MATH 532 Discussion Mon 05/10

Defn A <u>V-partition of n</u> is an $\mathbb{N}_{\geq 0}$-array

$$\begin{bmatrix} c & a_1 & a_2 & \cdots \\ & b_1 & b_2 & \cdots \end{bmatrix} \quad \text{s.t.} \quad \begin{array}{l} c + \sum a_i + \sum b_i = n \\ c \geq a_1 \geq a_2 \geq \cdots \\ c \geq b_1 \geq b_2 \geq \cdots \end{array}$$

Ex. $\begin{bmatrix} 7 & 4211 \\ & 732 \end{bmatrix}$ is a V-partition of 27

This is a composition w/ extra structure. $\#\text{V-par of } n \underset{\text{wrong}}{\leq} \#\text{comp of } n$

Ex. How many V-partitions of 4?

$4 \to \begin{bmatrix} 4 & \emptyset \\ & \emptyset \end{bmatrix}$

$3\,1 \to \begin{bmatrix} 3 & 1 \\ & \emptyset \end{bmatrix}$ or $\begin{bmatrix} 3 & \emptyset \\ & 1 \end{bmatrix}$

~~1 3~~

$2\,2 \to \begin{bmatrix} 2 & 2 \\ & \emptyset \end{bmatrix}$ or $\begin{bmatrix} 2 & \emptyset \\ & 2 \end{bmatrix}$

$2\,1\,1$, $1\,2\,1$, $1\,1\,2 \to \begin{bmatrix} 2 & 11 \\ & \emptyset \end{bmatrix}$ or $\begin{bmatrix} 2 & 1 \\ & 1 \end{bmatrix}$ or $\begin{bmatrix} 2 & \emptyset \\ & 11 \end{bmatrix}$

$1\,1\,1\,1 \to [1 \cdots]$ 4 ways.

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$$V(q) = \sum_{n \geq 0} v(n)\, q^n = 1 + q + 3q^2 + 6q^3 + 12q^4 + 21q^5 + 38q^6 + \cdots$$

$v(n)$ = # V-partitions of n

$$= \sum_{k \geq 0} \frac{q^k}{[k]!^2} \qquad \text{exercise}$$

$$P(q) = \sum_{n \geq 0} p(n)\, q^n = \prod_{i \geq 1} \frac{1}{1 - q^i}$$

$p(n)$ = # partitions of n

$M_1 : D(n) \twoheadrightarrow V(n)$

$\#D(n) = \#V(n) + \boxed{\#V_1(n)}$

$$D(q) = \sum_{n \geq 0} d(n)\, q^n = P(q) \cdot P(q) = \prod_{i \geq 1} \left(\frac{1}{1 - q^i} \right)^2$$

$d(n)$ = # pairs $\lambda, \mu \in \mathrm{Par}$, $|\lambda| + |\mu| = n$

$$V(n) = \left\{ \begin{bmatrix} c \begin{array}{l} \ge a_1 \ge a_2 \ge \cdots \\ \ge b_1 \ge b_2 \ge \cdots \end{array} \end{bmatrix} \;\middle|\; c + \sum a_i + \sum b_i = n \right\} \qquad V(q) = ??$$

$$D(n) = \left\{ \begin{bmatrix} a_1 \ge a_2 \ge \cdots \\ b_1 \ge b_2 \ge \cdots \end{bmatrix} \;\middle|\; \sum a_i + \sum b_i = n \right\} \qquad D(q) = \prod_{i \ge 1} \frac{1}{(1-q^i)^2}$$

Define a map $M_1 : D(n) \longrightarrow V(n)$ by

$$M_1\left(\begin{bmatrix} a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} \right) = \begin{cases} \begin{bmatrix} a_1 \begin{array}{l} a_2 \ge \cdots \\ b_1 \ge b_2 \ge \cdots \end{array} \end{bmatrix} & a_1 \ge b_1 \\ \begin{bmatrix} b_1 \begin{array}{l} a_1 \ge a_2 \ge \cdots \\ b_2 \ge \cdots \end{array} \end{bmatrix} & b_1 > a_1 \end{cases}$$

$$\begin{bmatrix} c \begin{array}{lll} a_1 & a_2 & \cdots \\ \ge b_1 & b_2 & \cdots \end{array} \end{bmatrix} \xrightarrow{M_1^{-1}} \left\{ \begin{array}{l} \begin{bmatrix} c & a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} \to c \ge b_1 \checkmark \\ \begin{bmatrix} a_1 & a_2 & \cdots \\ c & b_1 & b_2 & \cdots \end{bmatrix} \to \boxed{c > a_1} \end{array} \right.$$

$$V_1(n) \subseteq V(n) \text{ s.t. } c > a_1 \Rightarrow \#D(n) = \#V(n) + \underbrace{\#V_1(n)}_{\text{overcount}}$$

$$M_2 : D(n-1) \longrightarrow \underline{\underline{V_1(n)}} \text{ by}$$

$$M_2\left(\begin{bmatrix} a_1 \ge a_2 & \cdots \\ b_1 \ge b_2 & \cdots \end{bmatrix} \right) = \begin{cases} \begin{bmatrix} a_1{+1} \begin{array}{l} a_2 \ge \cdots \\ b_1 \ge b_2 \ge \cdots \end{array} \end{bmatrix} & a_1^{+1} \ge b_1 \\ \begin{bmatrix} b_1 \begin{array}{l} a_1^{+1} \ge a_2 \ge \cdots \\ b_2 \ge \cdots \end{array} \end{bmatrix} & b_1 > a_1 + 1 \end{cases}$$

$$\sum a_i + \sum b_i = n - 1$$

add it to a_1

$$\begin{bmatrix} c \begin{array}{l} > a_1 \ge a_2 \cdots \\ \ge b_1 \ge b_2 \cdots \end{array} \end{bmatrix} \xrightarrow{M_2^{-1}} \left\{ \begin{array}{ll} \begin{bmatrix} c-1 & a_1 & a_2 & \cdots \\ b_1 & b_2 & \cdots \end{bmatrix} & c \ge b_1 \checkmark \\ \begin{bmatrix} a_1 - 1 & a_2 & \cdots \\ c & b_1 & b_2 & \cdots \end{bmatrix} & \begin{array}{l} c > a_1 \\ a_1 - 1 \ge a_2 \\ \boxed{a_1 > a_2} \end{array} \end{array} \right.$$

$$V_2(n) \subseteq V(n) \quad c > a_1 > a_2$$

$$V(n) = D(n) - V_1(n) = D(n) - D(n-1) + V_2(n)$$

$$V_1(n) = D(n-1) - V_2(n)$$

$$M_3 : D(n-3) \to V_2(n) \text{ by}$$

$$\begin{cases} \begin{bmatrix} a_1+2 & a_2+1 & a_3 & \cdots \\ & b_1 & b_2 & \cdots \end{bmatrix} & a_1+2 \geq b_1 \\ \begin{bmatrix} b_1 & a_1+2 & a_2+1 & a_3 & \cdots \\ & b_2 & b_3 & \cdots & \end{bmatrix} & b_1 > a_1+2 \end{cases}$$

$$M_i : D\left(n - \binom{i}{2}\right) \to V_{i-1}(n)$$

where $\boxed{c > a_1 > \cdots > a_i}$ in

$$\#V(n) = \#D(n) - \#D(n-1) + \#D(n-3) - \#D(n-6)$$
$$+ \#D(n-10)$$
$$- \cdots$$

$$V(q) = \left(\sum_{n\geq 0} (-1)^n q^{\binom{n+1}{2}}\right) \prod_{i\geq 1} (1-q^i)^{-2}$$