MATH 532 Discussion FRIDAY 02/10

Unimodal Sequences

<u>Defn</u> a unimodal seq of weight n is $(u_1, u_2, \ldots, u_k)$ $u_i \in \mathbb{N}_{>0}$

① $\sum_i u_i = n$

② $\exists j$ s.t. $u_1 \leq u_2 \leq \cdots \leq u_j \geq u_{j+1} \geq \cdots \geq u_k$

<u>Ex.</u> sequence $u_i = \binom{n}{i}$ is unimodal & symmetric

<u>Ex.</u> List all unimodal seq of weights 1, 2, 3, 4, 5 ...

$u(n) =$ # unimodal seq of weight n

$u(0) = 0$ (because $\nexists j$)

$u(1) = 1$

$u(2) = 2$ (2) & (1,1)

$u(3) = 4$ (3), (12), (21), (111)

$u(4) = 8$ (4) (31) (13) (22) (211) (121) (112) (1111)

$u(5) = 15$

$$U(q) = \sum_{n \geq 0} u(n) q^n = q + 2q^2 + 4q^3 + 8q^4 + 15q^5 + 27q^6 + 47q^7 + 79q^8 + \cdots$$

<u>Prop.</u> $U(q) = \sum_{k \geq 1} \frac{q^k}{[k-1]_q! \, [k]_q!}$ $[k]_q = 1 + q + \cdots + q^{k-1}$

<u>Question</u> How do unimodal seq uences relate to partitions?

$$\sum_{n \geq 0} p(n) q^n = \sum_{k \geq 0} \frac{q^k}{[k]!} = \prod_{i \geq 1} \frac{1}{1-q^i}$$

<u>Defn</u> A V-partition of n is an array

$$\begin{bmatrix} c & a_1 & a_2 & \cdots \\ & b_1 & b_2 & \cdots \end{bmatrix} \qquad \begin{array}{l} c, a_i, b_i \in \mathbb{N}_{>0} \\ c + \sum_i a_i + \sum_i b_i = n \\ c \geq a_1 \geq a_2 \geq \cdots \\ c \geq b_1 \geq b_2 \geq \cdots \end{array}$$

We can count V-partitions using sieve methods.

$\mathcal{U}(n) = \{$ unimodal sequences of wt $n \}$

$V(n) = \{$ V-partitions of $n \}$

$D(n) = \{ \lambda, \mu$ partitions $\mid |\lambda| + |\mu| = n \}$

g.f. $\prod_{i \geq 1} \left(\frac{1}{1-q^i} \right)^2$

<u>Proposition</u> $\# \mathcal{U}(n) + \# V(n) = \# D(n)$

can figure out   Known

<u>Bijection</u>: $D(n) \xrightarrow{\sim} \mathcal{U}(n) \sqcup V(n)$

$$(\lambda, \mu) \longmapsto \begin{cases} \begin{pmatrix} \mu_1 & \geq \lambda_1 \; \lambda_2 \cdots \\ & \geq \mu_2 \; \mu_3 \cdots \end{pmatrix} V(n) & \lambda_1 \leq \mu_1 \\ \underbrace{\lambda_k \cdots \lambda_1 \mu_1 \cdots \mu_\ell}_{\mathcal{U}(n)} & \lambda_1 > \mu_1 \end{cases}$$

$\lambda_1 \geq \lambda_2 \geq \cdots$

$\mu_1 \geq \mu_2 \geq \cdots$

1. well-defn. ✓

2. image what I want? ✓

$n = 11$

Ex. $\begin{Bmatrix} (3\,1\,1) \\ (4\,2) \end{Bmatrix} \longrightarrow \begin{pmatrix} & 3\;1\;1 \\ 4 & \\ & 2 \end{pmatrix}$

any thing in $V(11)$ that goes back to (λ, μ) is this.

$\mathcal{U}(n)$ $\overline{1\;1\;3}\; \lambda_1 > \mu_1 \; \overline{4\;2}$

break at 1st strict descent.

$\begin{pmatrix} 2 \\ 4\,3\,1\,1 \end{pmatrix} \in D(n)$