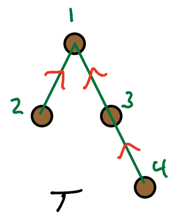
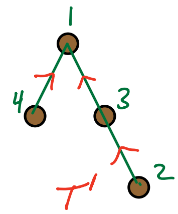
MATH 532 Discussion 13 Nov 2020





MATRIX-TREE THEOREM

Given a labelled tree T, assign a monomial x^T $\qquad x = x_1, x_2, x_3 \ldots$

$$x^T = x_1^{\deg(1)} x_2^{\deg(2)} \cdots$$

$$= \prod_{i-j \text{ edge of } T} x_i x_j$$

x^T: $x_1x_2 \cdot x_1x_3 \cdot x_3x_4$
$= x_1^2 x_2 x_3^2 x_4$

$\deg(x^T) = 2 \times \#\text{edges}$
$= 2(n-1)$

<u>Theorem</u> (Cayley) $\displaystyle\sum_{\substack{T \text{ tree} \\ \text{on } [n]}} x^T = x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}$

<u>Specialize</u>: $x_i = 1 \ \forall i$: $\displaystyle\sum_{\substack{T \text{ tree} \\ \text{on } [n]}} 1 = \#\text{trees on } [n] = (\overbrace{1+1+\cdots+1}^{n})^{n-2} = n^{n-2}$

<u>Comment</u>: x^T does not uniquely determine T.

$$x^{T'} = x_1^2 x_2 x_3^2 x_4$$

Change x^T as follows: $\displaystyle x^T = \prod_{i \to j \text{ edge}} x_{ij}$

Rooted trees
direct edges
toward the
root.

x^T: $x_{21} x_{31} x_{43}$ $\qquad x^{T'} = x_{41} x_{31} x_{23}$

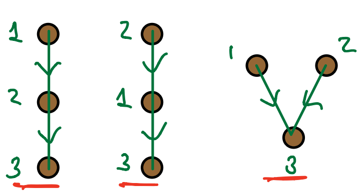
<u>NOTE</u>: Set $x_{ij} \mapsto x_i x_j$
get Cayley's result.

A <u>directed</u> (rooted) forest is uniquely determined by its monomial

Roots $= \{ i \mid i \text{ never first index} \}$

Given $R \subseteq [n]$ $\qquad \displaystyle F_{n,R}(x) = \sum_{\substack{F \text{ rooted forest} \\ \text{on } [n] \text{ w/ roots } R}} x^F$

<u>Ex.</u>
$R = \{3\}$
$n = [3]$



$$F_{3,\{3\}} = x_{12}x_{23} + x_{21}x_{13} + x_{13}x_{23}$$

homogeneous of degree $n - |R|$.

Theorem For $R \subseteq [n]$, we have

$$(*) \quad F_{n,R}(x) = \det\left(M_{n,R}(x)\right)$$

where

$$M_n(x) = \begin{bmatrix} (x_{12}+\cdots+x_{1n}) & -x_{12} & \cdots & -x_{1n} \\ -x_{21} & (x_{21}+x_{23}+\cdots+x_{2n}) & \cdots & -x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -x_{n1} & -x_{n2} & \cdots & (x_{n1}+x_{n2}+\cdots+x_{nn}) \end{bmatrix}$$

$$(M_n)_{ij} = \begin{cases} -x_{ij} & \text{if } i \neq j \\ \sum_{k \neq i} x_{ik} & i = j \end{cases}$$

$M_{n,R}$ means delete rows/cols in R.

Ex. $$|M_{3,\{3\}}| = \left|\begin{pmatrix} x_{12}+x_{13} & -x_{12} & -x_{13} \\ -x_{21} & x_{21}+x_{23} & -x_{23} \\ -x_{31} & -x_{32} & x_{31}+x_{32} \end{pmatrix}\right| = \left|\begin{matrix} (x_{12}+x_{13})(x_{21}+x_{23}) \\ -x_{12}x_{21} \end{matrix}\right| = x_{13}x_{21} + x_{13}x_{23} + x_{12}x_{23}$$

root

root

Proof. $R = \emptyset$ & $n > 0$

$F_{n,\emptyset} = 0$ (need a root!)

$M_{n,\emptyset} = M_n$ singular $\Rightarrow \det M_n = 0$

$R = [n]$

$F_{n,[n]} = 1$

$M_{n,[n]}$ empty matrix

$\det(\text{empty matrix}) = 1$.

(1) In each term, $\exists j$ s.t. x_{ij} never occurs for any i.

- $F_{n,R}$: j any leaf
- $M_{n,R}$: det. is homog. of degree $n-|R|$, & $R \neq \emptyset \Rightarrow$ degree $< n$. Pigeonhole principle.

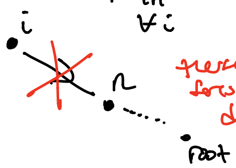
Enough to show $(*)$ holds when $x_{ij} \equiv 0$ for each possible j.

* By symmetry, enough to consider $j = n$.

$$F_{n,R}(x)\Big|_{\substack{x_{in}=0\\ \forall i}} \longrightarrow \begin{cases} F_{n-1,R\setminus\{n\}}(x) & \text{if } n \text{ root} \\ (x_{n1}+\cdots+x_{n\,n-1})\,F_{n-1,R}(x) & n \text{ not root} \end{cases}$$

Only count forest in which n is a leaf

x_{in} never occurs.



these forest die

leaf goes somewhere.

$$M_{n,R}(x)\Big|_{\substack{x_{in}=0\\ \forall i}} = \left[\begin{array}{ccc|c} & M_{n-1}(x) & & 0 \\ \hline \underbrace{-x_{n1} & \cdots & -x_{n\,n-1}}_{n-1 \text{ terms}} & x_{n1}+\cdots+x_{n\,n-1} \end{array}\right]$$

$$\det(M_{n,R}(x))\Big|_{\substack{x_{in}=0\\ \forall i}} = \begin{cases} \det M_{n-1,R\setminus\{n\}} & \text{if } n\in R \\ (x_{n1}+\cdots+x_{n\,n-1})\det M_{n-1}(x) \end{cases}$$

By induction, we are done! //