PROBLEMS: ANALYSIS III

(1) A marathon runner ran a 10 km course in 50 minutes. Prove that there is a 1 km stretch which he ran in exactly 5 min.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that every real number is a value of f.

(3) Prove that every convex compact surface (e.g. convex polygon) can be divided with two perpendicular lines into 4 regions all of equal areas.

(4) Prove that for all real positive a, b and $n \in \mathbb{N}_{\geq 1}$ we have

$$(n-1)a^n + b^n \geq na^{n-1}b.$$

(5) Let $a_1 < \ldots < a_n$ be n distinct real numbers and define

$$f(x) = \sum_{i=1}^{n} \frac{(x-a_1)\cdots(x-a_n)}{x-a_i},$$

which is a polynomial of degree $n-1$ in x. Show that $f(x) < 0$ for $x = a_{n-1}, a_{n-3}, a_{n-5}, \ldots$.

(6) Find all functions $f : \mathbb{R} \to \mathbb{R}$ differentiable at $x = 0$, such that

$$f(3x) + f(2x) + f(x) = e^{3x} + e^{2x} + e^{x}.$$

(7) Find (and solve, or ask me about) an interesting (and hard) problem from the Putnam (or similar) which fits the current topic. As sources you can use the Putnam Archive of Kiran Kedlaya, or IMO, or "AoPS", or any of the sources listed in the syllabus.