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Факультет математики і інформатики
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"Фокусуюче рівняння Вадаті-Конно-Ішікава на ненульовому фоні"

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Підпис керівника

Харків 2022 р.

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1 Introduction

As a well-known integrable equation, the nonlinear Schroedinger (NLS) equation:

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q = 0$$

has important research significance in the field of mathematical physics, which is widely used in optics, free water waves of ideal fluids, plasma waves, etc [8,12, 18, 24, 28, 29, 33, 43, 46, 49, 50]. When pulses with a short propagation time along the fiber are considered, the NLS equation with high-order nonlinear dispersion terms is further studied, which indeed plays an important role in great success of optical communication technology. The nonlinear term is added to make it balance with the linear term, which produce a stable soliton solution. The high-order nonlinear term can expand the category of Kerr-type nonlinearity and change the influence on the NLS equation of infinite short pulse duration when the input field strength or the number of soliton changes. Because the change of the refractive index, caused by the field, is controlled by high-order nonlinearity, from which we know that Kerr-type nonlinearity cannot describe short pulses and higher input peak pulse power. As a result, the photoinduced refractive index change becomes saturated at higher field strengths. In order to study the propagation of soliton in materials with saturable nonlinearity, the NLS equation with saturable nonlinearity [16, 30] is studied, which reads

$$iq_t + q_{xx} + \frac{|q|^2}{1 + r|q|^2} q = 0$$

where r is a constant. However, it is difficult to obtain an analytical solution to this equation because it is not integrable. For integrable models with saturable nonlinearities, people can trace back to the Wadati-Konno-Ichikawa (WKI) equation [32, 36], which is written as

$$iq_t + \left(\frac{q}{\sqrt{1 + |q|^2}} \right)_{xx} = 0$$

where q denotes a potential complex function related to x and t. As we all know, the WKI equation has very high nonlinearity and peak solutions. Therefore, more and more attention is paid to the research of this equation. For the WKI equation, there is a gauge transformation that enables mutual conversion between the Ablowitz, Kaup, Newell and Segur (AKNS) system and the WKI equation in [17]. Resorting to the gauge transformation, the Backlund transformation was studied in [2, 21, 38]. The classic inverse scattering method with the Schwartz initial value and finite density initial value for the WKI equation was reported in [32] and [7], respectively. In [34, 35], exact stationary solution is considered and its orbital stability is discussed. The existence of global solution for the WKI equation with small initial data was studied in [31] and the explicit theta function representations of solutions were studied in [23]. In [47], the Darboux transformation was used to investigate the WKI system and the breathe and rogue wave solution were obtained. Employing the Riemann-Hilbert problem to investigate the WKI equation under the condition that the scattering coefficient has simple poles and multiple higher-order poles shown in [25] and [48]. The purpose of this work is considering the WKI equation with the initial value condition

$$q(x,0) = q_0(x) \in (S_-(\mathbb{R}) + c_-) \cap (S_+(\mathbb{R}) + c_+)\,, -\infty < x < +\infty$$

where $S_\pm(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}), \forall n, m \in \mathbb{Z}_+ \sup_{x \in (0, \pm\infty)} \left| x^n f^{(m)}(x) \right| < \infty \right\}$.

As early as 1976, the exact expression of the long time asymptotic solution for the NLS equation with decaying initial value was given in [45]. In 1993, Deift and Zhou (DZ) put forward analytical methods to analyze the asymptoticity of the solution of oscillatory Riemann-Hilbert problem, that is, a nonlinear steepest descent method, which gave a rigorous proof process [10]. The basic idea of this method is to deform the jump contour of the RH problem, from the jump on the original real axis to the jump on the fastest-descending contour. In addition, it is necessary to ensure that the jump matrix on the fastest-descending contour converges to the identity matrix when time tends to infinity, and the vicinity of the phase point is processed by the classic parabolic cylinder model. Due to its effectiveness in the long-time asymptotic behavior of the solution for nonlinear integral equations, the nonlinear steepest descent method is widely used in many equations, such as the defocusing nonlinear Schroedinger equation [9], the modified nonlinear Schroedinger equation [19, 20], the focusing nonlinear Schroedinger equation [1], the Korteweg-de Vries equation [13], derivative nonlinear Schroedinger equation [42], Fokas-Lenells equation [41], the Spin-1 Gross-Pitaevskii equation [11], coupled Hirota equation [26], short pulse equation [39, 40], the Cammasa-Holm equation [3], Kundu-Eckhaus equation [14, 37], the coupled dispersive AB system [5], the sine-Gordon equation [6], extended modified Korteweg-de Vries equation [27], the "good" Boussinesq equation [4], nonlocal mKdV equation [15] etc.

2 Problem formulation

We are going to apply Riemann-Hilbert approach to constructing solutions to the Wadati–Konno–Ichikawa equation:

$$iq_t + \left(\frac{q}{\Phi}\right)_{xx} = 0, \Phi = \sqrt{1 + |q|^2} \tag{1}$$

$$q\,(x,0) = q_0\,(x) \tag{2}$$

We will also assume that:

$$q - c_-, q_0 - c_- \in \mathcal{S}\,(\mathbb{R}_-) \tag{3}$$

$$q - c_+, q_0 - c_+ \in \mathcal{S}\,(\mathbb{R}_+) \tag{4}$$

Therefore,

$$\lim_{x\to\pm\infty} q(x) \to c_\pm, \lim_{x\to\pm\infty} \Phi(x) \to \phi_\pm = \sqrt{1 + |c_\pm|^2} \tag{5}$$

We will turn our attention to the Lax pair equation for the WKI:

$$\begin{cases} \Psi_x = U\,(x,t,\lambda)\,\Psi \\ \Psi_y = V\,(x,t,\lambda)\,\Psi \end{cases} \tag{6}$$

$$U = \lambda \begin{pmatrix} -i & q \\ -q^* & i \end{pmatrix} \tag{7}$$

$$V = \begin{pmatrix} -\frac{2i\lambda^2}{\Phi} & \frac{2q}{\Phi}\lambda^2 + i\lambda\left(\frac{q}{\Phi}\right)_x \\ -\frac{2q^*}{\Phi}\lambda^2 + i\lambda\left(\frac{q^*}{\Phi}\right)_x & \frac{2i\lambda^2}{\Phi} \end{pmatrix} \tag{8}$$

3 Spectral analysis

3.1 Asymptotic analysis at $\lambda = 0$

To be able to write our matrix solutions as solutions for Volterra-type integral equations we have to perform two transformations. The first one:

$$\tilde{\Psi}^{(1)} = {P^{(1)}}^{-1}\Psi \tag{9}$$

in a way, so that our equations turn into

$$\begin{cases} \tilde{\Psi}^{(1)}_x + \theta^{(1)}_x\tilde{\Psi}^{(1)} = \tilde{U}^{(1)}\tilde{\Psi}^{(1)} \\ \tilde{\Psi}^{(1)}_t + \theta^{(1)}_t\tilde{\Psi}^{(1)} = \tilde{V}^{(1)}\tilde{\Psi}^{(1)} \end{cases} \tag{10}$$

with

$$\tilde{U}^{(1)}, \tilde{V}^{(1)} \to 0, x \to -\infty \tag{11}$$

$$\theta - diagonal \tag{12}$$

In order to do that, we introduce

$$\overline{U}^{(1)} = {P^{(1)}}^{-1} U P^{(1)} \tag{13}$$

$$\theta^{(1)}_x = -\overline{U}^{(1)}(-\infty) \tag{14}$$

$$\tilde{U}^{(1)} = \overline{U}^{(1)} + \theta^{(1)}_x \tag{15}$$

Since $U(-\infty), V(-\infty)$ commute, V diagonializes at infinity as well. So,

$$P^{(1)} = \begin{pmatrix} i(1+\phi_-) & -\frac{ic_-}{1+\phi_-} \\ c^*_- & 1 \end{pmatrix} \tag{16}$$

$$\theta^{(1)} = (i\lambda\phi_- x + 2i\lambda^2 t)\sigma_3 \tag{17}$$

$$\tilde{U}^{(1)} = \frac{i\lambda}{2\phi_-}\begin{pmatrix} c_-\left(c^*_- - q^*\right) + c^*_-\left(c_- - q\right) & \left(c_- - q\right) + \frac{c_-^2}{(1+\phi_-)^2}\left(q^* - c^*_-\right) \\ c^{*2}\left(q - c_-\right) - \left(1+\phi_-\right)^2\left(q^* - c^*_-\right) & c_-\left(q^* - c^*_-\right) + c^*_-\left(q - c_-\right) \end{pmatrix} \tag{18}$$

The second transformation is aimed directly at changing (10) into Volterra equations:

$$\mu = \tilde{\Psi}^{(1)} e^{\theta^{(1)}} \tag{19}$$

$$\begin{cases} \mu_x + \left[\theta^{(1)}_x, \mu\right] = \tilde{U}^{(1)}\mu \\ \mu_t + \left[\theta^{(1)}_t, \mu\right] = \tilde{V}^{(1)}\mu \end{cases} \tag{20}$$

Then,

$$\mu_-(x) = I + \int_{-\infty}^{x} e^{\theta^{(1)}(y) - \theta^{(1)}(x)}\tilde{U}^{(1)}(y)\,\mu_-(y)\,e^{\theta^{(1)}(x) - \theta^{(1)}(y)}dy \tag{21}$$

which implies:

$$\mu_-(x,t,\lambda) = I + \int_{-\infty}^{x} \tilde{U}^{(1)}(y,t,\lambda)\,dy + O\left(\lambda^2\right) \tag{22}$$

3.2 Asymptotic analysis at $\lambda = \infty$

In this case will perform 3 tranformations in ordred to be able to write the solutions in the similar way. First,

$$\overline{\Psi}^{(2)} = P_1 \Psi \tag{23}$$

$$P_1 = \sqrt{\frac{1+\Phi}{2\Phi}} \begin{pmatrix} 1 & \frac{iq}{\Phi+1} \\ \frac{iq^*}{\Phi+1} & 1 \end{pmatrix} \tag{24}$$

Again, (6) starts to take form:

$$\begin{cases} \overline{\Psi}_x^{(2)} + \theta_x^{(2)} \overline{\Psi}^{(2)} = \overline{U}^{(2)} \overline{\Psi}^{(2)} \\ \overline{\Psi}_t^{(2)} + \theta_t^{(2)} \overline{\Psi}^{(2)} = \overline{V}^{(2)} \overline{\Psi}^{(2)} \end{cases} \tag{25}$$

with

$$\theta_x^{(2)} = i\lambda\Phi\sigma_3 \tag{26}$$

$$\theta_t^{(2)} = 2i\lambda^2\sigma_3 + \lambda \frac{qq_x^* - q_x q^*}{2\Phi^2}\sigma_3 \tag{27}$$

The compatibility condition $\theta_{xt}^{(2)} = \theta_{tx}^{(2)}$ will be proven in the appendix.
From (26) and (27) we obtain:

$$\theta^{(2)}(x,t,\lambda) = \left(i\lambda\tilde{x}^{(2)} + 2i\lambda^2 t\right)\sigma_3 \tag{28}$$

$$\tilde{x}^{(2)} = \phi_- x + \int_{-\infty}^{x} \left(\Phi(y) - \phi_-\right) dy \tag{29}$$

$$\overline{U}^{(2)} = \begin{pmatrix} -\frac{qq_x^* - q_x q^*}{4\Phi(1+\Phi)} & -\frac{iq\left(\Phi(qq_x^* - q_x q^*) - |q|_x^2\right)}{4\Phi^2(\Phi^2-1)} \\ \frac{iq^*\left(\Phi(qq_x^* - q_x q^*) + |q|_x^2\right)}{4\Phi^2(\Phi^2-1)} & \frac{qq_x^* - q_x q^*}{4\Phi(1+\Phi)} \end{pmatrix} \tag{30}$$

$$\overline{V}^{(2)} = \begin{pmatrix} \overline{V}_{11}^{(2)} & \overline{V}_{12}^{(2)} \\ \overline{V}_{21}^{(2)} & -\overline{V}_{11}^{(2)} \end{pmatrix} \tag{31}$$

$$\overline{V}_{11}^{(2)} = -\frac{qq_t^* - q_t q^*}{4\Phi(1+\Phi)} \tag{32}$$

$$\overline{V}_{12}^{(2)} = -\frac{iq\left(q^*\left(qq_x^* - q^* q_x\right) - 2q_x^*(\Phi-1)\right)}{2(\Phi-1)\Phi^3 q^*}\lambda - \frac{iq\left(q^*\left(qq_t^* - q^* q_t\right) - 2q_t^*(\Phi-1)\right)}{4\Phi^2(\Phi-1)q^*} \tag{33}$$

$$\overline{V}_{21}^{(2)} = \frac{iq^*\left(q\left(qq_x^* - q^* q_x\right) + 2q_x^*(\Phi-1)\right)}{2(\Phi-1)\Phi^3 q}\lambda + \frac{iq^*\left(q\left(qq_t^* - q^* q_t\right) + 2q_t^*(\Phi-1)\right)}{4\Phi^2(\Phi-1)q} \tag{34}$$

Note, that both $\overline{U}^{(2)}$ and $\overline{V}^{(2)}$ tend to 0 as $x \to \pm\infty$. However, in order to control to solutions' of the Volterra integral equations behaviour as $\lambda \to \infty$, we will need the diagonal parts of $\tilde{U}^{(2)}, \tilde{V}^{(2)}$ to decay as $\lambda \to \infty$. If we want to achieve that, we introduce the second transformation:

$$\tilde{\Psi}^{(2)} = P_2 \overline{\Psi}^{(2)} \tag{35}$$

where P_2 is diagonal. This way, $\theta^{(2)}$ will remain unchanged and

$$\tilde{U}^{(2)} = P_2\overline{U}^{(2)}P_2^{-1} + (P_2)_x P_2^{-1} \tag{36}$$

So,

$$P_2(x,t) := e^{-\int\limits_{\infty}^{x} \frac{qq_x^* - q_x q^*}{4\Phi(1+\Phi)} dy\sigma_3} \tag{37}$$

which would eliminate the diagonal of $\tilde{U}^{(2)}$, and $\tilde{V}^{(2)}$ still tends to 0 as $x \to \pm\infty$.
To put it together,

$$\tilde{\Psi}^{(2)} = P^{(2)}\Psi \tag{38}$$

$$P^{(2)} = P_2P_1 \tag{39}$$

Lastly,

$$\chi = \tilde{\Psi}^{(2)}e^{\theta^{(2)}} \tag{40}$$

$$\begin{cases} \chi_x + \left[\theta_x^{(2)}, \chi\right] = \tilde{U}^{(2)}\chi \\ \chi_t + \left[\theta_t^{(2)}, \chi\right] = \tilde{V}^{(2)}\chi \end{cases} \tag{41}$$

so

$$\chi_\pm(x) = I + \int\limits_{\pm\infty}^{x} e^{\theta^{(2)}(y)-\theta^{(2)}(x)}\tilde{U}^{(2)}(y)\,\chi_\pm(y)\,e^{\theta^{(2)}(x)-\theta^{(2)}(y)}dy \tag{42}$$

From this definition the following properties of $\chi_\pm$ can be inferred (subscript denotes a column):

$$1)\ \det\chi_\pm = 1 \tag{43}$$

$$2)\ \chi_{+1} \text{ and } \chi_{-2} \text{ are analytic in } \mathbb{C}_+ \text{ and continuous in } \overline{\mathbb{C}_+} \tag{44}$$

$$3)\ \chi_{-1} \text{ and } \chi_{+2} \text{ are analytic in } \mathbb{C}_- \text{ and continuous in } \overline{\mathbb{C}_-} \tag{45}$$

$$4)\ \begin{pmatrix}\chi_{+1} & \chi_{-2}\end{pmatrix} \to I \text{ as } \lambda\to\infty \text{ from within } \overline{\mathbb{C}_+} \tag{46}$$

$$5)\ \begin{pmatrix}\chi_{-1} & \chi_{+2}\end{pmatrix} \to I \text{ as } \lambda\to\infty \text{ from within } \overline{\mathbb{C}_+} \tag{47}$$

$$6)\ \chi_\pm^*(x,t,\lambda^*) = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\chi_\pm(x,t,\lambda)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \tag{48}$$

Moreover, since $\chi_\pm e^{-\theta^{(2)}}$ solve the same Lax pair, they are related in the following way:

$$\chi_+ = \chi_- e^{-\theta^{(2)}}s(\lambda)\,e^{\theta^{(2)}} \tag{49}$$

where

$$s(\lambda) = \begin{pmatrix} a^*(\lambda) & b(\lambda)\\ -b^*(\lambda) & a(\lambda)\end{pmatrix} \tag{50}$$

$$a(\lambda) = \det\begin{pmatrix}\chi_{-1} & \chi_{+2}\end{pmatrix} \tag{51}$$

$$b(\lambda) = \det\begin{pmatrix}\chi_{+2} & \chi_{-2}\end{pmatrix}e^{2\theta^{(2)}} \tag{52}$$

in the light of the symmetries of $\chi_{\pm}$. From (43)-(48) and (50)-(52) we can obtain:

$$1)\ a(\lambda), b(\lambda) \text{ can be determined from } q_0(x) \text{ (by taking t=0)} \tag{53}$$

$$2)\ \det s(\lambda) = |a(\lambda)|^2 + |b(\lambda)|^2 = 1, \lambda \in \mathbb{R} \text{ (by taking } \lambda \in \mathbb{R}, \text{ x=}\infty) \tag{54}$$

$$3)\ a(\lambda) \to 1, a(\lambda) \to 0 \text{ as } \lambda \to \infty \tag{55}$$

$$4)\ a(\lambda) \text{ is analytic in } \mathbb{C}_+ \text{ and continuous in } \overline{\mathbb{C}_+} \tag{56}$$

$$5)\ b(\lambda) \text{ is continuous for } \lambda \in \mathbb{R} \tag{57}$$

4 Formulating the Riemann-Hilbert problem

Since χ_{-1} and χ_{+2} are analytical in $\mathbb{C}_+$ we compile the first part of the variable M of the RHP from them, but, in order to ensure the uniqueness of the solution we construct M with $\lim\limits_{\lambda\to\infty} M = I$. Thus,

$$M(x,t,\lambda) = \begin{cases} \begin{pmatrix} \chi_{-1}(x,t,\lambda) & \frac{\chi_{+2}(x,t,\lambda)}{a(\lambda)} \end{pmatrix} & Im\lambda > 0 \\ \begin{pmatrix} \frac{\chi_{+1}(x,t,\lambda)}{a^*(\lambda^*)} & \chi_{-2}(x,t,\lambda) \end{pmatrix} & Im\lambda < 0 \end{cases} \tag{58}$$

Now, from (49), (50), (58),

$$M_+ = M_- e^{-\theta^{(2)}} J(\lambda) e^{\theta^{(2)}} \tag{59}$$

$$J(\lambda) = \begin{pmatrix} 1 & \gamma \\ \gamma^* & 1 + |\gamma|^2 \end{pmatrix} \tag{60}$$

$$\gamma(\lambda) = \frac{a(\lambda)}{b(\lambda)} \tag{61}$$

However, now the matrix has singularities, but, according to our assumptions, just simple poles. Let $\{\lambda_j\}_{j=1}^n$ be the poles and $\{r_j\}_{j=1}^n$ the respective residues. Substituting λ_j into the jump property of χ, it turns into $\chi_+(\lambda_j) = \begin{pmatrix} b^*(\lambda_j)\chi_{-2}(\lambda_j) & -b(\lambda_j)\chi_{-1}(\lambda_j) \end{pmatrix} \Rightarrow Res_{\lambda=\lambda_j} M_{+2} = d_j M_{-1}(\lambda_j), Res_{\lambda=\lambda_j^*} M_{-1} = f_j M_{+2}(\lambda_j^*)$, where d_j and f_j are defined in terms of λ_j, r_j, a, b. But this definition requires using $\tilde{x}^{(2)}$ instead of x, so the **RHP takes the following form:**
Find a piecewise meromorphic function M, which satisfies:
1) The jump condition $M_+ = M_- e^{-\theta^{(2)}} J(\lambda) e^{\theta^{(2)}}, J(\lambda) = \begin{pmatrix} 1 & \gamma \\ \gamma^* & 1 + |\gamma|^2 \end{pmatrix}$
2) The residue conditions $Res_{\lambda=\lambda_j} M_{+2} = d_j M_{-1}(\lambda_j), Res_{\lambda=\lambda_j^*} M_{-1} = f_j M_{+2}(\lambda_j^*)$
3) The normalization condition $\lim\limits_{\lambda\to\infty} M(\lambda) = I$

5 Asymptotics of the RHP solution

Since ${P^{(2)}}^{-1}\chi_- e^{-\theta^{(2)}}, P^{(1)}\mu_- e^{-\theta^{(1)}}$ solve the same Lax pair, we have:

$$\chi_-(x,t,\lambda) = \left(P^{(2)}P^{(1)}\right)(x,t)\, \mu_-(x,t,\lambda)\, e^{-\theta^{(1)}} C(\lambda)\, e^{\theta^{(2)}} \tag{62}$$

Taking $x \to -\infty$ leads to:

$$I = \left(P^{(2)}P^{(1)}\right)(-\infty, t)\, I e^{-\theta^{(1)}(-\infty,t)} C(\lambda)\, e^{\theta^{(2)}(-\infty,t)} \tag{63}$$

Let

$$\left(P^{(2)}P^{(1)}\right)(-\infty) = P_\infty = I\sqrt{\frac{1+\phi_-}{2\phi_-}} \begin{pmatrix} 1 & \frac{i(\phi_- -1)}{c_-^*} \\ \frac{i(\phi_- -1)}{c_-} & 1 \end{pmatrix} \begin{pmatrix} i(1+\phi_-) & \frac{i(1-\phi_-)}{c_-^*} \\ c_-^* & 1 \end{pmatrix} \tag{64}$$

$$P_\infty = \sqrt{2\phi_-(1+\phi_-)} \begin{pmatrix} i & 0 \\ 0 & \frac{1}{1+\phi_-} \end{pmatrix} \tag{65}$$

So, from (63)-(65),

$$C(\lambda) = P_\infty^{-1} e^{\left(\theta^{(1)}-\theta^{(2)}\right)(-\infty,t)} = P_\infty^{-1} \tag{66}$$

Thus, from (62), (66),

$$\chi_-(x,t,\lambda) = \left(P^{(2)}P^{(1)}\right)(x,t)\, \mu_-(x,t,\lambda)\, P_\infty^{-1} e^{i\lambda\nu(x)\sigma_3} \tag{67}$$

with

$$\nu(x) = \int_{-\infty}^{x} \left(\Phi(y) - \phi_-\right) dy = \tilde{x}^{(2)} - \phi_- x \tag{68}$$

Asymptotically, from (22), (67):

$$\chi_-(x,t,\lambda) = \left(P^{(2)}P^{(1)}\right)(x,t)\, P_\infty^{-1} \left(I + \int_{-\infty}^{x} P_\infty \tilde{U}^{(1)}(y,t,\lambda)\, P_\infty^{-1} dy + i\lambda\nu(x)\sigma_3 + O\left(\lambda^2\right) \right) \tag{69}$$

Since,

$$\left(P_\infty \tilde{U}^{(1)} P_\infty^{-1}\right)_1 = \frac{i\lambda}{2\phi_-} \begin{pmatrix} c_-\left(c_-^* - q^*\right) + c_-^*\left(c_- - q\right) \\ -\frac{i}{1+\phi_-}\left({c_-^*}^2 (q - c_-) - (1+\phi_-)^2 \left(q^* - c_-^*\right)\right) \end{pmatrix} \tag{70}$$

Based on (58), (69), (70):

$$M_{+1} = \chi_{-1} = \begin{pmatrix} 1 - \frac{i\lambda}{2\phi_-}\left(c_- \hat{q}^*(x) + c_-^* \hat{q}(x)\right) + i\lambda\nu(x) \\ \frac{\lambda}{2\phi_-(1+\phi_-)}\left({c_-^*}^2 \hat{q}(x) - (1+\phi_-)^2 \hat{q}^*(x)\right) \end{pmatrix} + O\left(\lambda^2\right) \tag{71}$$

where

$$\hat{q}(x) = \int_{-\infty}^{x} \left(q(y) - c_-\right) dy \tag{72}$$

6 Extracting solution to the WKI from the RH problems solution's asymptotics

Provided that the specified Riemann-Hilbert problem has a solution $M(\tilde{x}^{(2)}, t, \lambda)$, a solution $q(x,t)$ to the Cauchy problem (1)-(4) for the Wadati-Konno-Ichikawa equation can be written in terms of $M(\tilde{x}^{(2)}, t, \lambda)$ as $\lambda \to \infty$. More specifically, let us define the functions $g_i(\tilde{x}^{(2)}, t), h_i(\tilde{x}^{(2)}, t), i = 1, 2$:

$$g_1(\tilde{x}^{(2)}, t) = \lim_{\lambda \to 0} \frac{2\phi_-}{i} \frac{\partial M_{+11}(\tilde{x}^{(2)}, t, \lambda)}{\partial \lambda} \tag{73}$$

$$g_2(\tilde{x}^{(2)}, t) = \lim_{\lambda \to 0} 2\phi_- (1 + \phi_-) \frac{\partial M_{+21}(\tilde{x}^{(2)}, t, \lambda)}{\partial \lambda} \tag{74}$$

$$\frac{\partial g_i}{\partial \tilde{x}^{(2)}}(\tilde{x}^{(2)}, t) = h_i(\tilde{x}^{(2)}, t) \tag{75}$$

Then, $\hat{q}(x,t) = \check{q}(\tilde{x}^{(2)}(x,t), t)$, where

$$\check{q}(\tilde{x}^{(2)}, t) = -\frac{c_-^2 g_2(\tilde{x}^{(2)}, t) + (1 + \phi_-)^2 g_2^*(\tilde{x}^{(2)}, t)}{4\phi_- (\phi_- + 1)^2} \tag{76}$$

Substituting (76) into (73):

$$g_1(\tilde{x}^{(2)}, t) = \frac{c_- {c_-^*}^2 g_2^*(\tilde{x}^{(2)}, t) + c_- (1 + \phi_-)^2 g_2(\tilde{x}^{(2)}, t)}{4\phi_- (\phi_- + 1)^2} + \frac{c_-^* c_-^2 g_2(\tilde{x}^{(2)}, t) + c_-^* (1 + \phi_-)^2 g_2^*(\tilde{x}^{(2)}, t)}{4\phi_- (\phi_- + 1)^2} \tag{77}$$

$$g_1(\tilde{x}^{(2)}, t) = \frac{c_- g_2(\tilde{x}^{(2)}, t) + c_-^* g_2^*(\tilde{x}^{(2)}, t)}{2(\phi_- + 1)} + 2\phi_- \tilde{x}^{(2)} - 2\phi_-^2 x \tag{78}$$

So, from (78),

$$x\left(\tilde{x}^{(2)}, t\right) = -\frac{g_1(\tilde{x}^{(2)}, t)}{2\phi_-^2} + \frac{c_- g_2(\tilde{x}^{(2)}, t) + c_-^* g_2^*(\tilde{x}^{(2)}, t)}{4\phi_-^2 (\phi_- + 1)} + \frac{\tilde{x}^{(2)}}{\phi_-} \tag{79}$$

And from (78), (79),

$$q(x,t) = \frac{\partial \hat{q}(x,t)}{\partial x} = \frac{\partial \check{q}\left(\tilde{x}^{(2)}(x,t), t\right)}{\partial \tilde{x}^{(2)}} \frac{\partial \tilde{x}^{(2)}(x,t)}{\partial x} \tag{80}$$

Therefore, $q(x,t) = \tilde{q}\left(\tilde{x}^{(2)}(x,t), t\right)$ with

$$\tilde{q}\left(\tilde{x}^{(2)}, t\right) = -\frac{c_-^2 h_2\left(\tilde{x}^{(2)}, t\right) + (1 + \phi_-)^2 h_2^*\left(\tilde{x}^{(2)}, t\right)}{4\phi_- (\phi_- + 1)^2} \left(-\frac{h_1\left(\tilde{x}^{(2)}, t\right)}{2\phi_-^2} + \frac{c_- h_2\left(\tilde{x}^{(2)}, t\right) + c_-^* h_2^*\left(\tilde{x}^{(2)}, t\right)}{4\phi_-^2 (\phi_- + 1)} + \frac{1}{\phi_-}\right)^{-1} \tag{81}$$

7 Appendix

7.1 Compatibility condition $\theta_{xt}^{(2)} = \theta_{tx}^{(2)}$

Let us examine $\theta_{xt}^{(2)}$ and $\theta_{tx}^{(2)}$:

$$\theta_{xt}^{(2)} = i\lambda\Phi_t\sigma_3 \tag{82}$$

$$\theta_{tx}^{(2)} = \lambda\left(\frac{qq_x^* - q_xq^*}{2\Phi^2}\right)_x\sigma_3 \tag{83}$$

We will begin rewriting (82):

$$i\Phi_t = \frac{qiq_t^* + q^*iq_t}{2\Phi} = \frac{1}{2}\left(\frac{q}{\Phi}\left(\frac{q^*}{\Phi}\right)_{xx} - \frac{q^*}{\Phi}\left(\frac{q}{\Phi}\right)_{xx}\right) = \left(\frac{1}{2}\left(\frac{q}{\Phi}\left(\frac{q^*}{\Phi}\right)_x - \frac{q^*}{\Phi}\left(\frac{q}{\Phi}\right)_x\right)\right)_x =$$
$$= \left(\frac{2qq_x^*\Phi^2 - qq^*\Phi_x - 2q^*q_x\Phi^2 + q^*q\Phi_x}{4\Phi^4}\right)_x = \left(\frac{qq_x^* - q_xq^*}{2\Phi^2}\right)_x$$

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