2.1 Basic Concepts

Learning objectives

- To study the basic concepts of ordinary differential equations.
- To form the differential equation of a given family of curves by eliminating the arbitrary constants.

  AND
- To practice the related problems.

2.1. Basic Concepts

Many practical problems in science and engineering are formulated by finding how one quantity is related to or depend upon one or more other quantities defined in the problem.

In many situations, it is easier to model a relation between the rates of changes in the variables rather than between the variables themselves. The study of this relationship gives rise to ***differential equations***. Many of the general laws of nature find their natural expression in the language of differential equations.

Differential Equation

An equation involving derivatives or differentials of one dependent variable with respect to one or more independent variables is called a ***Differential equation.***

For example: $dy = (x + \sin x)dx$ $\quad\ldots\ (1)$

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t \quad \ldots (2)$$

$$y = \sqrt{x}\frac{dy}{dx} + \frac{k}{\left(\frac{dy}{dx}\right)} \quad \ldots (3)$$

$$k\left(\frac{d^2y}{dx^2}\right) = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} \quad \ldots (4)$$

$$\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2 \quad \ldots (5)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \ldots (6)$$

Differential equations are classified as either ***Ordinary Differential Equations*** or ***Partial Differential Equations***, depending on whether one or more independent variables are involved.

Ordinary Differential Equation

*A differential equation involving derivatives w.r.t a single independent variable is called an **Ordinary Differential Equation (ODE)***

Equations (1), (2), (3) and (4) are ordinary differential equations.

Partial Differential Equation

*A differential equation involving partial derivatives w.r.t more than one independent variables is called a **Partial Differential Equation (PDE).***

Equations (5) and (6) are partial differential equations.

Order of a differential equation

*The order of a differential equation is the **Order** of the highest derivative present in the differential equation.*

Equations (1) and (3) are of the first order, (4) and (6) are of the second order, (5) is of the third order and the equation (2) is of the fourth order.

Degree of differential equation

*The **Degree** of a differential equation is the highest exponent of highest order derivative in the equation, after the equation is converted to the form free from radicals and fractions*.

Equations (1), (2) and (6) are of first degree, (5) is of second degree, (3) is second degree (after making it free from fraction), and (4) is of second degree (after making it from radicals)

Linear and Non-Linear differential equations

*A differential equation is called **Linear** if*

(i) The dependent variable and every derivative involved occurs in the first degree only and,
(ii) No products of the dependent variable and /or derivatives occur.

*A differential equation is called **Non-Linear** if it is not linear.*

Equation (1) and (6) are linear and (2), (3), (4) and (5) are non-linear.

This unit is devoted for the study of Ordinary Differential Equations and their solutions by different methods. In what follows a differential equation means an ordinary differential equation.

Homogeneous and Non-Homogeneous Differential Equations

An ordinary linear differential equation can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_0(x)y = g(x) \qquad \dots (7)$$

where $a_0(x), a_1(x), \dots, a_n(x)$ called ***coefficients*** and $g(x)$ called the ***right hand side function***, are all continuous real valued functions of x defined on an interval $\boldsymbol{I}$ and y is an n times differentiable function defined on $\boldsymbol{I}$. If $(x) \neq 0, \forall x \in \boldsymbol{I}$, then the differentiable equation (7) is called a ***Non-Homogeneous equation***; otherwise it is called ***Homogeneous equation***

Family of curves

*Let y and x be the dependent and independent variables respectively. The equation $f(x, y, c_1, c_2, \dots, c_n) = 0$ containing n arbitrary constants $c_1, c_2, \dots, c_n$ represents a family curves called **$\boldsymbol{n}$ parameter family of curves**.*

For example, the set of circles, defined by

$$(x - c_1)^2 + (y - c_2)^2 = c_3$$

is a three parameter family of curves if c_1, c_2 are arbitrary real numbers and c_3 is an arbitrary non-negative real number.

Solution of a Differential Equation

*Any relation between the dependent and independent variables, when substituted in the differential equation, reduces to an identity is called a **Solution** or **Integral** of the differential equation. The curves representing the solution are called **Integral curves**.*

General Solution (Complete primitive or complete integral)

To obtain a solution of a differential equation, we integrate it as many times as the order of the differential equation. Notice that, each integration reduces the order by of the differential equation by one and introduces one arbitrary constant in the solution. A solution of a differential equation of order n can have n arbitrary constants.

*A solution of a differential equation is called a **general solution** if it contains as many independent arbitrary constants as the order of the differential equation.*

General solution, Particular Solution and Singular Solution

Let $F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0$ ... (8)

be an n^{th} order ordinary differential equation.

A solution of (8) containing n independent arbitrary constants is called a ***general solution.***

A solution of (8) obtained from a general solution of (8) by giving particular values to one or more of the n independent arbitrary constants is called ***Particular solution (Particular integral)*** of (8).

A solution of (8) which cannot be obtained from any general solution of (8) by any choice of n independent arbitrary constants is called a ***Singular solution*** of (8).

Example 1: $y = c_1 e^x + c_2 e^{2x}$, where c_1 and c_2 are arbitrary constants is a general solution of $y'' - 3y' + 2y = 0$,since $y = c_1 e^x + c_2 e^{2x}$ satisfies the differential equation (verify!) and it contains as many arbitrary constants as the order of the differential equation , we get $y = e^x + e^{2x}$ is a particular solution of the differential equation $y'' - 3y' + 2y = 0$, since it is obtained by taking $c_1 = c_2 = 1$ in the general solution of the differential equation.

Example 2: $y = (x + c)^2$, where c is an arbitrary constant is a general solution of $\left(\frac{dy}{dx}\right)^2 - 4y = 0$ (verify!). Notice that $y = 0$ is also a solution of the differential equation and it cannot be obtained by any choice of c in the general solution. Thus, $y = 0$ is a singular solution of the differential equation under discussion.

Formation of Differential Equations

If a n parameter family of curves is given, then we can obtain an n^{th} order ordinary differential equation whose solution is the given family of curves.

Let $f(x, y, c_1, c_2, \dots, c_n) = 0$ $\quad\dots (9)$

where $c_1, c_2, \dots, c_n$ be n arbitrary constants.

Differentiating (9) successively n times, we get

$$f_1(x, y, y', c_1, c_2, \dots, c_n) = 0$$

$$f_2(x, y, y', y'', c_1, c_2, \dots, c_n) = 0$$

…

…

$$f_n\left(x, y, y', y'', \dots, y^{(n)}, c_1, c_2, \dots, c_n\right) = 0$$

Eliminating the n arbitrary constants $c_1, c_2, \dots, c_n$ from the above $(n+1)$ equations, we get the eliminant, the n^{th} order differential equation

$$F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0$$

The differential equation $F\left(x, y, y', y'', \dots, y^{(n)}\right) = 0$ obtained by eliminating the n parameters $c_1, c_2, \dots, c_n$ is called the

Differential Equation of the family of curves

$$\boldsymbol{f(x, y, c_1, c_2, \dots, c_n) = 0}$$

Example 3: *Obtain the differential equation for which*

$$xy = ae^x + be^{-x} + x^2$$

is a solution, where a, b are arbitrary constants.

Solution: Given $xy = ae^x + be^{-x} + x^2$ ... (i)

Since (i) has two arbitrary constants, we differentiate (i) successively two times. Then we get

$$xy' + y = ae^x - be^{-x} + 2x \quad \text{... } (ii)$$

$$xy'' + y' + y' = ae^x + be^{-x} + 2 \quad \text{... } (iii)$$

$$\Rightarrow xy'' + 2y' = (x\,y - x^2) + 2 \qquad \text{(using } (i)\text{)}$$

i.e., $xy'' + 2y' - x\,y = 2 - x^2$

is the required differential equation for which (i) is a general solution.

Example 4: *Find the differential equation of all circles of radius a.*

Solution: The equation of all circles of the given radius a is

$$(x - h)^2 + (y - k)^2 = a^2 \quad \text{... } (i)$$

where h and k are to be taken as arbitrary constants.

Since (i) has two arbitrary constants, we differentiate successively two times. Differentiating (i) we get,

$$2(x - h) + 2(y - k)y' = 0$$

$$\Longrightarrow (x - h) + (y - k)y' = 0 \quad \text{... } (ii)$$

Differentiating (ii), we get

$$1 + (y')^2 + (y - k)y'' = 0 \Longrightarrow y - k = -\frac{1 + (y')^2}{y''} \quad \text{... } (iii)$$

Now, from (ii) $x - h = -(y - k)y' = \{1 + (y')^2\}\frac{y'}{y''}$ and

$$(x - h)^2 + (y - k)^2 = a^2 \Longrightarrow \{1 + (y')^2\}^2\left(\frac{y'}{y''}\right)^2 + \left\{\frac{1 + (y')^2}{y''}\right\}^2 = a^2$$

$$\Longrightarrow \{1 + (y')^2\}^3 = a^2(y'')^2$$

is the differential equation of all circles of radius a.

Example 5: *Find the differential equation corresponding to* $y = ae^{2x} + bxe^{2x}$*, where* a, b *are arbitrary constants.*

Solution: Given $y = ae^{2x} + bxe^{2x} = (a + bx)e^{2x}$, where a, b are arbitrary constants. Since it has two arbitrary constants, we need to differentiate two times and eliminate a and b from these equations. Differentiating, we get

$$y' = 2(a + bx)e^{2x} + be^{2x} \Longrightarrow y' = 2y + be^{2x}$$

Differentiating the above, we get $y'' = 2y' + 2be^{2x}$

$$\Longrightarrow y'' = 2y' + 2(y' - 2y) \qquad \text{(from the above)}$$

Thus, $y'' - 4y' + 4y = 0$ is the differential equation corresponding to the given equation.

Initial Value Problem

The general solution of a n^{th} order differential equation contains n arbitrary constants and we need n conditions to obtain a particular solution. If all the n conditions are prescribed at a single point, say $x = x_0$, then the differential equation together with the n conditions is called an ***Initial Value Problem (IVP)*** and the conditions are called ***Initial Conditions***.

Boundary Value Problem

Suppose that the n^{th} order differential equation is valid in the interval $a \leq x \leq b$. If the required n conditions are prescribed at $x = a$ and $x = b$, then the differential equation together with these conditions is called the ***Boundary Value Problem (BVP)*** and the conditions are called the ***Boundary conditions***. Out of these n conditions, k conditions may be prescribed at $x = a$ and the remaining $n - k$ conditions may be prescribed at $x = b$

With these basic definitions, we are now ready for our main problem, that is, the problem of finding the solution of a given differential equation. In this unit we are concerned merely with an exposition of the methods of solving some differential equations and expressing their solutions by algebraic, trigonometric, hyperbolic, exponential and logarithmic functions.

2.1. Basic Concepts

EXERCISE

I. Determine the order and degree of the equation and classify it as linear or non-linear:

a. $y''' + 3y'' + 4y' - y = 0$
b. $y'' + 4y' + y = x\ sinx$
c. $y'' + x(y')^2 = coshx$
d. $(y'')^{\frac{3}{2}+xy'} = [(1+x)y']$
e. $y^{(4)} + x^2\sqrt{y} = 3 + x^2$
f. $y'' + y = \tan(y')$
g. $[1 + (y')^2]^{1/2} = x^2 + y$

II. Eliminate the arbitrary constants and obtain the differential equation satisfied by it.

a. $y = c\cos(pt - a)$; p is an arbitrary constant
b. $y = e^{-2x}(a\ cos2x + b\ sinx)$; a, b are arbitrary constants
c. $y = \frac{a}{x^2} + bx$; a, b is arbitrary constants
d. $(x - p)^2 + (y - q)^2 = a^2$; p, a, q arbitrary constants
e. $y = ae^{-x} + be^{-2x} + ce^{-3x}$; a, b, c are arbitrary constants
f. $x^2 - y^2 = a(x^2 - y^2)^2$; a is an arbitrary constants.

III. Verify that the given function satisfies the differential equation:

a. $y = ce^{-x^2}$, $y' + 2xy = 0$
b. $y = xlogx - x$, $y' = logx$
c. $y = sin^{-1}x$, $y'' = \frac{x}{(1-x^2)^{3/2}}$
d. $y = 2tan\frac{x}{2} - x$, $(1 + cosx)y' = 1 - cosx$
e. $y = secx + tanx$, $(1 - sinx)^2 y'' = cosx$

IV.

a) Find the differential equation of all circles which pass through the origin and whose centers are on the x-axis.

b) Find the differential equation of all circles which passes through the origin and whose centers on the y-axis.

c) Show that the differential equation of the family of circles of fixed radius r with center on y −axis is $(x^2 - r^2)\left(\frac{dy}{dx}\right)^2 + x^2 = 0$

P1.

Find the order and degree of the following differential equations. Also classify them as linear and non-linear.

a) $\boldsymbol{y = \sqrt{x}\frac{dy}{dx} + \frac{k}{\left(\frac{dy}{dx}\right)}}$ **b)** $\boldsymbol{y = x\frac{dy}{dx} + a\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}}$

Solution:

a) The given differential equation is $y = \sqrt{x}\frac{dy}{dx} + \frac{k}{\left(\frac{dy}{dx}\right)}$

Multiplying on both sides by $\frac{dy}{dx}$, we get

$$y\frac{dy}{dx} = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + k \qquad \dots (1)$$

which is of the first order and second degree because the degree of the highest derivative $\frac{dy}{dx}$ is 2. Here (1) is non-linear because the degree of $\frac{dy}{dx}$ is 2 and product $y\frac{dy}{dx}$ of dependent variable y and its derivative $\frac{dy}{dx}$ occurs.

b) The given differential equation is

$$y = x\frac{dy}{dx} + a\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} \Rightarrow y - x\frac{dy}{dx} = a\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$$

To get rid of radicals, square on both sides, we get

$$y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} = a^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}$$

which is of the first order and second degree because the degree of the highest derivative $\frac{dy}{dx}$ is 1. Since degree of $\frac{dy}{dx}$ is 2, the given differential equations is Non-Linear.

P2.

Show that $\boldsymbol{y = \frac{A}{x} + B}$ is a solution of the differential equation $\boldsymbol{\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are arbitrary constants.

Solution: Given that $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0$ ... (1)

Also given that $y = \frac{A}{x} + B$

Differentiating y w.r.t x , we get

$$\frac{dy}{dx} = -\frac{A}{x^2} \qquad \dots (2)$$

Differentiating (2) w.r.t x, we get

$$\frac{d^2y}{dx^2} = \frac{2A}{x^3} \qquad \dots (3)$$

Substitute (2) and (3) in (1), we get

$\frac{2A}{x^3} + \frac{2}{x}\left(-\frac{A}{x^2}\right) = 0 \Longrightarrow 0 = 0$ which is true.

Hence, $y = \frac{A}{x} + B$ is a solution (1).

P3.

Find the differential equation of the family of curves $\boldsymbol{y = e^x(A\cos x + B\sin x)}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are arbitrary constants.

Solution: Given that $y = e^x(Acosx + Bsinx)$ ... (1)

Differentiating (1) w.r.t x , we get

$$\frac{dy}{dx} = e^x(-Asinx + Bcosx) + e^x(Acosx + Bsinx)$$

$$\frac{dy}{dx} = e^x(-Asinx + Bcosx) + y \qquad \text{(Using (1))} \qquad \text{... (2)}$$

Differentiating (2) w.r.t x, we get

$$\frac{d^2y}{dx^2} = -e^x(Acosx + Bsinx) + e^x(-Asinx + Bcosx) + \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -y + \frac{dy}{dx} - y + \frac{dy}{dx} \qquad \text{(Using (2))} \qquad \text{... (3)}$$

Hence, by eliminating A and B from (1), (3) and (4), we get

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

P4:

Obtain the differential equation of the family of ellipses whose axes coincide with co-ordinate axes and the centre is at the origin.

Solution: The equation of family of ellipses whose axes coincide with the axes of co-ordinates and the centre at the origin is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \dots (1)$$

where a and b are parameters.

Differentiating (1) with respect to x, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{y}{x}\frac{dy}{dx} = -\frac{b^2}{a^2} \qquad \dots (2)$$

Differentiating (2) w.r.t x, we get

$\frac{y}{x}\frac{d^2y}{dx^2} + \frac{1}{x^2}\left(x\frac{dy}{dx} - y\right)\frac{dy}{dx} = 0 \Longrightarrow xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$, which is the required differential equation.

IP1.

Find the order and degree of the following differential equations. Also classify them as linear and non-linear.

a) $\boldsymbol{dy = (y + sinx)dx}$ **b)** $\boldsymbol{\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^5 + y = x^2}$

Solution:

a) The given differential equation is $dy = (y + sinx)dx$

$$\frac{dy}{dx} = y + sinx \qquad \text{... (1)}$$

which is of the first order and first degree because the degree of the highest derivative $\frac{dy}{dx}$ is 1. Here (1) is Linear differential equation because degree of $\frac{dy}{dx}$ is 1

b) The given differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^5 + y = x^2$, is of the second order and third degree because the degree of the highest derivative $\frac{d^2y}{dx^2}$ is 3. Since the degree of $\frac{d^2y}{dx^2}$ is 3, the given differential equation is Non-Linear

IP2.

Show that $\boldsymbol{y = a\cos(mx + b)}$ is a solution of the differential equation $\boldsymbol{\frac{d^2y}{dx^2} + m^2y = 0}$, where $\boldsymbol{a, b}$ are arbitrary constants.

Solution: Given that $\frac{d^2y}{dx^2} + m^2y = 0$ ... (1)

Also given that $y = a\,cos(mx + b)$

Differentiating y w.r.t , we get

$$\frac{dy}{dx} = -ma\,sin(mx + b) \quad \text{... (2)}$$

Differentiating (2) w.r.t x, we get

$$\frac{d^2y}{dx^2} = -m^2a\,cos(mx + b) \quad \text{... (3)}$$

Substitute (2) and (3) in (1), we get

$-m^2a\,cos(mx + b) + m^2a\,cos(mx + b) = 0 \implies 0 = 0$ which is true.

Hence, $y = a\,cos(mx + b)$ is a solution (1).

IP3.

Find the differential equation from the relation $\boldsymbol{y = a\ sinx + b\ cosx + xsinx}$ **where** $\boldsymbol{a}$ **and** $\boldsymbol{b}$ **are arbitrary constants.**

Solution: Given that $y = a\ sinx + b\ cosx + xsinx$ $\quad\quad ...(1)$

Differentiating (1) w.r.t x , we get

$$\frac{dy}{dx} = a\ cosx - b\ sinx + sinx + xcosx \quad\quad ...(2)$$

Differentiating (2) w.r.t x, we get

$$\frac{d^2y}{dx^2} = -a\ sinx - b\ cosx + 2cosx - x\ sinx$$

$$\frac{d^2y}{dx^2} = 2\ cosx - (a\ sinx + b\ cosx + x\ sinx) = 2\ cosx - y \quad\quad \text{(Using (1))}$$

Hence, by eliminating a and b from (1) and (2), we get

$\frac{d^2y}{dx^2} + y = 2\ cosx$, which is the required differential equation.

IP4:

Form the differential equation by eliminating the constant $'\boldsymbol{a}'$ from the relation $\boldsymbol{r = 2a(\sin\theta - \cos\theta)}$.

Solution: Given equation is $r = 2a(\sin\theta - \cos\theta)$ ... (1)

Differentiating (1) with respect to θ, we get

$$\frac{dr}{d\theta} = 2a(\cos\theta + \sin\theta) \quad \text{... (2)}$$

(2) ÷ (1) gives $\frac{\frac{dr}{d\theta}}{r} = \frac{2a(\cos\theta+\sin\theta)}{2a(\sin\theta-\cos\theta)} \Rightarrow \frac{dr}{d\theta} = \frac{r(\cos\theta+\sin\theta)}{\sin\theta-\cos\theta}$

This is the required differential equation.

2.2. First Order Differential Equations

Learning objectives

* To solve the first order first degree ordinary differential equations by the method of variables and separable.
* To solve homogeneous differential equations of first degree and first order.

AND

* To practice the related problems.

2.2. First Order Differential Equations

This module and the subsequent two modules are devoted for the solutions of differential equations of first order and first degree. In this module we present the solution by the method of variables separable and the method of solving homogeneous first order differential equations.

A differential equation of first order and first degree is of the form

$$y' = f(x, y)$$

We now consider these differential equations or an initial value problem associated with it.

Variables Separable

If the differential equation of first order and first degree

$$\frac{dy}{dx} = f(x, y) \qquad \dots (1)$$

can be expressed in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \quad \text{or} \quad h(y)dy = g(x)dx \text{ ,}$$

where g and f are continuous functions of x and y respectively, then the equation can be solved by integration. The general solution of (1) is

$$\int h(y)\, dy = \int g(x)\, dx + C \text{ ,}$$

where C is an arbitrary constant. Since the equation is solved by separating the variables, the method is called ***variables separable*.**

Example 1: *Solve:* $3e^x\, tan\, y\, dx + (1 - e^x)\, sec^2\, y\, dy = 0$

Solution: The given equation is $3e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$

Separating the variables, we get $3\frac{e^x}{1-e^x} dx + \frac{\sec^2 y}{\tan y} dy = 0$

Integrating, $\int 3\frac{e^x}{1-e^x}dx + \int \frac{sec^2 y}{tan\, y}dy = ln\, C$

$\Rightarrow -3\, ln|(1-e^x)| + ln|tan\, y| = ln\, C \Rightarrow \frac{tan\, y}{(1-e^x)^3} = C$

Therefore, the general solution of the given differential equation is

$$tan\, y = C(1-e^x)^3,$$

where C is an arbitrary constant

Example 2: *Solve the initial value problem* $L\frac{dI}{dt} + RI = 0$, $I(0) = I_0$ *where* I, R, L *are respectively, the current, the resistance and inductance in electrical circuits (*R, L *being constants).*

Solution: For $I \neq 0, L \neq 0$, we write the differential equation as

$$\frac{dI}{I} = -\frac{R}{L}dt$$

Integrating, $\int \frac{dI}{I} = -\int \frac{R}{L}dt + ln\, C \Rightarrow ln|I| = -\frac{R}{L}t + ln\, C$

$\Rightarrow ln|I| = ln\, Ce^{-\frac{R}{L}t} \Rightarrow I = Ce^{-\frac{R}{L}t}$, where C is an arbitrary constant.

We have, the initial condition $I = 0$ when $t = 0$. Applying the condition, we get $I_0 = C$. Thus the particular solution of the given initial value problem is $I = I_0 e^{-\frac{R}{L}t}$

Example 3: F*ind the equation of the curve passing through the point* $(1,1)$ *whose differential equation is* $(y - yx)dx + (x + xy)dy = 0$.

Solution: The given differential equation is

$$(y - yx)dx + (x + xy)dy = 0 \Rightarrow y(1-x)dx + x(1+y)dy = 0$$

Separating the variables, we get

$\frac{1-x}{x}dx + \frac{1+y}{y}dy = 0 \Rightarrow \left(\frac{1}{x} - 1\right)dx + \left(\frac{1}{y} + 1\right)dy = 0$

Integrating, $\int\left(\frac{1}{x}-1\right)dx+\int\left(\frac{1}{y}+1\right)dy=C$, where C is an arbitrary constant.

$\Rightarrow ln|x|-x+ln|y|+y=C \Rightarrow ln|xy|+y-x=C$

It represents one parameter of family of curves. Since the curve passing through $(1,1)$; put $y=1$ when $x=1$. Then we get, $C=0$. Therefore, the equation of the curve passing through the point $(1,1)$ is $\ln|xy|=x-y$. i.e., $xy=e^{x-y}$.

Example 4: *Reduce the differential equation $y''+e^y(y')^3=0$ to a lower order equation and solve.*

Solution: Set $y'=u$. Then $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx}=u\cdot\frac{du}{dy}$

The given equation now becomes $u\frac{du}{dy}+e^y u^3=0$

Separating the variables, we get $\frac{du}{u^2}=-e^y dy\ ,u\neq 0.$

Integrating we get; $\frac{1}{u}=e^y+C \Longrightarrow \frac{dx}{dy}=e^y+C \Longrightarrow dx=(e^y+C)dy.$

Integrating again, we get $x=e^y+Cy+D$, where C and D are arbitrary constants.

For $u=0$, i.e., $\frac{dy}{dx}=0$, we get $y=C_1$ and it is also a solution of the given differential equation.

Equations Reducible to separable form

The equations of the following form can be reduced to an equation in which variables can be separated.

Equations of the form $\frac{dy}{dx}=f(ax+by+c)$

Put $ax+by+c=u$, differentiating w.r.t x, we get

$$a+b\frac{dy}{dx}=\frac{du}{dx}\Rightarrow\frac{dy}{dx}=\frac{1}{b}\left(\frac{du}{dx}-a\right)$$

The given differential equation becomes

$$\frac{1}{b}\left(\frac{du}{dx}-a\right)=f(u)\text{ , i.e., }\frac{du}{dx}=a+bf(u)$$

Separating the variables, we get $\quad \frac{du}{a+bf(u)}=dx$

Integrating we get, $\int \frac{du}{a+bf(u)}=\int dx+C$ where C is an arbitrary constant

Example 5: *Solve:* $\frac{dy}{dx}=sin(x+y)+cos(x+y)$

Solution: Put $x+y=u$. Differentiating w.r.t x, we get,

$$1+\frac{dy}{dx}=\frac{du}{dx}\Rightarrow\frac{dy}{dx}=\frac{du}{dx}-1$$

The given differential equation now becomes,

$$\frac{du}{dx}-1=\sin u+\cos u\Longrightarrow\frac{du}{dx}=1+\sin u+\cos u$$

Separating the variables we get, $\quad \frac{du}{1+\sin u+\cos u}=dx$

Integrating we get, $\int \frac{du}{1+\sin u+\cos u}=x+C\Longrightarrow\ \int \frac{du}{2cos^2\frac{u}{2}+2\sin\frac{u}{2}\cos\frac{u}{2}}=x+C$

$$\Longrightarrow\ \int \frac{1}{2}\frac{sec^2\frac{u}{2}}{1+tan\frac{u}{2}}du=x+C\Longrightarrow ln\left(1+tan\frac{u}{2}\right)=x+C$$

$\Longrightarrow ln\left(1+\tan\left(\frac{x+y}{2}\right)\right)=x+C$ is the general solution of the given differential equation.

Homogeneous function

A function $f(x,y)$ of two variables is said to be a homogenous function of degree n in x and y if

$$f(\lambda x,\lambda y)=\lambda^n f(x,y)\ \textit{for all}\ \lambda>0.$$

The degree of homogeneity n can be an integer or any real number.

Examples

i) $f(x,y)=\frac{x\sqrt{x}+y\sqrt{y}}{x+y}$ is a homogeneous function of degree $\frac{1}{2}$. (Since for all $\lambda>0$,

we have $f(\lambda x+\lambda y)=\frac{\lambda x\sqrt{\lambda x}+\lambda y\sqrt{\lambda y}}{\lambda x+\lambda y}=\lambda^{\frac{1}{2}}\frac{x\sqrt{x}+y\sqrt{y}}{x+y}=\lambda^{\frac{1}{2}}f(x,y)\Big)$

ii) $f(x,y)=\frac{x^2+y^2+xy}{x^2-y^2},\ x\neq y$ is a homogeneous function of degree 0.

Note the following:

$$f(x,y)=\frac{1+\left(\frac{y}{x}\right)^2+\left(\frac{y}{x}\right)}{1-\left(\frac{y}{x}\right)^2}=g\left(\frac{y}{x}\right)\ \text{ and } f(x,y)=\frac{1+\left(\frac{x}{y}\right)^2+\left(\frac{x}{y}\right)}{\left(\frac{x}{y}\right)^2-1}=h\left(\frac{x}{y}\right)$$

Note: Every homogeneous function of degree zero can be written either in the form $g\left(\frac{y}{x}\right)$ or in the form $h\left(\frac{x}{y}\right)$.

Homogeneous first order differential equation

*A differential equation of first order and first degree $\frac{dy}{dx}=f(x,y)$ is said to be a **homogeneous differential equation**, if $f(x,y)$ is a homogeneous function of degree 0.*

That is the equation can be written in either of the form

$$\frac{dy}{dx}=g\left(\frac{y}{x}\right)\quad \text{or}\quad \frac{dy}{dx}=h\left(\frac{x}{y}\right)$$

If a first order and first degree differential equation is homogeneous, then the substitution of $y=vx$ $(or\ x=uy)$ reduces to a separable form.

Suppose that $\frac{dy}{dx}=f(x,y)$ is a homogeneous differential equation. That is, $f(x,y)$ is a homogeneous function of degree 0. Then $f(x,y)=g\left(\frac{y}{x}\right)$ or $h\left(\frac{x}{y}\right)$.

Let $\frac{dy}{dx}=g\left(\frac{y}{x}\right)$. Now, put $y=vx$.

Differentiating, $\frac{dy}{dx}=v+x\frac{dv}{dx}$. Now, the differential equation becomes

$$v + x\frac{dv}{dx} = g(v) \Longrightarrow x\frac{dv}{dx} = g(v) - v$$

Separating the variables, we get $\frac{dv}{g(v)-v} = \frac{dx}{x}$

Integrating, we obtain $\int \frac{dv}{g(v)-v} = ln\,|x| + C$

Now, replace v by $\frac{y}{x}$ to obtain the general solution of the given differential equation.

Note: To solve the homogeneous equation $y' = h\left(\frac{x}{y}\right)$, we use the substitution $x = uy$

Example 6: *Solve:* $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

Solution: Rewriting the differential equation, we have

$$\frac{dy}{dx} = \frac{x^2-4xy-2y^2}{2x^2+4xy-y^2}$$

Notice that $f(x, y) = \frac{x^2-4xy-2y^2}{2x^2+4xy-y^2}$ is a homogeneous function of degree 0.
Therefore, the given differential equation is a homogeneous differential equation.

Put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and the given differential equation becomes,

$$v + x\frac{dv}{dx} = \frac{1-4v-2v^2}{2+4v-v^2} \Longrightarrow x\frac{dv}{dx} = \frac{1-4v-2v^2}{2+4v-v^2} - v = \frac{1-6v-6v^2+v^3}{2+4v-v^2}$$

Separating the variables, we get

$$\frac{2+4v-v^2}{1-6v-6v^2+v^3}dv = \frac{dx}{x} \Longrightarrow \frac{3(v^2-4v-2)}{v^3-6v^2-6v+1}dv + 3\frac{dx}{x} = 0$$

Integrating we get, $\int \frac{3(v^2-4v-2)}{v^3-6v^2-6v+1}dv + 3\int \frac{dx}{x} = ln\,|C|$

$\Longrightarrow ln|v^3 - 6v^2 - 6v + 1| + 3\,ln|x| = ln|C|$

$\Longrightarrow x^3(v^3 - 6v^2 - 6v + 1) = C$, where $v = \frac{y}{x}$

The general solution of the given differential equation is

$y^3 - 6xy^2 - 6x^2y + x^3 = C$, where C is an arbitrary constant.

Non-homogenous Differential equations

Differential equations of the form $\frac{dy}{dx} = \frac{ax+by+c}{lx+my+n}$ …… (1), where a, b, c, l, m and n are real numbers. Notice that (1) is not homogenous. We consider the following two cases:

Case (i): If $\frac{a}{l} = \frac{b}{m} = s$, then (1) becomes $\frac{dy}{dx} = \frac{s(lx+my)+c}{lx+my+n}$

Put $lx + my = v \Longrightarrow l + m\frac{dy}{dx} = \frac{dv}{dx}$

Now, the above becomes $\frac{1}{m}\left(\frac{dv}{dx} - l\right) = \frac{sv+c}{v+n}$

$\Longrightarrow \frac{dv}{dx} = \frac{m(sv+c)}{v+n} + l = \frac{m(sv+c)+l(v+n)}{v+n}$

Separating the variables we get, $\frac{v+n}{m(sv+c)+l(v+n)}dv = dx$

Integrating and replacing v by $lx + my$, we obtain the general solution.

Case (ii): If $\frac{a}{l} \neq \frac{b}{m}$ then we substitute $X = x + h, y = y + k$ in (1) .Then $\frac{dy}{dx} = \frac{dy}{dY}.\frac{dY}{dX}.\frac{dX}{dx} = \frac{dY}{dX}$ and (1) becomes

$$\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{lX+mY+(lh+mk+n)}$$

Choose h and k such that $ah + bk + c = 0$ and $lh + mk + n = 0$. Then the above equation becomes

$$\frac{dY}{dX} = \frac{aX+bY}{lX+mY}$$

and this is a homogeneous equation in x and y. Solve this equation and substitute $X = x - h$ and $Y = y - k$ to get the general solution.

Example 7: *Solve:* $(x - 2y + 1)dy - (3x - 6y + 2)dx = 0$

Solution: The given equation is $\frac{dy}{dx} = \frac{3x-6y+2}{x-2y+1} = \frac{3(x-2y)+2}{(x-2y)+1}$

Put $x - 2y = v \Longrightarrow 1 - 2\frac{dy}{dx} = \frac{dv}{dx} \Longrightarrow \frac{dy}{dx} = \frac{1}{2}\left(1 - \frac{dv}{dx}\right)$

The given equation becomes $\frac{1}{2}\left(1 - \frac{dv}{dx}\right) = \frac{3v+2}{v+1} \Longrightarrow \frac{dv}{dx} = 1 - \frac{2(3v+2)}{v+2} = -\frac{5v+3}{v+1}$

Separating the variables we get,$\frac{v+1}{5v+3}dv = -dx$

$\Longrightarrow \frac{\frac{1}{5}(5v+3)+\frac{2}{5}}{5v+3}dv = -dx \Longrightarrow \left(\frac{1}{5} + \frac{2}{5(5v+3)}\right)dv = -dx$

Integrating, $\int\left(\frac{1}{5}+\frac{2}{2(5v+3)}\right)dv = -\int dx + C \Longrightarrow \frac{v}{5} + \left(\frac{2}{5}\right)\frac{1}{5}ln|5v + 3| = -x + C$

i.e.,$\frac{x-2y}{5} + \frac{2}{25}ln|5x - 10y + 3| = -x + C$, where C is an arbitrary constant and is the general solution of the given differential equation.

Example 8: *Solve:* $(2x + y - 3)\frac{dy}{dx} = x + 2y - 3$

Solution: The given differential equation is $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$.

Notice that $2x + y = k(x + 2y)$ for no $k \in \boldsymbol{R}$. Now, put $x = X + h$ and $y = Y + k$ in the equation. Then the equation becomes

$$\frac{dy}{dx} = \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)}$$

Choose h, k such that $h + 2k - 3 = 0$ and $2h + k - 3 = 0$ and solving them, we get $h = k = 1$. Thus, the above equation now becomes $\frac{dy}{dx} = \frac{X+2Y}{2X+Y}$

Note that $f(X, Y) = \frac{X+2Y}{2X+Y}$ is a homogenous function of degree zero.

Put $Y = vX \Longrightarrow \frac{dY}{dX} = v + X\frac{dv}{dX}$. Then, the equation now becomes

$$v + X\frac{dv}{dX} = \frac{1+2v}{2+v} \Longrightarrow X\frac{dv}{dX} = \frac{1+2v}{2+v} - v \Longrightarrow X\frac{dv}{dX} = \frac{1-v^2}{2+v}$$

Separating the variables we get, $\frac{2+v}{1-v^2}\,dv = \frac{dX}{X}$

Integrating we get, $\int\left(\frac{2+v}{1-v}\right)dv = \int\frac{dX}{X} + C \Rightarrow \int\frac{2}{1-v}dv + \int\frac{v}{1-v}dv = \int\frac{dX}{X} + C$

$\Rightarrow \int\left(\frac{1}{1+v} + \frac{1}{1-v}\right)dv + \frac{1}{2}\int\frac{2dv}{1-v} = \int\frac{dX}{X} \Rightarrow \ln\left|\frac{1+v}{1-v}\right| - \frac{1}{2}\ln|1-v| = \ln|X| + \ln C$

$\Rightarrow \frac{1+v}{1-v}\frac{1}{\sqrt{1-v^2}} = CX \Rightarrow \sqrt{1+v} = CX(1-v)^{3/2}$

$\Rightarrow 1 + v = C^3X^2(1-v)^3 \Rightarrow 1 + \frac{Y}{X} = kX^2\left(1 - \frac{Y}{X}\right)^3$

$\Rightarrow X + Y = k(X - Y)^3$, where $X = x - 1, Y = y - 1$

Therefore, the general solution is $x + y - 2 = k(x - y)^3$, where k is an arbitrary constant.

2.2. First Order Differential Equations

EXERCISE

I. Solve the following Differential Equations

a) $\frac{dy}{dx}+\sqrt{\frac{1+y^2}{1+x^2}}=0$ b) $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$

c) $3e^x tany\, dx+(1-e^x)\sec^2 y\, dy=0$

d) $(2ax+x^2)\,\frac{dy}{dx}=a^2+2ax$ f) $x\sqrt{1-x^2}\,dx+y\sqrt{1-y^2\,dy}=0$

e) $\frac{dy}{dx}=(4x+y+1)^2$ g) $\frac{dy}{dx}=\frac{x(2logx+1)}{siny+ycosy}$ h) $(x+y)^2\,\frac{dy}{dx}=a^2$

Answers

a) $sinh^{-1}x+sinh^{-1}y=c$ b) $e^y=\frac{x^3}{3}+e^x+c$

c) $tany=c(1-e^x)^3$ d) $x(x+2a)^3=Ce^{\frac{2y}{a}}$

e) $\sqrt{1-x^2}+\sqrt{1-y^2}+c=0$ f) $\tan^{-1}\left(\frac{4x+y+1}{2}\right)=2x+c$

g) $ysiny=x^2lnx+c$ h) $y=a\tan^{-1}\left(\frac{x+y}{a}\right)+c$

II. Solve the following Differential Equations

i) $x^2y\,dx-(x^3+y^3)dy=0$ ii) $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

iii) $2xy\frac{dy}{dx}=x^2+y^2$ iv) $y^2dx+(x^2+xy)dy=0$

v) $\left(x+\sqrt{xy}\right)dy=y\,dx$ vi) $(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$

Answers

i) $ln|y| - \frac{x^3}{3y^3} = c$ ii) $x + y\, e^{x/y} = c$ iii) $x = (x^2 - y^2)c$

iv) $xy^2 = c^2(x + 2y)$ v) $-2\sqrt{\frac{x}{y}} + lny = c$ vi) $x^3 + y^3 - 6xy^2 - 6x^2y = c$

III. Solve the following Differential Equations

1) $(x + 2y - 3)dy = (2x - y + 1)dx$
2) $(2x + 3y - 5)dy + (3x + 2y - 5)dx = 0$
3) $(y - x - 1)dy - (x - y - 1)dx = 0$
4) $(2x + y - 3)dy - (x + 2y - 3)dx = 0$

Answers

1) $y^2 + xy - x^2 + x - 3y = c$
2) $3(y - 1)^2 + 4(x - 1)(y - 1) + 3(y - 1)^2 = c$
3) $ln(x - y) = x + y + c$
4) $(x + y - 2) = e\,(y - x)^3$

IV. Solve the following Differential equations

i) $\frac{dy}{dx} = \frac{x+y+3}{2x+2y+1}$
ii) $(x + 2y - 1)dx - (2x + 4y + 2)dx = 0$
iii) $(2x + 2y + 3)dy - (x + y + 1)dx = 0$
iv) $(x + y)(dx - dy) = dx + dy$

Answers

i) $6y - 3x - 5ln\left(\frac{3x+3y+4}{3}\right) = c$
ii) $x + 2y + ln|x + 2y| = 2x + c$
iii) $6(x + y) + ln|2x + y - 1| = 3x + c$
iv) $x + y + ln|x + y| = 2x + c$

P1:

Solve: $\boldsymbol{y - x\,\frac{dy}{dx} = 2(y^2 + \frac{dy}{dx})}$

Solution: Given $y - x\,\frac{dy}{dx} = 2(y^2 + \frac{dy}{dx})$ ... (1)

$$\Rightarrow y - x\,\frac{dy}{dx} = 2y^2 + 2\frac{dy}{dx} \Rightarrow (y - 2y^2) = (x + 2)\frac{dy}{dx}$$

By separating the variables, we get $\frac{dy}{y(1-2y)} = \frac{dx}{x+2} \Rightarrow \left[\frac{1}{y} + \frac{2}{1-2y}\right] dy = \frac{dx}{x+2}$

Integrating on both sides, we get $\int\left(\frac{1}{y}+\frac{2}{1-2y}\right)dy = \int\frac{dx}{x+2}$

$$\Rightarrow ln|y| - ln\,|1 - 2y| = ln|x + 2| + lnC$$

$$\Rightarrow ln\left|\frac{y}{1-2y}\right| = ln|C\,(x + 2)| \Rightarrow \frac{y}{1-2y} = C\,(x + 2) \Rightarrow y = C\,(1 - 2y)\,(x + 2)$$

Therefore, the general solution of (1) is $y = C(1 - 2y)\,(x + 2)$, where C is an arbitrary constant.

P2.

Solve $\boldsymbol{x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx}$ and given that $\boldsymbol{y(\sqrt{3}) = 1}$

Solution: Given $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx \Rightarrow \frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ ... (1)

Notice that $f(x,y) = \frac{y+\sqrt{x^2+y^2}}{x}$ is a homogeneous function of degree 0.
Therefore, (1) is a homogeneous equation.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. The equation (1) now becomes

$$v + x\frac{dv}{dx} = v + \sqrt{1+v^2} \Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2}$$

Separating the variables, we get $\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

Integrating we get, $\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x} + \ln C$

$$\Rightarrow \ln\left|v + \sqrt{1+v^2}\right| = lnx + lnC \Rightarrow v + \sqrt{1+v^2} = Cx$$

Substituting $v = \frac{y}{x}$ in above equation, we get $y + \sqrt{x^2+y^2} = Cx^2$ ... (2)

This is the general solution of (1).

By hypothesis, we have initial condition $y(\sqrt{3}) = 1$.

i.e., $y = 1$ when $x = \sqrt{3}$. (2) $\Rightarrow 1 + 2 = 3C \Rightarrow C = 1$

Therefore, the particular solution of (1) is $(x-y)^2 = x^2 + y^2$

P3:

Solve: $(\boldsymbol{x-y-6})\boldsymbol{dy}-(\boldsymbol{x+y+4})\boldsymbol{dx}=\mathbf{0}$

Solution: Given $\frac{dy}{dx}=\frac{x+y+4}{x-y-6}=\frac{ax+by+c}{lx+my+n}$ ... (1)

Notice that $am-bl=-1-1=-2\neq 0$

Now, put $x=X+h,\ \ y=Y+k$ in equation (1), then the equation becomes

$$\frac{dY}{dX}=\frac{(X+Y)+(h+k+4)}{(X-Y)+(h-k-6)}$$

Choose h,k such that $h+k+4=0$ and $h-k-6=0$.

Solving, we get$h=1,k=-5$. Thus, the above equation now becomes

$$\frac{dY}{dX}=\frac{X+Y}{X-Y}$$

Note that $f(X,Y)=\frac{X+Y}{X-Y}$ is a homogeneous function of degree zero. Put $Y=vX\Rightarrow\frac{dY}{dX}=v+X\frac{dv}{dX}$ .The equation now becomes $v+X\frac{dV}{dX}=\frac{1+v}{1-v}$

Separating the variables, we get $\frac{dX}{X}=\frac{1-v}{1+v^2}dv=\frac{dv}{1+v^2}-\frac{vdv}{1+v^2}$

Integrating, $\int\frac{dX}{X}=\int\frac{dv}{1+v^2}-\int\frac{vdv}{1+v^2}\Rightarrow ln\,X=\tan^{-1}v-\frac{1}{2}ln(1+v^2)+\frac{1}{2}ln\,C$

$\Rightarrow 2\,ln\,X+ln\left(1+\frac{Y^2}{X^2}\right)-ln\,C=2\tan^{-1}\left(\frac{Y}{X}\right)$, where $v=\frac{Y}{X}$

$\Rightarrow ln\left(\frac{X^2+Y^2}{C}\right)=2\tan^{-1}\left(\frac{Y}{X}\right)\Rightarrow X^2+Y^2=Ce^{2\tan^{-1}\left(\frac{Y}{X}\right)}$

Substitute, $X=x-1,\ Y=y+5$ in the above equation, we obtain the general solution of (1),

$$(x-1)^2+(y+5)^2=Ce^{2\tan^{-1}\left(\frac{y+5}{x-1}\right)},$$

where C is an arbitrary constant.

P4.

Solve: $(\mathbf{4x+6y+5})\frac{\mathbf{dy}}{\mathbf{dx}}=\mathbf{3y+2x+4}$

Solution: Given $\frac{dy}{dx}=\frac{2x+3y+4}{4x+6y+5}=\left(\frac{ax+by+c}{lx+my+n}\right)$ $\quad\ldots(1)$

Notice that, $am-bl=12-12=0$

Put $2x+3y=u \Longrightarrow 2+3\frac{dy}{dx}=\frac{du}{dx} \Longrightarrow \frac{dy}{dx}=\frac{1}{3}\left(\frac{du}{dx}-2\right)$

Thus, (1) now becomes $\frac{1}{3}\left(\frac{du}{dx}-2\right)=\frac{u+4}{2u+5} \Longrightarrow \frac{du}{dx}=\frac{7u+22}{2u+5}$

Separating the variables, we obtain $\frac{2u+5}{7u+22}du=dx$

Integrating, $\int\frac{2u+5}{7u+22}du=\int dx \Rightarrow \int\left(\frac{2}{7}-\frac{9}{7}\frac{1}{7u+22}\right)du=x+C$

$\Longrightarrow \frac{2}{7}u-\frac{9}{7}\left(\frac{1}{7}ln|7u+22|\right)=x+C$

Now, put $u=2x+3y$, then the equation becomes

$14(2x+3y)-9ln|7(2x+3y)+22|=49x+C$

Therefore, the general solution of (1) is

$$21(2y-x)-9ln|14x+21y+22|=49x+C,$$

where C is an arbitrary constant.

IP1:

Solve: $\boldsymbol{\frac{dy}{dx} - x\,tan(y - x) = 1}$

Solution: Given $\frac{dy}{dx} - x\ tan(y - x) = 1$ ... (1)

Here we use substitution method to separate the variables

Put $y - x = v$, so that $\frac{dy}{dx} - 1 = \frac{dv}{dx}$

From (1) , we have $\frac{dv}{dx} + 1 - xtanv = 1 \Longrightarrow \frac{dv}{dx} = x\ tanv$

Separating the variables, we have $\frac{dv}{tanv} = x\ dx$

Integrating we get, $\int \frac{dv}{\tan v} = \int x\,dx + C \Longrightarrow ln|sinv| = \frac{x^2}{2} + C$

$\therefore$ The general solution of (1) is $ln\big(sin(y - x)\big) = \frac{x^2}{2} + C$, where C is the arbitrary constant.

IP2.

Solve: $(\boldsymbol{y\,dx + x\,dy})\boldsymbol{x\cos}\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right) = (\boldsymbol{x\,dy - y\,dx})\boldsymbol{x\sin}\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right)$

Solution: Given $(y\,dx + x\,dy)x\cos\left(\frac{y}{x}\right) = (x\,dy - y\,dx)x\sin\left(\frac{y}{x}\right)$

$$\Longrightarrow \frac{dy}{dx} = \frac{xy\cos\left(\frac{y}{x}\right)+y^2\sin\left(\frac{y}{x}\right)}{xy\sin\left(\frac{y}{x}\right)-x^2\cos\left(\frac{y}{x}\right)} \qquad \ldots(1)$$

Notice that $f(x,y) = \frac{xy\cos\left(\frac{y}{x}\right)+y^2\sin\left(\frac{y}{x}\right)}{xy\sin\left(\frac{y}{x}\right)-x^2\cos\left(\frac{y}{x}\right)}$ is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous equation.

Put $y = vx \Longrightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

The equation (1) now becomes

$$v + x\frac{dv}{dx} = \frac{v\cos v+v^2\sin v}{v\sin v-\cos v} \Longrightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v-\cos v}$$

Separating the variables, we get $\frac{dx}{x} = \frac{v\sin v-\cos v}{2v\cos v}dv$

Integrating,

$$2\int\frac{dx}{x} = \int\left(\frac{v\sin v-\cos v}{v\cos v}\right)dv \Rightarrow 2\int\frac{dx}{x} = \int\tan v\,dv - \int\frac{1}{v}dv$$

$$\Rightarrow \ln\left|x^2\right| = -\ln\left|\cos v\right| - \ln\left|v\right| + \ln C \Rightarrow x^2 = \frac{C}{v\cos v}$$

Substituting $v = \frac{y}{x}$, the equation now becomes $x^2\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right) = C$

Therefore, the general solution of (1) is $xy\cos\left(\frac{y}{x}\right) = C$, where C is an arbitrary constants.

IP3:

Solve: $(\mathbf{2x+y-1})\mathbf{dy}-(\mathbf{x-2y+5})\mathbf{dx}=\mathbf{0}$

Solution: Given $\frac{dy}{dx}=\frac{x-2y+5}{2x+y-1}=\frac{ax+by+c}{lx+my+n}$ ... (1)

Notice that $am-bl=1+4=5\neq 0$

Now, put $x=X+h,\ \ y=Y+k$ in equation (1), then the equation becomes

$$\frac{dY}{dX}=\frac{(X-2Y)+(h-2k+5)}{(2X+Y)+(2h+k-1)}$$

Choose h,k such that $h-2k+5=0$ and $2h+k-1=0$. Solving, we get $h=-\frac{3}{5}$, $k=\frac{11}{5}$. Thus, the above equation now becomes

$$\frac{dY}{dX}=\frac{X-2Y}{2X+Y}$$

Note that $f(X,Y)=\frac{X-2Y}{2X+Y}$ is a homogeneous function of degree zero.

Put $Y=vX\Rightarrow\frac{dY}{dX}=v+X\frac{dV}{dX}$. The equation now becomes $v+X\frac{dv}{dX}=\frac{1-2v}{2+v}$

Separating the variables, we get $\frac{dX}{X}=-\frac{1}{2}\left(\frac{2v+4}{v^2+4v-1}\right)dv$

Integrating, $\int\frac{dX}{X}=-\frac{1}{2}\int\frac{2v+4}{v^2+4v-1}dv\Rightarrow ln\,X^2+\log(v^2+4v-1)=ln\,C$

$\Rightarrow ln\left(X^2(v^2+4v-1)\right)=ln\,C\Rightarrow X^2(v^2+4v-1)=C$

Put $v=\frac{Y}{X}$, we obtain $X^2\left(\left(\frac{Y}{X}\right)^2+4\left(\frac{Y}{X}\right)-1\right)=C\Rightarrow Y^2+4XY-X^2=C$

Substitute, $X=x+\frac{3}{5},\ Y=y-\frac{11}{5}$ in the above equation, we obtain

$$x^2-y^2-4xy+10x+2y=C$$

Therefore, the general solution of (1) is $x^2-y^2-4xy+10x+2y=C$, where C is an arbitrary constant.

IP4.

Solve: $\boldsymbol{\frac{dy}{dx} = \frac{x+2y+1}{2x+4y+3}}$

Solution: Given $\frac{dy}{dx} = \frac{x+2y+1}{2x+4y+3} = \left(\frac{ax+by+c}{lx+my+n}\right)$ ... (1)

Notice that $am - bl = 4 - 4 = 0$. Put $x + 2y = z \implies \frac{dy}{dx} = \frac{1}{2}\left(\frac{dz}{dx} - 1\right)$

The equation (1) now becomes $\frac{1}{2}\left(\frac{dz}{dx} - 1\right) = \frac{z+1}{2z+1} \implies \frac{dz}{dx} = \frac{4z+5}{2z+3}$

Separating the variables, we get $\left(\frac{2z+3}{4z+5}\right) dz = dx$. Integrating,

$$\int\left(\frac{2z+3}{4z+5}\right)dz = \int dx + C \Rightarrow \int\left[\frac{1}{2} + \frac{1}{2(4z+5)}\right]dz = x + C$$

$$\implies \frac{z}{2} + \frac{1}{8} ln\,|4z + 5| = x + C$$

Put $z = x + 2y$, Then the above equation becomes

$$\frac{x+2y}{2} + \frac{1}{8} l\,n|4(x + 2y) + 5| = x + C$$

Therefore, the general solution of (1) is $8y + 7ln|4x + 8y + 5| = 8x + C$, where C is an arbitrary constant.

2.3. Exact First Order Differential Equations

Learning objectives

- To state the necessary and sufficient condition for the differential equation $M(x, y)dy + N(x, y)dx$ to be exact.
- To find the integrating factors and general solution of non-exact differential equations.

AND

- To practice the related problems.

2.3. Exact First Order Differential Equations

In this module we consider differential equations of first order and first degree of the form $M(x, y)dx + N(x, y)dy = 0$ and discuss their solutions

Exact differential equations of first order and first degree

Let $M\ (x, y)$ and $N(x, y)$ be real valued functions defined on some rectangle $R: \{(x, y) |\ |x - x_0| \le a, |y - y_0| \le b\}$. The differential equation

$$M(x, y)dx + N(x, y)dy = 0 \qquad \dots (1)$$

is said to be ***exact*** if there is a function $f(x, y)$ having continuous first order partial derivatives in R such that

$$d[f(x, y)] = M(x, y)dx + N(x, y)dy$$

$$\text{i.e.,}\quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = M(x, y)dx + N(x, y)dy$$

Example 1: Consider the differential equation $2xy + x^2dy = 0$.

Notice that there is a $f(x,\ y) = x^2y$ such that $\frac{\partial f}{\partial x} = 2xy, \frac{\partial f}{\partial y} = x^2$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 2xydx + x^2dy$$

Thus, the given differential equation is exact.

In practice, finding $f(x, y)$ is not easy. The methods out lined here will be often useful.

Theorem 1: Let $\boldsymbol{M(x, y)}$ *and* $\boldsymbol{N(x, y)}$ *be two real valued functions with continuous partial derivatives on a rectangle*

$$\boldsymbol{R = \{(x, y) |\ |x - x_0| \le a, |y - y_0| \le b\}}.$$

The differential equation $\boldsymbol{M(x, y)dx + N(x, y)dy = 0}$ is exact if and only if $\boldsymbol{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ in $\boldsymbol{R}$.

Proof:

(i) Suppose the differential equation $Mdx + Ndy = 0$ is exact. By definition there exists a function $f(x, y)$ having continuous first order partial derivatives such that

$$df = Mdx + Ndy \Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

$$\Rightarrow M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}$$

Now, $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$

Since M and N have continuous partial derivatives in R; $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are continuous in R. Therefore, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ in R

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in R. Thus, the condition is necessary.

(ii) We will now show that the condition is sufficient. Suppose that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in R.

Define $u(x, y) = \int^x M\, dx$, where. $\int^x M\, dx$ denotes, the integration w.r.t x treating y as constant.

Now, $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\int^x M\, dx\right) = M$ and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$ in R.

(Since M has continuous partial derivatives of first order in R; u_{xx} and u_{yx} are continuous in R. By a theorem in advanced calculus $u_{yx} = u_{xy}$ in R)

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial y \partial x} \text{ in } R = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

Integrating w.r.t x treating y as constant, we get

$$N = \frac{\partial u}{\partial y} + C = \frac{\partial u}{\partial y} + \emptyset(y)$$

(Since the arbitrary constant may be any function of y alone)

Thus, $M\,dx + N\,dy = \frac{\partial u}{\partial y}\,dx + \left[\frac{\partial u}{\partial y} + \emptyset(y)\right]dy = \frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial y}dy + \emptyset(y)dy$

$$= du + \emptyset(y)dy = d[u + \int \emptyset(y)\,dy]$$

Thus, there is a function $f(x, y) = u + \int \emptyset(y)\,dy$ such that $df = Mdx + Ndy$. Therefore, the differential equation $Mdx + Ndy = 0$ is exact. Hence the theorem

The general solution of the differential equation $Mdx + Ndy = 0$ when exact

In the above theorem we have found a function $f(x, y)$ such that

$$d[f(x, y)] = Mdx + Ndx$$

The general solution of the differential equation $Mdx + Ndy = 0$ is $f(x, y) = C$, where C is an arbitrary constant, where

$$f(x, y) = u + \int \emptyset(y)\,dy = \int^{x} M\,dx + \int \left(N - \frac{\partial u}{\partial y}\right)dy$$

Thus, the general solution of $Mdx + Ndy = 0$ is given by

$$\int^{x} \boldsymbol{M\,dx} + \int \left(\boldsymbol{N} - \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{y}}\right)\boldsymbol{dy} = \boldsymbol{C} \text{ , where } \boldsymbol{u} = \int^{X} \boldsymbol{M\,dx}$$

We can carry out the integration in the above equation as follows:

First integrate $M(x, y)$ w.r.t x treating y as constant, integrate those terms in $N(x, y)$ which are free from x w.r.t y and add these two expressions and equate to an arbitrary constant to obtain the general solution.

Example 2: *Solve:* $(ax + hy + 1)dx + (hx + by + f)dy = 0$

Solution: The given differential equation is of the form $Mdx + Ndy = 0$, where $M = ax + hy + g$ and $N = hx + by + f$

Now, $\frac{\partial M}{\partial y} = h = \frac{\partial N}{\partial x}$. Thus, the given equation is exact.

The general solution is given by

$$\int_{\left(\substack{treating\ y \\ as\ constant}\right)} M\,dx + \int (terms\ in\ N\ free\ from\ x)\,dy = K$$

i.e., $\int (ax + hy + g)dx + \int (by + f)dy = K$

i.e., $\frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = K$

i.e., $ax^2 + 2hxy + 2by^2 + 2gx + 2fy + C = 0$, where C is an arbitrary constant.

Integrating factor

Let the differential equation $M(x, y)dx + N(x, y)dy = 0$ be not exact. If it can be made exact by multiplying it by a suitable function $\mu(x, y) \neq 0$ then $\mu(x, y)$ is called an ***integrating factor (I.F.)***

Methods of finding integrating factors

Now, we present some methods for finding an integrating factor for a non exact differential equation given by

$$Mdx + Ndy = 0$$

Method-1: To find an integrating factor (I.F.) by inspection

Sometimes an I.F. can be found by inspection.

Example 3: *Solve:* $ydx - xdy = 0$

Solution: The given equation is of the form $Mdx + Ndy = 0$, where $M = y$ and $N = -x$. Now, $\frac{\partial M}{\partial y} = 1$; $\frac{\partial N}{\partial x} = -1$, and so $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Thus, the given differential equation (1) is not exact . By inspection, we notice that if we multiply $ydx - xdy$, by $\frac{1}{y^2}$ then it is $d\left(\frac{x}{y}\right)$. Thus, $\frac{1}{y^2}$ is an I.F. Multiplying the given differential equation by $\frac{1}{y^2}$, we get

$$\frac{ydx - xdy}{y^2} = 0 \Rightarrow d\left(\frac{x}{y}\right) = 0$$

Integrating, we get $\frac{x}{y} = C$, where C is an arbitrary constant and it is the general solution of the given differential equation.

Note: 1) Notice that $d\left(\log\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{xy}$ and $d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{x^2+y^2}$. Thus, $\frac{1}{xy}$ and $\frac{1}{x^2+y^2}$ are also integrating factors.

2) A first order differential equation $Mdx + Ndy = 0$ can admit more than one integrating factors.

Example 4: *Solve:* $(1 + xy)ydx + x(1 - xy)dy = 0$

Solution: The given equation is the form of $Mdx + Ndy = 0$,where

$$M = (1 + xy)y = y + xy^2\ , N = x - x^2y$$

Now, $\frac{\partial M}{\partial y} = 1 + 2xy$; $\frac{\partial N}{\partial y} = 1 - 2xy$

Clearly, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the given differential equation is not exact.

Rewrite the equation as $ydx + xdy + xy(ydx - xdy) = 0$,

$$\Rightarrow d(xy) + x^2y^2\left(\frac{dx}{x} - \frac{dy}{y}\right) = 0 \Rightarrow d(xy) + x^2y^2d\left(ln\left(\frac{x}{y}\right)\right) = 0$$

$$\Rightarrow \frac{1}{(xy)^2}d(xy) + d\left(ln\left(\frac{x}{y}\right)\right) = 0 \Rightarrow d\left(ln\left(\frac{x}{y}\right) - \frac{1}{xy}\right) = 0.$$

Integrating, we get $ln\left(\frac{x}{y}\right) - \frac{1}{xy} = C$.

The general solution of the given differential equation (2) is $ln\left(\frac{x}{y}\right) - \frac{1}{xy} = C$, where C is the arbitrary constant.

Remark: By re arranging terms of the given equation and/or dividing by a suitable function of x and y, the equation so obtained may contain parts which are integrable. Remembering the following exact differentials will help while solving by the inspection method.

$(i)\ d(xy) = xdy + ydx,$ $\quad(ii)\ d\left(\frac{x}{y}\right) = \frac{ydx-xdy}{y^2}$

$(iii)\ d\left(\frac{y}{x}\right) = \frac{xdy-ydx}{x^2},$ $\quad(iv)\ d\big(ln(xy)\big) = \frac{xdy+ydx}{xy}$

$(v)\ d\left(ln\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{xy}$, $\quad(vi)\ d\left(\frac{1}{2}ln(x^2+y^2)\right) = \frac{xdx+ydy}{x^2+y^2}$

$(vii)\ d\left(\tan^{-1}\left(\frac{x}{y}\right)\right) = \frac{ydx-xdy}{x^2+y^2}$, $\quad(viii)\ d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$

$(ix)\ d(\sin^{-1}(xy)) = \frac{xdy+ydx}{\sqrt{1-x^2y^2}}$

Method 2: If $M(x,y)dx + N(x,y)dy = 0$ is a homogeneous differential equation (i.e. ,$M(x,y) and\ N(x,y)$ are homogeneous functions the same degree) and $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is an integrating factor.

Example 5: Solve: $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

Solution: The given equation is of the form $Mdx + Ndy = 0$,where

$$M = x^2y - 2xy^2 \ \ ; \ \ N = -x^3 + 3x^2y$$

$$\Longrightarrow \frac{\partial M}{\partial y} = x^2 - 4xy \ ; \ \frac{\partial N}{\partial x} = -3x^2 + 6xy.$$

Clearly, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the given differential equation is not exact and we look for an I.F. Notice that M and N are homogeneous functions of the same degree 3 and

$$Mx + Ny = x(x^2y - 2xy^2) + y(-x^3 + 3x^2y) = x^2y^2 \neq 0$$

Therefore, $\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$ is an integrating factor, multiplying the given equation by the I.F. we get

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{-x}{y^2} + \frac{3}{y}\right)dy = 0$$

This is of the form $M_1dx + N_1dy = 0$ and it is exact. The general solution is given by

$$\int_{\left(\substack{\text{treating } y\\ \text{as constant}}\right)} M_1\,dx + \int (\text{terms in } N_1 \text{ free from } x)\,dy = C$$

$$i.e., \int\left(\frac{1}{y}-\frac{2}{x}\right)dx + \int \frac{3}{y}\,dy = \ln C \Rightarrow \frac{x}{y} - 2\ln x + 3\ln y = \ln C$$

$i.e., \frac{y^3}{x^2} = c.e^{\frac{-x}{y}} \Longrightarrow y^3 = cx^2e^{\frac{-x}{y}}$, where C is the arbitrary constant

Method 3: If the differential equation $M(x,y)dx + N(x,y)dy = 0$ is of the form $yf(xy)\,dx + xg(xy)\,dy = 0$ and $Mx - Ny \neq 0$, then $\frac{1}{Mx-Ny}$ is an I.F.

Example 6: Solve: $(x^3y^4 + x^2y^3 + xy^2 + y)dx + (x^4y^3 - x^3y^2 - x^2y + x)dy = 0$

Solution: The given differential equation is of the form $Mdx + Ndy = 0$

where $M = x^3y^4 + x^2y^3 + xy^2 + y \quad ; \quad N = x^4y^3 - x^3y^2 - x^2y + x$

$$\Longrightarrow \frac{\partial M}{\partial y} = 4x^3y^3 + 3x^2y^2 + 2xy + 1 \;\; ; \;\; \frac{\partial N}{\partial x} = 4x^3y^3 - 3x^2y^2 - 2xy + 1$$

Clearly, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.Therefore, the given differential equation is not exact and we look up for an I.F. The given differential equation can be written as

$yf(xy)dx + xg(xy)dy = 0$, where $f(xy) = x^3y^3 + x^2y^2 + xy + 1,$

$g(xy) = x^3y^3 - x^2y^2 - xy + 1,$

Notice that $Mx - Ny = xy(f(xy) - g(xy)) = 2x^2y^2(xy+1) \neq 0$

Therefore, $\frac{1}{Mx-Ny} = \frac{1}{2x^2y^2(xy+1)}$ is an integrating factor.

Multiplying the given differential equation by this I.F, we get

$$\frac{x^3y^3+x^2y^2+xy+1}{2x^2y^2(xy+1)}ydx + \frac{x^3y^3-x^2y^2-xy+1}{2x^2y^2(xy+1)}xdx = 0$$

$$\Longrightarrow \frac{x^2y^2+1}{2x^2y^2}ydx + \frac{(xy+1)(x^2y^2-xy+1)-xy(xy+1)}{2x^2y^2(xy+1)}xdx = 0$$

$$\Rightarrow \left(\frac{x^2y^2+1}{x^2y^2}\right) ydx + \frac{x^2y^2-2xy+1}{x^2y^2} xdx = 0$$

$$\Rightarrow \left(y + \frac{1}{x^2y}\right) dx + \left(x - \frac{2}{y} + \frac{1}{x^2y^2}\right) dx = 0$$

This is of the form $M_1dx + N_1dy = 0$ and is exact.

The general Solution is given by $\int_{\substack{\text{treating } y \\ \text{as constant}}} M_1\,dx + \int (\text{terms in } N_1 \text{ free from } x)\,dy = C$

i.e., $\int\left(y+\frac{1}{x^2y}\right)dx + \int\left(-\frac{2}{y^2}\right)dy = C \Rightarrow xy - \frac{1}{xy} - 2\ln y = C$, where C is any arbitrary constant.

Method 4: If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone say $f(x)$, then $e^{\int f(x)dx}$ is an integrating factor of the differential equation

$$M(x,y)dx + N(x,y)dy = 0.$$

Example 7: *Solve:* $\left(xy^2 - e^{\frac{1}{x^3}}\right) dx - x^2ydy = 0$

Solution: It is of the form $Mdx + Ndy = 0$, where

$$M = xy^2 - e^{\frac{1}{x^3}} \;\; ; \;\; N = -x^2y \Rightarrow \frac{\partial M}{\partial y} = 2xy \;\; ; \;\; \frac{\partial N}{\partial x} = -2xy.$$

Clearly, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the given differential equation is not exact. We look for an I.F. Notice that $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4xy}{-x^2y} = -\frac{4}{x} = f(x)$ (say)

Therefore, $I.F = e^{\int f(x)dx} = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = \frac{1}{x^4}$.

Multiplying the given differential equation by the I.F, we get

$$\frac{1}{x^4}\left(xy^2 - e^{\frac{1}{x^3}}\right) dx - \frac{1}{x^4}x^2ydy = 0 \Rightarrow \left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{\frac{1}{x^3}}\right) dx - \frac{y}{x^2}dy = 0$$

It is of the form $M_1dx + N_1dy = 0$ and it is exact. The general solution is given by

$$\int\limits_{\binom{treating\,y}{as\,constant}} M_1\,dx+\int(terms\,in\ N_1\ free\ from\,x)\,dy=C$$

$$\Rightarrow \int^{x}\left(\frac{y^2}{x^3}-\frac{1}{x^4}e^{\frac{1}{x^3}}\right)dx+\int 0\,dy=C \Rightarrow -\frac{y^2}{2x^2}-\int e^t\left(-\frac{1}{3}\right)dt=C\text{ , where } t=\frac{1}{x^3}$$

$$\Rightarrow -\frac{y^2}{2x^2}+\frac{e^{\frac{1}{x^3}}}{3}=C$$, where C is an arbitrary constant.

Method 5: If $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)$ is a function of y alone say $g(y)$, then $e^{\int g(y)dy}$ is an I.F of the differential equation $Mdx+Ndy=0$.

Example 8: *Solve:* $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$

Solution: It is of the form $Mdx+Ndy=0$, where

$$M=xy^3+y,\ N=x^2y^2+x+y^4 \Rightarrow \frac{\partial M}{\partial y}=3xy^2+1\ ,\frac{\partial N}{\partial x}=4xy^2+2$$

Clearly, $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. Therefore, the given differential equation is not exact. We look for an I.F. Notice that

$$\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{4xy^2+2-3xy^2-1}{xy^3+y}=\frac{xy^2+1}{y(xy^2+1)}=\frac{1}{y}=g(y)\text{ (say)}$$

Therefore, $I.F=e^{\int g(y)dy}=e^{\int\frac{1}{y}dy}=e^{\log y}=y$

Multiplying the given differential equation by the I.F, we get

$$(xy^4+y^2)dx+2(x^2y^3+xy+y^5)dy=0$$

It is of the form $M_1dx+N_1dy=0$ and it is exact.

Let, $u=\int^{x}M_1dx=\int^{x}(xy^4+y^2)dx=\frac{x^2y^4}{2}+xy^2$ and

$$\emptyset(y)=N_1-\frac{\partial u}{\partial y}=2x^2y^3+2xy+2y^5-2x^2y^3-2xy=2y^5$$

The general solution is given by $u+\int\emptyset(y)\,dy=K$

$\Longrightarrow \frac{x^2y^4}{2} + xy^2 + \int 2y^5\, dy = K \Longrightarrow \frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = K$

$\Longrightarrow 3x^2y^4 + 6xy^2 + 2y^6 = C$, where C is an arbitrary constant

Method 6: If the given differential equation $Mdx + Ndy = 0$ is not exact and can be put in the form

$$x^a y^b(my\, dx + nx\, dy) + x^c y^d(py\, dx + qx\, dy) = 0$$

where a, b, c, d, m, n, p, q are all constants, then the equation has an I.F. $x^h y^k$, where h, k are so chosen that after multiplying by $x^h y^k$, the equation is exact.

Example 9: *Solve:* $(3x + 2y^2)ydx + 2x(2x + 3y^2)dy = 0$

Solution: It is of the form $Mdx + Ndy = 0$, where

$M = 3xy + 2y^3$, $N = 4x^2 + 6xy^2 \Longrightarrow \frac{\partial M}{\partial y} = 3x + 6y^2$, $\frac{\partial N}{\partial x} = 8x + 12xy.$

Clearly, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, the given differential equation is not exact.

The given equation can be put in the form

$$x(3y\, dx + 4x\, dy) + y^2(2x\, dx + 6x\, dy) = 0$$

Let $x^h y^k$ be an I.F. Multiply the given equation by $x^h y^k$, we get,

$(3x^{h+1}y^{k+1} + 2x^h y^{k+3})dx + (4x^{h+2}y^k + 6x^{h+1}y^{k+2})dy = 0$

If this is exact, then we must have,

$\frac{\partial}{\partial y}(3x^{h+1}y^{k+1} + 2x^h y^{k+3}) = \frac{\partial}{\partial x}(4x^{h+2}y^k + 6x^{h+1}y^{k+2})$

$\Longrightarrow 3(k+1)x^{h+1}y^k + 2(k+3)x^h y^{k+2} = 4(h+2)x^{h+1}y^k + 6(h+1)x^h y^{k+2}$

$\Longrightarrow 3(k+1) = 4(h+2) and\ 2(k+3) = 6(h+1)$

$\Longrightarrow 3k - 4h = 5, k - 3h = 0$

Solving, we get $h = 1, k = 3$. Thus xy^3 is an I.F. Multiplying the given equation by xy^3 we get,

$(3x^2y^4 + 2xy^6)dx + (4x^3y^3 + 6x^2y^5)dy = 0$

It is of the form $M_1 dx + N_1 dy = 0$ and this is exact. The general solution is given by

$$\int_{\binom{\text{treating } y}{\text{as constant}}} M_1\,dx + \int (\text{terms in } N_1 \text{ free from } x)\,dy = C$$

$\Longrightarrow \int (3x^2y^4 + 2xy^6)dx + \int 0\,dy = C$

$\Longrightarrow x^3y^4 + x^2y^6 = C,$ i.e., $x^2y^4(x + y^2) = C$, where C is an arbitrary constant.

ANNEXURE

Partial Derivatives

The derivative of a function of several variables w.r.t one of the independent variables keeping all the other independent variables as constant is called the ***partial derivative*** of the function w.r.t that variable.

Let $z = f(x, y)$ be a function of two independent variables x and y defined in a domain of $xy -$ plane. The partial derivatives of f w.r.t x and y are denoted by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ respectively and are defined as below:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}; \quad \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

The ***total derivative*** of $f(x, y)$ is denoted by df and

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

We define the ***second order partial derivatives*** as

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}(x, y) = \lim_{\Delta x \to 0} \frac{f_x(x+\Delta x, y) - f_x(x,y)}{\Delta x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}(x, y) = \lim_{\Delta y \to 0} \frac{f_y(x, y+\Delta y) - f_y(x,y)}{\Delta y}$$

(Differentiate partially w.r.t x and then w.r.t y)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}(x,y) = \lim_{\Delta x \to 0} \frac{f_y(x+\Delta x,y) - f_y(x,y)}{\Delta x}$$

(Differentiate partially w.r.t y and then w.r.t x)

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}(x,y) = \lim_{\Delta y \to 0} \frac{f_y(x,y+\Delta y) - f_y(x,y)}{\Delta y}$$ if the limits exists.

Note: If f_{xy} and f_{yx} are continuous at the point $P(a,b)$ then at this point $f_{xy} = f_{yx}$. That is, the order of the differentiation is immaterial in this case.

Example 1: Let $f(x,y) = \ln(x^2+y^2) + tan^{-1}\left(\frac{y}{x}\right), (x,y) \neq (0,0)$

To get f_x differentiate f w.r.t x keeping y as constant.

$$f_x(x,y) = \frac{2x}{x^2+y^2} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = \frac{2x-y}{x^2+y^2}$$

To get f_y differentiate f w.r.t y keeping x as constant

$$f_y(x,y) = \frac{2y}{x^2+y^2} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{2y+x}{x^2+y^2}$$

$$f_{yx}(x,y) = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{2x-y}{x^2+y^2}\right) = \frac{(x^2+y^2)(-1)-(2x-y)2y}{(x^2+y^2)^2} = \frac{y^2-x^2-4xy}{(x^2+y^2)^2}$$

$$f_{xy}(x,y) = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{2y+x}{x^2+y^2}\right) = \frac{(x^2+y^2)1-(2y+x)2x}{(x^2+y^2)^2} = \frac{y^2-x^2-4xy}{(x^2+y^2)^2}$$

$$f_{xx}(x,y) = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{2x-y}{x^2+y^2}\right) = \frac{(x^2+y^2)2-(2x-y)2x}{(x^2+y^2)^2} = \frac{2y^2-2x^2+2xy}{(x^2+y^2)^2}$$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}\left(\frac{2y+x}{x^2+y^2}\right) = \frac{(x^2+y^2)2-(2y+x)2y}{(x^2+y^2)^2} = \frac{2x^2-2y^2-2xy}{(x^2+y^2)^2}$$

Notice that $f_{xy} = f_{yx}$. The total derivative

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \frac{2x-y}{(x^2+y^2)^2}dx + \frac{2y+x}{(x^2+y^2)^2}dy$$

2.3. Exact First Order Differential Equations

EXCERCISE

I. Solve the following differential equations

i) $(e^y + 1)cosxdx + e^y sinxdy = 0$

ii) $\left[y\left(1 + \frac{1}{x}\right) + cosy\right]dx + [x + logx - xsiny]dy = 0$

iii) $ysin2xdx - (1 + y^2 \cos^2 x)dy = 0$

iv) $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

v) $\frac{dy}{dx} + \frac{ycosx+siny+y}{sinx+xcosy+x} = 0$

ANSWERS

i) $(e^y + 1)sinx = C$

ii) $y(x + logx) + xcosy = C$

iii) $ycos2x + 3y + \frac{2}{3}y^3 = C$

iv) $x + ye^{\frac{x}{y}} = c$

v) $ysinx + (siny + y)x = C$

II. Solve the following differential equations.

i) $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$

ii) $y(2x^2y + e^x)dx = (e^x + y^3)dy$

iii) Find the equation of the curve passing through the point (1,1) whose differential equation is $(y - yx)dx + (x + xy)dy = 0$

ANSWERS:

i) $x + ye^{x^3} = Cy$

ii) $\frac{2}{3}x^3 + \frac{e^x}{y} - \frac{y^2}{2} = C$

iii) $ln|xy| = x - y$

III. Solve the following differential equations

i) $x^2dx - (x^3 + y^3)dy = 0$

ii) $(3xy^2 - y^3)dx - (2x^2y - xy^3)dy = 0$

iii) $(x^2y - 2xy^2)dx - (x^3 - 3x^2y) = 0$

iv) $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy)xdy = 0$

v) $y(x^2y^2+2)dx + x(2-2x^2y^2)dy = 0$
vi) $(2xy+1)ydx + (1+2xy-x^3y^3)xdy = 0$

ANSWERS

i) $3y^3 ln|y| - x^3 = 3cy^3$

ii) $3ln|x| - 2ln|y| + \frac{y}{x} = C$

iii) $x^3y^3C = e^{\frac{x}{y}}$

iv) $xy - \frac{1}{xy}C = e^{x/y}$

v) $\frac{1}{3}ln|x| - \frac{1}{3x^2y^2} - \frac{2}{3}ln|y| = C$

vi) $ln|y| + \frac{1}{x^2y^2} + \frac{1}{3x^3y^3} = C$

IV. Solve the following differential equations.

i) $(x^2+y^2+2x) + 2ydy = 0$

ii) $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \frac{1}{4}(x+xy^2)dy = 0$

iii) $\left(xy^2 + e^{1/x^3}\right)dx - x^2ydy = 0$

iv) $(xy^3+y)dx + 2(x^2y^2+x+y^4)dy = 0$

v) $(xy^2-x^2)dx + (3x^2y^2+x^2y-2x^3+y^2)dy = 0$

vi) $y(x+y+1)dx + x(x+3y+2)dy = 0$

ANSWERS

i) $e^x(x^2+y^2) = C$

ii) $3x^4y + x^4y^3 + x^6 = C$

iii) $2exp\left(\frac{1}{x^3}\right) - 3\left(\frac{y^2}{x^2}\right) = C$

iv) $3x^2y^4 + 6xy^2 + 2y^6 = C$

v) $e^{xy}\left(\frac{x^2y^2}{2} + \frac{y^2}{6} - \frac{x^3}{3} - \frac{y}{18} + \frac{1}{108}\right) = C$

vi) $xy^2(x+2y+2) = C$

V. Solve the following the differential equations

i) $(3x + 2y^2)y\,dx + 2x(2x + 3y^2)dy = 0$

ii) $x(3y\,dx + 2x\,dy) + 8y^4(y\,dx + 3x\,dy) = 0$

iii) $x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5xdy) = 0$

iv) $xy^3(y\,dx + 2x\,dy) + (3y\,dx + 5x\,dy) = 0$

ANSWERS

i) $x^3y^4 + x^2y^6 = C$

ii) $x^2y^3(x + 4y^4) = C$

iii) $x^4y^3 + x^3y^5 = C$

iv) $x^3y^5(xy^3 + 4) = 0$

P1:

Solve: $(\mathbf{2xy+y-\tan y})\boldsymbol{dx}+(\boldsymbol{x^2-x\tan^2 y+\sec^2 y})\boldsymbol{dy}=\mathbf{0}.$

Solution: Given $(2xy+y-\tan y)dx+(x^2-x\tan^2 y+\sec^2 y)dy=0$ ... (1)

Comparing (1) with $Mdx+Ndy=0$, we have

$$M=2xy+y-\tan y \ ; \ N=x^2-x\tan^2 y+\sec^2 y$$

Differentiating M partially w.r.t y, we get

$$\frac{\partial M}{\partial y}=2x+1-\sec^2 y=2x-\tan^2 y$$

Differentiating N partially w.r.t x, we get $\frac{\partial N}{\partial x}=2x-\tan^2 y$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, (1) is an exact equation.

Now, $u=\int^{x} Mdx=\int^{x}(2xy+y-\tan y)dx \Rightarrow u=x^2y+xy-x\tan y$

Differentiating u partially w.r.t y, we get $\frac{\partial u}{\partial y}=x^2+x-x\sec^2 y$

$$\emptyset(y)=N-\frac{\partial u}{\partial y}=x^2-x\tan^2 y+\sec^2 y-x^2-x+x\sec^2 y$$

$$=\sec^2 y+x(\sec^2 y-\tan^2 y)-x=\sec^2 y+x-x=\sec^2 y$$

$\therefore \emptyset(y)=\sec^2 y$ and $\int \emptyset(y)\,dy=\int \sec^2 y\,dy=\tan y$

Therefore, the general solution of (1) is $u+\int \emptyset(y)\,dy=C$

$$\Rightarrow x^2y+xy-x\tan y+\tan y=C$$

$\Rightarrow (x+1)xy+(1-x)\tan y=C$, where C is an arbitrary constant.

P2:

Solve: $\boldsymbol{y^2 dx + (x^2 - xy - y^2)dy = 0}$

Solution: Given, $y^2 dx + (x^2 - xy - y^2)dy = 0$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ... (1)

Comparing (1) with $Mdx + Ndy = 0$, we have

$$M = y^2 \;\;;\;\; N = x^2 - xy - y^2$$

Differentiating M partially w.r.t y, we get $\frac{\partial M}{\partial y} = 2y$

Differentiating N partially w.r.t x, we get $\frac{\partial N}{\partial x} = 2x - y$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, (1) is not an exact equation. But (1) is a homogeneous equation.

Now, the integrating factor (I.F) is $\frac{1}{Mx+Ny} = \frac{1}{xy^2+x^2y-xy^2-y^3} = \frac{1}{y(x^2-y^2)}$

Multiplying (1) by $\frac{1}{y(x^2-y^2)}$, we get

$$\frac{y}{x^2-y^2}dx + \frac{x^2-xy-y^2}{y(x^2-y^2)}dy = 0 \quad\quad \text{.... (2)}$$

Comparing (2) with $M_1 dx + N_1 dy = 0$, we have

$$M_1 = \frac{y}{x^2-y^2} \quad ; \quad N_1 = \frac{x^2-xy-y^2}{y(x^2-y^2)}$$

$$\frac{\partial M_1}{\partial y} = \frac{x^2+y^2}{(x^2-y^2)^2} \;\;;\;\; \frac{\partial N_1}{\partial y} = \frac{x^2+y^2}{(x^2-y^2)^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial y}$, (2) is an exact equation.

Now, $u = \int^x M_1 dx = \int^x \frac{y}{x^2-y^2}dx = \frac{1}{2}ln\left|\frac{x-y}{x+y}\right|$ and $\frac{\partial u}{\partial x} = \frac{x}{y^2-x^2}$

$\emptyset(y) = N_1 - \frac{\partial u}{\partial x} = \frac{x^2-xy-y^2}{y(x^2-y^2)} = \frac{1}{y}$

$\therefore \int \emptyset(y)\,dy = \int \frac{1}{y}dy = ln|y|$

Therefore, the general solution of (2) is $u + \int \emptyset(y)\, dy = lnC$

$$\Rightarrow \frac{1}{2} ln\left|\frac{x-y}{x+y}\right| + ln|y| = lnC \Rightarrow \left(\frac{x-y}{x+y}\right)^{\frac{1}{2}} y = C$$

$\Rightarrow (x - y)y^2 = C(x + y)$, where C is an arbitrary constant.

P3.

Solve: $\boldsymbol{2xy\,dy - (x^2 + y^2 + 1)dx = 0}$

Solution: Given $2xy\,dy - (x^2 + y^2 + 1)dx = 0$ ... (1)

Comparing (1) with $M\,dx + Ndy = 0$, we have

$$M = -(x^2 + y^2 + 1) \quad ; \quad N = 2xy$$

$$\frac{\partial M}{\partial y} = -2y \quad ; \quad \frac{\partial N}{\partial x} = 2y$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial y}$, (1) is not an exact

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}(-2y - 2y) = -\frac{2}{x} = f(x)$

$\therefore$ Integrating factor $= e^{\int f(x)dx} = e^{-\int \frac{2}{x}dx} = e^{-2\ln|x|} = \frac{1}{x^2}$

Multiplying (1) with $\frac{1}{x^2}$, we get

$$\frac{2y}{x}\,dy - \left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx = 0 \qquad \text{... (2)}$$

Comparing (2) with $M_1 dx + N_1 dy = 0$, we obtain

$$M_1 = -\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) \quad ; \quad N_1 = \frac{2y}{x} \Longrightarrow \frac{\partial M_1}{\partial y} = -\left(\frac{2y}{x^2}\right) = \frac{\partial N_1}{\partial x}$$

Thus, (2) is an exact.

Now, $u = \int^{x} M_1 dx = -\int^{x}\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx = -x + \frac{y^2}{x} + \frac{1}{x}$

$\frac{\partial u}{\partial y} = \frac{2y}{x}$ and $\emptyset(y) = N_1 - \frac{\partial u}{\partial y} = 0$

Therefore, the general solution of (2) is $u + \int \emptyset(y)\,dy = C$

$\Longrightarrow\ y^2 - x^2 + 1 = Cx$, where C is an arbitrary constant

P4.

Solve: $\boldsymbol{(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0}$

Solution: Given $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$ ... (1)

Comparing (1) with $Mdx + Ndy = 0$, we get

$$M = 3x^2y^4 + 2xy \quad ; \quad N = 2x^2y^3 - x^2$$

$$\frac{\partial M}{\partial y} = 12\,x^2y^3 + 2x \;\; ; \quad \frac{\partial N}{\partial x} = 6\,x^2y^3 - 2x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, (1) is not an exact equation

Now, $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{3x^2y^4+2xy}[6x^2y^3 - 2x - 12x^2y^3 - 2x]$

$$= \frac{1}{y(3x^2y^3+2x)}[-6x^2y^3 - 4x] = -\frac{2}{y}$$

$\therefore$ Integrating factor $= e^{\int f(y)dy} = e^{-2\int \frac{1}{y}dy} = \frac{1}{y^2}$

Multiplying (1) with $\frac{1}{y^2}$, we get

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0 \qquad \text{... (2)}$$

Comparing (2) with $M_1dx + N_1dy = 0$, we get

$$M_1 = 3x^2y^2 + \frac{2x}{y} \quad ; \quad N_1 = 2x^3y - \frac{x^2}{y^2}$$

$$\frac{\partial M_1}{\partial y} = 6x^2y - \frac{2x}{y^2} \quad ; \quad \frac{\partial N_1}{\partial x} = 6x^2y - \frac{2x}{y^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, (2) is an exact

$$u = \int^{X} M_1dx = \int^{X}\left(3x^2y^2 + \frac{2x}{y}\right)dx = x^3y^2 + \frac{x^2}{y}$$

$\frac{\partial u}{\partial y} = 2x^3y - \frac{x^2}{y^2}$ and

$\emptyset(y) = N_1 - \frac{\partial u}{\partial y} = 2x^3y - \frac{x^2}{y^2} - 2x^3y + \frac{x^2}{y^2} = 0 \Longrightarrow \int \emptyset(y)dy = 0$

Therefore, the general solution of (2) is $u + \int \emptyset(y)dy = C$

$\Longrightarrow x^3y^2 + \frac{x^2}{y} = C \Longrightarrow x^3y^2 + x^2 - Cy = 0$, where C is an arbitrary constant.

IP1:

Solve: $(\mathbf{1}+\boldsymbol{xy})\boldsymbol{xdy}+(\mathbf{1}-\boldsymbol{yx})\boldsymbol{ydx}=\mathbf{0}$

Solution: Given, $(1+xy)xdy+(1-yx)ydx=0$ ... (1)

$\Rightarrow xdy+ydx+xy(xdy-ydx)=0$

Multiplying with $\frac{1}{x^2y^2}$ (by inspection), we get

$$\Rightarrow \frac{xdy+ydx}{x^2y^2}+\frac{xdy-ydx}{xy}=0 \Rightarrow \frac{d(xy)}{(xy)^2}+d\left(ln\left(\frac{y}{x}\right)\right)=0$$

Integrating, we get $\int\frac{d(xy)}{(xy)^2}+\int d\left(ln\left(\frac{y}{x}\right)\right)=0$

$\Rightarrow -\frac{1}{xy}+ln\left|\frac{y}{x}\right|=C$, where C is an arbitrary constant.

IP2.

Solve: $(\boldsymbol{xy}\sin \boldsymbol{xy} + \cos \boldsymbol{xy})\boldsymbol{y}\, \boldsymbol{dx} + (\boldsymbol{xy}\sin \boldsymbol{xy} - \boldsymbol{cos\ xy})\boldsymbol{xdy} = \mathbf{0}$

Solution: Given differential equation is

$$(xy\sin xy + \cos xy)y\, dx + (xy\sin xy - cox\ xy)xdy = 0 \qquad \dots (1)$$

Comparing (1) with $Mdx + N\, dx = 0$, we have

$$M = (xy\, sin\, xy + cos\, xy)y\ ; N = (xy\, sin\, xy - cos\, xy)x$$

Notice that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Therefore, (1) is not exact.

(1) is of the form $yf(xy)\, dx + xg(xy)\, dy = 0$ and

$$Mx - Ny = x^2y^2\, sin\, xy + xy\, cos\, xy - x^2y^2\, sin\, xy + xy\, cos\, xy$$

$$= 2xy\, cos\, xy \neq 0$$

$$\therefore I.F. = \frac{1}{Mx-Ny} = \frac{1}{2xy\cos xy}$$

Multiplying (1) by $\frac{1}{2xy\cos xy}$, we get

$$\frac{1}{2}\left(y\tan xy + \frac{1}{x}\right) dx + \frac{1}{2}\left(xtan\, xy - \frac{1}{y}\right) dy = 0$$

Comparing (2) with $M_1dx + N_1dy = 0$, we have

$$M_1 = \frac{1}{2}\left(y\tan xy + \frac{1}{x}\right) \quad ; \quad N_1 = \frac{1}{2}\left(xtan\, xy - \frac{1}{y}\right)$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{2}[xy\, sec^2xy + \tan xy]\ ; \quad \frac{\partial N_1}{\partial y} = \frac{1}{2}[xy\, sec^2xy + \tan xy]$$

Since, $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial y}$, (2) is an exact equation

Now, $u = \int^x M_1 dx = \int^x \left(\frac{y}{2}\tan xy + \frac{1}{2x}\right) dx = \frac{1}{2} ln|sec\, xy| + \frac{1}{2} ln\, |x| + C$

$$\frac{\partial u}{\partial y} = \frac{x\sec xy.\tan xy}{2\sec xy} = \frac{1}{2}\ x\tan xy$$

$$\emptyset(y) = N_1 - \frac{\partial u}{\partial y} = \frac{1}{2}x\ tan\ xy - \frac{1}{2y} - \frac{1}{2}x \tan xy = -\frac{1}{2y}$$

$$\int \emptyset(y)\, dy = -\frac{1}{2} ln\, |y|$$

Therefore, the general solution of (2) is $u + \int \emptyset(y)\, dy = C$

$$\Longrightarrow \frac{1}{2} ln\, |\, sec\, xy\, | + \frac{1}{2} ln\, |x| - \frac{1}{2} ln\, |y| = ln\, C$$

$$\Longrightarrow ln \left|\frac{xsec\, xy}{y}\right| = ln\, C \Longrightarrow \frac{x \sec xy}{y} = C$$

$\Longrightarrow x \sec xy - Cy = 0$, where C is an arbitrary constant.

IP3.

Solve: $(\boldsymbol{3xy - 2ay^2})\boldsymbol{dx} + (\boldsymbol{x^2 - 2axy})\boldsymbol{dy} = \boldsymbol{0}$

Solution: Given $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$ ... (1)

Comparing (1) with $Mdx + Ndy = 0$, we get

$$M = 3xy - 2ay^2 \quad ; \quad N = x^2 - 2axy$$

$$\frac{\partial M}{\partial y} = 3x - 4ay \quad ; \quad \frac{\partial N}{\partial x} = 2x - 2ay$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, (1) is not an exact

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x^2 - 2axy}(3x - 4ay - 2x + 2ay) = \frac{1}{x} = f(x)$

$\therefore$ Integrating factor $= e^{\int f(x)dx} = e^{\int \frac{1}{x} dx} = x$

Multiplying (1) with x, we get

$$(3x^2y - 2axy^2)dx + (x^3 - 2ax^2y)dy = 0 \quad \text{... (2)}$$

Comparing (2) with $M_1dx + N_1dy = 0$, we get

$$M_1 = 3x^2y - 2axy^2 \; ; \quad N_1 = x^3 - 2ax^2y$$

$$\frac{\partial M_1}{\partial y} = 3x^2 - 4axy \quad ; \quad \frac{\partial N_1}{\partial x} = 3x^2 - 4axy$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, (2) is an exact.

$$u = \int^x M_1 dx = \int^x (3x^2y - 2axy^2)dx = x^3y - ax^2y^2$$

$\frac{\partial u}{\partial y} = x^3 - 2ax^2y$ and $\emptyset(y) = N_1 - \frac{\partial u}{\partial y} = x^3 - 2ax^2y - x^3 + 2ax^2y = 0$

$\therefore \int \emptyset(y)\, dy = 0$

Therefore, the general solution of (2) is $u + \int \emptyset(y)\, dy = C \Longrightarrow x^3y - ax^2y^2 = C$, where C is an arbitrary constant.

IP4.

Solve: $(\boldsymbol{y^4+2y})\boldsymbol{dx}+(\boldsymbol{xy^3+2y^4-4x})\boldsymbol{dy}=\boldsymbol{0}$

Solution: Given $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$ $\quad\ldots(1)$

Comparing (1) with $M\,dx+N\,dy=0$, we get

$$M=y^4+2y \quad ; \quad N=xy^3+2y^4-4x$$

$$\frac{\partial M}{\partial y}=4y^3+2 \quad ; \quad \frac{\partial N}{\partial x}=y^3-4$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, (1) is not an exact equation

Now, $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=\frac{1}{y(y^3+2)}(y^3-4-4y^3-2)=-\frac{3}{y}=g(y)$

$\therefore$ Integrating factor $=e^{\int g(y)dy}=e^{-3\int\frac{1}{y}dy}=\frac{1}{y^3}$

Multiplying (1) with $\frac{1}{y^3}$, we get

$$\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0 \quad\ldots(2)$$

Comparing (2) with $M_1dx+N_1dy=0$, we get

$$M_1=y+\frac{2}{y^2} \quad ; \quad N_1=x+2y-\frac{4x}{y^3}$$

$$\frac{\partial M_1}{\partial y}=1-\frac{4}{y^3} \quad ; \quad \frac{\partial N_1}{\partial x}=1-\frac{4}{y^3}$$

Since $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$, is an exact

$u=\int^x M_1dx=\int^x\left(y+\frac{2}{y^2}\right)dx=\left(y+\frac{2}{y^2}\right)x$ and $\frac{\partial u}{\partial y}=x\left[1-\frac{4}{y^3}\right]$,

$\emptyset(y)=N_1-\frac{\partial u}{\partial y}=x+2y-\frac{4x}{y^3}-x+\frac{4x}{y^3}=2y$

$\int\emptyset(y)\,dy=2\int y\,dx=y^2$

Therefore, the general solution of (2) is $u + \int x(y)dy = C$

$\Longrightarrow x\left(y + \frac{2}{y^2}\right) + y^2 = C$, where C is an arbitrary constant

2.4. Linear Differential Equation of First Order and Bernoulli's Form

Learning objectives

- To find the general solution of a linear differential equation of first order.
- To study the methods of finding solutions of the following non-linear differential equations of first order:
    - ✓ Bernoulli Equation
    - ✓ Riccati Equation
    - ✓ Clairaut Equation

AND

- To practice the related problems

2.4. Linear Differential Equation of First Order and Bernoulli's Form

Linear Differential Equation of First Order

A linear differential equation of first order in y is of the form

$$\boldsymbol{\frac{dy}{dx} + P(x)y = Q(x)} \qquad \ldots(1)$$

where $P(x)$ and $Q(x)$ are functions of x defined on an interval I.

If $Q(x) = 0$, then the equation is easily solved by separation of variables. We investigate for the solution in the case $P(x) \neq 0$ and $Q(x) \neq 0$.

An integrating factor of $(\mathbf{1})$

The given linear differential equation of first order can be written as

$$[P(x)y - Q(x)]dx + dy = 0.$$

It is of the form $Mdx + Ndy = 0$, where $M = P(x)y - Q(x)$, $N = 1$.

Now $\frac{\partial M}{\partial y} = P(x), \frac{\partial N}{\partial x} = 0$. Clearly, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Notice that $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = P(x)$, a function of x alone.

Therefore, $\boldsymbol{e^{\int P(x)dx}}$ is an I.F of (1).

The general solution of $(\mathbf{1})$

Multiplying (1) both sides by the I.F, we get

$$e^{\int P(x)dx}\left(\frac{dy}{dx} + P(x)y\right) = Q(x)e^{\int P(x)dx} \Rightarrow d\left(ye^{\int P(x)dx}\right) = Q(x)e^{\int P(x)dx}$$

Integrating both sides w.r.t x, we get the general solution of (1).

The general solution of (1) is

$$\boldsymbol{y e^{\int P(x)dx} = \int Q(x) e^{\int P(x)dx} dx + C}$$

where C is an arbitrary constant.

Note: A linear differential equation of first order in x is of the form

$$\frac{dx}{dy} + P(y)x = Q(y)$$

where $P(y)$ and $Q(y)$ are functions of y defined on an interval $\boldsymbol{I}$. An I.F. of this equation is $e^{\int P(y)dy}$ and the general solution is given by

$$\boldsymbol{x e^{\int P(y)dy} = \int Q(y) e^{\int P(y)dy} dy + C}$$

where C is an arbitrary constant.

Example 1: *Solve:* $cot\, 3x \frac{dy}{dx} - 3y = cos\, 3x + sin\, 3x\ , 0 < x < \frac{\pi}{2}$

Solution: The given differential equation can be written as

$$\frac{dy}{dx} - 3(tan\, 3x)y = sin\, 3x + sin^2\, 3x\, sec\, 3x$$

It is a linear differential equation of first order in y.

It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$, where

$$P(x) = -3\, tan\, 3x \text{ and } Q(x) = sin\, 3x + sin^2\, 3x\, sec\, 3x.$$

I.F $= e^{\int P(x)dx} = e^{\int -3\, tan\, 3x dx} = e^{ln\, cos\, 3x} = cos\, 3x$

Multiplying the given differential equation by the I.F and integrating, we get

$$y e^{\int P(x)dx} = \int Q(x) e^{\int P(x)dx} dx + C$$

i.e., $y\, cos\, 3x = \int cos\, 3x\, (sin\, 3x + sin^2\, 3x\, sec\, 3x) dx + C$

$$= \int (sin\, 3x\, cos\, 3x + sin^2\, 3x) dx + C$$

$$= \frac{1}{2} \int (sin\, 6x + 1 - cos\, 6x) dx + C = \frac{1}{2}\left[-\frac{cos\, 6x}{6} + x - \frac{sin\, 6x}{6}\right] + C$$

$$= \frac{1}{12}(6x - cos\,6x - sin\,6x) + C$$

Therefore, the general solution of the given differential equation is

$y\,cos\,3x = \frac{1}{12}(6x - cos\,6x - sin\,6x) + C$, where C is an arbitrary constant.

Example 2: Solve: $(1 + y^2)dx = (\tan^{-1} y - x)dy$

Solution: The given differential equation can be written as

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1} y}{1+y^2}$$

It is a linear differential equation of first order in x.

It is of the form $\frac{dx}{dy} + P(y)x = Q(y)$, where $P(y) = \frac{1}{1+y^2}$, $Q(y) = \frac{\tan^{-1} y}{1+y^2}$.

I.F $= e^{\int P(y)dy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}$

Multiplying the given differential equation by the I.F and integrating, we get

$$xe^{\int P(y)dy} = \int Q(y)e^{\int P(y)dy}dy + C$$

i.e., $xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y}dy + C$

Put $\tan^{-1} y = u \Rightarrow \frac{1}{1+y^2}dy = du$

$$= \int ue^u du + C = (u-1)e^u + C = (\tan^{-1} y - 1)e^{\tan^{-1} y} + C$$

The general solution of the given differential equation is

$xe^{tan^{-1} y} = (tan^{-1} y - 1)e^{tan^{-1} y} + C$, where C is an arbitrary constant

Example 3: The initial value problem governing the current i flowing in a series RL circuit when a voltage $V(t) = t$ is applied, is given by

$$iR + L\frac{di}{dt} = t\,,\ t \geq 0, i(0) = 0$$

where R and L are constants. Find the current $i(t)$ at time t.

Solution: The given equation is written as $\frac{di}{dt} + \frac{R}{L} i = \frac{1}{L} t$

It is a linear differential equation of first order in i. An integrating factor is $e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$.

Multiplying the given equation by the I.F and integrating, we get

$$i\, e^{\frac{R}{L}t} = \int \frac{1}{L} t \cdot e^{\frac{R}{L}t} dt + C = \frac{1}{L}\left[\frac{te^{\frac{R}{L}t}}{\left(\frac{R}{L}\right)} - \frac{1}{\left(\frac{R}{L}\right)} e^{\int \frac{R}{L}t} dt\right] + C \text{ (integrating by parts)}$$

$$= \frac{1}{R}\left(t - \frac{L}{R}\right) e^{\frac{R}{L}t} + C$$

Applying the initial condition $i(0) = 0$, we get $0 = -\frac{L}{R^2} + C \implies C = \frac{L}{R^2}$

The current $i(t)$ at time t is given by

$$i(t) = \frac{1}{R}\left(t - \frac{L}{R}\right) + \frac{L}{R^2} e^{-\frac{R}{L}t} \implies i(t) = \frac{t}{R} + \frac{L}{R^2}\left(e^{-\frac{R}{L}t} - 1\right)$$

Equations Reducible to Linear Form

Equations of the form $f'(y)\frac{dy}{dx} + P(x)f(y) = Q(x)$, where $P(x)$ and $Q(x)$ are functions of x, can be reduced to linear form. Put $f(y) = u$.

Then $f'(y)\frac{dy}{dx} = \frac{du}{dx}$ and the given equation becomes

$$\frac{du}{dx} + P(x)u = Q(x)$$

This is a linear differential equation of first order in u, whose solution can be obtained.

Example 4: *Solve:* $\frac{dy}{dx} + \frac{y}{x} \ln y = \frac{y}{x^2} (\ln y)^2$

Solution: Dividing the given equation by $y(\ln y)^2$, we get

$$\frac{1}{y(\ln y)^2}\frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{\ln y} = \frac{1}{x^2}$$

Put $\frac{1}{\ln y} = u \implies -\frac{1}{(\ln y)^2} \cdot \frac{1}{y}\frac{dy}{dx} = \frac{du}{dx}$. The equation now becomes

$$-\frac{du}{dx} + \frac{1}{x}\,u = \frac{1}{x^2}\ , i.e., \frac{du}{dx} - \frac{1}{x}u = -\frac{1}{x^2}$$

It is a linear differential equation of first order in u. I.F $= e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$.
Multiplying the above equation by the I.F and integrating, we get

$$\frac{u}{x} = \int -\frac{1}{x^2} \cdot \frac{1}{x} dx + C \implies \frac{u}{x} = \frac{1}{2x^2} + C \implies \frac{1}{x \ln y} = \frac{1}{2x^2} + C$$

The general solution of the given differential equation is $\frac{1}{x \ln y} = \frac{1}{2x^2} + C$, where C is an arbitrary constant.

Bernoulli Equation

An equation of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ is known as ***Bernoulli equation***, where n is a real number and $P(x)$ and $Q(x)$ are continuous functions of x on an interval $\boldsymbol{I}$**.**

If $n = 0$ or 1, the equation is linear. For all other values of n, the equation is non-linear. Note that if $n > 0$, then $y = 0$ is a solution.

To find the non-trivial solutions (when $n \neq 0,1$) divide the equation on both sides by y^n, then the equation becomes

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{1}{y^{n-1}} = Q(x)$$

Put $\frac{1}{y^{n-1}} = u \Rightarrow (1-n)\frac{1}{y^n}\frac{dy}{dx} = \frac{du}{dx} \Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$

The equation now becomes $\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$ and it is linear in u. After getting its solution, replace u by $\frac{1}{y^{n-1}}$, then we get the general solution of Bernoulli equation.

Example 5: *Solve:* $x\frac{dy}{dx} + y = y^2x^3 \cos x$

Solution: The given differential equation can be written as

$$\frac{dy}{dx} + \frac{1}{x}\, y = (x^2 \cos x) y^2$$

It is a Bernoulli equation. Divide throughout by y^2, we get

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y} = x^2 \cos x$$

Put $\frac{1}{y} = u \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$. The above equation now becomes

$$-\frac{du}{dx} + \frac{1}{x} \cdot u = x^2 \cos x \Rightarrow \frac{du}{dx} - \frac{1}{x}\, u = -x^2 \cos x,$$ which is linear in u.

I.F $= e^{\int -\frac{1}{x} dx} = \frac{1}{x}$. Multiplying the equation by the I.F and integrating, we get

$u \cdot \frac{1}{x} = -\int \left(\frac{1}{x}\right) x^2 \cos x\, dx + C = -\int x \cos x\, dx + C$

i.e., $\frac{1}{xy} = -[x \sin x - \int \sin x\, dx] + C = -x \sin x - \cos x + C.$

The general solution of the given differential equation is

$$\frac{1}{xy} = -x \sin x - \cos x + C,$$ where C is an arbitrary constant.

Riccati Equation

A first order differential equation of the form

$$\boldsymbol{y' = Q(x) y^2 + P(x) y + R(x)}$$

is called the ***Riccati Equation***.

It is an important first order non-linear differential equation. If $R(x) = 0$, then it reduces to the Bernoulli's equation with $n = 2$ and the trivial solution $y = 0$ is also a solution in this case.

When $R(x) \neq 0$, there is no simple method of solving the equation.

However, it can be reduced to a linear differential equation of first order in u, if one solution (i.e., a particular solution) say $y = v(x)$ is known by the substitution $y = v(x) + \frac{1}{u(x)}$.

Now, $\frac{dy}{dx} = \frac{dv}{dx} - \frac{1}{u^2} \cdot \frac{du}{dx}$

The Riccati equation now becomes

$$\frac{dv}{dx} - \frac{1}{u^2}\frac{du}{dx} = Q\left[v + \frac{1}{u}\right]^2 + P\left(v + \frac{1}{u}\right) + R$$

$$\Rightarrow \frac{dv}{dx} - \frac{1}{u^2}\frac{du}{dx} = Qv^2 + Pv + R + \left(\frac{2v}{u} + \frac{1}{u^2}\right) + \frac{P}{u} \Rightarrow -\frac{1}{u^2}\frac{du}{dx} = Q\left(\frac{2v}{u} + \frac{1}{u^2}\right) + \frac{P}{u}$$

(since v is a solution of Riccati equation)

$\Rightarrow \frac{du}{dx} + (2Qv + P)u = -Q$, which is a linear differential equation of first order in u and can be solved in the usual method.

Example 6: *Solve:* $y' = y^2 - (2x - 1)y + x^2 - x + 1$ *if* $y = x$ *is a solution of it.*

Solution: It is of the form $y' = Q(x)y^2 + P(x)y + R(x)$ and it is a Riccati equation. Given $y = x$ is a solution of it.

Put $y = x + \frac{1}{u(x)}$. Then $y' = 1 - \frac{1}{u^2}\frac{du}{dx}$.

Substituting in the given equation, we get

$$1 - \frac{1}{u^2}\frac{du}{dx} = \left(x + \frac{1}{u}\right)^2 - (2x - 1)\left(x + \frac{1}{u}\right) + x^2 - x + 1$$

$$= x^2 + \frac{2x}{u} + \frac{1}{u^2} - 2x^2 + x - \frac{2x}{u} + \frac{1}{u} + x^2 - x + 1$$

$$\Rightarrow -\frac{1}{u^2}\frac{du}{dx} = \frac{1}{u^2} + \frac{1}{u} \quad \Rightarrow \frac{du}{dx} + u = -1$$

It is a linear differential equation of first order in u. Its $I.F = e^{\int dx} = e^x$. Multiplying the above equation by the I.F and integrating, we get

$$ue^x = \int -e^x dx + C, \ i.e., \ u = Ce^{-x} - 1$$

The general solution of the given Riccati equation is

$$y = x + \frac{1}{u} = x + \frac{1}{Ce^{-x} - 1}\text{, where } C \text{ is an arbitrary constant.}$$

Clairaut's Equation

A first order differential equation of the form

$$\boldsymbol{y = xy' + f(y')} \;\; or \;\; \boldsymbol{y = px + f(p)},$$

where $p = y'$ is called the ***Clairaut's equation***.

It is a non-linear differential equation. This equation is an interesting equation since it admits a singular solution. Differentiating w.r.t x, we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx} \Rightarrow p = p + \left(x + f'(p)\right)\frac{dp}{dx} \Rightarrow \frac{dp}{dx} = 0 \text{ or } x + f'(p) = 0$$

If $\frac{dp}{dx} = 0$, then by integration w.r.t x, we get $p = c$.

Now, substituting in the given equation, we get the general solution as $y = Cx + f(C)$, where C is an arbitrary constant

If $x + f'(p) = 0$, then $x = -f'(p)$. The eliminant of p between the given Clairaut's equation $y = px + f(p)$ and $x = -f'(p)$ gives the singular solution.

Note: The parametric equations

$$x = -f'(t) \text{ and } y = f(t) - tf'(t)$$

define a solution (not involving any constant), which is the singular solution.

The general solution of the Clairaut's equation defines a one parameter family of straight lines. These straight lines are all tangential to the curve defined by the singular solution. The singular solution curve is the ***envelope*** of the family of straight lines defined by the general solution $y = Cx + f(C)$.

Example 7: *Obtain the general solution and singular solution of the non-linear differential equation $y = xy' + (y')^2$.*

Solution: The given differential equation is $y = px + p^2$, where $p = y'$ and it is a Clairaut's equation.

Differentiating w.r.t x, we get

$$p = p + x\frac{dp}{dx} + 2p\frac{dp}{dx} \text{ i.e., } \frac{dp}{dx}(x + 2p) = 0.$$

Setting $\frac{dp}{dx} = 0$, we get $p = C$. Therefore, the general solution of the given differential equation is $y = Cx + C^2$, where C is an arbitrary constant.

Setting $x + 2p = 0$, we get $x = -2p$. The eliminant of p between $x = -2p$ and $y = xp + p^2$ gives $y = x\left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2$ i.e., $x^2 + 4y = 0$.

Thus, the singular solution is $x^2 + 4y = 0$.

P1:

Solve: $\boldsymbol{x\cos x\frac{dy}{dx}+(x\,sinx+\cos x)y=1}$

Solution: The given differential equation can be written as

$$\frac{dy}{dx}+\frac{x\sin x+\cos x}{x\cos x}y=\frac{1}{x\cos x} \quad \ldots(1)$$

This is of the form $\frac{dy}{dx}+P(x)y=Q(x)$, and it is a linear differential equation of first order in y, where

$$P(x)=\frac{x\sin x+\cos x}{x\cos x}\ ; \quad Q(x)=\frac{1}{x\cos x}$$

$$I.F=e^{\int P(x)dx}=e^{\int\left(\tan x+\frac{1}{x}\right)dx}=e^{\ln|x\sec x|}=x\sec x$$

Multiplying the given differential equation by the I.F. and integrating, we get

$$y.e^{\int P(x)dx}=\int Q(x)\,e^{\int P(x)dx}+C$$

$$\Rightarrow\ xy\sec x=\int\frac{1}{x\cos x}.x\sec x\ dx+C=\int\sec^2 x\,dx+C$$

Therefore, the general solution of the given differential equation is

$xy\sec x=\tan x+C$, where C is an arbitrary constant.

P2:

Solve: $(\mathbf{1}+\boldsymbol{y}^{2})\boldsymbol{dx}+\left(\boldsymbol{x}-\boldsymbol{e}^{\tan^{-1}y}\right)\boldsymbol{dy}=\mathbf{0}$

Solution: The given differential equation can be written as

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{\tan^{-1}y}}{1+y^2} \qquad \ldots(1)$$

It is of the form $\frac{dx}{dy}+P(y)x=Q(y)$, where

$$P(y)=\frac{1}{1+y^2} \quad ; \quad Q(y)=\frac{e^{\tan^{-1}y}}{1+y^2}$$

It is a linear differential equation in x. I.F. $= e^{\int P(y)dy}=e^{\int\frac{1}{1+y^2}dy}=e^{tan^{-1}y}$

Multiplying the given differential equation by the I.F. and integrating, we get

$$x.e^{\int P(y)dy}=\int Q(y).e^{\int P(y)dy}dy+C$$

$$\Rightarrow xe^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}}{1+y^2}.e^{\tan^{-1}y}dy+C=\int\frac{e^{2\tan^{-1}y}}{1+y^2}.+C$$

Put $\tan^{-1}y=t\Rightarrow\frac{1}{1+y^2}dy=dt$

$$=\int e^{2t}dt+C=\frac{e^{2t}}{2}+C=\frac{e^{\tan^{-1}y}}{2}+C$$

Therefore, the general solution of (1) is

$xe^{tan^{-1}y}=\frac{e^{tan^{-1}y}}{2}+C$, where C is an arbitrary constant

P3:

Solve: $\boldsymbol{2ycosy^2 \frac{dy}{dx} - \frac{2}{x+1} siny^2 = (x+1)^3}$

Solution: The given differential equation is

$$2ycosy^2 \frac{dy}{dx} - \frac{2}{x+1} siny^2 = (x+1)^3 \qquad \ldots (1)$$

Taking $siny^2 = u$ we get $2ycosy^2 \left(\frac{dy}{dx}\right) = \frac{du}{dx}$. Substituting in (1) ,we get

$$\frac{du}{dx} - \frac{2}{x+1} u = (x+1)^3$$

This is a first order linear differential equation in u.

$I.F. = e^{-\int \frac{2}{1+x^2} dx} = e^{-2\log(x+1)} = \frac{1}{(x+1)^2}$

Multiplying the above equation by the I.F. and integrating, we get

$$u \frac{1}{(x+1)^2} = \int (x+1)^3 \frac{1}{(x+1)^2} dx + C$$

$$\Rightarrow \frac{\sin y^2}{(x+1)^2} = \int (x+1) dx + C = \frac{(x+1)^2}{2} + C$$

Therefore, the general solution of the given differential equation is

$siny^2 = \frac{(x+1)^4}{2} + C(x+1)^2$, where C is an arbitrary constant.

P4:

Find the general solution of the equation

$$\frac{dy}{dx} = 2xy^2 + (1-4x)y + 2x - 1, \text{ if } y = 1 \text{ is a solution of the equation}$$

Solution: The given differential equation is

$$\frac{dy}{dx} = 2xy^2 + (1-4x)y + 2x - 1$$

It is a Riccati equation. Given $y = 1$ is a particular solution.

Substitute $y = 1 + \frac{1}{u}$ in the above equation, we obtain

$$-\frac{1}{u^2}\frac{du}{dx} = 2x\left(1 + \frac{2}{u} + \frac{1}{u^2}\right) + (1-4x)\left(1 + \frac{1}{u}\right) + 2x - 1$$

$\Rightarrow \frac{du}{dx} + u = -2x$, which is a linear in u

$I.F. = e^{\int dx} = e^x.$

Multiplying the above equation by the I.F. and integrating, we get

$$ue^x = -\int 2xe^x\,dx + C = -2(x-1)e^x + C \;\Rightarrow u = Ce^{-x} - 2(x-1)$$

Therefore, the general solution of given differential equation is

$y = 1 + \frac{1}{u} = 1 + \frac{1}{Ce^{-x} - 2(x-1)}$, where C is an arbitrary constant.

IP1.

Obtain the equation of the curve satisfying the differential equation $(1 + x^2)\frac{dy}{dx} + (2xy - 4x^2) = 0$ and passing through the origin.

Solution: The given differential equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$$

It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$, where $P(x) = \frac{2x}{1+x^2}$; $Q(x) = \frac{4x^2}{1+x^2}$

It is a linear differential equation of first order in y.

$$\text{I.F.} = e^{\int P(x)dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln\left|1+x^2\right|} = 1 + x^2$$

Multiplying the above equation by the integrating factor and integrating, we get

$$y.e^{\int P(x)dx} = \int Q(x).e^{\int P(x)dx}dx + K$$

$$\Rightarrow y\left(1+x^2\right) = \int \frac{4x^2}{1+x^2}.\left(1+x^2\right)dx + K = \int 4x^2dx + C = \frac{4x^3}{3} + K$$

$$\Rightarrow 3y(1+x^2) + 3y - 4x^3 = C$$

Since the curve passes through the origin (0,0), we have

$0 + 0 + 0 = C \Rightarrow C = 0$

Required equation of the curve is $3y(1 + x^2) = 4x^3$

IP2.

Solve: $(\mathbf{1}+\boldsymbol{x}+\boldsymbol{x}\boldsymbol{y}^{2})\boldsymbol{dy}+(\boldsymbol{y}+\boldsymbol{y}^{3})\boldsymbol{dx}=\mathbf{0}$

Solution: The given differential equation can be written as

$$\frac{dx}{dy}+\frac{1+x(1+y^2)}{y(1+y^2)}=0 \Longrightarrow \frac{dx}{dy}+\frac{x}{y}=-\frac{1}{y(1+y^2)}$$

It is a linear differential equation in x. $I.F.=e^{\int\frac{1}{y}dy}=e^{ln|y|}=y.$

Multiplying the above equation by the I.F. and integrating, we get

$$xy=-\int\frac{1}{y(1+y^2)}.y\,dy+C=-\int\frac{1}{(1+y^2)}\,dy+C=-\tan^{-1}y+C$$

Therefore, the general solution of the given differential equation is

$$xy+\tan^{-1}y=C, \text{ where } C \text{ is an arbitrary constant}$$

IP3:

Solve: $\boldsymbol{\frac{dy}{dx} + \frac{y}{x} = y^2 x sinx}$

Solution: Given equation is $\frac{dy}{dx} + \frac{y}{x} = y^2 xsinx$ … (1)

This is a Bernoulli's differential equation with $n = 2$. Dividing throughout by y^2, we get

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}.\frac{1}{x} = xsinx$$

Put $\frac{1}{y} = u \Longrightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$. The above equation now becomes

$$-\frac{du}{dx} + u.\frac{1}{x} = xsinx \Longrightarrow \frac{du}{dx} - u\frac{1}{x} = -xsinx$$

This is a linear differential equation in u. I.F. $\int -\frac{1}{x}dx = e^{-logx} = \frac{1}{x}$

Multiplying the equation by the I.F. and integrating, we get

$$u.\frac{1}{x} = \int (xsinx)\frac{1}{x}dx + C \Longrightarrow \frac{1}{xy} = -\int (sinx)dx + C = cosx + C$$

Therefore, the general solution of the given differential equation is

$\frac{1}{xy} = cosx + C$, where C is an arbitrary constant.

IP4.

Find the general solution and singular solution of the differential equation $\boldsymbol{y = xy' - \frac{1}{y'}}$

Solution: The given equation is $y = xp - \frac{1}{p}$, where $p = y'$

It is of the form $y = px + f(p)$, which is a Clairaut's equation.

Differentiating w.r.t x, we get $p = x\frac{dp}{dx} + p + \frac{1}{p^2}\frac{dp}{dx} \Rightarrow \left(x + \frac{1}{p^2}\right)\frac{dp}{dx} = 0$

(i) $\frac{dp}{dx} = 0 \Rightarrow p = C$

The general solution is $y = Cx - \frac{1}{C}$, where C is an arbitrary constant.

(ii) $x + \frac{1}{p^2} = 0 \Rightarrow p^2 = -\frac{1}{x}$

Now, $y = xp - \frac{1}{p}$, squaring on both sides, we get

$y^2 = x^2p^2 + \frac{1}{p^2} - 2x$, Eliminating p, we have

$y^2 = x^2\left(-\frac{1}{x}\right) - x - 2x \Rightarrow y^2 + 4x = 0$, is the singular solution.