Interactive Lecture - Two

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Interactive web applications available at
https://apps.eeng.dcu.ie/ESOA/index.html

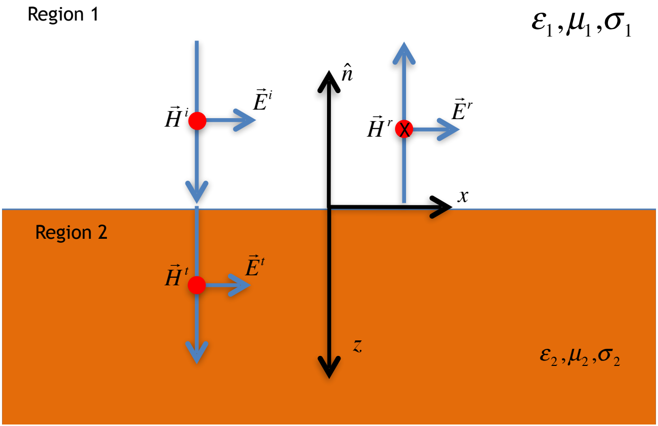
Applying boundary condition at $z = 0$ gives

Reflection Coefficient

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

Transmission Coefficient

$$T = 1 + \Gamma$$



Question One: Consider the scenario in figure (1). A unit amplitude plane wave is normally incident on a planar boundary between free space and a lossless medium. Given the data in the figure compute Γ, ϵ_r and T.

By inspection we see that $\Gamma = -\frac{2}{3}$ (Careful! The point given could lead one to conclude $\Gamma = \frac{2}{3}$ but looking at the boundary we see that the reflected wave has amplitude $-\frac{2}{3}$ that of the incident wave.) This allows us to compute the impedance of the reflecting material.

$$\frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} = \frac{-2}{3}$$

Using the fact that

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \sqrt{\frac{\mu_0}{\epsilon_r \epsilon_0}} = \frac{1}{\sqrt{\epsilon_r}}\eta_0$$

we can we-write this as:

$$\frac{\eta_0 \left(\psi - 1\right)}{\eta_0 \left(\psi + 1\right)} = \frac{-2}{3}$$

where $\psi = \frac{1}{\sqrt{\epsilon_r}}$. Rearranging gives

$$3\psi - 3 = -2\psi - 2$$

or $\psi = \frac{1}{5}$. From this we see that $\epsilon_r = 25$. $T = 1 + \Gamma = \frac{1}{3}$.

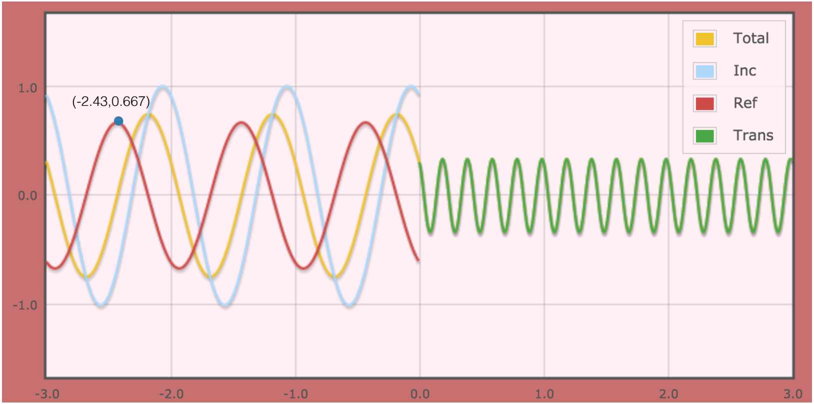


Figure: Figure for Q1

Question Two: When a plane wave is reflected from a planar surface the transmission coefficient satisfies

$$T = 1 + \Gamma$$

A reasonable, (but incorrect!), thought would be that in order to ensure that power is conserved at the boundary that the relationship should be $T + \Gamma = 1$. Prove that, with $T = 1 + \Gamma$, power is conserved when the wave reflects, i.e. the total power in the incident wave is present in the reflected and transmitted waves.

Let the incident wave be $E_0 e^{-\jmath\beta_0 z}$ (assume travelling in free space). The Poynting vector has magnitude

$$S^i_{av} = \frac{|E_0|^2}{2\eta_0}$$

The reflected wave has power

$$S^r_{av} = \frac{|E_0|^2 \Gamma^2}{2\eta_0}$$

while the transmitted wave has power

$$S^t_{av} = \frac{|E_0|^2 T^2}{2\eta_1}$$

The total power in the reflected and transmitted wave combined is:

$$S_{av} = \frac{|E_0|^2}{2\eta_0}\left(\Gamma^2 + \frac{T^2\eta_0}{\eta_1}\right)$$

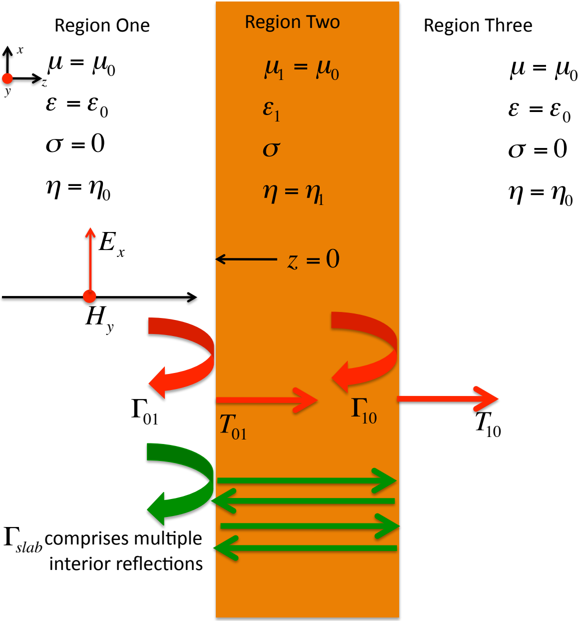
T can be written as

$$T = 1 + \Gamma = \frac{\eta_1 + \eta_0 + \eta_1 - \eta_0}{\eta_1 + \eta_0} = \frac{2\eta_1}{\eta_1 + \eta_0}$$

Thus we get

$$\begin{aligned}
S_{av} &= \frac{|E_0|^2}{2\eta_0}\left(\frac{(\eta_1 - \eta_0)^2}{(\eta_1 + \eta_0)^2} + \frac{4\eta_1\eta_0}{(\eta_1 + \eta_0)^2}\right) \\
&= \frac{|E_0|^2}{2\eta_0}\left(\frac{\eta_1^2 - 2\eta_1\eta_0 + \eta_0^2 + 4\eta_1\eta_0}{(\eta_1 + \eta_0)^2}\right) \\
&= \frac{|E_0|^2}{2\eta_0}\left(\frac{\eta_1^2 + 2\eta_1\eta_0 + \eta_0^2}{(\eta_1 + \eta_0)^2}\right) \\
&= \frac{|E_0|^2}{2\eta_0}\left(\frac{(\eta_1 + \eta_0)^2}{(\eta_1 + \eta_0)^2}\right) \\
&= \frac{|E_0|^2}{2\eta_0} = S_{av}^i
\end{aligned}$$

Reflection from a slab



Question Three: Derive expressions for $\vec{E}$ in regions 1, 2 and 3. Use the simulator to validate your expressions.

The field in the space to the left of the slab can be expressed as the following sum. The first term is the incident field while the remaining terms are reflections from the front face of the slab and various higher order terms involving multiple reflections within the slab. Recognising that $\Gamma_{01} = -\Gamma_{10}$ and calling $P_d = e^{-2\gamma_1 d}$ we can write

$$\begin{aligned} E &= e^{-\jmath\beta_0 z} + e^{\jmath\beta_0 z}\left(\Gamma_{01} + T_{01}T_{10}\left(-\Gamma_{10}\right)P_d + T_{01}T_{10}\left(-\Gamma_{10}\right)^3 P_d^2 + \dots\right) \\ &= e^{-\jmath\beta_0 z} + e^{\jmath\beta_0 z}\left(\Gamma_{01} + T_{01}T_{10}\left(-\Gamma_{10}\right)P_d\left(1 + \Gamma_{01}^2 P_d + \Gamma_{01}^4 P_d^2 + \dots\right)\right) \\ &= e^{-\jmath\beta_0 z} + e^{\jmath\beta_0 z}\left(\Gamma_{01} + \frac{T_{01}T_{10}\left(-\Gamma_{10}\right)P_d}{1 - \Gamma_{01}^2 P_d}\right) \end{aligned}$$

where we have used the fact that

$$1 + a + a^2 + a^3 + \dots = \frac{1}{1-a}$$

for $|a| < 1$.

Using the fact that $T_{01} = 1 + \Gamma_{01}$ and $T_{10} = 1 + \Gamma_{10} = 1 - \Gamma_{01}$ we can simplify:

$$\begin{aligned} E &= e^{-\jmath\beta_0 z} + e^{\jmath\beta_0 z}\left(\Gamma_{01} + \frac{\left(1-\Gamma_{01}^2\right)\left(-\Gamma_{10}\right)P_d}{1-\Gamma_{01}^2 P_d}\right) \\ &= e^{-\jmath\beta_0 z} + e^{\jmath\beta_0 z}\left(\frac{\Gamma_{01} - \Gamma_{01}^3 P_d + \left(1-\Gamma_{01}^2\right)\left(-\Gamma_{10}\right)P_d}{1-\Gamma_{01}^2 P_d}\right) \\ &= e^{-\jmath\beta_0 z} + e^{\jmath\beta_0 z}\left(\frac{\Gamma_{01} - \Gamma_{01} P_d}{1-\Gamma_{01}^2 P_d}\right) \end{aligned}$$

and the slab reflection coefficient is thus

$$\Gamma_{slab} = \frac{\Gamma_{01}\left(1-P_d\right)}{1-\Gamma_{01}^2 P_d}$$

as per the notes.

Question Four: A plane wave is normally incident on a lossless dielectric slab with permittivity ϵ_1 and thickness d. Derive an expression for d such that the total slab reflection coefficient equals 0.

The reflection coefficient is given by

$$\Gamma = \frac{\Gamma_{01} - \Gamma_{01} e^{-2\gamma_1 d}}{1 - \Gamma_{01}^2 e^{-2\gamma_1 d}}$$

To equal zero we require

$$\Gamma_{01} = \Gamma_{01} e^{-2\gamma_1 d}$$

or $e^{-2\gamma_1 d} = 1$. For lossy media this is impossible (unless in the trivial case $d = 0$) but for lossless slabs we can write this requirement as $e^{-2\jmath\beta_1 d} = 1$ and so

$$2\beta_1 d = 2\pi$$

or

$$\frac{2\pi}{\lambda_d} d = \pi$$

yielding $d = \frac{\lambda_d}{2}$.