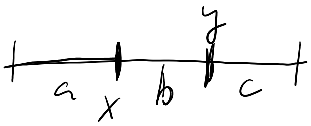
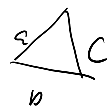
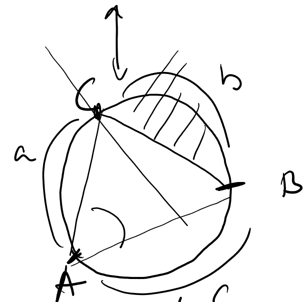
(1)







$b < a+c$ | $c < a+b$

$\angle A < 90^\circ$ | $\angle C < 90^\circ$

prob of an acute triangle → n class → 1/4.

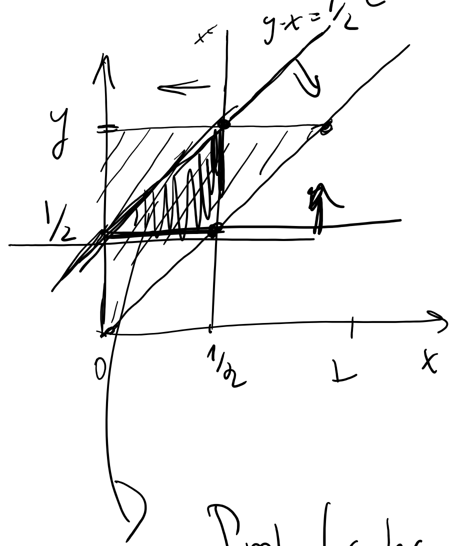
assume by symmetry $\boxed{x<y}$

$a = x,\ b = y-x,\ c = 1-y$

$\underline{x} < y-x+1-y = \underline{1-x}$ $\quad x < 1/2$

$\underbrace{y-x}_{b} < \underbrace{x+1-y}_{a+c} \iff y-x < 1/2$

$1-y < x+y-x \iff y > 1/2$



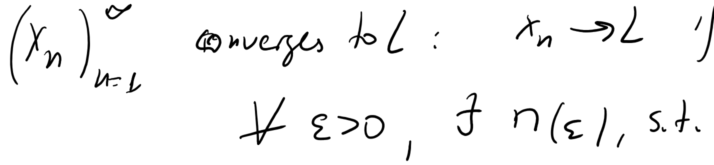
$$\text{Prob}(a\,b\,c \text{ form a } \Delta) = \frac{\text{area }(x,y)\text{ allowed}}{\text{total}} = 1/4.$$

Analysis: sequences, series and cont. functions.

$(x_n)_{n=1}^{\infty}$ converges to L: $x_n \to L$ iff

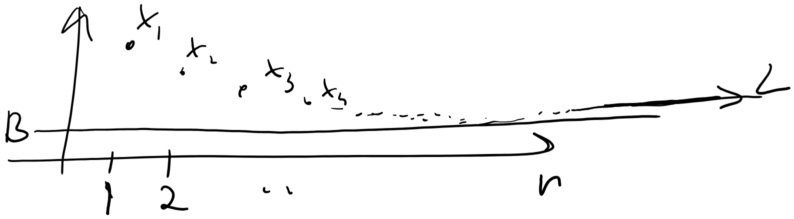
$\forall \varepsilon > 0,\ \exists\, n(\varepsilon)$, s.t. for every $n > n(\varepsilon)$

$|x_n - L| < \varepsilon$



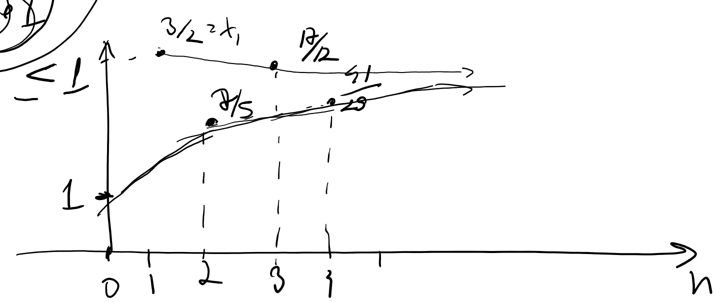
Squeeze thm: if $a_n \le \underline{b_n} \le c_n$, $a_n, c_n \to L \Rightarrow b_n \to L$

Weierstrass: if x_n is monotonic, and is bounded $\Rightarrow$ $x_n \to L$ some L.



$x_{n+1} = 1 + \frac{1}{x_n + 1}$, $x_0 = 1$, $x_1 = 3/2$, $x_2 = 7/5$, $x_3 = \frac{17}{12}$, $x_4 = \frac{41}{29}$

$x_n \in (1, 2)$



$a_n = x_{2n+1}$

$c_n = x_{2n}$

Claim: $a_n \downarrow$, $c_n \uparrow$, and also bounded.

$$\underbrace{x_{2n+1}}_{a_n} = 1 + \frac{1}{x_{2n}+1} = 1 + \frac{1}{1+\frac{1}{a_{n-1}+1}+1} = 1 + \frac{a_{n-1}+1}{2a_{n-1}+3}$$

$\Rightarrow a_n < a_{n-1} \iff 1 + \frac{a_{n-1}+1}{2a_{n-1}+3} < a_{n-1}$

$\iff a_{n-1}+1 < (a_{n-1}-1)(2a_{n-1}+3)$

$\cancel{a_{n-1}}+1 < 2a_{n-1}^2 + \cancel{a_{n-1}} - 3$

$\Rightarrow 4 < 2a_{n-1}^2 \iff a_{n-1} > \sqrt{2}$

By induction: $a_n > \sqrt{2}$ and $a_n \downarrow$, $c_n \uparrow$

$c_n < \sqrt{2}$

$\Rightarrow a_n \to L_1$

$c_n \to L_2$

$L_2 = \sqrt{2}$.

$a_n = 1 + \frac{a_{n-1}+1}{2a_{n-1}+3}$ $\to L_1$

lim $\downarrow n \to \infty$

$L_1 = 1 + \frac{L_1+1}{2L_1+3} \Rightarrow \boxed{L_1 = \sqrt{2}}$

Ex $a = \sqrt{1+\sqrt{2 \ldots}}$ does it converge!, where?

Ex $a_n = \sqrt{1+\sqrt{2+\sqrt{3+\dots+\sqrt{n}}}}$ does it converge? , where?

$\underbrace{1+\sqrt{2+\sqrt{3+\dots}}}_{}$ ... a_{n-1} $\Rightarrow$ $a_n > a_{n-1}$

$a_n^2 = 1 + \sqrt{\underset{\exists}{2}+\underbrace{\sqrt{3+\dots+\sqrt{n}}}_{1/2}} = 1+\sqrt{2}\cdot\underbrace{\sqrt{\tfrac{1}{2}+\sqrt{\tfrac{3}{4}+\sqrt{\dots \tfrac{n}{2^{\dots}}}}}}_{<\; a_{n-1}}$

suppose $\boxed{a_{n-1} < 2}$

$\Rightarrow a_n^2 < 1+\sqrt{2}\,a_{n-1}$

$\Rightarrow a_n^2 < 1+\underbrace{2\sqrt{2}}_{<3} < 4$

$\Rightarrow \boxed{a_n < 2}$

$\Rightarrow \quad 1 < a_n < 2 \;,\; a_n\uparrow \Rightarrow$ limit exists. what is it?

Ex. $a_n \downarrow$, $a_n \to 0$

$\boxed{S_n = a_1 - a_2 + a_3 - a_4 \dots + (-1)^{n-1} a_n}$ converges as well?

Cauchy's criterion $(x_n)_{n=1}^{\infty}$ converges iff $\forall \varepsilon>0$, $\exists n_\varepsilon$, s.t. $m,n > n_\varepsilon$, $|x_m - x_n| < \varepsilon$.

if $x_n \to L$, $\varepsilon' = \varepsilon/2 \Rightarrow \exists n(\varepsilon')$: $|x_n - L| < \underset{\varepsilon/2}{\varepsilon'}$ for $n > n(\varepsilon')$

for $n,m > n(\varepsilon') \Rightarrow |x_n - x_m| = |(x_n - L) - (x_m - L)| \le \underbrace{|x_n - L|}_{<\varepsilon/2} + \underbrace{|x_m - L|}_{<\varepsilon/2} = \varepsilon$

$|S_n - S_m| = |\overset{>0}{(a_{m+1} - a_{m+2})} + \overset{<0\;>0}{a_{m+3}} - \overset{<0}{\dots} + (-1)^{n-m} a_n| < \max(a_{m+1}, a_n) < \varepsilon$

$0 \leftarrow -a_n < \dots \le a_{m+1} \to 0$

for $n,m > n(\varepsilon)$

since $a_n \to 0$

$a_{m+1} \underbrace{- a_{m+2} + a_{m+3}}_{<0} \underbrace{- a_{m+4}}_{<0}$

$\underline{\qquad\qquad} < 0 \qquad \underline{\quad} < 0$

Ex. $a_n \downarrow 0$, $S_n = a_1 + \dots + a_n \to$ converging, $S_n \to L$, Show that $n a_n \xrightarrow[n\to\infty]{} 0$

Need to show: for $\varepsilon > 0$, $\exists\, n(\varepsilon)$: $|n a_n| < \varepsilon$ for $n > n(\varepsilon)$ $\uparrow$ ✓.

from Cauchy:

$|S_m - S_n| < \varepsilon'$ for $m, n > n_{\varepsilon'}$ $\qquad \varepsilon' = \varepsilon/2$

$\Downarrow$

$|S_{2n} - S_n| < \varepsilon'$

$\Downarrow$

$|a_{2n} + a_{2n+1} + \dots + a_{4n-1}| < \varepsilon'$ $\Rightarrow$ $n |a_{2n}| < \varepsilon'$

$\Downarrow$ $\qquad\qquad$ $|2n a_{2n}| < 2\varepsilon' = \varepsilon$

$n a_{2n}$ $\qquad \Rightarrow \quad \Rightarrow \quad |a_n| < 1/n$ for $n > n(\varepsilon)$

$|S_{2n-1} - S_n|$ . . .

Counterexample: $a_n = \frac{(-1)^n}{n}$; $S_n = -1 + 1/2 - 1/3 \dots \to L$ ✓.

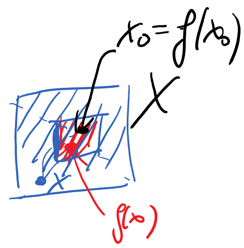
$|n a_n| = 1 \not\to 0$

Fixed point.

$f : X \to X$, $X \subset \mathbb{R}^n$

f – continuous function, : if $x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$

Suppose: $\|f(x) - f(y)\| \le c\,\|x - y\|$, $c < 1$



Theorem:

$\Rightarrow$ f has a unique fixed point : $x_0 = f(x_0)$

($f(x_0) = x_0$, $f(y_0) = y_0$ $\Rightarrow$ $\|\underbrace{x_0 - y_0}_{f(x_0) - f(y_0)}\| \le c\,|$