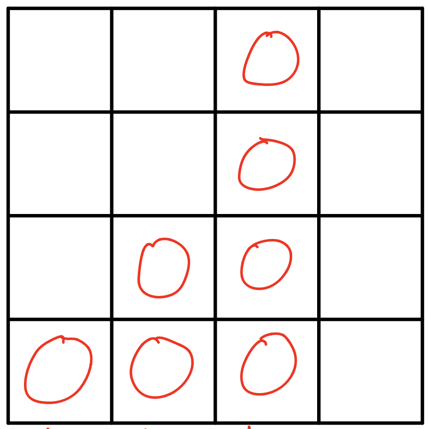
MATH 532 Discussion

$(s_i) = 1 \quad 1 \quad 2$



$b_1 \quad b_2 \quad b_3$

$b_1 - 0 \quad b_2 - 1 \quad b_3 - 2$

Rook Theory

r_k = # ways to place k nonattacking rooks onto a board $B \subseteq [n] \times [n]$

$B = (1, 2, 4)$ — reverse of a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m)$

Ferrers boards $b_1 \leq b_2 \leq \cdots \leq b_m$

Theorem B board of shape $(b_1 \leq b_2 \leq \cdots \leq b_m)$

Set $\boxed{s_i = b_i - \overbrace{i+1}}$. Then

$$\sum_k r_k \, (x)_{m-k} = \prod_{i=1}^{m} (x + s_i)$$

falling factorial

$$\underbrace{x(x-1)\cdots(x-(m-k-1))}_{m-k \text{ terms}}$$

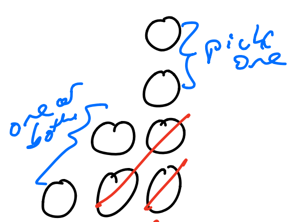
Ex. $$\sum_{k=0}^{3} r_k \, (x)_{3-k} = \prod_{i=1}^{3} (x + s_i)$$

$r_0 = 1$

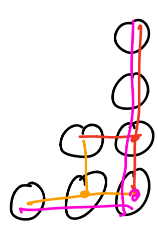
$r_1 = 7$

$r_2 = 10$

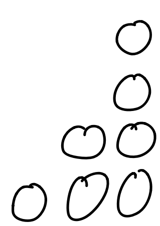
$r_3 = 2$



$1 + 2 \times 2 = 5$



$2 + 3 + 1 = 6$



$$1(x)_3 + 7(x)_2 + 10(x)_1 + 2(x)_0$$

$$x(x-1)(x-2) + 7x(x-1) + 10x + 2$$

$$(x^3 - 3x^2 + 2x) + (7x^2 - 7x) + 10x + 2$$

$$x^3 + 4x^2 + 5x + 2$$

$$(x+1)(x+1)(x+2)$$

$$(x^2+2x+1)(x+2)$$

$$x^3 + 4x^2 + 5x + 2$$

PROOF:

$$\sum_k r_k (x)_{m-k} = \prod_{i=1}^{m} (x+s_i)$$

FACT: If two polynomials in x agree for ∞ly many integer values of x, then they agree as polynomials in x in general.

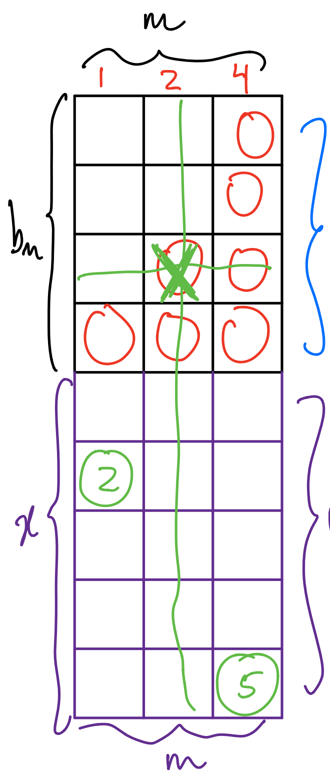
pf. $p(x) - q(x)$ has ∞ly many roots!

Regard x as a really big integer.
Interpret both sides as counting same thing.

$$\sum_{k=0}^{m} r_k \, (x)_{m-k} = \#\text{ways to place } m \text{ nonattacking rooks onto } B' = B \cup ([x] \times \underline{[m]})$$

place k rooks onto B

choose m-k distinct elements from $\underline{1,2,\dots,x}$ } place m-k rooks



k rooks.

k=1

m-k=2

m-k rooks anywhere.

$(x)_{m-k}$

only got to choose row

ordered left to right.

$$\prod_{i=1}^{m} (x + s_i) = \#\text{ways. place } m \text{ rooks col by col.}$$

rook 1: $x + b_1$

rook 2: $x + b_2 - 1$

rook 3: $x + b_3 - 2$

$s_1 = b_1 - 0$

$s_2 = b_2 - 1$

$s_3 = b_3 - 2$

$s_i = b_i - (i-1)$ //