MATH 532 Discussion 06 Nov 2020

1. Compute small examples $n = 0, 1, 2, 3, 4$ ✓
2. Look for recurrence. ✓
3. Derive gen. fn. ✓
4. Solve for coeffs.

Count the number of 0-1 sequences of length $2n$ with n 0's & n 1's s.t. always have at least as many 0's as 1's, (reading R to L).

→ Balanced words in $\{(,)\}$ with n-pairs.

$n = 0 : \emptyset \qquad C_0 = 1$

$n = 1 : () \qquad C_1 = 1$

$n = 2 : (()), ()() \qquad C_2 = 2$

$n = 3 :$ ((())), (()()), ()(()), ()()(), (())() $\qquad C_3 = 5$

$n = 4 : (\underline{\text{stuff}})\underline{\text{stuff}} \qquad C_4 = 14$

$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \qquad C_5 = 42$

$\underbrace{(\underbrace{()}\ \underbrace{()})}\underbrace{(\underbrace{()})}$

$\underbrace{(\underbrace{()}\ \underbrace{()})}\ )\ (\ ()$ NO ✗

$C_n = \#$ balanced words w/ n pairs

$$C_4 = \sum_{k=0}^{3} C_k C_{3-k}$$

$$= C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0$$

$$1\cdot 5 + 1\cdot 2 + 2\cdot 1 + 5\cdot 1 = 14$$

$()\underline{3}$, $(())\underline{2}$, $(\underline{2})()$, $(\underline{3})$

$$y = \sum_{n\geq 0} C_n x^n \qquad (C_0 = 1)$$

$$y - 1 = \sum_{n\geq 1} C_n x^n$$

$$y - 1 = x\Big(\sum_{n\geq 0} C_{n+1} x^n\Big)$$

$$\boxed{y - 1 = x y^2}$$

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

$$y^2 = \Big(\sum_{n\geq 0} C_n x^n\Big)^2$$

$$= \sum_{n\geq 0}\Big(\sum_{k=0}^{n} C_k C_{n-k}\Big)x^n$$

$$= \sum_{n\geq 0} C_{n+1} x^n$$

$$xy^2 - y + 1 = 0$$

$$y = \frac{1 \pm \sqrt{1-4x}}{2x} = \sum_{n\geq 0} c_n x^n$$

$$(1+u)^r = \sum_{n\geq 0} \binom{r}{n} u^n$$

$$(1-4x)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n} (-4)^n x^n$$

$$\binom{1/2}{n} = \frac{(1/2)(1/2-1)(1/2-2)\cdots(1/2-n+1)}{n!}$$

$\underline{n\geq 1}$

$$= \frac{(1/2)(-1/2)(-3/2)\cdots\left(-\frac{(2n-3)}{2}\right)}{n!}$$

$$= (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n \; n!}$$

$$\binom{1/2}{n}(-4)^n = -\frac{2^n\,(1\cdot 3\cdot 5\cdots(2n-3))}{n!}$$

$$= -2\frac{(2\cdot 4\cdot 6\cdots(2n-2))(1\cdot 3\cdot 5\cdots(2n-3))}{n!\;(n-1)!}$$

$$= -2\frac{(2n-2)!}{n!\,(n-1)!}$$

kills constant term

$$y = \frac{1-\sqrt{1-4x}}{2x} = \frac{1 - \left(1 + \sum_{n\geq 1} -\cancel{2}\frac{(2n-2)!}{n!\,(n-1)!} x^{n-1}\right)}{\cancel{2}\cancel{x}}$$

$$= \sum_{n\geq 1} \frac{(2n-2)!}{n!\,(n-1)!} x^{n-1} = \sum_{n\geq 0} \frac{(2n)!}{(n+1)!\,n!} x^n$$

C_n

$$= \sum_{n\geq 0} \boxed{\frac{1}{n+1}\binom{2n}{n}} x^n \qquad C_4 = \frac{1}{4+1}\binom{8}{4} = 14.$$