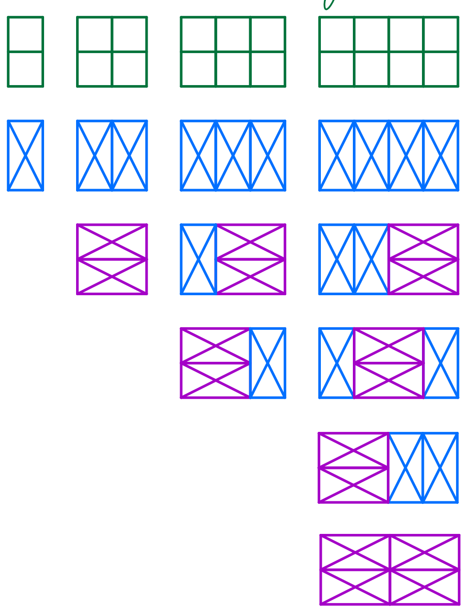
MATH 532 Lecture 1 How to Count

Basic Problem: Given an infinite collection of finite sets S_i where $i \in I$ (here I might be $\mathbb{N}$ or $\mathbb{Z}$ or some other countable set), what does it mean to count S_i?

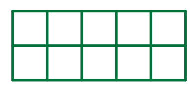
Possible Answers:

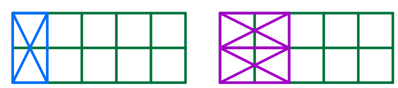
1. Closed formula for $f(i) = \# S_i$.
   ↳ uses well known functions, no summations
2. Recurrence relation for $f(j)$ in terms of $f(i)$ for $i<j$
   ↳ how many terms deep is acceptable?
3. Algorithm for computing $f(i)$
   ↳ what is the complexity? Should be less than $f(i)$.
4. Estimate for $f(i)$ asymptotically
   ↳ how accurate
5. Generating function closed form for $\sum_{i \in I} f(i) x^i$
   ↳ polynomial, rational, analytic

Example: S_i = # ways to fill $2 \times i$ plate with 1×2 LEGO bricks.



3. Generate them systematically.

2. Observe  tiled by one of two beginning moves:



And so $f(i) = f(i-1) + f(i-2)$ $i \geq 2$

4. Suppose $f(i) \sim 2^{ci}$ some c.

$$2^{ci} = 2^{c(i-1)} + 2^{c(i-2)} \quad \div 2^{(i-2)}$$

$$\Rightarrow 2^{2c} = 2^c + 1$$

quadratic formula

$$\Rightarrow 2^c = \frac{1 \pm \sqrt{5}}{2}$$

5. $F(x) = \sum_{i \geq 0} f(i) x^i$ satisfies $F(x) = x + \sum_{i \geq 2} f(i-1) x^i + f(i-2) x^i$

$$= x + x F(x) + x^2 F(x)$$

$$F(x) = \frac{x}{1 - x - x^2}$$

1. Use partial fractions to write $F(x) = \frac{a}{1 - \alpha x} + \frac{b}{1 - \beta x}$

$\alpha = \frac{1+\sqrt{5}}{2}$ $\beta = \frac{1-\sqrt{5}}{2}$

$a = \frac{1}{\sqrt{5}}$ $b = -\frac{1}{\sqrt{5}}$

$$= \sum_{i \geq 0} \frac{1}{\sqrt{5}} \left(\alpha^n - \beta^n \right) x^n$$