A Blind Cow Called Bessie

Theoretical Thinking 1

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Abstract

This is a short review of the paper "Searching in an Unknown Environment: An Optimal Randomized Algorithm for the Cow-Path Problem" by Ming-Yang Kao, John H. Reif, and Stephen R. Tate [2] which proposes an optimal randomized algorithm to solve a classic problem in search theory: the cow-path problem.

1 Introduction

The subject of this short review is a paper which uses randomized algorithms and competitive analysis to improve a classic problem in search theory: the cow-path problem. First introduced and solved by Baeza Yates et al., the problem in its simplest form considers a cow (she's usually called "Bessie") which is initially standing at a fork in the road (the original formulation does not actually use cows, but it has become de-facto standard). She knows a green pasture lies on just one of the two roads, but doesn't know how far away it is. Bessie also has poor eyesight, so she can't see any distance down the road. Clearly, she can't simply commit to just one road (if the pasture is on the other road, she'll be walking forever!). Her strategy must involve walking a distance down the left road then turning around and walking some distance down the right road — alternating until she finds the pasture. The cow-path problem asks the question: "What strategy minimizes the distance that Bessie must walk before finding the pasture"?

This is a fundamental problem in search theory because of its applications to a wide variety of problems. While most obviously useful for robotics applications, it also directly applies to other search tasks like hybrid algorithm execution (how much of an algorithm should you execute before trying another algorithm?). This problem is also interesting because it lends itself nicely to study by competitive analysis, where we're interested in a finding an *online* algorithm that always does well compared to some optimal *offline* algorithm. In other words, if Bessie exactly where the pasture was, say n steps away on the left road, she could walk directly towards it and arrive in exactly n steps. Bessie is blind though, so she must execute an *online* algorithm, where she doesn't have full knowledge of the system. We hope to find a strategy that Bessie can execute which is "good" compared to the offline algorithm. An algorithm is c-competitive if it terminates in no more than $cn + d$ steps for some constants c and d (where n is the number of steps required by an optimal offline algorithm for the same problem instance).

It turns out the optimal online deterministic algorithm is quite simple and intuitive: Bessie should walk 1 unit down the left road, then 2 units down the right road, 4 units down the left again, and so on — doubling the distance she travels every time. In round i, Bessie will travel 2^i units down the left (or right, if i is even) road. It is relatively simple to show that Bessie finds the pasture in at most $9n$ steps and that no algorithm can do better. One proof for the latter statement involves reasoning about where an adversary could "place" the pasture in order to maximize the distance Bessie must travel. That the adversary can effectively choose the worst position for the pasture stems directly from the fact that Bessie is following a deterministic algorithm — the adversary knows exactly what Bessie is going to do next at all times. What if the adversary didn't know this, though? What if Bessie's next move was unpredictable?

2 A Randomized Algorithm

One of the reasons I like this paper so much is that it does a good job at demonstrating how randomized algorithms can mitigate the damage worst-case scenarios can do to the performance of an online algorithm. Rather than walk a fixed amount down each path, Kao et al. suggest randomizing the order Bessie walks down the paths (left then right or right then left) and the distance she walks down each path (note these need only be randomly generated once at the beginning of the algorithm). In particular, ϵ is sampled uniformly randomly from the interval $[0, 1]$ and Bessie walks a distance $r^{i+\epsilon}$ down the appropriate path in round i (for some constant $r > 1$, which will be discussed later). Intuitively, this randomization hinders the adversary's ability to place the pasture in an inconvenient location, since it does not know how Bessie is going to behave.

Kao et al. find the optimal randomized algorithm by first presenting the competitive ratio of the randomized algorithm in terms of r and then finding the r which minimizes the ratio. They also study the w-path variant of the problem, where there are w paths (instead of just two) that must be explored. The deterministic algorithm for the w-path variant has a competitive ratio asymptotically equal to $1 + 2ew$ (and exactly 9 in the $w = 2$ case, as explained before). The randomized algorithm, however has a competitive ratio asymptotically equal to $\kappa w + o(w)$ (where $\kappa \approx 3.088$). For the case of $w = 2$, the competitive difference between the algorithms is most profound. The competitive ratio of the deterministic algorithm is 9, while that of randomized algorithm is approximately 4.511. The randomized algorithm is almost twice as competitive!

3 Conclusion

The cow-path problem is a fundamental problem in search theory, a field which has hundreds of publications every year is ever-broadening. The problem has been studied for different topologies (on the plane, the ring, in the graph, etc.) and (most interesting to me) under multi-agent paradigm. When considering multi-agent exploration, other questions arise: can agents communicate? if so, how (wirelessly, face-to-face)?, do agents have memory or are they oblivious?, do agents have a shared sense of location (i.e. GPS) or do they have separate coordinate systems? Most importantly, how do the answers to these questions affect the competitiveness of an online algorithm? These are just a few of the questions addressed by researchers who study search by mobile agents. By so dramatically improving the result (at least for $w = 2$) of the fundamental problem in this field, Kao et al. suggest that an answer to any of the aforementioned questions should be followed up by: "Okay, what about a randomized algorithm?"

References

[1] Ricardo A Baeza Yates, Joseph C Culberson, and Gregory JE Rawlins. Searching in the plane. *Information and computation*, 106(2):234–252, 1993.

[2] Ming-Yang Kao, John H Reif, and Stephen R Tate. Searching in an unknown environment: An optimal randomized algorithm for the cow-path problem. *Information and Computation*, 131(1):63–79, 1996.