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TOPICS IN NUMERICAL ANALYSIS FOR HIGH SCHOOL MATHEMATICS

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AT CASSVILLE High School during the 1968–69 school year, the mathematics department initiated an exploratory computer mathematics program and received a gratifying response from the students. Whether this response was based upon the popular fascination that computers seem to hold in our society or upon the intrinsic interest that working with a computer affords, we are not certain. However, we are certain that this high degree of motivation has held the attention of students to the extent that it has permitted a treatment of more sophisticated topics from numerical analysis than had been initially thought possible. Topics ranging from numerical integration to calculation of five-by-five determinants and Lagrangian interpolation have successfully been taught.

The participating students can be described as good, but not necessarily exceptional, with an interest in math and science. In particular, the students were academically in the top half of their class and had completed or were taking a second course in algebra. None of the students had studied calculus, and the class was not limited to seniors; juniors and even a few sophomores also participated.

The class used various computing facilities. Numerical analysis processing was carried out chiefly at a nearby computing center (the University of Arkansas), which utilizes an IBM 7040/44 computing system. Student-prepared coding forms were sent to the computing center where the programs were key punched and processed. The printed output was then returned to the students. Most programming was done with the FORTRAN IV language, although the BASIC language was occasionally used.

A teaching approach was used that tended to emphasize the role of an advisor or consultant over that of a lecturer, especially after basic programming had been introduced. An important element in teaching was that of salesmanship. Typically, in presenting a new topic, the instructor would point out the futility of trying to solve a problem *without* a computer, perhaps even challenging the students to try to do so, thus demonstrating the value of the computer in handling such problems.

A major duty of the teacher was selecting and presenting topics from numerical analysis that did not explicitly use calculus or differential equations. Unfortunately, since these topics have traditionally been taught in advanced college courses, a calculus or power series background is utilized in the textbooks commonly available. Therefore, it was necessary for the teacher to study such topics closely and where possible formulate a treatment that did not explicitly rely on calculus or other advanced mathematics. Surprisingly, there are many such topics that yield to a noncalculus treatment.

Among the varied techniques presented, the most modest topic was numerical integration. This presentation was made without specifically mentioning integration. Instead, the area under a curve was approached geometrically. Both the rectangle rule and the trapezoidal rule were derived graphically, and the students subsequently compared the results of using these techniques with several problems. In addition, Simpson's Rule was presented, but no attempt was made to derive it. Techniques for the addition and multiplication of matrices and the evaluation of determinants were discussed, and students wrote programs that executed these operations. While determinants and matrices are often presented in algebra courses, one rarely evaluates five-by-five determinants or multiplies ten-by-ten matrices as was possible with the computer. The theory of linear equations was also touched upon. With the help of teacher-prepared flow charts the students were able to program Gaussian Elimination and solve some linear programming problems. Another important topic of a somewhat different nature was Lagrangian interpolation, a numerical technique for fitting the best curve to a set of data points. In one application the students were given six specific values of the common logarithm between log 2 and log 6, and they were able to interpolate anywhere between these points with an accuracy of no less than four decimal places. The final major topic of numerical analysis presented was methods for solution of nonlinear equations. This topic will be treated in more detail after a summary of class reaction to numerical integration, matrices, linear programming, and Lagrangian interpolation.

Perhaps the topics most readily understood by the students were the addition and multiplication of matrices and the rectangle rule for integration. These topics had fewer applications than the others; thus students who had no previous experience with matrices tended to see little significance in what they were doing. The solution of nonlinear equations and Lagrangian interpolation, while somewhat more sophisticated, proved to be more rewarding for students who mastered them. For Lagrangian interpolation a general derivation of the interpolating polynomial of arbitrary degree was presented, but only the highly competent students fully understood its significance. However, all students were able to program the interpolation procedure and generally seemed to become convinced of its significance because of the impressive results obtained. Students who had experience with fitting curves to data in laboratory sciences were particularly impressed, and a few were allowed to analyze data from physics and chemistry experiments as a project for extra credit. The least successful topic was linear programming. Although we are still not certain of the reason for relative failure with this important subject, the programming of nontrivial linear programming problems does not involve advanced mathematics, but does in general require fairly elaborate use of algebra. Apparently it was difficult for the students to grasp and maintain interest in these rather lengthy algebraic and programming procedures. We have noted a somewhat analogous reaction to lengthy proofs in mathematics.

The solution of nonlinear equations was a topic that aroused special interest. Linear iteration, really quite simple to program, served impressively as a starting point. This was followed by the secant method for the solution of a single nonlinear equation, and by the time the school year ended students were tackling systems of nonlinear equations. The secant method is a topic of moderate difficulty with which we had success. As a specific illustration of a numerical analysis technique and our approach to teaching, this topic is presented in more detail.

```
C     SECANT METHOD FOR EQUATION IN THE FORM F(X)=0
C     INITIAL GUESSES X1=1. X2=1.5
      F(X)=SIN(X)-3. * ALOG10(X)
      Xl=1.
      X2=1.5
      DO 10 I=1, 25
      X3=X2-F(X2 * (X2-Xl) / (F(X2)-F(Xl) )
      Y=F(X3)
      WRITE (6,20) X3, Y
   20 FORMAT (1X, 2E16.6)
C     STOP IF SUCCESSIVE ROOTS LESS THAN .000001 APART
      IF(ABS(X3-X2). LT. .000001) STOP
      Xl=X2
      X2=X3
   10 CONTINUE
      WRITE (6, 30)
   30 FORMAT(36H FAILED TO CONVERGE IN 25 ITERATIONS)
      STOP
      END
```

FIGURE 1

The secant method was introduced by presenting an example which would be difficult or impossible for the students to solve without numerical techniques. They were asked to solve the deceptively simple equation

$$\sin x - 3 \log x = 0.$$

Initial confidence gave way as the students exhausted their limited algebraic techniques: factoring, quadratic formula, and so forth. When they had convinced themselves of their inability to solve this equation, the secant method algorithm was presented.

To solve an equation in the form $f(x) = 0$,

1. Make two initial guesses x_1, x_2 to the root.

2. Generate successive approximations using the formula

$$x_{i+2} = x_{i+1} - \frac{(x_{i+1} - x_i)f(x_{i+1})}{f(x_{i+1}) - f(x_i)} \quad i = 1, 2, 3, \ldots$$

3. Stop the calculation when the difference of two successive roots $|x_{i+2} - x_{i+1}|$ is less than the desired accuracy.

The following points were emphasized: (1) the equation must be written (or rewritten) in the form $f(x) = 0$, (2) any guesses x_1, x_2 can be tried, but an effort should be made to have them as good as can be easily estimated, (3) an iterated procedure is being used where two previous values are used to get a third improved value, and so on, and (4) sometimes the secant method will not work, so test the final approximation, say x_n, to see if $f(x_n) \simeq 0$. Although this may not seem to be a simple procedure, once programmed for a computer it can be applied with ease and will enable an equation to be solved which would be insoluble without techniques of numerical approximation. The teacher might demonstrate the first iteration using hand calculations, and finally present the results obtained by computer.

Figure 1 shows the FORTRAN IV program used to solve the equation, and table 1 lists the results in the tabular sequence

TABLE 1

	X	$F(X)$
FIRST ITERATION	2.13025	1.37747×10^{-1}
SECOND ITERATION	1.98722	1.98039×10^{-2}
THIRD ITERATION	2.00520	6.49855×10^{-4}
FOURTH ITERATION	2.00581	-3.30061×10^{-6}
FIFTH ITERATION	2.00581	2.23517×10^{-7}

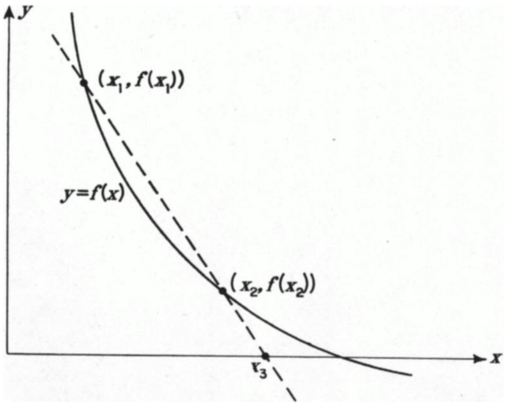

FIGURE 2

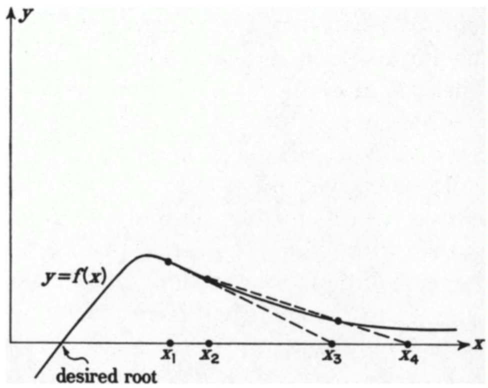

FIGURE 3

as printed out by the computer. The final result, 2.00581, is accurate to five decimal places. The students were actually never shown the program in figure 1; just the results. They were provided a flow chart and had to write their own programs for solving a series of assigned problems arranged in increasing order of difficulty. The problems ranged from single root equations for which any initial approximation would work, to multirooted equations that required computer graphing to discover starting approximations. An example of the more difficult problems was to find all the real roots $(-.62236, 63.84175)$ of $x^4 - 64x^3 + 10.02x^2 + 5.544x - 16.0083 = 0$ with no starting hints. All of the students were able to solve some of the equations, and many solved all of them.

After the students had worked problems and proved to themselves that the secant method does indeed work, a graphical derivation of the algorithm was presented. A secant line is drawn through the two points on $y = f(x)$ corresponding to the initial approximations x_1, x_2 as shown in figure 2. The x intercept, x_3, of the secant line is then considered to be the new approximation to the actual x intercept of $y = f(x)$. The slope between points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is the same as the slope between $(x_2, f(x_2))$ and $(x_3, 0)$ or, algebraically,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0 - f(x_2)}{x_3 - x_2}.$$

This can be rewritten in the form

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}.$$

By presenting this derivation in a more general form, the teacher can provide the result given in the algorithm.

In the presentation of the secant method, specific discussion of convergence criteria was avoided. It was found more useful to instruct the students to try the method and see if it worked. If the method did not work, it was suggested that the students try different initial guesses or perhaps even one of the other iteration methods that had been discussed in class. The most common reason for divergence of the secant method is the existence of turning points or inflections in the graph of $f(x)$ between the starting guesses x_1 and x_2 and the true

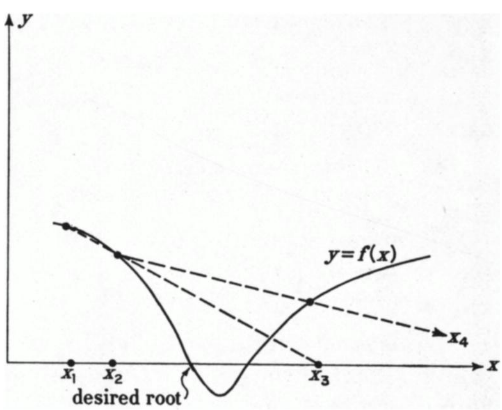

FIGURE 4

root, or between the starting guesses and the improved value x_3. Figures 3 and 4 illustrate cases of initial choices that lead to divergent sequences of iterations (x_1, x_2, x_3, . . .). In both cases divergence could have been prevented with initial guesses closer to the desired root. For curves with slopes of large magnitude, the calculation termination criterion of $|x_{i+2} - x_{i+1}|$ being small might still allow a relatively large error of $f(x_{i+2})$ compared to zero. In these cases a termination criterion of $|f(x_{i+2})|$ becoming small would be more suitable. Sometimes both criteria are used jointly, but if the approximated root x_{i+2} is to be used for further calculations, it is usually more important that its error be small than that $f(x_{i+2})$ be close to zero. For curves with slopes nearer zero, $|f(x_{i+2})|$ can be small with the error in x_{i+2} still large. In any case, for simplicity in the teacher's initial presentation, use of the $|x_{i+2} - x_{i+1}|$ termination criteria is suggested.

The secant method for solving nonlinear equations was only one of many topics of numerical analysis presented to the high school students. The contents of the course seemed less limited by the enthusiasm of the students than by the teacher's ability and time to find and adapt topics from the large field of numerical analysis.

BIBLIOGRAPHY

Conte, S. D. *Elementary Numerical Analysis.* New York: McGraw-Hill Book Co., 1965.

Householder, A. H. *Principles of Numerical Analysis.* New York: McGraw-Hill Book Co., 1953.