MATH 541A HW6

PROBLEM - 1 [Convergence of random variables]

1. Show that if $\sqrt{n}(Y_n - \mu) \to N(0, \sigma^2)$ in distribution, then $Y_n \to \mu$ in probability.

i.e. $\sqrt{n}(Y_n - \mu) \xrightarrow{D} N(0, \sigma^2)$

$$\Rightarrow \quad Y_n \xrightarrow{P} \mu.$$

Solution :- Let $Z_n = \sqrt{n}(Y_n - \mu)$

then $Y_n = \mu + \dfrac{Z_n}{\sqrt{n}}$

$Z_n \xrightarrow{D} N(0, \sigma^2)$ as given in question

Consider the sequence $A_n = \dfrac{1}{\sqrt{n}}$.

$\lim_{n\to\infty} A_n = 0$ & hence $P(\lim_{n\to\infty} A_n = 0) = 1$

which means $A_n \xrightarrow{a.s} 0 \Rightarrow A_n \xrightarrow{P} 0$

Constant

Now $Z_n \xrightarrow{D} N(0, \sigma^2)$

& $A_n \xrightarrow{P} 0$ (some constant)

Hence by slutsky's theorem

$$F_n := \frac{Z_n}{\sqrt{n}} = Z_n A_n \xrightarrow{D} 0$$

By theorem 6.11 in lecture notes,

$$F_n \xrightarrow{D} \underline{\underline{0}} \iff F_n \xrightarrow{P} \underline{\underline{0}}$$

some constant   same constant

Now applying continuous mapping theorem, via the function $g(x) = \mu + x$

$$g(F_n) \xrightarrow{P} g(0)$$

or $$\mu + \frac{X_n}{\sqrt{n}} \xrightarrow{P} \mu + \frac{0}{\sqrt{n}} = \mu$$

$$\Rightarrow Y_n \xrightarrow{P} \mu$$

② Prove by definition that if Y_n converges to a constant c in probability, then Y_n converges to c in distribution.

<u>Solution</u> <u>Definitions</u>

$$Y_n \xrightarrow{P} c \iff \lim_{n\to\infty} P(\underbrace{|Y_n - c| > \epsilon}_{\forall \epsilon}) = 0$$

$$=$$

$$\begin{cases} \boxed{\lim_{n\to\infty} P(Y_n - c > \epsilon) = 0}, & Y_n > c \\ \boxed{\lim_{n\to\infty} P(c - Y_n > \epsilon) = 0, \quad Y_n < c} \end{cases}$$

$$Y_n \xrightarrow{D} c \Rightarrow \lim_{n\to\infty} F_{Y_n}(y) = F_c(y)$$ point wise where F_c is continuous

But $F_c(y) = \begin{cases} 1 & \text{if } y \geq c \\ 0 & \text{if } y < c \end{cases}$

$F_{Y_n}(y) = P(Y_n \leq y)$ Pick ϵ s.t $y < c - \epsilon$

For $y \leq c$, $P(Y_n \leq y) \leq \boxed{P(Y_n < c - \epsilon)}$

taking $\lim_{n\to\infty}$ on both sides $\lim_{n\to\infty} P(Y_n \leq y) = \lim_{n\to\infty} F_{Y_n}(y) = 0$

$\therefore\ P(Y_n \le y) = 0 \quad \forall\ y \le c$

Let $y > c$, then

$F_{Y_n}(y) = P(Y_n \le y) \geq P(Y_n \le c+\epsilon)$

pick ϵ such that $c+\epsilon \le y$

Then $F_{Y_n}(y) \geq P(Y_n - \epsilon \le c)$

$$= P(Y_n - c \le \epsilon)$$

$$= 1 - P(Y_n - c > \epsilon)$$

Taking lim on both sides

$$\lim_{n\to\infty} F_{Y_n}(y) \geq 1 - \boxed{\lim_{n\to\infty} P(Y_n - c > \epsilon)}$$

$$\geq 1 - 0 = 1$$

$\Rightarrow \lim_{n\to\infty} F_{Y_n}(y) \geq 1$ for $y > c$

But $F_{Y_n}(y) \le 1$ as it is a CDF.

Q.E.D

Hence $F_{Y_n}(y) = 1$ for $y > c$

PROBLEM: 2 (Consistency and convergence in distribution)

Let $X_1 \cdots X_n$ be i.i.d and

$$X_{(n)} = \max_{1 \le i \le n} X_i .$$

1. If $X_i \sim \text{Beta}(1, \beta)$, find a value ν such that $n^\nu (1 - X_{(n)})$ converges in distribution

Sol^n For $\text{Beta}(\alpha, \beta)$, the pdf is

$$f(x) = \underbrace{c}_{\text{normalization const.}} \cdot (x)^{1-\alpha} (1-x)^{\beta-1}, \quad x \in [0,1]$$

For $\text{Beta}(1, \beta)$,

$$f(x) = c \cdot (x)^{1-1} (1-x)^{\beta-1}$$

$$= c (1-x)^{\beta-1}$$

Normalization $\int_0^1 f(x)\, dx = 1$

$$\Rightarrow c \int_0^1 (1-x)^{\beta-1} dx = 1$$

$$\Rightarrow c \int_0^1 (1-x)^{\beta-1} \, dx = 1$$

$$\Rightarrow c \left. \frac{(1-x)^{\beta-1+1}}{\beta-1+1} \times \frac{1}{(-1)} \right]_0^1 = 1$$

$$\Rightarrow c \cdot \left. \frac{(1-x)^{\beta}}{-\beta} \right]_0^1 = 1$$

$$\Rightarrow c \cdot \frac{(1-1)-(1-0)}{-\beta} = 1$$

$$\Rightarrow c = \beta .$$

$$\therefore f(x) = \beta (1-x)^{\beta-1}$$

$$F_X(x) = \int_0^x f(x) \, dx$$

$$= \beta \int_0^x (1-x)^{\beta-1} \, dx$$

$$= \cancel{\beta} \cdot \left. \frac{(1-x)^{\beta}}{-\cancel{\beta}} \right]_0^x$$

$$= -1 \cdot \left[(1-x)^{\beta} - (1-0)^{\beta} \right] = 1-(1-x)^{\beta}$$

So $F(x) = 1 - (1-x)^{\beta}$

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x)$$

$$= P(\max(X_1 \dots X_n) \le x)$$

$$= P(X_1 \le x, X_2 \le x \dots x_n \le x)$$

$$= \underbrace{P(X_1 \le x)\, P(X_2 \le x) \dots P(X_n \le x)}_{\text{because independence}}$$

$$= F(X_1 \le x) \cdot F(X_2 \le x) \dots F(X_n \le x)$$

$$= \left(1 - (1-x)^{\beta}\right)^n$$

Let $Y_n := n^{\nu}(1 - X_{(n)})$

$$P(Y_n \le y) = P(n^{\nu}(1 - X_{(n)}) \le y)$$

$$\parallel$$

$$P\left(X_{(n)} \ge 1 - \frac{y}{n^{\nu}}\right)$$

$$\parallel$$

$$1 - F_X\left(1 - \frac{y}{n^{\nu}}\right)$$

$$= 1 - \left(1 - \left(1 - \left(1 - \frac{y}{n^{\nu}}\right)\right)^{\beta}\right)^n$$

$$\Rightarrow P(Y_n \le y) = 1 - \left(1 - \left(1 - \left(1 - \frac{y}{n^\nu}\right)\right)^\beta\right)^n$$

$$= 1 - \left(1 - \left(\frac{y}{n^\nu}\right)^\beta\right)^n$$

For large n, $\left(\frac{y}{n^\nu}\right)^\beta \to 0$ & hence we can approximate using the identity

$$(1-z)^n \to e^{-nz}$$

$$\therefore P(Y_n \le y) = 1 - \exp\left(-n\left(\frac{y}{n^\nu}\right)^\beta\right)$$

$$= 1 - \exp\left(-n\, y^\beta\, n^{-\nu\beta}\right)$$

$$= 1 - \exp\left(-y^\beta\, n^{1-\nu\beta}\right)$$

For convergence in distribution, we want

$$\lim_{n\to\infty} F_{Y_n}(y) = \lim_{n\to\infty} P(Y_n \le y)$$

to converge to some function which then becomes the point wise limiting CDF of Y_n.

$$\lim_{n\to\infty} F_{Y_n}(y) = \lim_{n\to\infty} P(Y_n \le y)$$

$$\parallel$$

$$\lim_{n\to\infty} 1 - \exp\{-y^{\beta} n^{1-\beta v}\}$$

The only way this can converge is if $1-\beta v \le 0$ or $\frac{1}{\beta} \le v$

$$\lim_{n\to\infty} F_{Y_n} = \begin{cases} 1 & \text{if } v > \frac{1}{\beta} \\ 1-\exp\{-y^{\beta}\} & \text{if } v = \frac{1}{\beta} \end{cases}$$

$\therefore$ we can set $v = \frac{1}{\beta}$ to have $n^{v}(1-X_{(n)})$ converge in distribution to $1-\exp\{-y^{\beta}\}$.

2. If $X_i \sim \exp(1)$, find a sequence a_n so that $X_{(n)} - a_n$ converges in distribution

$f_X(x) = \lambda e^{-\lambda x}$ $x \ge 0$ for $\exp(\lambda)$

So here $f_X(x) = e^{-x}$

$$F_X(x) = \int_0^x e^{-x}\,dx = -e^{-x}\Big|_0^x$$

$$= -[e^{-x} - e^{-0}]$$

$$= -[e^{-x} - 1]$$

$$= 1 - e^{-x}$$

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x)$$

$$= P(\max(X_1 \ldots X_n) \le x)$$

independence

$$= P(X_1 \le x, X_2 \le x, \ldots X_n \le x)$$

$$= P(X_1 \le x)\,P(X_2 \le x) \ldots P(X_n \le x)$$

identically distributed

$$= F_{X_1}(x) \cdots F_{X_n}(x)$$

$$= F_X(x) \cdots F_X(x)$$

$$= (1 - e^{-x})^n$$

Let $Y_n = X_{(n)} - a_n$.

Then $P(Y_n \le y) = P(X_{(n)} - a_n \le y)$

$$F_{X_{(n)}}(y + a_n) = P(X_{(n)} \le y + a_n)$$

$$\Rightarrow P(Y_n \le y) = F_{X_{(n)}}(y + a_n)$$

$$= \left(1 - e^{-y-a_n}\right)^n$$

We want $Y_n := X_{(n)} - a_n$ to have distribution that converges as $n \to \infty$.

$$\lim_{n\to\infty} F_{Y_n}(y) = \lim_{n\to\infty} P(Y_n \le y)$$

$$= \lim_{n\to\infty} \left(1 - e^{-y-a_n}\right)^n$$

$$= \lim_{n\to\infty} \left(1 - e^{-y} e^{-a_n}\right)^n$$

$$= \exp \log \lim_{n\to\infty} \left(1 - e^{-y} e^{-a_n}\right)^n$$

log is continuous

$$= \exp \lim_{n\to\infty} \log \left(1 - e^{-y} e^{-a_n}\right)^n$$

$$= \exp\left(\lim_{n\to\infty} n \log\left(1 - e^{-y} e^{-a_n}\right)\right)$$

$$\Rightarrow \lim_{n\to\infty} F_{Y_n}(y)$$

$$= \exp\left(\lim_{n\to\infty} n \log\left(1 - e^{-y} e^{-a_n}\right)\right)$$

If $e^{-a_n} \to 0$ as $n \to \infty$,

then $e^{-y} e^{-a_n} \to 0$ as $n \to \infty$

& then $\log(1 - e^{-y} e^{-a_n}) \approx - e^{-y} e^{-a_n}$.

So $\lim_{n\to\infty} F_{Y_n}(y) = \exp\left\{\lim_{n\to\infty} -n e^{-y} e^{-a_n}\right\}$

$$= \exp\left\{-e^{-y} \lim_{n\to\infty} n e^{-a_n}\right\}$$

For $e^{-a_n} \to 0$ as $n \to \infty$, we must have $a_n \to \infty$ as $n \to \infty$.

Now the limit has to exist & better if its a non-zero costant. If zero, then the cdf is trivial. Converging to $\pm\infty$ will not give us a valid CDF.

$$\lim_{n\to\infty} n e^{-a_n} = \text{Const}\cdot k$$

$$\Rightarrow e^{-a_n} = \frac{k}{n}$$

$$\Rightarrow \ln e^{-a_n} = \ln \frac{k}{n}$$

$$\Rightarrow -a_n = \ln \frac{k}{n}$$

$$\Rightarrow a_n = -\ln \frac{k}{n}$$

$$\Rightarrow \boxed{a_n = \ln \frac{n}{k}}$$ Answer.

Check if $a_n \to \infty$ as $n \to \infty$ ✓ PASS

3. Let $X_i \sim \text{Uniform}(0, \theta)$. Show that $X_{(n)}$ is a consistent estimator of θ.

$$f_X(x) = 1/\theta \text{ for } x \in [0, \theta]$$

$$F_X(x) = \int_0^x f_X(x)\,dx = \frac{1}{\theta} x$$

$$F_{X_{(n)}}(x) = P(X_{(n)} \leq x)$$

$$= P(\max(X_1 \cdots X_n) \leq x)$$

$$= P(X_1 \leq x, \cdots X_n \leq x)$$

$$= P(X_1 \leq x) \cdots P(X_n \leq x)$$

$$= F_{X_1}(x) \cdots F_{X_n}(x)$$

$$= \left(\frac{x}{\theta}\right) \cdots \left(\frac{x}{\theta}\right)$$

$$= \left(\frac{x}{\theta}\right)^n$$

A sequence of estimators Y_n is said to be (weakly) consistent if $Y_n \xrightarrow{P} \theta$ where θ is the true parameter for all θ

We want to show $X_{(n)} \xrightarrow{P} \theta$

i.e. $\lim_{n \to \infty} P(|X_n - \theta| < \epsilon) = 1$

$$\|$$

$$\lim_{n \to \infty} P(-\epsilon < X_n - \theta < \epsilon) = 1$$

or $\lim_{n\to\infty} P(-\epsilon < X_n - \theta < \epsilon) = 1$

But $X_{(n)} - \theta < 0$ always

So $\lim_{n\to\infty} P(-\epsilon < X_n - \theta) = 1$

or $\lim_{n\to\infty} P(\theta - \epsilon < X_n) = 1$

or $\lim_{n\to\infty} 1 - P(X_n \leq \theta - \epsilon) = 1$

$$1 - \lim_{n\to\infty} F_{X_{(n)}}(\theta - \epsilon) = 1$$

$$\Rightarrow \quad - \lim_{n\to\infty} F_{X_{(n)}}(\theta - \epsilon) = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} F_{X_{(n)}}(\theta - \epsilon) = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} \left(\frac{\theta - \epsilon}{\theta}\right)^n = 0$$

Hence Assumption is true

$$\Rightarrow \quad \lim_{n\to\infty} \left(1 - \frac{\epsilon}{\theta}\right)^n = 0$$ TAUTOLOGY

But this is always true. Hence proved.

4. Let $X_i \sim$ Uniform$(0, \theta)$. Show that

$$\frac{2}{n}\sum_{i=1}^{n} X_i = 2\overline{X}$$ is a consistent estimator of θ.

Sol^n

$$E[2\overline{X}] = 2E[\overline{X}]$$
$$= 2 \cdot \frac{\theta}{2} = \theta$$

$$Var(2\overline{X}) = 4\,Var(\overline{X}) = 4\,Var\left(\frac{1}{n}\sum X_i\right)$$
$$= \frac{4}{n^2}\sum Var(X_i) = \underline{\frac{4}{n^2}\cdot n \cdot \frac{\theta^2}{12}}$$
$$= \frac{\theta^2}{3n}$$

$$Var(X_i) = E[X_i^2] - E[X_i]^2$$
$$= \int_0^\theta x^2 \frac{1}{\theta}\,dx - \left(\frac{\theta}{2}\right)^2$$
$$= \frac{1}{\theta}\int_0^\theta x^2\,dx - \left(\frac{\theta}{2}\right)^2$$
$$= \frac{1}{\theta}\left.\frac{x^3}{3}\right|_0^\theta - \frac{\theta^2}{4} = \frac{1}{\theta}\times\frac{\theta^3}{3} - \frac{\theta^2}{4}$$
$$= \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$$

$\therefore E[2\bar{x}] = \theta$ & $Var(2\bar{X}) = \dfrac{\theta^2}{3n}$

As $n \to \infty$ $Var(2\bar{X}) \to 0$

Using Chebyshev's inequality

$$P(|2\bar{x} - E(2\bar{x})| \geq \epsilon) \leq \frac{Var(2\bar{x})}{\epsilon^2}$$

$$\Rightarrow P(|2\bar{x} - \theta| \geq \epsilon) \leq \frac{Var(2\bar{x})}{\epsilon^2}$$

$$\Rightarrow \lim_{n \to \infty} P(|2\bar{x} - \theta| \geq \epsilon) \leq \lim_{n \to \infty} \frac{\theta^2}{3n\epsilon} = 0$$

$$\Rightarrow \lim_{n \to \infty} 1 - P(|2\bar{x} - \theta| < \epsilon) \leq 0$$

$$\Rightarrow 1 - \lim_{n \to \infty} P(|2\bar{x} - \theta| < \epsilon) \leq 0$$

$$\Rightarrow 1 \leq \lim_{n \to \infty} P(|2\bar{x} - \theta| < \epsilon)$$

But $P(|2\bar{x} - \theta| < \epsilon) \leq 1$ as it is a probability.

Hence $\lim_{n\to\infty} P(|2\bar{X}-\theta| < \epsilon) = 1$

$\Rightarrow \quad 2\bar{X} \xrightarrow{P} \theta$

Hence $2\bar{X}$ is a consistent estimator of θ.

PROBLEM 3 (CLT and Delta Method)

1. Let $X_1, \ldots, X_n$ be i.i.d Bernoulli(p) random variables. Let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

a) Show that for $p \neq \frac{1}{2}$, the estimate of variance $Y_n(1-Y_n)$ satisfies

$$\sqrt{n}\,[\,Y_n(1-Y_n) - p(1-p)] \to N(0, (1-2p)^2 p(1-p))$$

in distribution.

b) Show that for $p = \frac{1}{2}$,

Chi-Squared

$$n\left[Y_n(1-Y_n) - \frac{1}{4}\right] \longrightarrow -\frac{1}{4}\chi_1^2$$

in distribution.

$\underline{\underline{Sol^n}}$ a) <u>Delta method</u>

If Y_n is a sequence of r.v and

$$\sqrt{n}\,(Y_n - \theta) \xrightarrow{D} N(0, \sigma^2)$$

If g is function such that $g'(\theta)$ exists but $g'(\theta) \neq 0$, then

$$\sqrt{n}\left[g(Y_n) - g(\theta)\right] \xrightarrow{D} N(0, \sigma^2 [g'(\theta)]^2)$$

$$E[Y_n] = \sum \frac{1}{n} E[X_i] = \sum_{i=1}^{n} \frac{p}{n} = p.$$

$$Y_n \xrightarrow{D} N\left(p, \frac{1}{n} p(1-p)\right) \text{ by CLT}$$

$$\sqrt{n}\,(Y_n - p) \xrightarrow{D} N(0, \overbrace{p(1-p)}^{\sigma^2})$$

Now set $g(x) = x(1-x)$, $g'(x) = 1 - 2x$

Let by delta method

$$\sqrt{n}\,(Y_n(1-Y_n) - p(1-p)) \xrightarrow{D} N(0, \underbrace{p(1-p)}_{\sigma^2} \underbrace{(1-2p)^2}_{(g'(p))^2})$$

b) Second Order delta method

Given $\sqrt{n}(Y_n - \theta) \xrightarrow{d} N(0, \sigma^2)$

If $g'(\theta) = 0$, then the second order delta method says

$$n\left(g(Y_n) - g(1/2)\right) \xrightarrow{d} \frac{1}{2}\sigma^2 g''(\theta)\, \chi_1^2$$

provided $g''(\theta)$ exists & $g''(\theta) \neq 0$

$g(x) = x(1-x)$, $g'(x) = 1 - 2x$

$g''(x) = -2$

$g'(p = 1/2) = 1 - 2 \cdot \frac{1}{2} = 1 - 1 = 0$

$g''(p = 1/2) = -2$

From part ⓐ, setting $p = 1/2$

$\sqrt{n}(Y_n - 1/2) \xrightarrow{D} N(0, 1/4)$

Hence $n\left(Y_n(1 - Y_n) - \frac{1}{2} \cdot (1 - 1/2)\right)$

$$\xrightarrow{D} \frac{1}{2} \cdot \frac{1}{4} \cdot (-2)\, \chi_1^2 = \frac{-1}{4} \chi_1^2$$

Q.E.D

2. Let $X_1, \dots, X_n$ be i.i.d from Uniform$(1,2)$ and let

$$H_n = \frac{n}{X_1^{-1} + \dots + X_n^{-1}}$$

a) Show that $H_n \to c$ in probability, identifying the constant c

b) Show that $\sqrt{n}(H_n - c)$ converges in distribution, and identify the limit.

Sol^n $H_n = \left(\frac{1}{n}\sum \frac{1}{X_i}\right)^{-1}$ & $f_{X_i}(x) = \frac{1}{2-1}$

$$E\left[\frac{1}{X_i}\right] = \int_1^2 \frac{1}{x} f_{X_i}(x)\,dx = \int_1^2 \frac{1}{x} \cdot 1\,dx$$

$$= \ln x \Big|_1^2 = \ln 2 - \ln 1 = \ln 2$$

By Law of Large Numbers

$$\frac{1}{n}\sum \frac{1}{X_i} \xrightarrow{P} \ln 2$$

By continuous mapping theorem

$$H_n = \left(\frac{1}{n}\sum \frac{1}{X_i}\right)^{-1} \xrightarrow{P} \frac{1}{\ln 2}$$

Note that we used the $g(x) = 1/x$ which is continuous @ $x = \ln 2$

b) Let $Y_i = \frac{1}{X_i}$ such that $\bar{Y} = \frac{1}{n}\sum Y_i = \frac{1}{n}\sum \frac{1}{X_i}$

$$E[Y_i] = \ln 2$$

$$\mathrm{Var}(Y_i) = \mathrm{Var}\left(\frac{1}{X_i}\right) = E\left[\left(\frac{1}{X_i}\right)^2\right] - E\left[\frac{1}{X_i}\right]^2$$

$$E\left[\frac{1}{X_i}\right]^2 = E[Y_i]^2 = \ln^2 2$$

$$E\left[\left(\frac{1}{X_i}\right)^2\right] = \int_1^2 \frac{1}{x^2} f(x)\,dx = \int_1^2 \frac{1}{x^2}\cdot 1\,dx$$

$$= \left[\frac{-1}{x}\right]_1^2 = \left(\frac{1}{1} - \frac{1}{2}\right)$$

$$= 1/2$$

$$\mathrm{Var}\left(\frac{1}{X_i}\right) = \frac{1}{2} - \ln^2 2$$

Applying CLT

$$\sqrt{n}\left(\bar{Y}_n - \ln 2\right) \xrightarrow{D} N\left(0, \tfrac{1}{2} - \ln^2 2\right)$$

Apply delta method with $g(x) = \frac{1}{x}$, $g' = \frac{-1}{x^2}$

$$\sqrt{n}\left(H_n - \frac{1}{\ln 2}\right) \xrightarrow{D} N(0, \alpha)$$

where $\alpha = \left(\frac{1}{2} - \ln 2\right)^2 g'(\ln 2)^2$

$$\alpha = \left(\frac{1}{2} - \ln 2\right)^2 \left(-\frac{1}{(\ln 2)^2}\right)^2$$

$$= \left(\frac{1}{2} - \ln 2\right)^2 \left(\frac{1}{-(\ln 2)^2}\right)^2$$

$$= \left(\frac{1}{2} - \ln 2\right)^2 \frac{1}{(\ln 2)^4}$$

Q.E.D.