Math 541 A - Spring 2025

Midterm 2

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11:00-11:50 AM

Name: SAMPAD. B. MOHANTY

Student ID Number: 5679528317

- There are 3 problems in total. Make sure your exam contains all these questions.
- You are allowed to use one page of double-sided **hand-written** 8.5 by 11 inch notes.
- You must show your work on all problems. The correct answer with no supporting work may result in no credit.
- If you need more room, use the backs of the pages and indicate to the grader that you have done so.

Problem 1	20	
Problem 2	20	
Problem 3	20	
Total	60	

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1. (20 pts) Let $X_1, \ldots, X_n \sim N(\mu, 1)$ be i.i.d. samples. Recall the PDF for $N(\mu, 1)$ is

$$f(x|\mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}}.$$

(a) (7 pts) For one random variable $X \sim N(\mu, 1)$, compute its Fisher information

$$I_X(\mu) = \mathbb{E}\left(\frac{d}{d\mu} \log f(X|\mu)\right)^2.$$

(b) (6 pts) Find the the Cramé-Rao lower bound for any unbiased estimator (as a function of $X_1, \ldots, X_n$) of μ.

(c) (7 pts) Assume $X_1, \ldots, X_n \sim N(0, 1)$ be i.i.d. random variables. Show the following convergence in distribution holds:

$$\frac{\sum_{i=1}^n (X_i^2 - 1))}{\sqrt{2n}} \xrightarrow{d} N(0, 1).$$

a) $\frac{d}{d\mu} \log f(X|\mu) = \frac{d}{d\mu}\left[\log\frac{1}{\sqrt{2\pi}} - \frac{(x-\mu)^2}{2}\right] = \frac{\cancel{2}(x-\mu)}{\cancel{2}} \cdot 1 = x-\mu$

$\underset{X}{E}\left[\left(\frac{d}{d\mu}\log f(x|\mu)\right)^2\right] = E_X\left[(X-\mu)^2\right] = Var(X) = 1$

b) C.R lower bound $= \frac{1}{n I_X(\mu)} = \frac{1}{n \cdot 1} = \frac{1}{n}$

c) By CLT

$p^2 \cdot 1-p$
$+ (1-p)^2 p$
$p(1-p)$

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2. (20 pts) Let $X_1, \dots, X_n$ be i.i.d. Ber(p).

(a) (5 pts) Show that the following convergence in probability holds:

$$\frac{1}{n^2}(X_1 + \cdots + X_n)^2 \xrightarrow{P} p^2.$$

(b) (5 pts) Consider a risk function for any estimator $\delta(\mathbf{X})$ of p given by

$$R(p, \delta(\mathbf{X})) = \mathbb{E}_p|\delta(\mathbf{X}) - p|.$$

Let $\delta(\mathbf{X}) = X_1$ be the estimator, and $\pi(p) = \text{Uniform}(0,1)$ be the prior distribution on p. Find the Bayes risk

$$\int_0^1 R(p, \delta(\mathbf{X}))\pi(p)dp.$$

(c) (10 pts) Show that $\phi(\mathbf{X}) = X_1X_2$ is an unbiased estimator of p^2. Based on $\phi(\mathbf{X})$ and the Lehmann–Scheffé Theorem, find the best unbiased estimator $W = W(X_1, \dots, X_n)$ of p^2 with an explicit form.

a) $E[\frac{1}{n^2}(x_1 + \cdots x_n)^2] = Var(\frac{1}{n}(x_1 + \cdots x_n)) + E[\frac{1}{n}(x_1 + \cdots x_n)]^2$
$= \frac{1}{n} \cdot p(1-p) + p^2 = p^2 + \frac{p(1-p)}{n}$.

Also $var(\frac{1}{n^2}(x_1 + \cdots x_n)^2) < \infty$. $\therefore$ by WLL, $\frac{1}{n^2}(x_1 + \cdots x_n)^2 \xrightarrow{P} p^2$.

~~b)~~ $\cancel{\pi(p|x) = L(p;x) \cdot \pi(p) = f(x_1|p)\pi(p) = 1 \cdot \pi(p) = \pi(p)}$ $\cancel{1_{[x_1 \in (0,1)]}}$

~~Bayes risk for absolute error loss is median of posterior~~

b) $\pi(p|x) = L(p;x) \cdot \pi(p) = f(x_1|p)\pi(p) = p^{x_1}(1-x)^{1-p} \cdot 1_{[p \in [0,1]]}$

The minimizer is the median of posterior.

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(c) $\phi(X) = X_1 X_2$ is unbiased estimator for p^2.

$\cancel{f(\vec{x}\mid p) =}$ $\cancel{p^{x}(1-p)^{1-x}}$ $\cancel{\sum \binom{n}{k} p^{k}(1-p)^{n-k}}$

We ~~know that~~ need to find a complete sufficient statistics T of p^2 & by L. Scheffé.

$W = E[\phi(X) \mid T]$ is the UMVUE

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3. (20 pts) Let $X_1, \dots, X_n$ be i.i.d. $\text{Exp}(\lambda)$ with pdf $f(x;\lambda) = \lambda e^{-\lambda x}$ for $x > 0$. For each $i = 1, \dots, n$, define the indicator $Y_i = 1\{X_i > 1\}$.

   (a) (7 pts) Compute the log-likelihood function
   $$\log L(\lambda) = \log f(X_1, \dots, X_n|\lambda).$$

   (b) (7 pts) Consider the t-th iteration of an EM algorithm for approximating the MLE of λ. For the E-step, expected log-likelihood satisfies
   $$Q(\lambda|\lambda^{(t)}) = n \log \lambda - \lambda \sum_{i=1}^{n} \mathbb{E}[X_i|Y_i, \lambda^{(t)}]. \quad \leftarrow \text{given.} \tag{1}$$
   Given Equation (1), find the explicit solution for $\lambda^{(t+1)}$ in the M-step:
   $$\lambda^{(t+1)} = \arg\max_{\lambda} Q(\lambda|\lambda^{(t)}).$$
   Your answer should be a function of $\mathbb{E}[X_i|Y_i, \lambda^{(t)}]$.

   (c) (6 pts) Compute $\mathbb{P}(Y_1 = 1|\lambda)$ and $\mathbb{E}[X_1|Y_1 = 1, \lambda]$.

a) $\log L(\lambda) = \log f(x_1 \cdots x_n|\lambda)$

$= \log \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \sum_{i=1}^{n} \log \lambda e^{-\lambda x_i}$

$= n \log \lambda - \lambda \sum x_i$

b) $\frac{d}{d\lambda} Q(\lambda|\lambda^{(t)}) = \frac{n}{\lambda} - \sum E[x_i|Y_i, \lambda^{(t)}]$

c) $P(Y_1 = 1|\lambda) = P(x_i > 1|\lambda) = \int_1^\infty \lambda e^{-\lambda x} dx = \lambda \int_1^\infty e^{-\lambda x} dx$

$= \lambda \frac{e^{-\lambda x}}{-\lambda}\Big]_1^\infty = e^{-\lambda x}\Big]_\infty^1 = e^{-\lambda}$

$E[X_1|Y_1 = 1, \lambda] = \int x f_X(x|Y_1 = 1, \lambda) dx = \int x f_x(x|x>1, \lambda) dx$

$= \int_1^\infty x \lambda e^{-\lambda(x-1)} dx$

$\boxed{f_x(x|x>1) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda}} = \lambda e^{-\lambda(x-1)}}$

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$$= \lambda \int_1^{\infty} x\, e^{-\lambda(x-1)}\, dx = \lambda \left[x \int e^{-\lambda(x-1)} dx - \int \int e^{-\lambda(x-1)} dx \right]$$

$$= \lambda \left[x \frac{e^{-\lambda(x-1)}}{-\lambda} \Big]_1^{\infty} - \int_1^{\infty} \frac{e^{-\lambda(x-1)}}{-\lambda} dx \right]$$

$$= \cancel{\lambda} \left[\frac{1}{\cancel{\lambda}} + \frac{1}{\cancel{\lambda}} \int_1^{\infty} e^{-\lambda(x-1)} dx \right]$$

$$= \left[1 + \int_1^{\infty} e^{-\lambda(x-1)} dx \right]$$

$$= 1 + \frac{e^{-\lambda(x-1)}}{-\lambda} \Big]_1^{\infty} = 1 - \frac{e^{-\lambda(x-1)}}{\lambda} \Big|_1^{\infty}$$

$$= 1 - \left(0 - \frac{1}{\lambda}\right) = 1 + 1/\lambda .$$