MATH 532 Discussion 09/10/2020

Sign-Reversing Involutions.

$$A = A^+ \uplus A^-$$ notion of positive & negative.

$\mathrm{sgn}(a) = +1$, $\mathrm{sgn}(a) = -1$

evaluate $\displaystyle\sum_{a \in A} (-1)^{\mathrm{sgn}(a)} = \#A^+ - \#A^-$

Construct an involution $\tau : A \circlearrowleft$ $(\tau^2 = \mathrm{id})$ s.t.

① If $\tau(x) \neq x$, then $\mathrm{sgn}(\tau(x)) \neq \mathrm{sgn}(x)$

② If $\tau(x) = x$, then $x \in A^+$.

$$\#\text{ fixed pts of } \tau = \#A^+ - \#A^-$$

Ex. $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$

A = collection of subsets of $[n]$.

For $S \subseteq [n]$ $(S \in A)$ $\mathrm{sgn}(S) = (-1)^{\#S}$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{S \subseteq [n]} (-1)^{\#S}$$

Define $\tau : A \circlearrowleft$ $\tau(S) = \begin{cases} S \setminus \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}$

$\#\tau(S) = \#S \pm 1$ sign-reversing involution w/ 0 fixed pts.

EXERCISE. Construct a sign-reversing involution to evaluate $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2$

EXAMPLE: $n = 3$

$$(-1)^0\binom{3}{0}^2 - \binom{3}{1}^2 + \binom{3}{2}^2 - \binom{3}{3}^2$$

$$1 - 9 + 9 - 1 = 0$$

$n = 2$ $\quad 1^2 - 2^2 + 1^2 = 2$

Odd n: $\{(A,B) \mid A \subseteq [n],\ B \subseteq [n],\ \#A = \#B\}$

$$\iota(A,B) = ([n]\setminus A,\ [n]\setminus B)$$

$2m+1$

- involution ✓
- toggle sgn? $(-1)^{\#A} \to (-1)^{n - \#A}$ differ b/c n odd
- FIXED PTS: NONE

$$\Rightarrow \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k}^2 = 0$$

General CASE.

$$(-1)^k\binom{n}{k}^2 = (-1)^k\binom{n}{k}\binom{n}{n-k}$$

$\{(A,B) \mid \#A + \#B = n\}$ $\quad \mathrm{sgn}(A,B) = (-1)^{\#A}$

Move one element from A to B (or back)

smallest elt of $A\setminus B$ or $B\setminus A$ (symmetric diff)

FIXED PTS? When is this not possible?

$A = B$ $\quad \#A + \#B = n$

\# fixed pts? $\binom{n}{n/2}$ if n even, 0 else