Recursions: $f(n+1) = c_1 f(n) + c_2 f(n-1) + \dots + c_k f(n-k+1)$

k - fixed, $c_i \to$ constants, e.g. Fibonacci

$f(n+1) = c_1 f(n) + c_2 f(n-1)$, $c_1 = 1$, $c_2 = 1$

General formula: $x^k = c_1 x^{k-1} + c_2 x^{k-2} + \dots + c_k x^0$ ← characteristic polynomial equation

if it has k distinct roots $\to \alpha_1, \alpha_2, \dots, \alpha_k$

$$\Rightarrow \quad f(n) = \underbrace{A_1 \alpha_1^n + A_2 \alpha_2^n + \dots + A_k \alpha_k^n}$$

constants, determine from initial conditions

(need at least k initial values $f(0), \dots, f(k-1)$)

Why it works?

check if recursion holds?:

$A_1 \alpha_1^{n+1} + \dots + A_k \alpha_k^{n+1} = c_1 (A_1 \alpha_1^n + \dots) + c_2 (A_1 \alpha_1^{n-1} + \dots) + \dots + c_k (A_i \alpha_i^{n-k+1})$

③ $\Uparrow$ $\sum_i \dots = \sum_i \dots$ $\Uparrow$ follows from:

for every i: $A_i \alpha_i^{n+1} = c_1 A_i \alpha_i^n + c_2 A_i \alpha_i^{n-1} + \dots + c_k A_i \alpha_i^{n-k+1}$

$\Uparrow \times \alpha_i^{n-k+1}$ $\Updownarrow$

② $\cancel{A_i}\, \alpha_i^k = c_1 \cancel{A_i}\, \alpha_i^{k-1} + \dots + c_k \cancel{A_i}$

proof $\Uparrow$ $\Updownarrow$

① $\underline{\alpha_i^k = c_1 \alpha_i^{k-1} + \dots + c_k}$ ← holds because α_i is a root

Find the A_i's:

init. cond. $\to$ $f(0) = A_1 + A_2 + \dots + A_k$

$f(1) = A_1 \alpha_1 + A_2 \alpha_2 + \dots + A_k \alpha_k$ $\iff$ solve $\begin{bmatrix} 1 & \dots & 1 \\ \alpha_1 & \dots & \alpha_k \\ \vdots & & \vdots \\ \alpha_1^{k-1} & \dots & \alpha_k^{k-1} \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_k \end{bmatrix} = \begin{bmatrix} f(0) \\ \vdots \\ f(k-1) \end{bmatrix}$

$\vdots$

$f(k-1) = A_1 \alpha_1^{k-1} + A_2 \alpha_2^{k-1} + \dots + A_k \alpha_k^{k-1}$

$\uparrow$ $\alpha_i \neq \alpha_j$ $\forall\, i \neq j$

$$f(k-1) = A_1\alpha_1^{k-1} + A_2\alpha_2^{k-1} + \dots + A_k\alpha_k^{k-1}$$

$(\alpha_1^{k-1} \dots \alpha_k^{k-1})$ $(k \times k)$

$\alpha_i \neq \alpha_j \ \forall i \neq j$

this is invertible.

det Vandermonde $= \prod_{i<j} (\alpha_i - \alpha_j) \neq 0$

$$f(n+1) = 3f(n) - 2f(n-1), \quad f(0) = 1, \ f(1) = 5$$

$x^2 \leftarrow \quad x^1 \leftarrow \quad x^0$

$$x^2 = 3x - 2 \Rightarrow \alpha_1 = 1, \ \alpha_2 = 2$$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$

$$f(n) = A_1 1^n + A_2 2^n$$

find A_1 & A_2.

$$f(0) = 1 = A_1 1^0 + A_2 . 2^0 = A_1 + A_2$$

$$f(1) = 5 = A_1 . 1^1 + A_2 . 2^1 = A_1 + 2A_2$$

$$\longleftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

$$\Rightarrow A_2 = 5 - 1 = 4$$

$$\Rightarrow A_1 = 1 - 4 = -3$$

$$\Rightarrow \boxed{f(n) = 4 . 2^n - 3}$$

If $x^k = c_1 x^{k-1} + \dots + c_k$ has repeated roots:

$\alpha_1 \rightarrow$ multiplicity m_1

$\alpha_2 \rightarrow$ m_2

$\vdots$

$$f(n) = \alpha_1^n . \left(p_1(n) \right) + \alpha_2^n \left(p_2(n) \right) + \dots$$

$p_1(n)$: poly in $\underline{n}$ of degree $m_1 - 1$ $\quad$ $p_2(n)$: poly in n of deg. $m_2 - 1$

$$f(n+1) = 4f(n) - 4f(n-1) \longrightarrow x^2 = 4x - 4 \Leftrightarrow (x-2)^2 = 0$$

$$\alpha_1 = 2, \ m_1 = 2$$

$f(n) = 2^n (a_1 n + a_0)$ for some constants a_0, a_1.

$$f(n) = 2^n \cdot (a_1 \cdot n + a_0) \quad \text{for some constants } a_0, a_1.$$

check?

$$4f(n) - 4f(n-1) = 4 \cdot 2^n (a_1 n + a_0) - 4\, 2^{n-1}(a_1(n-1) + a_0) =$$

$$= a_1 \cdot \Big(\underbrace{2^{n+2} n - 2^{n+1}(n-1)}_{2^{n+1}(\underbrace{2n - n + 1}_{n+1})}\Big) + a_0 \Big(\underbrace{2^{n+2} - 2^{n+1}}_{2^{n+1}}\Big) =$$

$$= 2^{n+1}\big(a_1(n+1) + a_0\big) \quad ✓$$

$$f(n+1) = c_1 f(n) + \dots + c_k f(n+1-k)$$

need $\underline{f(0), \dots, f(k-1)}$, then $f(k) = \underbrace{c_1 f(k-1) + \dots + c_k f(0)}$

↑ determined by $f(0), \dots, f(k-1)$

→ for problem 1 on HW1.

Problems:

① Show that $x^2 + y^2 = z^n$ has a solution $x, y, z \in \mathbb{N}_{>0}$ for every $n \geq 1$. e.g. $n = 2$: $x^2 + y^2 = z^2$ Pytagorean tripple.

$3^2 + 4^2 = 5^2$

$n = 3$: $x^2 + y^2 = z^3$? $(2, 2, 2)$

by induction on n.

② Prove that if $S \subset \{1, 2, \dots, 2n\}$, $\#S = n+1$, then there exist $a, b \in S$, s.t. $a \mid b$.

$n = 3$, e.g. $S = \{2, 3, 5, 4\}$

$2 \mid 4$ ✓

---

binomial coefficient.

$$\#\{S \subset [n];\ \#S = k\} = \#\{(a_1, \dots, a_k) : 1 \leq a_1 < a_2 < \dots < a_k \leq n\} =: \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$[n] = \{1, 2, \dots, n\}$

$m! := m(m-1) \cdots 1$

want to prove that.

Recursion: $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \leftarrow \text{Pascals triangle}$$

$\binom{0}{0} = \binom{n}{0} = 1$

$\#\{S \subset [n];\ \#S = k\}$

$n = 0$ 1

$\#\{S \subset [n] \,;\, \#S = k\}$

$n \in S$	$n \notin S$
$S' = S \setminus \{n\} \subset [n-1]$	$S \subset [n-1]$
$\#S' = k-1$	$\#S = k$
$\binom{n-1}{k-1}$ +	$\binom{n-1}{k}$

$\binom{0}{0} = \binom{1}{0} = 1$

$n=0$: 1

$n=1$: 1 1 ($k=0$, $k=1$)

$n=2$: 1 2 1

$n=3$: 1 3 3 1

$n=4$: 1 4 6 4 1 $\quad \binom{4}{2}$

Prove the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Statement: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ for every $0 \le k \le n$

base case: $n=1$, $n=2$ ✓

Ind. steps $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{n!}{k!(n-k+1)!}(n-k+1+k)$

$n+1$

$\frac{(n+1)!}{k!(n-k+1)!}$ = for every k ✓.

$k=0$ $\binom{n+1}{0} = 1$ by def.

$1 = \frac{(n+1)!}{0!(n+1-0)!}$

Ex. Prove that $1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$ (how to do with integrals?)

By induction on n: $n = 1 \to 1 < 2\sqrt{1}$ ✓

$n=2$ $\quad 1 + \frac{1}{\sqrt{2}} < 2\sqrt{2}$

$\Updownarrow$

$\sqrt{2} + 1 < 4$

$\Updownarrow$

$\sqrt{2} < 3$

Ind. hypothesis.. $1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$ holds for some n.

Ind. step ? $\quad$? $\Downarrow$

$1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1}$?

$\underbrace{1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}}_{2\sqrt{n}} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}}$ is this $\overset{?}{<} 2\sqrt{n+1}$

Is: $\quad 2\sqrt{n} + \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1}$?

$\Updownarrow$

$2\sqrt{n(n+1)} + 1 < 2(n+1)$

$\Uparrow \quad 2\sqrt{n(n+1)} + 1 < 2(n+1)$

$\Uparrow \qquad \Updownarrow$

$2\sqrt{n^2+n} < 2n+1$

$\Uparrow \qquad \Updownarrow \quad (\;)^2$

$4n^2+4n < 4n^2+4n+1$ ✓ end of pr. □

$\frac{1}{\sqrt{n+1}} < 2\sqrt{n+1} - \cancel{2\sqrt{n}}$ → holds for every n:

$\frac{1}{\sqrt{n}} < \cancel{2\sqrt{n}} - \cancel{2\sqrt{n-1}}$ → $1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n+1}} < 2\sqrt{n+1}$

$+$ $\qquad$ $+$

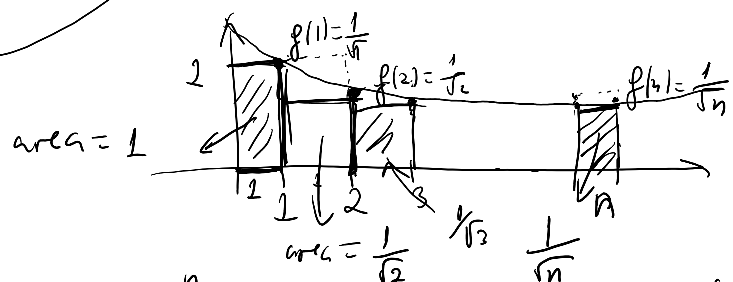
$\frac{1}{\sqrt{n-1}} < \cancel{2\sqrt{n-1}} - \cancel{2\sqrt{n-2}}$

$\vdots$

$\frac{1}{\sqrt{2}} < \cancel{2\sqrt{2}} - \cancel{2\sqrt{1}}$

$1 < \cancel{2\cdot 1}$

$f(x) = \frac{1}{\sqrt{x}}$



$\Rightarrow \int_0^n \frac{1}{\sqrt{x}}\,dx$ = area under $f(x)$ > $\sum$ areas rectangels $= 1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}$

$2\sqrt{x}\Big|_0^n$

$2\sqrt{n}$

$\ln(n-1) < \cancel{1}+\frac{1}{2}+\dots+\frac{1}{n} < \ln(n)$