MATH 532 Discussion WED 19/08

$\mathbb{N} = \{0,1,2,\dots\}$ Natural numbers $[n] = \{1,2,\dots,n\}$
$\mathbb{P} = \{1,2,3,\dots\}$ Positive integers

Let's get a formula for $f(k,n)$ where $k,n \in \mathbb{P}$ defined by

$$f(k,n) = \#\{(S_1,\dots,S_k) \mid S_i \subseteq [n] \ \& \ S_1 \cap S_2 \cap \cdots \cap S_k = \phi\}.$$

$\underline{k=1}$ $S_1 \subseteq [n]$ $S_1 = \phi$ $f(1,n) = 1$

$\underline{n=1}$ $S_i \subseteq [1]$ $f(k,1) =$

$S_i = \begin{cases} \phi \\ [1] \end{cases}$ one of S_i's $= \phi$
others anything.

$S_1 = \phi$ 2^{k-1} choices for S_i $i = 2,\dots,k.$

$S_1 \neq \phi$

<u>Better</u>: only bad thing is if $S_i = [1]$ $\forall i$.

$\underline{f(k,1) = 2^k - 1}$

$$f(2,n) = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}$$

choose S_1 freely.

choose S_2 from $[n] \setminus S_1$.

$f(k,n)$

Choose $S_1,\dots,S_{k-1}$ freely (?)

$\boxed{S_1 \cap \cdots \cap S_{k-1} = T} \leftarrow \#T = i$

Choose S_k from $[n] - T$. $\leftarrow$ 2^{n-i}

$$f(k,n) = \sum_{i=0}^{n} \binom{n}{i} f(k-1,n-i)\, 2^{n-i}$$

choose T

choose $S_1,\dots,S_{k-1}$ s.t. $S_1 \cap \cdots \cap S_{k-1} = T$.

choose S_k s.t. $S_k \cap T = \phi$

$$\overbrace{(S_1 \setminus T) \cap \cdots \cap (S_{k-1} \setminus T) = \phi}^{f(k-1,n-i)}$$

$$f(k,n) = \#\{(S_1,\dots,S_k) \mid S_i \subseteq [n] \text{ \& } S_1 \cap S_2 \cap \cdots \cap S_k = \emptyset\}.$$

$$= \sum_{i=0}^{n} \binom{n}{i} f(k-1, n-i)\, 2^{n-i}$$

Choose $T \subseteq [n]$ of size i

Choose $S_1, \dots, S_{k-1}$ s.t. $S_1 \cap \cdots \cap S_{k-1} = T$

Choose S_k s.t. $S_k \cap T = \emptyset$

Aside:
$f(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ $\quad g(x) = \sum_{n\geq 0} g_n \frac{x^n}{n!}$

Convolution product:

$$f(x)\, g(x) = \sum_{n\geq 0} \left(\sum_{i=0}^{n} \binom{n}{i} f_{n-i}\, g_i \right) x^n$$

Take $g(x) = e^x = \sum_{n\geq 0} \frac{x^n}{n!}$

$g_i = 1.$

$$f(1,n) = 1.$$

$$F_k(x) = \sum_{n\geq 0} f(k,n) \frac{x^n}{n!}$$

$$F_k(x) = e^x F_{k-1}(2x) \qquad F_1(x) = e^x$$

$$= e^{(x + 2x + 4x + \cdots + 2^{k-1}x)} = e^{((2^k-1)x)}$$

$$= \sum_{n\geq 0} (2^k-1)^n \frac{x^n}{n!}$$

$$F_1(x) = e^x$$

$$F_2(x) = e^x F_1(2x) = e^x e^{2x} = e^{x+2x}$$

$$F_3(x) = e^x F_2(2x) = e^x e^{2x+4x} = e^{x+2x+4x}$$

$$f(k,n) = (2^k-1)^n$$

For each $i = 1, \dots, n$, put i into any of $S_1, \dots, S_k$ but not all of them

$2^k - 1$