Theorem Given a simple (no loops, no multiple edges) connected graph G on n vertices, the following are equivalent

(1) G has no cycles
(2) G has $n-1$ edges
(3) G is minimally connected

Proof $(3 \Rightarrow 1)$ Removing any edge results in >1 component. Suppose G has a cycle, then removing an edge of that cycle cannot disconnect G. $\rightarrow\leftarrow$.

(not3 $\Rightarrow$ not 1) Let e be an edge s.t. $G \setminus e$ connected. If $e = \{v_1, v_2\}$, a path v_1 to v_2 in $G \setminus e$ is a cycle in G.

$(1 \Rightarrow 2)$ By induction on n. If $n=1$, get 0 edges b/c no loops. For $n>1$, pick v_0 a vertex, follow edges until stuck, say at v_1, b/c no cycles this happens. Then v_1 is degree 1, so remove it an its edge to get a connected tree. Done by induction. //

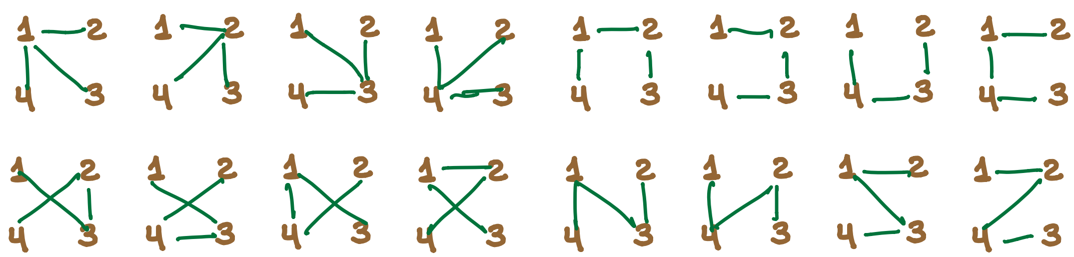
$(2 \Rightarrow 1)$ exercise!

$t_n = \#$ trees on $[n]$

Defn A *tree* is a connected, simple graph with no loops.
A *forest* is a simple graph with no loops.

$n = 1, 2, 3, 4$
$1, 1, 3, 16$

Trees on [4]



Trees have a recursive structure that make gen fn techniques highly amenable.

A tree is rooted by choosing a distinguished vertex.
A forest is rooted if each of its constituent trees is rooted.

$t_n = n \hat{t}_n$ (root)

$t_n = \#$ rooted spanning trees on $[n]$ $\quad (t_0 = 0)$ ← need a root!

$$T(x) = \sum_{n \geq 0} t_n \frac{x^n}{n!}$$

n	0	1	2	3	4
t_n	0	1	2	9	64
		1	2^1	3^2	4^3

guess: $t_n = n^{n-1}$

Proposition $T(x) = x e^{T(x)}$ $\qquad F(x) = e^{T(x)} = \sum_{n \geq 0} \frac{(T(x))^n}{n!}$

Proof. Rooted tree ... Rooted forest w/ each root attached to 0 //

$$\left[T(x) = x R(x) \;\&\; R(x) = e^{T(x)} \right]$$

$\underline{F}(X) = \{ \text{functions } X \to X \}$ $\qquad |F([n])| = n^n$

$$\underline{F}(x) = \sum_{n \geq 0} n^n \frac{x^n}{n!}$$

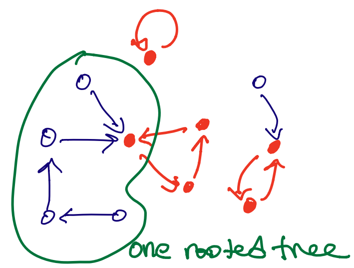
Prop. $F(x) = \dfrac{1}{1-T(x)}$. $\qquad T(x) = \dfrac{1}{1-F(x)}$.

$F = \text{Permutation}(\text{Tree})$

ex gen fn $\sum_{n \geq 0} n! \frac{x^n}{n!} \to \frac{1}{1-x}$

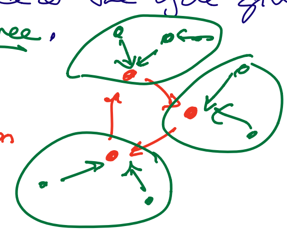
Proof. Draw a "functional diagram"

X



Each connected component has a cycle, and each node of the cycle gives a rooted tree.

Inner: tree
Outer: permutation



//

$T(x) = x e^{T(x)}$

$T'(x) = e^{T(x)} + x T'(x) e^{T(x)}$

$$x T'(x) = x \frac{d}{dx}\left(x e^{T(x)}\right) = x\left(e^{T(x)} + x T'(x) e^{T(x)}\right)$$

$$= x e^{T(x)} + x T'(x)\, x e^{T(x)}$$

$$= T(x) + x T'(x) T(x)$$

$$\Rightarrow x T'(x)\left(1 - T(x)\right) = T(x) \Rightarrow x T'(x) = \frac{T(x)}{1-T(x)}$$

So... $F(x) = \dfrac{1}{1-T(x)} = 1 + \dfrac{T(x)}{1-T(x)} = 1 + x T'(x)$

$$\sum_{n \geq 0} n^n \frac{x^n}{n!} \qquad\qquad 1 + x \sum_{n \geq 1} n\, t(n) \frac{x^{n-1}}{n!}$$

$$1 + \sum_{n \geq 1} n\, t(n) \frac{x^n}{n!}$$

$n \geq 1 \qquad n^n = n\, t(n)$
$\qquad\qquad n^{n-1} = t(n)$

Theorem $\quad t(n) = n^{n-1}$ $\qquad$ unrooted trees $t_n = n^{n-2}$ (CAYLEY's Thm)

Going back: $\boxed{T(x) e^{-T(x)} = x} \Rightarrow T(x) = \left(x e^{-x}\right)^{\langle -1 \rangle}$ compositional inverse.

$T(x) = x e^{T(x)}$

Lagrange Inversion lets us compute functional inverses of exp gen fns!