Monday, March 29, 2021 6:00 PM

Continuous probability in analysis (integrals)

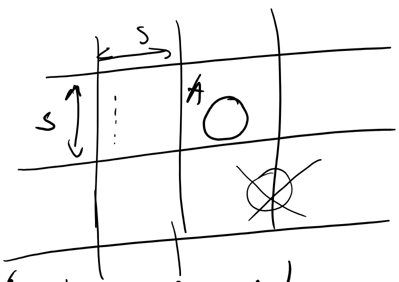
in class..

① $\int \frac{\sin x}{\sin x + \cos x}\,dx = ?$

② $\int (1+2x^2)e^{x^2}\,dx = ?$

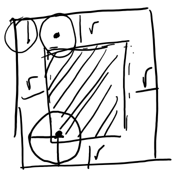
③ Putnam 2010, A6:
$f : [0,\infty) \to [0,+\infty)$, continuous,
$f \downarrow$ (strictly decreasing), $f \xrightarrow[x\to\infty]{} 0$.
Prove that $\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\,dx$ diverges.

①

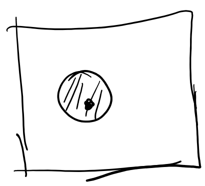


Prob (coin is entirely in one square)

- center in square A



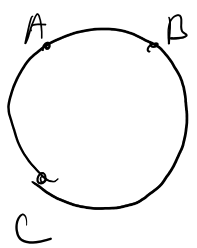
$\frac{\text{Area}(\,\square\,)}{s^2} = \frac{(s-2r)^2}{s^2}$

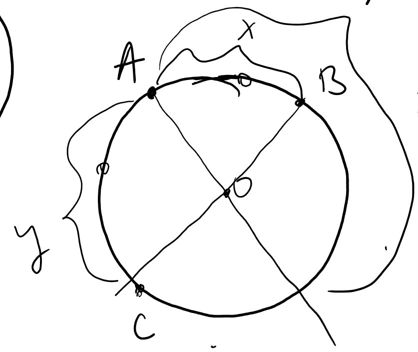


point inside circle.

$\frac{\text{area (circle)}}{s^2} = \frac{\pi r^2}{s^2}$.

② pick 3 points, $P(\triangle ABC \text{ is acute angled}) = ?$



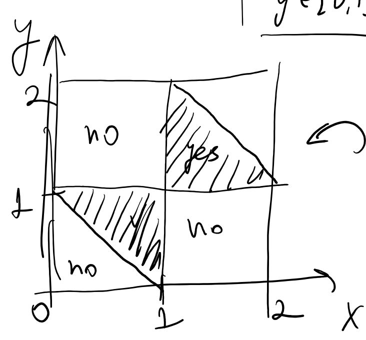


$x + y > 1$

$\int_{\substack{x+y>1 \\ x\in[0,1] \\ y\in[0,1]}} 1\,dx\,dy$

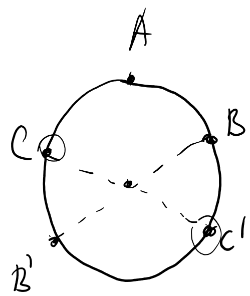
$x \in [0,2]$
$y \in [0,2]$

if $x \in [1,2]$
$y \in [1,2]$



integrate over

$\frac{\quad}{2\times 2} = \frac{1}{4}$.



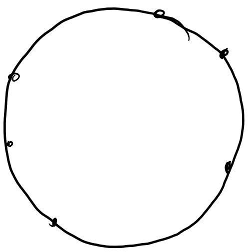
exactly one acute triangle

4 choices

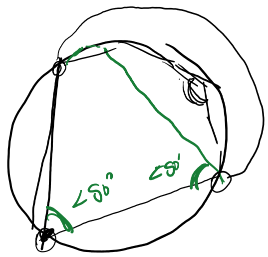
$AB'C'$ → $\frac{1}{4}$

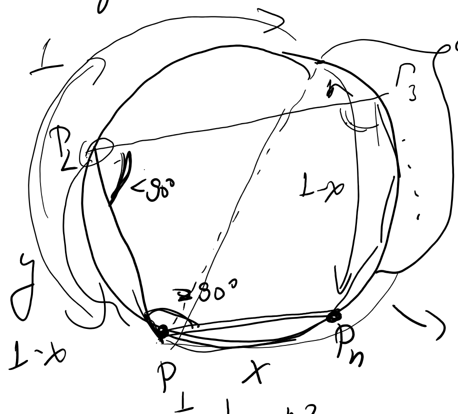
③ Prob. $n \geq 4$ points $P_1, \dots, P_n$ → polygon they form has at least 1 acute angle.

Prob. points $P_1, \dots, P_n \to$ polygon they form has at least 1 acute angle.



at most 2 acute angles:





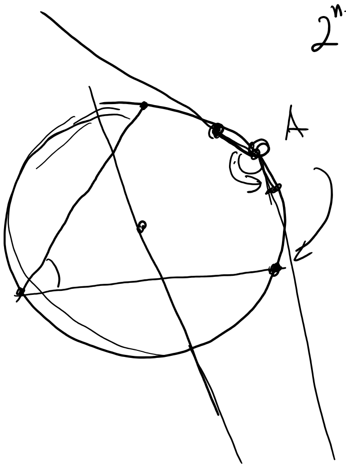
$$\int_0^1 \frac{(1-x)^{n-3}}{2^{n-3}} \frac{(1-x)}{2} dx$$

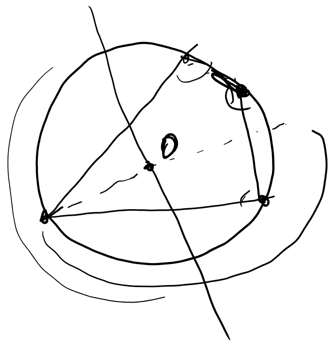
$y \leq 1-x$

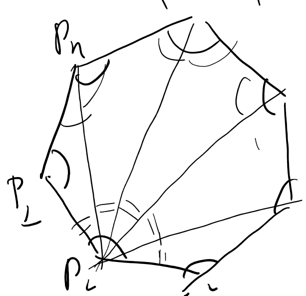
$$\frac{1}{2^{n-2}} \int_0^1 \underbrace{(1-x)}_{t}^{n-2} dx \quad x = 1-t \quad = \frac{1}{2^{n-2}} \int_0^1 t^{n-2} dt = \boxed{\frac{1}{(n-1)\, 2^{n-2}}}.$$

$$\frac{1}{n-1} t^{n-1} \Big|_0^1$$

$$\frac{n-2}{(n-1)\, 2^{n-2}}$$ choice of points.







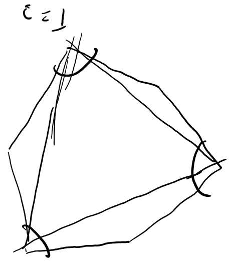
Convex n-gon.

$$\sum_{i=1}^{n} \measuredangle P_i = 180^\circ (n-2)$$

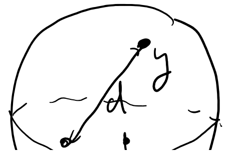
$\measuredangle P_1 < 90^\circ, \measuredangle P_2 < 90^\circ, \measuredangle P_3 < 90^\circ, \measuredangle P_3 < 90^\circ \to \boxed{\text{no 4 acute angles.}}$

$$180^\circ(n-2) = \sum_{i=1}^{n} \measuredangle P_i < 90^\circ \cdot \cancel{4} + (n-\cancel{4}) \cdot \underbrace{180^\circ} \to \underline{no}.$$

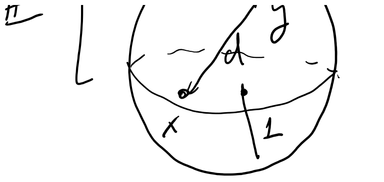
3 acute angles if not inscribed in a circle.



$$\mathbb{E}\left[d^2 \right] = \mathbb{E}\left[\|\vec{x} - \vec{y}\|^2 \right]$$



$\|\vec{x} - \vec{y}\|^2 + \|\vec{x} + \vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2 = 4$



Sphere radius $\underline{\underline{1}}$

$$\|\vec{x}-\vec{y}\|^2+\|\vec{x}+\vec{y}\|^2 = \underline{2\|\vec{x}\|^2+2\|\vec{y}\|^2} = 4$$

$$\|\vec{x}\|^2 - \cancel{2\vec{x}\vec{y}} + \|\vec{y}\|^2$$

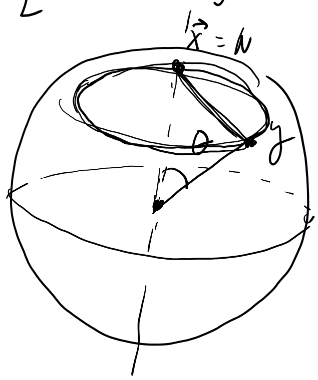
$$\mathbb{E}\left[\|\vec{x}-\vec{y}\|^2+\|\vec{x}+\vec{y}\|^2\right] = 4$$

$$\mathbb{E}\left[\|\vec{x}-\vec{y}\|^2\right] + \mathbb{E}\left[\|\vec{x}+\vec{y}\|^2\right] \quad \Rightarrow \quad \underline{\mathbb{E}\left[\|\vec{x}-\vec{y}\|^2\right] = 2}$$

$\vec{x}$ or distance $-\vec{y}$

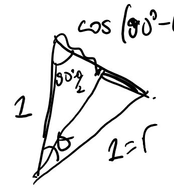
$$\mathbb{E}\left[\|\vec{x}-\vec{y}\|\right] \stackrel{?}{=}$$

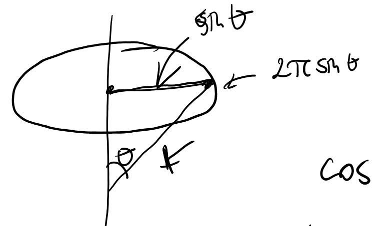
$$\mathbb{E}\left[\|\vec{x}-\vec{y}\|\right] = \int \|\vec{x}-\vec{y}\| \, d\,\mathrm{Prob}(x,y), \quad d\,\mathrm{Prob}(y)$$



$$\|\vec{x}-\vec{y}\| = 2\overset{r}{\sin}\,\theta/2$$

$$\cos(90^\circ-\theta/2) = \sin\theta/2$$





$$= \frac{1}{\cancel{4\pi r^2}} \int_0^{\pi} \cancel{2r}\sin\theta/2 \cdot \cancel{2\pi r}\sin\theta \, d\theta$$

$$= \int_0^{\pi} \sin\theta/2 \cdot \sin\theta \, d\theta$$

$$\cos\theta = 1 - 2\sin^2\theta/2$$

$$\sin\theta = \underline{2\sin\theta/2\cos\theta/2}$$

$$= \int_0^{\pi} \underbrace{\sin\theta/2}_{t} \cdot 2\underbrace{\sin\theta/2}_{t} \cdot \underbrace{\cos\theta/2\, d\theta}_{2\cdot d(\sin\theta/2)}$$

$t = \sin\theta/2$

$$= 4\int_0^1 t^2\,dt = 4\cdot\frac{1}{3}t^3\Big|_0^1 = 4/3$$

Integrals:

$$\int f(u(x))\,\underline{u'(x)dx} = \int f(u)\,du \quad , \quad \int u\,dv = uv - \int v\,du$$

$$d\tan x = \frac{1}{\cos^2 x}dx$$

(1) $I = \int \frac{\sin x}{\ldots}dx = ?$

① $I_1 = \int \frac{\sin x}{\sin x + \cos x}\,dx = ?$

$I_2 = \int \frac{\cos x}{\sin x + \cos x}\,dx$

$\underline{I_1 + I_2} = \int 1\,dx = \underline{x + C}$

$\underline{I_2 - I_1} = \int \frac{\boxed{(\cos x - \sin x)\,dx}}{\sin x + \cos x}$

$d\sin x = \cos x\,dx$

$d\cos x = -\sin x\,dx$

$d(\cos x + \sin x) = \underline{(-\sin x + \cos x)\,dx}$

$\int \frac{d(\overbrace{\sin x + \cos x}^{t})}{\underbrace{\sin x + \cos x}_{t}} = \int \frac{dt}{t} = \ln t + C_1 = \underline{\ln|\sin x + \cos x| + C_1}$

$I_1 = \frac{I_1 + I_2 - (I_2 - I_1)}{2} = \frac{x - \ln|\sin x + \cos x|}{2} + C_3$

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② $\underline{\int (1 + 2x^2) e^{x^2} dx} = \overbrace{\int e^{x^2} dx}^{F(x)} + \underline{\int 2x^2 e^{x^2} dx}$

erf $\quad \underline{F(x) = \int e^{x^2} dx} \Rightarrow$ not an elementary functions.

$\int_{-\infty}^{\infty} e^{x^2} dx = ?$

$= x\, e^{x^2} \underbrace{2x\,dx}_{dx^2}$

$\int x\, e^{x^2} d(x^2) \quad t \qquad e^t dt = de^t$

$\int x\, d e^{x^2}$

$= x e^{x^2} - \underbrace{\int e^{x^2} dx}_{F(x)}$

$\cancel{F(x)} + x e^{x^2} - \cancel{F(x)} = \underline{x e^{x^2}}$