Math 395: problem set 4

Due Mon 02/15 by 6pm, uploaded in blackboard

Instructions: Solve (with proof) and submit 2 of the following problems, arranged in order of increasing difficulty.

Notation: $[n] := \{1, \ldots, n\}$, the set of the first n positive integers.

$\#\{...\}$ denotes "the number of elements in the given set. Note that in many problems the set consists of sets.

Useful facts:

Euler's formula for planar graphs: $V - E + F = 2$.

Handshake lemma:

$$\sum_{v \in V} deg(v) = 2E.$$

For planar graphs the faces also have "degrees" – the number of edges around them, and again if we count the number of edges around every face then every edge of the graph is counted twice being part of two faces.

$$\sum_{f \in F} deg(f) = 2E.$$

1. How many different planar graphs are there such that every face is a triangle and every vertex has degree:

   a) 3.

   b) 4.

2. Prove that K_5 (complete graph on 5 vertices) is not planar.

3. Prove that every planar graph has at least one vertex of degree less than 6.

4. Tournaments: a tournament is a sports competition where everyone plays a game with everyone else, and for this problem every match has exactly one winner and one looser (no draws). We represent tournaments on n players as a directed complete graph on n vertices, i.e. for every two distinct vertices i and j we have either the edge $i \to j \in E$ or $j \to i \in E$. Prove that for any given tournament (i.e. after all games have been played) it is possible to label the players as $w_1, \ldots, w_n$, so that w_i won the match against w_{i+1} for every i, or in graph terms: such that $w_1 \to w_2 \to \cdots \to w_n$.

5. Let G be an undirected graph on n vertices such that every vertex has degree $\geq n/2$. Prove that there is a Hamiltonian circuit, i.e. a labeling of the vertices $v_1, \ldots, v_n$, such that $(v_i, v_{i+1}) \in E$ and $(v_1, v_n) \in E$. (here G is simple – no loops, no double edges)

6. Let G be a connected simple graph with k edges. Is it possible to label the edges with the numbers $1, 2...k$ (each number appearing exactly once), so that for every vertex the greatest common divisor (gcd) of the numbers on the edges adjacent to that vertex is 1.

   E.g. if $G = K_4$ then $k = 6$ and we can label $(1,2) \to 1, (1,3) \to 2, (1,4) \to 6, (2,3) \to 4, (2,4) \to 5, (3,4) \to 3$, then the gcd of the labels at vertex 4 is $gcd(6,5,3) = 1$ etc.

7. For the Putnam trainees: find (and solve, or ask me about) an interesting (and hard) problem from the Putnam (or similar) which fits the current topic. As sources you can use the Putnam Archive of Kiran Kedlaya, or IMO, or "AoPS", or any of the sources listed in the syllabus.