Basic definitions

A sequence $a_1, a_2, \ldots, a_k$ of distinct integers is **alternating** if

$$a_1 > a_2 < a_3 > a_4 < \cdots ,$$

and **reverse alternating** if

$$a_1 < a_2 > a_3 < a_4 > \cdots .$$

Euler numbers

$\mathfrak{S}_n$: symmetric group of all permutations of $1, 2, \ldots, n$

Euler number:

$$\begin{aligned} E_n &= \#\{w \in \mathfrak{S}_n : w \text{ is alternating}\} \\ &= \#\{w \in \mathfrak{S}_n : w \text{ is reverse alternating}\} \end{aligned}$$

(via $a_1 \cdots a_n \mapsto n+1-a_1, \ldots, n+1-a_n$)

E.g., $E_4 = 5 : \ 2143, 3142, 3241, 4132, 4231$

André’s theorem

Theorem (**Désiré André**, 1879)

$$y := \sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

$$= \mathbf{1} + \mathbf{1}x + \mathbf{1}\frac{x^2}{2!} + \mathbf{2}\frac{x^3}{3!} + \mathbf{5}\frac{x^4}{4!} + \mathbf{16}\frac{x^5}{5!}$$

$$+ \mathbf{61}\frac{x^6}{6!} + \mathbf{272}\frac{x^7}{7!} + \cdots$$

E_{2n} is a **secant number**.

E_{2n+1} is a **tangent number**.

Proof of André's theorem

$$y := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

Choose $\boldsymbol{S} \subseteq \{1, 2, \ldots, n\}$, say $\#S = k$.

Choose a reverse alternating permutation $\boldsymbol{u} = a_1 a_2 \cdots a_k$ of S.

Choose a reverse alternating permutation $\boldsymbol{v} = b_1 b_2 \cdots b_{n-k}$ of $[n] - S$.

Let $\boldsymbol{w} = a_k \cdots a_2 a_1, \boldsymbol{n+1}, b_1 b_2 \cdots b_{n-k}$.

Proof (continued)

$$w = a_k \cdots a_2 a_1, \mathbf{n+1}, b_1 b_2 \cdots b_{n-k}$$

Given k, there are:

- $\binom{n}{k}$ choices for $\{a_1, a_2, \ldots, a_k\}$
- E_k choices for $a_1 a_2 \cdots a_k$
- E_{n-k} choices for $b_1 b_2 \cdots b_{n-k}$.

We obtain each alternating and reverse alternating $w \in \mathfrak{S}_{n+1}$ once each.

Completion of proof

$$\Rightarrow 2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k},\ n \geq 1$$

Multiply by $x^{n+1}/(n+1)!$ and sum on $n \geq 0$:

$$2y' = 1 + y^2, \quad y(0) = 1.$$

$$\Rightarrow y = \sec x + \tan x.$$

A new subject?

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

Define

$$\begin{aligned} \tan x &= \sum_{n\geq 0} E_{2n+1} \frac{x^{2n+1}}{(2n+1)!} \\ \sec x &= \sum_{n\geq 0} E_{2n} \frac{x^{2n}}{(2n)!}. \end{aligned}$$

$\Rightarrow$ **combinatorial trigonometry**

The boustrophedon array

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & \rightarrow & 1 & & & & & \\
 & & & & 1 & \leftarrow & 1 & \leftarrow & 0 & & & & \\
 & & & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 2 & & & \\
 & & 5 & \leftarrow & 5 & \leftarrow & 4 & \leftarrow & 2 & \leftarrow & 0 & & \\
 & 0 & \rightarrow & 5 & \rightarrow & 10 & \rightarrow & \mathbf{14} & \rightarrow & 16 & \rightarrow & 16 & \\
61 & \leftarrow & 61 & \leftarrow & 56 & \leftarrow & \mathbf{46} & \leftarrow & \mathbf{32} & \leftarrow & 16 & \leftarrow & 0. \\
 & & & & & & \cdots & & & & & &
\end{array}
$$

Boustrophedon entries

- last term in row n: E_{n-1}
- sum of terms in row n: E_n
- kth term in row n: number of alternating permutations in $\mathfrak{S}_n$ with first term k, the **Entringer number** $\boldsymbol{E_{n-1,k-1}}$.

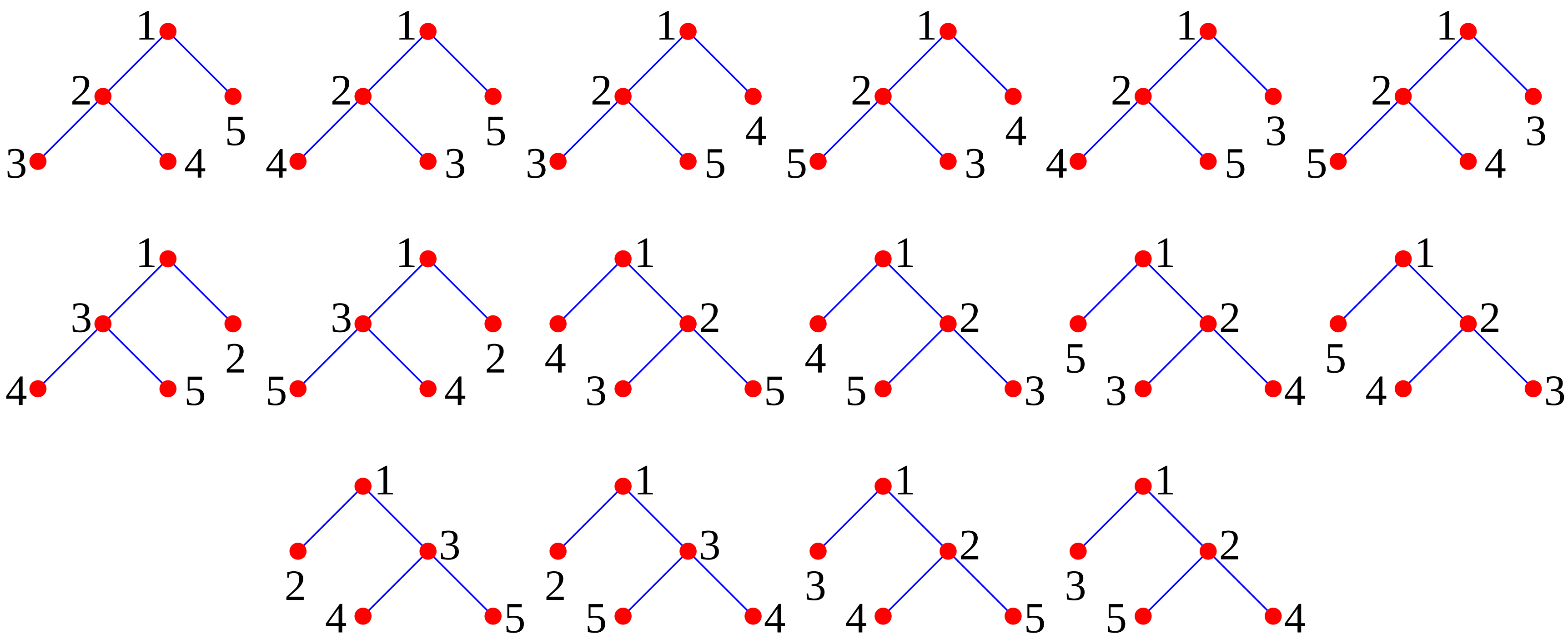
$$\sum_{m\geq 0}\sum_{n\geq 0} E_{m+n,[m,n]}\frac{x^m}{m!}\frac{y^n}{n!} = \frac{\cos x+\sin x}{\cos(x+y)},$$

$$[m,n] = \begin{cases} m, & m+n \text{ odd} \\ n, & m+n \text{ even.} \end{cases}$$

Some occurrences of Euler numbers

(1) E_{2n+1} is the number of complete increasing binary trees on the vertex set $[2n+1] = \{1, 2, \ldots, 2n+1\}$.

Five vertices



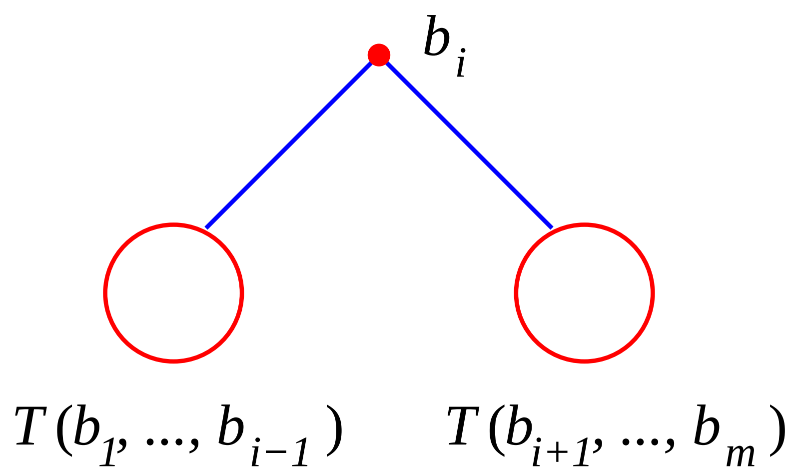
Slightly more complicated for E_{2n}

Proof for $2n + 1$

$$\boldsymbol{b_1 b_2 \cdots b_m} : \text{ sequence of distinct integers}$$

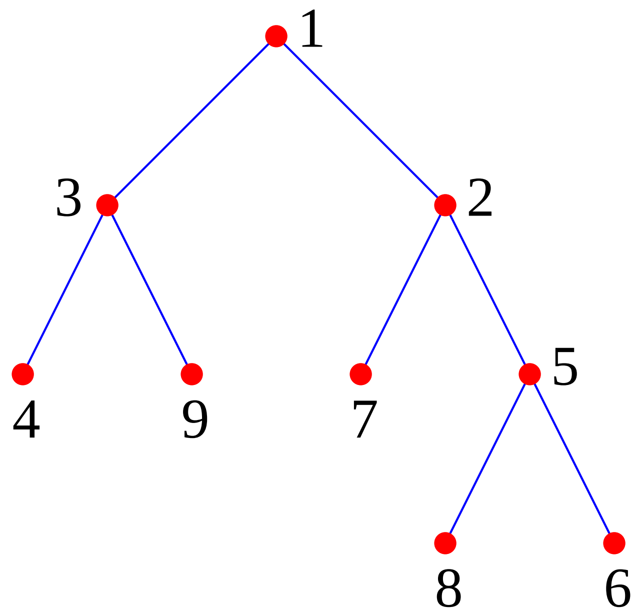
$$\boldsymbol{b_i} = \min\{b_1, \dots, b_m\}$$

Define recursively a binary tree $\boldsymbol{T(b_1, \dots, b_m)}$ by



Completion of proof

Example. 439172856



Let $\boldsymbol{w} \in \mathfrak{S}_{2n+1}$. Then $T(w)$ is complete if and only if w is alternating, and the map $w \mapsto T(w)$ gives the desired bijection. $\square$

Orbits of mergings

(2) Start with n one-element sets $\{1\}, \ldots, \{n\}$.

Merge together two at a time until reaching $\{1, 2, \ldots, n\}$.

$$1{-}2{-}3{-}4{-}5{-}6, \quad 12{-}3{-}4{-}5{-}6, \quad 12{-}34{-}5{-}6$$
$$125{-}34{-}6, \quad 125{-}346, \quad 123456$$

$\mathfrak{S}_n$ acts on these sequences.

Theorem. *The number of $\mathfrak{S}_n$-orbits is E_{n-1}.*

Proof. Exercise. $\square$

Orbit representatives for $n = 5$

$12-3-4-5$	$123-4-5$	$1234-5$
$12-3-4-5$	$123-4-5$	$123-45$
$12-3-4-5$	$12-34-5$	$125-34$
$12-3-4-5$	$12-34-5$	$12-345$
$12-3-4-5$	$12-34-5$	$1234-5$

Volume of a polytope

(3) Let $\mathcal{E}_n$ be the convex polytope in $\mathbb{R}^n$ defined by

$$\begin{aligned} x_i &\geq 0, \quad 1 \leq i \leq n \\ x_i + x_{i+1} &\leq 1, \quad 1 \leq i \leq n-1. \end{aligned}$$

Theorem. *The volume of $\mathcal{E}_n$ is $E_n/n!$.*

Naive proof

$$\mathrm{vol}(\mathcal{E}_n) = \int_{x_1=0}^{1} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1\, dx_2 \cdots dx_n$$

$$\boldsymbol{f_n(t)} := \int_{x_1=0}^{\boldsymbol{t}} \int_{x_2=0}^{1-x_1} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_1\, dx_2 \cdots dx_n$$

$$\begin{aligned} f_n'(t) &= \int_{x_2=0}^{1-t} \int_{x_3=0}^{1-x_2} \cdots \int_{x_n=0}^{1-x_{n-1}} dx_2\, dx_3 \cdots dx_n \\ &= f_{n-1}(1-t). \end{aligned}$$

$F(y)$

$$f_n'(t) = f_{n-1}(1-t), \quad f_0(t) = 1, \quad f_n(0) = 0 \ (n > 0)$$

$$\begin{aligned} \boldsymbol{F(y)} &= \sum_{n\geq 0} f_n(t) y^n \\ \Rightarrow \frac{\partial^2}{\partial t^2} F(y) &= -y^2 F(y), \end{aligned}$$

etc.

Conclusion of proof

$$F(y) = (\sec y)(\cos(t-1)y + \sin ty)$$

$$\Rightarrow F(y)|_{t=1} = \sec y + \tan y.$$