Q2 1. $X_1, X_2 \ldots X_n$ are random samples from $f(x|\theta)$. Find a minimal statistic for θ.

a) $f(x|\theta) = (2\pi)^{-1/2} e^{-(x-\theta)^2/2}$

Sol^n $f(x|\theta) = (2\pi)^{-1/2} e^{-\left(\frac{x^2}{2} - \theta x + \frac{\theta^2}{2}\right)}$

$$= \underbrace{(2\pi)^{-1/2} \exp\left\{-\frac{x^2}{2}\right\}}_{h(x)} \exp\left\{ \underset{\eta(\theta)\ T(x)}{\theta x} - \underset{A(\theta)}{\frac{\theta^2}{2}} \right\}$$

$$= h(x) \exp\{\eta(\theta) T(x) - A(\theta)\}$$

Hence, this distribution is from the exponential family.

We have seen in class that $S_j = \sum_{i=1}^{N} \vec{T}_j(x_i)$ form the components of the sufficient statistic $\vec{S}$.

Here we just have one component of $\vec{T}(x)$ i.e. it is a scalar. So $\sum_{i=1}^{N} x_i$ is sufficient for θ.

We claim that $\bar{x} = \sum_{i=1}^{N} x_i$ is also minimal for θ.

By Lehmann - Scheffé theorem (Thm 2.16 in class notes)

$\dfrac{f(x_1, x_2 \cdots x_N | \theta)}{f(y_1, y_2 \cdots y_N | \theta)}$ is constant as a function of $\theta \Leftrightarrow T(x) = T(y)$.

i.e. constant w.r.t θ

$$\frac{f(x_1 x_2 \dots x_n \mid \theta)}{f(y_1 y_2 \dots y_n \mid \theta)} = \frac{\prod_{i=1}^{N} h(x_i) \exp(\theta \bar{x} - nA(\theta))}{\prod_{i=1}^{N} h(y_i) \exp(\theta \bar{y} - nA(\theta))}$$

Clearly, the above ratio is independent w.r.t θ if and only if $\bar{x} = \bar{y}$. Hence $\bar{x} = \sum_{i=1}^{n} x_i$ is sufficient and minimal Statistic.

b) $f(x \mid \theta) = e^{-(x-\theta)} = \exp$

$$f(x_1, x_2 \dots x_n \mid \theta) = \prod_{i=1}^{N} e^{-(x_i - \theta)} = e^{-(\sum x_i - n\theta)}$$

Define $\underbrace{T(x_1 \dots x_n)}_{T(\vec{x})} = \bar{x} = \sum_{i=1}^{N} x_i$.

Then,
$$\frac{f(x_1 \dots x_n \mid \theta)}{f(y_1 \dots y_n \mid \theta)} = \frac{e^{-(T(\vec{x}) - n\theta)}}{e^{-(T(\vec{y}) - n\theta)}}$$

$$= \underbrace{e^{T(\vec{y}) - T(x)}}_{\text{Independent of } \theta}.$$

Hence by the Lehmann–Scheffé theorem, $T(x) = \bar{x}$ is sufficient.

c) $f(x|\theta) = \dfrac{e^{(x-\theta)}}{[1+e^{-(x-\theta)}]^2}$

This is a distribution from the location family and the minimal & sufficient statistic is the order statistic which contains all the samples. [Note that there are some distributions from local family that have additional structure, for example the Normal distribution which is also from the exponential family. For such distribution, we can have better compression & the minimal sufficient statistic does not need to be the order statistic.

d) $f(x|\theta) = \dfrac{1}{\pi[1+(x-\theta)^2]}$.

This is also from a location family. This is from class note example 2.19 and we already saw that the order statistic is the minimal sufficient statistic.

e) $f(x|\theta) = \frac{1}{2} e^{-|x-\theta|}$

$$f(x_1 \dots x_n|\theta) = \left(\frac{1}{2}\right)^n e^{-\sum_{i=1}^{n} |x_i - \theta|}$$

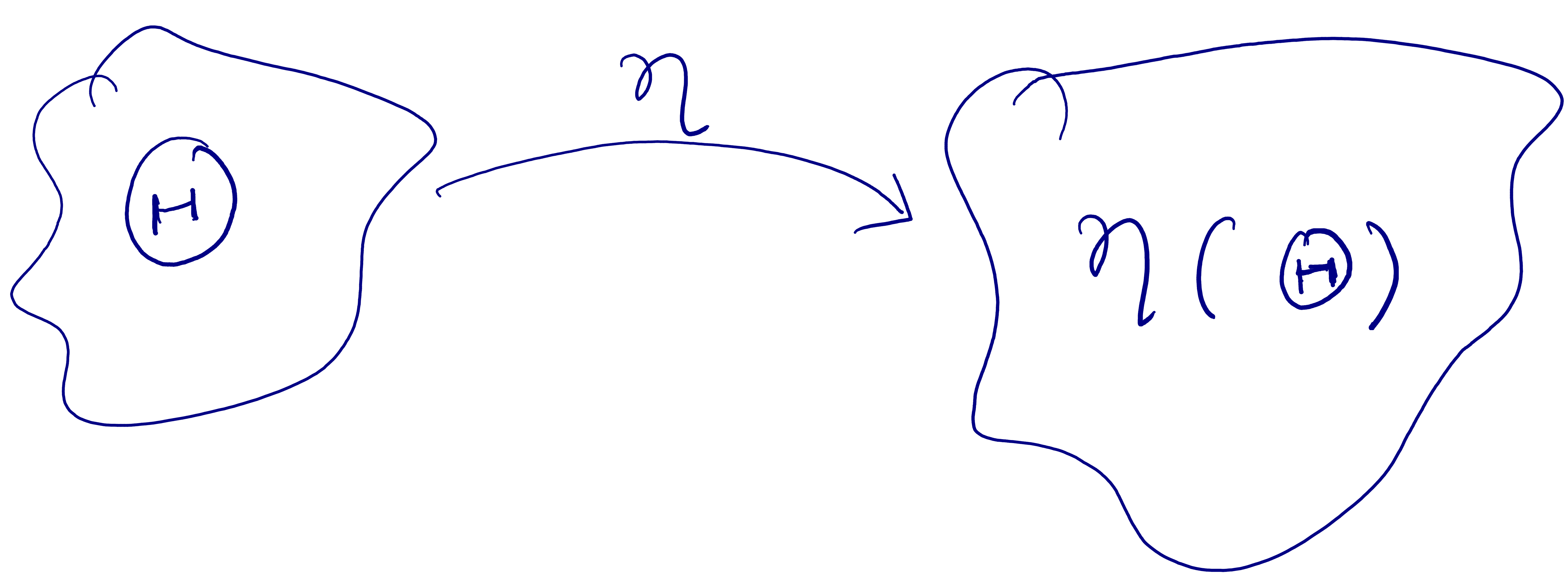
The minimal sufficient statistic is the order statistic.

2) $f(x|\theta) = h(x) \exp\left\{ \underset{\mathbb{R}^k}{\vec{\eta}(\theta)} \cdot \overset{\mathbb{R}^k}{\vec{T}(x)} - A(\theta) \right\}$

where $\theta \in \Theta$

Define the difference set D as

$$D = \{\eta(\theta_1) - \eta(\theta_2) \mid \theta_1, \theta_2 \in \Theta\}$$
$$= \{\eta_1 - \eta_2 \mid \eta_1, \eta_2 \in \eta(\Theta)\}$$



$$\frac{f(x_1, \dots, x_n|\theta)}{f(y_1, \dots, y_n|\theta)} = \prod_{i=1}^{n} \frac{h(x_i)}{h(y_i)} e^{\vec{\eta}(\theta)\left(\sum_{i=1}^{n} \vec{T}(x_i) - \vec{T}(y_i)\right)}$$

Show that if $\text{span}(D) = \mathbb{R}^k$, then $T(X) = (t_1(X), t_2(X) \ldots t_k(X))$ is a minimal sufficient statistic.

where $t_j(X) = \sum_i t_j(x_i)$

Sol^n We want to show that for any two samples $X = \{x_1 \ldots x_n\}$ & $Y = \{y_1 \ldots y_n\}$,

$\dfrac{f(x_1 \ldots x_n \mid \theta)}{f(y_1 \ldots y_n \mid \theta)}$ is independent of $\theta \iff T(X) = T(Y)$

This is for all $\theta \in \Theta$

$\dfrac{f(x_1 \ldots x_n \mid \theta)}{f(y_1 \ldots y_n \mid \theta)}$ ~~is~~ is independent of θ when

when $\vec{\eta}(\theta) \cdot \{\vec{T}(x) - \vec{T}(y)\} = 0$

This should imply $T(x) = T(y)$ if $T(x)$ is a minimal sufficient statistic {and this should be for all $\theta \in \Theta$}

Assume $\vec{\eta}(\Theta)$ contain k elements $b_1, b_2 \ldots b_k$ ($b_i = \eta(\theta_i)$ for some $\theta_i \in \Theta$), which form a basis in $\mathbb{R}^k$.

Now we require $\forall\, \theta \in \Theta$,

$$\vec{\eta}(\theta) \cdot [\vec{T}(x) - \vec{T}(y)] = 0$$

$$\Rightarrow \quad \vec{b}_i \cdot [\vec{T}(x) - \vec{T}(y)] = 0 \;\; \forall i$$

$$\Rightarrow \quad \alpha_i \vec{b}_i [\vec{T}(x) - \vec{T}(y)] = 0 \;\; \forall i$$

$$\forall\, \vec{\alpha}_i \in \mathbb{R}$$

$$\Rightarrow \quad \sum_{i=1}^{k} \alpha_i \vec{b}_i \; [\vec{T}(x) - \vec{T}(y)] = 0$$

$$\Rightarrow \quad \underbrace{\left(\sum_{i=1}^{k} \alpha_i \vec{b}_i\right)}_{\vec{v} \in \eta(\Theta)} [\vec{T}(x) - \vec{T}(y)] = 0$$

$$\therefore \quad \vec{v} \cdot (\vec{T}(x) - \vec{T}(y)) = 0 \quad \forall\, \vec{v} \in \eta(\Theta)$$

$$\Rightarrow \quad T(x) = T(y).$$

So our assumption was we have a basis in $\eta(\Theta)$ i.e. $\mathrm{span}(\eta(\Theta)) = \mathbb{R}^k$

Realize that $\mathrm{Span}(D) = \mathbb{R}^k \Leftrightarrow \mathrm{span}(\eta(\Theta)) = \mathbb{R}^k$

because $\text{Span}(D) \subseteq \text{Span}(\Theta)$.

Hence $\text{Span}(D) = \mathbb{R}^k \subseteq \text{Span}(\eta(\Theta))$

we also have $\text{span}(\eta(\Theta)) \subseteq \mathbb{R}^k$

Hence $\text{span}(\eta(\Theta)) = \mathbb{R}^k$.

Since $\text{span}(D) = \mathbb{R}^k \Rightarrow \text{Span}(\eta(\Theta)) = \mathbb{R}^k$ which is what is required such that $T(x)=T(y)$ for all $\theta \in \Theta$, we have proved the given statement of the question.

Q3 1. $X_1 \ldots X_n$ from Gamma(α, β)

$\alpha, \beta > 0$

$$f(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x > 0$$

Derive the methods of moments estimators for α & β.

Solution $E[X] = \frac{\alpha}{\beta}$ & $E[X^2] = \frac{\alpha(\alpha+1)}{\beta^2}$ for the Gamma(α, β) distribution.

Also the sample means & sample 2nd moment are $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ & $M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

We equate the sample moments to the theoretical moments to estimate the parameters

$$E[X] = \bar{X} \quad \& \quad E[X^2] = M_2$$

$$\Rightarrow \frac{\alpha}{\beta} = \bar{X} \quad \& \quad \frac{\alpha(\alpha+1)}{\beta^2} = M_2$$

$$\Downarrow$$

$$\frac{\alpha}{\beta}\left(\frac{\alpha}{\beta} + \frac{1}{\beta}\right) = M_2$$

$$\Downarrow$$

$$\bar{X}\left(\bar{X} + \frac{1}{\beta}\right) = M_2$$

$$\Downarrow$$

$$\beta = \left(\frac{M_2}{\bar{X}} - \bar{X}\right)^{-1}$$

Hence $\alpha = \bar{X}\beta = \dfrac{\bar{X}}{\dfrac{M_2}{\bar{X}} - \bar{X}}$

$$\therefore \alpha = \frac{1}{\dfrac{M_2}{\bar{X}} - \bar{X}} \quad \text{and} \quad \beta = \frac{\bar{X}}{\dfrac{M_2}{\bar{X}} - \bar{X}}$$

$$\text{or} \quad \alpha = \frac{\bar{X}}{M_2 - \bar{X}^2} \quad \text{and} \quad \beta = \frac{\bar{X}^2}{M_2 - \bar{X}^2}$$

2. $f(x|\theta) = \theta x^{-2}, \quad 0 < \theta < x < \infty$

a) find method of moments estimator of θ

b) find M.L.E estimator of θ.

Sol^n

$$E[X] = \int_{\theta}^{\infty} x f(x \; \theta)\, dx$$

$$= \int_{\theta}^{\infty} x \; \theta x^{-2}\, dx = \int_{\theta}^{\infty} \theta x^{-1}\, dx$$

$$= \theta \big[\log x\big]_{\theta}^{\infty} \longrightarrow \infty.$$

So this has infinite mean. But we can try to use the 2nd moment (This is a problem of method of moments as the estimate of θ depends on which moments we choose to equate)

$$E[X^2] = \int_{\theta}^{\infty} x^2 f(x|\theta)\, dx = \int_{\theta}^{\infty} x^2 \theta x^{-2}\, dx$$

$$= \int_{\theta}^{\infty} \theta\, dx \rightarrow \infty.$$

Meh!!! What were we thinking. ofcourse $\int x^2 f(x|\theta)$ wont converge finitely if $\int x f(x|\theta)\, dx$ didnt.

So we cannot use method of moments in this case.

b) Let's try M.L.E.

Note that $f(x|\theta) = \theta x^{-2} \mathbb{I}_{\{x>\theta\}} \mathbb{I}_{\{\theta>0\}}$

as it is zero if $x < \theta$ and $\theta < 0$

$$f(\underbrace{x_1, x_2 \ldots x_n}|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \prod_{i=1}^{n} \theta x_i^{-2} \mathbb{I}_{\{x_i>\theta\}} \mathbb{I}_{\{\theta>0\}}$$

$$L(\theta|x_1, x_2 \ldots x_n) = f(x_1 \ldots x_n|\theta)$$

$$= \mathbb{I}_{\{\theta>0\}} \theta^n \underbrace{\left(\prod_{i=1}^{n} x_i^{-2}\right)}_{A} \left(\prod_{i=1}^{n} \mathbb{I}_{\{x_i>\theta\}}\right)$$

$$= A \theta^n \prod_{i=1}^{n} \mathbb{I}_{\{x_i>\theta\}} \mathbb{I}_{\{\theta>0\}}$$

$$\mathbb{I}_{\{x_i>\theta\}} = U(x_i - \theta)$$

where $U(x)$ is the unit step function

i.e $U(x) = \begin{cases} 1 & ; x \geq 0 \\ 0 & ; 0 < x \end{cases}$

$\because L(\theta|x_1 \dots x_n) = \mathbb{I}_{\{\theta>0\}} A\, \theta^n \prod_{i=1}^{n} U(x_i - \theta)$

$$\mathbb{I}_{\{\theta>0\}} A\, \theta^n\, U(\min_i x_i - \theta)$$

Clearly $A\theta^n$ is increasing $\forall\, \theta > 0$

Hence $L(\theta | x_1 \dots x_n)$ is maximized

when $\theta = \min_i X_i$.

3) $\underbrace{P(X_i \le x | \alpha, \beta)}_{C.D.F = F(x|\alpha,\beta)} = \begin{cases} 0 & ; x < 0 \\ (x/\beta)^\alpha & ; 0 \le x \le \beta \\ 1 & ; x > \beta \end{cases}$

Find M.L.E for (α, β).

Sol^n $f(x|\alpha,\beta) = \frac{d}{dx} F(x|\alpha,\beta)$

$$= \begin{cases} 0, & x < 0 \\ \frac{\alpha}{\beta^\alpha} x^{\alpha-1} & 0 \le x \le \beta \\ 0, & x > \beta \end{cases}$$

$$L(\alpha,\beta\mid x_1,\dots x_n) = \prod_{i=1}^{n} \frac{\alpha}{\beta^\alpha}\, x_i^{\alpha-1}\, \mathbb{I}_{\{0<x_i<\beta\}}$$

$$= \left(\frac{\alpha}{\beta^\alpha}\right)^n \underbrace{\left(\prod_{i=1}^{n} x_i\right)}_{A}^{\alpha-1} \underbrace{\prod_{i=1}^{n} \mathbb{I}_{\{0<x_i<\beta\}}}_{\mathbb{I}_{\{\max_i(x_i)<\beta\}}\,\mathbb{I}_{\{0<\min_i(x_i)\}}}$$

$$= \left(\frac{\alpha}{\beta^\alpha}\right)^n (A)^{\alpha-1}\, \mathbb{I}_{\{\max_i(x_i)<\beta\}} \times \mathbb{I}_{\{0<\min_i(x_i)\}}$$

Assuming $\alpha>0$, Clearly as β increases, L decreases. Hence β should be as small as possible. At the same time we want $\max_i(x_i)<\beta$ as otherwise L becomes zero which is a minimum & not maximum.

$\therefore$ we MLE of $\hat{\beta} = \max_i\{x_i\}$

$$L = \left(\frac{\alpha}{\beta^\alpha}\right)^n A^{\alpha-1} \mathbb{I}_{\{\max_i x_i < \beta\}} \mathbb{I}_{\{0 < \min_i x_i\}}$$

$$\log L = n\left(\log \alpha - \alpha \log \beta\right) + (\alpha - 1)\log A + \log \mathbb{I}_{\{..\}} + \log \mathbb{I}_{\{..\}}$$

$$\frac{\partial \log L}{\partial \alpha} = n\left(\frac{1}{\alpha} - \log \beta\right) + \log A = 0$$

$$\Rightarrow \frac{1}{\alpha} - \log \beta = -\frac{\log A}{n}$$

$$\Rightarrow \frac{1}{\alpha} = \log \beta - \log A^{1/n}$$

$$\Rightarrow \alpha = \frac{1}{\log \beta - \log A^{1/n}}$$

$$\Rightarrow \hat{\alpha} = \frac{1}{\log\left(\max_i x_i\right) - \log \underbrace{\left(\prod x_i\right)^{1/n}}_{\text{Geometric mean}}}$$

Q1 Ancillary and complete statistics

1.

2. Show $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$ is an ancillary statistic of $N(\mu, \sigma)$.

Sol^n Let $X_i = \mu + Y_i$ where $Y_i \sim N(0, \sigma^2)$

$$S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2 = \frac{1}{n-1} \sum \left[(\mu + Y_i) - (\mu + \bar{Y}) \right]^2$$

$$= \frac{1}{n-1} \sum (Y_i - \bar{Y})^2$$

where $\bar{Y} = \bar{X} - \mu$

$\therefore$ S^2 is a function of $\{Y_i\}$ alone & does not depend on μ.

Y_i are i.i.d from $N(0, \sigma^2)$.

The distribution of $\sum (Y_i - \bar{Y})^2$ is known to be from χ^2 (chi-squared) distribution and does not depend on μ (only on σ^2)

Hence S^2 is ancillary.

3. $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ ← show this is a complete Statistic for μ

σ^2 is constant in

$$f(x \mid \mu, \sigma) = \frac{1}{\sigma} f\left((x-\mu)/\sigma \right)$$

Sol^n Def^n of complete statistic.

$$E[g(T)] = 0 \;\; \forall \; \theta \Rightarrow P(g(T) = 0) = 1$$

4) Show that $\bar{X}, S^2$ are independent using Basu's theorem when μ, σ^2 are known

& $\mu \in \mathbb{R}, \sigma^2 > 0$

Soln

$$T(x) = \left(\bar{X}, \frac{1}{n}\sum_{i=1}^{n}(X - \bar{X})^2\right)$$

is Complete & sufficient.

Consider the centered data

$$U = (X_1 - \bar{X}, X_2 - \bar{X}, \dots X_n - \bar{X})$$

distribution of U is not dependent on μ or σ.

Hence U is ancilliary.

By Basu's theorem Complete & Ancillary statistic are independent.

$\therefore \; X \perp\!\!\!\perp S^2.$