Exercise Sheet 3

These exercises are mainly about

- the use of the inference rules of classical propositional logic,
- the use of derived rules and facts as lemmas (global hypotheses) in proofs.

You can freely use $\phi_1, \phi_2 \dots, \phi_n \vdash \phi$ instead of $\vdash \phi_1 \to (\phi_2 \to \dots \to (\phi_n \to \phi) \dots)$ and vice versa. You can also substitute logically equivalent subformulas in proofs (after having solved Question 3.1) and derive your own lemmas to be used in more complex proofs.

Question 3.1 Using the definition and the derived rules for the biconditional from the lecture notes, show that

(a) $\phi \leftrightarrow \psi, \psi \leftrightarrow \chi \vdash \phi \leftrightarrow \chi$,

(b) $\phi \leftrightarrow \psi \vdash \phi \vee \chi \leftrightarrow \psi \vee \chi$,

(c) $\phi \leftrightarrow \psi \vdash \phi \wedge \chi \leftrightarrow \psi \wedge \chi$,

(d) $\phi \leftrightarrow \psi \vdash \neg\phi \leftrightarrow \neg\psi$.

Question 3.2 Prove that $\vdash ((\varphi \to \psi) \to \varphi) \to \varphi$.

Question 3.3 Show that the connectives $\vee$, $\wedge$ and $\neg$ can be expressed in terms of $\to$ and $\bot$.

(a) Show that $\neg\phi \dashv\vdash \phi \to \bot$, $\phi \vee \psi \dashv\vdash \neg\phi \to \psi$ and $\phi \wedge \psi \dashv\vdash \neg(\neg\phi \vee \neg\psi)$.

(b) Derive the natural deduction rules for $\vee$ and $\wedge$ from those for $\neg$ and $\to$. You may use the classical rules in derivations. Express negations, conjunctions and disjunctions in terms of $\neg$ and $\to$. For the first conjunction elimination rule, for instance, show that $\neg(\phi \to \neg\psi) \vdash \phi$.

Question 3.4 Show that the three ways of extending intuitionistic propositional logic to classical propositional logic are equivalent in the sense that the same formulas are derivable in all three extensions.