COM2003

Automata, Logic and Computation

Slides originally written by Lucia Specia
based on those by M.S. Moorthy
who relied on slides by Prof Costas Busch

Reading weeks are week 5 and 12.

Regular Expressions

Regular Expressions

Instead an **automaton**, one can use a RE as a way to describe a regular language

Example: $(a+b\cdot c)*$ = also written: $(a+b\circ c)*$

$(a+bc)*$

describes the language

$$\{a,bc\}* = \{\varepsilon, a, bc, aa, abc, bca, \ldots\}$$

Recursive Definition

Primitive regular expressions: $\emptyset, \quad \varepsilon, \quad \alpha$

Given regular expressions r_1 and r_2

$$\left.\begin{array}{l} r_1 + r_2 \xrightarrow{\text{also written as}} r_1 \cup r_2 \\ r_1 \cdot r_2 \\ r_1{}^* \\ (r_1) \end{array}\right\}$$

Are regular expressions

Examples

A regular expression: $(a+b\cdot c)^{*}\cdot(c+\varnothing)$

Not a regular expression
(syntactically incorrect): $(a+b+)$

Other regular expressions:

$$\Sigma^{*} \qquad \Sigma^{*}1 \qquad (0\Sigma^{*})\cup(\Sigma^{*}1)$$

$$(a\cup b\cdot c)^{*} \qquad \Sigma\Sigma^{*}1 = \Sigma^{+}1$$

1 or more times

Examples

Precedence is just like in arithmetic:

5+3x4 not the same as (5+3)x4

In RE, unless () indicate otherwise:

$* \; + \quad \circ \quad +$ (union)

E.g.: $(a+b\circ c)^*$

$(a+b)\circ c^*$

$a+b\circ c^*$

Important: difference between r^+ and r_1+r_2

Why do we care about RE?

- Programs involving **text**:
  - Search for strings that follow certain patterns, e.g. negative words in product reviews: il-, im-, in-, ir-, non-, un-
  - E.g.: happy vs unhappy
  - Part of modern languages like Perl and Python, plus AWK and GREP in Unix
- **Compilers** for programming languages:
  - Tokens (variable names and constants) may be described by REs, based on which automatic systems can generate a **lexical analyzer** – first step of a compiler

Lexical analyser I use in research to process Erlang terms

```java
public static Lexer buildLexer(String whatToParse) {
    return new Lexer("(\\s*\\{\\s*)|" + // erlTupleBegin
            "(\\s*}\\s*)|" + // erlTupleEnd
            "(\\s*\\[\\s*)|" + // erlListBegin
            "(\\s*]\\s*)|" + // erlListEnd
            "(\\s*<<\\s*)|" + // erlBitStrBegin
            "(\\s*>>\\s*)|" + // erlBitStrEnd
            "(\')|" + // erlAtomQuote
            "(\")|" + // erlString
            "(\\s*\\+?\\d+\\s*)|" + // erlPositiveNumber
            "(\\s*-\\d+\\s*)|" + // erlNegativeNumber
            "(\\\\)|" + // erlBackslash
            "(\\s*,\\s*)|" + // erlComma
            "(\\s*\\|\\s*)|" + // erlBar
            "(\\s+)|" + // erlSpaces
            "(>)|" + // erlGT
            "(<)|" + // erlLT
            "(:)|" + // erlCol
            "(-)|" + // erlMinus
            "(\\+)|" + // erlMinus
            "(/)|" + // erlSlash
            "(\\.)|" + // erlDot
            "([eE])|" + // erlE
            "([^\\\\\\"\',\\|\\[\\]{}<>:\\-/ ]+)" // erlText, together with
                                                  // spaces but none of the
                                                  // special characters above.
    , whatToParse);
}
```

Tuple is {a,b},
List is [a,b]
BitString « … »
Atom starts with '
String with "

Unlike what you see in COM2003, RE in practice have an order, in that expressions earlier in the list have a priority over the later ones.

https://github.com/kirilluk/statechum/blob/master/src/statechum/analysis/Erlang/ErlangLabel.java

Languages of Regular Expressions

$L(r)$: language of regular expression r

Example

$$L((a+b\cdot c)^*)=\{\varepsilon,a,bc,aa,abc,bca,\ldots\}$$

Definition

For primitive regular expressions:

$$L(\emptyset)=\emptyset$$

$$L(\varepsilon)=\{\varepsilon\}$$

$$L(a)=\{a\}$$

Definition (continued)

For regular expressions r_1 and r_2

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

$$L((r_1)) = L(r_1)$$

Example

Regular expression: $(a+b)\cdot a^*$

$$
\begin{aligned}
L((a+b)\cdot a^*) &= L((a+b))\,L(a^*) \\
&= L(a+b)\,L(a^*) \\
&= (L(a)\cup L(b))\,(L(a))^* \\
&= (\{a\}\cup\{b\})\,(\{a\})^* \\
&= \{a,b\}\,\{\varepsilon,a,aa,aaa,\ldots\} \\
&= \{a,aa,aaa,\ldots,b,ba,baa,\ldots\}
\end{aligned}
$$

Example

Regular expression $r = (a+b)^*(a+bb)$

$$= (\{a\} \cup \{b\})^* (\{a\} \cup \{bb\})$$

$$= (\{a,b\})^* (\{a,bb\})$$

$$L(r) = \{a, bb, aa, abb, ba, bbb, ...\}$$

Example

Regular expression $r = (aa)^*(bb)^*b$

$$L(r) = \{a^{2n}b^{2m}b: \quad n, m \geq 0\}$$

Example

Regular expression $r = (0+1)^* 00 (0+1)^*$

$L(r)$ = { all strings containing substring 00 }

Example

Regular expression $r = (1 + 01)^* (0 + \varepsilon)$

$L(r)$ = { ??? }

Example

Regular expression $r = (1 + 01)^*(0 + \varepsilon)$

$L(r)$ = { all strings without substring 00 }

Equivalent Regular Expressions

Definition:

Regular expressions r_1 and r_2

are **equivalent** if $L(r_1) = L(r_2)$

Example

L = { all strings without substring 00 }

$$r_1 = (1 + 01)^*(0 + \varepsilon)$$

$$r_2 = (1^*011^*)^*(0 + \varepsilon) + 1^*(0 + \varepsilon)$$

$$L(r_1) = L(r_2) = L \Rightarrow$$

r_1 and r_2 are equivalent regular expressions

Regular Expressions and Regular Languages (and thus, Finite Automata!)

REs and **DFA/NFA** are equivalent in their description power: RE can be converted into finite automata that recognizes the same (regular) language and vice versa.

Theorem

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Generated by}\\ \text{Regular Expressions}\end{array}\right\} = \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

Proof:

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

If a language is described by a RE, then it is regular

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

If a language is regular, then it described by a RE

Proof - Part 1

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

If a language is described by a RE, then it is regular

For any regular expression r

the language $L(r)$ is regular

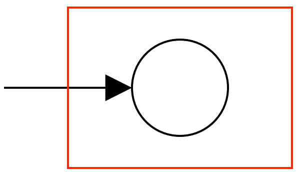
Proof by induction: inductive proof means defining RE in terms of smaller REs

Induction Basis

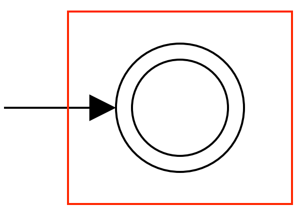
Primitive Regular Expressions: $\emptyset$, ε, α

Corresponding

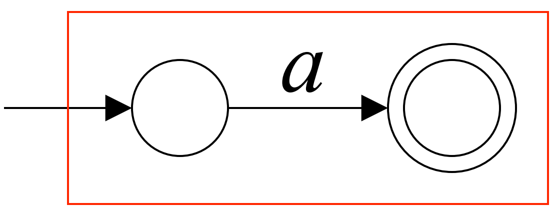
NFAs



$$L(M_1) = \emptyset = L(\emptyset)$$



$$L(M_2) = \{\varepsilon\} = L(\varepsilon)$$



$$L(M_3) = \{a\} = L(a)$$

regular languages

Inductive Hypothesis

Suppose

that for regular expressions r_1 and r_2,

$L(r_1)$ and $L(r_2)$ are regular languages

Inductive Step

We will prove:

$$L(r_1 + r_2)$$

$$L(r_1 \cdot r_2)$$

$$L(r_1 *)$$

$$L((r_1))$$

Are regular Languages

By definition of regular expressions:

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

$$L((r_1)) = L(r_1)$$

By inductive hypothesis we know:

$L(r_1)$ and $L(r_2)$ are regular languages

We also know:

Regular languages are closed under:

Union	$L(r_1) \cup L(r_2)$
Concatenation	$L(r_1)\, L(r_2)$
Star	$(L(r_1))^*$

Therefore:

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\,L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

Are regular languages

$$L((r_1)) = L(r_1)$$

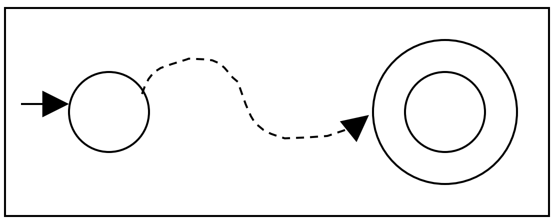
is trivially a regular language (by induction hypothesis)

End of Proof-Part 1

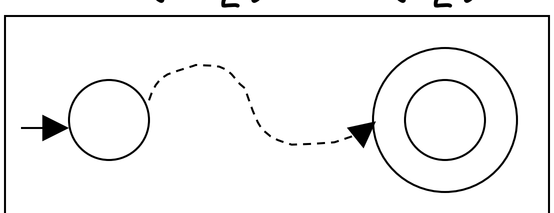
Using the regular closure of these operations, we can construct recursively the NFA M that accepts $L(M) = L(r)$

Example: $r = r_1 + r_2$

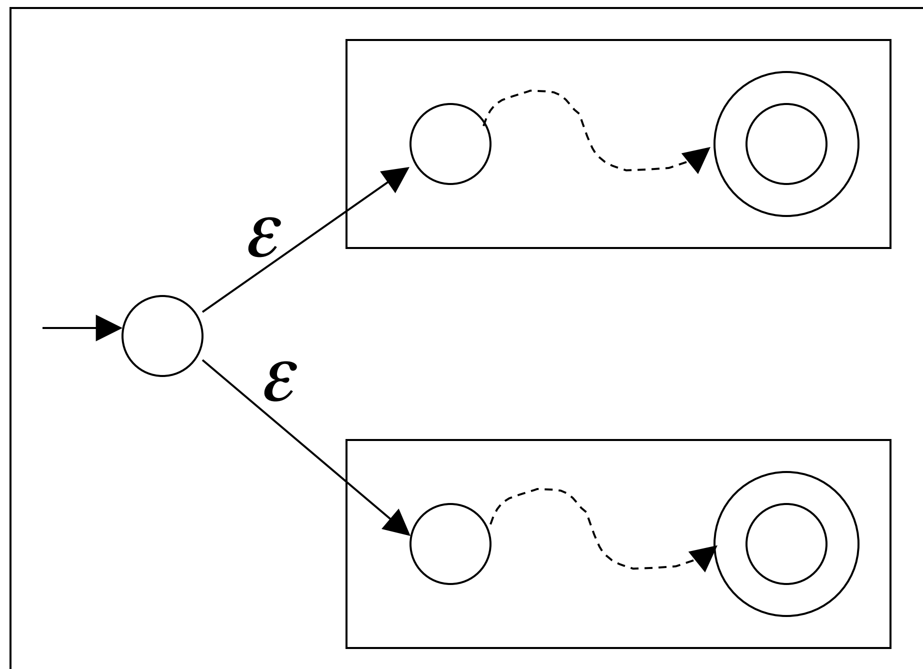
$L(M_1) = L(r_1)$



$L(M_2) = L(r_2)$



$L(M) = L(r)$



Proof - Part 2

$$\left\{\begin{array}{l}\text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular} \\ \text{Languages}\end{array}\right\}$$

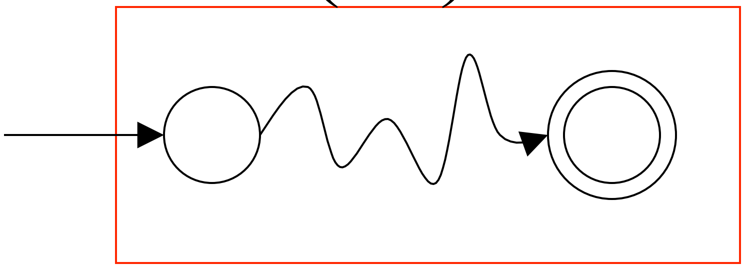
If a language is regular, then it described by a RE

For any regular language L there is a regular expression r with $L(r) = L$

We will convert an NFA that accepts L to a regular expression

Since L is regular, there is a NFA M that accepts it

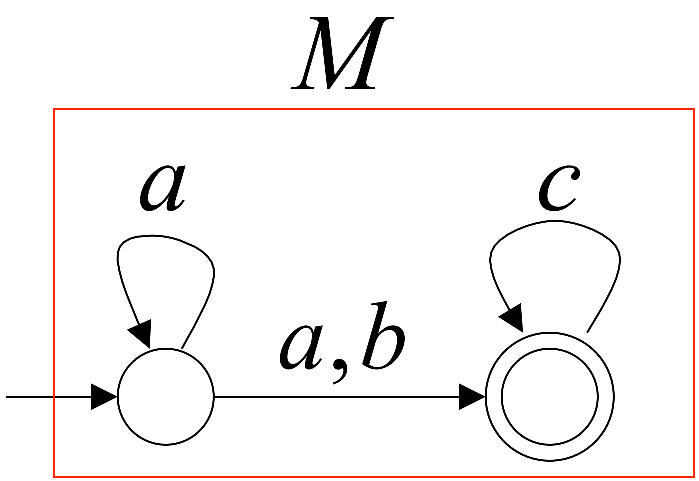
$$L(M) = L$$

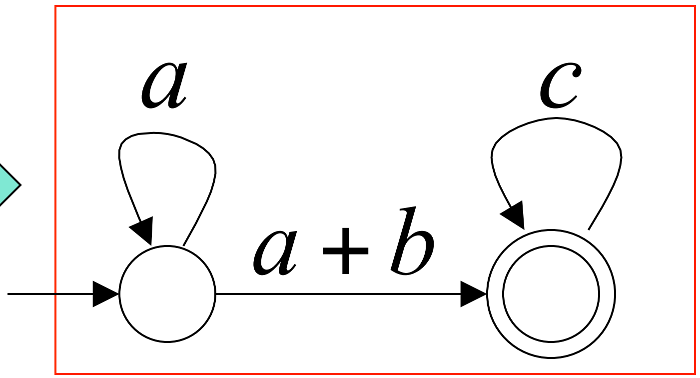


Take it with a single final state (for many initial/final states, it is a '+' of expressions for each start/final pair)

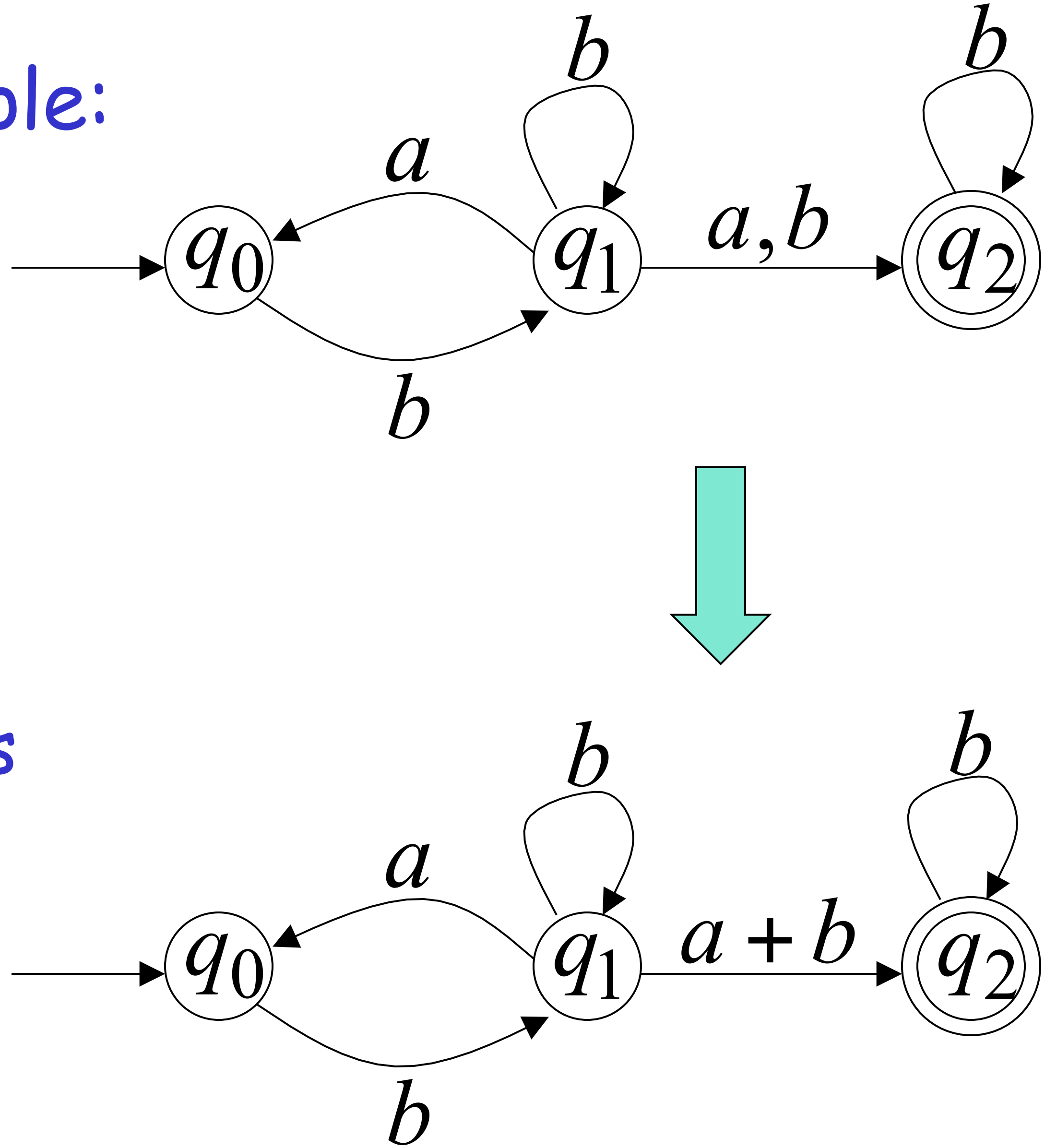
From M construct the equivalent **Generalized Transition Graph** in which transition labels are regular expressions

Example:



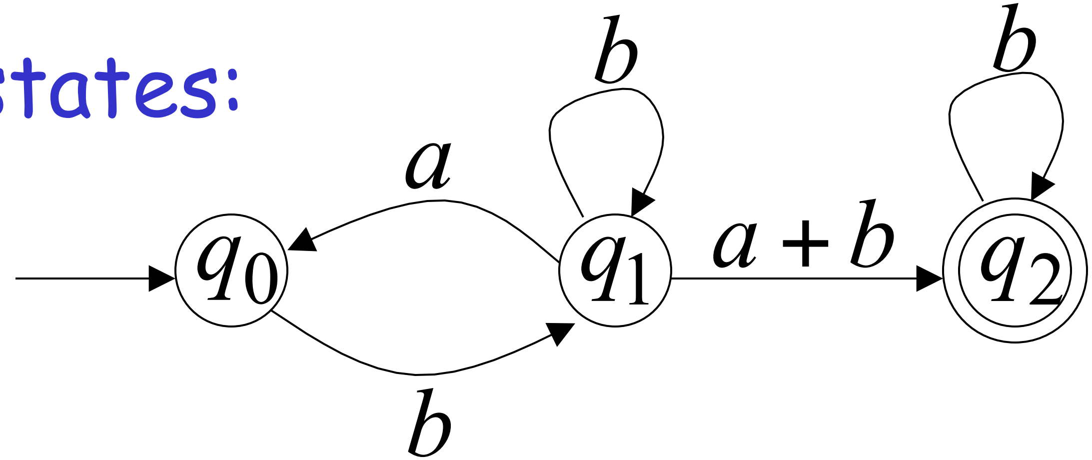


Another Example:

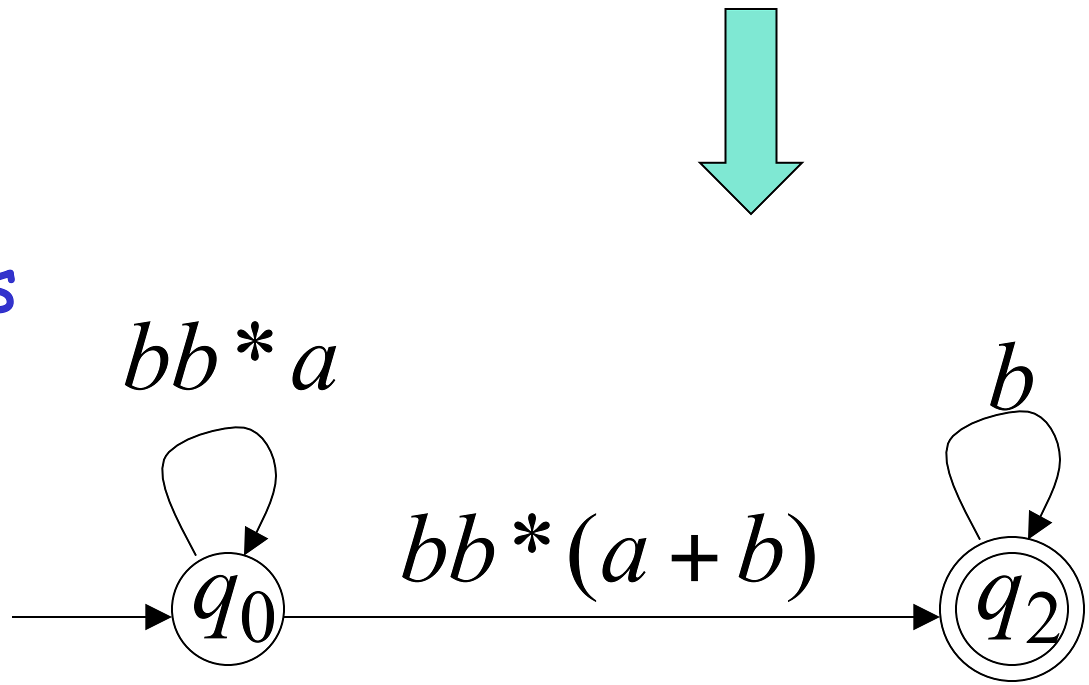


Transition labels are regular expressions

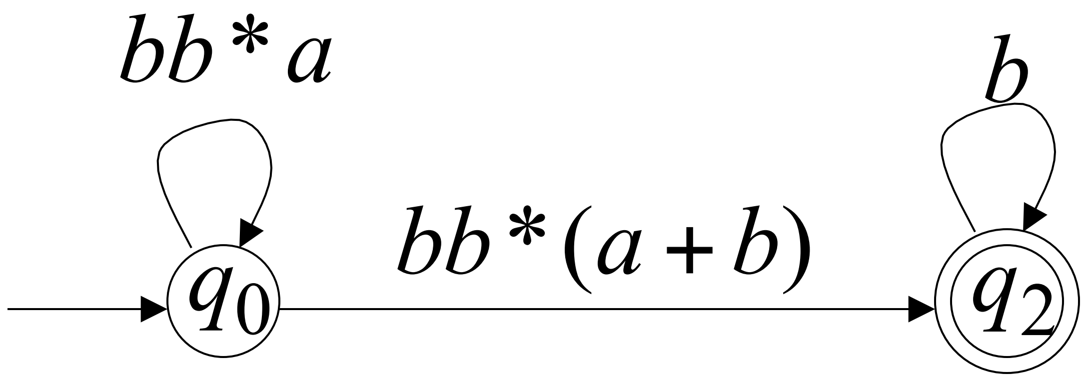
Reducing the states:



Transition labels are regular expressions



Resulting Regular Expression:

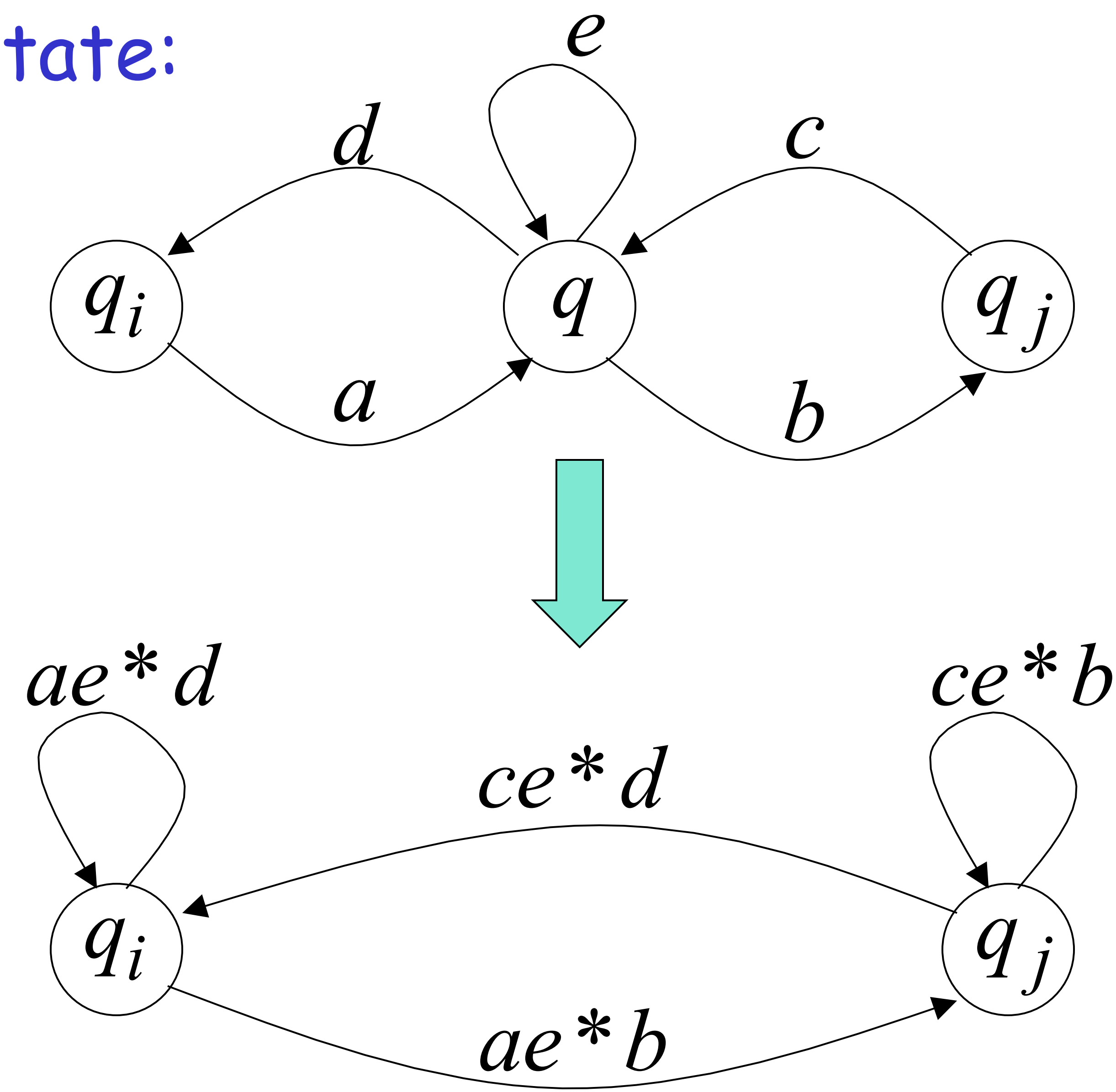


$$r = (bb^*a)^*bb^*(a+b)b^*$$

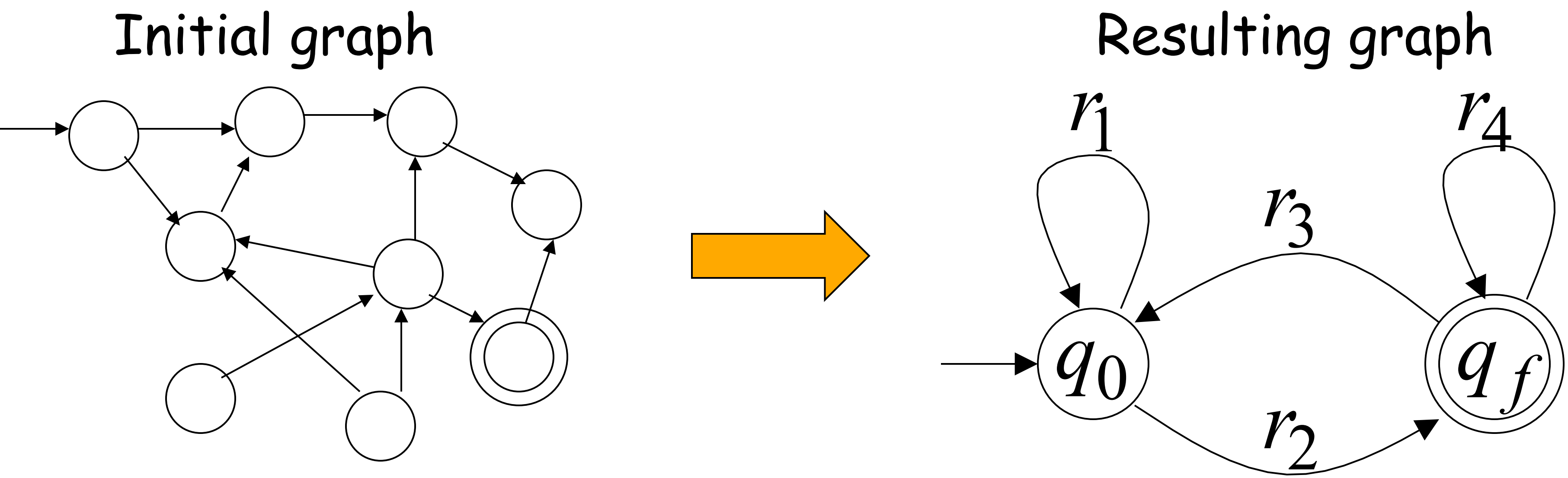
$$L(r) = L(M) = L$$

In General

Removing a state:



By repeating the process until two states are left, the resulting graph is



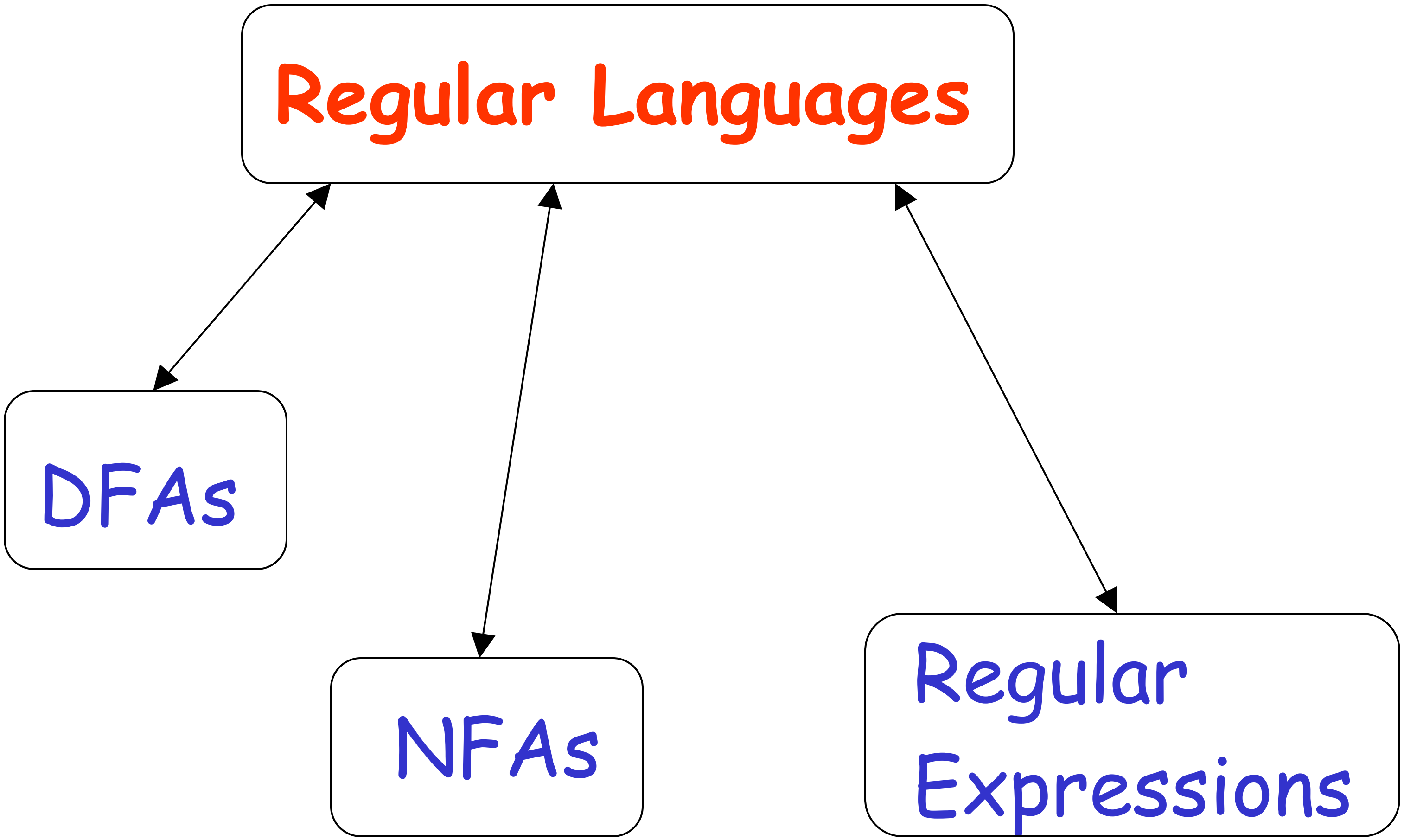
The resulting regular expression:

$$r = r_1^* r_2 (r_4 + r_3 r_1^* r_2)^*$$

$$L(r) = L(M) = L$$

End of Proof-Part 2

Standard Representations of Regular Languages



When we say: We are given a Regular Language L

We mean: Language L is in a standard representation

(DFA, NFA, or Regular Expression)