CSCI 670: Theoretical Thinking Assignment 2

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In my senior year of college, we were assigned to develop an educational application as our graduation project. My friends and I had designed and developed an app to help students form project or study groups. I wanted to study the game-theoretical properties of our app, which led to two papers on formation of partnerships [1] and teams [2]. The first one is about a model for general partnerships that was formulated in the 1980's [3]. The second one, which I would like to discuss in this assignment, is about a related model that I came up with, called *hedonic expertise games (HEGs)*.

1. What is the practical background of your problem?

Consider a team formation setting where agents have varying levels of expertise in a global set of required skills. For instance, the required skills for a class project assignment may be `(Python, Java, SQL)` where the expertise of two students, say Alice and Bob, in these skills are `(1, 3, 3)` and `(3, 3, 1)` respectively. We measure the success of a team by how well the expertise of teammates complement each other. For instance, notice that Alice may compensate the lack of expertise of Bob in `SQL`, just as Bob may compensate Alice in `Python`. We say that a team's *joint expertise* in some skill is the maximum level of expertise of its members in that skill, and its *joint utility* is the sum of its joint expertise in each skill. The team formed by Alice and Bob would have a joint expertise of 3 in each skill, and thus a joint utility of 9.

Suppose that every agent wants to maximize the joint utility of the team they are in. Then, a natural question to ask is how can we efficiently partition agents into "stable" teams, assuming there exists a limit on the sizes of teams? (If there was not a size constraint then notice that the grand team would be trivially stable but this is not so practical. For instance, it would not make sense if the whole class formed a single team, in the classroom example above.)

2. What is the mathematical definition of your problem?

HEGs can be formally defined as follows: We have a set of *agents* N and a set of *skills* S. For each agent $i \in N$, we have a non-negative integer-valued *expertise function* $e_i : S \to \mathbb{Z}_{\geq 0}$, where $e_i(s)$ denotes the expertise that agent $i \in N$ has in skill $s \in S$. Additionally, we have an upper bound of κ on the sizes of coalitions, i.e., teams of size greater than κ are disqualified. We denote an HEG instance by $\mathcal{G} = (N, S, e, \kappa)$. For each coalition C, we define the *joint expertise function* $E_C : S \to \mathbb{Z}_{\geq 0}$ as $E_C(s) = \max_{i \in C} e_i(s)$, i.e., $E_C(s)$ is the maximum expertise level that a member of coalition C has in skill s. Lastly, we define the *joint utility* of a coalition C as $U(C) = \sum_{s \in S} E_C(s)$, i.e., $U(C)$ is the sum of the maximum expertise that a member of coalition C has in each skill $s \in S$. A solution of an HEG instance is a partition π over the set of agents N where for each coalition $C \in \pi$ we have $|C| \leq \kappa$. We use $\pi(i)$ to denote the coalition containing agent $i \in N$ in partition π.

In hedonic (coalition formation) games, such as HEGs, the notion of stability can be defined in numerous ways based on assumptions on capabilities of agents to coordinate their strategies. Due to space constraints, I will only talk about Nash stability and core stability concepts here. Which are, respectively, the main stability notions based on individual and group deviations.

Nash Stability A partition π is *Nash stable (NS)* if no agent $i \in N$ can benefit from moving to a coalition $C \in \pi$ such that $|C| < \kappa$, i.e., $U(\pi(i)) \geq U(C \cup \{i\})$ for all $C \in \pi$ where $|C| < \kappa$

Core Stability A coalition C is said to *block* π, if $U(C) > U(\pi(i))$ for all agents $i \in C$, i.e., any agent $i \in C$ is better off in C than she is in her coalition $\pi(i)$. A partition π is *core stable (CS)* if no coalition blocks π.

3. What are the interesting algorithmic question concerning your problem?

In HEGs, there always exists a partition which is both Nash stable and core stable. However, is it reasonable to assume that agents can reach such a stable partition? The proof of existence uses a potential function argument, i.e., if agents take successive deviations then they will eventually reach an NS and CS partition. This process is referred to as *better response dynamics*. Since better response dynamics of HEGs converge, agents can reach a stable partition via natural game play but it is not clear how many steps this would require. Convergence rate of better response dynamics are, of course, constricted by the computational complexity of finding such a stable partition. For example, in the paper, we show that finding a CS partition is NP-hard, and thus, better response dynamics require exponential number of moves to reach a CS-partition. On the other hand, using the submodularity of joint utility function U, it is possible to greedily find a $(1 - 1/e)$-approximate CS partition. See [4], for a detailed discussion of better response dynamics of hedonic games.

4. What is the state of the art for modeling and solving this problem?

We were unable to determine the convergence rate of better response dynamics of HEGs to an NS-partition, which is discussed in the next question. However, we discovered that agents can shortcut their way into a stable partition by imitating others' deviations, which we refer to as *imitative better response dynamics*.

Imitative Better Response Dynamics Given a partition π, suppose that an agent i benefits from moving to an existing coalition $C \in \pi$ such that $|C| < \kappa$, i.e., $U(C \cup \{i\}) > U(\pi(i))$. Suppose that agent i takes the above better response. If $|C \cup \{i\}| < \kappa$, then notice that another agent $i' \in \pi(i) \setminus \{i\}$ also benefits from moving to $C \cup \{i\}$, since $U(C \cup \{i\}) > U(\pi(i)) \geq U(\pi(i) \setminus \{i\})$. That is, if the size of the coalition that the last agent i has moved to did not reach the upper bound of κ, then an agent $i' \in \pi(i) \setminus \{i\}$ simply *"imitates"* the last agent i by moving to the same coalition. Otherwise, an arbitrary agent takes an improving deviation. Note that if no agent has an improving deviation, then we must have reached an NS partition.

Since the above decentralized algorithm is a restricted version of better response dynamics, it is guaranteed to converge to an NS partition. Moreover, we showed that it terminates in a linear number of moves, if the number of levels of expertise is bounded above by a constant. There exists such a bound for most practical purposes because the expertise in some real-life skill is most commonly measured by a small number of levels such as `(0: None, 1: Beginner, 2: Intermediate, 3: Advanced)`.

5. What are interesting open questions inspired by your formulation?

As mentioned earlier, the convergence rate of better response dynamics of HEGs to an NS partition remains open. More generally, we do not even know the computational complexity of finding an NS partition of a given HEG instance. We know that if the level of expertise is bounded by a constant, then an NS partition can be found in linear time using imitative better response dynamics. We also show, in the paper, that a contractually[1] NS partition can be found in polynomial time. However, our efforts remained short for the general setting. Whether this problem is polynomial time solvable or PLS-complete[2] is still a mystery to me, and I find both possibilities equally likely.

References

[1] Caskurlu, B., Kizilkaya, F. E. 2019. On hedonic games with common ranking property. *In Proceedings of the 11th International Conference on Algorithms and Complexity (CIAC)*, 11485, 137-148.

[2] Caskurlu, B., Kizilkaya, F. E., Ozen, B. 2021. Hedonic expertise games. *In Proceedings of the 14th International Symposium on Algorithmic Game Theory (SAGT)*, 12885, 314-328.

[3] Farrell, J. Scotchmer, S. 1988. Partnerships. *The Quarterly Journal of Economics*, 103(2), 279-297.

[4] Brandt, F., Bullinger, M., Wilczynski, A. 2021. Reaching individually stable coalition Structures in hedonic Games. *Proceedings of the AAAI Conference on Artificial Intelligence*, 35(6), 5211-5218.

[1] In this stability concept, agents are assumed to not leave their coalition if it causes harm to other members.

[2] Since finding an NS partition can be defined as a local search problem using better response dynamics and we can check whether a partition is NS efficiently, this problem is in PLS.