MATH 532 Discussion 12/10/2020

Partially Ordered SETs

Equivalence Relations on S

- reflexive: for all $x \in S$, $x = x$
- symmetric: for all $x, y \in S$, if $x = y$, then $y = x$
- transitive: $\forall x, y, z \in S$, if $x = y$ & $y = z$ then $x = z$

A partial order on a set S is a relation $\preceq$ (\prec) (\preceq)

- reflexive: $\forall x \in S$, $x \preceq x$
- antisymmetry: if $x \preceq y$ & $y \preceq x$ then $x = y$
- transitive: $\forall x, y, z \in S$ if $x \preceq y$ & $y \preceq z$ then $x \preceq z$.

NOTATION: $P = (\overset{\text{set}}{S}, \overset{\text{ordering}}{\preceq})$ $\qquad \preceq_P$

EXAMPLES

- $P = ([n], \leq)$ $\quad i \preceq_P j$ if $i \overset{\text{usual}}{\leq} j$

  $\{1 < 2 < 3 < \cdots < n\}$ TOTAL / LINEAR ORDER

- B_n = Boolean poset on $2^{[n]}$

  Objects are subsets of $[n]$
  order is containment.

  $n=3$: $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1 2\}$, $\{1 3\}$, $\{2 3\}$, $\{1 2 3\}$

  $\{2\} \leq \{1, 2\}$

  $\{2\}$ & $\{1, 3\}$ are incomparable

$(2^{[n]}, S \leq T \text{ if } \#S \leq \#T) \cong [n+1]$

• $D_n = ([n],\ i \le_{D_n} j$ iff j is evenly divisible by $i)$
$j \bmod i \equiv 0$

$\underline{n=12}$ What #s are ≤ 12? 1, 2, 3, 4, 6, 12
What #s are ≤ 8? 1, 2, 4, 8

incomparable: i incomp j iff $\gcd(i,j) \neq i$ or j

1 is the unique minimal element. $\hat{0}$

• Π_n partial order on set partitions of $[n]$ given by refinement.

$(\{1,3,4\}, \{2,6\}, \{5\})$ of $[6]$

$\le$

$\{1,3,4,5\}\ \{2,6\}$

$\{B_i\} \le \{C_i\}$ if $\forall i$ $B_i \subseteq C_j$ some j.

minimal element? $\{1\}\{2\} \ldots \{n\}$

maximal element? $\{1, 2, \ldots, n\}$.

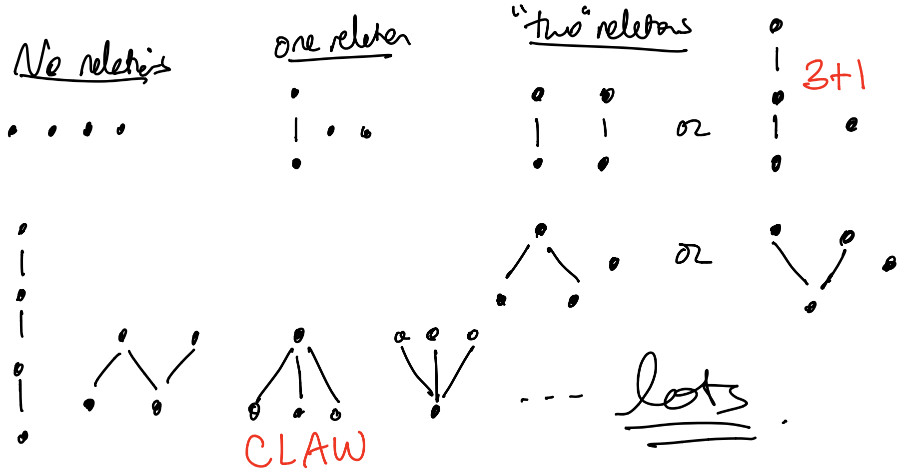
• $B_n(q)$ all subspaces of $\mathbb{F}_q^n$ ordered by inclusion.

HASSE DIAGRAM

b
$|$ if $a \le b$ and $\nexists c$ s.t. $a \le c \le b$
a $a \neq b$

List All Posets on 4 Objects

No relations one relation "two" relations



... lots.

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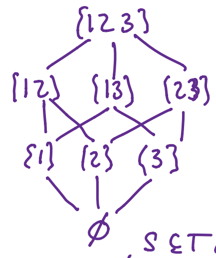
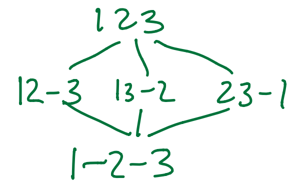
Hasse Diagram has node for every elt of P, edge $\begin{smallmatrix} t \\ | \\ s \end{smallmatrix}$ iff
- $s \le t$, $s \neq t$
- $\forall r \in P$ s.t. $s \le r \le t$, either $r = s$ or $r = t$.

"cover relations"

Ex. $([n], \le)$ $B_n = (S \subseteq [n], \subseteq)$ $\Pi_n = (B \vdash [n], \text{refinement})$





$F_p(x) = \sum_{i=0}^{n-1} x^i$

$= (x)_n$

$S \lessdot T$ if $S \subseteq T$ & $\# T \setminus S = 1$

$F_p(x) = \sum_{i=0}^{n} \binom{n}{k} x^k$

$= (1+x)^n$

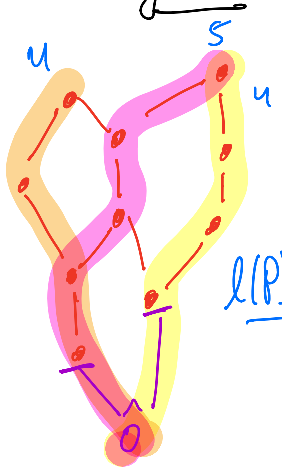
$B \lessdot C$ if merge 2 blocks

$rk(B) = n - \#$ blocks

$F_p(x) = \sum_{i=0}^{n-1} S(n, n-i)\, x^i$

- Say P has $\hat{0}$ if $\exists$ $\hat{0} \in P$ s.t. $\hat{0} \le t$ $\forall t \in P$.
  $\hat{1}$: $\hat{1} \in P$, $t \le \hat{1}$ $\forall t \in P$.
- Chain in P is a seq $(t_1, \dots, t_k)$ s.t. $t_i \le t_{i+1}$
  - Maximal if it's not properly contained in another chain.
  - saturated if $\nexists u \in P$ s.t. $t_i < u < t_{i+1}$ some i
- Rank of a finite poset P is $\ell(P) =$ max length saturated chain in P.

P is graded if all max'l chains have same length.



$\ell(P) = 5$

If P is graded, then for $t \in P$

$\ell(t) = rk(t) =$ max'l length chain $s_1 \lessdot \cdots \lessdot s_k < t$

$$F_p(x) = \sum_{i=0}^{n} p_i x^i$$

where $p_i = \#$ elts of P of rk i

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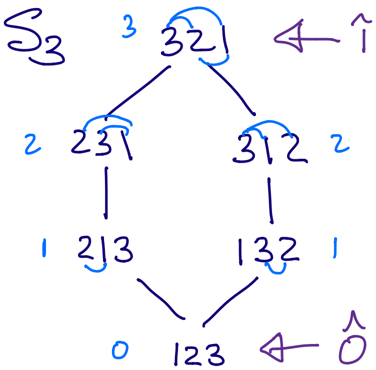
Weak order on S_n

$u \lessdot w$ if get w by swapping adjacent letters $u_i u_{i+1}$ when $u_i < u_{i+1}$

S_3



$\hat{1}$ reverse of identity

$\hat{0}$ identity

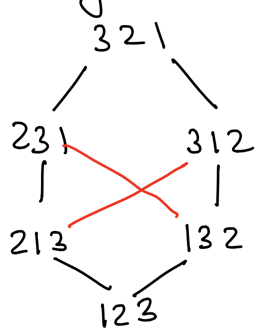
- Is there always $\hat{0}$? $\hat{1}$?
- What is the rank function? ↳ $rk(u) = inv(u) = \#\{(i<j) \mid u_i > u_j\}$.
- Anything else?

$$F_{S_n}(q) = \sum_{w \in S_n} q^{rk(w)} \;?$$

$$= 1 + 2q + 2q^2 + q^3$$

$$= (1+q)(1+q+q^2) = [n]_q!$$

Strong order on S_n



$u \lessdot w$ if Ⓐ $u_i < w_{i+1}$ swap u_i & w_{i+1}

Ⓑ i left of $i+1$ swap $\boxed{i \,\&\, i+1}$

$21\underline{3} \not\lessdot 231$

$213 \lessdot 312$

$\hat{0}$ same b/c still sort

$\hat{1}$? same

position = value at identity $\underline{u_i = i}$

An atom of a ranked poset w/ $\hat{0}$ is an element with only $\hat{0}$ below it.

Is $rk(u) = inv(u)$? $u \lessdot w$ $inv(w) \neq inv(u) + 1$

$\cdots\; \overset{b}{i} \;\cdots\cdots\; i+1 \;\cdots\cdots$ gain $(i+1, i)$

swap (i,b) for $(i+1, b)$

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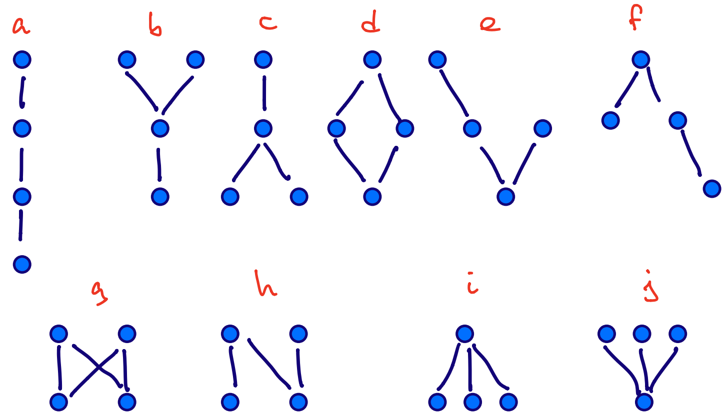
Connected Posets on 4 elements



Defn An upper bound for $s,t \in P$ is an elt $u \in P$ s.t. $s \leq u$ & $t \leq u$.

A least upper bound for $s,t \in P$ is an upper bound u s.t. if u' is also an upper bound, then $u \leq u'$.

Defn A lattice is a poset in which every pair of elts has a lub & glb.

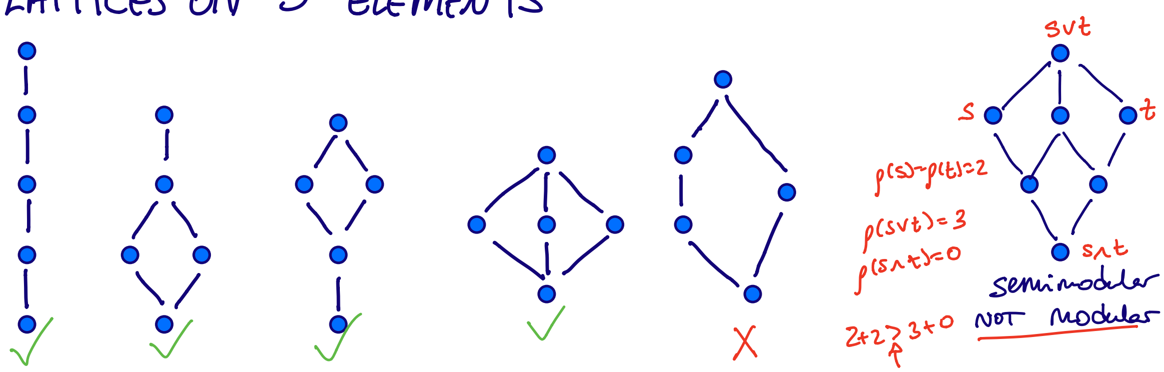
Q: Which of these posets is a lattice?

a & d

NOTATION: lub (least upper bound): $s \vee t$ "join"
glb (greatest lower bound): $s \wedge t$ "meet"

OBSERVE: A (finite) lattice always has $\hat{0}$ & $\hat{1}$.

LATTICES ON 5 ELEMENTS



$\rho(s)=\rho(t)=2$

$\rho(s \vee t)=3$

$\rho(s \wedge t)=0$

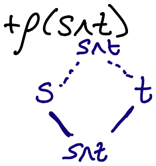
semimodular

$2+2 \geq 3+0$ NOT modular

Prop. L finite lattice. TFAE

(i) L is graded and the rk fn ρ satisfies $\rho(s)+\rho(t) \geq \rho(s \vee t)+\rho(s \wedge t)$

(ii) If both s & t cover $s \wedge t$, then $s \vee t$ covers s & t.



Defn A lattice satisfying (i) or (ii) is called semi-modular.

A modular lattice is graded with $\rho(s)+\rho(t)=\rho(s \vee t)+\rho(s \wedge t)$.

Prop L modular iff $s \vee (t \wedge u) = (s \vee t) \wedge u$

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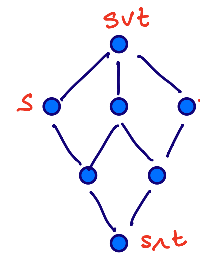
Finite Poset



- meet semilattice ($s \wedge t$ exists) $\Rightarrow \hat{0}$
- join semilattice ($s \vee t$ exists) $\Rightarrow \hat{1}$
- lattice : meets & joins
  - semimodular : graded & $\rho(s)+\rho(t) \geq \rho(s\wedge t)+\rho(s \vee t)$
    s, t cover $s\wedge t$ & both covers $\Rightarrow$ s, t covered by $s\vee t$ ← also covers.
  - modular : graded & $\rho(s)+\rho(t) = \rho(s\wedge t)+\rho(s\vee t)$
  - atomic : an atom is any $s \in \mathcal{L}$ s.t. $\hat{0} \lessdot s$
    ↳ every $u \in \mathcal{L}$ can be written $u = s_1 \vee s_2 \vee \cdots \vee s_k$ s_i's atoms
  - a lattice is complemented if $\forall s \in \mathcal{L}$ $\exists t \in \mathcal{L}$ s.t. $s \vee t = \hat{1}$ & $s \wedge t = \hat{0}$.

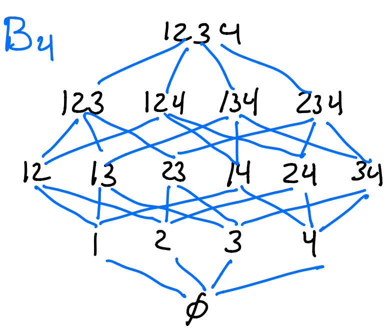
geometric lattice is a finite, semimodular atomic lattice.

Classify the following:

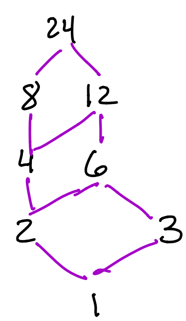
- $([n], \leq)$ modular lattice (not atomic, not complemented)
- $B_n = (S \subseteq [n], \subseteq)$ modular, complemented, atomic, geometric
- $D_n = ([n], i \leq j$ if $j \equiv 0 \bmod i)$
- Π_n = (set partitions of $[n]$, refinement) atomic geometric not modular.
- $B_n(q) = (\text{subspaces of } \mathbb{F}_q^n, \subseteq)$

BONUS weak order on S_n

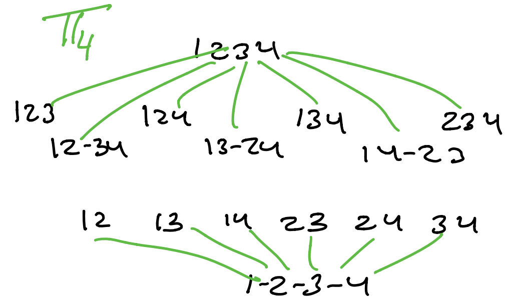
B_4



D_{24}



Π_4



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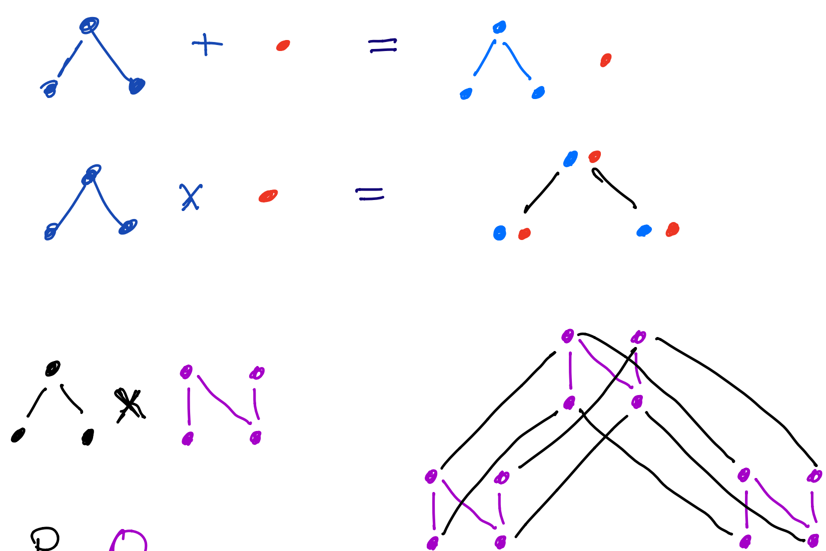
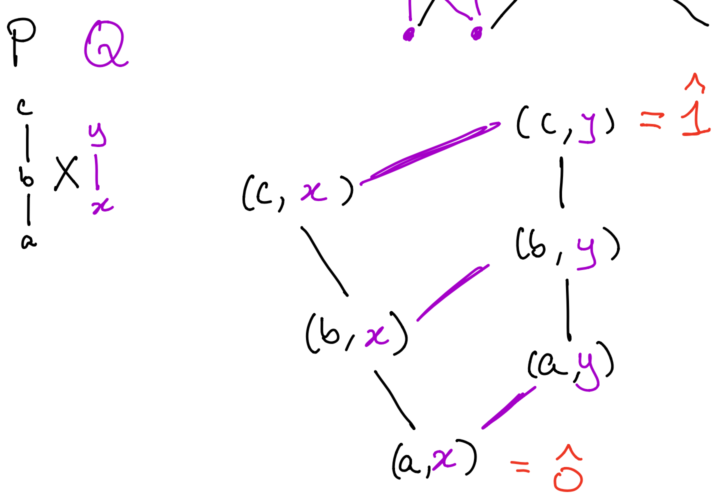
P, Q posets

$P+Q$ ($P \cup Q$): $s \leq t$ if $s,t \in P$ & $s \leq t$, $s,t \in Q$ & $s \leq t$

$P \times Q$ ($(p,q) \in P \times Q$): $(p,q) \leq (p',q')$ if $p \leq p'$ and $q \leq q'$

EXAMPLES



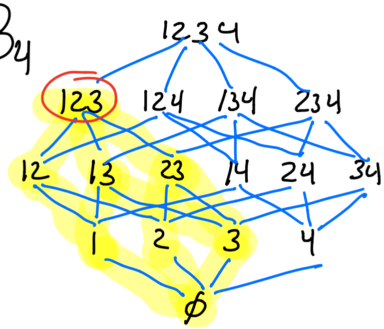


(b,x) (a,y)

$b >_P a$ $x <_Q y$

Defn An order ideal of A in P $(A \subseteq P)$ is
$$I_A = \{ s \mid s \leq t \text{ for some } t \in A \}$$

B_4



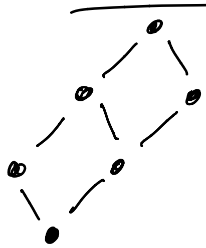
$$\mathcal{I}_{123} \cong B_3 \cong \mathcal{I}_{134}$$

234

$$\mathcal{I}_{12,234} = \left\{ \begin{matrix} 12 & 23 \; 24 \; 34 \\ 1 \quad 2 & 3 \quad 4 \\ \emptyset & \end{matrix} \right\} \cong \mathcal{I}_{12}$$

Defn Given a poset P, define the lattice of order ideals $J(P)$ with inclusion as relation.

P

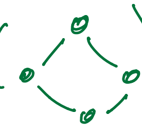


$J(P)$

lattice

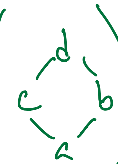
WARM UP:

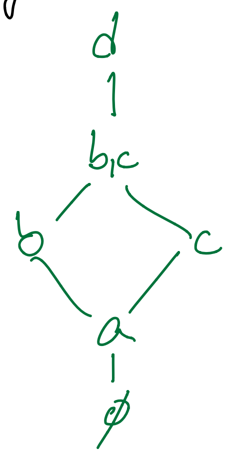
$\to J($  $)$

$\to J($  $)$

Defn An antichain is a subset $A \subseteq P$ s.t. x, y incomparable $\forall\, x, y \in A$ distinct.

$$J\begin{pmatrix} c \\ | \\ b \\ | \\ a \end{pmatrix} = \begin{matrix} \mathcal{I}_c = \{a,b,c\} \\ | \\ \mathcal{I}_b = \{a,b\} \\ | \\ \mathcal{I}_a = \{a\} \\ | \\ \mathcal{I}_\emptyset = \emptyset \end{matrix}$$

$J($  $) =$



Defn A lattice $\mathcal{L}$ is distributive if

$$s \vee (t \wedge u) = (s \vee t) \wedge (s \vee u)$$
$$s \wedge (t \vee u) = (s \wedge u) \vee (s \wedge t)$$

B_n is dist.
$B_n(q)$ is not.

Theorem $\mathcal{L}$ finite distributive lattice. Then $\exists!$ poset P s.t. $\mathcal{L} \cong J(P)$.

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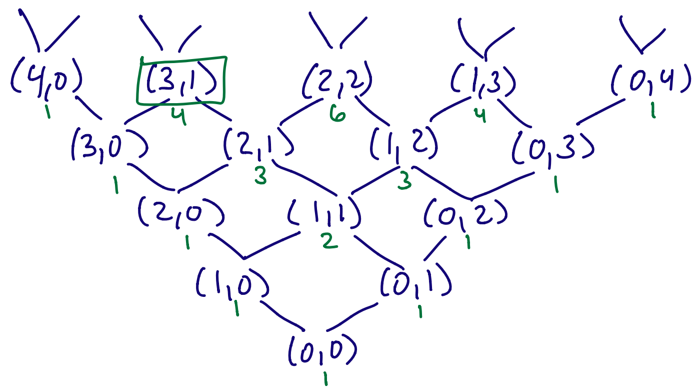
$P = \mathbb{N} \times \mathbb{N}$ $\{(a,b) \mid a,b \in \mathbb{N}\}$

$(a,b) \le (a',b')$ if $\begin{matrix} a \le a' \\ b \le b' \end{matrix}$



- cover relations?
  add 1 to x or y comp.
- rank function?
  $\rho(a,b) = a+b$
- how many elts of rk i? $i=4$
  $i+1$.

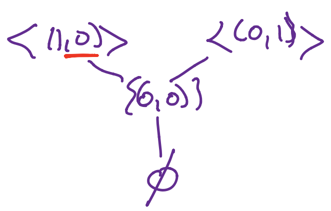
Count SATURATED CHAINS from $\hat{0}$ to t.

(SATURATED: skip no steps; CHAINS: Linear order (group!))

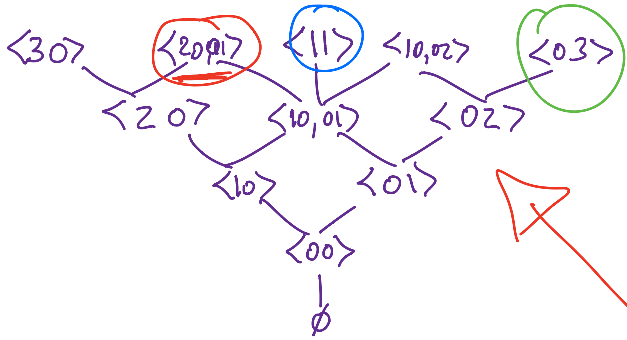
antichain

Recall an order ideal for a subset $A \subseteq P$ is
$$\{s \in P \mid s \le t \text{ for some } t \in A\}$$
A principle order ideal is where $A = \{t\}$.
The lattice of order ideals of P is $J(P)$,
the set of order ideals ordered by inclusion as sets

- order ideals $\leftrightarrow$ antichains → set of pairwise incomp elts.

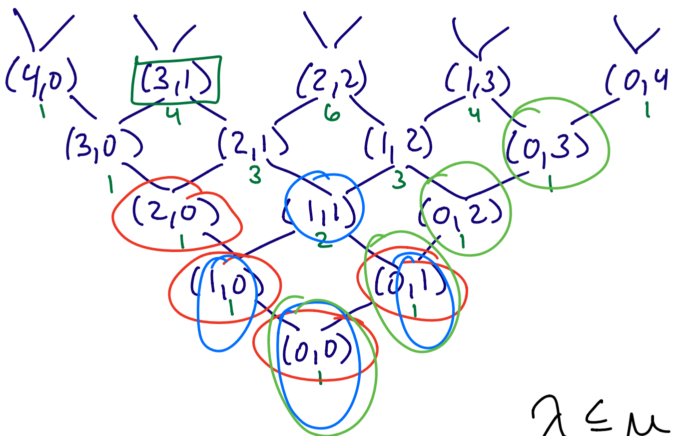


- Draw ≥ 2 more levels
- cover relations?
- rank function?
- how many elts of each rank?
- how many saturated chains?

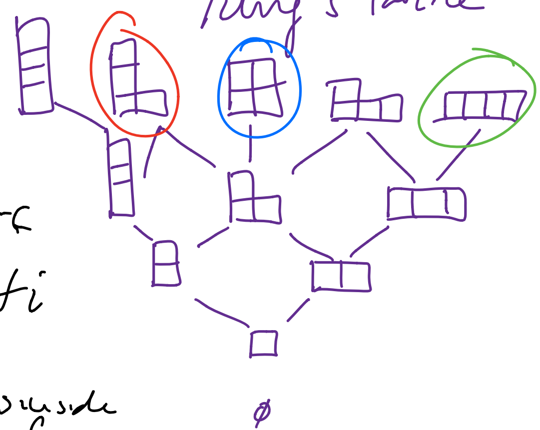


If $s \le t$ in P,
is $I_s \le I_t$ in $J(P)$?

NO... $\langle 01 \rangle \not\le \langle 11 \rangle$



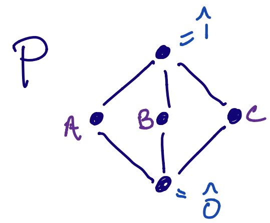
Young's lattice



$\lambda \le \mu$ if
$\lambda_i \le \mu_i \ \forall i$

diagram of λ fits inside
diagram of μ

MATH 532 Discussion 28/10/2020



Möbius function of a poset P.

$$\mu(s,s) = 1 \quad \forall s \in P$$

$$\mu(s,u) = -\sum_{s \le t < u} \mu(s,t)$$

$s < u \in P$

$$\mu(\hat{0},\hat{0}) = 1$$
$$\mu(\hat{0},A) = -1$$
$$\mu(\hat{0},B) = -1$$
$$\mu(\hat{0},C) = -1$$
$$\mu(\hat{1},\hat{1}) = 1$$

$$\mu(A,A) = 1$$
$$\mu(A,\hat{1}) = -1$$
$$\mu(B,B) = 1$$
$$\mu(B,\hat{1}) = -1$$
$$\mu(C,C) = 1$$
$$\mu(C,\hat{1}) = -1$$

$$\mu(\hat{0},\hat{1}) = -\sum_{\hat{0} \le t < \hat{1}} \mu(\hat{0},t)$$
$$= -\left(\mu(\hat{0},\hat{0}) + \mu(\hat{0},A) + \mu(\hat{0},B) + \mu(\hat{0},C)\right)$$
$$= -(1 + -3) = 2$$

Suppose we have set A, B, C and suppose

$$D = A \cap B = A \cap C = B \cap C.$$

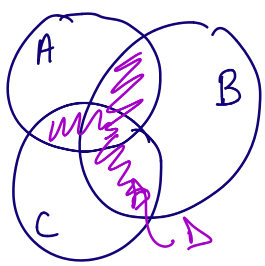
Compute $|A \cup B \cup C| = |A| + |B| + |C|$

$- |A \cap B| - |A \cap C| - |B \cap C|$

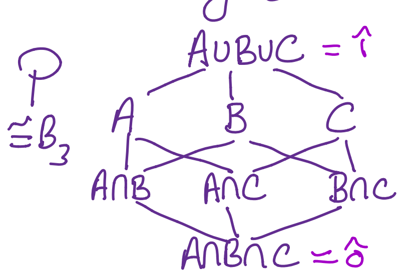
$+ |A \cap B \cap C|$

$= |A| + |B| + |C| - 3|D| + |D|$

$= |A| + |B| + |C| - 2|D|$



Given finite sets A, B, C, let P be the poset on intersections ordered by containment. Empty intersection is $A \cup B \cup C$.

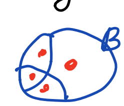


For $T \in P$, define functions:

$g(T) = \#T$ (e.g. $g(\hat{1})$)

$f(T) = \#\{t \in T \mid t \notin T' \text{ any } T' \lneq_P T\}$

NOTE: $g(T) = \sum_{s \le t} f(s)$.



Theorem (Möbius Inversion)

Let P poset s.t. every poi is finite. Let $f, g : P \to K$ (any field). Then

$$g(t) = \sum_{s \le t} f(s) \quad \forall t \in P \quad \text{iff} \quad f(t) = \sum_{s \le t} g(s)\mu(s,t) \quad \forall t \in P.$$

Return to example Compute $g(\hat{1}) = |A \cap B \cap C|$.

Know $g(T) = \sum_{S \in T} f(S)$. By Möbius inversion

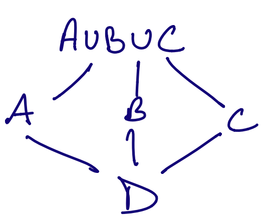
$$0 = f(\hat{1}) = \sum_{T \in P} g(T)\mu(T,\hat{1})$$

$$0 = g(\hat{1})\mu(\hat{1},\hat{1}) + \sum_{T < \hat{1}} g(T)\mu(T,\hat{1})$$

$$g(\hat{1}) = -\sum_{T < \hat{1}} \#T \cdot \mu(T,\hat{1}) \to \{(-1)^{\text{distance } T \text{ to } \hat{1}}$$

b/c alt sum of binom coeffs is 0.

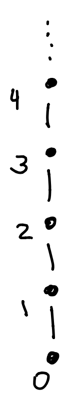
TOP EX



$$|A \cup B \cup C| = -\big((-1)|A| + (-1)|B| + (-1)|C| + (+2)|D|\big)$$

$$= |A| + |B| + |C| - 2|D|.$$

EXAMPLE Let P be $\mathbb{N}$. Compute $\mu(i,j)$.



$\mu(3,5) = 0$

$\mu(3,4) = -1$

$\mu(3,3) = 1$

$\mu(2,3) = -1$

$\mu(1,3) = 0$

$\mu(0,3) = 0$

$$\mu(i,j) = \begin{cases} +1 & i = j \\ -1 & i+1 = j \\ 0 & \text{else} \end{cases}$$

$$g(n) = \sum_{i=0}^{n} f(i) \quad \forall n > 0$$

iff

$$f(n) = g(n) - g(n-1) \ \&\ f(0) = g(0)$$