Monday, March 22, 2021 5:31 PM

$$p_0 + p_1 + p_2 + p_3 + \dots = 1$$

$$\mathbb{E}[X] = p_1 + 2p_2 + \dots = \cancel{4}\,1$$

$$\mathbb{E}[X^2] = p_1 + 4p_2 + \dots = 2$$

$$\mathbb{E}[X^3] = p_1 + 8p_2 + 27p_3 + \dots = 4$$

Inequality: $(k-1)(k-2)(k-3) \geq 0$ if $k \in \mathbb{N}_{\geq 0}$

$k \geq 4$ ≥ 0

$\Updownarrow$

$$\sum_k p_k (k^3 - 6k^2 + 11k - 6) \geq 0 \quad \text{for } \forall k = 1, 2, \dots$$

$$\Rightarrow \underbrace{\mathbb{E}[X^3] - 6\mathbb{E}[X^2] + 11\mathbb{E}[X]}_{} - 6\underbrace{(p_1 + p_2 + \dots + p_k + \dots)}_{1-p_0} \geq 0$$

$$\sum_{k=1}^{\infty} (k^3 - 6k^2 + 11k - 6) p_k \geq 0$$

$$4 - 6.2 + 11.1 - 6(1-p_0) \geq 0 \Rightarrow 1 - p_0 \leq 1/2$$

$$3 - 6(1-p_0) \geq 0 \quad \rightarrow \boxed{p_0 \geq 1/2}$$

Can we achieve $\boxed{p_0 = 1/2}$? $\rightarrow$ all $\geq$ turn into "="

$\Rightarrow p_k = 0$ if $k \geq 4$

$\rightarrow$ solve

$$\left.\begin{aligned} p_1 + p_2 + p_3 &= 1/2 \; (1-p_0) \\ p_1 + 2p_2 + 3p_3 &= 1 \\ p_1 + 4p_2 + 9p_3 &= 2 \\ p_1 + 8p_2 + 27p_3 &= 4 \end{aligned}\right| \rightarrow \text{solve} \quad p_1 = p_3 = 0, \ \boxed{p_2 = 1/2}$$

$$P(r,g,b) = \frac{r}{r+g+b} P(r-1, g, b) + \frac{g}{r+g+b} P(r, g-1, b) + \frac{b}{r+g+b} P(r, g, b-1)$$

$$P(r, 0, *) = 0 \quad (r > 0)$$

$$P(r, *, 0) = 0 \quad (r > 0)$$

$$P(0, g, b) = 1$$

check if

$$\frac{g\,b}{r+g+b}\left(\frac{1}{r+b} + \frac{1}{r+g}\right)$$

check?

$$\mathbb{P}(0, g, b) = 1 \qquad \underset{>0}{g},\ \underset{>0}{b}$$

$$\frac{gb}{r+g+b}\left(\frac{1}{r+b} + \frac{1}{r+g}\right)$$

$$\cancel{\frac{gb}{r+g+b}}\;\frac{2r+g+b}{(r+b)(r+g)} = \frac{r}{\cancel{r+g+b}}\cdot\frac{\cancel{gb}}{r+g+b-1}\left(\frac{1}{b+r-1} + \frac{1}{g+r-1}\right) + \frac{\cancel{g}(g-1)\cancel{b}}{\cancel{(r+g+b)}(r+g+b-1)}(\dots)$$

$$+ \frac{\cancel{b\,g}(b-1)}{\cancel{(r+g+b)}(r+g+b-1)}(\dots)$$

$$\frac{(r+g+b-1)(2r+g+b)}{(r+b)(r+g)} = \frac{r(b+g+2r-2)}{(b+r-1)(g+r-1)} + \frac{(g-1)(2r+g+b-1)}{(r+b)(r+g-1)} + \frac{(b-1)(2r+g+b-1)}{(r+g)(r+b-1)}$$

$$\mathbb{E}[\underbrace{\#\text{ of tosses till get } \underbrace{HH\dots H}_{n}}_{X_n}]$$

e.g. $n=2$.

$T\,H\,T\,T\,H\,H$

$X_2 = 6$

$n=1$

$$\mathbb{E}[X_1] \quad \xrightarrow{\frac{1}{2}T} 1 + \mathbb{E}[X_1] \qquad \xrightarrow{\frac{1}{2}H} 1 \leftarrow \#\text{ on tosses.}$$

$$\Rightarrow \mathbb{E}[X_1] = \tfrac{1}{2}\cdot(1 + \mathbb{E}[X_1]) + \tfrac{1}{2}\cdot 1$$

$$\Rightarrow \mathbb{E}[X_1] = 2$$

$$\mathbb{E}[X_1] = \sum_{k} k\cdot\left(\tfrac{1}{2}\right)^k \qquad \underbrace{TT\dots T}_{k-1}H$$

$$\mathbb{E}[X_2] \quad \xrightarrow{\frac{1}{2}T} 1 + \mathbb{E}[X_2] \qquad \xrightarrow{\frac{1}{2}H} H \quad \xrightarrow{\frac{1}{2}T} 2 + \mathbb{E}[X_2] \qquad \xrightarrow{\frac{1}{2}H} 2$$

$$\Rightarrow \mathbb{E}[X_2] = \tfrac{1}{2}(1 + \mathbb{E}[X_2]) + \tfrac{1}{2}\cdot\left(\tfrac{1}{2}(2 + \mathbb{E}[X_2]) + \tfrac{1}{2}\cdot 2\right)$$

$$\mathbb{E}[X_2] = 6$$

$$\mathbb{E}[X_3] = 14$$

$n=$	1	2	3
	2	6	14

check OEIS → guess $\boxed{2^{n+1} - 2}$

until we get $\underbrace{H\dots H}_{n}$

$$\mathbb{E}[X_{n+1}] \quad \xrightarrow{\frac{1}{2}H} 1 + \mathbb{E}[X_n] \qquad \xrightarrow{\frac{1}{2}T} \mathbb{E}[X_n] + 1 + \mathbb{E}[X_{n+1}]$$

$$\underbrace{TH\dots\underbrace{H\dots H}_{n}}_{X_n}T\ \underbrace{\qquad}_{\text{new}}$$

$$\Rightarrow \mathbb{E}[X_{n+1}] = \tfrac{1}{2}(1 + \mathbb{E}[X_n]) + \tfrac{1}{2}(\mathbb{E}[X_n] + 1 + \mathbb{E}[X_{n+1}])$$

$$\frac{1}{2}\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n] + 1$$

$2^{n+1} - 2$

check

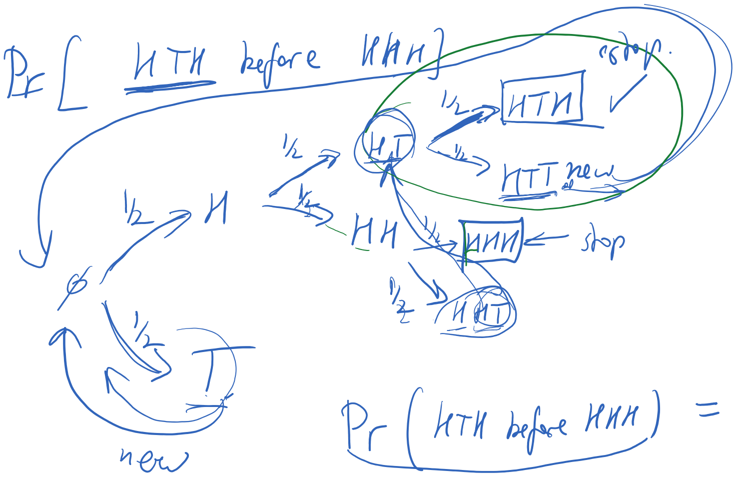
$\underbrace{+H \ldots \underbrace{H \ldots H}_{n}}_{X_n} \quad \underline{\ \ }_{\text{new}}$ $\quad\quad \frac{1}{2}\mathbb{E}[\ldots] = \mathbb{E}[\ldots]$

check $\quad\quad 2^{n+1}-2$

$\Rightarrow \mathbb{E}[X_{n+1}] = 2^{n+2}-2$

$$\mathbb{E}[\underline{HTH}] = 10 \neq \mathbb{E}[HHH] = 14$$

$\Pr[\underline{HTH} \text{ before } HHH]$



$$\underbrace{\Pr\left(HTH \text{ before } HHH\right)}_{X} = \frac{1}{2}\cdot X + \frac{1}{2}\Big(\overset{HT}{\left(\tfrac{1}{2}+\tfrac{1}{2}\right)}\left(\tfrac{1}{2}+\tfrac{1}{2}X\right) + \tfrac{1}{4}\cdot \underset{HHH}{0}\Big) =$$

$$X = \frac{1}{2}X + \frac{3}{16}(1+X) \Rightarrow X = 3/5$$

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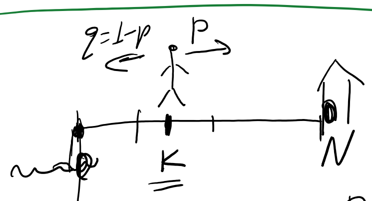
(1)



steps: prob p $\longrightarrow$, $q=1-p$ $\longleftarrow$

$p_K = \Pr($ 🚶 goes 🏠 $)$

$$\underline{p_0 = 0, \ p_N = 1}$$

$p = q$

$\Rightarrow 2p_k = p_{k+1} + p_{k-1}$

$0 \quad 1/N \quad 2/N \quad k/N \quad N$

$$1 = (p+q)\,p_k = p \cdot p_{k+1} + q \cdot p_{k-1}$$

$p \neq q \quad \Rightarrow \quad p_{k+1} - p_k = \frac{q}{p}(p_k - p_{k-1}) = \frac{q^k}{p^k}(p_1 - p_0)$

$$1 = p_N - p_0 = (p_N - p_{N-1}) + (p_{N-1} - p_{N-2}) + \ldots + (p_1 - p_0) =$$

$$= \left(\left(\frac{q}{p}\right)^{N-1} + \ldots + 1\right) p_1 - p_0 \Rightarrow p_1 - p_0 = \frac{1 - q/p}{1 - (q/p)^N}$$

$$\Rightarrow \quad p_K = \frac{\frac{q^K}{p^K} - 1}{\frac{q^N}{p^N} - 1} \xrightarrow{q=p} \frac{K}{N}$$

$\mathbb{E}[\text{time to reach home}] = \ldots$

$$\mathbb{E}\left[\underline{|X_H - X_T|}\right] = \frac{2}{2^n} \sum_{k=0}^{n/2} \underbrace{(n-2k)}_{X_H - X_T} \cdot \binom{n}{k} \overset{n \to \text{odd}}{=} \frac{1}{2^{n-1}} \cdot \left(n \underbrace{\sum_{k=0}^{n/2} \binom{n}{k}}_{1/2 \cdot 2^n} - \sum_{k=0}^{n/2} 2k \binom{n}{k} \right)$$

$\underbrace{HTT\ldots H}_{n}$ $\quad X_H - \# H$

$X_T - \# T$

$$\frac{\cancel{k} \cdot n!}{\cancel{k!}\,(n-k)!} = n \cdot \binom{n-1}{k-1}, \quad (k-1)!, \quad k-1 \to m$$

$$0 = \mathbb{E}[\underbrace{X_H}_{k} - \underbrace{X_T}_{n-k}] = \sum_{k=0}^{n} (2k-n) \cdot \binom{n}{k} \cdot \frac{1}{2^n}$$

$k \geq n-k$

$$= \frac{1}{2^{n-1}} \left(n \cdot 2^{n-1} - 2n \sum_{m=0}^{n/2 - 1} \binom{n-1}{m} \right)$$

$$\ldots \quad \frac{2^{n-1} - \binom{n-1}{\frac{n-1}{2}}}{2}$$

$$\to \frac{n \cdot \binom{n-1}{\frac{n-1}{2}}}{2^{n-1}}$$

$$\mathbb{E}[(X_H - X_T)^2]$$