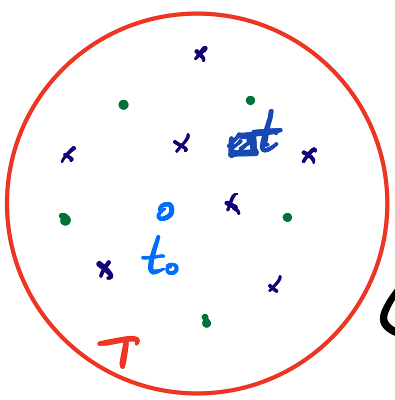
(T,d) metric space $\qquad$ T is finite

Th $\{X_t, t\in T\}$ - sub-Gaussian (wrt $d(\cdot,\cdot)$)

$$\gamma_2(T,d) = \inf_{\{T_j\}_{j\geq 0}} \sup_{t\in T} \sum_{j\geq 0} 2^{j/2} d(t,T_j)$$

$$T_0 = \{t_0\}, \quad |T_j| \leq 2^{2^j}, \quad T_j \subseteq T_{j+1} \quad \forall j$$

$$\left(\mathbb{E} \sup_t |X_t - X_{t_0}|^p\right)^{1/p} \leq C\gamma_2(T,d) + 2\sup_{t\in T}\left(\mathbb{E}|X_t - X_{t_0}|^p\right)^{1/p}$$



$$X_t - X_{t_0} = X_{\pi_\ell(t)} - X_{t_0} + \sum_{j\geq \ell}\left(X_{\pi_{j+1}(t)} - X_{\pi_j(t)}\right)$$

$$\pi_j(t) = \operatorname{argmin}_{s\in T_j} d(t,s)$$

$$\left(\mathbb{E}\sup_t |X_t - X_{t_0}|^p\right)^{1/p} \leq \left(\mathbb{E}\sup_t |X_{\pi_\ell(t)} - X_{t_0}|^p\right)^{1/p}$$

$$+ \left(\mathbb{E}\sup_t \Big|\sum_{j\geq \ell} X_{\pi_{j+1}(t)} - X_{\pi_j(t)}\Big|^p\right)^{1/p}$$

$\leq$?

$C\gamma_2(T,d)$

"Nice" event:

$$\mathcal{E} = \bigcap_{j\geq \ell}\bigcap_{t\in T}\left\{ |X_{\pi_{j+1}(t)} - X_{\pi_j(t)}| \leq u\cdot 2^{\frac{j+1}{2}} d(\pi_{j+1}(t), \pi_j(t))\right\}$$

Assume that $\mathcal{E}$ holds. Then

$$\sup_t \left| \sum_{j\geq \ell} X_{\pi_{j+1}(t)} - X_{\pi_j(t)} \right| \leq \sup_t \sum_{j\geq \ell} \left| X_{\pi_{j+1}(t)} - X_{\pi_j(t)} \right|$$

$$\leq \sup_{t\in T} \sum_{j\geq \ell} u \cdot 2^{\frac{j+1}{2}} d(\pi_{j+1}(t), \pi_j(t)) \leq (*)$$

$$d(\pi_{j+1}(t), \pi_j(t)) \leq d(t, \pi_j(t)) + d(t, \pi_{j+1}(t))$$

$$(*) \leq \sup_t u \sum_{j\geq \ell} 2^{\frac{j+1}{2}} \left(d(t, T_j) + d(t, T_{j+1}) \right)$$

$$= \sup_t \left(u \sum_{j\geq \ell} 2^{\frac{j+1}{2}} d(t, T_{j+1}) + u\sqrt{2} \sum_{j\geq \ell} 2^{j/2} d(t, T_j) \right)$$

$$\leq \boxed{u(1+\sqrt{2})\, \gamma_2(T, d)}$$

$$\mathcal{E} = \bigcap_{j\geq \ell} \bigcap_{t\in T} \left\{ \left| X_{\pi_{j+1}(t)} - X_{\pi_j(t)} \right| \leq u \cdot 2^{\frac{j+1}{2}} d(\pi_{j+1}(t), \pi_j(t)) \right\}$$

$$P(\mathcal{E}^c) = P\left(\bigcup_{j\geq \ell} \bigcup_{t\in T} \left\{ \left| X_{\pi_{j+1}(t)} - X_{\pi_j(t)} \right| > u \cdot 2^{\frac{j+1}{2}} d(\pi_{j+1}(t), \pi_j(t)) \right\} \right)$$

$$\leq \sum_{j\geq \ell} \sum_{\substack{t\in T_j \\ s\in T_{j+1}}} P\left(\left| X_{\pi_{j+1}(t)} - X_{\pi_j(t)} \right| > u \cdot 2^{\frac{j+1}{2}} d(\pi_{j+1}(t), \pi_j(t)) \right)$$

$$P(|X_t - X_s| > z) \leq 2e^{-z^2/2d^2(t,s)}$$

$$\leq \sum_{j \geq l} \sum_{\substack{t \in T_j \\ s \in T_{j+1}}} 2 \exp\left(- \frac{u^2 \cdot 2^{j\cancel{+1}} \cancel{d^2(\pi_{j+1}(t), \pi_j(t))}}{\cancel{2}\, \cancel{d^2(\pi_{j+1}(t), \pi_j(t))}} \right)$$

$$\leq 2 \sum_{j \geq l} |T_j| \cdot |T_{j+1}|\, e^{-u^2 \cdot 2^j} \leq 2 \sum_{j \geq l} 2^{2^j + 2^{j+1}} \cdot e^{-u^2 \cdot 2^j}$$

$$= 2 \sum_{j \geq l} e^{(3 \log 2 - u^2) \cdot 2^j} \leq 2 \sum_{j \geq l} e^{-2^j \cdot u^2/2}$$

If $u^2 \geq 6 \log 2 \Rightarrow u^2 - 3 \log 2 \geq u^2/2$

$$2 \sum_{j \geq l} e^{-2^j \cdot u^2/2} = 2 e^{-2^l \cdot u^2/2} \left(1 + e^{-2 \cdot u^2/2} + \dots + e^{-2^j u^2/2} + \dots \right)$$

$$\leq 2 e^{-2^l u^2/2} \Big(\underbrace{\sum_{j \geq 1} e^{-2^j \cdot 3 \log 2} + 1}_{C} \Big) = 2C e^{-2^l u^2/2} \leq 2C e^{-p u^2/4}$$

$l = \lfloor \log_2 p \rfloor$

$$\boxed{P\Big(\underbrace{\sup_t |X_t - X_{\pi_l(t)}|}_{Z} \geq (\sqrt{2}+1)\, u \cdot \underset{u^2 \geq 6 \log 2}{\underline{\gamma_2(T,d)}} \Big) \leq 2C e^{-p u^2/4}}$$

Assume that Z is s.t.

$$P(Z \geq \alpha \cdot u \cdot D) \leq C' e^{-p u^2}, \quad u \geq u_0$$

$$\Rightarrow \left(\mathbb{E} |Z|^p \right)^{1/p} \leq D \cdot C(\alpha, u_0)$$

$$\mathbb{E}\, Z^p = p\int_0^\infty t^{p-1}\, P(Z \geq t)\, dt$$

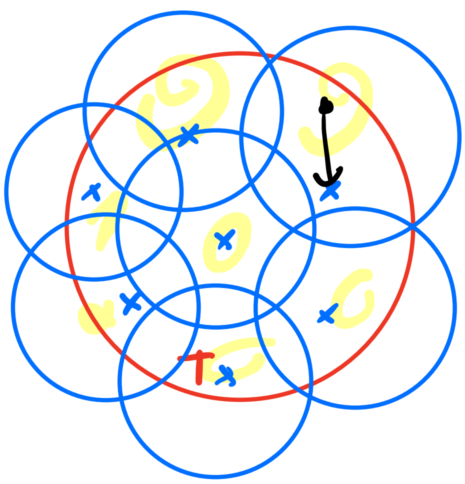
$$\leq p\int_0^{u_0} t^{p-1}\, 1\, dt + p\int_{u_0}^{\infty} t^{p-1} e^{-pu^2/4}\, dt \; \ldots$$

$$\Rightarrow \left(\mathbb{E} \sup_t |X_t - X_{\pi_\ell(t)}|^p\right)^{1/p} \leq C\, \gamma_2(T,d)$$

The metric entropy and Dudley's integral

(T, d)

$$N(T,d,\varepsilon) = \min\left\{ m \geq 1 : \exists\, t_1, \ldots, t_m \text{ s.t. } \bigcup_{j=1}^{m} B(t_j, \varepsilon) \supset T \right\}$$

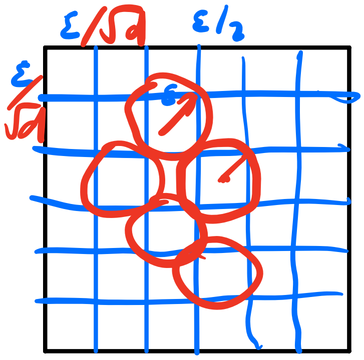


$$B(t,\varepsilon) = \{ s \in T : d(s,t) \leq \varepsilon \}$$

$H(\varepsilon) = \log(N(\varepsilon))$ - the metric entropy

Theorem $$\gamma_2(T,d) \leq C \int_0^{\operatorname{diam}(T)} \sqrt{H(\varepsilon)}\, d\varepsilon = C\int_0^\infty \sqrt{H(\varepsilon)}\, d\varepsilon$$

$$\varepsilon \geq \operatorname{diam}(T) \Rightarrow N(\varepsilon) = 1 \Rightarrow \log N(\varepsilon) = 0$$



$$[0,1]^d \to \left(\frac{1}{\varepsilon}\right)^d$$

$$N(\varepsilon) \le \left(\frac{\sqrt{d}}{\varepsilon}\right)^d$$

Exercise ? $N(\varepsilon) \ge \left(\frac{c\sqrt{d}}{\varepsilon}\right)^d$

(compare volumes)

$$H(\varepsilon) \approx d \log\left(\frac{\sqrt{d}}{\varepsilon}\right)$$

$$\mathbb{E} \sup_t (X_t - X_{t_0}) \le C \int_0^\infty \sqrt{H(\varepsilon)}\, d\varepsilon$$

$$c_1 \gamma_2(T,d) \le \mathbb{E} \sup_t (X_t - X_{t_0}) \le C_1 \gamma_2(T,d)$$

Pf: $\gamma_2(T,d) = \inf_{\{T_j\}} \sup_{t\in T} \sum_{j\ge 0} 2^{j/2} d(t,T_j)$

$$\le \inf_{\{T_j\}} \sum_{j\ge 0} \sup_{t\in T} 2^{j/2} d(t,T_j)$$

$$\varepsilon_j = \min\{\varepsilon > 0 : N(T,d,\varepsilon) \le 2^{2^j}\}$$

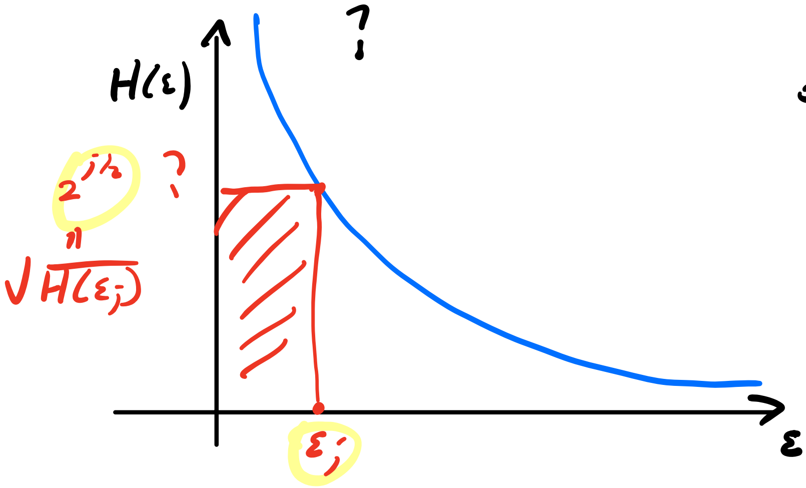
$T_j = \{t_1, t_2, \dots, t_{2^{2^j}}\}$ - centers of the balls.

$$N(T,d,\varepsilon_j) = 2^{2^j} \iff H(T,d,\varepsilon_j) = 2^j$$

$$\Rightarrow \varepsilon_j = H^{-1}(2^j) \qquad H(\varepsilon_j) = 2^j \Rightarrow \sqrt{H(\varepsilon_j)} = 2^{j/2}$$

$$\gamma_2(T,d) \le \sum_{j\ge 0} 2^{j/2} \overbrace{\sup_t d(t,T_j)}^{\le \varepsilon_j} \le \sum_{j\ge 0} 2^{j/2} \varepsilon_j$$

$$= \sum_{j\ge 0} 2^{j/2} H^{-1}(2^j) \lesssim C \int_0^\infty \sqrt{H(\varepsilon)}\, d\varepsilon$$



$$\int_0^\infty \sqrt{H(\varepsilon)}\, d\varepsilon$$