1.1

$$f(x|p) = \frac{N! \cdot p_1^{x_1} \cdot p_2^{x_2} \cdots p_{k-1}^{x_{k-1}} \cdot}{x_1!\, x_2! \cdots x_{k-1}!\, (N-S)!} (1-q)^S$$

where $S = \sum_{i=1}^{k-1} x_i$ and $q = \sum_{i=1}^{k-1} p_i$

$\therefore$ $f(x_1 \ldots x_{k-1} | p_1 \ldots p_{k-1}) = f(x|p)$ and is given in the formula above.

1.2

We can represent the above formula in canonical form as below :-

$$\frac{N! \prod_{i=1}^{k-1} p_i^{x_i} \, q^S}{\prod_{i=1}^{k-1} x_i!\, (N-S)!} = \frac{N!}{\prod_{i=1}^{k-1} x_i! \cdot (N-S)!} \exp\left(\sum_{i=1}^{k-1} x_i \log p_i + S \log(1-q)\right)$$

$$= \frac{N!}{\prod_{i=1}^{k-1} x_i!\, (N-S)!} \exp\left\{\sum_{i=1}^{k-1} x_i \log p_i + \left(N - \sum_{i=1}^{k-1} x_i\right) \log(1-q)\right\}$$

$$= \frac{N!}{\prod_{i=1}^{k-1} x_i!\, (N-S)!} \exp\left\{\sum_{i=1}^{k-1} x_i \log p_i + N \log(1-q) - \sum_{i=1}^{k-1} x_i \log(1-q)\right\}$$

$$= \frac{N!}{\prod_{i=1}^{k-1} x_i!\, (N-S)!} \exp\left\{\sum_{i=1}^{k-1} x_i \left[\log p_i - \log(1-q)\right] + N \log(1-q)\right\}$$

$$= \frac{N!}{\prod_{i=1}^{k-1} x_i! \, (N-S)!} \exp\left\{ \sum_{i=1}^{k-1} x_i \left[\log \frac{p_i}{1-q} \right] + N \log(1-q) \right\}$$

Set $\eta_i(\vec{p}) = \log\left(\frac{p_i}{1-q}\right) = \log\left(\frac{p_i}{1 - \sum_{i=1}^{k-1} p_i}\right)$

Set $t_i(\vec{x}) = x_i$

$A(\vec{p}) = -N \log(1-q) = N \log\left(\frac{1}{1-q}\right)$

$\boxed{h(\vec{x}) = \frac{N!}{\prod_{i=1}^{k-1} x_i! \, (N-S)!}}$ (Base measure)

$\therefore \; f(x_1 \ldots x_{k-1} \mid p_1 \cdot p_2 \cdot p_3 \cdots p_{k-1}) = h(x) \exp\left\{ \sum_{i=1}^{k-1} \eta_i \cdot t_i - A(\vec{p}) \right\}$

☆ Now we want to express $f(x_1 \ldots x_{k-1} \mid p_1 \ldots p_{k-1})$ in terms of canonical parameters.

$$\eta_i = \log\left(\frac{p_i}{1 - \sum_{i=1}^{k-1} p_i}\right) \Rightarrow e^{\eta_i} = \frac{p_i}{1-q}$$

$$\Rightarrow \sum_{i=1}^{k-1} e^{\eta_i} = \frac{1}{1-q} \sum_{i=1}^{k-1} p_i = \frac{q}{1-q}$$

$$\Rightarrow 1 + \sum_{i=1}^{k-1} e^{\eta_i} = \frac{q}{1-q} + 1 = \frac{q + 1 - q}{1-q}$$

$$= \frac{1}{1-q}$$

$$\Rightarrow 1 + \sum_{i=1}^{k-1} e^{\eta_i} = \frac{1}{1-q}$$

$$\Rightarrow N \log\left(1 + \sum_{i=1}^{k-1} e^{\eta_i}\right) = \underbrace{N \log\left(\frac{1}{1-q}\right)}_{A(\vec{p})}$$

$$\Rightarrow A(\vec{p}) = \underbrace{N \log\left(1 + \sum_{i=1}^{k-1} e^{\eta_i}\right)}_{A^*(\vec{\eta})}$$

$$\therefore A^*(\vec{\eta}) = N \log\left(1 + \sum_{i=1}^{k-1} e^{\eta_i}\right)$$

Hence $p(x_1 \ldots x_{k-1} \mid p_1 p_2 \ldots p_{k-1})$

$$= h(x) \exp\left\{ \vec{\eta}(\vec{p}) \cdot \vec{T}(\vec{x}) - A^*(\vec{\eta}) \right\}$$

where $T(\vec{x}) = [x_1\ x_2 \ldots x_{k-1}]$

$$\eta_i(\vec{p}) = \log \frac{p_i}{1 - \sum_{i=1}^{k-1} p_i}$$

and $h(x) = \dfrac{N!}{\prod_{i=1}^{k-1} x_i!\,(\underbrace{N-S}_{x_k})!} = \dfrac{N!}{\prod_{i=1}^{k} x_i!}$

1.3

$$A^*(\vec{\eta}) = N \log\left(1 + \sum_{i=1}^{k-1} e^{\eta_i}\right)$$

as shown in last part 1.2

The natural parameter space is defined as

$$\mathcal{H} = \{\eta : A^*(\eta) < \infty, \eta \in \mathbb{R}^{k-1}\}$$

i.e. the natural parameter space is the set that keeps A^* well defined.

This is so because of the following reason ⇒

$$p(\vec{x} \mid \vec{\eta}) = h(\vec{x}) \exp\{\vec{\eta} \cdot \vec{T}(\vec{x}) - A^*(\vec{\eta})\}$$

$$= \frac{h(\vec{x})}{e^{A^*(\vec{\eta})}} \exp\{\vec{\eta} \cdot \vec{T}(\vec{x})\}$$

If $A^*(\vec{\eta}) \longrightarrow \infty$, $p(\vec{x} \mid \vec{\eta}) \longrightarrow 0$

In our question, $A^*(\eta) = N \log\left(1 + \sum_{i=1}^{k-1} e^{\eta_i}\right)$

$$A^*(\eta) = \infty \Rightarrow \sum_{i=1}^{k-1} e^{\eta_i} = \infty \Rightarrow \eta_i \neq \infty$$

$\therefore$ The natural parameter space is

$$\mathcal{H} = \{ \eta_i < \infty \} = \{ \eta_i \in \mathbb{R} \}.$$

$$\therefore \mathcal{H} = \{ \vec{\eta} \in \mathbb{R}^{k-1} \}$$

1.4 $$A(\vec{\eta}) = N \log \left(1 + \sum_{i=1}^{k-1} e^{\eta_i} \right)$$

$$E(T_i(\vec{x})) = E[X_i] = \frac{\partial A}{\partial \eta_i} = \frac{N e^{\eta_i}}{1 + \sum_{i=1}^{k-1} e^{\eta_i}}$$

$$\boxed{\because T_i(\vec{x}) = x_i}$$

Observation :-

Recall that $e^{\eta_i} = \dfrac{p_i}{1-q}$

where $q = \sum_{i=1}^{k-1} p_i$

$$\sum_{i=1}^{k-1} e^{\eta_i} = \frac{1}{1-q} \sum_{i=1}^{k-1} p_i = \frac{q}{1-q}$$

$$\Rightarrow 1 + \sum_{i=1} e^{\eta_i} = \frac{1}{1-q}$$

$$\therefore \quad e^{\eta_i} = \frac{p_i}{1-q} = p_i\left(1+\sum_{i=1}^{k-1} e^{\eta_i}\right)$$

$$\Rightarrow \quad p_i = \frac{e^{\eta_i}}{1+\sum_{i=1}^{k-1} e^{\eta_i}}$$

$$\therefore \quad E[T_i(\vec{x})] = E[X_i]$$

$$= N \frac{e^{\eta_i}}{1+\sum_{i=1}^{k-1} e^{\eta_i}} = N p_i$$

$$\operatorname{Cov}(T_i(\vec{x}), T_j(\vec{x})) = \operatorname{Cov}(X_i, X_j)$$

$$\boxed{\because T_i(\vec{x}) = x_i}$$

$$= \frac{\partial^2 A}{\partial \eta_j \partial \eta_i} = \frac{\partial}{\partial \eta_j}\left(\frac{\partial A}{\partial \eta_i}\right) = \frac{\partial}{\partial \eta_j}(N p_i)$$

$$= N \frac{\partial p_i}{\partial \eta_j} = N \frac{\partial}{\partial \eta_j}\left(\frac{e^{\eta_i}}{1+\sum_{i=1}^{k-1} e^{\eta_i}}\right)$$

$$= N e^{\eta_i} \frac{-1 \cdot e^{\eta_j}}{\left(1+\sum_{i=1}^{k-1} e^{\eta_i}\right)^2} = -N \frac{e^{\eta_i} e^{\eta_j}}{\left(1+\sum_{i=1}^{k-1} e^{\eta_i}\right)^2}$$

$$\mathrm{Cov}(x_i, x_j) = -N \frac{e^{\eta_i}}{\left(1+\sum_{i=1}^{k-1} e^{\eta_i}\right)} \cdot \frac{e^{\eta_j}}{\left(1+\sum_{i=1}^{k-1} e^{\eta_i}\right)}$$

$$= -N p_i p_j$$

For Variance, $\mathrm{Var}(x_i) = \dfrac{\partial^2 A}{\partial \eta_i^2}$

$$\frac{\partial^2 A}{\partial \eta_i^2} = \frac{\partial}{\partial \eta_i}\left(\frac{\partial A}{\partial \eta_i}\right) = \frac{\partial}{\partial \eta_i}\left(\frac{N e^{\eta_i}}{1+\sum_{l=1}^{k-1} e^{\eta_l}}\right)$$

$$= N\left[\frac{\left(1+\sum_{l=1}^{k-1} e^{\eta_l}\right) e^{\eta_i} - e^{\eta_i} \cdot e^{\eta_i}}{\left(1+\sum_{l=1}^{k-1} e^{\eta_l}\right)^2}\right]$$

$$= N\left[\frac{e^{\eta_i}\left\{\left(1+\sum_{l=1}^{k-1} e^{\eta_l}\right) - e^{\eta_i}\right\}}{\left(1+\sum_{l=1}^{k-1} e^{\eta_l}\right)^2}\right]$$

$$= N\left[\frac{e^{\eta_i}}{\left(1+\sum_{l=1}^{k-1} e^{\eta_l}\right)} \cdot \frac{\left(1+\sum_{l=1}^{k-1} e^{\eta_l}\right) - e^{\eta_i}}{\left(1+\sum_{l=1}^{k-1} e^{\eta_l}\right)}\right]$$

$$\Rightarrow \mathrm{Var}(X_i) = N\left[p_i \cdot (1 - p_i) \right]$$

$$\therefore \mathrm{Cov}(X_i, X_j) = \begin{cases} N p_i (1-p_i) \; ; \; i = j \\ -N p_i p_j \; ; \; i \neq j \end{cases}$$

Q2 CUMULANT GENERATING FUNCTION

$$K_n(X) = \frac{d^n}{dt^n} K_X(t) \Big|_{t=0}$$

2.1 $K_1(X) = \mathbb{E}X$ ← to show

$$LHS = K_1(X) = \frac{d}{dt} K_X(t) \Big|_{t=0}$$

$$\frac{d}{dt} \log \mathbb{E}[e^{tX}] = \mathbb{E}\left[\frac{\partial}{\partial t} e^{tX}\right] \times \frac{1}{\mathbb{E}[e^{tX}]}$$

$$= \frac{\mathbb{E}[X e^{tX}]}{\mathbb{E}[e^{tX}]}$$

@ $t = 0$, $$\frac{d}{dt} K_X(t) \Big|_{t=0} = \frac{\mathbb{E}[X]}{\mathbb{E}[1]} = \mathbb{E}[X]$$

$= RHS$

$$K_2(X) = Var(X) = \left.\frac{d^2}{dt^2} K_X(t)\right|_{t=0}$$

$$\Rightarrow K_2(X) = \frac{d}{dt}\left\{\frac{d}{dt} K_X(t)\right\}$$

$$= \frac{d}{dt} \frac{\mathbb{E}[Xe^{tX}]}{\mathbb{E}[e^{tX}]}$$

$$= \frac{\mathbb{E}[X^2 e^{tX}]\,\mathbb{E}[e^{tX}] - \mathbb{E}[Xe^{tX}]\,\mathbb{E}[Xe^{tX}]}{\mathbb{E}[e^{tX}]^2}$$

$$= \frac{\mathbb{E}[X^2 e^{tX}]\,\mathbb{E}[e^{tX}] - \mathbb{E}[Xe^{tX}]^2}{\mathbb{E}[e^{tX}]^2}$$

@ $t=0$, $K_2(X) = \dfrac{\mathbb{E}[X^2]\,\mathbb{E}[1] - \mathbb{E}[X]^2}{\mathbb{E}[1]^2}$

$$\Rightarrow K_2(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Let $E[X] = \mu$ (a constant), then

$$K_2(X) = E[X^2] - \mu^2$$

$$= E[X^2 - \mu^2]$$

$$= E[X^2 - 2X\mu + \mu^2 + 2X\mu - 2\mu^2]$$

$$= E[x^2 - 2x\mu + \mu^2 + 2x\mu - 2\mu^2]$$

$$= E[(x-\mu)^2] + E[2x\mu - 2\mu^2]$$

$$= \underbrace{E[(x-\mu)^2]}_{Var(x)} + 2\underbrace{E[x]}_{\mu}\mu - E[2\mu^2]$$

$$= Var(x) + \cancel{2\mu^2} - \cancel{2\mu^2}$$

$$= Var(x)$$

2.2 $X \sim N(\mu, \sigma^2)$

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$\Rightarrow M_x(t) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$K_x(t) = \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) + \log \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$K_X(t) = \underbrace{\log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)}_{\text{Constant }(C)} + \log \int_{-\infty}^{\infty} e^{tx - \frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

$$= C + \log \int_{-\infty}^{\infty} e^{\frac{2\sigma^2 t x - (x-\mu)^2}{2\sigma^2}}\, dx$$

$$= C + \log \int_{-\infty}^{\infty} e^{\frac{2\sigma^2 t x - x^2 - \mu^2 + 2x\mu}{2\sigma^2}}\, dx$$

$$= C + \log \int_{-\infty}^{\infty} e^{-\mu^2/2\sigma^2} \cdot e^{\frac{-x^2 + 2x(\sigma^2 t + \mu)}{2\sigma^2}}\, dx$$

$$= \underbrace{C + \log e^{-\mu^2/2\sigma^2}}_{\text{Constant }(D)} + \log \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left(x^2 - 2x(\sigma^2 t + \mu)\right)}\, dx$$

$$= D + \log \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[\left\{x - (\sigma^2 t + \mu)\right\}^2 - (\sigma^2 t + \mu)^2\right]}\, dx$$

$$= D + \log \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left\{x - (\sigma^2 t + \mu)^2\right\}} \cdot e^{\frac{1}{2\sigma^2}(\sigma^2 t + \mu)^2}\, dx$$

$$= D + \log e^{\frac{1}{2\sigma^2}(\sigma^2 t + \mu)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left\{x - (\sigma^2 t + \mu)^2\right\}}\, dx$$

$$= D + \log e^{\frac{1}{2\sigma^2}(\sigma^2 t + \mu)^2} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\{x - (\sigma^2 t + \mu)^2\}} dx}$$

$$= D + \frac{(\sigma^2 t + \mu)^2}{2\sigma^2} +$$

this integral doesn't depend on t as this is the integration of pdf of normal distribution with mean $\sigma^2 t + \mu$ & variance σ^2.

Its integral is $\sqrt{2\pi\sigma^2}$

$$\Rightarrow K_x(t)$$

$$D + \frac{(\sigma^2 t + \mu)^2}{2\sigma^2} + \log\left(\sqrt{2\pi\sigma^2}\right)$$

$$\Rightarrow \frac{d}{dt} K_x(t) = \frac{1}{\cancel{2}\cancel{\sigma^2}} \times \cancel{2}(\sigma^2 t + \mu) \times \cancel{\sigma^2}$$

$$@\ t = 0,\quad \frac{d}{dt} K_x(t)\Big|_{t=0} = \mu = K_1(x)$$

Similarly, $\dfrac{d^2}{dt^2} K_x(t) = \dfrac{d}{dt}\left(\dfrac{d}{dt} K_x(t)\right)$

$$= \frac{d}{dt}(\sigma^2 t + \mu) = \sigma^2 = K_2(x)$$

$$\because \left.\frac{d^2 K_x(t)}{dt^2}\right|_{t=0} = \sigma^2$$

$$\frac{d^3 K_x(t)}{dt^3} = \frac{d}{dt}\left(\frac{d^2 K_x(t)}{dt^2}\right)$$

$$= \frac{d}{dt}(\sigma^2) = 0$$

$$\therefore \frac{d^n K_x(t)}{dt^n} = 0 \quad \forall\, n \geq 3$$

$$\therefore \left.\frac{d^n K_x(t)}{dt^n}\right|_{t=0} = 0$$

$$\therefore K_n(x) = \left.\frac{d^n K_x(t)}{dt^n}\right|_{t=0} = 0 .$$

2.3

Let $Y = X + c$

$$K_Y(t) = \log \mathbb{E}[e^{tY}]$$

$$= \log \mathbb{E}[e^{t(X+c)}]$$

$$= \log \mathbb{E}[e^{tc} e^{tX}]$$

$$= \log e^{tc} E[e^{tX}]$$

$$= tc + \log \mathbb{E}[e^{tX}]$$

$$K_Y(t) = tc + K_X(t)$$

$$\Rightarrow \frac{d}{dt} K_Y(t) = c + \frac{d}{dt} K_X(t)$$

$$\Rightarrow \frac{d^2}{dt^2} K_Y(t) = \frac{d^2 K_X(t)}{dt^2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\Rightarrow \frac{d^n}{dt} K_Y(t) = \frac{d^2 K_X(t)}{dt^2}$$

$$\Rightarrow K_n(Y) = \frac{d^n}{dt} K_Y(t)\Big|_{t=0} = \frac{d^n}{dt} K_X(t)\Big|_{t=0} = K_n(X)$$

Let $Z = cX$

$$K_Z(t) = \log \mathbb{E}[e^{Zt}]$$

$$= \log \mathbb{E}[e^{cXt}]$$

$$= \log \mathbb{E}[e^{Xct}]$$

$$= \log \mathbb{E}[e^{Xq}] \; ; \; q = ct.$$

$$= K_X(q)$$

$$\Rightarrow \frac{d}{dt} K_Z(t) = \frac{d}{dt} K_X(q) = \frac{d}{dq} K_X(q) \times \frac{dq}{dt}$$

$$\Rightarrow \frac{d}{dt} K_Z(t) = \frac{d}{dq} K_X(q) \times c$$

$$\Rightarrow \frac{d^2 K_Z(t)}{dt^2} = c \frac{d}{dt}\left(\frac{d}{dq} K_X(q)\right)$$

$$= c \frac{d}{dq}\left(\frac{d}{dq} K_X(q)\right) \frac{dq}{dt}$$

$$= c^2 \frac{d^2 K_X(q)}{dq^2}$$

Continuing the same way, we get :-

$$\frac{d^n K_Z(t)}{dt^n} = c^n \frac{d^n K_X(q)}{dq^n}$$

$$\Rightarrow \left.\frac{d^n K_Z(t)}{dt^n}\right|_{t=0} = c^n \left.\frac{d^n}{dq^n} K_X(q)\right|_{t=0}$$

$\Rightarrow q = ct = 0$

$$\Rightarrow K_n(Z) = c^n K_n(X)$$

$$\Rightarrow K_n(cX) = c^n K_n(X)$$

2.4

$$Y = X_1 + X_2 \cdots X_k = \sum_{i=1}^{k} X_i$$

$$K_Y(t) = \log E[e^{tY}] = \log E[e^{t\sum_{i=1}^{k} X_i}]$$

$$= \log \mathbb{E}[e^{\sum_{i=1}^{k} tX_i}]$$

$$= \log \mathbb{E}\left[\prod_{i=1}^{k} e^{tX_i}\right]$$

$$= \log \prod_{i=1}^{k} \mathbb{E}[e^{tX_i}]$$ (using independence)

$$\Rightarrow K_Y(t) = \log \prod_{i=1}^{k} \mathbb{E}[e^{tx_i}]$$

$$= \sum_{i=1}^{k} \log \mathbb{E}[e^{tx_i}]$$

$$= \sum_{i=1}^{k} K_{x_i}(t)$$

$$\Rightarrow \frac{d}{dt} K_Y(t) = \sum_{i=1}^{k} \frac{d}{dt} K_{x_i}(t)$$

$$\Rightarrow \frac{d^n}{dt^n} K_Y(t) = \sum_{i=1}^{k} \frac{d^n}{dt^n} K_{x_i}(t)$$

$$\Rightarrow \left.\frac{d^n}{dt^n} K_Y(t)\right|_{t=0} = \left.\sum_{i=1}^{k} \frac{d^n}{dt^n} K_{x_i}(t)\right|_{t=0}$$

$$\Rightarrow K_n(Y) = \sum_{i=1}^{k} K_n(x_i)$$

3.1 If $T(X)$ is sufficient for θ, factorization theorem says

$$f(x_1 \dots x_n \mid \theta) = g(T,\theta)\, h(x_1 \dots x_n)$$

$$f(x_1 \dots x_n \mid \theta) = \prod_{i=1}^{n} f_i(x_i \mid \theta)$$

$$= \prod_{i=1}^{n} e^{(i\theta - x_i)} \underbrace{\mathbb{I}_{\{x_i \geq i\theta\}}}_{\text{indicator function}}$$

$$= e^{\sum_{i=1}^{n}(i\theta - x_i)} \prod_{i=1}^{k} \mathbb{I}_{\{x_i \geq i\theta\}}$$

$$= e^{\theta \frac{n(n+1)}{2}} e^{-\sum_{i=1}^{n} x_i} \prod_{i=1}^{k} \mathbb{I}_{\{x_i \geq i\theta\}}$$

Let $T = \min_i \frac{X_i}{i}$

Note that if $T \geq \theta$, then $X_i \geq i\theta \ \forall i$

$$\therefore \prod_{i=1}^{k} \mathbb{I}_{\{x_i \geq i\theta\}} = \mathbb{I}_{\{T \geq \theta\}}$$

$$\therefore f(x_1 \dots x_n \mid \theta) = \mathbb{I}_{\{T \geq \theta\}}\, e^{\frac{\theta n(n+1)}{2}} \cdot e^{-\sum_{i=1}^{n} x_i}$$

$$\therefore f(x_1 \ldots x_n|\theta) = \underbrace{\mathbb{I}_{\{T \geq \theta\}} e^{\frac{\theta n(n+1)}{2}}}_{g(T,\theta)} \cdot \underbrace{e^{-\sum_{i=1}^{n} x_i}}_{h(x_1 \ldots x_n)}$$

$$= g(T,\theta)\, h(x_1, x_2 \ldots x_n)$$

Since we can factorize the pdf in this form, T must be a sufficient statistic for θ.

4.2

$$f(x_1 \ldots x_n | \alpha, \beta) = \prod_{i=1}^{n} \frac{\beta^\alpha}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\beta x_i}$$

$$= \left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right)^n \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\beta \sum_{i=1}^{n} x_i}$$

$$= \frac{\beta^{n\alpha}}{\Gamma(\alpha)^n} \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\beta \sum_{i=1}^{n} x_i}$$

Set $T_1(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} x_i$

$T_2(x_1 \ldots x_n) = \sum_{i=1}^{n} x_i$

$$\Rightarrow f(x_1 \dots x_n \mid \alpha, \beta)$$

$$= \frac{\beta^{n\alpha}}{\Gamma(\alpha)^n} (T_1)^{\alpha-1} e^{-\beta T_2}$$

Set $g(T, \alpha, \beta) = \frac{\beta^{n\alpha}}{\Gamma(\alpha)^n} (T_1)^{\alpha-1} e^{-\beta T_2}$

and $h(x_1 \dots x_n) = 1$

Hence $T_1 = \prod_{i=1}^{n} x_i$ & $T_2 = \sum_{i=1}^{n} x_i$ are sufficient for α, β.

$$T(\vec{x}) = \begin{bmatrix} \prod_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i \end{bmatrix}$$

4.3

$$f(x_1 \dots x_n \mid \mu, \sigma)$$

$$= \prod_{i=1}^{n} \frac{1}{\sigma} e^{-(x-\mu)/\sigma} \mathbb{I}_{\{x_i > \mu}$$

$$f(x_1 \dots x_n \mid \mu, \sigma) = \left(\frac{1}{\sigma}\right)^n e^{-\frac{1}{\sigma}\sum_{i=1}^{n}(x_i-\mu)} \times \prod_{i=1}^{n} \mathbb{I}_{\{x_i > \mu\}}$$

Set $T_1 = \min\{x_1 \dots x_n\}$

$$f(x_1 \dots x_n \mid \mu, \sigma) = \left(\frac{1}{\sigma}\right)^n e^{-\frac{1}{\sigma}\sum_{i=1}^{n} x_i} e^{\frac{n\mu}{\sigma}} \times \prod_{i=1}^{n} \mathbb{I}_{\{x_i > \mu\}}$$

Set $T_1 = \min_i \{x_i\}$

$$T_2 = \sum_{i=1}^{n} x_i$$

$$f(x_1 \dots x_n \mid \mu, \sigma) = \left(\frac{1}{\sigma}\right)^n e^{-\frac{1}{\sigma}T_2} e^{\frac{n\mu}{\sigma}} \cdot \mathbb{I}_{\{T_1 > \mu\}}$$

$$h(\vec{T}, \mu, \sigma) = \left(\frac{1}{\sigma}\right)^n e^{-\frac{1}{\sigma}T_2} e^{n\mu/\sigma} \cdot \mathbb{I}_{\{T_1 > \mu\}}$$

$$g(x_1 \dots x_n) = 1$$