Ex. Cycle Index $Z_n(t_1,\dots,t_n) = \frac{1}{n!}\sum_{w\in S_n} t^{\mathrm{type}(w)}$

$\mathrm{type}(w) = (c_1, c_2, \dots, c_n)$ $c_i = \#\, i\text{-cycles of } w$

$t^c = t_1^{c_1} t_2^{c_2} \cdots t_n^{c_n}$

Ex. $Z_4 = \frac{1}{24}\left(t_1^4 + 6t_1^2 t_2 + 8t_1 t_3 + 3t_2^2 + 6t_4\right)$

$1 + 6 + 8 + 3 + 6 = 24$

A permutation statistic is a map $\alpha : S_n \to \mathbb{N}$

$F(q) = \sum_{w\in S_n} q^{\alpha(w)} \Rightarrow F(1) = n!$

In Z_n, set $t_i = q$, $\alpha = \#$ cycles of w.

$Z_4 = \frac{1}{24}\left(q^4 + 6q^3 + \underline{8q^2 + 3q^2} + 6q\right)$

Inversions: $\mathrm{inv}(w) = \#\{(i<j) \mid w_i > w_j\}$.

Exercise: $\sum_{w\in S_n} q^{\mathrm{inv}(w)} = I_n(q)$ NOTE: $I_n(1) = n!$

$I_2 = 1 + q$

$I_3 = 1 + 2q + 2q^2 + q^3$

$= \underbrace{(1+q)}_{I_2(q)}(1+q+q^2)$

Inserting n into a permutation of $[n-1]$ branches q^j $j = 0$ to $n-1$ based on position of n inserted.

$(*)\ I_{n+1}(q) = \left(\sum_{k=1}^{n+1} q^{n-k+1}\right) I_n(q)$

Theorem $\sum_{w\in S_n} q^{\mathrm{inv}(w)} = 1\,(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$

Proof1: By induction. $I_1(q)=1$ ✓ (*) //

Proof2: Each permutation has a Lehmer code $L(w)=(a_1,\dots,a_n)$
$a_i := \#\, j>i \mid w_i > w_j$

$$\sum_{w\in S_n} q^{\mathrm{inv}(w)} = \sum_{a_1=0}^{n-1}\sum_{a_2=0}^{n-2}\cdots\sum_{a_n=0}^{0} q^{a_1+a_2+\cdots+a_n}$$
$$= \left(\sum_{a_1=0}^{n-1} q^{a_1}\right)\left(\sum_{a_2=0}^{n-2} q^{a_2}\right)\cdots\left(\sum_{a_n=0}^{0} q^{a_n}\right) //$$

$$[K]_q = \frac{1-q^K}{1-q} = 1+q+q^2+\cdots+q^{K-1}$$

$$[K]_q! = [1]_q\,[2]_q\cdots[K]_q$$

$n! = \#$ sequences $(\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = [n])$ $\#S_i = i$

$[n]_q! = \#$ sequences $(0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{F}_q^n)$ $\dim V_i = i$

Prop. $\mathrm{inv}(w) = \mathrm{inv}(w^{-1})$.

$w = 4\,2\,1\,5\,3$ $\qquad w^{-1} = 3\,2\,5\,1\,4$

$\mathrm{inv}(w) = 5$ $\qquad \mathrm{inv}(w^{-1}) = 5$

$(1<3)$ $\underset{4}{w_1} > \underset{1}{w_3}$ $\qquad \underset{w_3}{3} > \underset{w_1}{1}$ $(1,4)$ inversion here.

$i<j$ inversion $\Leftrightarrow$ w_i, w_j inv of w^{-1}
$w_i > w_j$