MATH 532 Discussion 2 Nov 2020

Generating Functions

Ex. F_n = n^{th} Fibonacci number $\quad F_{n+1} = F_n + F_{n-1} \quad F_0 = 0,\ F_1 = 1$

$$\sum_{n\geq 0} F_n x^n = \frac{x}{1-x-x^2} \qquad 1-x-x^2 = (1-\varphi x)(1-\bar{\varphi} x)$$

$$\varphi = \frac{1+\sqrt{5}}{2} \qquad \bar{\varphi} = \frac{1-\sqrt{5}}{2}$$

$$-1 < \bar{\varphi} < 0$$

$$F_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}} = \text{nearest integer to } \frac{\varphi^n}{\sqrt{5}}$$

NOTICE: $F_{n+1} - F_n - F_{n-1} = 0$

Theorem Let $\alpha_1, \dots, \alpha_d \in \mathbb{C}$, $d \geq 1$ $\alpha_d \neq 0$

Consider all functions $f : \mathbb{N} \to \mathbb{C}$. TFAE

(i) $$\sum_{n\geq 0} f(n) x^n = \frac{P(x)}{Q(x)}$$

where $Q(x) = 1 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_d x^d$ ← recurrence
$P(x)$ poly of degree $< d$.

(ii) $\forall n \geq 0$

$$f(n+d) + \alpha_1 f(n+d-1) + \alpha_2 f(n+d-2) + \dots + \alpha_d f(n) = 0.$$

degree d recurrence. What is $P(x)$? It's the initial conditions!

Proof. $V_1 = \{f : \mathbb{N} \to \mathbb{C} \mid (i) \text{ holds}\}$ $\dim V_1 = d$ b/c choose $P(x)$ & $\deg(P(x)) < d$.

$V_2 = \{f : \mathbb{N} \to \mathbb{C} \mid (ii) \text{ holds}\}$ $\dim V_2 = d$ b/c pick $f(0), \dots, f(d-1)$.

Suppose $f \in V_1$. $Q(x)\left(\sum_{n\geq 0} f(n) x^n\right) = P(x)$

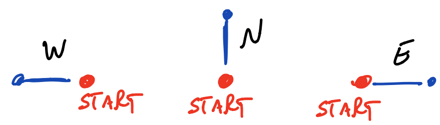
$\Rightarrow f \in V_2 \Rightarrow V_1 \subseteq V_2 \Rightarrow V_1 = V_2$ by dimension. //

Ex. Count non-intersecting paths from $(0,0)$ that go N, E, or W at each step.

G_n = # n-step non-intersecting walks of N, E or W steps.

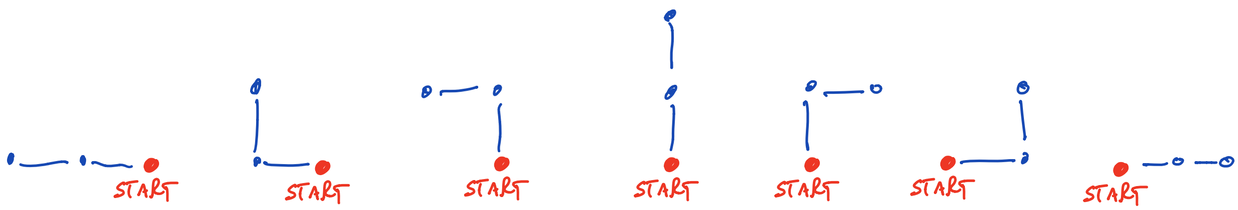
$n=0 \qquad G_0 = 1$

$n=1 \qquad G_1 = 3$



Compute by example G_2. $=7$

Find a recurrence for G_n.



NOT ALLOWED: EW or WE

Write path as a word in W, N, E

could end w/ letter N: $G(n-1)$ prefixes

could end w/ letter NW $\quad G(n-2)$ prefixes

WW

NE

EE

Exercise (HW) finish this example:

$$\sum_{n\geq 0} G(n)x^n = \frac{1+x}{1-2x-x^2}$$

$$G(n) = \frac{1}{2}\left((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}\right)$$