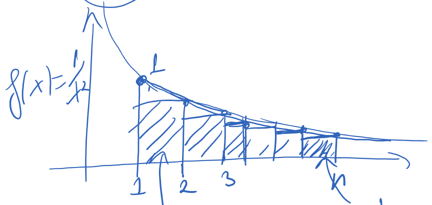
$1 + \frac{1}{2^2} + \dots + \frac{1}{n^2} < 2 - \frac{1}{n}$



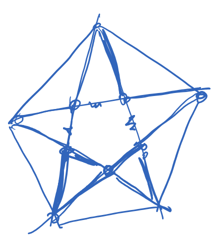
area $\frac{1}{4} + \dots \frac{1}{n^2} < \int_1^n \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^n = 1 - \frac{1}{n}$

$1 + \frac{1}{4} + \dots + \frac{1}{n^2} < 2 - \frac{1}{n}$.

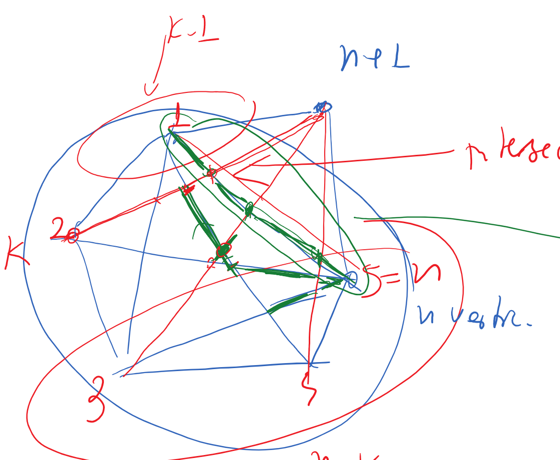
$\frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n}$ ← use for $n = 2, \dots, n$

$$
+\begin{pmatrix}
1 & \leq & 1 \\
\frac{1}{2^2} & < & \frac{1}{1} - \cancel{\frac{1}{2}} \\
\frac{1}{3^2} & < & \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \\
\vdots & & \\
\frac{1}{n^2} & < & \cancel{\frac{1}{n-1}} - \frac{1}{n}
\end{pmatrix} = 2 - \frac{1}{n}
$$

Problem 4:



$f(5) = 15$, $f(n) = 2\binom{n}{4} + \binom{n}{2} - n$



intersections $(k-1)(n-k)$

$$\sum_{k=2}^{n-1} (k-1)(n-k) = A$$

$+ 1$ (for each diagonal intersected by a $(i, n+1)$)

$\left(\binom{n}{2} - n\right)$ $+ n - 1$

↑ no sides

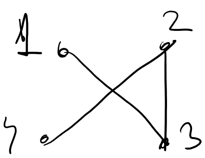
$$f(n+1) = f(n) + A + \binom{n}{2} - n + n - 1$$

$g(n+1) = g(n) + A + \binom{n}{2} - n + n - 1$

Graphs and Pigeonhole principle

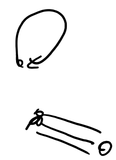
$G = (V, E)$

V: vertice, E: edges $\in (V \times V)$

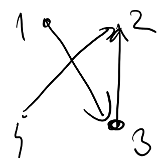


$E = \{(1,3), (2,3), (2,4)\}$.

undirected, simple

no 

no



$E = 1 \to 3,$
$3 \to 2$
$4 \to 2$

directed graph.

$|V| = n$, $|E| \le \binom{n}{2}$

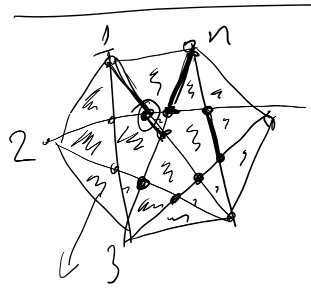
$\deg(v) = \#\{(v,w) \in E\}$.

$\deg(2) = 2$, $\deg(3) = 2$ . . .

Handshake lemma:

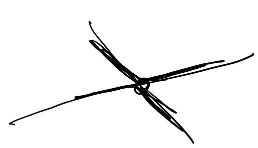
$$\sum_{v \in V} \deg(v) = 2|E| = \sum_{v,w \in V} \begin{cases} 1 & \text{if } (v,w) \in E \\ 0 & \text{o/w.} \end{cases} = \sum_{v \in V} \underbrace{\sum_{w \text{ s.t. } (v,w) \in E} 1}_{= \deg(v)}$$

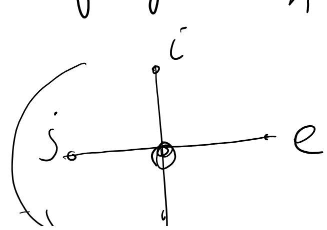
$1 + 2 + 2 + 1 = 2 \cdot 3$ ✓



regions

V = intersection points $\to \binom{n}{4}$
+ vertices of n-gon. $\to n$





regions

$$\underbrace{2|E|}_{\text{segments} + n} = \sum_{v\in V} \deg(v) = \sum_{v \to \text{intersectn.}}^{k} 4 + \sum_{v \to \text{vertex}} (n-1) =$$

↑ sides of n-gon

$$= 4\binom{n}{4} + n(n-1)$$

$$\Rightarrow 2\,\text{segments} + 2n = 4\binom{n}{4} + n(n-1)$$

$$\Rightarrow \text{segments} = 2\binom{n}{4} + \binom{n}{2} - n$$

Pigeonhole principle:

elements $e_1, e_2 \ldots e_n$ (pigeons)

sets $B_1, B_2 \ldots B_m$ (holes)

If $m < n \Rightarrow \exists i : B_i = \{e_{i_1}, e_{i_2}, \ldots\}$ ← at least 2 of these elements.

"there exists"

If $\boxed{mk < n}$ for some k $\rightarrow \exists i : B_i = \{e_{i_1}, \ldots, e_{i_{k+1}}, \ldots\}$ ← at least $k+1$ elements.

Proof: By contradiction:

Assume not true: $\rightarrow \forall i \ \#B_i \leq k$

$\Rightarrow n = \sum_{i=1}^{m} \#B_i \leq k \cdot m$ → not true

→ contradiction.

---

ex. 1 $S \subset \{1, 2, \ldots, 2n\}$, $\#S = n+1$

$\Rightarrow x, y \in S$, $x - y = 1$

$S = \{x_1, x_2 \ldots x_{n+1}\}$ x goes to $\{2i-1, 2i\}$ iff $x = 2i-1$ or $x = 2i$

$\underbrace{\{1,2\}}_{1}$ $\underbrace{\{3,4\}}_{2}$ . . . $\underbrace{\{2n-1, 2n\}}_{n}$ sets

$n+1$ elements x_i → into n sets

, $\{x_i, x_j\} = \{2r-1, 2r\}$ some r

$n+1$ elements $x_i \to$ into n sets

$\Rightarrow x_i, x_j \to$ same set. $\to \{x_i, x_j\} = \{2r-1, 2r\}$ some r

ex. 2 $\quad \#S = n+1 \subset \{1, 2, \dots, 2n\}$

$\Rightarrow$? $\exists\, x, y \in S$, s.t. $x+y = 2n+1$

$S = \{x_1, x_2, \dots, x_{n+1}\}$ $n+1$ elements.

$\downarrow$

$\{1, 2n\}, \{2, 2n-1\}, \dots, \{n, n+1\} \leftarrow n$ sets.

2 elements $\to$ same set $\{i, 2n+1-i\}$

$x, y \quad \Rightarrow x = i, y = 2n+1-i \Rightarrow x+y = 2n+1$

ex. 3 $\quad a_1, \dots, a_{n+1} \in \mathbb{N}$, Prove that $\exists$ j and k : $n \mid a_j + a_{j+1} + \dots + a_{j+k}$

e.g. $\quad 1, 1, 1, 8, 7, 1 \qquad 5 \mid 8+7$

$5 \mid 1+1+8$

$a_1, a_1+a_2, a_1+a_2+a_3, \dots, a_1+a_2+\dots+a_{n+1}$

$1, 2, 3, 11, 18, 19$

same residue by div. by 5

$5 \mid 11 - 1 = \cancel{1}+1+1+8-\cancel{1} = \underbrace{1+1+8}$

residues by div by $n \to n$ possible.

$0, 1, \dots, n-1$

$n+1$ sums $\to$ 2 have the same residue.

$j > i$

$n \mid (\cancel{a_1 + a_2 + \dots} + a_j) - (\cancel{a_1 + a_2 + \dots + a_i}) =$

$= \underline{a_j + a_{j-1} + \dots + a_{i+1}} \leftarrow$ sum of consecutive integers div by n.

ex. 4 $\quad$ Suppose that we have a sequence $b_1, b_2, \dots, b_n$ of n integers from $\{a_1, a_2, \dots, a_m\}$

$\boxed{n > 2^m} \Rightarrow$ Prove that $\exists\, b_j \cdot b_{j+1} \cdots b_{j+k} = A^2$ for some $A \in \mathbb{N}$

e.g. $\quad \{a_1, a_2\} = \{2, 6\}$

e.g. $\{a_1, a_2\} = \{2, 6\}$

$n \geq 5$: $b_1 \to 2, \underbrace{6, 2, 6, 2}$ $\to b_2 b_3 b_4 b_5 = 6.2.6.2 = 12^2$ ✓

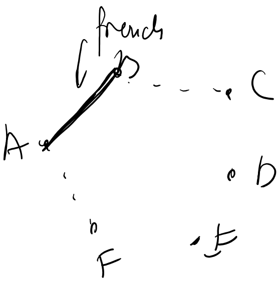
$\boxed{\text{ex.5}}$ Let $N = 123456789$, show that there exists an n : $N \mid F_n$ ← Fibonacci
$n > 0$

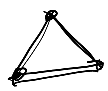
$(F_0 = 0, F_1 = 1, F_2 = 1, \ldots)$

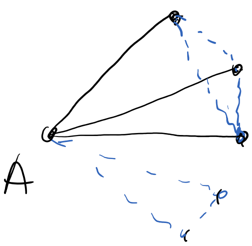
Back to graphs and Pigeonhole principle. → Ramsey theory.

6 people at a party, some are friends

⇒ either 3 → all friends, or 3 → no 2 know each other.

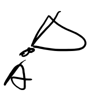






$5 =$ edges + nonedges from A

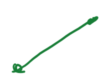
at least one is ≥ 3

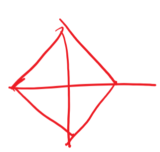
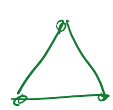
↖ 3 edges, if one edge →  ✓.

if no edge: →  → 3 of nonedges

if 3 nonedges ---

Problem: A graph on 9 vertices, ← all $\binom{9}{2}$ edges present (complete graph / K_9)

→ if each edge:  or  ⇒ then there is

either:  or 

$R(4,3)$

(if 8 vertices: → not nec. true)