MATH 532 Discussion

Partially Ordered SETs

Equivalence Relations on S

- reflexive for all $x \in S$ $x = x$
- symmetric forall $x, y \in S$ if $x = y$, then $y = x$
- transitive $\forall x, y, z \in S$, if $x = y$ & $y = z$ then $x = z$

A partial order on a set S is a relation $\preceq$ (\prec \preceq)

- reflexive $\forall x \in S$ $x \preceq x$
- antisymmetry if $x \preceq y$ & $y \preceq x$ then $x = y$
- transitive $\forall x, y, z \in S$ if $x \preceq y$ & $y \preceq z$ then $x \preceq z$.

NOTATION $P = (S, \preceq)$ set, ordering. $\preceq_P$

EXAMPLES • $P = ([n], \leq)$ $i \preceq_P j$ if $i \leq j$ usual

$\{1 < 2 < 3 < \cdots < n\}$ TOTAL / LINEAR ORDER

- B_n = Boolean poset on $2^{[n]}$

  Objects are subsets of $[n]$
  order is containment.

  $n=3$: $\emptyset, \{1\}, \{2\}, \{3\}, \{12\}, \{13\}, \{23\}, \{123\}$

  $\{2\} \preceq \{1,2\}$

  $\{2\}$ & $\{1,3\}$ are incomparable

$(2^{[n]}, S \preceq T \text{ if } \#S \leq \#T) \cong [n+1]$

• $D_n = \left([n],\ i \underset{D_n}{\leq} j \text{ iff } j \text{ is evenly divisible by } i\right)$
$j \bmod i \equiv 0$

$\underline{n=12}$ What #s are ≤ 12? $1,2,3,4,6,12$
What #s are ≤ 8? $1,2,4,8$

incomparable: i incomp j iff $\gcd(i,j) \neq i$ or j

1 is the unique minimal element. $\hat{0}$

• Π_n partial order on set partitions of $[n]$ given by refinement.

$(\{1,3,4\}, \{2,6\}, \{5\})$ of $[6]$

$\wedge |$

$\{1,3,4,5\}\ \{2,6\}$

$\{B_i\} \leq \{C_i\}$ if $\forall i\ B_i \subseteq C_j$ some j.

minimal element? $\{1\}\{2\} \ldots \{n\}$
maximal element? $\{1,2,\ldots,n\}$.

• $B_n(q)$ all subspaces of $\mathbb{F}_q^n$ ordered by inclusion.

HASSE DIAGRAM
b
$|$ if $a \leq b$ and
a $\nexists c$ s.t. $a \leq c \leq b$
$a \neq b$

List All Posets on 4 Objects