8a - Dimensionality Reduction I

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Lecture Objectives

In this lecture we will,

- Consider the problems of using very large feature vector.
- Explain what is meant by ‘the curse of dimensionality’.
- We will try to develop an intuition for high dimensional spaces by comparing 1-d and 2-d distributions.
- Introduce the idea of dimensionality reduction.

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Recap

- Last week we looked at Feature Selection.
- Using large numbers of features is problematic (more on this today)
  - May increase the number of training examples we need for good performance.
  - For some classifiers adds to the memory costs.
  - Nearly always increases computational cost.
- Feature Selection is the process of reducing the feature vector size by choosing a small subset of the raw feature set.

Recap: a feature selection system

- We want to select a set of features that maximises classification performance (i.e. produces few classification errors).
- There are typically two components to a feature selection system: a **class separability measure**, and the **feature selection algorithm**.
- The separability measure is a measure of how separated the classes are given a particular set of features (e.g. divergence).
- We use a separability measure because we can't usually find the set of features by trying them on the test data directly.
- For a well-designed separability measure a high separability will be a good predictor of loss classification error rate.
- Designing a good separability measure is not easy.

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Recap: feature selection algorithms

- **Rank features** according to the class separability that each provides and take the best N.
- **Forward sequential selection**: build the vector up starting with 1 feature and picking the vector that maximises separability at each size.
- **Backward sequential selection**: prune the vector down starting with all the features and picking the vector the maximises separability at each size.
- **Brute force**: try all (or large proportion of all) possible feature vectors. Only really possible for toy problems.
- Many other more complicated variants.

overview

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- **Dimensionality Reduction**
  - Feature Selection is one particular type of *dimensionality reduction*, i.e. we just throw away the least useful dimensions.
  - This week we will be looking at a more general way of reducing dimensionality.
- **Today**: Further discussion of problems caused by high dimensionality.
- **Next Lecture**: Principle Components Analysis; Linear Discriminant Analysis

Face Recognition

- Attach a name label to an unidentified face image (classification task).
- Classifier is trained using a number of labelled examples for each individual.

Many security applications:

Example 1

Example 2

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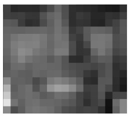
Representation of an image

- Assume all face images have been rescaled to a fixed size, e.g., 17 by 17 pixel.
- For a grey scale image, each pixel is represented by an 8-bit number (i.e.,0-255) corresponding to the grey level.
  - typically $0 =$ black, $255 =$ white



Representation of an image

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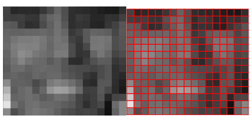
- If we zoom in we can see the individual pixels.



- The n by n grid of grey levels can be laid out into one long feature vector containing n^2 elements.
- For a 17 by 17 image the feature vector would contain 289 features, $\mathbf{x} = \{x_1, x_2, \ldots, x_{289}\}$.

Aside: Comparing face recognition and character recognition

- Characters = 1 of 26; Faces = 1 of 100's or 1000's.
- Characters, 2-d images; Faces 2-d projections of a 3-d form.
- Characters, pixels essentially black/white; Faces, grey level is important.
- Characters, low resolution may suffice; Faces, probably need higher resolution (i.e. more pixels)
- Characters, separable by local features; Faces, holistic, c.f. Gestalt perception
- Characters, within-class variability probably low; Face, within-class variability very high.
- Characters, class clearly separable; Faces, classes possibly less separable.
- Conclusion: Face recognition is a harder problem. Would our simple feature selection strategy be effective?

Curse of dimensionality

- Large feature vectors are bad news...
- Remember, classification depends on comparing $p(\mathbf{x})$ for each class of data.
- $p(\mathbf{x})$ – the true distribution – isn't ever known precisely. It is estimated from example training data.
- Now, if $\mathbf{x}$ has$n$ elements then $p(\mathbf{x})$ is an n dimensional distribution, e.g. $p(\mathbf{x})$ for the raw face images would be a $17 \times 17 = 289$ dimensional distribution.
- Let's consider the implications of this.

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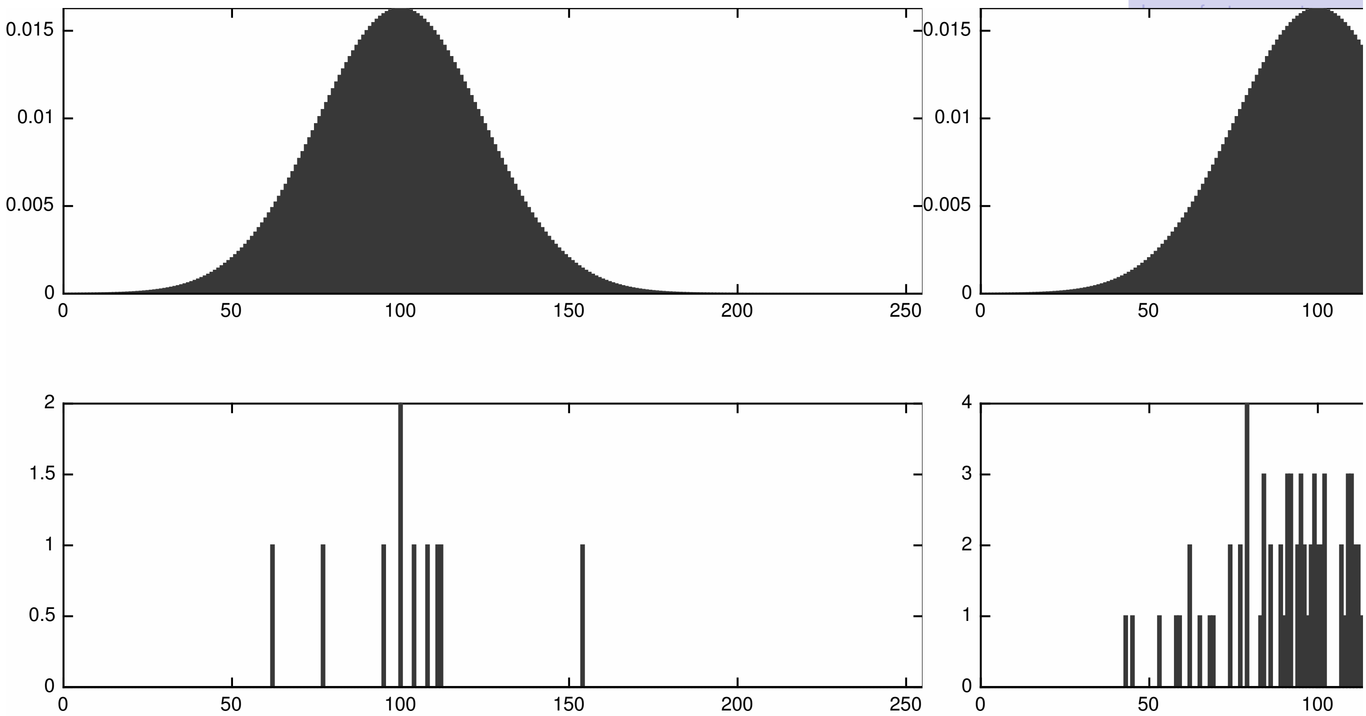
Estimating a 1-d distribution

- Consider the distribution of gray levels for a single pixel, $p(x)$
- x can have any one of 256 values, so the true $p(x)$ can be represented exactly by a histogram with 256 bins.
- We estimate the true $p(x)$ by constructing a histogram from training examples
- $p(x = n)$ = number of times x has the value n in the training data divided by total number of training examples
- The more training examples that are used the more accurate the histogram becomes, i.e. the closer the estimated $p(x)$ will be to the true $p(x)$.

Estimating a 1-d distribution

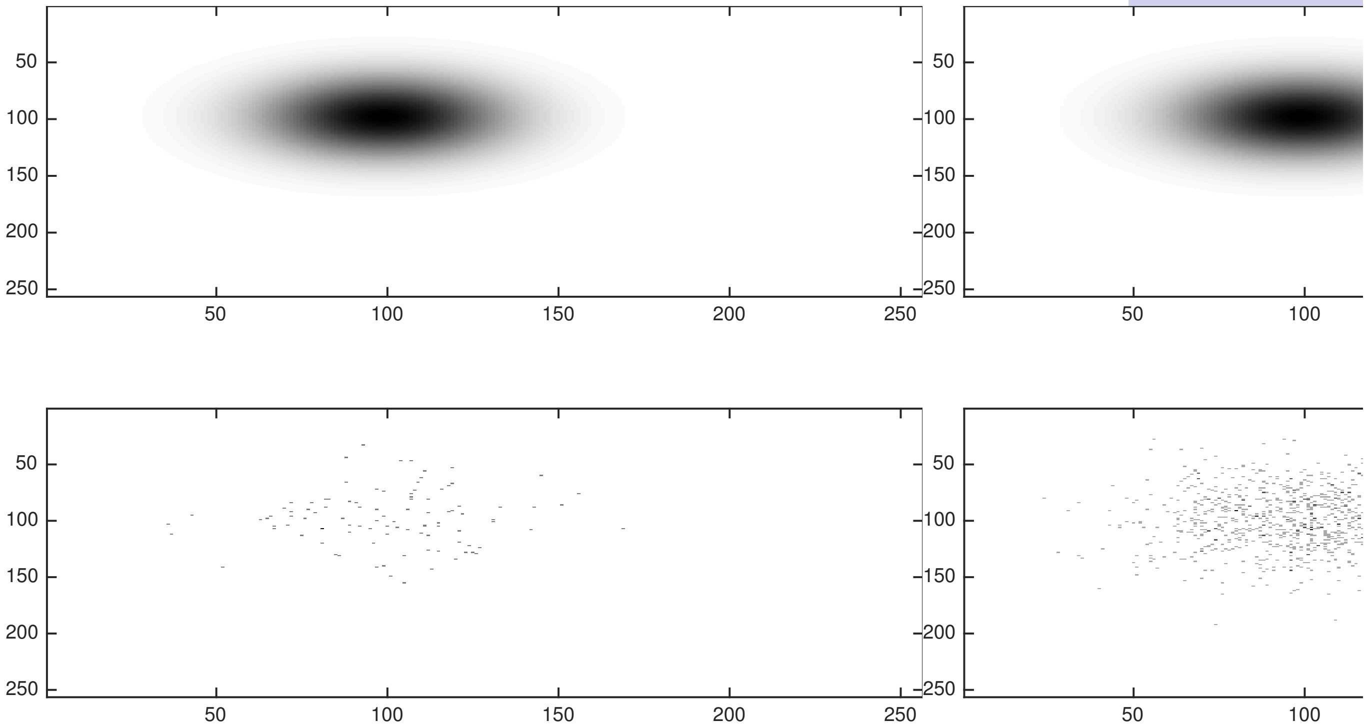
- Estimating the true histogram using progressively more samples



- 10 samples
- 100 samples
- 1,000 samples

Estimating a 2-d distribution

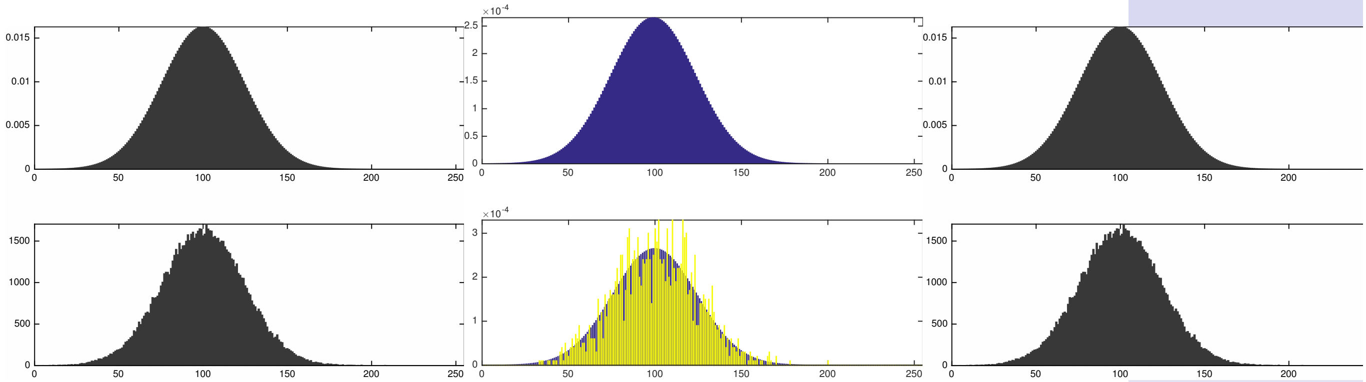
- 2-D histogram represented as an image



- 100 samples
- 1,000 samples
- 10,000 samples
- 100,000 samples

1-d slice through 2-d histogram

- Whereas 100,000 samples produced a good approximation to the 1-d distribution, it is far less good for the 2-d distribution.
- Left: 1d; Right: slice through 2d



- 100,000 samples versus 100,000 samples
- 100,000 samples versus 1,000,000 samples

Using a parametric model

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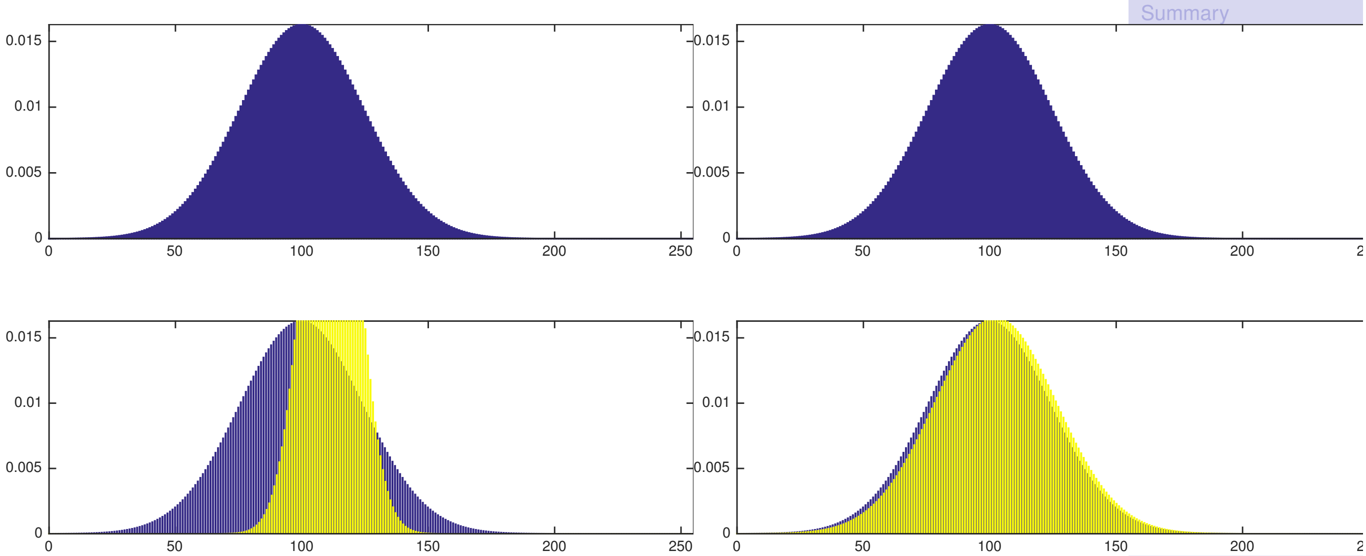
- In practice, we would rarely estimate the histogram directly.
- Instead we would use a parametric model: here we make an assumption about the parametric form of the true distribution (e.g. Gaussian, Laplacian etc) and then attempt to estimate the model parameters (e.g. mean and variance for a Gaussian)



- 3 samples
- 10 samples
- 100 samples

Using a parametric model

- Even with a parametric model we need more samples to estimate the model parameters as the dimensionality increases
- Number of free model parameters increases as dimensionality increases
  - 1-d gaussian has 2 parameters (mean, variance)
  - 2-d gaussian has 5 parameters (2 means, 2 variances, 1 covariance)
  - 3-d gaussian has 9 parameters (3 means, 3 variances, 3 covariances)
  - n-d has $\frac{n}{2}(n+3)$ i.e., $\mathrm{O}\left(n^2\right)$

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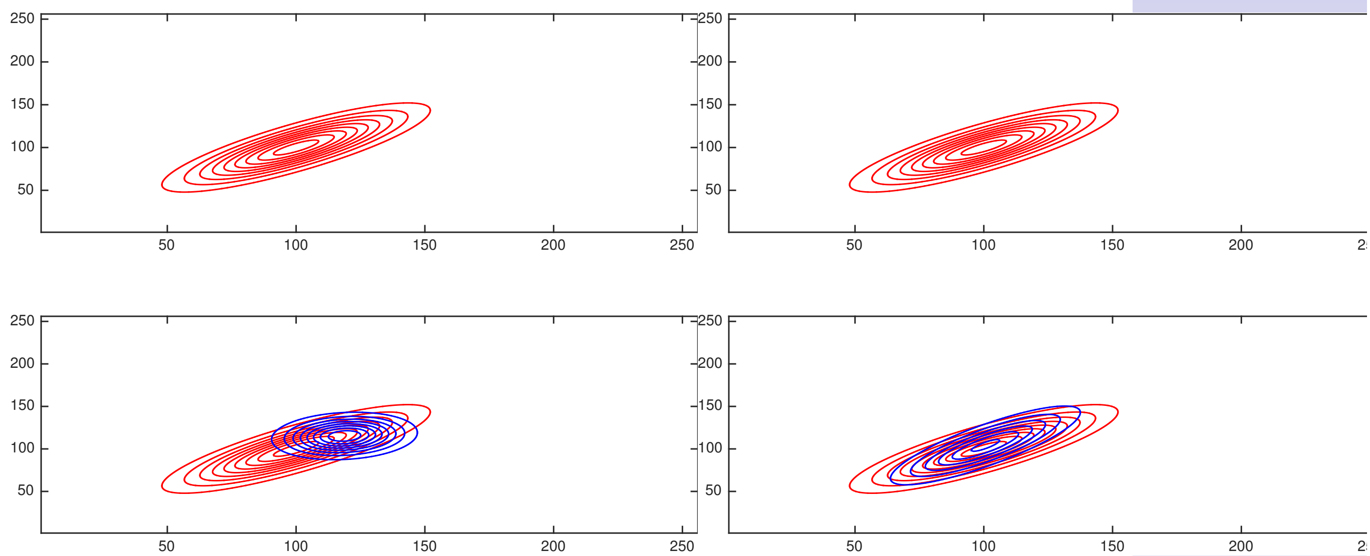
Estimating a 2 D gaussian

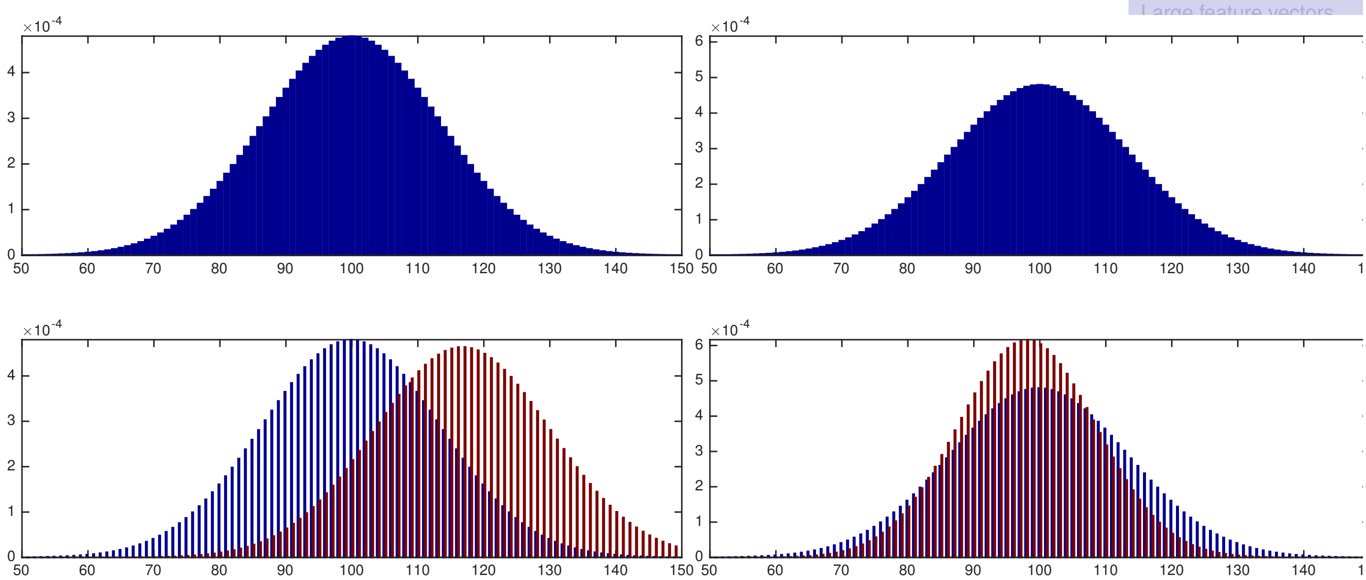
- Contours of true distribution versus estimate using increasing number of samples



- 3 samples
- 10 samples
- 100 samples
- 500 samples

1-D slice through 2-D gaussian

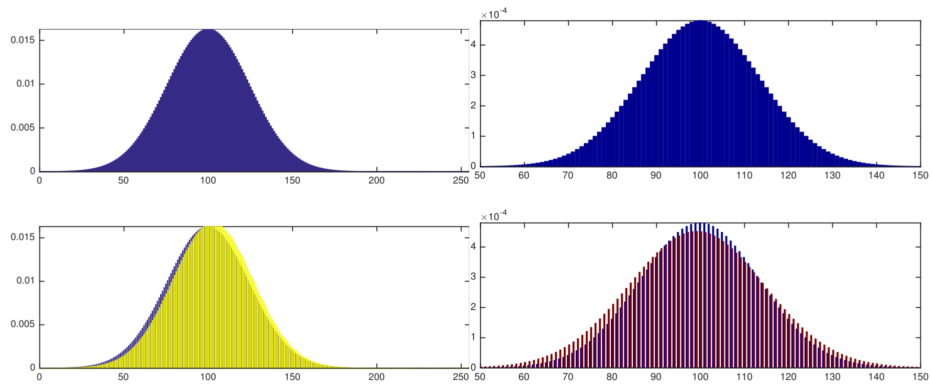
- Estimate converges on true distribution as number of samples increases



- 3 samples
- 10 samples
- 100 samples
- 500 samples
- Match becomes close at around 500 samples

Comparing fit for 1-D and 2-D distributions

- Using 100 samples parameters of 1-D distribution well-estimated, whereas those of 2-D distribution are not.
- Left: 1d; Right: slice through 2d



How can we reduce dimensionality of $\mathbf{x}$

Consider our face data.

- Select some subset of elements, e.g. keep just a line of pixels down the center of the image
  - Will loose information.
  - How do we select which pixels to keep. . . ?
- Use feature selection techniques like those discussed last week
  - But are any individual pixels likely to discriminate between classes?
  - People can't be identified by looking at individual pixels.
  - Need to find features that are less 'local'.
- We are going to exploit the correlation between the features
  - Note, adjacent pixels tend to have similar values – they are correlated
  - A pair of correlated features hold less information that a pair of independent features
  - Intuitively, the 'effective' dimensionality of the face images in less than 17×17

Summary

- If $\mathbf{x}$ has a small number of elements, then p($\mathbf{x}$) can be modelled with a smaller number of parameters
  - the parameters will require less storage
  - the computation of $p(\mathbf{x})$ will be faster
  - the parameters can then be robustly estimated with a smaller number of training examples
- Feature selection is a form of dimensionality reduction.
- However, it is not always appropriate: e.g. faces probably can't be robustly classified on the basis of a small number of pixel values.

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