MATH 547: HOMEWORK 3
DUE ON: MONDAY, OCTOBER 21, 9AM.

Note: this is the longest homework of the semester. I strongly recommend that you start working on it early!

Problem 1, 10 points:

Let ξ be a Rademacher random variable, namely,

$$\mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = 1/2.$$

Show directly from the definition of sub-Gaussian random variables that $\xi \in \mathrm{SG}(1)$.

Problem 2, 10 points:

Let $X_1, \ldots, X_n$ be i.i.d. normal random variables with mean 0 and variance σ^2. Prove that

$$\mathbb{E} \max_{j=1,\ldots,n} X_j \geq c\sigma\sqrt{\log(n)},$$

where $c > 0$ is some numerical constant (independent of n).

[Hint: the most straightforward (but not the only) way to proceed is to use the fact that, for a nonnegative random variable Z, $\mathbb{E}Z \geq \sup_{\delta>0} \delta P(Z \geq \delta)$].

Problem 3, 10 points:

Let $X_1, \ldots, X_n$ be (not necessarily independent) random variables such that

$$\max_{j=1,\ldots,n} \left(\mathbb{E}|X_j|^p\right)^{1/p} = \sigma_p < \infty$$

for some $p > 1$. Prove that

$$\mathbb{E} \max_{j=1,\ldots,n} |X_j| \leq n^{1/p}\sigma_p.$$

(hint: follow the similar argument that we used in class for sub-Gaussian random variables, but replace $t \mapsto e^{\lambda t}$ but another (which?) function)

Problem 4, 25 points:

Let $X \in \mathrm{SG}(\sigma^2)$ be a sub-Gaussian random variable. Prove that

(a) (5pts) $\mathrm{Var}(X) \leq \sigma^2$;

(b) (10pts) Show that for any integer $p \geq 1$, $\mathbb{E}|X|^p \leq p\, 2^{p/2}\, \sigma^p\, \Gamma(p/2)$, and deduce that

$$\|X\|_p := \left(\mathbb{E}|X|^p\right)^{1/p} \leq C\sigma\sqrt{p} \tag{1}$$

for an absolute constant C that does not depend on σ or p (give a specific value of C). Here, $\Gamma(x)$ is the gamma function defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

(c) (10pts) Conversely, show that if X is centered (that is, $\mathbb{E}X = 0$) and satisfies (1) for all $p \geq 1$, then X is sub-Gaussian with sub-Gaussian parameter $C_1\sigma$ for some constant $C_1 > 0$.

Problem 5, 15 points:

Let X be a random variable such that $X \geq 0$ with probability 1. Next, let $p \geq 1, \gamma > 0, \alpha > 0$ and assume that

$$\mathbb{P}(X \geq \gamma u) \leq C_1 \exp\left(-C_2 p u^2\right)$$

for all $u \geq u_0 > 0$ and some positive absolute constants C_1, C_2 (u_0 is a fixed number). Prove that

$$(\mathbb{E}|X|^p)^{1/p} \leq \gamma(u_0 + C_3)$$

for another absolute constant $C_3 > 0$ (we will use this fact in the proof of the generic chaining theorem in class). You can start with the formula

$$\mathbb{E}|X|^p = p \int_0^\infty x^{p-1} \mathbb{P}(X \geq x) dx.$$

Problem 6, 20 points:

Deduce Dudley's entropy integral bound directly from the generic chaining bound, namely, show that for an absolute constant $C > 0$,

$$\gamma_2(T, d) \leq C \int_0^\infty \sqrt{\log(N(T, d, \varepsilon))} d\varepsilon,$$

and therefore the generic chaining bound implies that

$$\mathbb{E} \sup_{t \in T} X_t \leq C_1 \int_0^\infty \sqrt{\log(N(T, d, \varepsilon))} d\varepsilon.$$

Problem 7 (extra credit): Dudley's theorem and the Brownian motion, 30 points:

Remark: this problem showcases "original" application of chaining/Dudley's integral techniques for establishing continuity of the paths of Gaussian processes. We will cover the required background material during the week of October 7.

Let $\{W_t,\ t \in [0,1]\}$ be the standard Brownian motion (i.e., for any $k \geq 1$ and $t_1, \ldots, t_k$, $(W_{t_1}, \ldots, W_{t_k})$ has multivariate normal distribution, the mean $\mathbb{E}W_t = 0$ for all t, and the covariance is defined by $\mathbb{E}W(t)W(s) = \min(t, s)$).

(a) Define the *increment process* $X_{t,s}$ via $X_{t,s} := W_t - W_s,\ (t, s) \in T_\delta$, where

$$T_\delta = \{(u, v) \in [0,1] \times [0,1] :\ |u - v| \leq \delta\}.$$

Check that X is sub-Gaussian with respect to

$$d^2((t, s), (u, v)) := c(|t - u| + |s - v|),$$

where c is a suitable numerical constant.

(hint: find an upper bound for the variance of $X_{t,s} - X_{u,v}$)

(b) Derive an upper bound on the covering number $N(T_\delta, d, \varepsilon)$ of T_δ with respect to the metric d.

(c) Use Dudley's entropy integral to get an upper bound on the "modulus of continuity"

$$\mathbb{E} \sup_{|t-s| \leq \delta} |W_t - W_s| = \mathbb{E} \sup\{X_{t,s} : (t, s) \in T_\delta)\}.$$

In particular, show that this expectation converges to 0 as $\delta \to 0$.

Remark. *The logic of this problem is the following: since the process $X_{t,s}$ is Gaussian, it is also sub-Gaussian with respect to the "canonical" distance*

$$\tilde{d}((t, s), (u, v)) = \sqrt{Var(X_{t,s} - X_{u,v})}.$$

The only benefit of using a different distance $d(t, s)$ (as suggested in the problem) is the fact that it yields simpler covering number estimates compared to using $\tilde{d}$ directly. For part (c), recall that the limits of integration are between 0 and the diameter of T_δ (with respect to which metric?)