Midterm 2

Graded

Student

Sampad Mohanty

Total Points

37 / 60 pts

Question 1

Problem 1

15 / 20 pts

- **+ 20 pts** Correct
- ✔ **+ 7 pts** (a) correct
- **+ 6 pts** (a) minor error. We ask $I_X(\mu)$ for one variable X not for $X_1, \ldots, X_n$
- **+ 3 pts** (a) Partial credit for incorrect answers
- ✔ **+ 6 pts** (b) correct
- **+ 5 pts** (b) minor error
- **+ 7 pts** (c) correct
- ✔ **+ 2 pts** Only stated a CLT for $\sum_{i=1}^{n} X_i$
- **+ 4 pts** (c) used the fact X_i^2 are i.i.d., and tried to prove a CLT, but the variance is not correctly computed or (ii) directly claim $\chi^2(n)$ is Gaussian without justification
- **+ 6 pts** (c) almost correct except showing $\mathrm{Var}(X_i^2) = 2$ in details.

Question 2

Problem 2

6 / 20 pts

+ **5 pts** (a) Correct
+ **1 pt** (a) no theorem is used. Only match the expectation
+ ✔ **3 pts** (a) Incorrect theorem used
+ **5 pts** (b) Correct
+ **4 pts** (b) minor mistake
+ **2 pts** partial correct for trying to compute $R(p, \delta(X)$
+ **10 pts** (c) correct
+ ✔ **3 pts** (c)Used Lehmann-Scheffe's thm but the complete suffciient statistics is incorrect or didn't write down a correct, explicit anwer
+ **9 pts** (c) only minor mistake
+ **2 pts** (c) only showed ϕ is unbiased.
+ **5 pts** (c) correct application of Lehmann-Scheffe and suffient statistics is correct. Error in computing $\mathbb{E}[X_1 X_2 | \sum_i X_i]$.
+ **0 pts** Incorrect

1 why? WLLN you need a sum of i.i.d. random variables.

Question 3

Problem 3

16 / 20 pts

+ **20 pts** Correct
+ ✔ **7 pts** (a) correct
+ **4 pts** (a) incorrect answer involving Y_i
+ **7 pts** (b) correct
+ **6 pts** (b) minor error in the final answer
+ ✔ **3 pts** partial credit for taking the derivative with respect to λ
+ ✔ **6 pts** (c) correct
+ **5 pts** (c) only one minor error in the final answer
+ **4 pts** (c)one answer is correct
+ **2 pts** (c) both answers are incorrect but with some correct intermediate steps
+ **0 pts** (c) incorrect

Math 541 A - Spring 2025
Midterm 2
April 9, 2025
11:00-11:50 AM

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- There are 3 problems in total. Make sure your exam contains all these questions.
- You are allowed to use one page of double-sided **hand-written** 8.5 by 11 inch notes.
- You must show your work on all problems. The correct answer with no supporting work may result in no credit.
- If you need more room, use the backs of the pages and indicate to the grader that you have done so.

Problem 1	20	
Problem 2	20	
Problem 3	20	
Total	60	

Sampled

Name: ID:

1. (20 pts) Let $X_1, \ldots, X_n \sim N(\mu, 1)$ be i.i.d. samples. Recall the PDF for $N(\mu, 1)$ is

$$f(x|\mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}}.$$

(a) (7 pts) For one random variable $X \sim N(\mu, 1)$, compute its Fisher information

$$I_X(\mu) = \mathbb{E}\left(\frac{d}{d\mu}\log f(X|\mu)\right)^2.$$

(b) (6 pts) Find the the Cramé-Rao lower bound for any unbiased estimator (as a function of $X_1, \ldots, X_n$) of μ.

(c) (7 pts) Assume $X_1, \ldots, X_n \sim N(0, 1)$ be i.i.d. random variables. Show the following convergence in distribution holds:

$$\frac{\sum_{i=1}^n (X_i^2 - 1))}{\sqrt{2n}} \xrightarrow{d} N(0, 1).$$

a) $\frac{d}{d\mu} \log f(x|\mu) = \frac{d}{d\mu}\left[\log\frac{1}{\sqrt{2\pi}} - \frac{(x-\mu)^2}{2}\right] = \frac{\cancel{2}(x-\mu)}{\cancel{2}} \cdot 1 = x - \mu$

$\underset{X}{E}\left[\left(\frac{d}{d\mu}\log f(x|\mu)\right)^2\right] = E_X\left[(X-\mu)^2\right] \xrightarrow{} = Var(X) = \underline{\underline{1}}$ ~~E[X]-E[μ] = E[X]-μ = μ-μ = 0~~

b) C.R lower bound $= \frac{1}{n I_X(\mu)} = \frac{1}{n \cdot 1} = \underline{\underline{\frac{1}{n}}}$

c) By CLT

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$p^2 \cdot 1-p$
$+ (1-p)^2 p$
$p(1-p)$

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2. (20 pts) Let $X_1, \ldots, X_n$ be i.i.d. Ber(p).

   (a) (5 pts) Show that the following convergence in probability holds:

   $$\frac{1}{n^2}(X_1 + \cdots + X_n)^2 \xrightarrow{p} p^2.$$

   (b) (5 pts) Consider a risk function for any estimator $\delta(\mathbf{X})$ of p given by

   $$R(p, \delta(\mathbf{X})) = \mathbb{E}_p|\delta(\mathbf{X}) - p|.$$

   Let $\delta(\mathbf{X}) = X_1$ be the estimator, and $\pi(p) = \text{Uniform}(0,1)$ be the prior distribution on p. Find the Bayes risk

   $$\int_0^1 R(p, \delta(\mathbf{X}))\pi(p)dp.$$

   (c) (10 pts) Show that $\phi(\mathbf{X}) = X_1X_2$ is an unbiased estimator of p^2. Based on $\phi(\mathbf{X})$ and the Lehmann–Scheffé Theorem, find the best unbiased estimator $W = W(X_1, \ldots, X_n)$ of p^2 with an explicit form.

a) $E[\frac{1}{n^2}(X_1 + \ldots X_n)^2] = Var(\frac{1}{n}(X_1 + \ldots X_n)) + E[\frac{1}{n}(X_1 + \cdots X_n)]^2$

$= \frac{1}{n} \cdot p(1-p) + p^2 = p^2 + \frac{p(1-p)}{n}.$

Also $var(\frac{1}{n^2}(X_1 + \ldots X_n)^2) < \infty$. $\therefore$ by WLL, $\frac{1}{n^2}(X_1 + \ldots X_n)^2 \xrightarrow{p} p^2.$

~~b)~~ $\cancel{\pi(p|x) = L(p; x) \cdot \pi(p) = f(x_1|p)\pi(p) = \mathbb{1}_{[x_1 \in (0,1)]} \cdot \pi(p) = \pi(p)}$

~~Bayes risk for absolute error loss is median of posterior;~~

b) $\pi(p|x) = L(p; x) \cdot \pi(p) = f(x_1|p)\pi(p) = x^p + (1-x)^{1-p} \cdot \mathbb{1}_{[p \in [0,1]]}$

The minimizer is the median of posterior.

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(c) $\phi(X) = X_1 X_2$ is Unbiased estimator for p^2.

~~$f(\vec{X}|p)$~~ $\cancel{p^{x}(1-p)^{1-x}}$ $\cancel{\sum \binom{n}{k} p^k (1-p)^{n-k}}$

We ~~know that~~ need to find a complete sufficient Statistics T

of p^2 & by L. Scheffé.

$W = E[\phi(X) | T]$ is the UMVUE

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3. (20 pts) Let $X_1, \ldots, X_n$ be i.i.d. Exp(λ) with pdf $f(x;\lambda) = \lambda e^{-\lambda x}$ for $x > 0$. For each $i = 1, \ldots, n$, define the indicator $Y_i = 1\{X_i > 1\}$.

(a) (7 pts) Compute the log-likelihood function

$$\log L(\lambda) = \log f(X_1, \ldots, X_n|\lambda).$$

(b) (7 pts) Consider the t-th iteration of an EM algorithm for approximating the MLE of λ. For the E-step, expected log-likelihood satisfies

$$Q(\lambda|\lambda^{(t)}) = n\log\lambda - \lambda\sum_{i=1}^{n}\mathbb{E}[X_i|Y_i, \lambda^{(t)}]. \quad \leftarrow \text{given.} \tag{1}$$

Given Equation (1), find the explicit solution for $\lambda^{(t+1)}$ in the M-step:

$$\lambda^{(t+1)} = \arg\max_{\lambda} Q(\lambda|\lambda^{(t)}).$$

Your answer should be a function of $\mathbb{E}[X_i|Y_i, \lambda^{(t)}]$.

(c) (6 pts) Compute $\mathbb{P}(Y_1 = 1|\lambda)$ and $\mathbb{E}[X_1|Y_1 = 1, \lambda]$.

a) $\log L(\lambda) = \log f(X_1 \cdots X_n|\lambda)$

$= \log \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \sum_{i=1}^{n} \log \lambda e^{-\lambda x_i}$

$= n \log \lambda - \lambda \sum x_i$

b) $\frac{d}{d\lambda} Q(\lambda|\lambda^{(t)}) = \frac{n}{\lambda} - \sum E[X_i|Y_i, \lambda^{(t)}]$

c) $P(Y_1 = 1|\lambda) = P(X_i > 1|\lambda) = \int_1^\infty \lambda e^{-\lambda x}\,dx = \lambda \int_1^\infty e^{-\lambda x}\,dx$

$= \lambda \left.\frac{e^{-\lambda x}}{-\lambda}\right]_1^\infty = \left.e^{-\lambda x}\right]_\infty^1 = e^{-\lambda}$

$E[X_1|Y_1 = 1, \lambda] = \int x f_X(x|Y_1 = 1, \lambda)\,dx = \int x f_X(x|x>1, \lambda)\,dx$

$\boxed{f_X(x|x>1) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda}} = \lambda e^{-\lambda(x-1)}}$

$= \int_1^\infty x \lambda e^{-\lambda(x-1)}\,dx$

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$$= \lambda \int_1^\infty x\, e^{-\lambda(x-1)}\, dx = \lambda \left[x \int e^{-\lambda(x-1)} dx - \int \int e^{-\lambda(x-1)} dx \right]$$

$$= \lambda \left[x \frac{e^{-\lambda(x-1)}}{-\lambda} \Big]_1^\infty - \int_1^\infty \frac{e^{-\lambda(x-1)}}{-\lambda} dx \right]$$

$$= \cancel{\lambda} \left[\frac{1}{\cancel{\lambda}} + \frac{1}{\cancel{\lambda}} \int_1^\infty e^{-\lambda(x-1)} dx \right]$$

$$= \left[1 + \int_1^\infty e^{-\lambda(x-1)} dx \right]$$

$$= 1 + \frac{e^{-\lambda(x-1)}}{-\lambda} \Big]_1^\infty = 1 - \frac{e^{-\lambda(x-1)}}{\lambda} \Big|_1^\infty$$

$$= 1 - \left(0 - \frac{1}{\lambda}\right) = 1 + 1/\lambda .$$