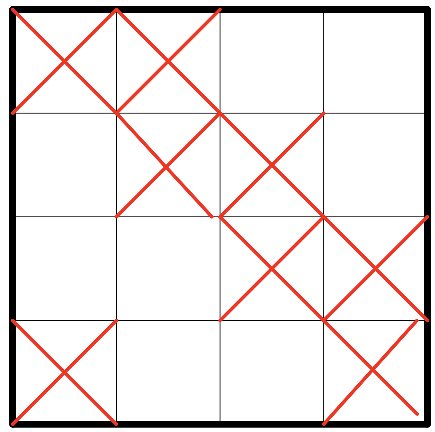
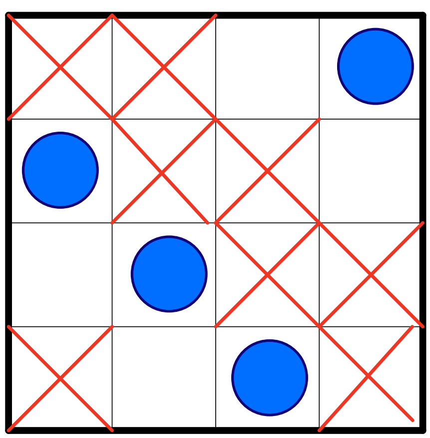
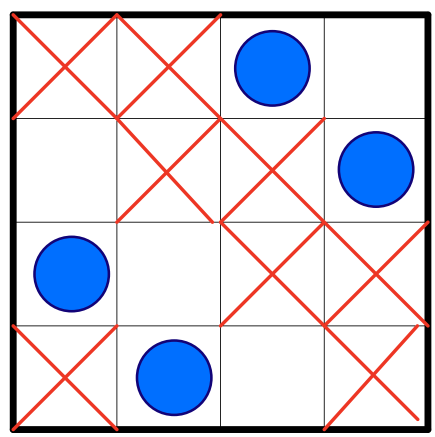
Rook Theory

Place n non-attacking rooks onto an $n\times n$ chess board avoiding squares in B.



$M(n) = \#$ rook placements on $[n]\times[n]$ avoiding
$B = \{(1,1)(2,2)\cdots(n,n), (1\,2)(2\,3)\cdots(n{-}1\; n)(n\,1)\}$





$B \subseteq [n]\times[n]$ board of allowed squares

$G(\omega) = \{(i,\omega(i)) \mid i\in[n]\}$ ← perm "matrix" rook placement.

$N_j = \#\{\omega\in S_n \mid j = \#(B\cap G(\omega))\}$

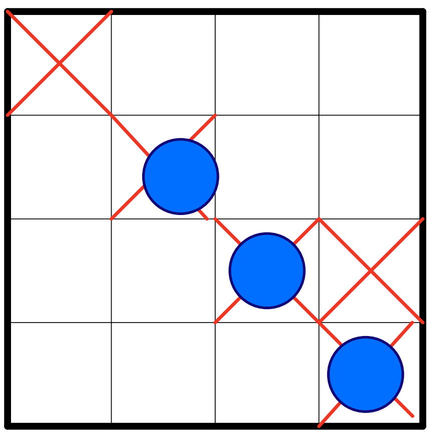
r_k = # ways to place k nonattacking rooks on B
= # k-subsets of B s.t. no two have common coordinate.

rook number

Rook polynomials

$r_B(x) = \sum_k r_k x^k$

Also. $N_n(x) = \sum_j N_j x^j$

$B =$ 

$N_0 = 6$	$r_0 = 1$
$N_1 = 9$	$r_1 = 5$
$N_2 = 7$	$r_2 = 8$
$N_3 = 1$	$r_3 = 3 + 2 = 5$
$N_4 = 1$	$r_4 = 1$

N_j = # placements w/ j rooks on red.

r_k = # ways place k rooks on red board.

Theorem $N_n(x) = \sum_{k=0}^{n} r_k (n-k)! (x-1)^k$

In particular $N_0 = N_n(0) = \sum_{k=0}^{n} (-1)^k r_k (n-k)!$