PROBLEMS: ANALYSIS II

(1) Let $f(x) : \mathbb{R}_{>0} \to \mathbb{R}$ be a function satisfying
$$f(f(x)) = x^2 - 2x + 3 - 1/x.$$
Prove that is has a unique fixed point.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(3x) - f(x) = x$ for every x, and $f(0) = 2$. Find $f(x)$.

(3) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$f(f(f(f(x)))) = x^4 \qquad \text{and} \qquad f(f(f(f(f(f(x)))))) = x^6.$$
Find f or show that it does not exist.

(4) How many real positive roots does the equation
$$e^x - x/2 - \sin(x)/3 = 100$$
have?

(5) Does there exist a nonconstant function $f : (1, +\infty) \to \mathbb{R}$, such that
$$f(x) = f(\frac{x^2 + 2}{3})$$
such that the limit $\lim_{x \to \infty} f(x)$ exists?

(6) Find (and solve, or ask me about) an interesting (and hard) problem from the Putnam (or similar) which fits the current topic. As sources you can use the Putnam Archive of Kiran Kedlaya, or IMO, or "AoPS", or any of the sources listed in the syllabus.