CSCI 670: Theoretical Thinking Assignment 1

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Selected Paper: Resolving the Optimal Metric Distortion Conjecture *(FOCS 2020)*

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1. How did you find this paper?

In my first meeting with David Kempe, he told me about the problems that he is currently interested in, and this paper was related to one of them. I was curious about it also because the conjecture that has been resolved in this paper was proposed by Elliot Anshelevich, who is the former PhD advisor of my former MS advisor.

2. Give a brief, but clear, discussion of the problem as well as the main result of the paper.

When a group of people with different preferences and/or conflicting interests need to reach a collective decision, often the sensible thing to do is to resort to voting among possible options. The options might be candidates for mayor of a town, or different amounts to spend on a new building, or several versions of an immigration reform bill.

If there are only two options, the rules of voting are straightforward. May's Theorem tells us that the only reasonable voting rule is to choose the option with a plurality (greatest number) of votes [1]. But when there are more than two options, there is no obvious voting rule that chooses the "best" option. There are numerous voting rules, and a great deal of controversy surrounds the discussion of which voting rule is most fair.

For a long time, the main concern in voting theory was to identify a notion of fairness axiomatically (i.e., by aiming for a set of desirable properties) that can be always achieved by a voting rule. However, this axiomatic approach resulted in a series of impossibility results such as the famous Arrow's Impossibility Theorem, which can be stated as follows: if a voting rule is not affected by candidates entering or leaving the race (unless they win) then it is a dictatorship [2].[1]

[1] Another well-known impossibility result is Gibbard-Satterthwaite Theorem, which asserts that every voting rule other than a dictatorship fails to be strategyproof [3, 4].

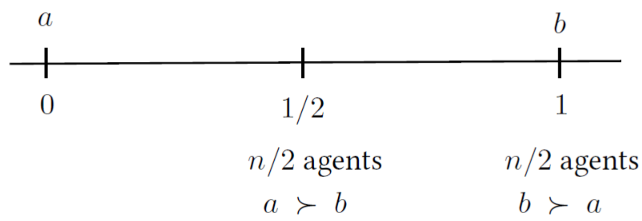

Figure 1: Since $n/2$ voters prefer a to b and $n/2$ voters prefer b to a, without loss of generality, any rule may select a as the winner for a social cost of $n/4 + n/2 = 3n/4$. On the other hand, the optimal candidate b has social cost $n/4$.

Anshelevich et al. proposed another approach to compare voting rules in 2015 [5]. Think of the voters and the candidates as points in a metric space, and suppose that each voter prefers candidates that are closer to her to ones that are further away, with the motivation being that voters usually prefer candidates that are closer to their own personal beliefs, and thus the preferences are determined by an "ideological distance".[2]

In this setting, it is often desirable to select a candidate that minimizes the sum of distances to the voters (i.e., the utilitarian social cost).[3] But finding such a candidate is not as easy as it sounds, since coming up with the exact distances between the corresponding points is a cognitively demanding task. Therefore, common voting rules operate on only voters' preference rankings, and may be unable to find an optimal candidate.

A natural measure of the quality of a voting rule is then its *distortion*, which is defined as the worst-case ratio between the social cost of a candidate selected by the voting rule and that of an optimal one. Thus, distortion measures how good a voting rule (using only ordinal information) is at approximating a candidate with minimum social cost.

It is easy to observe that no deterministic voting rule can achieve a distortion better than 3 (see Figure 1). Anshelevich et. al conjectured that the lower bound of 3 must be tight for deterministic rules, and claimed that the ranked pairs rule achieves this bound, but they were only able to show that the Copeland rule has distortion at most 5 [5].

Kempe showed later on that ranked pairs has distortion $\Theta(\sqrt{m})$ where m is the number of candidates [6]. In 2019, Munagala and Wang introduced a novel voting rule which improves the upper bound from 5 to 4.236 [7]. They also identified important properties that would suffice to achieve the conjectured bound of 3. In this paper, the conjecture was finally positively resolved by designing a simple novel voting rule called *plurality-matching* which satisfies these properties. The plurality-matching rule can be stated as follows:

For every candidate a, construct a bipartite graph $G(a)$ with the nodes of both sides corresponding to voters where an edge between voters i and j indicates that i prefers a to the top choice of j. Return a candidate a for which $G(a)$ admits a perfect matching. Interestingly enough, such a candidate is always guaranteed to exist.

[2]For example, one can imagine ideological axes from "liberal" to "conservative", or from "libertarian" to "authoritarian", and the preferences of the agents as points in the space defined by these axes.

[3]Another prominent measure of social cost is the median, under which the well-known Copeland rule is optimal among deterministic voting rules [5].

3. Briefly discuss the reason that you find this paper interesting and result(s) important.

The main result of this paper is clearly important since it resolves a long-standing and well-studied conjecture. However, the most interesting aspect, to me, of this paper is how they show that there always exists a candidate a for which $G(a)$ has a perfect matching. This result follows from their so-called *ranking-matching lemma* — which, in and of itself, is an interesting combinatorial fact, and does not even require rankings to be consistent with a metric space. This lemma can be stated as follows:

Given a candidate a, and normalized weight vectors p and q, define a vertex weighted bipartite graph $G_{p,q}(a)$ with one side corresponding to voters, and the other corresponding to candidates (weighted by vectors p and q respectively) where an edge between voter i and candidate c indicates that i prefers c over a. For any weight vectors p and q, there always exists a candidate a, for which $G_{p,q}(a)$ has a fractional perfect matching.[4]

Notice that if $p(i)$ is uniform for each voter i and $q(c)$ is proportional to the plurality of candidate c, then ranking-matching lemma implies that there exists a voter a for which $G(a)$ has a perfect matching. This implies that candidate a satisfies a sufficient condition for achieving the conjectured bound of 3 that was given by Munagala & Wangfor [7].[5]

That is, the ranking-matching lemma actually proves much more than what one needs to resolve the optimal metric distortion conjecture. I believe that this is a nice example of how sometimes proving a more general result might be easier while using induction as the proof technique. In this particular case, the proof requires the lemma to hold for all choices of p and q in smaller elections.

4. State one major open question posted or inspired by the paper. What might be your initial idea to address this problem?

What is also interesting about ranking matching lemma is that it gives rise to a family of voting rules by choosing different values for weight vectors p and q. For example, choosing both p and q as uniform proves the existence of a candidate a such that in any x% of the votes, for any x, candidate a weakly defeats at least x% of the candidates.

This guarantee is eerily reminiscent of the committee selection setting where the voting rule is supposed to choose k candidates, instead of just one. A "stable" committee is defined as k candidates that weakly defeat all candidates in any n/k votes (where n is the number of voters). Thus, ranking-matching lemma might be useful to approximate stable committees as well.

Getting a little bit of insight about what ranking-matching lemma is telling might be useful for related problems, since its proof does not give a clue about which candidates have such a matching. I played around with the peer selection setting where voters submit rankings over themselves. Given a candidate a, since both sides of $G(a)$ correspond to candidates in this setting, we can use a directed graph instead of a bipartite graph where we will have a cycle decomposition instead of a perfect matching. These disjoint cycles seem to be nicely capturing a notion of centrality.

[4] A fractional perfect matching is a generalization of perfect matching to vertex weighted graphs.

[5] They also provide a direct proof of this bound that is much shorter and simpler in this paper.

Since the conjecture is resolved for deterministic voting rules, the remaining big open problem in this line of research is determining the optimal metric distortion of randomized voting rules. Strangely, the random dictatorship rule which selects a voter uniformly at random and outputs her favorite candidate has distortion $3 - 2/n$, which beats any deterministic voting rule [8]. It is open whether one can achieve distortion better than 3 (while n or m goes to infinity) using a randomized rule. I find it quite curious that both axiom-based and distortion-based approaches currently fail to identify a voting rule that is better than a dictatorship.

References

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