

Homework 2

● Graded

Student

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Total Points

52 / 60 pts

Question 1

Problem 1

12 / 20 pts

+ 5 pts part (a) Correct

✓ + 5 pts part (b) Correct

+ 5 pts part(c) Correct

✓ + 5 pts part(d) Correct

💬 + 2 pts for your part (c), you need to show your work

Question 2

Problem 2

20 / 20 pts

✓ + 10 pts part (1) Complete

✓ + 10 pts part (2) Complete

Question 3

Problem 3

20 / 20 pts

✓ + 7 pts part (1) Correct

✓ + 6 pts part (2) Complete

✓ + 7 pts part (3) Correct

Question assigned to the following page: [1](#)

Q2 1. x_1, x_2, \dots, x_n are random samples from $f(x|\theta)$. Find a minimal statistic for θ .

a) $f(x|\theta) = (2\pi)^{-1/2} e^{-(x-\theta)^2/2}$

Solⁿ $f(x|\theta) = (2\pi)^{-1/2} e^{-\left(\frac{x^2}{2} - \theta x + \frac{\theta^2}{2}\right)}$ $A(\theta)$
 \uparrow

$$= \underbrace{(2\pi)^{-1/2} \exp\left\{-\frac{x^2}{2}\right\}}_{h(x)} \exp\left\{\downarrow \theta x - \frac{\theta^2}{2}\right\} T(\theta)$$

$$= h(x) \exp\{n(\theta)T(x) - A(\theta)\}$$

Hence, this distribution is from the exponential family.

We have seen in class that $s_j = \sum_{i=1}^N \vec{T}_j(x_i)$ form the components of the sufficient statistic \vec{s} .

Hence we just have one component of $\vec{T}(x)$ i.e. it is a scalar. So $\sum_{i=1}^N x_i$ is sufficient for θ .

We claim that $\bar{x} = \sum_{i=1}^N x_i$ is also minimal for θ .

By Lehmann-Scheffé theorem (Thm 2.16 in class notes)

$\frac{f(x_1, x_2, \dots, x_N | \theta)}{f(y_1, y_2, \dots, y_N | \theta)}$ is constant as a function of $\theta \Leftrightarrow T(x) = T(y)$.
 \downarrow
i.e. constant wrt θ

Question assigned to the following page: [2](#)

$$\frac{f(x_1, x_2, \dots, x_n | \theta)}{f(y_1, y_2, \dots, y_n | \theta)} = \frac{\prod_{i=1}^N h(x_i) \exp(\theta \bar{x} - nA(\theta))}{\prod_{i=1}^N h(y_i) \exp(\theta \bar{y} - nA(\theta))}$$

Clearly, the above ratio is independent w.r.t θ if and only if $\bar{x} = \bar{y}$. Hence $\bar{x} = \sum_{i=1}^n x_i$ is sufficient and minimal statistic.

b) $f(x | \theta) = e^{-(x-\theta)} = \exp$

$$f(x_1, x_2, \dots, x_n | \theta) = \prod_{i=1}^N e^{-(x_i-\theta)} = e^{-(\sum x_i - n\theta)}$$

Define $\underbrace{T(x_1, \dots, x_n)}_{T(\vec{x})} = \bar{x} = \sum_{i=1}^n x_i$.

$$\begin{aligned} \text{Then, } \frac{f(x_1, \dots, x_n | \theta)}{f(y_1, \dots, y_n | \theta)} &= \frac{e^{-(T(\vec{x}) - n\theta)}}{e^{-(T(\vec{y}) - n\theta)}} \\ &= e^{\underbrace{T(\vec{y}) - T(\vec{x})}_{\text{Independent of } \theta}}. \end{aligned}$$

Hence by the Lehmann-Scheffé theorem, $T(\vec{x}) = \bar{x}$ is sufficient.

Question assigned to the following page: [3](#)

$$c) f(x|\theta) = \frac{e^{(x-\theta)}}{[1+e^{-(x-\theta)}]^2}$$

This is a distribution from the location family and the minimal & sufficient statistic is the order statistic which contains all the samples [Note that there are some distributions from local family that have additional structure, for example the Normal distribution which is also from the exponential family. For such distribution, we can have better compression if the minimal sufficient statistic does not need to be the order statistic.]

$$d) f(x|\theta) = \frac{1}{\pi [1+(x-\theta)^2]}.$$

This is also from a location family. This is from class note example 2.19 and we already saw that the order statistic is the minimal sufficient statistic.

No questions assigned to the following page.

$$e) f(x|\theta) = \frac{1}{2} e^{-|x-\theta|}$$

$$f(x_1, \dots, x_n | \theta) = \left(\frac{1}{2}\right)^n e^{-\sum_{i=1}^n |x_i - \theta|}$$

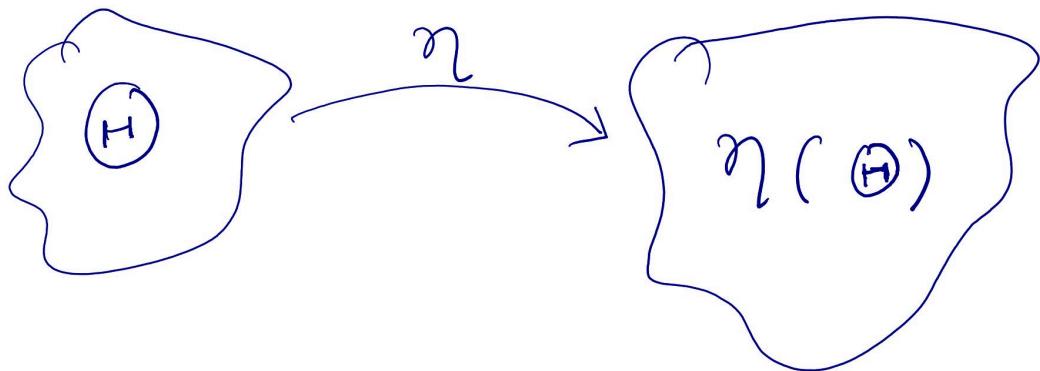
The minimal sufficient statistic is the order statistic.

$$2) f(x|\theta) = h(x) \exp \left\{ \vec{n}(\theta) \cdot \vec{T}(x) - A(\theta) \right\}$$

where $\theta \in \mathbb{H}$

Define the difference set D as

$$\begin{aligned} D &= \left\{ n(\theta_1) - n(\theta_2) \mid \theta_1, \theta_2 \in \mathbb{H} \right\} \\ &= \left\{ n_1 - n_2 \mid n_1, n_2 \in n(\mathbb{H}) \right\} \end{aligned}$$



$$\frac{f(x_1, \dots, x_n | \theta)}{f(y_1, \dots, y_n | \theta)} = \prod_{i=1}^n \frac{h(x_i)}{h(y_i)} e^{\vec{n}(\theta) \left(\sum_{i=1}^n \vec{T}(x_i) - \vec{T}(y_i) \right)}$$

No questions assigned to the following page.

Show that if $\text{span}(D) = \mathbb{R}^k$,
then $T(X) = (t_1(x), t_2(x), \dots, t_k(x))$
is a minimal sufficient statistic.

where $t_j(x) = \sum_i t_j(x_i)$

Soln We want to show that for any two
samples $X = \{x_1, \dots, x_n\}$ & $Y = \{y_1, \dots, y_n\}$,

$\frac{f(x_1, \dots, x_n | \theta)}{f(y_1, \dots, y_n | \theta)}$ is independent of $\theta \Leftrightarrow T(X) = T(Y)$

This is for all $\theta \in \Theta$

$\frac{f(x_1, \dots, x_n | \theta)}{f(y_1, \dots, y_n | \theta)}$ is independent of θ when

when $\vec{n}(\theta) \cdot \{\vec{T}(x) - \vec{T}(y)\} = 0$

This should imply $T(X) = T(Y)$ if $T(X)$ is a
minimal sufficient statistic {and this should be for
all $\theta \in \Theta$ }

Assume $\vec{n}(\Theta)$ contain k elements

b_1, b_2, \dots, b_k ($b_i = \vec{n}(\theta_i)$ for some $\theta_i \in \Theta$),
which form a basis in \mathbb{R}^k .

No questions assigned to the following page.

Now we require $\forall \theta \in \Theta$,

$$\vec{n}(\theta) \cdot [\vec{T}(x) - \vec{T}(y)] = 0$$

$$\Rightarrow \vec{b}_i \cdot [\vec{T}(x) - \vec{T}(y)] = 0 \quad \forall i$$

$$\Rightarrow \alpha_i \vec{b}_i \cdot [\vec{T}(x) - \vec{T}(y)] = 0 \quad \forall i$$

$\forall \vec{\alpha}_i \in \mathbb{R}$

$$\Rightarrow \sum_{i=1}^k \alpha_i \vec{b}_i \cdot [\vec{T}(x) - \vec{T}(y)] = 0$$

$$\Rightarrow \underbrace{\left(\sum_{i=1}^k \alpha_i \vec{b}_i \right)}_{\vec{v} \in \mathcal{N}(\Theta)} \cdot [\vec{T}(x) - \vec{T}(y)] = 0$$

$$\therefore \vec{v} \cdot (\vec{T}(x) - \vec{T}(y)) = 0 \quad \forall \vec{v} \in \mathcal{N}(\Theta)$$

$$\Rightarrow T(x) = T(y).$$

So our assumption was we have a basis in

$$\mathcal{N}(\Theta) \text{ i.e. } \text{Span}(\mathcal{N}(\Theta)) = \mathbb{R}^k$$

Realize that $\text{Span}(D) = \mathbb{R}^k \Leftrightarrow \text{Span}(\mathcal{N}(\Theta)) = \mathbb{R}^k$

No questions assigned to the following page.

because $\text{Span}(D) \subseteq \text{Span}(\Theta)$.

Hence $\text{Span}(D) = \mathbb{R}^k \subseteq \text{Span}(\mathcal{N}(\Theta))$

we also have $\text{Span}(\mathcal{N}(\Theta)) \subseteq \mathbb{R}^k$

Hence $\text{Span}(\mathcal{N}(\Theta)) = \mathbb{R}^k$.

Since $\text{Span}(D) = \mathbb{R}^k \Rightarrow \text{Span}(\mathcal{N}(\Theta)) = \mathbb{R}^k$

which is what is required such that $T(x)=T(y)$ for all $\theta \in \Theta$, we have proved the given statement of the question.

Q3 1. X_1, \dots, X_n from $\text{Gamma}(\alpha, \beta)$

$$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad \alpha, \beta > 0, \quad x > 0$$

Derive the methods of moments estimators for α & β .

Solution $E[x] = \frac{\alpha}{\beta}$ & $E[x^2] = \frac{\alpha(\alpha+1)}{\beta^2}$ for

the $\text{Gamma}(\alpha, \beta)$ distribution.

Also the sample means & sample 2nd moment are $\bar{X} = \frac{1}{n} \sum_{i=1}^n x_i$ & $M_2 = \frac{1}{n} \sum_{i=1}^n x_i^2$

No questions assigned to the following page.

We equate the sample moments to the theoretical moments to estimate the parameters

$$E[x] = \bar{x} \quad \& \quad E[x^2] = M_2$$

$$\Rightarrow \frac{\alpha}{\beta} = \bar{x} \quad \& \quad \frac{\alpha(\alpha+1)}{\beta^2} = M_2$$

↓

$$\frac{\alpha}{\beta} \left(\frac{\alpha}{\beta} + \frac{1}{\beta} \right) = M_2$$

$$\bar{x} \left(\bar{x} + \frac{1}{\beta} \right) = M_2$$

$$\beta = \left(\frac{M_2 - \bar{x}}{\bar{x}} \right)^{-1}$$

Hence $\alpha = \bar{x}$ $\beta = \frac{\bar{x}}{\frac{M_2}{\bar{x}} - \bar{x}}$

$$\therefore \alpha = \frac{1}{\frac{M_2}{\bar{x}} - \bar{x}} \quad \text{and} \quad \beta = \frac{\bar{x}}{\frac{M_2}{\bar{x}} - \bar{x}}$$

$$\text{or } \alpha = \frac{\bar{x}}{M_2 - \bar{x}^2} \quad \text{and} \quad \beta = \frac{\bar{x}^2}{M_2 - \bar{x}^2}$$

No questions assigned to the following page.

$$2. \quad f(x|\theta) = \theta x^{-2}, \quad 0 < \theta < x < \infty$$

a) find method of moments estimator of θ

b) find M.L.E estimator of θ .

$$\begin{aligned} \text{SOLN} \quad E[X] &= \int_{\theta}^{\infty} x f(x|\theta) dx \\ &= \int_{\theta}^{\infty} x \theta x^{-2} dx = \int_{\theta}^{\infty} \theta x^{-1} dx \\ &= \theta [\log x]_{\theta}^{\infty} \rightarrow \infty. \end{aligned}$$

So this has infinite mean. But we can try to use the 2nd moment (This is a problem of method of moments as the estimate of θ depends on which moments we choose to equate)

$$\begin{aligned} E[X^2] &= \int_{\theta}^{\infty} x^2 f(x|\theta) dx = \int_{\theta}^{\infty} x^2 \theta x^{-2} dx \\ &= \int_{\theta}^{\infty} \theta dx \rightarrow \infty. \end{aligned}$$

Meh!!! What were we thinking. Of course $\int x^2 f(x|\theta) dx$ won't converge finitely if $\int x f(x|\theta) dx$ didn't.

No questions assigned to the following page.

So we cannot use method of moments in this case.

b) Let's try M.L.E.

$$\text{Note that } f(x|\theta) = \theta x^{-2} \mathbb{I}_{\{x > \theta\}} \mathbb{I}_{\{\theta > 0\}}$$

as it is zero if $x < \theta$ and $\theta < 0$

$$\begin{aligned} f(x_1, x_2, \dots, x_n | \theta) &= \prod_{i=1}^n f(x_i | \theta) \\ &= \prod_{i=1}^n \theta x_i^{-2} \mathbb{I}_{\{x_i > \theta\}} \mathbb{I}_{\{\theta > 0\}} \end{aligned}$$

$$\begin{aligned} L(\theta | x_1, x_2, \dots, x_n) &= f(x_1, \dots, x_n | \theta) \\ &= \mathbb{I}_{\{\theta > 0\}} \theta^n \underbrace{\left(\prod_{i=1}^n x_i^{-2} \right)}_A \left(\prod_{i=1}^n \mathbb{I}_{\{x_i > \theta\}} \right) \\ &= A \theta^n \prod_{i=1}^n \mathbb{I}_{\{x_i > \theta\}} \mathbb{I}_{\{\theta > 0\}} \end{aligned}$$

$$\mathbb{I}_{\{x_i > \theta\}} = U(x_i - \theta)$$

where $U(x)$ is the unit step function

$$\text{i.e. } U(x) = \begin{cases} 1 & ; x > 0 \\ 0 & ; 0 < x \end{cases}$$

No questions assigned to the following page.

$$\therefore L(\theta | x_1, \dots, x_n) = \mathbb{I}_{\{\theta > 0\}} A^{\theta^n} \prod_{i=1}^n \mathbb{U}(x_i - \theta)$$

$$\mathbb{I}_{\{\theta > 0\}} A^{\theta^n} \cup (\min_i x_i - \theta)$$

Clearly A^{θ^n} is increasing $\forall \theta > 0$

Hence $L(\theta | x_1, \dots, x_n)$ is maximized

when $\theta = \min_i x_i$.

$$3) P(x_i \leq x | \alpha, \beta) = \begin{cases} 0 & ; x < 0 \\ (x/\beta)^\alpha & ; 0 \leq x \leq \beta \\ 1 & ; x > \beta \end{cases}$$

$\underbrace{\quad}_{C.D.F = F(x|\alpha, \beta)}$

Find M.L.E for (α, β) .

$$\text{Soln} \quad f(x | \alpha, \beta) = \frac{d}{dx} F(x | \alpha, \beta)$$

$$= \begin{cases} 0, & x < 0 \\ \frac{\alpha}{\beta^\alpha} x^{\alpha-1} & 0 \leq x \leq \beta \\ 0, & x > \beta \end{cases}$$

No questions assigned to the following page.

$$\begin{aligned}
 L(\alpha, \beta | x_1, \dots, x_n) &= \prod_{i=1}^n \frac{\alpha}{\beta^\alpha} x_i^{\alpha-1} \mathbb{I}_{\{0 < x_i < \beta\}} \\
 &= \left(\frac{\alpha}{\beta^\alpha} \right)^n \underbrace{\left(\prod_{i=1}^n x_i \right)^{\alpha-1}}_A \underbrace{\prod_{i=1}^n \mathbb{I}_{\{0 < x_i < \beta\}}}_{\mathbb{I}_{\{\max_i(x_i) < \beta\}}} \\
 &\quad \times \mathbb{I}_{\{0 < \min_i(x_i)\}}
 \end{aligned}$$

$$\begin{aligned}
 &= \left(\frac{\alpha}{\beta^\alpha} \right)^n (A)^{\alpha-1} \mathbb{I}_{\{\max_i(x_i) < \beta\}} \\
 &\quad \times \mathbb{I}_{\{0 < \min_i(x_i)\}}
 \end{aligned}$$

Assuming $\alpha > 0$, Clearly as β increases, L decreases. Hence β should be as small as possible. At the same time we want $\max_i(x_i) < \beta$ as otherwise L becomes zero which is a minimum & not maximum.

\therefore we MLE of $\hat{\beta} = \max_i \{x_i\}$

No questions assigned to the following page.

$$L = \left(\frac{\alpha}{\beta}\right)^n A^{\alpha-1} \mathbb{I}_{\{\max_i x_i < \beta\}} \mathbb{I}_{\{0 < \min_i x_i\}}$$

$$\begin{aligned} \log L &= n(\log \alpha - \alpha \log \beta) + (\alpha - 1) \log A \\ &\quad + \log \mathbb{I}_{\xi_{-s}} + \log \mathbb{I}_{\xi_{+s}} \end{aligned}$$

$$\frac{\partial \log L}{\partial \alpha} = n\left(\frac{1}{\alpha} - \log \beta\right) + \log A = 0$$

$$\Rightarrow \frac{1}{\alpha} - \log \beta = -\frac{\log A}{n}$$

$$\Rightarrow \frac{1}{\alpha} = \log \beta - \frac{\log A}{n}$$

$$\Rightarrow \alpha = \frac{1}{\log \beta - \log A/n}$$

$$\Rightarrow \hat{\alpha} = \frac{1}{\log(\max_i x_i) - \log(\underbrace{\prod x_i}_{\text{Geometric mean}})^{1/n}}$$

No questions assigned to the following page.

Q1 Ancillary and complete statistics

1.

No questions assigned to the following page.

2. Show $S^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$ is an ancillary statistic of $N(\mu, \sigma^2)$.

Soln Let $x_i = \mu + y_i$ where $y_i \sim N(0, \sigma^2)$

$$\begin{aligned} S^2 &= \frac{1}{n-1} \sum (x_i - \bar{x})^2 = \frac{1}{n-1} \sum [(\mu + y_i) \\ &\quad - (\mu + \bar{y})]^2 \\ &= \frac{1}{n-1} \sum (y_i - \bar{y})^2 \end{aligned}$$

$$\text{where } \bar{Y} = \bar{x} - \mu$$

$\therefore S^2$ is a function of $\{y_i\}$ alone & does not depend on μ .

y_i are i.i.d from $N(0, \sigma^2)$.

The distribution of $\sum (y_i - \bar{y})^2$ is known to be from χ^2 (chi-squared) distribution and does not depend on μ (only on σ^2)

Hence S^2 is ancillary.

No questions assigned to the following page.

3. $\bar{X} = \frac{1}{n} \sum_{i=1}^n x_i$ ← show this is a complete statistic for μ
 σ^2 is constant in

$$f(x|\mu, \sigma) = \frac{1}{\sigma} f((x-\mu)/\sigma)$$

Solⁿ Defⁿ of complete statistic.

$$E[g(\tau)] = 0 \neq \theta \Rightarrow P(g(\tau)=0)=1$$

No questions assigned to the following page.

4) Show that \bar{X}, S^2 are independent using Basu's theorem when μ, σ^2 are known

$$f \quad \mu \in \mathbb{R}, \sigma^2 > 0$$

Soln

$$T(x) = \left(\bar{X}, \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \right)$$

is Complete & Sufficient.

Consider the centred data

$$U = (x_1 - \bar{x}, x_2 - \bar{x}, \dots, x_n - \bar{x})$$

distribution of U is not dependent on μ or σ .

Hence U is ancillary.

By Basu's theorem complete & Ancillary statistic are independent.

$$\therefore \bar{X} \perp\!\!\!\perp S^2.$$