PROBLEMS: POLYNOMIALS

Useful facts: Every polynomial (with integer/rational/real/complex coefficients) of degree n has exactly n complex (including real) roots, so $p(x) = a(x-r_1)\cdots(x-r_n)$

If r is a root of a polynomial, then $p(x) = (x-r)q(x)$ for some polynomial $q(x)$.

If $P(x) = Q(x)$ for more than n distinct values of x then $P = Q$ as polynomials, i.e. their coefficients coincide.

Binomial theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

DO NOT submit problems 1 or 2 if you have taken Math 395 before.

(1) Find all polynomials $P(x)$, such that

$$(x+1)P(x) = (x-2)P(x+1)$$

for all $x \in \mathbb{R}$.

(2) Find all polynomials $P(x)$ such that $(P(x))^2 = P(x^2)$.

(3) Find all solutions (real and complex) of

$$x(x+2)(x^2-1) - 24 = 0.$$

(4) Using polynomial identities modulo p (p is a prime number) show that:
   a) $\binom{p}{k} \equiv 0 \pmod{p}$ for $k = 1, 2, \ldots, p-1$.
   b)* $\binom{2p}{p} \equiv 2 \pmod{p}$.

(5) Let $d \in \mathbb{N}$ be fixed. Prove that if we define

$$f_d(n) = 1^d + \cdots + n^d,$$

for every n, then there is a polynomial $p_{d+1}(x) \in \mathbb{Q}[x]$ of degree $d+1$, such that $f_d(n) = p_{d+1}(n)$ for every n. Find the leading term of this polynomial.

(6) Let $n \in \mathbb{N}$. Find all polynomials $p(x), q(x) \in \mathbb{R}[x]$, such that $deg(p) > deg(q)$ and

$$(p(x))^2 + (q(x))^2 = x^{2n} + 1$$

for every x.

(7) Find (and solve, or ask me about) an interesting (and hard) problem from the Putnam (or similar) which fits the current topic. As sources you can use the Putnam Archive of Kiran Kedlaya, or IMO, or "AoPS", or any of the sources listed in the syllabus.