MATH 532 Discussion Monday 14/09

Def. A descent of a permutation $w = w_1 w_2 \cdots w_n$ is an index i s.t. $w_i > w_{i+1}$.

Ex. $w = 6\ \underset{2}{8} > 3\ \underset{4}{9} > \underset{5}{4} > 1\ \underset{7}{7} > 2\ 5 \in S_9$

$$\mathrm{Des}(w) = \{ i \mid w_i > w_{i+1} \} \subseteq [n-1]$$
$$= \{2, 4, 5, 7\}$$

$D = \{d_1 < \cdots < d_k\}$

Subsets of $[n-1]$ $\overset{\sim}{\longleftrightarrow}$ strong comp of n

$d_{i+1} - d_i$

$\{2,4,5,7\} \subseteq [8] \longmapsto (2, 4-2, 5-4, 7-5, 9-7)$

$(2, 2, 1, 2, 2)$

Eulerian numbers $A(d,k) = \#\{ w \in S_d \mid \underbrace{\#\mathrm{Des}(w)}_{\mathrm{des}(w)} = k-1 \}$

1. Compute $A_4(x)$ where $A_d(x) = \sum_{w \in S_d} x^{1+\mathrm{des}(w)}$

$$= \sum_{k=1}^{d} A(d,k) x^k$$

2. $\mathrm{Exc}(w) = \{ i \mid w(i) \geq i \}$ $\quad \mathrm{exc}(w) = \#\mathrm{Exc}(w)$

Ex. $\underset{1}{\overset{6}{\vee}}\ \underset{2}{\overset{8}{\vee}}\ \underset{3}{\overset{3}{\mathrm{VI}}}\ \underset{4}{\overset{9}{\vee}}\ 4\ 1\ \underset{7}{\overset{7}{\mathrm{VI}}}\ 2\ 5$ $\quad \mathrm{exc}(w) = 5$

Relate #descents to #weak excedences.

$$A_d(x) = \sum_{k=1}^{d} A(d,k)\,x^k = \sum_{w\in S_d} x^{1+\mathrm{des}(w)}$$

$A_0(x) = 1$

$A_1(x) = x$

$A_2(x) = x + x^2$

$A_3(x) = x + 4x^2 + x^3$

$A_4(x) = x + 11x^2 + 11x^3 + x^4$

$A_5(x) = x + 26x^2 + 66x^3 + 26x^4 + x^5$

These are palindromes!

$f : S_d \longrightarrow S_d$

$\mathrm{des}(w) = n - \mathrm{des}(f(w))$

$$\begin{array}{ccccccccccccccccccc} 6&8&3&9&4&1&7&2&5&\mapsto&5&2&7&1&4&9&3&8&6\\ &2&&4&5&&7&&&&1&&3&&&6&&8& \end{array}$$

$\mathrm{Des}(f(w)) = [n-1] \setminus \mathrm{Des}(w)$

$\mathrm{des}(w) = \#\{i \mid w_i > w_{i+1}\}$ $\quad\quad$ $\mathrm{exc}(w) = \#\{i \mid w_i \geq i\}$

Proposition $\quad A(d,k) = \#\{w \in S_d \mid \mathrm{exc}(w) = k\}$.

$\psi : S_d \longrightarrow S_d$

$\mathrm{des}(w) = d - \mathrm{exc}(\psi(w))$

$$\begin{array}{ccccccccccccccc} & 1 & 2 & & 3 & 4 & & 5 & & 6 & 7 & & 8 & 9 \\ w = & 6 & 8 & > & 3 & 9 & > & 4 & > & 1 & 7 & > & 2 & 5 \end{array}$$

$\mathrm{des}(w) = 4$

cycle notation $\quad (1\ 6)(2\ 8)(3)(4\ 9\ 5)(7)$

$(3)(6\ 1)(7)(8\ 2)(9\ 5\ 4)$

Lemma. Heads of cycles are weak exceedences.

$$\begin{array}{cccccccccc} \psi(w) = & 3 & 6 & 1 & 7 & 8 & 2 & 9 & 5 & 4 \\ & \geq & \geq & & \geq & \geq & & \geq & & \\ & 1 & 2 & & 4 & 5 & & 7 & & \end{array}$$

$\mathrm{exc} = 5$ $\quad\quad$ $9 - 5 = 4$