Math 395: problems 2

Due Mon 02/01 by 6pm, uploaded in blackboard

Instructions: Solve (with proof) and submit 2 of the following problems, arranged in order of increasing difficulty.

Notation: $[n] := \{1, \ldots, n\}$, the set of the first n positive integers.

$\#\{...\}$ denotes "the number of elements in the given set. Note that in many problems the set consists of sets.

1. Let the sequence a_n be defined recursively by

$$a_{n+1} = 3a_n - 4a_{n-2} \quad \text{for all } n \geq 1$$

   with initial conditions $a_0 = 1, a_1 = 3, a_2 = 9$. Find a closed form formula for a_n.

   Hint: what are the roots of $x^3 - 3x^2 + 4$.

2. Prove by induction on n that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}.$$

   What other proofs can you find?

   Curious fact: the above series converges to $\pi^2/6$ as $n \to \infty$, and is known as Basel's problem.

3. (Do not do this problem if you've taken Math 395 before).

   The Fibonacci numbers are given by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Prove that for every n we have

$$F_{2n} = F_n F_{n-1} + F_n F_{n+1}.$$

4. Let $f(n)$ be the number of segments obtained by the intersection of the diagonals of an n-gon (non regular, no 3 diagonals intersect at a point). Find (and prove) a closed form formula for $f(n)$.

   Note: we consider segments AB, such that both A and B are intersection points of diagonals, AB lies on one diagonal, and there are no other intersection points inside AB.

5. Show that for every $n \in \mathbb{N}$ the following equation has solution with x, y, z all positive integers:

$$x^2 + y^2 = z^n.$$

6. For the Putnam trainees: find (and solve, or ask me about) an interesting (and hard) problem from the Putnam (or similar) which fits the current topic. As sources you can use the Putnam Archive of Kiran Kedlaya, or IMO, or "AoPS", or any of the sources listed in the syllabus.