

Homework 1

● Graded

Student

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Total Points

55 / 60 pts

Question 1

Problem 1

20 / 20 pts

✓ + 5 pts part (1) Correct

✓ + 5 pts part (2) Correct

✓ + 5 pts part (3) Correct

✓ + 5 pts part (4) Correct

Question 2

Problem 2

20 / 20 pts

✓ + 5 pts complete part (1)

✓ + 5 pts complete part (2)

✓ + 5 pts complete part (3)

✓ + 5 pts complete part (4)

Question 3

Problem 3

15 / 20 pts

✓ + 5 pts part (1) Correct

✓ + 5 pts part (2) Correct

✓ + 5 pts part (3) Correct

+ 5 pts part (4) Correct

💬 didn't find your solution for part (4)

No questions assigned to the following page.

1.1

$$f(x|p) = \frac{N! \cdot p_1^{x_1} \cdot p_2^{x_2} \cdots p_{k-1}^{x_{k-1}}}{x_1! x_2! \cdots x_{k-1}! (N-S)!} (1-q)^S$$

where $S = \sum_{i=1}^{k-1} x_i$ and $q = \sum_{i=1}^{k-1} p_i$

$\therefore f(x_1, \dots, x_{k-1} | p_1, \dots, p_{k-1}) = f(x|p)$ and
is given in the formula above.

1.2 We can represent the above formula in canonical form as below :-

$$\begin{aligned} & \frac{N! \prod_{i=1}^{k-1} p_i^{x_i} q^S}{\prod_{i=1}^{k-1} x_i! (N-S)!} = \frac{N!}{\prod_{i=1}^{k-1} x_i! \cdot (N-S)!} \exp \left(\sum_{i=1}^{k-1} x_i \log p_i + S \log (1-q) \right) \\ &= \frac{N!}{\prod_{i=1}^{k-1} x_i! (N-S)!} \exp \left\{ \sum_{i=1}^{k-1} x_i \log p_i + \left(N - \sum_{i=1}^{k-1} x_i \right) \log (1-q) \right\} \\ &= \frac{N!}{\prod_{i=1}^{k-1} x_i! (N-S)!} \exp \left\{ \sum_{i=1}^{k-1} x_i \log p_i + N \log (1-q) - \sum_{i=1}^{k-1} x_i \log (1-q) \right\} \\ &= \frac{N!}{\prod_{i=1}^{k-1} x_i! (N-S)!} \exp \left\{ \sum_{i=1}^{k-1} x_i [\log p_i - \log (1-q)] + N \log (1-q) \right\} \end{aligned}$$

No questions assigned to the following page.

$$= \frac{N!}{\prod_{i=1}^{k-1} x_i! (N-s)!} \exp \left\{ \sum_{i=1}^{k-1} x_i \left[\log \frac{p_i}{1-q} \right] + N \log(1-q) \right\}$$

Set $\eta_i(\vec{p}) = \log \left(\frac{p_i}{1-q} \right) = \log \left(\frac{p_i}{1 - \sum_{i=1}^{k-1} p_i} \right)$

Set $t_i(\vec{x}) = x_i$

$$A(\vec{p}) = -N \log(1-q) = N \log \left(\frac{1}{1-q} \right)$$

$$h(\vec{x}) = \frac{N!}{\prod_{i=1}^{k-1} x_i! (N-s)!} \quad (\text{Base measure})$$

$$\therefore f(x_1, \dots, x_{k-1} | p_1, p_2, p_3, \dots, p_{k-1}) = h(x) \exp \left\{ \sum_{i=1}^{k-1} \eta_i \cdot t_i - A(\vec{p}) \right\}$$

* Now we want to express $f(x_1, \dots, x_{k-1} | p_1, \dots, p_{k-1})$ in terms of canonical parameters.

$$\eta_i = \log \left(\frac{p_i}{1 - \sum_{j=1}^{k-1} p_j} \right) \Rightarrow e^{\eta_i} = \frac{p_i}{1-q}$$

$$\Rightarrow \sum_{i=1}^{k-1} e^{\eta_i} = \frac{1}{1-q} \sum_{i=1}^{k-1} p_i = \frac{q}{1-q}$$

$$\begin{aligned} \Rightarrow 1 + \sum_{i=1}^{k-1} e^{\eta_i} &= \frac{q}{1-q} + 1 = \frac{q+1-q}{1-q} \\ &= \frac{1}{1-q} \end{aligned}$$

No questions assigned to the following page.

$$\Rightarrow 1 + \sum_{i=1}^{k-1} e^{n_i} = \frac{1}{1-q}$$

$$\Rightarrow N \log \left(1 + \sum_{i=1}^{k-1} e^{n_i} \right) = \underbrace{N \log \left(\frac{1}{1-q} \right)}_{A(\vec{p})}$$

$$\Rightarrow A(\vec{p}) = N \log \left(1 + \sum_{i=1}^{k-1} e^{n_i} \right)$$

$\underbrace{\qquad\qquad\qquad}_{A^*(\vec{n})}$

$$\therefore A^*(\vec{n}) = N \log \left(1 + \sum_{i=1}^{k-1} e^{n_i} \right)$$

$$\text{Hence } P(x_1, \dots, x_{k-1} | p_1, p_2, \dots, p_{k-1})$$

$$= h(x) \exp \left\{ \vec{n}(\vec{p}) \cdot \vec{T}(\vec{x}) - A^*(\vec{n}) \right\}$$

where $\vec{T}(\vec{x}) = [x_1, x_2, \dots, x_{k-1}]$

$$n_i(\vec{p}) = \log \frac{p_i}{1 - \sum_{j=1}^{k-1} p_j}$$

$$\text{and } h(x) = \frac{N!}{\prod_{i=1}^{k-1} x_i! \underbrace{(N-s)!}_{x_k}} = \frac{N!}{\prod_{i=1}^k x_i!}$$

No questions assigned to the following page.

1.3

$$A^*(\vec{n}) = N \log \left(1 + \sum_{i=1}^{k-1} e^{n_i} \right)$$

as shown in last part 1.2

The natural parameter space is defined as

$$\mathcal{H} = \left\{ \vec{n} : A^*(\vec{n}) < \infty, \vec{n} \in \mathbb{R}^{k-1} \right\}$$

i.e. the natural parameter space is the set that keeps A^* well defined.

This is so because of the following reason

$$\begin{aligned} p(\vec{x} | \vec{n}) &= h(\vec{x}) \exp \left\{ \vec{n} \cdot \vec{T}(\vec{x}) - A^*(\vec{n}) \right\} \\ &= \frac{h(\vec{x})}{e^{A^*(\vec{n})}} \exp \left\{ \vec{n} \cdot \vec{T}(\vec{x}) \right\} \end{aligned}$$

If $A^*(\vec{n}) \rightarrow \infty$, $p(\vec{x} | \vec{n}) \rightarrow 0$

In our question, $A^*(\vec{n}) = N \log \left(1 + \sum_{i=1}^{k-1} e^{n_i} \right)$

$$A^*(\vec{n}) = \infty \Rightarrow \sum_{i=1}^{k-1} e^{n_i} = \infty \Rightarrow n_i \neq \infty$$

No questions assigned to the following page.

\therefore The natural parameter space is

$$\mathcal{H} = \{ n_i < \infty \} = \{ n_i \in \mathbb{R} \}.$$

$$\therefore \mathcal{H} = \{ \vec{n} \in \mathbb{R}^{k-1} \}$$

1.4 $A(\vec{n}) = N \log \left(1 + \sum_{i=1}^{k-1} e^{n_i} \right)$

$$E(T_i(\vec{x})) = E[x_i] = \frac{\partial A}{\partial n_i} = \frac{N e^{n_i}}{1 + \sum_{i=1}^{k-1} e^{n_i}}$$

$\boxed{\because T_i(\vec{x}) = x_i}$

Observation :-

Recall that $e^{n_i} = \frac{p_i}{1-q}$

where $q = \sum_{i=1}^{k-1} p_i$

$$\sum_{i=1}^{k-1} e^{n_i} = \frac{1}{1-q} \sum_{i=1}^{k-1} p_i = \frac{q}{1-q}$$

$$\Rightarrow 1 + \sum_{i=1}^{k-1} e^{n_i} = \frac{1}{1-q}$$

No questions assigned to the following page.

$$\therefore e^{n_i} = \frac{p_i}{1-q} = p_i \left(1 + \sum_{i=1}^{k-1} e^{n_i} \right)$$

$$\Rightarrow p_i = \frac{e^{n_i}}{1 + \sum_{i=1}^{k-1} e^{n_i}}$$

$$\begin{aligned}\therefore E[T_i(\vec{x})] &= E[x_i] \\ &= N \frac{e^{n_i}}{1 + \sum_{i=1}^{k-1} e^{n_i}} = N p_i\end{aligned}$$

$$\text{Cov}(T_i(\vec{x}), T_j(\vec{x})) = \text{Cov}(x_i, x_j)$$

\$\because T_i(\vec{x}) = x_i\$

$$= \frac{\partial^2 A}{\partial n_j \partial n_i} = \frac{\partial}{\partial n_j} \left(\frac{\partial A}{\partial n_i} \right) = \frac{\partial}{\partial n_j} (N p_i)$$

$$= N \frac{\partial p_i}{\partial n_j} = N \frac{\partial}{\partial n_j} \left(\frac{e^{n_i}}{1 + \sum_{i=1}^{k-1} e^{n_i}} \right)$$

$$= N e^{n_i} \frac{-1 \cdot e^{n_j}}{\left(1 + \sum_{i=1}^{k-1} e^{n_i} \right)^2} = -N \frac{e^{n_i} e^{n_j}}{\left(1 + \sum_{i=1}^{k-1} e^{n_i} \right)^2}$$

No questions assigned to the following page.

$$\text{Cov}(x_i, x_j) = -N \frac{e^{n_i}}{\left(1 + \sum_{l=1}^{k-1} e^{n_l}\right)} \cdot \frac{e^{n_j}}{\left(1 + \sum_{l=1}^{k-1} e^{n_l}\right)}$$

$$= -N p_i p_j$$

For Variance, $\text{Var}(x_i) = \frac{\partial^2 A}{\partial n_i^2}$

$$\frac{\partial^2 A}{\partial n_i^2} = \frac{\partial}{\partial n_i} \left(\frac{\partial A}{\partial n_i} \right) = \frac{\partial}{\partial n_i} \left(\frac{N e^{n_i}}{1 + \sum_{l=1}^{k-1} e^{n_l}} \right)$$

$$= N \left[\frac{\left(1 + \sum_{l=1}^{k-1} e^{n_l}\right) e^{n_i} - e^{n_i} \cdot e^{n_i}}{\left(1 + \sum_{l=1}^{k-1} e^{n_l}\right)^2} \right]$$

$$= N \left[\frac{e^{n_i} \left\{ \left(1 + \sum_{l=1}^{k-1} e^{n_l}\right) - e^{n_i} \right\}}{\left(1 + \sum_{l=1}^{k-1} e^{n_l}\right)^2} \right]$$

$$= N \left[\frac{e^{n_i}}{\left(1 + \sum_{l=1}^{k-1} e^{n_l}\right)} \cdot \frac{\left(1 + \sum_{l=1}^{k-1} e^{n_l}\right) - e^{n_i}}{\left(1 + \sum_{l=1}^{k-1} e^{n_l}\right)} \right]$$

No questions assigned to the following page.

$$\Rightarrow \text{Var}(X_i) = N \left[p_i \cdot (1-p_i) \right]$$

$$\therefore \text{Cov}(X_i, X_j) = \begin{cases} N p_i (1-p_i); i=j \\ -N p_i p_j; i \neq j \end{cases}$$

Q2 CUMULANT GENERATING FUNCTION

$$K_n(x) = \frac{d^n}{dt^n} K_x(t) \Big|_{t=0}$$

2.1 $K_1(x) = \mathbb{E}X \leftarrow \text{to show}$

$$\text{LHS} = K_1(x) = \frac{d}{dt} K_x(t) \Big|_{t=0}$$

$$\begin{aligned} \frac{d}{dt} \log \mathbb{E}[e^{tx}] &= \mathbb{E} \left[\frac{\partial}{\partial t} e^{tx} \right] \times \frac{1}{\mathbb{E}[e^{tx}]} \\ &= \frac{\mathbb{E}[xe^{tx}]}{\mathbb{E}[e^{tx}]} \end{aligned}$$

$$\text{@ } \underline{t=0}, \quad \frac{d}{dt} K_x(t) \Big|_{t=0} = \frac{\mathbb{E}[x]}{\mathbb{E}[1]} = \mathbb{E}[x] = \text{RHS}$$

No questions assigned to the following page.

$$\star K_2(x) = \text{Var}(x) = \left. \frac{d^2}{dt^2} K_x(t) \right|_{t=0}$$

$$\Rightarrow K_2(x) = \frac{d}{dt} \left\{ \frac{d}{dt} K_x(t) \right\}$$

$$= \frac{d}{dt} \frac{\mathbb{E}[xe^{tx}]}{\mathbb{E}[e^{tx}]}$$

$$= \frac{\mathbb{E}[x^2 e^{tx}] \mathbb{E}[e^{tx}] - \mathbb{E}[x e^{tx}] \mathbb{E}[x e^{tx}]}{\mathbb{E}[e^{tx}]^2}$$

$$= \frac{\mathbb{E}[x^2 e^{tx}] \mathbb{E}[e^{tx}] - \mathbb{E}[x e^{tx}]^2}{\mathbb{E}[e^{tx}]^2}$$

$$@ t=0, K_2(x) = \frac{\mathbb{E}[x^2] \mathbb{E}[1] - \mathbb{E}[x]^2}{\mathbb{E}[1]^2}$$

$$\Rightarrow K_2(x) = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

Let $\mathbb{E}[x] = \mu$ (a constant), then

$$K_2(x) = \mathbb{E}[x^2] - \mu^2$$

$$= \mathbb{E}[x^2 - \mu^2]$$

$$= \mathbb{E}[x^2 - 2x\mu + \mu^2 + 2x\mu - 2\mu^2]$$

No questions assigned to the following page.

$$\begin{aligned}
&= E[x^2 - 2x\mu + \mu^2 + 2x\mu - 2\mu^2] \\
&= E[(x-\mu)^2] + E[2x\mu - 2\mu^2] \\
&= \underbrace{E[(x-\mu)^2]}_{\text{Var}(x)} + 2\underbrace{E[x]\mu}_{\mu} - E[2\mu^2] \\
&= \text{Var}(x) + 2\cancel{\mu^2} - 2\cancel{\mu^2} \\
&= \text{Var}(x)
\end{aligned}$$

2.2 $X \sim N(\mu, \sigma^2)$

$$\begin{aligned}
M_X(t) &= E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x) dx \\
\Rightarrow M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&\quad \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx \\
K_X(t) &= \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) + \log \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx
\end{aligned}$$

No questions assigned to the following page.

$$\begin{aligned}
 K_x(t) &= \underbrace{\log\left(\frac{1}{\sqrt{2\pi}\sigma^2}\right)}_{\text{Constant}(C)} + \log \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx \\
 &= C + \log \int_{-\infty}^{\infty} e^{\frac{2\sigma^2 tx - (x-\mu)^2}{2\sigma^2}} dx \\
 &= C + \log \int_{-\infty}^{\infty} e^{\frac{2\sigma^2 t x - x^2 - \mu^2 + 2x\mu}{2\sigma^2}} dx \\
 &= C + \log \int_{-\infty}^{\infty} e^{-\frac{\mu^2}{2\sigma^2}} \cdot e^{-\frac{x^2 + 2x(\sigma^2 t + \mu)}{2\sigma^2}} dx \\
 &= C + \underbrace{\log e^{-\frac{\mu^2}{2\sigma^2}}}_{\text{Constant}(D)} + \log \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2 - 2x(\sigma^2 t + \mu))} dx \\
 &= D + \log \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[(x - (\sigma^2 t + \mu))^2 - (\sigma^2 t + \mu)^2]} dx \\
 &= D + \log \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{x - (\sigma^2 t + \mu)^2\}} \cdot e^{\frac{1}{2\sigma^2}(\sigma^2 t + \mu)^2} dx \\
 &= D + \log e^{\frac{1}{2\sigma^2}(\sigma^2 t + \mu)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{x - (\sigma^2 t + \mu)^2\}} dx
 \end{aligned}$$

No questions assigned to the following page.

$$= D + \log e^{\frac{1}{2\sigma^2}(\sigma^2 t + \mu)^2} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{x - (\sigma^2 t + \mu)^2\}} dx$$

$$= D + \frac{(\sigma^2 t + \mu)^2}{2\sigma^2} +$$

this integral doesn't
 depend on t as this is
 the integration of pdf of
 normal distribution
 with mean $\sigma^2 t + \mu$
 & variance σ^2 .

Its integral is $\sqrt{2\pi\sigma^2}$

$$\Rightarrow K_x(t)$$

$$D + \frac{(\sigma^2 t + \mu)^2}{2\sigma^2} + \log(\sqrt{2\pi\sigma^2})$$

$$\Rightarrow \frac{d}{dt} K_x(t) = \cancel{\frac{1}{2\sigma^2}} \times \cancel{2} (\sigma^2 t + \mu) \times \cancel{\sigma^2}$$

$$@) t = 0, \left. \frac{d}{dt} K_x(t) \right|_{t=0} = \mu = K_1(x)$$

$$\text{Similarly, } \frac{d^2}{dt^2} K_x(t) = \frac{d}{dt} \left(\frac{d}{dt} K_x(t) \right)$$

$$= \frac{d}{dt} (\sigma^2 t + \mu) = \sigma^2 = K_2(x)$$

No questions assigned to the following page.

$$\therefore \frac{d^2 K_x(t)}{dt^2} \Big|_{t=0} = \tau^2$$

$$\begin{aligned} \frac{d^3 K_x(t)}{dt^3} &= \frac{d}{dt} \left(\frac{d^2 K_x(t)}{dt^2} \right) \\ &= \frac{d}{dt} (\tau^2) = 0 \end{aligned}$$

$$\therefore \frac{d^n K_x(t)}{dt^n} = 0 \quad \forall n \geq 3$$

$$\therefore \frac{d^n K_x(t)}{dt^n} \Big|_{t=0} = 0$$

$$\therefore K_n(x) = \frac{d^n K_x(t)}{dt^n} \Big|_{t=0} = 0 .$$

No questions assigned to the following page.

2.3

$$\text{Let } Y = X + c$$

$$\begin{aligned}
 K_Y(t) &= \log E[e^{tY}] \\
 &= \log E[e^{t(X+c)}] \\
 &= \log E[e^{tc} e^{tx}] \\
 &= \log e^{tc} E[e^{tx}] \\
 &= tc + \log E[e^{tx}]
 \end{aligned}$$

$$K_Y(t) = tc + K_X(t)$$

$$\Rightarrow \frac{d}{dt} K_Y(t) = c + \frac{d}{dt} K_X(t)$$

$$\Rightarrow \frac{d^2}{dt^2} K_Y(t) = \frac{d^2}{dt^2} K_X(t)$$

$$\Rightarrow \frac{d^n}{dt^n} K_Y(t) = \frac{d^n}{dt^n} K_X(t)$$

$$\Rightarrow K_n(Y) = \left. \frac{d^n}{dt^n} K_Y(t) \right|_{t=0} = \left. \frac{d^n}{dt^n} K_X(t) \right|_{t=0} = K_n(X)$$

No questions assigned to the following page.

Let $Z = cX$

$$\begin{aligned}K_Z(t) &= \log \mathbb{E}[e^{Zt}] \\&= \log \mathbb{E}[e^{cXt}] \\&= \log \mathbb{E}[e^{Xct}] \\&= \log \mathbb{E}[e^{Xq}] ; q = ct \\&= K_X(q)\end{aligned}$$

$$\Rightarrow \frac{d}{dt} K_Z(t) = \frac{d}{dt} K_X(q) = \frac{d}{dq} K_X(q) \times \underbrace{\frac{dq}{dt}}$$

$$\Rightarrow \frac{d}{dt} K_Z(t) = \frac{d}{dq} K_X(q) \times C$$

$$\Rightarrow \frac{d^2 K_Z(t)}{dt^2} = C \frac{d}{dt} \left(\frac{d}{dq} K_X(q) \right)$$

$$= C \frac{d}{dq} \left(\frac{d}{dq} K_X(q) \right) \frac{dq}{dt}$$

$$= C^2 \frac{d^2 K_X(q)}{dq^2}$$

No questions assigned to the following page.

Continuing the same way, we get :

$$\frac{d^n K_Z(t)}{dt^n} = C^n \frac{d^n K_X(a)}{da^n}$$

$$\Rightarrow \left. \frac{d^n K_Z(t)}{dt^n} \right|_{t=0} = C^n \frac{d^n}{da^n} K_X(a) \Big|_{a=0}$$

$$\Rightarrow q = C t = 0$$

$$\Rightarrow K_n(Z) = C^n K_n(X)$$

$$\Rightarrow K_n(CX) = C^n K_n(X)$$

2.4

$$Y = X_1 + X_2 + \dots + X_k = \sum_{i=1}^k X_i$$

$$K_Y(t) = \log E[e^{tY}] = \log E\left[e^{t \sum_{i=1}^k X_i}\right]$$

$$= \log E\left[e^{\sum_{i=1}^k t X_i}\right]$$

$$= \log \prod_{i=1}^k E[e^{t X_i}]$$

$$= \log \prod_{i=1}^k E[e^{t X_i}] \quad (\text{using independence})$$

No questions assigned to the following page.

$$\Rightarrow K_Y(t) = \log \prod_{i=1}^k E[e^{tX_i}]$$

$$= \sum_{i=1}^k \log E[e^{tX_i}]$$

$$= \sum_{i=1}^k K_{X_i}(t)$$

$$\Rightarrow \frac{d}{dt} K_Y(t) = \sum_{i=1}^k \frac{d}{dt} K_{X_i}(t)$$

$$\Rightarrow \frac{d^n}{dt^n} K_Y(t) = \sum_{i=1}^k \frac{d^n}{dt^n} K_{X_i}(t)$$

$$\Rightarrow \left. \frac{d^n}{dt^n} K_Y(t) \right|_{t=0} = \left. \sum_{i=1}^k \frac{d^n}{dt^n} K_{X_i}(t) \right|_{t=0}$$

$$\Rightarrow K_n(Y) = \sum_{i=1}^k K_n(X_i)$$

No questions assigned to the following page.

3.1 If $T(x)$ is sufficient for θ ,
factorization theorem says

$$f(x_1, \dots, x_n | \theta) = g(T, \theta) h(x_1, \dots, x_n)$$

$$f(x_1, \dots, x_n | \theta) = \prod_{i=1}^n f_i(x_i | \theta)$$

$$= \prod_{i=1}^n e^{(i\theta - x_i)} \underbrace{\mathbb{I}_{\{x_i > i\theta\}}}_{\text{indicator function}}$$

$$= e^{\sum_{i=1}^n (i\theta - x_i)} \prod_{i=1}^k \mathbb{I}_{\{x_i > i\theta\}}$$

$$= e^{\theta \frac{n(n+1)}{2}} e^{-\sum_{i=1}^n x_i} \prod_{i=1}^k \mathbb{I}_{\{x_i > i\theta\}}$$

$$\text{Let } T = \min_i \frac{x_i}{i}$$

Note that if $T > \theta$, then $x_i > i\theta \forall i$

$$\therefore \prod_{i=1}^k \mathbb{I}_{\{x_i > i\theta\}} = \mathbb{I}_{\{T > \theta\}}$$

$$\therefore f(x_1, \dots, x_n | \theta) = \mathbb{I}_{\{T > \theta\}} e^{\theta \frac{n(n+1)}{2}} \cdot e^{-\sum_{i=1}^n x_i}$$

No questions assigned to the following page.

$$\begin{aligned}\therefore f(x_1, \dots, x_n | \theta) &= \underbrace{\mathbb{I}_{\{T \geq \theta\}} e^{\frac{\theta n(n+1)}{2}}}_{g(T, \theta)} \cdot \underbrace{e^{-\sum_{i=1}^n x_i}}_{h(x_1, \dots, x_n)} \\ &= g(T, \theta) h(x_1, x_2, \dots, x_n)\end{aligned}$$

Since we can factorize the pdf in this form, T must be a sufficient statistic for θ .

$$\begin{aligned}4.2 \quad f(x_1, \dots, x_n | \alpha, \beta) &= \prod_{i=1}^n \frac{\beta^\alpha}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\beta x_i} \\ &= \left(\frac{\beta^\alpha}{\Gamma(\alpha)} \right)^n \prod_{i=1}^n x_i^{\alpha-1} e^{-\beta \sum_{i=1}^n x_i} \\ &= \frac{\beta^{n\alpha}}{\Gamma(\alpha)^n} \left(\prod_{i=1}^n x_i \right)^{\alpha-1} e^{-\beta \sum_{i=1}^n x_i}\end{aligned}$$

$$\text{Set } T_1(x_1, x_2, \dots, x_n) = \prod_{i=1}^n x_i$$

$$T_2(x_1, \dots, x_n) = \sum_{i=1}^n x_i$$

No questions assigned to the following page.

$$\Rightarrow f(x_1, \dots, x_n | \alpha, \beta) \\ = \frac{\beta^{n\alpha}}{\Gamma(\alpha)^n} (T_1)^{\alpha-1} e^{-\beta T_2}$$

Set $g(T, \alpha, \beta) = \frac{\beta^{n\alpha}}{\Gamma(\alpha)^n} (T_1)^{\alpha-1} e^{-\beta T_2}$

and $h(x_1, \dots, x_n) = 1$

Hence $T_1 = \prod_{i=1}^n x_i$ & $T_2 = \sum_{i=1}^n x_i$ are

sufficient for α, β .

$$T(\vec{x}) = \begin{bmatrix} \prod_{i=1}^n x_i \\ \sum_{i=1}^n x_i \end{bmatrix}$$

4.3

$$f(x_1, \dots, x_n | \mu, \sigma) \\ = \prod_{i=1}^n \frac{1}{\sigma} e^{-(x_i - \mu)/\sigma} \mathbb{I}_{\{x_i > \mu\}}$$

No questions assigned to the following page.

$$f(x_1, \dots, x_n | \mu, \sigma) = \left(\frac{1}{\sigma}\right)^n e^{-\frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu)} \\ \times \prod_{i=1}^n \mathbb{I}_{\{x_i > \mu\}}$$

Set $T_1 = \min \{x_1, \dots, x_n\}$

$$f(x_1, \dots, x_n | \mu, \sigma) = \left(\frac{1}{\sigma}\right)^n e^{-\frac{1}{\sigma^2} \sum_{i=1}^n x_i} e^{\frac{n\mu}{\sigma}} \\ \times \prod_{i=1}^n \mathbb{I}_{\{x_i > \mu\}}$$

Set $T_1 = \min_i \{x_i\}$

$$T_2 = \sum_{i=1}^n x_i$$

$$f(x_1, \dots, x_n | \mu, \sigma) = \left(\frac{1}{\sigma}\right)^n e^{-\frac{1}{\sigma^2} T_2} e^{\frac{n\mu}{\sigma}}$$

$$h(T_1, \mu, \sigma) = \left(\frac{1}{\sigma}\right)^n e^{-\frac{1}{\sigma^2} T_2} e^{n\mu/\sigma} \cdot \mathbb{I}_{\{T_1 > \mu\}}$$

$$g(x_1, \dots, x_n) = 1$$