Math 395: problem set 3

Due Mon 02/08 by 6pm, uploaded in blackboard

Instructions: Solve (with proof) and submit 2 of the following problems, arranged in order of increasing difficulty.

Notation: $[n] := \{1, \ldots, n\}$, the set of the first n positive integers.

$\#\{...\}$ denotes "the number of elements in the given set. Note that in many problems the set consists of sets.

1. Show that if there are n people at a party, then there are two of them who know the same number of people. (we assume that if A knows then B knows A)

2. Let x and n be integers, such that $gcd(x, n) = 1$. Prove that there is a positive integer $m < n$, such that $n | x^m - 1$.

   (You should apply only the pigeonhole principle, do not use theorems like Fermat's little theorem or Euler's totient function)

3. Prove that if $S = \{x_1, \ldots, x_7\} \subset \{1, 2, \ldots, 23\}$, then there exist two distinct subsets of S with the same sum of their elements.

4. Prove that if $b_1, \ldots, b_n$ is a sequence, whose elements $b_i \in \{a_1, \ldots, a_m\}$, then if $n > 2^m$ we can always find several consecutive terms, whose product is a perfect square of an integer.

   e.g. if $a_1 = 2$ and $a_2 = 3$ and $b = 2, 3, 2, 3, 2$, then $b_2 b_3 b_4 b_5 = 6^2$.

5. Let a_n be a sequence given by $a_0 = 0, a_1 = 2, a_2 = 5$ and $a_{n+1} = a_n + 3a_{n-1} - a_{n-2}$ for $n \geq 2$. Prove that there exists an $n > 0$, such that $123456789 | a_n$.

6. Prove that if we color every edge of the complete graph on 9 vertices (so all possible edges present) in red or green, then there will either be a green triangle, or a red quadrangle (i.e. 4 points, all edges between them are red).

7. For the Putnam trainees: find (and solve, or ask me about) an interesting (and hard) problem from the Putnam (or similar) which fits the current topic. As sources you can use the Putnam Archive of Kiran Kedlaya, or IMO, or "AoPS", or any of the sources listed in the syllabus.