Math 395: problems 1

Due Tue 01/26 by 9:30am, uploaded in blackboard

Instructions: Solve (with proof) and submit 2 of the following problems, arranged in order of increasing difficulty.

Notation: $[n] := \{1, \ldots, n\}$, the set of the first n positive integers.

$\#\{...\}$ denotes "the number of elements in the given set. Note that in many problems the set consists of sets.

1. Let the sequence a_n be defined recursively by
$$a_{n+1} = 3a_n - 2a_{n-1} \quad \text{for all } n \geq 1$$
with initial conditions $a_0 = 2, a_1 = 3$. Prove that $a_n = 2^n + 1$.

2. Let $f(n) := \#\{S \subset [n],$ such that there are no $x, y \in S$ for which $x - y = 1\}$. What is the sequence $f(n)$ (write initial values, recursion and identify it with a famous sequence)?

   Example: $f(3) = 5$ because
$$\{S \subset [3], \text{ such that there are no } x, y \in S \text{ for which } x - y = 1\} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}\}.$$

3. Let $b_r(n) := \#\{S \subset [2n], \text{s.t. there are no } x, y \in S \text{ for which } x - y = r\}$. Find a formula for $b_n(n)$. Bonus: what can you say about $b_2(n)$ and $b_{n-1}(n)$.

4. Find and prove [by induction] a [closed-form] formula for $g(n)$ – the sum of the squares of the first n odd numbers, i.e.
$$g(n) = 1^2 + 3^2 + \cdots + (2n - 1)^2.$$

5. Let $h_k(n) = \#\{S \subset [n],$ such that $3|(\#S - k)\}$ for $k = 0, 1, 2$ and any n, i.e. $h_0(n)$ is the number of subsets whose size (number of elements) is divisible by 3 and so on. We can express them as
$$h_k(n) = \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n}{3i + k}.$$
Find a closed form formula in terms of n (and note that the answer would depend on the residue of n by division by 6).

   Hint: Try to find recurrences and prove by induction.

6. For the Putnam trainees: find (and solve, or ask me about) an interesting (and hard) problem from the Putnam (or similar) which fits the current topic. As sources you can use the Putnam Archive of Kiran Kedlaya, or IMO, or "AoPS", or any of the sources listed in the syllabus.