Analysis III

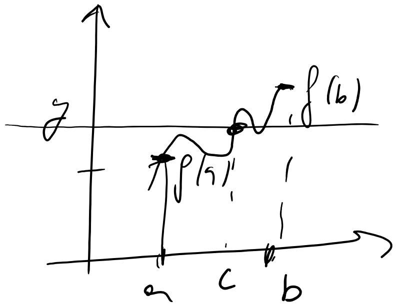
$f \to$ cont. functions

Intermediate Value Theorem:

$f : [a,b] \to [a_1, b_1]$, cont.

if $y \in [f(a), f(b)]$ (or $[f(b), f(a)]$)

$\Rightarrow \exists\, c \in [a,b] \Rightarrow f(c) = y.$



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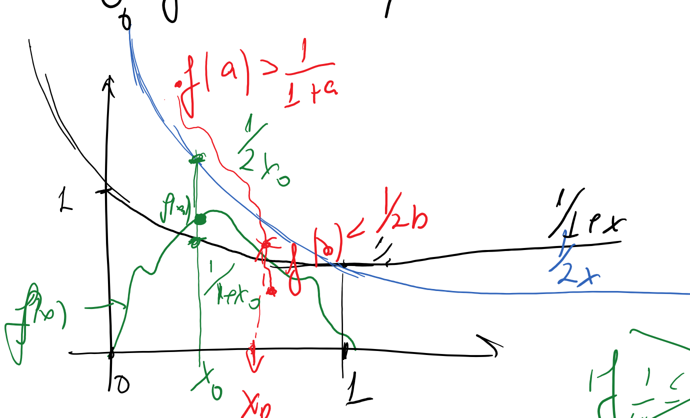
ex. $\int_0^1 f(x) = \pi/4$. Show that $\exists\, x_0 \in [0,1]$, s.t.

$$\frac{1}{1+x_0} \le f(x_0) \le \frac{1}{2x_0}$$



$$\frac{1}{2x} > \frac{1}{1+x} \quad \text{for all } x \in (0,1)$$

$\cancel{\text{if } \frac{1}{1+x} \le f(x) \le \frac{1}{2x}}$ for any x

$\Rightarrow f(x) < \frac{1}{1+x} \quad \forall x \in (0,1)$, or $f(x) > \frac{1}{2x}$ for all $x \in (0,1)$

$\Downarrow$

$$\pi/4 = \int_0^1 f(x) < \int_0^1 \frac{1}{1+x}\,dx = \ln(x+1)\Big|_0^1 = \ln(2) - \underbrace{\ln(1)}_{0} = 0.69\ldots < \pi/4 = 0.78$$

$\Rightarrow$ contradiction. ✓.

$$+\infty = \int_0^1 \frac{1}{2x}\,dx < \int_0^1 f(x)\,dx = \pi/4 \to \text{impossible.}$$

$\frac{1}{2}\ln x \Big|_0^1$

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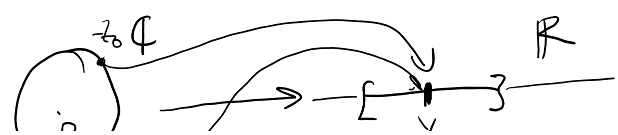
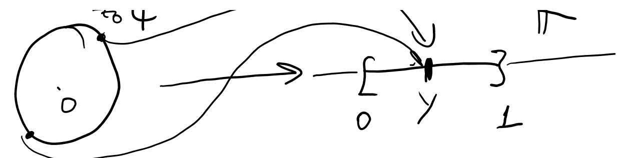
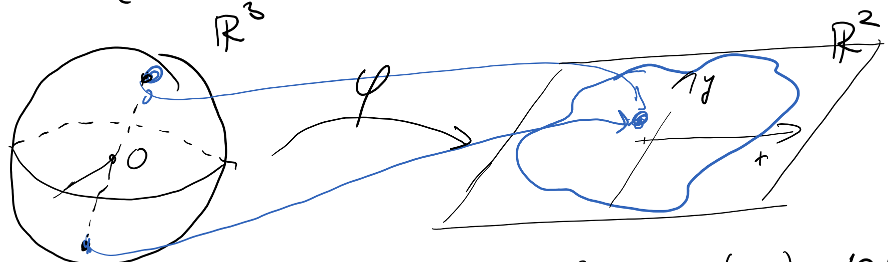
ex. f:  $\Rightarrow \exists\, z_0 \in S^1$, s.t.

(*) f:  $\Rightarrow \exists z_0 \in S^1$, s.t.

$$f(z_0) = f(-z_0) \quad (1)$$

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$$



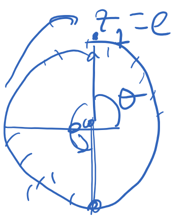
$S^2 \Rightarrow \exists z_0 \in S^2: \ \varphi(z_0) = \varphi(-z_0)$

Borsuk-Ulam theorem.

Proof of (*): $\quad g(z) := f(z) - f(-z) \quad (1) \Leftrightarrow g(z_0) = 0.$

say $g(z_1) > 0$ for some $z_1 \in S^1$

$$g(-z_1) = f(-z_1) - f(-(-z_1)) = -g(z_1) < 0$$

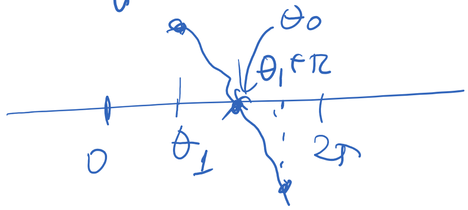


$z_1 = e^{\theta_1 i}$

$z = e^{\theta_2 i}$

$-z_1 = e^{(\theta+\pi)i}$

$g: \theta \in [0, 2\pi) \to [0, 1)$ z_0



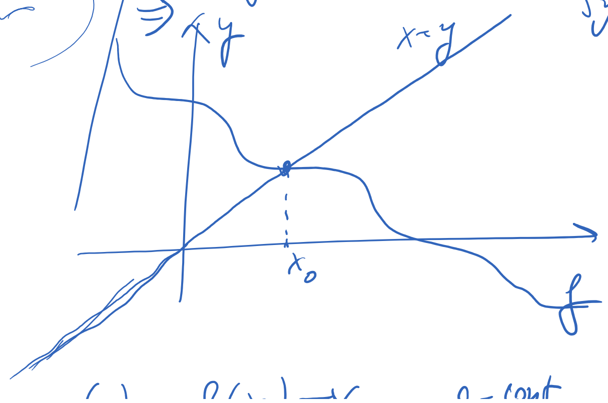
$\to g(e^{\theta_0 \cdot i}) = 0$

$$f(z_0) = f(-z_0).$$

$f: \mathbb{R} \to \mathbb{R}$ cont., $f \downarrow$, $f_{x \to -\infty} \to +\infty$, $f_{x \to +\infty} \to -\infty$

find all solutions (x, y, z) to this system.

$$\begin{cases} x = f(y) \\ y = f(z) \\ z = f(x) \end{cases}$$

$\Rightarrow$ 

$f(x_0) = x_0$

$\Rightarrow x = y = z = x_0$

$\Downarrow$

$\Rightarrow x = y = z = x_0$

$$\boxed{\begin{aligned} x_0 &= f(x_0) \\ x_0 &= f(x_0) \\ x_0 &= f(x_0) \end{aligned}}$$

$g(x) = f(x) - x$, g - cont.

$g(-\infty) = +\infty$, $g(+\infty) = -\infty \Rightarrow \underline{g(x_0) = 0}$

$f(x_0) - x_0 = 0$, $f(x_1) - x_1 = 0$, $x_0 < x_1$

$\Downarrow$

$f(x_0) < f(x_1)$

$\Rightarrow f(x_0) = x_0 < f(x_1) = x_1$

$\rightarrow$ Contradiction w/ $\underline{f\text{- decreasing}}$.

$\Rightarrow$ no

$x = f(y)$
$y = f(z)$
$z = f(x)$

$\Rightarrow x = \underbrace{f(f(f(x)))}$ $\rightarrow$ fixed point of $f^{(3)}$

$x > y$
$f(x) < f(y)$
$f(f(x)) > f(f(y))$
$f(f(f(x))) < f(f(f(y)))$

f decreasing

$\Downarrow$

$f^{(3)}$ is decreasing.

$f^{(3)}(x) \xrightarrow[x\to+\infty]{} -\infty$

$f^{(3)}(x) \xrightarrow[x\to-\infty]{} +\infty$

$\Rightarrow h(x) = f(f(f(x))) - x$ $\Downarrow$

$h(+\infty) = -\infty$, $h(-\infty) = +\infty$ $\rightarrow$ $\underline{h(x_0) = x_0}$ unique fixed point. $\rightarrow$ same fixed pt.

$f : C \subset \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$\underline{z \in \mathbb{C}}$

$f(z) = z^2 \Rightarrow$ fixed pts $z^2 = z \rightarrow \underline{z = 0}, \underline{1}$

$f^{(2)}(z) = f(f(z)) = f(z^2) = z^4$

fixed pts: $z = z^4 \rightarrow z(z^3 - 1) = 0$

$\Downarrow$

$\underline{z = 0}$, $\underline{z = 1}$, $z = e^{\frac{2}{3}\pi i}$, $e^{-\frac{2}{3}\pi i}$

$-\frac{1}{2} + \frac{\sqrt{3}}{2} i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2} i$

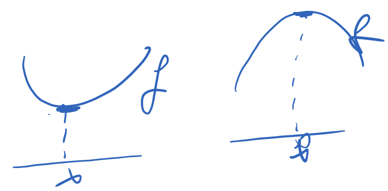
Derivatives

$f : \mathbb{R} \to \mathbb{R}$, differentiable: $f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$ $\to 0$ as $\varepsilon \to 0$

($\Rightarrow$ continuity)

$f'(x) > 0 \to f \nearrow$ there, $f'(x) = 0 \to$

$< 0 \to f \searrow$



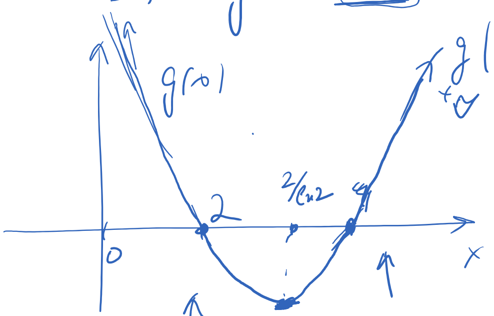
$x > 0$

Ex. $2^x = x^2 \to x = 2 \mid x = 4$

$\Updownarrow$

$x \ln 2 = 2 \ln x$

$\Rightarrow g(x) = x \ln 2 - 2 \ln(x)$, find $x > 0$: $g(x) = 0$.



$g(x \to 0) = +\infty$, $g(x \to +\infty) = +\infty$

$g'(x) = \ln 2 - \frac{2}{x} > 0$ iff $x > \frac{2}{\ln 2}$

< 0 iff $x < \frac{2}{\ln 2}$

$= 0$ $x = 2/\ln 2$.

one intersection each side $\to$ all solutions.

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hw (5) $e^x = \underbrace{100 + x/2 + \frac{\sin x}{3}}_{f(t)}$, $|f(t) - f(s)| < c|t - s|$

t

$\Rightarrow f$ has a unique fixed point by theorem ...

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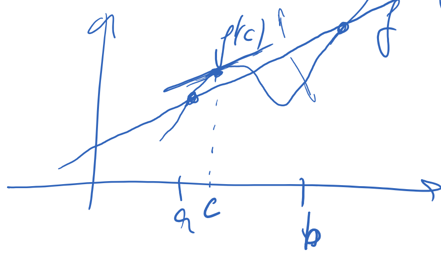
difficult w/ derivatives?

Mean value theorem: $f : [a,b] \to \mathbb{R}$

$\Rightarrow \exists c \in [a,b] : f'(c) = \frac{f(b) - f(a)}{b - a}$

slope

$\Rightarrow \exists\, c \in [a,b]: \quad f'(c) = \frac{f(b)-f(a)}{b-a}$ ⟶ slope



$f'(c) \to$ slope of tangent line ~~to~~ f at $(c, f(c))$

$4^x + 6^{x^2} = 5^x + 5^{x^2}$ ⟵ find x, $\quad x=0,\ x=1$

$f(t) = t^x + (10-t)^{x^2} \quad (\text{fix } x)$

$f(4) = f(5)$, analyse $f \to \left.\frac{d f(t)}{dt}\right|_{t=c\in[4,5]} = 0 \to$

$x c^{x-1} + x^2 (10-c)^{x^2-1} = 0$

$\Rightarrow c^{x-1} = x(10-c)^{x^2-1}$ ⟵ no solution

$<$ for $c \in [4,5]$

$x \neq 0$
$x \neq 1$

$f(x + f'(x)) \geq f(x)$ $\quad$ if $\quad f'' > 0 \quad (\Rightarrow f' \nearrow)$

$f'(x_0) = 0$

$f'(x) \leq 0$, $x < x_0$ ⟶ $f \downarrow$ ⟶ ≤ 0: $x + f'(x) \leq x$ (f decreasing)

$f(x + f'(x)) \geq f(x)$

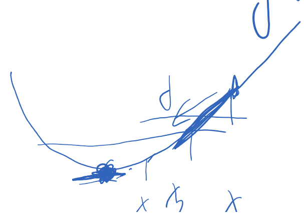
$f'(x) \geq 0$, $x > x_0$ ⟶ $f \nearrow$ for $x > x_0$ $\Rightarrow x + f'(x) \geq x$ (f increasing)

$\Downarrow$

$f(x + f'(x)) \geq f(x)$

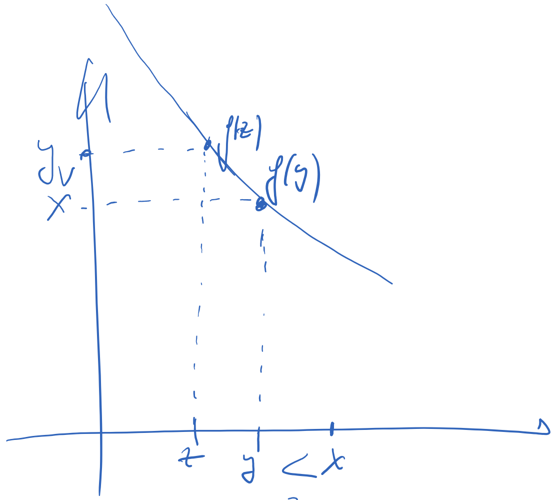
$f'' < 0 \Rightarrow f' \downarrow$

$\Rightarrow f(x + f'(x)) \leq f(x)$



$f'' > 0$

$x_n \to x_{n-1} - f'(x_{n-1})$



$x > y > z$

$x_n \to x_{n-1} - f'(x_{n-1})$

$f'(x) = 0$

$g(x) = x - f'(x)$

$g(x) = x \Leftrightarrow f'(x) = 0 \to$ local minimum

$x \to$ zero

$\to$ $y < z$, $y > z$.