CSCI 670: Advanced Analysis of Algorithms
6^{th} Assignment

Fall 2021

Problem 1. Let G be a d-regular connected graph, i.e. each vertex has degree d. Let A be the adjacency matrix where $A_{i,j} = 1$ if vertex v_i and v_j are adjacent. Prove that the largest eigenvalue $\lambda_1 = d$.

Solution 1. Consider the all 1's vector ν. Observe that $A\nu = d\nu$, and so d is an eigenvalue of A and $\mathbf{1}$ is the corresponding eigenvector.

Now, we need to prove that d is the largest possible eigenvalue. Let $\mathbf{x} = (x_1, \ldots x_n)$ be the eigenvector for the largest eigenvalue λ. Let $x_j = \max_i |x_i|$. Then,

$$|\lambda||x_j| = |(A\mathbf{x})_j| = \left| \sum_{v_i \sim v_j} x_i \right| \leq \deg(v_j)|x_j| = dx_j$$

where $v_i \sim v_j$ denotes that v_i and v_j are adjacent in the graph. Therefore, $\lambda_1 = d$.

Problem 2. Let G be a connected bipartite graph and A be its adjacency matrix. Show that if $\lambda > 0$ is an eigenvalue of A then $-\lambda$ is also an eigenvalue of A.

Solution 2. If we relabel vertices we can obtain the following adjacency matrix.

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

Let λ be an eigenvalue and $\nu = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}$ be its eigenvector. Then, observe that

$$\lambda\nu = A\nu = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \cdot \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}$$

thus

$$B\nu_2 = \lambda\nu_1, \qquad \text{and} \qquad B^T\nu_1 = \lambda\nu_2.$$

Define $\nu' = \begin{bmatrix} -\nu_1 \\ \nu_2 \end{bmatrix}$ and observe $A\nu' = -\lambda\nu'$. Therefore $-\lambda$ is an eigenvalue of A with eigenvector of ν'.

Problem 3. Let G be a connected bipartite graph. Prove that G does not contain any odd cycles. How do you apply the result of Problem 2?

Solution 3. Let k be any odd positive integer. Recall from your Linear Algebra classes that if A has eigenvalues $\lambda_1, \ldots \lambda_n$, then A^k has eigenvalues $\lambda_1^k, \lambda_2^k \ldots \lambda_n^k$. By Problem 2 we know that if $\lambda > 0$ is an eigenvalue of A, $\lambda' = -\lambda$ is also an eigenvalue of A with the same multiplicity (why?). Thus,

$$\sum_{i=1}^{n} \lambda_i^k = 0$$

Recall from Linear Algebra classes that $\sum_{i=1}^{n} \lambda_i^k = \mathrm{tr}(A^k)$ equals to the trace of A^k. Since A^k is a non-negative matrix, we conclude that $(A^k)_{i,i} = 0$ for any $i \in \{1, \dots n\}$. Notice that $(A^k)_{i,i}$ is the number of length k paths from i to itself. Therefore, there is no odd cycle in a bipartite graph.

Problem 4 (Optional)**.** Let $A, B \in \mathbb{R}^{n\times n}$ be two positive semidefinite matrices. Prove that their element-wise summation $A + B$ is also positive semidefinite.

Solution 4.

Definition 1. A matrix $M \in \mathbb{R}^{n\times n}$ is positive semidefinite (PSD) if it is symmetric matrix all of whose eigenvalues are nonnegative.

First of all, observe that $A+B$ is symmetric if both A and B are symmetric. So, it remains to prove that $\mathbf{u}^T(A+B)\mathbf{u}$ is nonnegative for any vector $\mathbf{u} \in \mathbb{R}$.

$$\mathbf{u}^T(A+B)\mathbf{u} = \mathbf{u}^T(A\mathbf{u} + B\mathbf{u}) = \mathbf{u}^T A\mathbf{u} + \mathbf{u}^T B\mathbf{u} \geq 0$$

where equalities follow by distributive properties of matrix multiplication. The last inequality follows by the assumption that both A and B are PSD.

Problem 5. For a given graph $G = (V, E)$, let L be the Laplacian matrix defined as

$$L_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

Prove that L is positive semidefinite. (Hint: you may use edge decomposition of L and result of Problem 4)

Solution 5. In class we defined the edge laplacian L_e for $e = (v_1, v_2)$ as follows.

$$L_e = \begin{pmatrix} 1 & -1 & 0 & & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Moreover, we proved that $L = \sum_{e\in E} L_e$. Observe for $e = (v_1, v_2)$ that L_e is a rank 1 matrix, so it has single non-zero eigenvalue. Morover, the vector $\nu = \begin{bmatrix} 1 \\ -1 \\ \mathbf{0} \end{bmatrix}$ is an eigenvector of L_e with eigenvalue 2. Therefore, L_e is a PSD matrix. By Problem 4, L is also PSD since it is a summation of PSD matrices.

Problem 6. Let $T_k = (V, E)$ be the complete binary tree of depth k. Let $n = 2^k - 1$ be the size of the tree and λ_2 be the second smallest eigenvalue of the Laplacian of T_n. Show that $\lambda_2 \leq O(1/n)$. (Hint: Consider the Rayleigh quotient of the vector with left subtree goes to -1 and right subtree goes to 1, and root goes to 0.)

Solution 6. First of all, we know that L is PSD, therefore all eigenvalues are non-negative. Moreover, we can easily show that 0 is an eigenvalue of L with eigenvector $\mathbf{1}$. One can write the second smallest eigenvalue as follows.

$$\lambda_2 = \min_{\nu:\nu\perp\mathbf{1}} \frac{\nu^T L\nu}{\nu^t\nu}$$

So, we can find an upperbound for λ_2 by testing a vector ν. Let T^L_{k-1} and T^R_{k-1} be two subtrees that lives below the left and right child of the root r of T_k. Define ν as follows

$$\nu_i = \begin{cases} 0, & v_i = r \\ 1, & v_i \in T^L_{k-1} \\ -1, & v_i \in T^R_{k-1} \end{cases}.$$

It is easy to check that $\nu \cdot \mathbf{1} = 0$. Therefore, $\lambda_2 \leq \frac{\nu^T L \nu}{\nu^t \nu}$.

One show that $\nu^T \nu = n - 1$, so it remains to compute $\nu^T L \nu$. We can use the edge decomposition of the Laplacian matrix to compute that quantity.

$$\begin{aligned} \nu^T L \nu &= \sum_{e=(v_i,v_j)\in E} \nu^T L_e \nu \\ &= \sum_{e=(v_i,v_j)\in E} \nu_i(\nu_i - \nu_j) + \nu_j(-\nu_i + \nu_j) \\ &= \sum_{e=(v_i,v_j)\in E} (\nu_i - \nu_j)^2 \\ &= 2 && \textit{(why?)} \end{aligned}$$