Chapter 4 Linear Model

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This part corresponds to **Chapter 1,3,4 of PRML, Chapter of UML**, and mainly answers the following questions:

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Contents

1 Linear classification

In the last chapter, we stops at the linear classification of binary classification task,

$$y = h(\mathbf{x}) = \mathbf{w}^\top \mathbf{x} + w_0 \tag{1}$$

in which $\mathbf{w}$ is weight vector, and w_0 is bias. The input vector is assigned to class C_1 iff. $h(\mathbf{x}) \geq 0$ and to class C_2 otherwise.

Consider two points $\mathbf{x}_1, \mathbf{x}_2$ on the decision boundary, i.e., $\mathbf{w}^\top(\mathbf{x}_1 - \mathbf{x}_2) = 0$, hence $\mathbf{w}$ is orthogonal to the decision boundary. And the distance from the origin to the decision boundary is

$$\frac{\mathbf{w}^\top \mathbf{x}}{\|\mathbf{w}\|} = \frac{-w_0}{\|\mathbf{w}\|} \tag{2}$$

It is usually convenient to use a more compact notation in which we introduce an additional input value $x_0 = 1$ and then define $\tilde{\mathbf{w}} = (w_0, \mathbf{w})$ and $\tilde{\mathbf{x}} = (x_0, \mathbf{x})$ so that $y = f(\mathbf{x}) = \tilde{\mathbf{w}}^\top \tilde{\mathbf{x}}$.

1.1 Extend to multiple classes

- *one-versus-the-rest* For each class $k = 1, 2, ..., K$, each classifier judge whether an example is C_k or not. So there are K classifiers needed.
- *one-versus-one* An alternative is to introduce $K(K-1)/2$ binary discriminant functions, one for every pair of classes (but will lead to ambiguous region).

1.2 Fisher's linear discriminant

One way to view a linear classification model is in terms of dimensionality reduction. By adjusting the components of the weight vector $\mathbf{w}$, we can select a projection that maximizes the class separation. To begin with, consider a two-class problem in which there are N_1 points of class C_1 and N_2 points of class C_2, so that the mean vectors of the two classes are given by

$$\mathbf{m}_1 = \frac{1}{N_1} \sum_{\mathbf{x}_n \in C_1} \mathbf{x}_n, \qquad \mathbf{m}_2 = \frac{1}{N_2} \sum_{\mathbf{x}_n \in C_2} \mathbf{x}_n \tag{3}$$

The simplest measure of the separation of the classes, when projected onto $\mathbf{w}$, is the separation of the projected class means. This suggests that we might choose w so as to maximize

$$m_2 - m_1 = \mathbf{w}^\top(\mathbf{m}_2 - \mathbf{m}_1) \tag{4}$$

where $m_k = \mathbf{w}^\top \mathbf{m}_k$ is the mean of the projected data from class C_k.

This expression can be made arbitrarily large simply by increasing the magnitude of $\mathbf{w}$. To solve this problem, we could constrain $\mathbf{w}$ to have unit length, i.e., $\|\mathbf{w}\|_2 = 1$. Using a Lagrange multiplier, it turns to maximize $\mathbf{w}^\top(\mathbf{m}_2 - \mathbf{m}_1) + \lambda(1 - \|\mathbf{w}\|_2)$, which leads to $\mathbf{w} \propto \mathbf{m}_2 - \mathbf{m}_1$

2 Linear regression

$$\min_{\mathbf{w}} L_S(h) = \sum_{i=1}^{m} (h(\mathbf{x}_i) - y_i)^2 = \sum_{i=1}^{m} (\mathbf{w}\mathbf{x}_i - y_i)^2 \tag{5}$$

Suppose the fitting error $\epsilon_i = y_i - \mathbf{w}\mathbf{x}_i$ is Gaussian noise, i.e., $\epsilon_i \sim \mathcal{N}(0, \beta)$. Then the log likelihood function of the training sequence is:

$$\log \mathcal{L} = -\frac{m}{2} \log 2\pi\beta - \sum_{i=1}^{m} \frac{(y_i - \mathbf{w}\mathbf{x}_i)^2}{2\beta} \tag{6}$$

Obviously, MLE is equivalent to linear regression.

remark1: Since linear regression is not a binary prediction task, we cannot analyse its sample complexity using the VC-dimension. One possible analysis of the sample complexity of linear regression is by relying on the "discretization trick". However, to apply the sample complexity bounds from Chapter 2 we also need that the loss function will be bounded.

over-fitting

2.1 Ridge regression

Ridge regression addresses on over-fitting by penalizing the l_2-norm of weight vector $\mathbf{w}$,

$$\min_{\mathbf{w}} \sum_{i=1}^{N} (\mathbf{w}\mathbf{x}_i - y_i)^2 + \lambda \|\mathbf{w}\|_2^2$$

If we assume a Gaussian prior for the weight vector, $\mathbf{w} \sim \mathcal{N}(0, \alpha^{-1}\mathbf{I})$, then the posterior of the training sequence is:

$$p(\mathbf{w}|S) \propto p(\mathbf{w})p(S|\mathbf{w}) \propto \exp\left(-\frac{\alpha}{2}\mathbf{w}^\top \mathbf{w}\right) \cdot \prod_{i=1}^{N} \exp\left(-\frac{(y_i - \mathbf{w}\mathbf{x}_i)^2}{2\beta}\right) \tag{7}$$

Maximizing the log posterior function is equivalent to the ridge regression.

2.2 Lasso

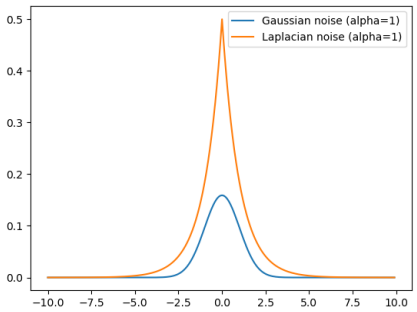
Lasso addresses on over-fitting by penalizing the l_1-norm of weight vector $\mathbf{w}$,

$$\min_{\mathbf{w}} \sum_{i=1}^{N} (\mathbf{w}\mathbf{x}_i - y_i)^2 + \lambda \|\mathbf{w}\|_1$$

If we assume a Laplace prior for the weight vector, $p(\mathbf{w}) = \frac{1}{2\alpha} \exp\left(-\frac{\|\mathbf{w}\|_1}{\alpha}\right)$, then the posterior of the training sequence is:

$$p(\mathbf{w}|S) \propto p(\mathbf{w})p(S|\mathbf{w}) \propto \exp\left(-\frac{\|\mathbf{w}\|_1}{\alpha}\right) \cdot \prod_{i=1}^{N} \exp\left(-\frac{(y_i - \mathbf{w}\mathbf{x}_i)^2}{2\beta}\right) \tag{8}$$

Maximizing the log posterior function is equivalent to the Lasso model.
remark2:



3 Generalized linear model