MATH 532 Discussion 9 Nov 2020

Card shuffling. GRS → Gilbert, Reeds, Shannon

Riffle Shuffle

N-card deck.

Cut deck

$$\text{Prob}\left(\begin{matrix}\text{cut } k \\ \text{cards} \\ \text{off} \\ \text{top}\end{matrix}\right) = \frac{\binom{N}{k}}{2^N}$$

Drop cards

$$\text{Prob}\left(\begin{matrix}\text{drop from} \\ L \text{ hand}\end{matrix}\right) = \frac{\#L}{\#L + \#R}$$

Silly Example

$N=3$ AKQ (A / K / Q)

- $(\emptyset, \text{AKQ})$: $1/8$
  - $\xrightarrow{1}$ Q $(\emptyset, \text{AK})$
  - $\xrightarrow{1}$ KQ $(\emptyset, \text{A})$
  - $\xrightarrow{1}$ AKQ $(\emptyset, \emptyset)$
  - AKQ: $\underbrace{\tfrac{1}{8}\cdot 1\cdot 1\cdot 1}_{1/8}$
- (A, KQ): $3/8$
- (AK, Q): $3/8$
  - $\xrightarrow{2/3}$ K (A, Q)
    - $\xrightarrow{1/2}$ AK $(\emptyset, \text{Q})$ $\xrightarrow{1}$ QAK $(\emptyset, \emptyset)$: $\underbrace{\tfrac{3}{8}\cdot\tfrac{2}{3}\cdot\tfrac{1}{2}\cdot 1}_{1/8}$
    - $\xrightarrow{1/2}$ QK $(\text{A}, \emptyset)$ $\rightarrow$ AQK $(\emptyset, \emptyset)$: $\underbrace{\tfrac{3}{8}\cdot\tfrac{2}{3}\cdot\tfrac{1}{2}\cdot 1}_{1/8}$
  - $\xrightarrow{1/3}$ Q $(\text{AK}, \emptyset)$
    - $\xrightarrow{1}$ KQ (A, Q) $\rightarrow$ AKQ $(\emptyset, \emptyset)$: $\underbrace{\tfrac{3}{8}\cdot\tfrac{1}{3}\cdot 1\cdot 1}_{1/8}$
- $(\text{AKQ}, \emptyset)$: $1/8$

$Q_2(\sigma)$ = prob (σ results from a single shuffle).

	AKQ	AQK	KAQ	KQA	QAK	QKA
$Q_2(\sigma)$	$1/2$	$1/8$	$1/8$	$1/8$	$1/8$	0
$U(\sigma)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$n=3$ $TV = 1/3$

TOTAL VARIATION ≤ 1

$$\|Q_2 - U\|_{TV} = \frac{1}{2} \sum_{\sigma \in S_n} |Q_2(\sigma) - U(\sigma)|$$

≥ 0

Q_a an a-shuffle

Cut: $\frac{1}{a^n}\binom{n}{k_1 k_2 \cdots k_a}$ Drop: $\frac{\#H_i}{\sum_{j=1}^{a} \#H_j}$

Theorem (Bayer-Diaconis 1992)

$$Q_a * Q_b = Q_{ab}$$

Ex. 3-shuffles w/2 heads = 1 shuffle w/8 heads.

$$Q_2 * Q_2 * Q_2 = Q_8$$

$$Q_a(\sigma) = \binom{n+a-r}{n}\frac{1}{2^a}$$

r = # rising sequences in σ