violation: $\underline{x+y=n+1}$

$$\begin{array}{ccccc} \underline{1} & \underline{0} & \underline{0} & \underline{1} & \underline{0} \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

$$\underline{\ }_1\ \underline{\ }_2\ \underline{\ }_3\ \underline{\ }_i\ \underline{\ }_{\frac{n+1}{2}}\ \underline{\ }_{n+1-i}\ \underline{\ }_{n-1}\ \underline{\ }_n$$

for every $i \le \lfloor n/2 \rfloor$

no $\to \underline{1}_i\ \underline{1}_{n+1-i}$; yes: $\underline{1}\ \underline{0}$, $\underline{0}\ \underline{0}$, $\underline{0}\ \underline{1}$

if $x+y=n+1 \Leftrightarrow y=n+1-x$

$\underline{\lfloor n/2 \rfloor \text{ pairs}}$ $\ \underline{\ }_i \smile \underline{\ }_{n+1-i} \to \begin{array}{cc} 0 & 0 \\ 1 & 0 \\ \text{or } 0 & 1 \end{array}$ | 3 distinct choices, independent $\underline{\text{of the others}}$

$\Rightarrow$ 3 choices (1 — n) 3 choices (2 — $n-1$) $\ldots \to \underline{3^{\lfloor n/2 \rfloor}\cdot \underset{\text{choices}}{2}}$ for $\frac{n+1}{2}$ if $n \to$ odd $\square$.

(2) $\#\{S\subset[n],\ \#S - \text{even}\} =: h_e(n)$, $\#\{S\subset[n],\ \#S - \text{odd}\} =: h_o(n)$

$h_e(n)+h_o(n)=2^n$ $(*)$

(2) $\#\{S \subset [n],\ \#S\text{-even}\} = h_e(n)$, $\#\{ \dots \}$,

$h_e(n) + h_o(n) = 2^n$ (*)

Induction: $n=1$ $h_e(1) = 1$ $(S=\emptyset)$

$n=2$ $\emptyset, \cancel{1}, \cancel{2}, 12 \to h_e(2) = 2$

$n=3$ $\emptyset, \cancel{1}, \cancel{2}, \cancel{3}, 12, 13, 23, \cancel{123} \to h_e(3) = 4$

Guess: $h_e(n) = 2^{n-1}$

Proof: by induction: $\to$ find a recursion:

$h_e(n+1) = \#\{S \subset [n+1],\ \#S\text{-even}\}$

$n+1 \in S$: $S = S' \cup \{n+1\}$, $S' \subset [n]$, $\#S' = \#S - 1$ - odd → $h_o(n)$

$n+1 \notin S$: $S \subset [n]$, $\#S$ - even → $h_e(n)$

no need for induction

$\Rightarrow h_e(n+1) = h_o(n) + h_e(n) = 2^n = 2^{(n+1)-1}$