PROBLEMS: POLYNOMIALS

Useful facts:

Conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{\sum P(B|A_i)P(A_i)},$$

where the last equality holds whenever $A_1 \cup A_2 \cdots \cup A_n = \Omega$, Ω is the space of all outcomes and $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

Independent events: $P(A \cap B) = P(A)P(B)$.

Expectation of a random (integral) variable X:

$$E[X] = \sum_k kP(X = k).$$

Linearity of expectation: $E[X + Y] = E[X] + E[Y]$.

(1) You throw three (standard, fair) dice one after the other. What is the probability that the numbers on top appear in a strictly increasing sequence?

(2) There is a clay (so nontransparent) jar with r red, g green and b blue jelly beans. Each turn you reach in the jar and take one jelly bean (so uniformly at random from what's inside). What is the probability that you'll take all red jellybeans before all the green and all the blue ones? Assume $0 < r < b < g$.

(3) Same setup as above. What is the expected number of beans you'd take out until you've gotten all the red ones out?

(4) You toss a fair coin multiple times, and record the outcomes creating a sequence of T and H. What is the expected number of tosses until the coin lands Heads 3 times in a row, i.e. HHH appears in the sequence? What is the expected number of tosses until THH appears in the sequence?

What is the probability that HHT appears before HTH?

(5) A random variable X takes only nonnegative integer values. Suppose that $E[X] = 1$, $E[X^2] = 2$ an $E[X^3] = 4$. What is the smallest possible value of the probability $P(X = 0)$.

(6) Find (and solve, or ask me about) an interesting (and hard) problem from the Putnam (or similar) which fits the current topic. As sources you can use the Putnam Archive of Kiran Kedlaya, or IMO, or "AoPS", or any of the sources listed in the syllabus.