Exercise Sheet 4

Question 4.1 Prove the following claims by using the principle of proof by contradiction. In each case suppose that there is a *minimal* element that does not have the desired property.

1. The sum of the first n odd positive integers is a perfect square.
2. A round robin tournament with n players (and no draws) has a cycle of length i, for $i \leq n$, if there are players $p_1, \dots, p_i$ such that player p_1 beats player p_2, player p_2 beats player p_3, ..., and p_i beats p_1. Then whenever the tournament has a cycle, it has a cycle of length 3.

Question 4.2 Consider the formula $(p \wedge (q \vee (\neg r)))$.

1. Compute its height by expanding the recursive definition of h step by step.
2. Let P be a set of propositional variables and $v : P \to \{0, 1\}$ a fixed valuation (here we use 0 instead of F and 1 instead of T). Define a recursive function $i : \Phi \to \{0, 1\}$ that interprets all formulas defined by

   $$\Phi ::= p \in P \mid (\neg\Phi) \mid (\Phi \vee \Phi) \mid (\Phi \wedge \Phi).$$

   For this, write down equations $i(p) = \dots$, $i((\neg\phi)) = \dots$, $i((\phi \vee \psi)) = \dots$, $i((\phi \wedge \psi)) = \dots$.
3. Apply your recursive interpretation function step by step to the above formula for the valuation $v(p) = 1$ and $v(q) = v(r) = 0$.

Question 4.3 The *length* $l : \Phi \to \mathbb{N}_{>0}$ of a formula can be defined recursively as

$$l(p) = 1, \qquad l((\neg\phi)) = 3 + l(\phi), \qquad l((\phi \vee \psi)) = 3 + l(\phi) + l(\psi) = l((\phi \wedge \psi)),$$

where Φ is given by the grammar from Q 4.2. Prove by structural induction that, for all $\phi \in \Phi$,

$$l(\phi) \geq h(\phi).$$