MATH 532 Discussion 30/10/2020

Recall Möbius function of P:

$\mu(s,s) = 1$

$\mu(s,u) = -\sum_{s \le t < u} \mu(s,t)$

Möbius Inversion

$g(t) = \sum_{s \le t} f(s)$ iff

$f(t) = \sum_{s \le t} g(s)\mu(s,t)$

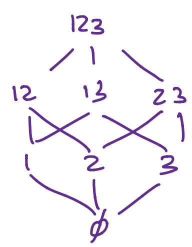
Ex. $P = \mathbb{N}$

$$\mu(i,j) = \begin{cases} 1 & i=j \\ -1 & i+1=j \\ 0 & \text{else} \end{cases}$$

$g(n) = \sum_{i=0}^{n} f(i)$ iff $f(n) = g(n) - g(n-1)$

Fundamental Theorem of Calculus

Ex. B_n



$\mu(S,T) = (-1)^{\#(T-S)}$

$g(T) = \sum_{S \subseteq T} f(S)$ iff

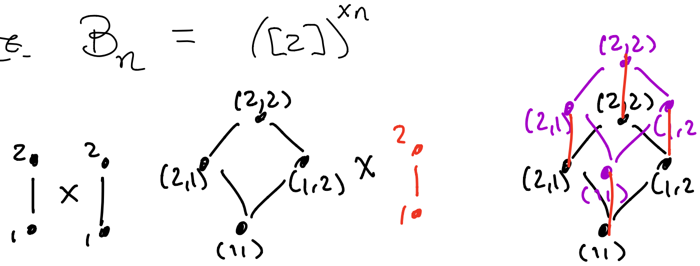
$f(T) = \sum_{S \subseteq T} (-1)^{\#(T-S)} g(S)$

Principle of Inclusion-Exclusion

Proposition P, Q finite posets. Suppose $s \le_P s'$ & $t \le_Q t'$

$$\mu_{P \times Q}((s,t),(s',t')) = \mu_P(s,s')\mu_Q(t,t')$$

Note. $B_n = ([2])^{\times n}$



for $[2]$:

$\mu(1,1) = 1$

$\mu(2,2) = 1$

$\mu(1,2) = -1$

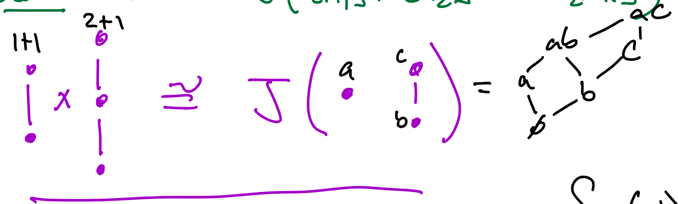
for $[2]^n$:

$\mu(S,T) = (-1)^{\#T - \#S}$

Ex. $P = [n_1+1]\times[n_2+1]\times\cdots\times[n_k+1]$

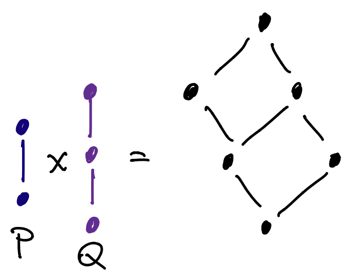
product of k chains of lengths $n_1, n_2, \dots, n_k$.

Claim $P \cong J([n_1]+[n_2]+\cdots+[n_k])$



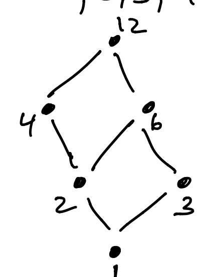
$$\mu_P((a_1,\dots,a_k),(b_1,\dots,b_k)) = \begin{cases} (-1)^{\sum(b_i-a_i)} & b_i-a_i=0 \text{ or } 1 \ \forall i \\ 0 & \text{else}\end{cases}$$

$\hat{P}$ $a_i \le b_i$



$D_N = \{ i \mid N \}$ ordered by divisibility.

$D_{12} = \{1,2,3,4,6,12\}$



Let $p_1,\dots,p_k$ be distinct primes.

Set $N = p_1^{n_1}\cdots p_k^{n_k}$. Claim $D_N \cong [n_1+1]\times\cdots\times[n_k+1]$

$$\mu(r,s) = \begin{cases} (-1)^t & \text{if } s/r \text{ is a product of } t \text{ distinct primes} \\ 0 & \text{else}\end{cases}$$

$$g(n) = \sum_{d|n} f(d) \quad \text{iff} \quad f(n) = \sum_{d|n} g(d)\mu(n/d)$$

Möbius fn from Number Theory

Theorem (Hall's Theorem) P poset, $\hat{P} = P$ w/ $\hat{1}$ & $\hat{0}$ adjoined

$c_i = \#$ chains $\hat{0}=t_0<t_1<\cdots<t_i=\hat{1}$.

$c_0=0$ $c_1=1$: $\hat{0}<\hat{1}$

$$\mu_{\hat{P}}(\hat{0},\hat{1}) = c_0 - c_1 + c_2 - c_3 + \cdots$$

reduced Euler characteristic of a simplicial complex.