Theorem Let S be an n-set.
Let V be the 2^n-dim v.s. over $\mathbb{C}$ of fns $f : 2^S \to \mathbb{C}$

Define $\varphi : V \to V$ linear trans.

$$\varphi(f(T)) = \sum_{S \supseteq U \supseteq T} f(U) \qquad \text{for all } T \subseteq S.$$

S ↑ fixed; T ↑ input of fn $\phi(f)$

Then φ^{-1} exists and

$$\varphi^{-1}(f(T)) = \sum_{S \supseteq U \supseteq T} (-1)^{\#(U-T)} f(U) \quad \forall\, T \subseteq S$$

S ↑ fixed; T ↑ input

Application Think of S as a set of properties, some objects might or might not have.

For $T \subseteq S$, $f_=(T) =$ # objects that have exactly properties of T.

Ex. $S = [4] = \{1,2,3,4\}$ regard i as a property on S_4
P_i means i is a fixed pt.

$T = \{1,3\} \subseteq S$

$f_=(T) =$ # perm w/ fixed points exactly 1 & 3.

relate these.

$f_\geq(T) =$ # perm w/ at least fixed pts 1 & 3.

$$f_\geq(T) = \sum_{Y \supseteq T} f_=(Y) \overset{\text{thm}}{\Longrightarrow} f_=(T) = \sum_{Y \supseteq T} (-1)^{\#(Y-T)} f_\geq(Y)$$

$$f_=(\emptyset) = \sum_{Y} (-1)^{\#Y} f_\geq(Y) \quad \leftarrow \text{derangements.}$$

Derangements $S = [n]$ $\mathfrak{S}_n$

$f_=(T) = \#\{\omega \in \mathfrak{S}_n \mid \text{fixed pts of } \omega \text{ are } T\}$
$\omega(i)=i$ iff $i \in T$

$f_\geq(T) = \#\{\omega \in \mathfrak{S}_n \mid \text{fixed pts of } \omega \text{ include } T\}$
$\omega(i)=i$ if $i \in T$

$n=5$ $f_\geq(\{1,3,4\}) = \#\left\{\begin{matrix} 1 & 5 & 3 & 4 & 2 \\ 1 & 2 & 3 & 4 & 5 \end{matrix}\right\}$

$f_\geq(T) = (n - \#T)!$ permute pts not forced to be fixed

$$d_n = f_=(\emptyset) = \sum_{[n] \supseteq Y \supseteq \emptyset} (-1)^{\#Y} f_\geq(Y)$$

$$= \sum_{Y} (-1)^{\#Y} (n - \#Y)!$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

if $f_=$ & $f_\geq$ depend only on $\#T$, this happens.

$$= \sum_{k=0}^{n} n! \frac{(-1)^k}{k!}$$

$$= n! \boxed{\sum_{k=0}^{n} \frac{(-1)^k}{k!}} \sim \frac{1}{e} \sim \frac{1}{3}$$

Exercise $h_n = \#$ perm of $\{\{1,1,2,2,\ldots,n,n\}\}$ w/ no consecutive equal entries.

h_0	h_1	h_2
1	0	1212
		2121
		2

set up derangement formalism.

$f_\geq(T) = \#\{\omega \mid ii \text{ appears in } \omega \text{ if } i \in T\}$.

$f_\geq(T) = (2n-k)! / 2^{n-k}$ $n-k$ repeats
$\#T = k$ take out second i for $i \in T$

$S = [n]$

$f_\geq([n]) = n!$

$$f_=(T) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2n-k)! / 2^{n-k}.$$