# Chapter *4* Linear Model

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This part corresponds to **Chapter 1,3,4 of PRML, Chapter of UML**, and mainly answers the following questions:

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## Contents

## 1 Linear classification

In the last chapter, we stops at the linear classification of binary classification task,

$$y = h(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + w_0 = \sum_{j=1}^{d} w_j x_j + w_0 \tag{1}$$

in which $\mathbf{w}$ is weight vector, and $w_0$ is bias. The input vector is assigned to class $C_1$ iff. $h(\mathbf{x}) \geq 0$ and to class $C_2$ otherwise.

Consider two points $\mathbf{x}_1, \mathbf{x}_2$ on the decision boundary, i.e., $\mathbf{w}^\top(\mathbf{x}_1 - \mathbf{x}_2) = 0$, hence $\mathbf{w}$ is orthogonal to the decision boundary. And the distance from the origin to the decision boundary is

$$\frac{\mathbf{w}^\top \mathbf{x}}{\|\mathbf{w}\|} = \frac{-w_0}{\|\mathbf{w}\|} \tag{2}$$

It is usually convenient to use a more compact notation in which we introduce an additional input value $x_0 = 1$ and then define $\tilde{\mathbf{w}} = (w_0, \mathbf{w})$ and $\tilde{\mathbf{x}} = (x_0, \mathbf{x})$ so that $h(\mathbf{x}) = \tilde{\mathbf{w}}^\top \tilde{\mathbf{x}}$. For simplification, we neglect the 'tilde' symbol below.

### 1.1 Extend to multiple classes

- *one-versus-the-rest* For each class $k = 1, 2, ..., K$, each classifier judge whether an example is $C_k$ or not. So there are $K$ classifiers needed.
- *one-versus-one* An alternative is to introduce $K(K-1)/2$ binary discriminant functions, one for every pair of classes (but will lead to ambiguous region).

### 1.2 Fisher's linear discriminant

One way to view a linear classification model is in terms of dimensionality reduction. By adjusting the components of the weight vector $\mathbf{w}$, we can select a projection that maximizes the class separation. To begin with, consider a two-class problem in which there are $N_1$ points of class $C_1$ and $N_2$ points of class $C_2$, so that the mean vectors of the two classes are given by

$$\mathbf{m}_1 = \frac{1}{N_1} \sum_{\mathbf{x}_n \in C_1} \mathbf{x}_n, \qquad \mathbf{m}_2 = \frac{1}{N_2} \sum_{\mathbf{x}_n \in C_2} \mathbf{x}_n \tag{3}$$

The simplest measure of the separation of the classes, when projected onto $\mathbf{w}$, is the separation of the projected class means. This suggests that we might choose $w$ so as to maximize

$$m_2 - m_1 = \mathbf{w}^\top(\mathbf{m}_2 - \mathbf{m}_1) \tag{4}$$

where $m_k = \mathbf{w}^\top \mathbf{m}_k$ is the mean of the projected data from class $C_k$.

This expression can be made arbitrarily large simply by increasing the magnitude of $\mathbf{w}$. To solve this problem, we could constrain $\mathbf{w}$ to have unit length, i.e., $\|\mathbf{w}\|_2 = 1$. Using a Lagrange multiplier, it turns to maximize $\mathbf{w}^\top(\mathbf{m}_2 - \mathbf{m}_1) + \lambda(1 - \|\mathbf{w}\|_2)$, which leads to $\mathbf{w} \propto \mathbf{m}_2 - \mathbf{m}_1$

## 2 Linear regression

In linear regression model, the model is the same except that the learning target $y$ is continuous but not discrete. And the learning goal is the sum-of-square (SSE) loss

$$\min_{\mathbf{w}} L_S(h) = \sum_{i=1}^{m} (h(\mathbf{x}_i) - y_i)^2 = \sum_{i=1}^{m} (\mathbf{w}^\top \mathbf{x}_i - y_i)^2 \tag{5}$$

Suppose the fitting error $\epsilon_i = y_i - \mathbf{w}\mathbf{x}_i$ is Gaussian noise, i.e., $\epsilon_i \sim \mathcal{N}(0, \beta)$. Then the log likelihood function of the training sequence is

$$\log \mathcal{L} = -\frac{m}{2} \log 2\pi\beta - \sum_{i=1}^{m} \frac{(y_i - \mathbf{w}^\top \mathbf{x}_i)^2}{2\beta} \tag{6}$$

Obviously, MLE is equivalent to linear regression.

remark1: *Since linear regression is not a binary prediction task, we cannot analyse its sample complexity using the VC-dimension. One possible analysis of the sample complexity of linear regression is by relying on the "discretization trick". However, to apply the sample complexity bounds from Chapter 2 we also need that the loss function will be bounded.*

## 2.1 Generalized linear regression

The model is just a linear function of the input variables, and this imposes significant limitations on it. Therefore, extended model considers **linear** combination of fixed **non-linear** functions of the input variables, of the form

$$h(\mathbf{x}) = w_0 + \sum_{j=1}^{d} w_j \phi_j(x) \tag{7}$$

where $\phi_j(x)$ are known as *basis functions*. Again, denote $\phi_0(\mathbf{x}) = 1$ so that $h(\mathbf{x}) = \tilde{\mathbf{w}}^\top \phi(\mathbf{x})$. For Simplification, we also neglect the 'tilde' symbol from now on.

Now, consider the closed-form solution for The gradient of the log likelihood. The gradient of the SSE loss takes the form

$$\nabla L_S(h) = \sum_{i=1}^{m} \left\{ y_i - \mathbf{w}^\top \phi(\mathbf{x}_i) \right\} \phi^\top(\mathbf{x}_i)$$

Setting it to zero gives

$$\mathbf{w} = \mathbf{\Phi}^\dagger \mathbf{y} = (\mathbf{\Phi}^\top \mathbf{\Phi})^{-1} \mathbf{\Phi}^\top \mathbf{y} \tag{8}$$

Here $\mathbf{\Phi}$ is an *nxd* matrix, whose elements are given by $\mathbf{\Phi}_{nj} = \Phi_j(\mathbf{x}_n)$. And $\mathbf{\Phi}^\dagger$ is *Moore-Penrose pseudo-inverse.*

remark2, multiple-outputs: *A more general case is multiple outputs, i.e.,* $\mathbf{y}_i \in \mathcal{R}^k, k > 1$. *However, the solution to multiple-outputs regression problem decouples between the different target variables so we do not discuss it here.*

remark3, on-line learning: Batch techniques involve processing the entire training set in one go, can be computationally costly for large data sets. For linear regression, stochastic gradient descent algorithm updates parameter using

$$\mathbf{w}^{t+1} = \mathbf{w}^t - lr * \nabla L_{(\mathbf{x}_t, y_t)}(h) = \mathbf{w}^t + lr * (y_i - (\mathbf{w}^t)^\top \phi(\mathbf{x}_t)) \phi(\mathbf{x}_t)$$

in which $lr$ is the learning rate.

remark3, basis: *There are three popular basis function: 1.* ***Polynomial basis****, 2.* ***Radical basis****, 3.* ***Fourier basis****.*

## 2.2 Regularization *a.k.a* Bayesian linear regression

Recall the closed-form solution (Eq. 8) for linear regression problem, over-fitting

### 2.2.1 Ridge regression

Ridge regression addresses on over-fitting by penalizing the $l_2$-norm of weight vector $\mathbf{w}$,

$$\min_{\mathbf{w}} \sum_{i=1}^{m} (\mathbf{w}\phi_{\mathbf{i}}(\mathbf{x}) - y_i)^2 + \lambda \|\mathbf{w}\|_2^2$$

If we assume a Gaussian prior for the weight vector, $\mathbf{w} \sim \mathcal{N}(0, \alpha^{-1}\mathbf{I})$, then the posterior of the training sequence is:

$$p(\mathbf{w}|S) \propto p(\mathbf{w})p(S|\mathbf{w}) \propto \exp\left(-\frac{\alpha}{2}\mathbf{w}^\top \mathbf{w}\right) \cdot \prod_{i=1}^{N} \exp\left(-\frac{(y_i - \mathbf{w}\mathbf{x}_i)^2}{2\beta}\right) \tag{9}$$

Maximizing the log posterior function is equivalent to the ridge regression.

### 2.2.2 Lasso

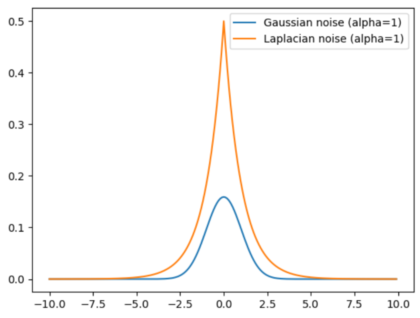
Lasso addresses on over-fitting by penalizing the $l_1$-norm of weight vector $\mathbf{w}$,

$$\min_{\mathbf{w}} \sum_{i=1}^{m} (\mathbf{w}\phi_{\mathbf{i}}(\mathbf{x}) - y_i)^2 + \lambda \|\mathbf{w}\|_1$$

If we assume a Laplace prior for the weight vector, $p(\mathbf{w}) = \frac{1}{2\alpha} \exp\left(-\frac{\|\mathbf{w}\|_1}{\alpha}\right)$, then the posterior of the training sequence is:

$$p(\mathbf{w}|S) \propto p(\mathbf{w})p(S|\mathbf{w}) \propto \exp\left(-\frac{\|\mathbf{w}\|_1}{\alpha}\right) \cdot \prod_{i=1}^{N} \exp\left(-\frac{(y_i - \mathbf{w}\mathbf{x}_i)^2}{2\beta}\right) \tag{10}$$

Maximizing the log posterior function is equivalent to the Lasso model.
*<u>remark2</u>:*


Gaussian noise (alpha=1)
Laplacian noise (alpha=1)