MATH 532 Discussion FRIDAY 04/09

Stirling Numbers of 1st Kind

$s(n,k) = (-1)^k \#\{w \in S_n \mid w \text{ has } k \text{ cycles}\}$

$c(n,k) = |s(n,k)|$

Ex. $c(4,2) = 11$

$(\_\,\_\,\_)(\_) \leftarrow \binom{4}{3}\,2!$

$(\_\,\_)(\_\,\_) \leftarrow \binom{4}{2,2}\,1!\,1!$

Stirling Numbers of 2nd Kind

$S(n,k) = \#$ set partitions of $[n]$ into k nonempty blocks.

Ex. $S(4,2) = 7$

$\{\_\,\_\,\_\}\{\_\} \leftarrow \binom{4}{3}$

$\{\_\,\_\}\{\_\,\_\} \leftarrow 3$

A set partition of $[n]$ is a collection $\pi = \{B_1, \ldots, B_k\}$ s.t.

- $B_i \neq \emptyset$ for $i = 1, \ldots, k$
- $B_i \cap B_j = \emptyset$ for all $i \neq j$
- $B_1 \cup B_2 \cup \cdots \cup B_k = [n]$

$\left(S(0,0) = 1 \text{ by convention}\right)$

$$c(n,k) = \underbrace{(n-1)\,c(n-1,k)}_{\text{place } n \text{ after any letter of } w \in S_{n-1}} + \underbrace{c(n-1,k-1)}_{n \text{ in a cycle of its own}}$$

$$S(n,k) = \underbrace{k\,S(n-1,k)}_{\text{place } n \text{ into an existing block}} + \underbrace{S(n-1,k-1)}_{n \text{ in its own block.}}$$

Lemma $\displaystyle\sum_{k=0}^{n} c(n,k)\,x^k = x(x+1)\cdots(x+n-1)$

Lemma $\displaystyle\sum_{k=0}^{n} S(n,k)\,(x)_k = x^n$

$(x)_k = x(x-1)(x-2)\cdots(x-k+1)$

Theorem $\displaystyle\sum_{k \geq 0} S(m,k)\,s(k,n) = \delta_{mn}$

Proof $x^n = \#\{f : [n] \to [x]\}$

$$\begin{pmatrix}1\\ \vdots\\ n\end{pmatrix} \longrightarrow \begin{pmatrix}1\\ \vdots\\ x\end{pmatrix}$$

$$\sum_{k=0}^{n} k!\,S(n,k)\,\underbrace{\binom{x}{k}}_{\text{choose image}}$$

$$\left[s(n,k)\right]_{n,k=0}^{4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & -6 & 11 & -6 & 1 \end{pmatrix}$$

$$\left[S(n,k)\right]_{n,k=0}^{4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 7 & 6 & 1 \end{pmatrix}$$