Interactive Lecture - Three

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Interactive web applications available at
https://apps.eeng.dcu.ie/ESOA/index.html

Question One: A perpendicularly polarised wave travelling in free space impinges on a medium with $\epsilon_r = 5$ at an angle of 45°. Compute the reflection and transmission coefficient. Does the reflection coefficient change if the wave is instead travelling from the medium $\epsilon_r = 5$ to free-space?

In this problem medium 1 is free space and medium 2 has relative permittivity $\epsilon_r = 5$. The transmission angle satisfies

$$\begin{aligned}
\sin\theta_t &= \frac{\gamma_1}{\gamma_2}\sin\theta_i \\
&= \frac{\jmath\omega\sqrt{\mu_0\epsilon_0}}{\jmath\omega\sqrt{5\mu_0\epsilon_0}}\frac{1}{\sqrt{2}} \\
&= \frac{1}{\sqrt{10}}
\end{aligned}$$

and so $\theta_t = 18.435^\circ$ and $\cos\theta_t = \sqrt{\frac{9}{10}}$.

The reflection coefficient is

$$
\begin{aligned}
\Gamma_\perp &= \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \\
&= \frac{\frac{\eta_0}{\sqrt{5}}\frac{1}{\sqrt{2}} - \eta_0\sqrt{\frac{9}{10}}}{\frac{\eta_0}{\sqrt{5}}\frac{1}{\sqrt{2}} + \eta_0\sqrt{\frac{9}{10}}} \\
&= \frac{\frac{1}{\sqrt{10}} - \sqrt{\frac{9}{10}}}{\frac{1}{\sqrt{10}} + \sqrt{\frac{9}{10}}} \\
&= \frac{1-3}{1+3} \\
&= -0.5
\end{aligned}
$$

The transmission coefficient is:

$$
\begin{aligned}
T_{\perp} &= \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \\
&= \frac{2\frac{\eta_0}{\sqrt{5}}\frac{1}{\sqrt{2}}}{\frac{\eta_0}{\sqrt{5}}\frac{1}{\sqrt{2}} + \eta_0\sqrt{\frac{9}{10}}} \\
&= \frac{2\frac{1}{\sqrt{10}}}{\frac{1}{\sqrt{10}} + \sqrt{\frac{9}{10}}} \\
&= \frac{2}{1+3} \\
&= 0.5
\end{aligned}
$$

Question Two: Repeat the above exercise for a parallel polarised wave.

For a parallel polarised wave:

$$
\begin{aligned}
\Gamma_{\parallel} &= \frac{-\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} \\
&= \frac{-\eta_0 \frac{1}{\sqrt{2}} + \frac{\eta_0}{\sqrt{5}}\sqrt{\frac{9}{10}}}{\eta_0 \frac{1}{\sqrt{2}} + \frac{\eta_0}{\sqrt{5}}\sqrt{\frac{9}{10}}} \\
&= \frac{\frac{-5}{\sqrt{50}} + \frac{\sqrt{9}}{\sqrt{50}}}{\frac{5}{\sqrt{50}} + \frac{\sqrt{9}}{\sqrt{50}}} \\
&= \frac{-5+3}{5+3} \\
&= -\frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
T_{\parallel} &= \frac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} \\
&= \frac{2\frac{1}{\sqrt{5}}\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}} + \frac{\sqrt{9}}{\sqrt{50}}} \\
&= \frac{\sqrt{5}}{4}
\end{aligned}
$$

Question Three: At what angle will the reflection coefficient go to zero? Investigate this for both polarisations and scenarios. Where possible derive an expression for the angle that produces zero reflection (Brewster angle). Assume in all cases that $\mu = \mu_0$ for all materials.

Parallel polarisation:

$$\Gamma_{\parallel} = \frac{-\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}$$

This equals zero when:

$$\begin{aligned}
\eta_1 \cos\theta_i &= \eta_2 \cos\theta_t \\
\implies \frac{\mu_1}{\epsilon_1}\left(1-\sin^2\theta_i\right) &= \frac{\mu_2}{\epsilon_2}\left(1-\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i\right) \\
\implies 1-\sin^2\theta_i &= \frac{\epsilon_1}{\epsilon_2}-\frac{\epsilon_1^2}{\epsilon_2^2}\sin\theta_i^2 \\
\implies \sin^2\theta_i &= \frac{\epsilon_1-\epsilon_2}{\epsilon_2}\frac{\epsilon_2^2}{\epsilon_1^2-\epsilon_2^2} = \frac{\epsilon_2}{\epsilon_1+\epsilon_2}
\end{aligned}$$

and so we get zero reflection when

$$\theta_i = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1+\epsilon_2}}$$

Perpendicular polarisation:

$$\Gamma_{\perp} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

This equals zero when:

$$\begin{aligned}
\eta_2 \cos\theta_i &= \eta_1 \cos\theta_t \\
\Longrightarrow \cos^2\theta_i &= \frac{\eta_1^2}{\eta_2^2} \cos^2\theta_t \\
\Longrightarrow 1 - \sin^2\theta_i &= \frac{\mu_1}{\mu_2}\frac{\epsilon_2}{\epsilon_1}\left(1 - \sin^2\theta_t\right) \\
&= \frac{\mu_1}{\mu_2}\frac{\epsilon_2}{\epsilon_1}\left(1 - \frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i\right) \\
&= \frac{\epsilon_2}{\epsilon_1} - \sin^2\theta_i
\end{aligned}$$

Can only be zero in the trivial case when $\epsilon_1 = \epsilon_2$.

Question Four: At what angle will the reflection coefficient go to one? Investigate this for both polarisations and scenarios. Where appropriate derive an expression for the angle that produces unity reflection (critical angle).

Let's consider perpendicular polarisation. For a reflection coefficient of 1 we need

$$\left| \frac{\sqrt{\frac{\mu_2}{\epsilon_2}} \cos\theta_i - \sqrt{\frac{\mu_1}{\epsilon_1}} \cos\theta_t}{\sqrt{\frac{\mu_2}{\epsilon_2}} \cos\theta_i + \sqrt{\frac{\mu_1}{\epsilon_1}} \cos\theta_t} \right| = 1$$

This requires that the term

$$\sqrt{\frac{\mu_1}{\epsilon_1}} \cos\theta_t$$

which can be written as

$$\jmath\sqrt{\frac{\mu_1}{\epsilon_1}}\sqrt{\frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_i - 1}$$

be completely imaginary. which in turn requires that

$$\sin^2\theta_i \geq \frac{\epsilon_2}{\epsilon_1}$$

(assuming $\mu_1 = \mu_2 = \mu_0$). Thus total reflection occurs when

$$\theta_i \geq \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

Note that the wave must be propagating from a more dense to a less dense medium for this to be possible (i.e. $\epsilon_1 \geq \epsilon_2$). The same critical angle applies to parallel polarisation.

Question Five: Investigate the behaviour of the reflection coefficients as the incident wave approaches grazing, ie. $\theta_i \to 90^\circ$. Let's look at perpendicular

polarisation. We have

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

At $\theta_i = 90^\circ$ we get

$$\begin{aligned}\Gamma_\perp &= \frac{0 - \eta_1\sqrt{1 - \frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}}{0 + \eta_1\sqrt{1 - \frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}}} \\ &= -1\end{aligned}$$

provided that $\epsilon_1 \neq \epsilon_2$ (i.e. that there is some actual impedance mismatch across the boundary). A similar analysis holds for parallel polarisation.

Question Six: Derive the expressions for Γ and T for both polarisations. This can be done by enforcing continuity of the tangential components of both $\vec{E}$ and $\vec{H}$ along the boundary ($z = 0$).

Perpendicular polarisation: Referring to the diagram in the notes depicting perpendicular polarisation. The tangential fields on the boundary are the E_y, H_x fields. The incident, reflected and transmitted waves satisfy

$$
\begin{aligned}
E_y^i &= e^{-\gamma_1(x\sin\theta_i - z\cos\theta_i)} \\
E_y^r &= \Gamma e^{-\gamma_1(x\sin\theta_r + z\cos\theta_r)} \\
E_y^t &= T e^{-\gamma_2(x\sin\theta_t - z\cos\theta_t)} \\
H_y^i &= \frac{-\cos\theta_i}{\eta_1} e^{-\gamma_1(x\sin\theta_i - z\cos\theta_i)} \\
H_y^r &= \frac{\Gamma\cos\theta_r}{\eta_1} e^{-\gamma_1(x\sin\theta_r + z\cos\theta_r)} \\
H_y^t &= \frac{-T\cos\theta_t}{\eta_2} e^{-\gamma_2(x\sin\theta_t - z\cos\theta_t)}
\end{aligned}
$$

Equating tangential components at the boundary $z = 0$ yields

$$
\begin{aligned}
e^{-\gamma_1 x \sin\theta_i} + \Gamma e^{\gamma_1 x \sin\theta_r} &= T e^{-\gamma_2 x \sin\theta_t} \\
\frac{1}{\eta_1}\left(-\cos\theta_i e^{-\gamma_1 x \sin\theta_i} + \cos\theta_r \Gamma e^{\gamma_1 x \sin\theta_r}\right) &= \frac{-T\cos\theta_t}{\eta_2} e^{-\gamma_2 x \sin\theta_t}
\end{aligned}
$$

Equating real and imaginary parts of the above and noting that they must be satisfied for all values of x forces us to conclude the following physical requirements on the directions that the reflected and transmitted waves must propagate in

$$
\begin{aligned}
\theta_i &= \theta_r \\
\gamma_1 \sin\theta_i &= \gamma_2 \sin\theta_t
\end{aligned}
$$

Enforcing these constraints yields

$$
\begin{aligned}
1 + \Gamma &= T \\
\frac{\cos\theta_i}{\eta_1}(-1 + \Gamma) &= -\frac{\cos\theta_t}{\eta_2} T
\end{aligned}
$$

These can be solved simultaneously to yield the expressions for Γ and T in the notes. A similar process yields the expressions for parallel reflection and transmission coefficients.

Question Seven: Demonstrate that the expression for the reflection coefficient from a slab for oblique incidence reduces to that for normal incidence when $\theta_i = 0$. For oblique incidence:

$$\Gamma = \frac{\Gamma_{01}\left(1 - P_d^2 P_a\right)}{1 - \Gamma_{01}^2 P_d^2 P_a}$$

where

$$\begin{aligned} P_d &= e^{-\gamma_1 \frac{d}{\cos\theta_i}} \\ P_a &= e^{2\jmath\beta_0 l \sin\theta_i \sin\theta_t} \end{aligned}$$

At normal incidence $\theta_i = 0$ and

$$\begin{aligned} P_d &= e^{-\gamma_1 d} \\ P_a &= 1 \end{aligned}$$

and

$$\Gamma = \frac{\Gamma_{01}\left(1 - e^{-2\gamma_1 d}\right)}{1 - \Gamma_{01}^2 e^{-2\gamma_1 d}}$$