Homework 4

Graded

Student

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Total Points

54 / 60 pts

Question 1

Problem 1

17 / 20 pts

- ✔ **+ 10 pts** part (1) complete
- ✔ **+ 5 pts** part (2) (a) Correct
- **+ 5 pts** part (2) (b) Correct
- **+ 2 pts** Point adjustment

Question 2

Problem 2

20 / 20 pts

- ✔ **+ 5 pts** part 1 (a) Correct
- ✔ **+ 5 pts** part 1 (b) Complete
- ✔ **+ 5 pts** part 2 (a) Correct
- ✔ **+ 5 pts** part 2 (b) Correct

Question 3

Problem 3

17 / 20 pts

- ✔ **+ 5 pts** part 1 complete
- ✔ **+ 5 pts** part 2(a) Correct
- ✔ **+ 5 pts** part 2(b) Correct
- **+ 5 pts** part 3 Correct
- **+ 2 pts** Point adjustment

Question assigned to the following page: 1

HOMEWORK-4

PROBLEM 1

1. If $f(x|\theta)$ satisfies

$$\frac{d}{d\theta}\mathbb{E}_\theta\left(\frac{\partial}{\partial\theta}\log f(X|\theta)\right) = \int \frac{\partial}{\partial\theta}\left[\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)f(x|\theta)\right]dx$$

Show that

$$\mathbb{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(X|\theta)\right)^2\right] = -\mathbb{E}_\theta\left(\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right)$$

Solution: Let $\ell(\theta) = \log f(x|\theta)$, then $\ell'(\theta) = \frac{\partial}{\partial\theta}\log f(X|\theta) = \frac{\frac{\partial}{\partial\theta}f(x|\theta)}{f(x|\theta)}$

Now, the given information above can be expressed as

$$\frac{d}{d\theta}E_\theta[\ell'(\theta)] = \int \frac{\partial}{\partial\theta}[\ell'(\theta)\cdot\ell(\theta)]\,dx$$

Also we have $\frac{\partial \ell}{\partial\theta} = \ell'(\theta) = \frac{\partial}{\partial\theta}\log f(X|\theta) = \frac{1}{f(x|\theta)}\cdot\frac{\partial}{\partial\theta}f(x|\theta)$

$$\Rightarrow \ell(\theta)f(X|\theta) = \frac{\partial}{\partial\theta}f(X|\theta)$$

$$\therefore\quad E_\theta[\ell'(\theta)] = \int\left(\frac{\partial\ell}{\partial\theta}\right)f(x|\theta)\,dx = \int\frac{1}{\cancel{f(x|\theta)}}\cdot\frac{\partial}{\partial\theta}f(x|\theta)\cdot\cancel{f(x|\theta)}\,dx$$

$$= \int\frac{\partial}{\partial\theta}f(x|\theta)\,dx = \frac{\partial}{\partial\theta}\underbrace{\int f(x|\theta)\,dx}_{=1} = \frac{\partial}{\partial\theta}(1) = 0$$

$$\therefore\quad E_\theta[\ell'(\theta)] = 0$$

Now $\frac{\partial}{\partial\theta}E_\theta[\ell'(\theta)] = \frac{\partial}{\partial\theta}(0) = 0$

$$\Rightarrow \frac{\partial}{\partial\theta}\int \ell'(\theta)f(X|\theta)\,dx = 0$$

$$\Rightarrow \int\frac{\partial}{\partial\theta}[\ell'(\theta)f(x|\theta)]\,dx = 0.$$

$$\frac{\partial}{\partial\theta}\ell'(\theta)f(x|\theta) = \ell'(\theta)\cdot\frac{\partial}{\partial\theta}f(X|\theta) + \ell''(\theta)f(x|\theta)$$

$$= \frac{1}{f(x|\theta)}\cdot\frac{\partial}{\partial\theta}f(x|\theta)\,\frac{\partial}{\partial\theta}f(x|\theta) + \ell''(\theta)f(x|\theta)$$

$$= \frac{1}{f(x|\theta)}\left[\frac{\partial}{\partial\theta}f(x|\theta)\right]^2 + \ell''(\theta)\,f(x|\theta$$

Question assigned to the following page: 1

$$\because \quad 0 = \int \frac{\partial}{\partial\theta}[\ell'(\theta) f(x|\theta)]\,dx = \int \frac{1}{f(x|\theta)}\left[\frac{\partial}{\partial\theta} f(x|\theta)\right]^2 dx + \underbrace{\int \ell''(\theta) f(x|\theta)\,dx}_{E_\theta[\ell''(\theta)]}$$

$$\Rightarrow 0 = \int \frac{1}{f(x|\theta)}\left[\frac{\partial}{\partial\theta} f(x|\theta)\right]^2 dx + E[\ell''(\theta)]$$

$$\Rightarrow 0 = \int \frac{1}{f(x|\theta)} \cdot \frac{1}{f(x|\theta)} \cdot \left[\frac{\partial}{\partial\theta} f(x|\theta)\right]^2 f(x|\theta)\,dx + E_\theta[\ell''(\theta)]$$

$$\Rightarrow 0 = \int \left[\frac{\partial}{\partial\theta} \log f(x|\theta)\right]^2 f(x|\theta)\,dx + E_\theta[\ell''(\theta)]$$

$$\Rightarrow 0 = \underbrace{\int \left[\frac{\partial}{\partial\theta} \ell(\theta)\right]^2 f(x|\theta)\,dx}_{E_\theta[\{\ell'(\theta)\}^2]} + E_\theta[\ell''(\theta)]$$

$$\Rightarrow 0 = E_\theta[(\ell')^2] + E_\theta[\ell''(\theta)]$$

$$\Rightarrow E_\theta[(\ell')^2] = -E_\theta[\ell''(\theta)]$$

$$\Rightarrow E_\theta\left[\left\{\frac{\partial}{\partial\theta} \log f(x|\theta)\right\}^2\right] = -E_\theta\left[\frac{\partial^2}{\partial\theta^2} \log f(x|\theta)\right]$$

Q.E.D

2. For each of the following distributions, let $X_1, \ldots, X_n$, be a random sample. Is there a function of θ, say $g(\theta)$, for which there exists an unbiased estimator whose Variance attains the Cramér Rao lower bound? If so, find it. If not, show why not.

a) $f(x|\theta) = \theta x^{\theta-1}, \quad x \in (0,1), \quad \theta > 0$

b) $f(x|\theta) = \frac{\log\theta}{\theta-1}\theta^x, \quad x \in (0,1), \quad \theta > 1$

Question assigned to the following page: 1

Solutions a) $f(x|\theta) = \theta x^{\theta-1}$, $\ell(\theta) = \log f(x|\theta) = \log \theta x^{\theta-1}$

$$\Rightarrow \ell(\theta) = \log\theta + (\theta-1)\log x.$$

Log Likelihood := $\log L(\theta) = \log\left(\prod_{i=1}^{n} f(x_i|\theta)\right) = \sum_{i=1}^{n} \log f(x_i|\theta)$

$$= \sum_{i=1}^{n} \log(\theta) + (\theta-1)\log x_i$$

$$= n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i$$

$$\frac{\partial}{\partial\theta}\log L(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i \quad \Big\| \quad \frac{\partial^2}{\partial\theta^2}\log L(\theta) = -\frac{n}{\theta^2}$$

Fisher Information $I(\theta) := -E_x\left[\frac{\partial^2}{\partial\theta^2}\log L(\theta)\right]$

$$= -E_x\left[-\frac{n}{\theta^2}\right] = +\frac{n}{\theta^2}$$

CR inequality says $Var_\theta(\hat{g}) \geq \frac{[g'(\theta)]^2}{I(\theta)}$

where $I(\theta)$ is the Fisher Information.

Equality can hold only if

$$\hat{g}(X) = g(\theta) + c(\theta)[\text{score function}]$$

Here, score function $= s(x,\theta) = \frac{\partial}{\partial\theta}\log \ell(\theta)$

$$\Rightarrow s(x,\theta) = \frac{1}{\theta} + \ln x$$

$$\therefore \hat{g}(X) = g(\theta) + c(\theta)\left[\frac{1}{\theta} + \ln x\right]$$

But $\hat{g}(X)$ cannot depend explicitly on θ. Hence $c(\theta)$ must be zero to suppress the dependence on $\frac{1}{\theta}$.

Then $\hat{g}(X) = g(\theta)$ which means $\hat{g}(X)$ doesn't depend on X at all but is a constant w.r.t X.

$\therefore$ we donot have any $g(\theta)$ other than a constant that can achieve the CR lower bound.

Question assigned to the following page: 1

(b) $f(x|\theta) = \frac{\log(\theta)}{\theta - 1} x^{\theta}$, $0 < x < 1$; $\theta > 1$

$S(x, \theta) = \frac{1}{\theta \log \theta} - \frac{1}{\theta - 1} + \ln x$. which depends on both x & θ.

Similarly to part a, $C(\theta) = 0$ as

$$\hat{g}(x) = g(\theta) + C(\theta)\left[\frac{1}{\theta \log \theta} - \frac{1}{\theta - 1} + \ln x\right]$$

to suppress the explicit dependence of $\hat{g}(x)$ on θ

Hence no non-constant $g(\theta)$ can achieve the CR-lower bound.

Question assigned to the following page: 2

PROBLEM 2: (Unbiased Estimators)

1. Let $X_1, \dots X_{n+1}$ be i.i.d $Ber(p)$ and define

$$h(p) = \mathbb{P}\left(\sum_{i=1}^{n} X_i > X_{n+1} \,|\, p\right)$$

a) Show that $T(X_1 \dots X_{n+1}) = \begin{cases} 1 & ; \sum_{i=1}^{n} X_i > X_{n+1} \\ 0 & ; \text{otherwise} \end{cases}$

is an unbiased estimator of $h(p)$

b) Find the best unbiased estimator of $h(p)$.

Solution: a) Clearly, T is an indicator random variable & hence

$$E[T] = Pr\left(\sum_{i=1}^{n} X_i > X_{n+1}\right) = h(p)$$

And that is the definition for an estimator to be unbiased i.e. the expectation of the estimator must match the estimand which is $h(p)$ here.

b) Best unbiased estimator of $h(p)$ i.e. UMVUE of $h(p)$

The best unbiased estimator is obtained by conditioning T on the minimal sufficient statistic for p. (M.S.S)

For Berenoulli Random variables, this M.S.S is $S = \sum_{i=1}^{n+1} X_i$.

Hence the UMVUE is

$$\hat{h}_{UMVUE}(S) = E[T|S] = Pr\left(\frac{1}{\left\{\sum_{i=1}^{n} X_i > X_{n+1}\right\}} \middle| S\right)$$

Now, when $X_{n+1} = 0$, $\sum_{i=1}^{n} X_i > X_{n+1} = 0$

or $\sum_{i=1}^{n} X_i \geq 1$

or $\sum_{i=1}^{n} X_i + 0 \geq 1$

or $\sum_{i=1}^{n} X_i + X_{n+1} \geq 1$ or $\sum_{i=1}^{n+1} X_i \geq 1$ or $S \geq 1$ or $S > 0$

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Similarly, when $X_{n+1} = 1$, $\sum_{i=1}^{n} X_n > X_{n+1} = 1$

$$\Rightarrow \sum_{i=1}^{n} X_n \geq 2$$

$$\Rightarrow \sum_{i=1}^{n} X_n + \underline{\underline{1}} \geq 3$$

$$\Rightarrow \sum_{i=1}^{n} X_n + \underline{\underline{X_{n+1}}} \geq 3$$

$$\Rightarrow \sum_{i=1}^{n+1} X_n \geq 3 \text{ or } S \geq 3.$$

$$\therefore E[T|S] = \frac{\mathbb{1}_{\{S>0\}}\binom{n}{S} + \mathbb{1}_{\{S\geq 3\}}\binom{n}{S-1}}{\binom{n}{S} + \binom{n}{S-1}}$$

$$= \frac{\mathbb{1}_{\{S>0\}}\binom{n}{S} + \mathbb{1}_{\{S\geq 3\}}\binom{n}{S-1}}{\binom{n+1}{S}}$$

$$\therefore \hat{h}_{UMVUE}(S) = \begin{cases} 0, & S = 0 \text{ or } \sum_{i=1}^{n+1} X_i = 0 \\ \frac{n}{n+1}, & S = 1 = \sum_{i=1}^{n+1} X_i \\ \frac{n-1}{n+1} & S = 2 \\ 1 & S \geq 3 \end{cases}$$

is the best unbiased estimator for $h(p)$ among all unbiased estimators of $h(p)$.

Question assigned to the following page: 2

2. Let $X_1 \dots X_n$ be iid exponential (λ) with pdf $f(x|\lambda) = \frac{1}{\lambda}\exp(-x/\lambda)$, $x>0$.

a) Find an unbiased estimatore of λ based only on $Y = \min\{X_1 \dots X_n\}$

b) Find a better estimator than the one in part (a). Prove that it is better.

Solution : $f(x|\lambda) = \frac{1}{\lambda}\exp(-x/\lambda)$, $x>0$

$$Y := \min\{X_1 \dots X_n\}$$

Claim:- $f_Y(y) = \frac{n}{\lambda}e^{-ny/\lambda}$, $y>0$

Proof:- $f_X(x) = \frac{1}{\lambda}\exp(-x/\lambda)$

$$F_X(x) = \int_0^x \frac{1}{\lambda}\exp(-x/\lambda)$$
$$= \cancel{\frac{1}{\lambda}} \times (-\cancel{\lambda})\left[\exp\left(\frac{-x}{\lambda}\right)\right]_0^x$$
$$= -\left[\exp\left(\frac{-x}{\lambda}\right) - \exp(0)\right]$$
$$= 1 - \exp\left(\frac{-x}{\lambda}\right)$$
$$\Rightarrow P(X \le x) = 1 - \exp\left(\frac{-x}{\lambda}\right)$$
$$\Rightarrow P(X > x) = 1 - \left(1 - \exp\left(\frac{-x}{\lambda}\right)\right)$$
$$= \exp\left(\frac{-x}{\lambda}\right) \Rightarrow$$

$$\Rightarrow F_Y(y) = P(Y \le y) = 1 - P(Y \ge y)$$
$$\Rightarrow F_Y(y) = 1 - P(\min\{X_1 \dots X_n\} \ge y)$$
$$= 1 - P(X_1 \ge y, X_2 \ge y, \dots X_n \ge y)$$
$$= 1 - P(X_1 \ge y) \dots P(X_n \ge y)$$
$\because$ independence.
$$= 1 - \left(e^{-y/\lambda}\right) \dots \left(e^{-y/\lambda}\right)$$
$$= 1 - e^{-yn/\lambda}.$$
$$\Rightarrow f_Y(y) = \frac{\partial}{\partial y}F_Y(y) = -e^{-\frac{yn}{\lambda}} \cdot \frac{-n}{\lambda} = \frac{n}{\lambda}e^{-ny/\lambda}$$
$$\Rightarrow f_Y(y) = \frac{n}{\lambda}e^{-ny/\lambda}. \quad \text{Q.E.D}$$

Now $E[Y] = \int y \cdot f_Y(y)\,dy = \int_0^\infty y \frac{n}{\lambda}e^{-ny/\lambda}\,dy = \frac{n}{\lambda}\int_0^\infty y e^{-ny/\lambda}\,dy$

Set $u = \frac{+ny}{\lambda} \Rightarrow du = \frac{+n}{\lambda}dy$ | $y \to 0, u \to 0$
or $y = \frac{u\lambda}{+n}$ | $y \to \infty, u \to \infty$

$$\therefore E[Y] = \frac{\cancel{n}}{\cancel{\lambda}}\int \frac{u\cancel{\lambda}}{\cancel{+n}} \cdot e^{-u}\frac{\lambda}{+n}du = \frac{\lambda}{n}\int_0^\infty u e^{-u}\,du = \frac{\lambda}{n} \cdot 1 = \frac{\lambda}{n}.$$

$$\int_0^\infty u e^{-u}\,du = u\int e^{-u}du\Big|_0^\infty - \int_0^\infty \int e^{-u}du = -ue^{-u}\Big|_0^\infty - e^{-u}\Big|_0^\infty = (0-0) - (0-1) = 1$$

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$\therefore E[Y] = \frac{\lambda}{n}$.

$\Rightarrow E[nY] = \lambda$.

Note: $\text{exponential}(\lambda) = \frac{1}{\lambda}\exp(\frac{-x}{\lambda})$ has $E[\text{exponential}(\lambda)] = \lambda$, $\text{Var}(\text{exponential}(\lambda)) = \lambda^2$.

$\therefore$ If we define $T_1 := nY$, then T_1 is an unbiased estimator of λ and is based only on n.

Solution b) We know that the mean of the exponential distribution is λ. We claim that the emperical mean, which we define as $T_2 = \bar{X} = \frac{1}{n}\sum x_i$ is a better estimator of λ in the sense that it has lower variance than the estimator $T_1 = nY$.

Recall the $f_Y(y) = \frac{n}{\lambda} e^{-ny/\lambda} = \overset{\frac{n}{\lambda}}{\alpha} e^{-\alpha y} = \text{exponential}(\alpha)$

$$\text{Var}(\text{exponential}(\tfrac{1}{\alpha})) = \frac{1}{\alpha^2} = \left(\frac{1}{n/\lambda}\right)^2 = \frac{\lambda^2}{n^2}.$$

$$\therefore \text{Var}(T_1) = \text{Var}(nY) = n^2\,\text{Var}(Y) = n^2\frac{\lambda^2}{n^2} = \lambda^2.$$

$$\begin{aligned}
\text{Var}(T_2) = \text{Var}(\bar{X}) &= \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \\
&= \frac{1}{n^2}\text{Var}\left(\sum_{i=1}^{n} x_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(x_i) \quad (\because \text{independence}) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(\text{exponential}(\lambda)) \\
&= \frac{1}{n^2}\, n \,.\, \text{Var}(\text{exponential}(\lambda)) \\
&= \frac{1}{n^2}\cdot n \cdot (\lambda)^2 \\
&= \frac{\lambda^2}{n} = \text{Var}(T_2)
\end{aligned}$$

Clearly $\underset{\text{Var}(T_1)}{\lambda^2} > \frac{\lambda^2}{n}$, & hence T_2 is a better estimator.

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PROBLEM 3

1. Let $X_1 \dots X_n$ be i.i.d Ber(p). Show that the variance of $\bar{X}$ attains the Cramer-Rao lower bound, and hence $\bar{X}$ is the best unbiased estimator of p.

Solution:- For a single random variable X, the log-likelihood is

$$l(p;x) = x \log p + (1-x)\log(1-p)$$

Fisher Information of a single observation $= I_1(p) = E_x\left[\left\{\frac{\partial}{\partial p} l(p;x)\right\}^2\right]$

$$\Rightarrow I_1(p) = E_x\left[\left(\frac{x}{p} + \frac{1-x}{1-p}\right)^2\right]$$

$$= p\cdot\left(\frac{1}{p} + \frac{1-1}{1-p}\right)^2 + (1-p)\cdot\left(\frac{0}{p} + \frac{1-0}{1-p}\right)^2$$

$$= p\cdot\frac{1}{p^2} + (1-p)\frac{1}{(1-p)^2} = \frac{1}{p} + \frac{1}{1-p}$$

$$= \frac{1-p+p}{p(1-p)} = \frac{1}{p(1-p)}$$

$\therefore$ With n i.i.d observations, the Fisher information is

$$I_n(p) = n\, I_1(p) = \frac{n}{p(1-p)}$$

The C-R lower bound for any unbiased estimator

is $\operatorname{Var}(\hat{p}) \geq \dfrac{1}{I_n(p)} = \dfrac{p(1-p)}{n}$

Now, $\operatorname{Var}(\bar{X}) = \operatorname{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right)$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \underbrace{\operatorname{Var}(X_i)}_{p(1-p) \text{ for Bernouilli}(p)} \quad (\because \text{independence})$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} p(1-p) = \frac{1}{n^2}\cdot n\cdot p(1-p)$$

$$= \frac{p(1-p)}{n}$$

and hence $\bar{X}$ hits or achieves the C-R-lower bound & hence the best.

Question assigned to the following page: 3

2. Let $X_1, \dots X_n$ be a random sample from a population with mean μ and variance σ^2.

a) Show that the estimator $\sum_{i=1}^{n} a_i X_i$ is an unbiased estimator of μ if $\sum_{i=1}^{n} a_i = 1$

b) Among all unbiased estimators of this form (linear unbiased estimators), find the one with minimum variance and calculate the variance.

Solution a) Let $T_1 = \sum_{i=1}^{n} a_i X_i$ where $\sum_{i=1}^{n} a_i = 1$

$$E[T_1] = E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} E[a_i X_i] = \sum_{i=1}^{n} a_i E[X_i]$$

$$= \sum_{i=1}^{n} a_i \mu = \mu \sum_{i=1}^{n} a_i = \mu(1) = \mu.$$

Hence T_1 is unbiased estimator of μ.

b) $\text{Var}(T_1) = \text{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} \text{Var}(a_i X_i)$ $\because$ independence

$$\Rightarrow \text{Var}(T_1) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) = \sum_{i=1}^{n} \text{Var}(X_i) \cdot a_i^2$$

$$\sum_{i=1}^{n} \sigma^2 a_i^2 = \sigma^2 \sum_{i=1}^{n} a_i^2$$

Now, we want to minimize $\text{Var}(T_1) = \sigma^2 \sum_{i=1}^{n} a_i^2$ because it is a function of a_i, $i = 1, 2, \dots n$.

Let $f(a_1, \dots a_n) = \sum_{i=1}^{n} a_i^2$.

We also have the constraint $\sum_{i=1}^{n} a_i = 1$.

Because of symmetry, all the variables a_i are equivalent and the universe has no reason to pick one variable over the other. Hence $a_i = a_j \ \forall\, i,j$ in $\{1, 2, \dots n\}$ at minima.

$\therefore a_i = a_j = \dots a_n = 1/n$.

Hence the best unbiased estimator is $T_2 = \frac{1}{n}\sum_{i=1}^{n} x_i$.

and has $Var(T_2) = \frac{1}{n}\sigma^2$.

Alternatively, we can also prove this minima using AM $\geq$ GM inequality (also can be done using Cauchy-Schwartz).

We know Arithmetic Mean $\geq$ Geometric Mean.

$$\Rightarrow \frac{(x-a_i)+(a_i)}{2} \geq \sqrt{(x-a_i)a_i}$$

$$\Rightarrow \left(\frac{x}{2}\right)^2 \geq (x-a_i)a_i \Rightarrow x^2 \geq 4(x-a_i)a_i$$

Now $\sum_{i=1}^{n} x^2 \geq \sum_{i=1}^{n} 4(x-a_i)a_i$

$$\Rightarrow nx^2 \geq \sum_{i=1}^{n} 4(a_i x - a_i^2) = 4\Big(x \underbrace{\sum a_i}_{=1} - \sum a_i^2\Big)$$

$$\Rightarrow nx^2 \geq 4\Big(x - \sum_{i=1} a_i^2\Big) = 4x - 4\sum_{i=1}^{n} a_i^2$$

$$\Rightarrow \underbrace{1}_{a} \cdot x^2 - \underbrace{\frac{4}{n}}_{b} x + \underbrace{\frac{4}{n}\sum_{i=1}^{n} a_i^2}_{c} \geq 0$$

Since this quadratic eq^n is always $\geq 0 \ \forall x$,

$b^2 - 4ac \leq 0 = b^2 \leq 4ac \Rightarrow \left(\frac{4}{n}\right)^2 \leq 4 \cdot 1 \cdot \frac{4}{n}\sum_{i=1}^{n} a_i^2$

$$\Rightarrow \frac{\cancel{4^2}}{n^{\cancel{2}}} \leq \frac{\cancel{4^2}}{\cancel{n}} \sum_{i=1}^{n} a_i^2$$

$$\Rightarrow \sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}$$

So we got a lower bound. And setting $a_i = \frac{1}{n}$, we achieve this lower bound because $\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n}\left(\frac{1}{n}\right)^2 = n\frac{1}{n^2} = \frac{1}{n}$.

Question assigned to the following page: 3

3. Let $X_1, \ldots X_n$ be i.i.d Gamma(α, β) with α known. Find the best unbiased estimator of $1/\beta$.

Sol^n: $X \sim$ Gamma(α, β)

$$\therefore f(x|\beta) = \frac{x^{\alpha-1} e^{-\beta x} \beta^{\alpha}}{\Gamma(\alpha)}$$

Let $T = \sum_{i=1}^{n} X_i$. Then $T \sim$ Gamma$(n\alpha, \beta)$

Now $\frac{T}{\beta} \sim$ Gamma$(n\alpha, 1)$

It is known that if $Y \sim$ Gamma$(\gamma, 1)$ with $\gamma > 1$, then $E[1/Y] = \frac{1}{\gamma - 1}$. Let $Y := \frac{T}{\beta}$ so that $\gamma = n\alpha$

$$\therefore E\left[\frac{1}{T/\beta}\right] = \frac{1}{n\alpha - 1} \quad \text{when } n\alpha > 1.$$

$$\therefore E\left[\frac{n\alpha - 1}{T/\beta}\right] = 1 \text{ or } E\left[\frac{n\alpha-1}{T}\right] = \frac{1}{\beta}.$$

So $T_1 = \frac{n\alpha - 1}{T} = \frac{n\alpha - 1}{\sum_{i=1}^{n} X_i}$ is an unbiased estimator of $\frac{1}{\beta}$.

Now, because T is a sufficient and Complete statistics for β in Gamma$(n\alpha, \beta)$, $\beta > 0$, any unbiased estimator that is a function of T must be the unique minimum variance unbiased estimator.

Hence $\frac{n\alpha - 1}{\sum_{i=1}^{n} X_i}$ is the UMVUE by Lehmann-Scheffé theorem.