

Homework 6

● Graded

14 Hours, 4 Minutes Late

Student

Sampad Mohanty

Total Points

59 / 60 pts

Question 1

Problem 1

20 / 20 pts

✓ + 10 pts part 1 Correct

✓ + 10 pts part 2 Correct

+ 0 pts not correct

Question 2

Problem 2

19 / 20 pts

✓ + 5 pts part 1 Correct

✓ + 5 pts part 2 Correct

✓ + 5 pts part 3 Correct

✓ + 5 pts part 4 Correct

+ 0 pts not correct

– 1 pt part 2, no k

Question 3

Problem 3

20 / 20 pts

✓ + 5 pts 1 (a) complete

✓ + 5 pts 1 (b) complete

✓ + 5 pts 2 (a) complete

✓ + 5 pts 2 (b) complete

+ 0 pts not complete

Question assigned to the following page: [1](#)

MATH 541A HWL

PROBLEM - 1 [Convergence of random variables]

1. Show that if $\sqrt{n}(Y_n - \mu) \rightarrow N(0, \sigma^2)$ in distribution, then $Y_n \rightarrow \mu$ in probability.

i.e. $\sqrt{n}(Y_n - \mu) \xrightarrow{D} N(0, \sigma^2)$

$$\Rightarrow Y_n \xrightarrow{P} \mu.$$

Solution :- Let $Z_n = \sqrt{n}(Y_n - \mu)$

then $Y_n = \mu + \frac{Z_n}{\sqrt{n}}$

$Z_n \xrightarrow{D} N(0, \sigma^2)$ as given in question

Consider the sequence $A_n = \frac{1}{\sqrt{n}}$.

$$\lim_{n \rightarrow \infty} A_n = 0 \quad \text{& hence } P(\lim_{n \rightarrow \infty} A_n = 0) = 1$$

which means $A_n \xrightarrow{a.s.} 0 \Rightarrow A_n \xrightarrow{P} \underline{\text{Constant}} 0$

Question assigned to the following page: [1](#)

Now $Z_n \xrightarrow{D} N(0, \sigma^2)$

& $A_n \xrightarrow{P} 0$ (some constant)

Hence by Slutsky's theorem

$$F_n := \frac{Z_n}{\sqrt{n}} = Z_n A_n \xrightarrow{D} 0$$

By theorem 6.11 in lecture notes,

$$F_n \xrightarrow{D} \underline{0} \underset{\text{Some constant}}{\Leftrightarrow} F_n \xrightarrow{P} \underline{0} \underset{\text{Same constant}}{=}$$

Now applying Continuous mapping theorem,

Via the function $g(x) = \mu + x$

$$g(F_n) \xrightarrow{P} g(0)$$

$$\text{or } \mu + \frac{X_n}{\sqrt{n}} \xrightarrow{P} \mu + \frac{0}{\sqrt{n}} = \mu$$

$$\Rightarrow Y_n \xrightarrow{P} \mu$$

Question assigned to the following page: [1](#)

② Prove by definition that if Y_n converges to a constant c in probability, then Y_n converges to c in distribution.

Solution Definitions

$$Y_n \xrightarrow{P} c \Leftrightarrow \lim_{n \rightarrow \infty} P(|Y_n - c| > \epsilon) = 0$$

$$\left\{ \begin{array}{l} \boxed{\lim_{n \rightarrow \infty} P(Y_n - c > \epsilon) = 0}, Y_n > c \\ \boxed{\lim_{n \rightarrow \infty} P(c - Y_n > \epsilon) = 0}, Y_n < c \end{array} \right.$$

$$Y_n \xrightarrow{D} c \Rightarrow \lim_{n \rightarrow \infty} F_{Y_n}(y) = F_c(y) \quad \text{Point wise}$$

$$\text{But } F_c(y) = \begin{cases} 1 & \text{if } y \geq c \\ 0 & \text{if } y < c \end{cases} \quad \text{where } F_c \text{ is continuous}$$

$$F_{Y_n}(y) = P(Y_n \leq y) \quad \left[\text{Pick } \epsilon \text{ s.t. } y < c - \epsilon \right]$$

$$\text{For } y \leq c, P(Y_n \leq y) \leq \boxed{P(Y_n < c - \epsilon)}$$

$$\text{Taking lim on both sides } \lim_{n \rightarrow \infty} P(Y_n \leq y) = \lim_{n \rightarrow \infty} F_{Y_n}(y) = 0$$

Question assigned to the following page: [1](#)

$$\therefore P(Y_n \leq y) = 0 \quad \forall y \leq c$$

Let $y > c$, then

$$F_{Y_n}(y) = P(Y_n \leq y) \geq P(Y_n \leq c+\epsilon)$$

pick ϵ such
that $c+\epsilon < y$

$$\text{Then } F_{Y_n}(y) \geq P(Y_n - \epsilon \leq c)$$

"

$$P(Y_n - c \leq \epsilon)$$

"

$$1 - P(Y_n - c > \epsilon)$$

Taking lim on both sides

$$\lim_{n \rightarrow \infty} F_{Y_n}(y) \geq 1 - \boxed{\lim_{n \rightarrow \infty} P(Y_n - c > \epsilon)}$$



$$\geq 1 - 0 = 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} F_{Y_n}(y) \geq 1 \text{ for } y > c$$

But $F_{Y_n}(y) \leq 1$ as it is a CDF. Q.E.D

Hence $F_{Y_n}(y) = 1$ for $y > c$

Question assigned to the following page: [2](#)

PROBLEM: 2 (Consistency and convergence in distribution)

Let X_1, \dots, X_n be i.i.d and

$$X_{(n)} = \max_{1 \leq i \leq n} X_i.$$

1. If $X_i \sim \text{Beta}(1, \beta)$, find a value v such that $n^v(1 - X_{(n)})$ converges in distribution

Solⁿ For $\text{Beta}(\alpha, \beta)$, the pdf is

$$f(x) = \frac{C \cdot (x)^{\alpha-1} (1-x)^{\beta-1}}{\text{normalization const.}}, \quad x \in [0, 1]$$

For $\text{Beta}(1, \beta)$,

$$\begin{aligned} f(x) &= C \cdot (x)^{1-1} (1-x)^{\beta-1} \\ &= C (1-x)^{\beta-1} \end{aligned}$$

Normalization $\int_0^1 f(x) dx = 1$

$$\Rightarrow C \int_0^1 (1-x)^{\beta-1} dx = 1$$

Question assigned to the following page: [2](#)

$$\Rightarrow C \int_0^1 (1-x)^{\beta-1} dx = 1$$

$$\Rightarrow C \left[\frac{(1-x)^{\beta-1+1}}{\beta-1+1} \times \frac{1}{(-1)} \right]_0^1 = 1$$

$$\Rightarrow C \cdot \left[\frac{(1-x)^\beta}{-\beta} \right]_0^1 = 1$$

$$\Rightarrow C \cdot \frac{(1-1) - (1-0)}{-\beta} = 1$$

$$\Rightarrow C = \beta \cdot$$

$$\therefore f(x) = \beta (1-x)^{\beta-1}$$

$$F_X(x) = \int_0^x f(x) dx$$

$$= \beta \int_0^x (1-x)^{\beta-1} dx$$

$$= \cancel{\beta} \cdot \left[\frac{(1-x)^\beta}{-\beta} \right]_0^x$$

$$= -1 \cdot \left[(1-x)^\beta - (1-0)^\beta \right] = 1 - (1-x)^\beta$$

Question assigned to the following page: [2](#)

$$S_0 \quad F(x) = 1 - (1-x)^\beta$$

$$\begin{aligned}
 F_{X(n)}(x) &= P(X_{(n)} \leq x) \\
 &= P(\max(X_1, \dots, X_n) \leq x) \\
 &= P(X_1 \leq x, X_2 \leq x, \dots, X_n \leq x) \\
 &= \underbrace{P(X_1 \leq x) P(X_2 \leq x) \dots P(X_n \leq x)}_{\text{because independence}} \\
 &= F(X_1 \leq x) \cdot F(X_2 \leq x) \dots F(X_n \leq x) \\
 &= (1 - (1-x)^\beta)^n
 \end{aligned}$$

$$\text{Let } Y_n := n^{\frac{1}{\beta}}(1 - X_{(n)})$$

$$P(Y_n \leq y) = P(n^{\frac{1}{\beta}}(1 - X_{(n)}) \leq y)$$

$$P(X_{(n)} \geq 1 - \frac{y}{n^{\frac{1}{\beta}}})$$

$$1 - F_X(1 - \frac{y}{n^{\frac{1}{\beta}}})$$

$$= 1 - (1 - (1 - (1 - \frac{y}{n^{\frac{1}{\beta}}}))^\beta)^n$$

Question assigned to the following page: [2](#)

$$\Rightarrow P(Y_n \leq y) = 1 - \left(1 - \left(1 - \left(1 - \frac{y}{n^v}\right)\right)^B\right)^n$$

$$= 1 - \left(1 - \left(\frac{y}{n^v}\right)^B\right)^n$$

For large n , $\left(\frac{y}{n^v}\right)^B \rightarrow 0$ & hence

we can approximate using the identity

$$(1-z)^n \rightarrow e^{-nz}$$

$$\therefore P(Y_n \leq y) = 1 - \exp\left(-n\left(\frac{y}{n^v}\right)^B\right)$$

$$= 1 - \exp\left(-ny^B n^{-vB}\right)$$

$$= 1 - \exp\left(-y^B n^{1-vB}\right)$$

For convergence in distribution, we want

$$\lim_{n \rightarrow \infty} F_{Y_n}(y) = \lim_{n \rightarrow \infty} P(Y_n \leq y)$$

to converge to some function which
then becomes the point wise limiting

CDF of Y_n .

Question assigned to the following page: [2](#)

$$\lim_{n \rightarrow \infty} F_{Y_n}(y) = \lim_{n \rightarrow \infty} P(Y_n \leq y)$$

$$\lim_{n \rightarrow \infty} 1 - \exp\left\{-y^\beta n^{1-\beta v}\right\}$$

The only way this can converge is

$$\text{if } 1 - \beta v \leq 0 \text{ or } \frac{1}{\beta} \leq v$$

$$\lim_{n \rightarrow \infty} F_{Y_n} = \begin{cases} 1 & \text{if } v > \frac{1}{\beta} \\ 1 - \exp\left\{-y^\beta\right\} & \text{if } v = \frac{1}{\beta} \end{cases}$$

\therefore we can set $n = \frac{1}{\beta}$ to have $n^v(1 - x_{(n)})$

converge in distribution to $1 - \exp\{-y^\beta\}$.

2. If $X_i \sim \exp(1)$, find a sequence a_n

so that $X_{(n)} - a_n$ converges in distribution

$$f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0 \text{ for } \exp(\lambda)$$

$$\text{So here } f_X(x) = e^{-x}$$

Question assigned to the following page: [2](#)

$$\begin{aligned}
 F_x(x) &= \int_0^x e^{-x} dx = -e^{-x} \Big|_0^x \\
 &= -[e^{-x} - e^0] \\
 &= -[e^{-x} - 1] \\
 &= 1 - e^{-x}
 \end{aligned}$$

$$\begin{aligned}
 F_{X_{(n)}}(x) &= P(X_{(n)} \leq x) \\
 &= P(\max(X_1, \dots, X_n) \leq x) \\
 &\stackrel{\text{independence}}{=} P(X_1 \leq x, X_2 \leq x, \dots, X_n \leq x) \\
 &\rightarrow = P(X_1 \leq x) P(X_2 \leq x) \dots P(X_n \leq x)
 \end{aligned}$$

$$\begin{aligned}
 &\stackrel{\text{identically distributed}}{=} F_{X_1}(x) \dots F_{X_n}(x) \\
 &\rightarrow = F_X(x) \dots F_X(x) \\
 &= (1 - e^{-x})^n
 \end{aligned}$$

$$\text{Let } Y_n = X_{(n)} - a_n.$$

$$\text{Then } P(Y_n \leq y) = P(X_{(n)} - a_n \leq y)$$

$$F_{X_{(n)}}(y + a_n) = P(X_{(n)} \leq y + a_n)$$

Question assigned to the following page: [2](#)

$$\Rightarrow P(Y_n \leq y) = F_{X_{(n)}}(y + a_n)$$

$$= (1 - e^{-y - a_n})^n$$

We want $Y_n := X_{(n)} - a_n$ to have distribution that converges as $n \rightarrow \infty$.

$$\lim_{n \rightarrow \infty} F_{Y_n}(y) = \lim_{n \rightarrow \infty} P(Y_n \leq y)$$

$$\lim_{n \rightarrow \infty} (1 - e^{-y - a_n})^n$$

!!

$$\lim_{n \rightarrow \infty} (1 - e^{-y} e^{-a_n})^n$$

$$= \exp \log \lim_{n \rightarrow \infty} (1 - e^{-y} e^{-a_n})^n$$

log is continuous

$$= \exp \lim_{n \rightarrow \infty} \log (1 - e^{-y} e^{-a_n})^n$$

$$= \exp \left(\lim_{n \rightarrow \infty} n \log (1 - e^{-y} e^{-a_n}) \right)$$

Question assigned to the following page: [2](#)

$$\Rightarrow \lim_{n \rightarrow \infty} F_{Y_n}(y) \\ = \exp\left(\lim_{n \rightarrow \infty} n \log(1 - e^{-y} e^{-a_n})\right)$$

If $e^{-a_n} \rightarrow 0$ as $n \rightarrow \infty$,

then $e^{-y} e^{-a_n} \rightarrow 0$ as $n \rightarrow \infty$

& then $\log(1 - e^{-y} e^{-a_n}) \approx -e^{-y} e^{-a_n}$.

$$\text{So } \lim_{n \rightarrow \infty} F_{Y_n}(y) = \exp\left\{\lim_{n \rightarrow \infty} -n e^{-y} e^{-a_n}\right\}$$

$$= \exp\left\{-e^{-y} \lim_{n \rightarrow \infty} n e^{-a_n}\right\}$$

For $e^{-a_n} \rightarrow 0$ as $n \rightarrow \infty$, we

must have $a_n \rightarrow \infty$ as $n \rightarrow \infty$.

Now the limit has to exist & better if its a non-zero constant. If zero, then the Cdf is trivial. Converging to $\pm \infty$ will not give us a valid CDF.

Question assigned to the following page: [2](#)

$$\lim_{n \rightarrow \infty} n e^{-an} = \text{Const.} \cdot k$$

$$\Rightarrow e^{-an} = \frac{k}{n}$$

$$\Rightarrow \ln e^{-an} = \ln \frac{k}{n}$$

$$\Rightarrow -an = \ln \frac{k}{n}$$

$$\Rightarrow a_n = -\ln \frac{k}{n}$$

$$\Rightarrow \boxed{a_n = \ln \frac{n}{k}}$$

Answer.

Check if $a_n \rightarrow \infty$ as $n \rightarrow \infty$ ✓ PASS

3. Let $X_i \sim \text{Uniform}(0, \theta)$ - Show

that $\bar{X}_{(n)}$ is a consistent estimator of θ .

$$f_X(x) = \frac{1}{\theta} \text{ for } x \in [0, \theta]$$

$$F_X(x) = \int_0^x f_X(x) dx = \frac{x}{\theta}$$

Question assigned to the following page: [2](#)

$$\begin{aligned}
F_{X(n)}(n) &= P(X_{(n)} \leq n) \\
&= P(\max(X_1, \dots, X_n) \leq n) \\
&= P(X_1 \leq n, \dots, X_n \leq n) \\
&= P(X_1 \leq n) \cdots P(X_n \leq n) \\
&= F_{X_1}(n) \cdots F_{X_n}(n) \\
&= \left(\frac{n}{\theta}\right) \cdots \left(\frac{n}{\theta}\right) \\
&= \left(\frac{n}{\theta}\right)^n
\end{aligned}$$

A sequence of estimators \hat{Y}_n is said to be
 (weakly) consistent if $\hat{Y}_n \xrightarrow{P} \theta$ where
 θ is the true parameter for all θ

We want to show $X_{(n)} \xrightarrow{P} \theta$
 i.e. $\lim_{n \rightarrow \infty} P(|X_n - \theta| < \epsilon) = 1$

$$\lim_{n \rightarrow \infty} P(-\epsilon < X_n - \theta < \epsilon) = 1$$

Question assigned to the following page: [2](#)

$$\text{or } \lim_{n \rightarrow \infty} P(-\epsilon < X_n - \theta < \epsilon) = 1$$

But $X(n) - \theta < 0$ always

$$\text{So } \lim_{n \rightarrow \infty} P(-\epsilon < X_n - \theta) = 1$$

$$\text{or } \lim_{n \rightarrow \infty} P(\theta - \epsilon < X_n) = 1$$

$$\text{or } \lim_{n \rightarrow \infty} 1 - P(X_n \leq \theta - \epsilon) = 1$$

$$1 - \lim_{n \rightarrow \infty} F_{X(n)}(\theta - \epsilon) = 1$$

$$\Rightarrow - \lim_{n \rightarrow \infty} F_{X(n)}(\theta - \epsilon) = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} F_{X(n)}(\theta - \epsilon) = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(\frac{\theta - \epsilon}{\theta} \right)^n = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left(1 - \frac{\epsilon}{\theta} \right)^n = 0] \text{ TAUOLOGY}$$

Hence
Assumption
↑ is
true

But this is always true. Hence proved.

Question assigned to the following page: [2](#)

4. Let $X_i \sim \text{Uniform}(0, \theta)$. Show that

$\frac{2}{n} \sum_{i=1}^n X_i = 2\bar{X}$ is a consistent estimator of θ .

$$\begin{aligned} \underline{\text{Soln}} \quad E[2\bar{X}] &= 2E[\bar{X}] \\ &= 2 \cdot \frac{\theta}{2} = \theta \end{aligned}$$

$$\begin{aligned} \text{Var}(2\bar{X}) &= 4\text{Var}(\bar{X}) = 4\text{Var}\left(\frac{1}{n} \sum x_i\right) \\ &= \frac{4}{n^2} \sum \text{Var}(x_i) = \frac{4}{n^2} \cdot n \cdot \frac{\theta^2}{12} \end{aligned}$$

$$\begin{aligned} \text{Var}(x_i) &= E[x_i^2] - E[x_i]^2 \quad \frac{\theta^2}{3n} \\ &= \int_0^\theta x^2 \frac{1}{\theta} dx - \left(\frac{\theta}{2}\right)^2 \\ &= \frac{1}{\theta} \int_0^\theta x^2 dx - \left(\frac{\theta}{2}\right)^2 \\ &= \frac{1}{\theta} \left. \frac{x^3}{3} \right|_0^\theta - \frac{\theta^2}{4} = \frac{1}{\theta} \times \frac{\theta^3}{3} - \frac{\theta^2}{4} \\ &= \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12} \end{aligned}$$

Question assigned to the following page: [2](#)

$$\therefore E[2\bar{X}] = \theta \text{ & } \text{Var}(2\bar{X}) = \frac{\theta^2}{3n}$$

As $n \rightarrow \infty$ $\text{Var}(2\bar{X}) \rightarrow 0$

Using Chebychev's inequality

$$P(|2\bar{X} - E(2\bar{X})| \geq \epsilon) \leq \frac{\text{Var}(2\bar{X})}{\epsilon^2}$$

$$\Rightarrow P(|2\bar{X} - \theta| \geq \epsilon) \leq \frac{\text{Var}(2\bar{X})}{\epsilon^2}$$

$$\Rightarrow \lim_{n \rightarrow \infty} P(|2\bar{X} - \theta| \geq \epsilon) \leq \lim_{n \rightarrow \infty} \frac{\theta^2}{3n\epsilon^2} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} 1 - P(|2\bar{X} - \theta| < \epsilon) \leq 0$$

$$\Rightarrow 1 - \lim_{n \rightarrow \infty} P(|2\bar{X} - \theta| < \epsilon) \leq 0$$

$$\Rightarrow 1 \leq \lim_{n \rightarrow \infty} P(|2\bar{X} - \theta| < \epsilon)$$

But $P(|2\bar{X} - \theta| < \epsilon) \leq 1$ as it is a probability.

Questions assigned to the following page: [3](#) and [2](#)

Hence $\lim_{n \rightarrow \infty} P(|2\bar{X} - \theta| < \epsilon) = 1$

$$\Rightarrow 2\bar{X} \xrightarrow{P} \theta$$

Hence $2\bar{X}$ is a consistent estimator of θ .

PROBLEM 3 (CLT and Delta Method)

1. Let X_1, \dots, X_n be i.i.d Bernoulli(p)

random variables. Let $Y_n = \frac{1}{n} \sum_{i=1}^n X_i$.

a) Show that for $p \neq \frac{1}{2}$, the estimate of variance $Y_n(1-Y_n)$ satisfies

$$\sqrt{n} \left[Y_n(1-Y_n) - p(1-p) \right] \xrightarrow{\text{in distribution}} N(0, (1-2p)^2 p(1-p))$$

in distribution.

b) Show that for $p = \frac{1}{2}$,

$$n \left[Y_n(1-Y_n) - \frac{1}{4} \right] \xrightarrow{\text{in distribution}} -\frac{1}{4} \chi_1^2$$

in distribution.

Question assigned to the following page: [3](#)

Solⁿ a) Delta method

If Y_n is a sequence of \mathcal{X} . v and

$$\sqrt{n}(Y_n - \theta) \xrightarrow{D} N(0, \sigma^2)$$

If g is function such that $g'(\theta)$ exists but $g'(\theta) \neq 0$, then

$$\sqrt{n}[g(Y_n) - g(\theta)] \xrightarrow{D} N\left(0, \sigma^2[g'(\theta)]^2\right)$$

$$E[Y_n] = \sum \frac{1}{n} E[x_i] = \sum_{i=1}^n \frac{p}{n} = p.$$

$$Y_n \xrightarrow{D} N(p, \frac{1}{n} p(1-p)) \text{ by CLT}$$

$$\sqrt{n}(Y_n - p) \xrightarrow{D} N(0, \underbrace{p(1-p)}_{\sigma^2})$$

Now set $g(x) = x(1-x)$, $g'(x) = 1-2x$

Let by delta method

$$\sqrt{n}(Y_n(1-Y_n) - p(1-p)) \xrightarrow{D} N\left(0, \underbrace{p(1-p)}_{\sigma^2} \underbrace{\frac{(1-2p)^2}{(g'(p))^2}}_{\sigma^2}\right)$$

Question assigned to the following page: [3](#)

b) Second Order delta method

$$\text{Given } \sqrt{n}(Y_n - \theta) \xrightarrow{d} N(0, \sigma^2)$$

If $g'(\theta) = 0$, then the second order delta method says

$$n(g(Y_n) - g(\frac{1}{2})) \xrightarrow{d} \frac{1}{2}\sigma^2 g''(\theta) X_1^2$$

provided $g''(\theta)$ exists & $g''(\theta) \neq 0$

$$g(x) = x(1-x), \quad g'(x) = 1-2x$$

$$g''(x) = -2$$

$$g'(\theta = \frac{1}{2}) = 1 - 2 \cdot \frac{1}{2} = 1 - 1 = 0$$

$$g''(\theta = \frac{1}{2}) = -2$$

From part (a), setting $\theta = \frac{1}{2}$

$$\sqrt{n}(Y_n - \frac{1}{2}) \xrightarrow{D} N(0, \frac{1}{4})$$

$$\text{Hence } n(Y_n(1-Y_n) - \frac{1}{2} \cdot (1-\frac{1}{2}))$$

$$\xrightarrow{D} \frac{1}{2} \cdot \frac{1}{4} \cdot (-2) X_1^2 = -\frac{1}{4} X_1^2$$

Q.E.D

Question assigned to the following page: [3](#)

2. Let X_1, \dots, X_n be i.i.d from Uniform(1, 2) and let

$$H_n = \frac{n}{X_1^{-1} + \dots + X_n^{-1}}$$

a) Show that $H_n \rightarrow c$ in probability, identifying the constant c

b) Show that $\sqrt{n}(H_n - c)$ converges in distribution, and identify the limit.

$$\text{Soln} \quad H_n = \left(\frac{1}{n} \sum \frac{1}{x_i} \right)^{-1} \text{ & } f_{X_i}(x) = \frac{1}{2-1}$$

$$E\left[\frac{1}{x_i}\right] = \int_1^2 \frac{1}{x} f_{X_i}(x) dx = \int_1^2 \frac{1}{x} \cdot 1 dx$$

$$= \ln x \Big|_1^2 = \ln 2 - \ln 1 = \ln 2$$

By Law of Large Numbers

$$\frac{1}{n} \sum \frac{1}{x_i} \xrightarrow{P} \ln 2$$

Question assigned to the following page: [3](#)

By continuous mapping theorem

$$H_n = \left(\frac{1}{n} \sum \frac{1}{x_i} \right)^{-1} \xrightarrow{P} \frac{1}{\ln 2}$$

Note that we used the $g(x) = \frac{1}{x}$
which is continuous @ $x = \ln 2$

b) Let $Y_i = \frac{1}{x_i}$ such that $\bar{Y} = \frac{1}{n} \sum Y_i$

$$E[Y_i] = \ln 2 \quad \frac{1}{n} \sum \frac{1}{x_i}$$

$$\text{Var}(Y_i) = \text{Var}\left(\frac{1}{x_i}\right) = E\left[\left(\frac{1}{x_i}\right)^2\right] - E\left[\frac{1}{x_i}\right]^2$$

$$E\left[\frac{1}{x_i}\right]^2 = E[Y_i]^2 = \ln^2 2$$

$$\begin{aligned} E\left[\left(\frac{1}{x_i}\right)^2\right] &= \int_1^2 \frac{1}{x^2} f(x) dx = \int_1^2 \frac{1}{x^2} \cdot 1 dx \\ &= \left[\frac{-1}{x} \right]_1^2 = \left(\frac{1}{1} - \frac{1}{2} \right) \\ &= \frac{1}{2} \end{aligned}$$

$$\text{Var}\left(\frac{1}{x_i}\right) = \frac{1}{2} - \ln^2 2$$

Question assigned to the following page: [3](#)

Applying CLT

$$\sqrt{n} \left(\bar{Y}_n - \ln 2 \right) \xrightarrow{D} N\left(0, \frac{1}{2} - \ln^2 2\right)$$

Apply delta method with $g(x) = \frac{1}{x}, g' = -\frac{1}{x^2}$

$$\sqrt{n} \left(H_n - \frac{1}{\ln 2} \right) \xrightarrow{D} N(0, \alpha)$$

where $\alpha = \left(\frac{1}{2} - \ln 2\right)^2 g'(\ln 2)^2$

$$\alpha = \left(\frac{1}{2} - \ln 2\right)^2 \left(-\frac{1}{(\ln 2)^2}\right)^2$$

$$= \left(\frac{1}{2} - \ln 2\right)^2 \left(-\frac{1}{(\ln 2)^2}\right)^2$$

$$= \left(\frac{1}{2} - \ln 2\right)^2 \frac{1}{(\ln 2)^4}$$

Q.E.D.