# Complexity Analysis

**Theorem 1.** The optimization problem of finding the optimal state $S^*$, as defined in Law IV of the Fundamental Laws of Network Dynamics, belongs to complexity class **P**.

*Proof.* Consider the objective function from Law IV:

$$F(S_t) = \alpha(S_t)\, O(S_t) + \beta(S_t)\, L(S_t),$$

where:

- $\alpha(S_t) = \dfrac{|\{(i,j) : R(i,j) \neq D\}|}{|\Omega|(|\Omega|-1)/2}$ is the fraction of non-disjunctive relations;
- $\beta(S_t) = 1 - \alpha(S_t)$ is the fraction of disjunctive relations;
- $O(S_t) = \sum_{i<j} \Lambda(i,j)$ is the realized value from cohesive relations;
- $L(S_t) = \sum_{\substack{i<j \\ R(i,j)=D}} (V_i I_i + V_j I_j)$ is the latent value from disjunctive relations.

Since $V_i \geq 0$ and $I_i \geq 0$ for all nodes $X_i$ (as they represent capacities/propensities), the weights

$$w_{ij} = V_i I_i + V_j I_j$$

are non-negative.

**Extreme cases:**

1. **All relations cohesive ($C$):** $\alpha = 1$, $\beta = 0$, so

$$F(S) = O(S) = \sum_{i<j} w_{ij}.$$

2. **All relations disjunctive ($D$):** $\alpha = 0$, $\beta = 1$, so

$$F(S) = L(S) = \sum_{i<j} w_{ij}.$$

3. **Mixed assignments:** For a mix of $C$ and $D$ (ignoring $N$ for now),

$$F(S) = \alpha \cdot \left( \sum_{C} w_{ij} \right) + (1 - \alpha) \cdot \left( \sum_{D} w_{ij} \right).$$

   Since the total $\sum w_{ij}$ is fixed and weights are non-negative, this weighted average cannot exceed the total sum.

Neutral relations ($N$) do not increase $F(S_t)$, since they contribute to neither $O(S_t)$ nor $L(S_t)$, while reducing $\alpha$ and $\beta$ away from their extremes.

Thus, the maximum is:

$$\max F(S_t) = \sum_{i<j} w_{ij}.$$

**Polynomial-time algorithm:**

1. Compute the total weight

$$W = \sum_{i<j} w_{ij},$$

   which requires $O(n^2)$ operations for $n$ nodes.

2. Return either the all-$C$ or all-$D$ assignment (constant time decision).

Additionally, simulating the dynamics for a polynomial number of steps (per Laws V and VII) requires updating $O(n^2)$ relations per step, which is also polynomial in $n$.

Hence, the problem is solvable in polynomial time, i.e. in class $\mathbf{P}$.