MATH 532 Discussion 4 Nov 2020

$$\sum_{n\geq 0} f(n)x^n = \frac{P(x)}{Q(x)}$$

$Q(x)$ gives recurrence
$P(x)$ gives initial conditions.

$Q(x) = 1 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_d x^d$
$P(x) = 1 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_e x^e$

where $\beta_k = 0$ for $k > e$

$$f(n+d) + \alpha_1 f(n+d-1) + \alpha_2 f(n+d-2) + \cdots + \alpha_d f(n) + \beta_n = 0.$$

Ex. $f : \mathbb{N} \to \mathbb{C}$ want to compute $f(i)$ for $i = 0, 1, 2, \ldots, 5$

$$\sum_{n\geq 0} f(n)x^n = \frac{1 - 2x + 4x^2 - x^3}{1 - 3x + 3x^2 - x^3}$$

initial conditions $f(0)$, $f(1)$, $f(2)$

$$f(n+3) = 3f(n+2) - 3f(n+1) + f(n) + \beta_{n+3} \qquad n \geq 0$$

$f(0) = 1$
$f(1) = 3f(0) + \beta_1 = 3 \cdot 1 - 2 = 1$
$f(2) = 3f(1) - 3f(0) + \beta_2 = 4$
$f(3) = 3f(2) - 3f(1) + f(0) + \beta_3 = 9$
$f(4) = 3f(3) - 3f(2) + f(1) = 16$
$f(5) = 3f(4) - 3f(3) + f(2) = 25$

Guess: $f(n) = n^2$ for $n \geq 1$

$$\frac{P(x)}{Q(x)} = 1 + \frac{x + x^2}{(1-x)^3} = 1 + \sum_{n\geq 0} n^2 x^n$$

$$1 - 3x + 3x^2 - x^3 = (1-x)^3$$

Corollary $f: \mathbb{N} \to \mathbb{C}$, $d \in \mathbb{N}$ TFAE

(i) $$\sum_{n \geq 0} f(n) x^n = \frac{P(x)}{(1-x)^{d+1}}$$ where $P(x) \in \mathbb{C}[x]$, $\deg P(x) \leq d$

(ii) $\forall n \geq 0$ $$\sum_{i=0}^{d+1} (-1)^{d+1-i} \binom{d+1}{i} f(n+i) = 0$$

(iii) $f(n)$ is a polynomial fn of n of degree $\leq d$.

Define an operator Δ on fns $f: \mathbb{N} \to \mathbb{C}$

$$\Delta f(n) = f(n+1) - f(n)$$

$$E f(n) = f(n+1) \qquad (\Delta = E - \mathbb{1})$$

operator.

$$\Delta^k f(n) = (E - \mathbb{1})^k f(n)$$
$$= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} E^i f(n)$$
$$= \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(n+i) \text{: } ii \Leftrightarrow iii \text{ above.}$$

$$\Delta^k f(0) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(i) \leftarrow \text{only need } f(0), \dots, f(k)$$

Combinatorial analog of e^x? $\frac{d}{dx} e^x = e^x$

$\Delta f(n) = f(n)$ $\qquad f(n) = 2^n$

$\hookrightarrow f(n+1) - f(n)$

$2^{n+1} - 2^n = 2^n(2-1) = 2^n = f(n)$

Ex.

$$n^4 = \binom{n}{1} + 14\binom{n}{2} + 36\binom{n}{3} + 24\binom{n}{4} + 0\binom{n}{5} + \dots$$

Basis $\{1, n, n^2, n^3, n^4\}$

$\left\{\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \binom{n}{4}\right\}$

$$n^4 = \sum_{k=0}^{4} k!\, S(4,k) \binom{n}{k}.$$

Prop. (a) $f: \mathbb{N} \to \mathbb{C}$ is a poly of degree $\leq d$ iff
$\Delta^{d+1} f(n) = 0$ (iff $\Delta^d f(n)$ constant)

(b) Write $f(n)$ in basis $\left\{\binom{n}{k}\right\}$ $0 \leq k \leq d$,
the coeffts are $\Delta^k f(0)$

$$f(n) = \sum_{k=0}^{d} \Delta^k f(0) \binom{n}{k}$$

(c) $f(n) = n^d$, then
$$\Delta^k f(0) = k!\, S(d,k).$$