

Chapter 2 VC-dimension

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This part corresponds to **Chapter 2-5 in UML**, and mainly answers the following questions:

- The necessary and sufficient condition of PAC learnability.
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1 The VC-dimension

1.1 Shattering

Consider the set of threshold functions over the real line $\mathcal{H} = \{h_a(x) = \mathbb{1}_{[x \leq a]}, a \in \mathbb{R}\}$. Let a^* be the threshold such that $L_{\mathcal{D}}(h^*) = 0$. Let $a_0 < a^* < a_1$ such that:

$$\mathbb{P}_{x \sim \mathcal{D}_x} [x \in (a_0, a^*)] = \mathbb{P}_{x \sim \mathcal{D}_x} [x \in (a^*, a_1)] = \epsilon$$

If $\mathcal{D}_x(-\infty, a^*) \leq \epsilon$, we set $a_0 = -\infty$, and similarly for a_1 .

Given a training set S , let $b_0 = \max\{x : (x, 1) \in S\}$ (if no example is positive then $b_0 = -\infty$), and $b_1 = \min\{x : (x, 0) \in S\}$ (if no example is negative then $b_1 = \infty$). Let b_S be the threshold of an ERM hypothesis h_S , which implies $b_S \in (b_0, b_1)$, then we have

$$\mathbb{P}_{S \sim \mathcal{D}^m} [L_{\mathcal{D}}(h_S) < \epsilon] \leq \mathbb{P}_{S \sim \mathcal{D}^m} [b_0 < a_0] + \mathbb{P}_{S \sim \mathcal{D}^m} [b_1 > a_1]$$

Each term on the right-side is bounded by $(1 - \epsilon)^m \leq e^{-\epsilon m}$. Let $m > \log(2/\delta)/\epsilon$, then the left-side is bounded by δ . As a result, the hypothesis class is PAC-learnable.

The example above shows that: **finiteness is not a necessary condition for learnability**, and hence we turn to the definition of **shattering**, which describes the ability of a hypothesis set to cover the training set.

The definition of VC-dimension is motivated from the No-Free-Lunch theorem: without restricting the hypothesis class, for any learning algorithm, an **adversary** can construct a distribution for which the learning algorithm will perform poorly, while there is another learning algorithm that will succeed on the same distribution. To make any algorithm fail, the **adversary** used the power of choosing a target function from the set of all possible labelling functions.

When considering PAC learnability of a hypothesis class \mathcal{H} , the **adversary** is restricted to constructing distributions for which some hypothesis $h \in \mathcal{H}$ achieves a zero risk. Since we are considering distributions that are concentrated on elements of C , we should study how $h \in \mathcal{H}$ behaves on C .

Definition (Restriction of \mathcal{H} to C): The restriction of \mathcal{H} to C is the set of functions from C to $\{0, 1\}$ that can be derived from \mathcal{H} . That is,

$$\mathcal{H}_C = \{(h(c_1), \dots, h(c_m)) : h \in \mathcal{H}\} \quad (1)$$

where we represent each function from C to $\{0, 1\}$ as a vector in $\{0, 1\}^{|C|}$.

Definition (Shattering): A hypothesis class \mathcal{H} shatters a finite set $C \in \mathcal{X}$ if the restriction of \mathcal{H} to C is the set of all functions from C to $\{0, 1\}$. That is, $|\mathcal{H}_C| = 2^{|C|}$.

Corollary 1 Let \mathcal{H} be a hypothesis class of functions from \mathcal{X} to $\{0, 1\}$. Let m be a training set size. Assume that there exists a set $C \subset \mathcal{X}$ of size $2m$ that is shattered by \mathcal{H} . Then, for any learning algorithm, A , there exist a distribution \mathcal{D} over $\mathcal{X} \times \{0, 1\}$ and a predictor $h \in \mathcal{H}$ such that $L_{\mathcal{D}}(h) = 0$ but with probability of at least $1/7$ over the choice of $S \sim \mathcal{D}^m$ we have that $L_{\mathcal{D}}(A(S)) \geq 1/8$.

The corollary shows that **whenever if \mathcal{H} shatters some set C of size $2m$, then we cannot learn \mathcal{H} by using m examples**. This leads us directly to the definition of the VC dimension.

1.2 The VC-dimension

Definition (VC-dimension): The VC-dimension of a hypothesis class \mathcal{H} , denoted $\text{VCdim}(\mathcal{H})$, is the maximal size of a set $C \subset \mathcal{X}$ that can be shattered by \mathcal{H} . If \mathcal{H} can shatter sets of arbitrarily large size we say that \mathcal{H} has infinite VC-dimension.

Theorem 1 If \mathcal{H} is a class of infinite VC-dimension, then \mathcal{H} is not PAC learnable.

1.2.1 Examples

To calculate the VC-dimension for a hypothesis set, we should show that:

- There **exists** a subset of size d that can be shattered;
- **Every** subset of size $d + 1$ can not be shattered.

1 Threshold functions

$$\mathcal{H} = \{\mathbb{1}_{x \leq a} : a \in \mathbb{R}\}$$

For an arbitrary set $C = \{c\}$, \mathcal{H} shatters C , therefore $\text{VCdim}(\mathcal{H}) \geq 1$; for an arbitrary set $C = \{c_1, c_2\}$, where $c_1 \leq c_2$, any threshold that assigns 0 to c_1 must assign 0 to c_2 . In other words, not all functions from C to $\{0, 1\}$ are included by \mathcal{H}_C . So, \mathcal{H} does not shatter C .

2 Intervals

$$\mathcal{H} = \{\mathbb{1}_{x \in (a,b)} : a < b, a, b \in \mathbb{R}\}$$

Denote the set $C = \{c_1, c_2\}$. If we take $a > c_2$ or $b < c_1$, then we have $h_{a,b}(c_1) = 0, h_{a,b}(c_2) = 0$; if we take $c_1 < a < c_2 < b$, then we have $h_{a,b}(c_1) = 0, h_{a,b}(c_2) = 1$; if we take $a < c_1 < b < c_2$, then we have $h_{a,b}(c_1) = 1, h_{a,b}(c_2) = 0$; if we take $a < c_1 < c_2 < b$, then we have $h_{a,b}(c_1) = 1, h_{a,b}(c_2) = 1$. Therefore, \mathcal{H}_C is the set of all functions from C to $\{0, 1\}^2$.

Take the set $C = \{c_1, c_2, c_3\}$, without loss of generalization, let the labels be $(1, 0, 1)$, therefore \mathcal{H} does not shatter C .

Hence, $\text{VCdim}(\mathcal{H}) = 2$.

3 Axis Aligned Rectangles

$$\mathcal{H} = \{\mathbb{1}_{(a_1 \leq x_1 \leq a_2, b_1 \leq x_2 \leq b_2)} : a_1 < a_2, b_1 < b_2\}$$

Any set with 4 points can be shattered. Take the set with 5 points. Suppose that there is 1 point (labelled as 0) surrounded by 4 points (labelled as 1), it cannot be shattered. Hence, $\text{VCdim}(\mathcal{H}) = 4$.

4 Finite class

Let \mathcal{H} be a finite class. Then, clearly, for any set C we have $|\mathcal{H}_C| \leq |\mathcal{H}|$ and thus it cannot be shattered if $|\mathcal{H}| < 2^{|C|}$. This implies that $\text{VCdim}(\mathcal{H}) < \log_2 |\mathcal{H}|$.

remark1: In the previous examples, the VC-dimension happened to equal the number of parameters defining. This is not always true. See exercise ? for detail.

2 Fundamental theorem of PAC learning

Theorem 2 Let \mathcal{H} be a hypothesis class of functions from a domain \mathcal{X} to $\{0, 1\}$ and let the loss function be the 0-1 loss. Then, the following are equivalent:

1. The hypothesis class has uniform convergence property.
2. Any ERM rule is a successful agnostic PAC learner for the hypothesis class.
3. The hypothesis class is agnostic PAC learnable.
4. The hypothesis class is PAC learnable.
5. Any ERM rule is a successful PAC learner for the hypothesis class.
6. The hypothesis class has a finite VC-dimension.

3 Effective size of a hypothesis class

4 Non-uniform learnability

“non-uniform learnability” allows the sample size to be non-uniform with respect to the different hypotheses with which the learner is competing.

A hypothesis is (ϵ, δ) -competitive with another if

5 Summary

6 Exercises and solutions

To be continue...

Chapter 3. Bayesian-PAC

Chapter 4. Generalization in Deep Learning