Separable Functions and the Generalization of Matricial Structure

Author(s): Milton Rosenberg

Source: *Mathematics Magazine*, Vol. 42, No. 4 (Sep., 1969), pp. 175-186
Published by: Taylor & Francis, Ltd. on behalf of the Mathematical Association of America
Stable URL: https://www.jstor.org/stable/2688535
Accessed: 12-11-2024 21:44 UTC

SEPARABLE FUNCTIONS AND THE GENERALIZATION OF MATRICIAL STRUCTURE

MILTON ROSENBERG, Kansas University

1. Introduction. Our main concern in this paper shall be the *separability* of a function in accordance with the following definition. The reader is assumed to have some background in linear algebra.

DEFINITION 1. *Let B and T be sets and let $f(b, t)$ be a function from $B \times T$ into a field F. We say that f is separable if it can be expressed in the form*

$$\text{(I)} \qquad \sum_{i=1}^{n} g_i(b) h_i(t)$$

where the g_i's and h_i's are functions defined on B and T respectively, with their values in F.

We may call (I) a *dyadic representation*. A dyadic representation of a $k \times m$ matrix A is given by $A = \sum_{i=1}^{n} g_i \cdot h_i'$, where the g_i's and h_i's are k-tuple and m-tuple column vectors, respectively ("$'$" means "transpose"). The function $f(b, t)$ may be viewed as a matrix where the b's parametrize the rows and the t's, the columns. Thus it is natural that we next define its rank. From Lemma 1, (Section 4), it will follow that this is an extension of the usual definition for the rank of a matrix.

DEFINITION 2. (a) *f is separable of rank zero if $f(b, t) \equiv 0$. (Notation, $rank(f) = 0$).*

(b) *f, $\not\equiv 0$, is separable of rank n if f is separable and if n is the smallest integer such that f can be represented in the form* (I). *(Notation, $rank(f) = n$).*

(c) *f is called inseparable and is said to have rank infinity ($rank(f) = \infty$) if f is not separable.*

Remark. In Definitions 1 and 2, we logically should have used the terms "F-separable" and "F-separable of rank n," with the corresponding notation "$\operatorname{rank}_F(f) = n$." However, it will be shown in Section 4, Theorem 1, that if we replace F by any other field K containing the range of f, then $\operatorname{rank}_K(f) = \operatorname{rank}_F(f)$.

2. Our results. In Section 3 (Contexts) we shall see that the topological (i.e., continuity) and the analytical (i.e., differentiability) properties of separable functions are often important. However, this paper solely concerns itself with the algebraic nature of separability. First we summarize basic characterizations of separability (Section 4, Lemma 1), prove Theorem 1 justifying the remark of Section 1, and then give some examples. We investigate the ranks of sums and products of separable functions in Lemma 4. Next we develop a general method for proving inseparability (Section 5, Theorem 3). In Section 6 we completely solve the problem of the separability of quotients of polynomial functions (Theorem 4). In Section 7 we mention the problem of pure separability and pure inseparability. Finally, we conclude the paper (Section 8).

3. Contexts. Although it is only in integral equations that separable functions are distinguished by a special name, "degenerate kernels" (cf. [2]), they occur in:

(1) *Ordinary differential equations.* The equation $db/dt = g(b)h(t)$ is commonly called "separable" (cf. [7] for its history). In [14, p. 8] the solution of $dy/dx = \sin(xy)$ with initial conditions $(x_0, y_0) = (0, 0)$ is approximated by the solution of $dy/dx = xy$; by a general theorem, it is seen that the error is less than $(.6/192)|x|$ for $|x| < \frac{1}{2}$.

(2) *Partial differential equations.* Solutions are often obtained by the "method of separation of variables" (cf. [5], [2] vol. 2). For example [2, vol. 2, 18–20], to solve $(\partial u/\partial x)^2 + (\partial u/\partial y)^2 = 1$, try $u = \varphi(x) + \psi(y)$. To solve the heat equation $u_{xx} = u_y$, set $u = \varphi(x) \cdot \psi(y)$, obtaining the solutions $(\cos \nu x) \cdot \exp(-\nu^2 y)$; differentiation, summation, or integration over the parameter ν again gives us solutions, e.g.,

$$u = \int_{-\infty}^{\infty} e^{-v^2 y} \cos vx dv = \sqrt{(\pi/y)} e^{-x^2/4y}$$

is the "fundamental solution."

(3) *Integral equations.* In the theory of the Fredholm integral equation, $\varphi(s) - \lambda \int_a^b K(s, t)\varphi(t)dt = f(s)$ (cf. [2] vol. I, p. 115 and [14] p. 119) where it is desired to obtain a solution $\varphi(t)$, general results are obtained by approximating the continuous kernel $K(s, t)$ by continuous degenerate kernels and taking limits.

(4) *Statistics* (Multiple regression, (cf. [10] p. 343)). This topic gave rise to the present paper. Assume we are observing the x-component of a motion given by a known separable time function, $x(b, t)$, in the form (I), where b is an unknown parameter. Assume we make our observations $s(b, t) = x(b, t) + e(t)$ ($e(t) =$ error) at a sequence of times $t_1, \cdots, t_N (N \geqq n)$. We have

$$s = \begin{bmatrix} s(b,t_1) \\ \vdots \\ s(b,t_N) \end{bmatrix} = \begin{bmatrix} h_1(t_1) & h_2(t_1) & \cdots & h_n(t_1) \\ \vdots & \vdots & & \vdots \\ h_1(t_N) & h_2(t_N) & \cdots & h_n(t_N) \end{bmatrix} \cdot \begin{bmatrix} g_1(b) \\ \vdots \\ g_n(b) \end{bmatrix} + \begin{bmatrix} e(t_1) \\ \vdots \\ e(t_N) \end{bmatrix};$$

i.e., $s = H \cdot g + e$. If the errors are uncorrelated with identical variance and if $\operatorname{rank}(H) = n$, then the best unbiased linear estimates of the $g_i(b)$'s and of their linear combinations exist and are given by

$$\hat{g} = (H' \cdot H)^{-1} H' \cdot s, \quad \hat{x}(b, t) = [h_i(t) \cdots h_n(t)] \cdot \hat{g}, \quad \text{etc.}$$

In general, estimation of the true value of the parameter b is still a problem, even when $x(b, t)$ is separable.

4. Generalities and examples. Parts (a), (e), and (f) of Lemma 1 tell us that either the column and row ranks of $f(b, t)$, viewed as a matrix, are both finite and equal to the rank of f, or that the row and column ranks are both infinite. Only part (b) seems to be clearly stated in the literature (cf. [2] vol. I, p. 114).

LEMMA 1. (Separability). *Let $f \neq 0$ be a function from $B \times T$ into a field F.*

Then the following conditions (a), (b), (c), (d), (e), *and* (f) *are equivalent*:

(a) *f is separable of rank n.*

(b) *There exist n linearly independent functions $g_i(b)$ and n linearly independent functions $h_i(t)$ such that f can be expressed in the dyadic form* (I).

(c) *The maximum rank of the* (*square or rectangular*) *matrices $[f(b_i, t_j)]$ for all possible finite choices of distinct b_i's and t_j's is equal to n.*

(d) *There exist $b_1, \cdots, b_n, t_1, \cdots, t_n$ such that the matrix $M=[f(b_i, t_j)]$ is nonsingular, and for any such $b_1, \cdots, b_n, t_1, \cdots, t_n$ the following formula holds*:

$$\text{(II)} \qquad f(b, t) = [f(b, t_1) \cdots f(b, t_n)] \cdot M^{-1} \cdot \begin{bmatrix} f(b_1, t) \\ \vdots \\ f(b_n, t) \end{bmatrix}.$$

(e) *The vector space over F spanned by the functions $\{f(\cdot, t)\}_{t \in T}$ has dimension n.*

(f) *The vector space over F spanned by the functions $\{f(b, \cdot)\}_{b \in B}$ has dimension n.*

Proof. We show first that (a)$\Rightarrow$(b): f is separable of rank n means we can express f in the form (I). Suppose the g's are linearly dependent, say $g_1 = \sum_{j=2}^n a_j g_j$. Then

$$f(b, t) = \sum_{j=2}^{n} g_j(b) \{ h_j(t) + a_j h_1(t) \},$$

and hence f is separable of rank $n-1$ or less, a contradiction.

Next, (c)$\Rightarrow$(d): Let

$$Q = \left[\begin{array}{c|ccc} f(b_1, t) & f(b_1, t_1) & \cdots & f(b_1, t_n) \\ \vdots & \vdots & & \\ f(b_n, t) & f(b_n, t_1) & \cdots & f(b_n, t_n) \\ \hline f(b, t) & f(b, t_1) & \cdots & f(b_1 t_n) \end{array}\right] = \left[\begin{array}{c|c} L & M \\ \hline f(b, t) & N \end{array}\right]$$

where M has rank n. Then the last row of Q is linearly dependent on the other rows. Therefore, there exists a unique row vector $[a_1 \cdots a_n]$ such that $[f(b,t) | N] = [a_1 \cdots a_n] \cdot [L | M]$. Thus $N = [a_1 \cdots a_n] \cdot M$ and hence $[a_1 \cdots a_n] = N \cdot M^{-1}$. Therefore $f(b, t) = [a_1 \cdots a_n] \cdot L = N \cdot M^{-1} \cdot L$.

Next, (d)$\Rightarrow$(e) ($d \Rightarrow f$ is similar): We note by equation (II) that $\{f(\cdot, t_1), \cdots, f(\cdot, t_n)\}$ spans $\{f(\cdot, t)\}_{t \in T}$. The linear independence of the $f(\cdot, t_j)$ follows from the nonsingularity of the matrix $M = [f(b_i, t_j)]$, i.e., the linear independence of its column vectors.

Now, for use in the proof of "(e)$\Rightarrow$(a)", we prove

LEMMA (A). *If $f(b, t) = \sum_{i=1}^n g_i(b) h_i(t)$, then for all finite choices of b_j, t_k we have*

$$\text{(1)} \qquad \max \textit{rank } [f(b_j, t_k)] = \tilde{n} \leqq n.$$

Proof of Lemma. (1) follows because $[f(b_j, t_k)] = [\sum_{i=1}^n g_i(t_j)h_i(b_k)] = G\cdot H$, where the ranks of the matrices G and H are at most n. (The rank of a product of matrices is less than or equal to the rank of each factor in the product (cf. [6]).

(e)$\Rightarrow$(a) (f$\Rightarrow$a is similar): Let $\{f(\cdot, t_1), \cdots, f(\cdot, t_n)\}$ span $\{f(\cdot, t)\}_{t\in T}$. Then for each $t \in T$ there exists a unique set of coefficients $h_1(t), \cdots, h_n(t)$ such that

$$f(\cdot, t) = \sum_{i=1}^{n} f(\cdot, t_i)h_i(t).$$

On defining $f(\cdot, t_i) = g_i(\cdot)$ we have $f(b, t) = \sum_{i=1}^n g_i(b)h_i(t)$. We now show that n is the smallest integer such that f may be written in the form (I). Assume there is a smaller integer $m < n$. Then, by Lemma(A), max rank $[f(b_j, t_k)] = \bar{n} \leqq m$. Since (c)$\Rightarrow$(d)$\Rightarrow$(e), we obtain $\bar{n}(<n)$ as the dimension of the space spanned by $\{f(\cdot, t)\}_{t\in T}$, a contradiction.

For proving (b)$\Rightarrow$(c) we shall use

LEMMA (B). *Let $\{g_i(b)\}$, $i = 1, \cdots, n$, be a set of linearly independent functions over the field of scalars F. Then*

(i) *The space S spanned by the n-tuples $\{g_1(b), g_2(b), \cdots, g_n(b)\}$ for $b \in B$ has dimension n.*

(ii) *There exist $b_1, \cdots, b_n$ such that the matrix $[g_i(b_j)]$, $i, j = 1, \cdots, n$, is nonsingular, i.e., has rank n.*

Proof. We shall prove (i) by contradiction. (For the fact that $\dim(S) = m \leqq n$, we refer to [1, pp. 166–168].) Now, suppose m is less than n. Let $b_1, \cdots, b_m$ parametrize a basis $\{g_1(b_j), \cdots, g_n(b_j)\}$, $j = 1, \cdots, m$, of S. Then, by deletion of columns and relabeling we may form the matrix

$$Q = \left[\begin{array}{c|c} g_i(b_1) & g_1(b_1) \cdots g_m(b)_1 \\ \vdots & \vdots \\ g_i(b_m) & g_1(b_m) \cdots g_m(b_m) \\ \hline g_i(b) & g_1(b) \cdots g_m(b) \end{array}\right] = \left[\begin{array}{c|c} L & M \\ \hline g_i(b) & N \end{array}\right]$$

where M is nonsingular. As in the proof that (c)$\Rightarrow$(d), we deduce

$$g_i(b) = [g_1(b) \cdots g_m(b)]\cdot M^{-1}\cdot L, \qquad i = m+1, \cdots, n,$$

contradicting the linear independence of the g's. (ii) follows immediately from (i).

(b) $\Rightarrow$ (c): From Lemma (A) we have max rank $[f(b_j, t_k)] \leqq n$.

Let $b_1, \cdots, b_n$ be chosen as in Lemma (B) (ii); similarly let $t_1, \cdots, t_n$ be chosen. Then $[f(b_j,t_k)] = [\sum_{i=1}^n g_i(b_j)h_i(t_k)]$, $j, k = 1, \cdots, n$, is the product of two nonsingular matrices and thus has rank n; i.e., max rank $[f(b_j, t_k)] = n$.

Thus we have shown

$$\begin{array}{ccccccccc} (a) & \Rightarrow & (b) & \Rightarrow & (c) & \Rightarrow & (d) & \Rightarrow & (e) \\ & & & & & & \Downarrow & & \Downarrow \\ & & & & & & (f) & \Rightarrow & (a) \end{array}$$

We mention that in the preceding proof, (c)$\Rightarrow$(d) is an immediate consequence of the "bordering method" of inverting a matrix [**3**, p. 105] and that in essence its proof is also contained in [**9**, p. 11] and [**12**, p. 762].

For completeness, we state the proof of (a)$\Rightarrow$(b) as

LEMMA 2. *f, $\not\equiv 0$, is separable of order n implies that for any representation of f in the form* (I) *with integer n, the g's and h's, respectively, are linearly independent.*

Also, we may strengthen Lemma(B) by the following:

COROLLARY (to Lemma 1). *The dimension of the linear space spanned by the functions $\{f_i(a)\}$, $i=1, \cdots, n$, a in A, and the dimension of the linear space spanned by the n-tuples $\{f_1(a), f_2(a), \cdots, f_n(a)\}$ for $a\in A$ are equal.*

Proof. Follows immediately from Lemma (1) (e) and (f), on noting that $f(i, a)$ defined equal to $f_i(a)$ is separable.

We now justify the remark of Section 1. The smallest field containing the range of $f(b, t)$ is the (set)-intersection of all the fields which contain the range of $f(b, t)$.

THEOREM 1. *Let F be the smallest field containing the range of f and let K be any other field such that range$(f)\subseteq K$. Then $\mathrm{rank}_F(f)=n \Leftrightarrow \mathrm{rank}_K(f)=n$.*

Proof. We may assume $f\not\equiv 0$ and that $n<\infty$.

$\Leftarrow$: Suppose $f(b, t)$ is K-separable with rank n, i.e., $\mathrm{rank}_K(f)=n<\infty$. Then applying Lemma 1(d), equation II, we may write

$$f(b, t) = \sum_{i=1}^{n} g_i(b)\cdot h_i(t), \tag{2}$$

where, say, $g_i(b)=f(b, t_i)$ and $[h_i(t), \cdots, h_n(t)]=[f(b_i, t) \cdots f(b_n, t)]\cdot(M^{-1})'$, i.e., range $g_i(b)\subseteq F$ and range $h_i(t)\subseteq F$. By Lemma 2, the g_i's and the h_i's are linearly independent, respectively, over K and hence over $F\subseteq K$. Thus by Lemma 1(b), $\mathrm{rank}_F(f)=n<\infty$.

$\Rightarrow$: Suppose $\mathrm{rank}_F(f)=n<\infty$. Then we may express f in the form of equation (I) where the conditions of Lemma 1(b) hold. In order to show that $\mathrm{rank}_K(f)=n$, we need only show that these g_i's and h_i's, respectively, are linearly independent functions over K. We do this for the g_i's. Suppose $k_1, k_2, \cdots, k_n$ contained in K are such that

$$\sum_{i=1}^{n} k_i g_i(b) \equiv 0, \qquad b \text{ in } B. \tag{3}$$

By Lemma (B), there exist $b_1, b_2, \cdots, b_n$ in B so that $[g_i(b_j)]$, $i, j=1, \cdots, n$, is a nonsingular matrix. Substituting these b_j's in equation (3), we deduce, by

Cramer's Rule for the solution of a system of linear equations, that $k_i=0$ for each i.

Before presenting examples of separable and inseparable functions, we quote an invaluable result.

LEMMA 3 (Vandermonde determinant, cf. [1] and [8]). *Let $x_1, x_2, \cdots, x_n$ be n values from a field F, and let*

$$D=\begin{bmatrix} 1 & x_1 & x_1^2 \cdots x_1^{n-1} \\ 1 & x_2 & x_2^2 \cdots x_2^{n-1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & x_n & x_n^2 \cdots x_n^{n-1} \end{bmatrix} \qquad \text{(Vandermonde matrix).}$$

Then (a) det $(D)=\Pi(x_k-x_j)$, $1\leqq j<k\leqq n$;

(b) *the x_i's are distinct$\Leftrightarrow$the columns of D are linearly independent.*

Example 1. $f(b, t)=b_0+b_1t+\cdots+b_{n-1}t^{n-1}$ where $b=(b_0, \cdots, b_{n-1})$, b_i's and t are real, is a separable function of rank n.

Proof. Let $g_i(b)=b_{i-1}(i=1, \cdots, n)$, $h_i(t)=t^{i-1}(i=1, \cdots, n)$. Then the g's are linearly independent, i.e.,

$$\sum_1^n c_ig_i(b) = 0 \qquad \text{all } b'\text{s} \Rightarrow c_i = 0 \quad (i = 1, \cdots, n).$$

For proof, take b with all components except the ith equal to zero. The linear independence of the h's follows by noting that they are linearly independent if their domain is restricted to n distinct t_i's, in particular, that the matrix $[h_j(t_i)]_{i=1}^{n}{}_{j=1}^{n}$ is the Vandermonde matrix, cf. Lemma 3(b). Thus by Lemma 1(b), since $f(b, t)=\Sigma g_i(b)h_i(t)$, we are done.

Example 2. $f(a, t)=1+at+a^2t^2+\cdots+a^{n-1}t^{n-1}$, where a and t are real, is separable of rank n. The proof is obvious from the previous example.

Example 3. $f(n, t)=t^n$ ($t^0\equiv 1$), $n=0, 1, 2, \cdots$, and t takes countably many distinct values $t_1, t_2, \cdots, t_i, \cdots$ in a field F, is inseparable (i.e., rank$(t^n)=\infty$).

Proof. We note by Lemma 3(b) that the matrix $[(t_i)^j]$, $i=1, \cdots, m$, $j=0, \cdots, m-1$ has rank m for each positive integer m. But if f were separable, these ranks would have to be bounded by a fixed integer. (Cf. Lemma 1(c).)

For subsequent use, we now investigate rank properties of combinations of separable functions.

LEMMA 4. *Let $f(b, t)$, $g(b, t)$ be separable functions of ranks m and n respectively ($m\geqq n$), defined for b in B, t in T and taking their values in the same field F. Then*

(a) *the functions $f(b, t)\cdot g(b, t)$ and $f(b, t)+g(b, t)$ are separable,*

(*b*) $0\leqq rank(f\cdot g)\leqq m\cdot n$,

(*c*) *for any never-vanishing function with rank one, say $r(b)\cdot s(t)$, we have $rank(r(b)s(t)f(b, t))=rank\ f(b, t)$,*

(*d*) $m-n\leqq rank(f+g)\leqq m+n$.

Proof. (a) and (b) follow easily from Definition 1 by observing the forms involved.

(c) Consider $f(b, t)$ as a matrix. $s(t)f(b, t)$ is the new matrix obtained by multiplying each column of $f(b, t)$ by a nonzero constant. Thus, since the spaces spanned by the columns of these two matrices have the same dimension, it follows by Lemma 1(e) that $\operatorname{rank}(s(t)f(b,t)) = \operatorname{rank} f(b,t)$. Similarly, $\operatorname{rank}(r(b)s(t)f(b,t)) = \operatorname{rank}(s(t)f(b,t))$. Thus we obtain (c).

(d) We note by (c) that $\operatorname{rank}(g) = \operatorname{rank}(-g)$. That $\operatorname{rank}(f+g)$ is $\leqq m+n$ follows immediately from the form of $f+g$. Using these two facts, we obtain

$$m = \operatorname{rank}(f) = \operatorname{rank}[(f+g) - g] \leqq \operatorname{rank}(f+g) + \operatorname{rank}(-g) = \operatorname{rank}(f+g) + n,$$

i.e., $m \leqq \operatorname{rank}(f+g)+n$, from which the first inequality of (d) follows.

Although we shall not subsequently need it, we conclude this section with a theorem on the rank of the usual type matrix product.

THEOREM 2. *Let* (i) *$f(a, b)$, $g(b, t)$ be separable functions of ranks m and n, respectively, taking their values in the same field F, defined for a in A, b in B, and t in T,*

(ii) *$b_1, b_2, \cdots, b_k$ be a finite sequence of distinct b's,*

(iii) *$h(a, t)$ be defined equal to $\Sigma_{i-1}^{k} f(a, b_i)g(b_i,t)$. Then $\operatorname{rank}(h) \leqq \min(k, m, n)$.*

Proof. Follows easily from Definition 2,

5. A general approach. The linear space spanned by a set of vectors $\{f_\alpha\}$ for $\alpha \in A$ over a field F shall be denoted by $\mathcal{G}_F\{f_\alpha\}_{\alpha\in A}$. The following theorem gives us a general method of attack for proving a function is inseparable.

THEOREM 3. *Let* (i) *$f(b, t)$ be a function from $B \times T$ into a field F,*

(ii) *$\mathcal{L}$ be an additive homogeneous (i.e., linear) mapping from $\mathcal{G}_F\{f(\cdot, t)\}_{t\in T}$ into a vector space of F-valued functions $g(a)$ defined on a set A,*

(iii) *$\mathcal{L}(f(\cdot, t)) = g(\cdot, t)$.*

Then

$$g(a, t) \text{ is inseparable} \Rightarrow f(b, t) \text{ is inseparable.} \tag{4}$$

Proof. The contrapositive form of (4) is

$$f(b, t) \text{ is separable} \Rightarrow g(a, t) \text{ is separable.} \tag{4'}$$

From Lemma 1(e), it follows that (4′) is equivalent to

$$\text{dimension } \mathcal{G}\{f\cdot, t)\}_{t\in T} < \infty \Rightarrow \text{dimension } \mathcal{G}\{g(\cdot, t)\}_{t\in T} < \infty. \tag{5}$$

We prove (5). Let $W = \mathcal{G}\{f(\cdot, t)\}_{t\in T}$. Then $\mathcal{L}(W) = \mathcal{G}\{g(\cdot, t)\}_{t\in T}$. Suppose $\dim(W) < \infty$, in particular that W is spanned by $f_1(b), \cdots, f_n(b)$. Since $\mathcal{L}(\Sigma_1^n c_i f_i(b) = \Sigma_1^n c_i \mathcal{L}(f_i(b))$, it follows that any element in $\mathcal{L}(W)$ is a linear combination of the $\mathcal{L}(f_i)$ $(i = 1, \cdots, n)$, i.e., $\dim(\mathcal{L}(W))$ is finite.

Clearly, in applications, we shall always use an infinite set for A. For, if A were finite, $g(a, t)$ would always be separable (cf. Lemma 1(f)). Most often it will be reasonable to take $A = \{0, 1, 2, \cdots\}$, where $g(a)$ will be a sequence $(g_0, g_1, g_2, \cdots)$. $\mathcal{L}$ generally will be some sort of differentiation, indefinite integration, or (infinite)-matrix multiplication (cf. Theorem 2). We illustrate the use of Theorem 3 with two examples.

Example 4. $f(b, t)=e^{bt}$, b and t real, is inseparable.

Proof. Let $\mathfrak{L}(f(\cdot, t))=D(\cdot, t)=D(t)$, where $D(t)$ is the sequence of derivatives (starting with the zero-th) of $f(\cdot, t)$ evaluated at $b=0$. Then $D(t)=(1, t, t^2, \cdots, t^n, \cdots)$, i.e., $D(n, t)=t^n$, $n=0, 1, 2, \cdots$. By Example 3, t^n is inseparable and thus, by Theorem 3, e^{bt} is inseparable.

Example 5. $f(b, t)=\sqrt{(b+t)}$, $b>0$, $t>0$ is inseparable.

Proof. Let $b_0>0$ be a fixed number. Similarly to Example 4, let $\mathfrak{L}(f(\cdot, t))=D(\cdot, t)=D(t)$ where $D(t)$ is the sequence of derivatives of $f(\cdot, t)$ evaluated at $b=b_0$. Then

$$D(t) = ((b_0+t)^{1/2}, \tfrac{1}{2}(b_0+t)^{-1/2}, -\tfrac{1}{4}(b_0+t)^{-3/2}, \cdots, \qquad c(n)(b_0+t)^{1/2}(1/(b_0+t)_1^n, \cdots)$$

i.e., $D(n, t) = c(n)(b_0+t)^{1/2}(1/(b_0+t))^n = c(n)\cdot d(t)\cdot F(n, t)$, where $F(n, t)=(1/(b_0+t))^n$. By Lemma 4(c), D is inseparable if F is inseparable. But the inseparability of F is easily established by noting that $s=1/(b_0+t)$ runs through infinitely many distinct values and that s^n is inseparable (cf. Example 3).

6. Polynomial quotients. Here (Theorem 4) we shall completely solve the problem of the separability of quotients of polynomial functions. Our approach is motivated by the following.

Example 6. If $f(b,t)=1/(b-t)$, where b and t each take countably many values in an arbitrary field, then f is inseparable.

Proof (by contradiction). Suppose f is separable, say of rank m. We note that $g_n(b, t)=b^n-t^n$ is separable of rank 2 for each $n\geqq 1$. By Lemma 4(b), we have, for each $n\geqq 1$, $\operatorname{rank}(f\cdot g_n)\leqq\operatorname{rank}(f)\cdot\operatorname{rank}(g_n)=2m$. But $f(b, t)\cdot g_n(b, t)=(b^n-t^n)/(b-t)=b^{n-1}\cdot 1+b^{n-2}\cdot t+\cdots+bt^{n-2}+1\cdot t^{n-1}$ has rank n since $\{b^i\}$, $i=0, \cdots, n-1$ and $\{t^i\}$, $i=0, \cdots, n-1$, are each sets of linearly independent functions (cf. Example 1 and Lemma 1(*b*)). Choosing $n=2m+1$, we have the contradiction $2m+1\leqq 2m$.

For subsequent use we generalize Example 4 into the following obvious

LEMMA 5. *Let $f(b, t)=1/(b-\beta(t))$ be defined for countably many b's in F and countably many values of t so that $\beta(t)$ takes countably many values in F. Then $1/(b-\beta(t))$ is inseparable.*

The following lemma emphasizes the fact that the rank of a function in polynomial form basically depends on the polynomial form.

LEMMA 6. *Let $f(b, t)$ be in polynomial form in t, b over a field F, defined for b and t in countably infinite subsets B and T, respectively, of F. Then the rank of $f(b, t)$ is independent of the particular subsets B and T which are used.*

Proof. Trivial, if $f\equiv 0$. Otherwise, suppose $\operatorname{rank}(f)=n$. By Lemma 1, equation (II), we may write $f(b, t) = \sum_{i=1}^{n} g_i(b)h_i(t)$ where the g's and h's, respectively, are linearly independent functions in polynomial form. The "g's are linearly independent for b in B" means

$$(6) \qquad \sum_{i=1}^{n} c_i g_i(b) \equiv 0 \qquad \text{all } b \text{ in } B \Rightarrow c_i = 0 \quad (i = 1, \cdots, n).$$

Since a polynomial which vanishes for infinitely many variable (b)-values must be the zero polynomial, (6) says that the polynomials in b, $g_i(b)$ $(i = 1, \cdots, n)$ are linearly independent. Clearly, the g's would be linearly independent, considered as functions over any infinite domain B. Similar reasoning holds with respect to the h's. Thus, by Lemma 1(b), we are done. We shall need the following obvious result for Theorem 4.

LEMMA 7. *Let $f(b, t)$ be defined for b in B and t in T. Let B' be contained in B and T' be contained in T. Then*

$$\textit{rank } (f) = \infty,\ (b, t) \textit{ in } B' \times T' \Rightarrow \textit{rank } (f) = \infty,\ (b, t) \textit{ in } B \times T.$$

We are now almost ready to prove Theorem 4, but first we need a lemma on factoring.

LEMMA 8. *Let* (i) $f(b, t) = \sum_{i=0}^{N} h_i(t)b^i = \sum_{i=0}^{M} r_i(b)t^i$, *where* $f \not\equiv 0$ *and the h_i's and r_i's are in polynomial form over a field F, be defined for countably many b's and t's respectively, in F,*

(ii) *$h_N(t)$ and $r_M(b)$ be nonvanishing over their domains,*

(iii) *the field F be extended* (cf. [1] p. 428) *to a larger field K, if necessary, so that in K for each t, f splits into a product of linear factors* $f(b, t) = f_t(b) = h_N(t)[(b - \alpha_1(t))(b - \alpha_2(t)) \cdots (b - \alpha_N(t))]$,

(iv) *$l(b)$ be the greatest common polynomial divisor of the coefficients $r_i(b)$* (cf. [1] p. 396, Theorem 2).

Then

$$(7) \qquad \begin{cases} f(b, t) = l(b) \cdot h(b, t) \\ \qquad\quad\ = l(b) \cdot h_N(t)(b - \beta_1(t))(b - \beta_2(t)) \cdots (b - \beta_{N-r}(t)), \end{cases}$$

where: (A) *rank*$(f) =$ *rank*(h); (*B*) $r =$ *degree of* $l(b)$; *and* (C) *none of the* $\beta_i(t)$ *take the same value more than M times.*

Proof. (A) follows immediately from Lemma 4(c). (B) is clear. It remains to prove (C): We note by (i) and (iv) that $h(b, t)$ may be written as

$$(8) \qquad h(b, t) = \sum_{i=0}^{M} S_i(b)t^i,$$

where the greatest common divisor of the polynomials $S_i(b)$ is a constant. If one of the $\beta_i(t)$'s takes the same value β_0 for more than M distinct t's, then it follows that the polynomial in t, $h(\beta_0, t)$ must be identically zero, i.e., $S_i(\beta_0) = 0$ for each i. But this implies that the $S_i(b)$'s have a nontrivial polynomial factor, contradicting (iv).

Now we prove the main result of this section.

THEOREM 4 (Polynomial Quotients). *Let* (i) $f(b, t)$ *and* $g(b, t)$, *in polynomial form in* b, t *over a field* F, *be defined for infinitely many* b's *and* t's *respectively in* F,

(ii) $rank(f) \geqq 2$, $rank(g) \geqq 1$, $f \equiv nonzero$,

(iii) $f(b, t) = l(b) \cdot h(b, t)$ *be the factorization of* f *in accordance with Lemma* 8, *Equation* (7).

Then $rank(g(b, t)/f(b, t)) = \infty \Leftrightarrow$ *the polynomial* $h_t(b) = h(b, t)$ *does not divide* $g_t(b) = g(b, t)$ *exactly* (*both considered as polynomials in the one variable* b).

Proof. $\Rightarrow$: The contrapositive of this implication is "$h_t(b)$ divides $g_t(b)$ exactly $\Rightarrow g(b, t)/f(b, t)$ is separable." We show this. By hypothesis

$$g_t(b)/h_t(b) = \sum_{i=0}^{m} p_i(t) b^i \tag{9}$$

where the $p_i(t)$ are rational fractional forms in t over F. Thus it follows that

$$g_t(b)/f_t(b) = \sum_{i=1}^{m} \{p_i(t)(b^i/l(b))\} \text{ is separable.} \tag{10}$$

$\Leftarrow$: In this case

$$g_t(b)/h_t(b) = \sum_{i=1}^{n} p_i(t) b^i + \gamma_t(b)/h_t(b), \tag{11}$$

where $\gamma_t(b)$ is of lower degree in b than $h_t(b)$ and has rational fractional forms in t as coefficients. Without loss of generality (cf. Lemma 7) we may assume that Equation (7) of Lemma 8 holds, i.e., $h_t(b) = h_N(t)(b - \beta_1(t)) \cdots (b - \beta_{N-r}(t)) = h_N(t) q(b, t)$.

We shall show that $\gamma_t(b)/q(b, t)$ is inseparable. From this by use of Lemma 4(c) and the easily proven fact that the sum of a separable function and an inseparable function is inseparable, it will immediately follow that $g(b, t)/f(b, t)$ is inseparable.

First, divide the polynomial in b, $b - \beta_1(t)$, into $\gamma_t(b)$, getting

$$\frac{\gamma_t(b)}{b - \beta_1(t)} = Q(b, t) + \frac{r_1(t)}{b - \beta_1(t)}; \tag{12}$$

hence

$$\frac{\gamma_t(b)}{q(b, t)} = \frac{r_1(t)}{(b - \beta_1(t)) \cdots (b - \beta_{N-r}(t)} + \frac{Q(b, t)}{(b - \beta_2(t)) \cdots (b - \beta_{N-r}(t))}. \tag{13}$$

Next, divide $Q(b, t) = Q_t(b)$ by $b - \beta_2(t)$, etc., finally obtaining

$$\begin{aligned} \frac{\gamma_t(b)}{q(b, t)} = {} & \frac{r_1(t)}{(b - \beta_1(t)) \cdots (b - \beta_{N-r}(t))} + \frac{r_2(t)}{(b - \beta_2(t)) \cdots (b - \beta_{N-r}(t))} + \cdots \\ & + \frac{r_{N-r}(t)}{b - \beta_{N-r}(t)}. \end{aligned} \tag{14}$$

We claim that at least one of the functions $r_i(t)$ must be nonzero for infinitely many of our t's. For otherwise $\gamma_t(b)$ would be zero for a fixed b and infinitely many t's and thus would be identically zero, contradicting our hypothesis. Now let j be the smallest integer so that $r_j(t)$ is nonzero infinitely often. Without loss of generality (cf. Lemma 7) we may discard the finitely many t-values for which $r_i(t)$, $i<j$, do not vanish. So

$$\frac{\gamma_t(b)}{q(b,t)}=\frac{r_j(t)}{(b-\beta_j(t))\cdots(b-\beta_{N-r}(t))}+\cdots+\frac{r_{N-r}(t)}{b-\beta_{N-r}(t)}\,. \tag{15}$$

We show (15) is inseparable by contradiction. Note that

$$k(b,t)=(b-\beta_{j+1}(t))\cdot(b-\beta_{j+2}(t))\cdots(b-\beta_{N-r}(t)) \tag{16}$$

is separable. Hence if (15) were separable, then the product of (15) and (16) would be separable. But

$$\begin{aligned} k(b,t)\cdot\frac{\gamma_t(b)}{q(b_1t)}=\frac{r_j(t)}{b-\beta_j(t)}&+r_{j+1}(t)+\cdots \\ &+r_{N-r}(t)(b-\beta_{j+1}(t))\cdots(b-\beta_{N-r+1}(t)), \end{aligned} \tag{17}$$

where by Lemma 5, the first term on the right hand side of (17) is inseparable and hence (17) is inseparable.

We point out, in our proof of Theorem 4, our assumption that $\text{rank}(f)\geqq 2$ forces the existence of at least one $\beta_i(t)$ (cf. (12) and (13)).

Although our hypotheses in Theorem 2 are in a one-sided form, we state the nicely symmetric

COROLLARY. *Let $f(b,t)$ be in polynomial form over the field F, defined for infinitely many b's and t's respectively, in F, and not vanish. Then $\text{rank}(f)\geqq 2$ $\Leftrightarrow 1/f(b,t)$ is inseparable.*

We note immediately from the corollary that forms such as

$$1/(b\cdot t^2+1\cdot t+b^2\cdot 1), \qquad 1/(b^2+t^3).$$

etc., will give us inseparable functions if we can choose infinite sequences of distinct b's and t's, respectively, from a field F so that the denominators will not vanish for any pair (b,t). If F is an infinite field, this may always be done inductively: First choose a b_1 so that $f(b_1,t)\not\equiv 0$. Then choose t_1 so that $f(b_1,t_1)\neq 0$. Then choose b_2 so that $f(b_2,t_1)\neq 0$. Then choose t_2, etc. These choices always exist since a polynomial in one variable can have at most a finite number of distinct roots.

7. Pureness. Let rank $(f)=n$, $1\leqq n\leqq\infty$. We say that f has *pure rank n* if every square $m\times m$ matrix $[f(b_i,t_j)]$, $i,j=1,\cdots,m$ where the b_i's and t_i's respectively run through distinct values, and where m is finite, $\leqq n$, has rank m (i.e., is of full rank).

Example 7. $f(b,t)=1/(b+t)$, where b and t respectively take countably many values from a field F, has pure rank $n=\infty$.

Proof. We refer to the following theorem due to A. Cauchy (cf. [4] [11]).

THEOREM. *Let* $b_1, \cdots, b_m, t_1, \cdots, t_m$ *be 2m numbers. Then*

$$\det\left[\frac{1}{b_i+t_j}\right]_{i=1}^{m}\,{}_{j=1}^{m} = \frac{\prod_{1\leq i<j\leq m}(b_j-b_i)(t_j-t_i)}{\prod_{i,j=1}^{m}(b_i+t_j)}.$$

Clearly, for distinct b's and t's, it follows that det $[1/(b_i+t_j)]\neq 0$ and thus that the matrix $[1/(b_i+t_j)]$ has rank m for each $m<\infty$.

With the posing of the general question, "Which functions have pure rank?" we end this section.

8. Conclusion. Although we have assumed a previous knowledge of matrices in the development of this paper, it is obvious that this is not necessary and that one may make our Definitions 1 and 2 the basis for the development of properties of matrices. Indeed the symmetry of our approach makes it seem more appealing than the usual definition of matricial rank in terms of row or column rank. We close with the hope that perhaps our small store of results also clears up various problems in the possibilities of approximating one function of two variables by another.

Mr. William G. Collar collaborated with the author in a preliminary version of this paper, comprising Lemma 1 and Theorem 3, written at the U. S. Naval Weapons Lab in 1964.

References

1. G. Birkhoff and S. MacLane, A Survey of Modern Algebra, Macmillan, New York, 1953.
2. R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. 1, Interscience, New York, 1953, vol. 2, 1962.
3. V. N. Fadeeva, Computational Methods of Linear Algebra, Dover, New York, 1959.
4. C. Goffman and G. Pedrick, First Course in Functional Analysis, Prentice-Hall, Englewood Cliffs, New Jersey, 1965, p. 204.
5. D. Greenspan, Introduction to Partial Differential Equations, McGraw-Hill, New York, 1961.
6. G. Hadley, Linear Algebra, Addison-Wesley, Reading, 1961, p. 138.
7. E. L. Ince, Ordinary Differential Equations, Longmans, Green, London, 1927.
8. G. Kowalewski, Einführung in die Determinantentheorie, Chelsea, New York, 1948.
9. W. E. Milne et al., Mathematics for Digital Computers, vol. 1, Multivariate Interpolation, WADC TR-57-556, U. S. Dept. of Commerce, (OT 5), 1958.
10. A. M. Mood and F. A. Graybill, Introduction to the Theory of Statistics, McGraw-Hill, New York, 1963.
11. R. Savage and E. Lukacs, Tables of inverses of finite segments of the Hilbert Matrix, Nat. Bur. Standards. Appl. Math. Ser., 39 (1954) 105.
12. H. C. Thacher, Jr., Derivation of interpolation formulas in several independent variables, Ann. New York Acad. Sci., 86 (1960) 677–874, 758.
13. B. L. van der Waerden, Modern Algebra, vol. 1, Ungar, New York, 1953.
14. K. Yosida, Lectures on Differential and Integral Equations, Interscience, New York, 1960.