ADDIS ABABA UNIVERSITY

COLLEGE OF NATURAL AND COMPUTATIONAL SCIENCES

DEPARTMENT OF MATHEMATICS

Applied Mathematics IB (Math 1041)

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PART I: INTRODUCTION TO LINEAR ALGEBRA

Chapter 1: Vectors and Vector Spaces

Introduction

Some of the things we measure are determined by their magnitudes. To record mass, length or time for example , We need only to write down a number and name an appropriate unit of measure.But we need more information to describribe force , displacement or velocity. To describe force we need to record the direction in which it acts as well as how large it is . To describe a body's disp[alcement , we have to say in what direction it moves as well as how far. To describe a body's velocity, we have to know where the body is headed as well as how fast it is going.

Quantities that have direction as well as magnitude are usually represented by arrows that point in the direction of the action and whose lengths give the magnitude of the action in terms of a suitably chosen unit.. When we discuss these arrows abstractly , we think of them as directed line segments and we call them vectors and their study comprises this chapter. Here we focus on the in **n**-vectors *i.e* **n**-dimensional vectors $\Re^n$.

The concept of a vector is basic for the whole module. It provides geometric motivation for everything that follows. Hence the properties of vectors, both algebraic and geometric, will be discussed in full. That is points in $\Re^n$. Scalar product and vector product are also studied here. So points in $\Re^2$ and $\Re^3$ can be treated as a special case for $\Re^n$. Geometrical interpretations are also presented whenever necessary.

The cross (vector) product is included for the sake of completeness. It is the only aspect of the theory of vectors which is valid only in 3-dimensional space (not 2, nor 4, nor *n*-dimensional space)

In the final section of this unit, you will study how to write equation of a line and equation of a plane in 2 and 3- dimensional vector space.

At the end of this unit , you will be able to

- Plot a point in space
- Interpret vectors geometrically
- Determine whether two or more vectors are parallel or not
- Find the norm of a vector

- Find the angle between two vectors
- Find the projection of one vector onto the other
- Describe vectors and their properties
- Calculate the scalar product and vector (cross) product of vectors
- Calculate norm of a vector, angle between vectors and projection of vectors
- Write equation of a line and equation of a plane
- Calculate area of a parallelogram and volume of parallelepiped using dot product and vector product

1.1 Definition of points in n-space

A natural way in which vectors arise is in simultaneous study of several characteristics of an individual. Suppose we want to study a_1 = height , a_2 = weight ,a_3 = age and a_4= blood pressure of an individual. The values of all these can together be represented by an ordered 4-tuple A= (a_1,a_2,a_3,a_4) which can be viewed as an element of $\Re^4$.

Definition 1.1.1:Suppose $\mathbf{A} = (a_1 , a_2 , \ldots,a_n)$ and $\mathbf{B} = (b_1 , b_2 , \ldots,b_n)$ be two points in n-space and $c\in\Re$.

a) $\mathbf{A} + \mathbf{B} = (a_1+ b_1, a_2 + b_2 , \ldots, a_n + b_n)$
b) $c\mathbf{A} = (ca_1 , ca_2 , \ldots,ca_n)$

Example 1.1.1 : Let $\mathbf{A} = (-1,3,6,2)$, $\mathbf{B} = (0,-5,-1,4)$, $\mathbf{C} = (2,-1,3)$ and $\mathbf{D} = (0,0,0,0)$

Find **a) A** + **B**
b) A + **C**
c) -2**A**
d) B + **D**

Solution

a) $\mathbf{A} + \mathbf{B} = (-1+0,3+-5,6+-1,2+4) = (-1,-2,5,6)$
b) **A** + **C** is not defined because **A** is a 4 dimensional space while **C** is a 3 dimensional space.
c) $-2\mathbf{A} = -2(-1,3,6,2) = ((-2)(-1) , (-2)(3) , (-2)(6) , (-2)(2)) = (2,-6,-12,-4)$
d) $\mathbf{B}+ \mathbf{D} = (0,-5,-1,4) + (0,0,0,0) = (0 + 0 , -5 + 0 , -1 + 0 , 4 + 0) = (0,-5,-1,4)$

Definition 1.1.2: Suppose **A** and **B** be two points in n-space , we define $\mathbf{A} - \mathbf{B} = \mathbf{A} +(-\mathbf{B})$

Example 1.1.2: Let $\mathbf{A} = (4,1,0,2,3)$, $\mathbf{B} = (0,1,-4,8,-3)$. Find **A** – **B**
Solution

By definition ,

A - **B** = **A** +(- **B**)

-**B** =(-1)**B**= (0,-1,4,-8,3)

A – **B** = **A** + (-**B**)

= (4,1,0,2,3) + (0,-1,4,-8,3)

= (4,0,4,-6,6)

Activity 1.1.1

a.) Graph the **XY-** plane and draw the points on a square sheet of paper.

i) (-1,3) **ii)** (0,4) **iii)** (1,5)

b.)Define what do we mean by there is a one to one correspondence between a point in a plane and ordered pair of real numbers .

c) Graph a three dimensional space in your square sheet of paper and draw the points

i) (1,0,0) **ii)** (0,3,0) **iii)** (-1,1,0) **iv)** (-1,1,1)

Theorem 1.1.1 Properties

Suppose **A** , **B and C** are points represented by n tuple and $\alpha ,\beta \in \Re$

1) A+**B**=**B**+**A**

2) A +(**B**+**C**)=(**A** +**B**)+**C**

3) α(**A** +**B**)= α **A** +α **B** =(**A** + **B**) α

4) $(\alpha+\beta)$**A** =α **A** +β **A**

5) $\alpha(\beta$ **A**)= $(\alpha\beta)$**A**

6) A +**0** =**0**+ **A** = **A**

7)1. **A** = **A** and **-1**. **A** =- **A**

8) A +(- **A**)=**0**

Proof :

Let **A** = $(a_1 , a_2 , \ldots,a_n)$ and **B** = $(b_1 , b_2 , \ldots,b_n)$ and C = $(c_1 , c_2 , \ldots,c_n)$ be three points in n-space and $\alpha ,\beta \in \Re$.

1. **A**+**B =** $(a_1+ b_1 , a_2 + b_2 , \ldots, a_n + b_n)$ by definition

   =$(b_1+ a_1 , b_2 +a_2 , \ldots, b_n + a_n)$ Since addition is commutative in the set of

   real numbers

   = **B** + **A** by definition

2. **A** +(**B**+**C**) = $(a_1 , a_2 , ..,a_n) +(b_1+ c_1 , b_2 +c_2 , \ldots, b_n + c_n)$

   = $(a_1 +[b_1+ c_1] , a_2 + [b_2 +c_2], .., a_n+[b_n + c_n])$

   = $([a_1 +b_1] +c_1 , [a_2 +b_2]+c_2 , .., [a_n+b_n]+ c_n)$

   = $(a_1 +b_1 , a_2 +b_2 , .., ,a_n+b_n) + (c_1 , c_2 , ..,, c_n)$

   = (A + B) + C

3. α(**A** +**B**) = $\alpha(a_1+ b_1 , a_2 + b_2 , \ldots, a_n + b_n)$ by definition

$= (\alpha[a_1+ b_1], \alpha[a_2 + b_2], \ldots, \alpha[a_n+b_n])$
$= (\alpha a_1+\alpha b_1, \alpha a_2 + \alpha b_2, \ldots, \alpha a_n+\alpha b_n)$
$= (\alpha a_1, \alpha a_2 ,.., \alpha a_n) + (\alpha b_1, \alpha b_2, \ldots,\alpha b_n)$
$= \alpha(a_1, a_2 ,.., a_n) + \alpha (b_1, b_2, \ldots,b_n)$
$= \alpha A + \alpha B$

4. $(\alpha+\beta)\mathbf{A} = (\alpha +\beta)(a_1 , a_2 , ..,a_n)$
$= ([\alpha +\beta]a_1 , ([\alpha +\beta] a_2 , ..,[\alpha +\beta]a_n)$
$= (\alpha a_1 +\beta a_1 , \alpha a_2 +\beta a_2, ..,\alpha a_n +\beta a_n)$
$= (\alpha a_1 , \alpha a_2 , ..,\alpha a_n) + (\beta a_1 ,\beta a_2, ..,\beta a_n)$
$= \alpha(a_1 , a_2 , .., a_n) +\beta(a_1 , a_2, .., a_n)$
$= \alpha A + \beta A$

5. $\alpha(\beta \mathbf{A})= \alpha(\beta a_1 ,\beta a_2, ..,\beta a_n)$
$= (\alpha[\beta a_1] , \alpha[\beta a_2], .., \alpha[\beta a_n])$
$= ([\alpha\beta]a_1 , [\alpha\beta]a_2 , \ldots, [\alpha\beta]a_n)$
$= \alpha\beta(a_1 , a_2 , .., a_n)$
$= (\alpha\beta)A$

6 , **7** and **8** are left as an exercise for the student.

<u>1.2 Vectors in-space; Geometric interpretation in 2-and 3-spaces</u>.

<u>Activity 1.2.1</u>

Draw a vector whose initial point is **A** and terminal point **B** .

a) **A** = (1,2) and **B** = (-2,0)
b) **A** = (0,-2) and **B** = (1, 1)

Let us consider a vector in a plane. Let **A** = (a_1 ,a_2) and **B** = (b_1 ,b_2) where $a_1 ,a_2 , b_1 ,b_2 \in \Re$

$b_1 = a_1 + (b_1 - a_1)$
$b_2 = a_2 + (b_2 - a_2)$

B $= (b_1 ,b_2) = (a_1 + (b_1 - a_1) , a_2 + (b_2 - a_2))$
$= (a_1 ,a_2) + (b_1 - a_1 , b_2 - a_2)$
$= (a_1 ,a_2) + [(b_1 , b_2) + -1 (a_1 , a_2)$
$= \mathbf{A} + (\mathbf{B} - \mathbf{A})$

Therefore **B** = **A** + (**B** – **A**)

<u>Activity 1.2.2 :</u> Let **A** = (1,4) , **B** = (-1,5) , **C** = (2,1) and **D**=(0,2)
Find
a) B – A **b) D - C**
Draw the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ on the coordinate plane.

Definition 1.2.1 : A vector $\overrightarrow{AB}$ is determined by the points **A** and **B** where **A** is the intial point and **B** is a terminal point

Definition 1.2.2: Suppose $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be two vectors.We say $\overrightarrow{AB}$ is **equivalent** to $\overrightarrow{CD}$ written $\overrightarrow{AB} \cong \overrightarrow{CD}$ iff **B** – **A** = **D** – **C**

Example 1.2.1: Let **A** = (4,3) **B** = (2,-1) **C** = (1,2) and **D** = (-1,-2)

Find **a) B- A**

b) D – **C**

c) Determine whether or not $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$.

d) Find a point P such that $\overrightarrow{OP} \cong \overrightarrow{AB}$

e) Can we always reduce any vector to equivalent vector whose initial point is the origin?

Solution:

a) B – **A** = (2,-1) + -1(4,3) = (2 - 4, -1-3) = (-2,-4)

b) D – **C** = (-1,-2)+-1(1,2) = (-1-1,-2 -2) = (-2,-4)

c) Since **B** – **A** = **D** – **C** , $\overrightarrow{AB} \cong \overrightarrow{CD}$

d) Let P = (x,y). $\overrightarrow{OP} \cong \overrightarrow{AB}$ if and only if **B- A** = **P - O**

B- A = (-2,-4) while **P - O** = (x,y)

B – **A** = **P - O** if and only if (-2,-4) = (x ,y)

That means **P** = (-2,-4)

e) yes

Remark : Every vector is equivalent to a vector whose initial point is the origin.

Example 1.2.2: Show that $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$ where **A** = (-1,0) , **B** = (0,1), **C** =(0,0) and **D** = (1,1)

Solution: **B** – **A** = (0,1) – (-1,0) = (1,1)

D – **C** = (1,1) – (0,0) = (1,1)

Since **B** – **A** = **D** – **C** , $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$

Definition 1.2.3: Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be **parallel** if there exists $\alpha \in \Re$

such that **B** – **A** = α(**D** – **C**)

Example 1.2.3 : Show that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ where **A** = (-1,0) , **B** = (0,1), **C** = (0,0) and **D** = (1,1)

Solution: **B** – **A** = (0,1) – (-1,0) = (1,1)

$\mathbf{D} - \mathbf{C} = (1,1) - (0,0) = (1,1)$

Since $\mathbf{B} - \mathbf{A} = 1(\mathbf{D} - \mathbf{C})$, $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

Remark : Equivalent vectors are parallel.where $\alpha=1$

Example 1.2.4 : Let **A** = (1,6,4) , **B** = (3,4,-2) , **C** = (4,8,-1) and **D** = (1,11,8).
Determine whether or not $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$

Solution

$\mathbf{B} - \mathbf{A} = (3,4,-2) - (1,6,4) = (3-1, 4-6, -2-4) = (2,-2,-6)$

$\mathbf{D} - \mathbf{C} = (1,11,8) - (4,8,-1) = (1-4, 11-8, 8-(-1)) = (-3,3,9)$

$\mathbf{B} - \mathbf{A} = -2/3(\mathbf{D} - \mathbf{C}) \quad \alpha = -2/3$

$\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$

Activity 1.2.3: Let **A** = (-1,4,8) , **B** = (3,-4, 0) , **C** = (-5,3,1) and **D** = (-4, 1,-1).
Show that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$

Remark :

i) if $\alpha > 0$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have the same direction.

ii)If $\alpha < 0$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have opposite direction.

1.3 The Scalar product

Definition 1.3.1 Let $\mathbf{A} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{B} = (b_1, b_2, \ldots, b_n)$ be two vectors define the scalar product or dot product A. **B** as

$\mathbf{A.B} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$

$c\mathbf{A} = (ca_1, ca_2, \ldots, ca_n)$

Example 1.3.1 : Let **A** = (2,0,-1,2) , **B** = (1,-1,3,5) and **C** = (1,-3 , 1)

Find **a) A. B**
b) B.A
c) A.C
d) (**A.B**)**C**
e) (**A** – **B**).**A**

Solution :

a) A.**B** = 2.1 + 0.(-1) + (-1)3 + 2.5 = 2 + 0 – 3 + 10 = 9
b) **B**.A = 1.2 + (-1)0 + 3.(-1) + 5.2 = 2 + 0 – 3 + 10 = 9
c) **A.C** is not defined
d) (**A**. **B**) **C** = 9(1,-3,1) = (9,-27,9)
e) (**A**- **B**)**A**
First let us find **A** – **B.**

$\mathbf{A} - \mathbf{B} = (2,0,-1,2) - (1,-1,3,5) = (1,1,-4,-3)$

$(\mathbf{A} - \mathbf{B}).\mathbf{A} = (1,1,-4,3).(2,0,-1,2)$

$= (1.2,\ 1.0, (-4)(-2),\ 3.2)$

$= (2,0,8,6)$

<u>**Theorem1.3.1:**</u> Let **A**, **B** and **C** be vectors in n dimensional space and $\alpha \in \Re$ then

a) **A**. **B** = **B** .**A**

b) $\mathbf{A}.(\mathbf{B} + \mathbf{C}) = \mathbf{A.B} + \mathbf{A.C} = (\mathbf{B} + \mathbf{C}).\mathbf{A} = \mathbf{B.A} + \mathbf{C.A}$

c) $(\alpha\mathbf{A}).\mathbf{B} = \alpha(\mathbf{A}.\mathbf{B}) = \mathbf{A}(\alpha\mathbf{B}) = \alpha(\mathbf{A.B})$

d) $\mathbf{A.A} \geq 0$ and $\mathbf{A.A} = 0$ if and only if $\mathbf{A} = \mathbf{0}$.

<u>**Proof:**</u>

Let $\mathbf{A} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{B} = (b_1, b_2, \ldots, b_n)$ and $\mathbf{C} = (c_1, c_2, \ldots, c_n)$ be three vectors in n-dimensional space and $\alpha \in \Re$.

a) $\mathbf{A.B} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$ by definition

$= b_1 a_1 + b_2 a_2 + \ldots + b_n a_n$ multiplication is commutative

$=$ B.A by definition

b) $\mathbf{A}.(\mathbf{B} + \mathbf{C}) = a_1 (b_1 + c_1) + a_2 (b_2 + c_2) \ldots + a_n (b_n + c_n)$

$= a_1b_1 + a_1 c_1 + a_2 b_2 + a_2c_2 + \ldots + a_nb_n + a_nc_n$

$= (a_1b_1 + a_2 b_2 + .+ a_nb_n) + (a_1 c_1 + a_2c_2 + \ldots + a_nc_n)$

$= A.B + A.C$

c) $(\alpha\mathbf{A}).\mathbf{B} = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n).b_1, b_2, \ldots, b_n)$

$= (\alpha a_1)b_1 + (\alpha a_2)b_2 . + \ldots + (\alpha a_n)b_n$

$. = a_1 (\alpha b_1) + a_2 (\alpha b_2) . + \ldots + a_n (\alpha b_n)$

. $= A.(\alpha B)$

d) $A.A = a_1 a_1 + a_2 a_2 + \ldots + a_n a_n$ by definition

$= a_1^2 + a_2^2 + \ldots + a_n^2 \geq 0$ since it is a sum of non negative numbers.

$A.A = 0$ if and only if each of the a_i 's are zero . that is $a_i = 0$ for $i = 1,2,\ldots n$. hence $A = 0$.

<u>**Activity 1.3.1:**</u> Let **A** = (0,4) and **B** = (1,0).Draw the vectors whose initial point is **O** and terminal point is at **A** and **B** respectively

Find **a) A. B**

b) The angle between **A** and **B**

c) Find two non zero vectors **A** and **B** such that **A**. **B**

Definition1.3.2: Two vectors **A** and **B** are said to be **perpendicular** (orthogonal) Iff **A**.**B**= 0.

Example 1.3.1 : Show that the vectors **A** = (1,3,2) and **B** = (-4,2,-1) are perpendicular.

Solution : **A**. **B** = 1.(-4) + 3.2 + 2.(-1) = 0
Hence **A** is perpendicular to **B**

Activity 1.3.2: Find the value of x such that the vectors **A**= (-1,4,5,2) and **B** =(3,x,1,0) are perpendicular.

Activity 1.3.3: Which of the following pairs of vectors are perpendicular?
a) (-5,2,7) and (3,1,-2) **b)** (-1,1,1) and (3,2,1)

The norm of a vector

Definition1.3.3: Let **A** = (a_1 , a_2 , …..,a_n).The **norm** of the vector **A** denoted by ||**A**|| is defined as ||**A**|| = $\sqrt{A.A}$ = $\sqrt{a_1{}^2 + a_2{}^2 + ... + a_n{}^2}$

Example 1.3.2: Let **A** = (-1,3,0,5) .Find the norm of **A**

Solution : Let **A** = (a_1 , a_2 ,a_3 , a_4) = (-1,3,0,5)
$\Rightarrow$ a_1 = -1, a_2 = 3 , a_3 = 0 and a_4 = 5
||**A**|| = $\sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2 + a_4{}^2}$ = $\sqrt{(-1)^2 + 3^2 + 0^2 + 5^2}$ = $\sqrt{1+9+0+25}$ = $\sqrt{35}$

Example 1.3.3: Let **A** = (x,y,z)
||**A**|| = $\sqrt{x^2 + y^2 + z^2}$
Let w = $\sqrt{x^2 + y^2}$ ||**A**|| = $\sqrt{w^2 + z^2}$ = $\sqrt{x^2 + y^2 + z^2}$

Theorem1.3.2 : Prove that
a) ||**A**|| $\neq$ 0 if **A** $\neq$ 0
b) ||**A**|| = ||-**A** ||
c) ||α**A**|| = |α|||**A**|| where $\alpha \in \Re$

Proof :
Let A = (a_1 , a_2 , …..,a_n). and $\alpha \in \Re$

a) ||**A**|| = $\sqrt{A.A}$ = $\sqrt{a_1{}^2 + a_2{}^2 + ... + a_n{}^2}$
Since $a_1^2 + a_2^2 + \ldots + a_n^2$ is a non negative number ||**A**||=0 if and only if a_i,=0 for i =

1,2,…n. Hence **A** =0

b) ||-A|| = $\sqrt{(-a_1)^2+(-a_2)^2+....+(-a_n)^2} = \sqrt{a_1^{\ 2}+a_2^{\ 2}+....+a_n^{\ 2}}$ =||**A**||

c) α**A =** α(a_1 , a_2 , …..,a_n) =(αa_1, αa_2 ….. αa_n**)**

||α**A**|| = $\sqrt{(\alpha a_1)^2+(\alpha a_2)^2+....+(\alpha a_n)^2}$

$= \sqrt{\alpha^2 a_1^{\ 2}+\alpha^2 a_2^{\ 2}+....+\alpha^2 a_n^{\ 2}}$

$= \sqrt{\alpha^2(a_1^{\ 2}+a_2^{\ 2}+....+a_n^{\ 2})}$

=|α| $\sqrt{a_1^{\ 2}+a_2^{\ 2}+....+a_n^{\ 2}}$

=|α|||**A**||

<u>Definition 1.3.4:</u> A **unit vector** is a vector whose norm is 1 unit.
That is **A** is a unit vector iff || **A**|| = 1

<u>Example 1.3.4:</u> Let **A** = (√2/2 , 0 , -√2/2)

||**A**|| = $\sqrt{(\sqrt{2}/2)^2 + (-\sqrt{2}/2)^2} = \sqrt{1/2 + 1/2}$ = 1

<u>Example1.3.5 :</u> Let **A**≠ 0 . || **A**|| ≠ 0 . Let a = ||**A**||

Consider the vector A/a . || A/a || = $1/a$|||**A**|| = $1/a$.a = 1

So $1/a$**A** = $\frac{A}{a}$ is a unit vector in the direction of **A** because $1/a$ >0

And - $1/a$**A** is a unit vector in the opposite direction of **A**.

<u>Example 1.3.6 :</u> Find two unit vectors which are parallel to **A** where

a) **A** = (-3,0,2) **b)** **A**= (1,-4,2,1)

<u>Solution</u>

a) ||**A**|| = $\sqrt{(-3)^2+0^2+2^2} = \sqrt{13}$

$A/\|A\|$ = $1/\sqrt{13}$ (-3,0,2)

and $-A/\|A\|$ = $-1/\sqrt{13}$ (-3,0,2) are unit vectors which are parallel to **A** = (-3,0,2)

b) ||**A**|| = $\sqrt{1^2+(-4)^2+2^2+1^2} = \sqrt{22}$

$A/\|A\|$ = $1/\sqrt{22}$ (1,4,2,1) and

$-A/\|A\|$ = $-1/\sqrt{22}$ (1,4,2,1) are unit vectors which are parallel to **A**= (1,-4,2,1)

Definition 1.3.5 : The **distance** between two points **A** and **B** in n space is defined as $\|\mathbf{A}-\mathbf{B}\| = \sqrt{(A-B)(A-B)}$

Example 1.3.7: Let **A** = (-1,2,-2) and **B** =(2,3,-1)

Solution: d = $\|\mathbf{A}-\mathbf{B}\| = \sqrt{(A-B)(A-B)}$

$\mathbf{A}-\mathbf{B} = (-1,2,-2)-(2,3,-1) = (-1-2,\ 2-3,\ -2-(-1)) = (-3,-1,-1)$

$d = \|\mathbf{A}-\mathbf{B}\| = \sqrt{(A-B)(A-B)} = \sqrt{(-3,-1,-1)(-3,-1,-1)} = \sqrt{9+1+1} = \sqrt{11}$

Example 1.3.8 : Let **A** = (1,2) and **B** =(3,-1).Find the distance between **B** and **A**.
Solution: d = $\|\mathbf{B}-\mathbf{A}\|$

$\mathbf{B}-\mathbf{A} = (3,-1)-(1,2) = (2,-3)$

$d = \|\mathbf{B}-\mathbf{A}\| = \sqrt{2^2+(-3)^2} = \sqrt{4+9} = \sqrt{13}$

Theorem 1.3.3: Given **A** and **B** points in n- space $\|\mathbf{A}+\mathbf{B}\| = \|\mathbf{A}-\mathbf{B}\|$ iff $\mathbf{A}\perp\mathbf{B}$
Proof :

($\Rightarrow$) Suppose $\|\mathbf{A}+\mathbf{B}\| = \|\mathbf{A}-\mathbf{B}\|$

$$\Rightarrow \sqrt{(A+B)(A+B)} = \sqrt{(A-B)(A-B)}$$

Squaring both sides

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) = (\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})$$
$$\Rightarrow \mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2 = \mathbf{A}^2-2\mathbf{AB}+\mathbf{B}^2$$
$$\Rightarrow 4\mathbf{A}.\mathbf{B} = 0$$
$$\Rightarrow \mathbf{A}.\mathbf{B} = 0$$
$$\Rightarrow \mathbf{A}\perp\mathbf{B}$$

($\Leftarrow$) Suppose $\mathbf{A}\perp\mathbf{B}$. $\mathbf{A}.\mathbf{B} = 0$

$$\Rightarrow 2\mathbf{AB} = -2\mathbf{AB}$$
$$\Rightarrow \mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2 = \mathbf{A}^2-2\mathbf{AB}+\mathbf{B}^2$$
$$\Rightarrow (\mathbf{A}+\mathbf{B})^2 = (\mathbf{A}-\mathbf{B})^2$$
$$\Rightarrow \sqrt{(A+B)(A+B)} = \sqrt{(A-B)(A-B)}$$
$$\Rightarrow \|\mathbf{A}+\mathbf{B}\| = \|\mathbf{A}-\mathbf{B}\|$$

Theorem 1.3.4 : **a)** If $\mathbf{A}\perp\mathbf{B}$, then $\|\mathbf{A}+\mathbf{B}\|^2 = \|\mathbf{A}\|^2+\|\mathbf{B}\|^2$

b) If $\mathbf{A}\perp\mathbf{B}$, then $\mathbf{A}\perp\alpha\mathbf{B}$, $\alpha\in\Re$

Proof : a) $\|\mathbf{A}+\mathbf{B}\|^2 = \mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2$

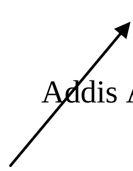
$= \mathbf{A}^2+\mathbf{B}^2$ Since $\mathbf{A}\perp\mathbf{B}$

$= \|\mathbf{A}\|^2+\|\mathbf{B}\|^2$

b) $\mathbf{A}.(\alpha\mathbf{B}) = \alpha(\mathbf{A}.\mathbf{B}) = \alpha.0 = 0$

.Hence **A** and α**B** are perpendicular

Example 1.3.9 : Let **A** = (3,4).Find the angles **A** makes with positive X,Y axes
Solution :



$\|\mathbf{A}\| = 5$

Let α be the angle the vector **A** makes with the x axis

$\text{Cos}\alpha = \frac{3}{5} \Rightarrow \alpha = 37^0$

Definition 1.3.6: The **angle** between two non zero vectors a and b defined to be the angle θ where $0 \leq \theta \leq \pi$, formed by the corresponding directed line segments whose initial points are the origin .

Theorem 1.3.5 : Let $\mathbf{a} = a_1i + a_2j + a_3k$ and $\mathbf{b} = b_1i + b_2j + b_3k$ be two non zero vectors and let θ be the angle between **a** and **b** , then $\mathbf{a.b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$

$$\text{i.e } \cos\theta = \frac{a.b}{\|a\|\|b\|}$$

Proof :

Use the law of cosines on the triangle formed by **a** , **b** and **b - a**

$\|\mathbf{b} - \mathbf{a}\|^2 = \|\mathbf{b}\|^2 + \|\mathbf{a}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$ where θ be the angle between **a** and **b**

$\mathbf{b} - \mathbf{a} = (b_1 - a_1)i + (b_2 - a_2)j + (b_3 - a_3)k$

$\|\mathbf{b} - \mathbf{a}\|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2$

$= (b_1^2 - 2a_1b_1 + a_1^2) + (b_2^2 - 2a_2b_2 + a_2^2) + (b_3^2 - 2a_3b_3 + a_3^2)$

$\|\mathbf{b}\|^2 = b_1^2 + b_2^2 + b_3^2$

$\|\mathbf{a}\|^2 = a_1^2 + a_2^2 + a_3^2$

$\|\mathbf{b} - \mathbf{a}\|^2 = \|\mathbf{b}\|^2 + \|\mathbf{a}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$

$\Rightarrow (b_1^2 - 2a_1b_1 + a_1^2) + (b_2^2 - 2a_2b_2 + a_2^2) + (b_3^2 - 2a_3b_3 + a_3^2)$

$= (b_1^2 + b_2^2 + b_3^2) + (a_1^2 + a_2^2 + a_3^2) - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$

$\Rightarrow -2a_1b_1 + -2a_2b_2 + -2a_3b_3 = -2\|a\|\|b\|\cos\theta$

$\Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$

$\Rightarrow \mathbf{a.b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$

Corollary 1.3.6 : Two non zero vectors **a** and **b** are perpendicular iff $\mathbf{a.b} = 0$

$\text{Cos}90^0 = 0$

Example 1.3.10 : Find the angle between the vector $\mathbf{a} = i + \sqrt{2}j + k$ and

a) Positive x – axis

b) Negative x-axis

Solution :

a) .Let $\mathbf{a} = i + \sqrt{2}j + k$ and $\mathbf{b} = \alpha i$ where $\alpha > 0$ be the positive x-axis

$\|\mathbf{a}\| = 2$, $\|\mathbf{b}\| = \alpha$

$\mathbf{a.b} = \alpha$

$\mathbf{a.b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta \Rightarrow \alpha = 2\alpha\cos\theta \Rightarrow \cos\theta = ½ \Rightarrow \theta = 60^0$

b) .Let $\mathbf{a} = i + \sqrt{2}j + k$ and $\mathbf{b} = \alpha i$ where $\alpha < 0$ be the negative x-axis

$\|\mathbf{a}\| = 2$, $\|\mathbf{b}\| = |\alpha|$

a.b = α

a.b = ||**a**||||**b**|| cos$\theta \Rightarrow \alpha = 2|\alpha|\cos\theta \Rightarrow \cos\theta = -½ \Rightarrow \theta = 120^0$

Example 1.3.11 : Find the angle between the vector i+ $\sqrt{2}$j + k and

a. Positive y – axis

b. Negative y-axis

Solution :

a).Let **a** = i+ $\sqrt{2}$j + k and **b** = αj where $\alpha > 0$

||**a**|| = 2 , ||**b**|| = α

a.b =$\alpha\sqrt{2}$

a.b = ||a||||b|| cos$\theta \Rightarrow \alpha\sqrt{2} = 2\alpha\cos\theta \Rightarrow \cos\theta = \sqrt{2}/2 \Rightarrow \theta = 45^0$

b).Let **a** = i+ $\sqrt{2}$j + k and **b** = αj where $\alpha < 0$

||**a**|| = 2 , ||**b**|| = $|\alpha|$

a.b =$\alpha\sqrt{2}$

a.b = ||**a**||||**b**|| cos$\theta \Rightarrow \alpha\sqrt{2} = 2|\alpha|\cos\theta \Rightarrow \cos\theta = -\sqrt{2}/2 \Rightarrow \theta = 135^0$

Activity 1.3.4:

For what values of c are the vectors 3i-2j and 2i+3j + ck are perpendicular

Example1.3.12 :

Find the equation of the circle having the points $\mathbf{P}_1 = (x_1, y_1)$ and $\mathbf{P}_2 = (x_2, y_2)$

Solution :

Let **P** =(x,y) be a point on the circle

OP = xi + yj , $\mathbf{OP_1} = x_1 i + y_1 j$, $\mathbf{OP_2} = x_2 i + y_2 j$

$\overrightarrow{P_1P_2} \cdot \overrightarrow{P_2P} = 0$

That is $(x - x_1, y - y_1).(x - x_2, y - y_2) = 0$

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Particular case $P_1 = (\sqrt{2}, \sqrt{2})$ and $P_2 = (-\sqrt{2}, -\sqrt{2})$

$(x - \sqrt{2})(x + \sqrt{2}) + (y + \sqrt{2})(y - \sqrt{2}) = 0$

$\Rightarrow (x^2 - 2) + (y^2 - 2) = 0 \Rightarrow x^2 + y^2 = 4$

Directional Cosines

The angles α,β and γ($0 \le \alpha, \beta, \gamma \le 180^0$) that a non zero vector **A** makes with the positive **X**,**Y**and **Z** axes respectively are called the **direction angles** of **A**.

A = $a_1 i + a_2 j + a_3 k$

positive x – axis i = (1,0,0)

A.i = ||**A**||||i||Cosα

$a_1 = ||\mathbf{A}|| \cos\alpha \Rightarrow \cos\alpha = {a_1}/{||A||}$

Similarly

positive y – axis j = (0,1,0)

A.j = ||**A**||||j|| Cosβ

$a_2 = \|\mathbf{A}\| \cos\beta \Rightarrow \text{Cos}\beta = {}^{a_2}\!/_{\|A\|}$

Similarly

positive Z– axis k = (0,0,1)

$\mathbf{A}.k = \|\mathbf{A}\|\|k\| \text{Cos}\gamma$

$a_3 = \|\mathbf{A}\| \cos\gamma \Rightarrow \text{Cos}\gamma = {}^{a_3}\!/_{\|A\|}$

Cosα , Cos β and Cos γ are called the **directional cosines** of A

<u>Example1.3.13 :</u> Let **A** = 4i – 3k . Find the angle the vector a makes with the positive X,Y Z axes .

<u>Solution :</u> **A** = 4i – 3k , ||A|| = 5

$\text{Cos}\alpha = {}^{4}\!/_{5} \Rightarrow \alpha = 37^0$

$\text{Cos}\ \beta = {}^{0}\!/_{5} \Rightarrow \beta = 90^0$

$\text{Cos}\ \gamma = {}^{-3}\!/_{5} \Rightarrow \gamma = 143^0$

Hence A makes 37^0, 90^0 and 143^0 with the positive X,Y Z axes .

<u>Example 1.3.14:</u> Let **A** = (1,-4,3)

Find **a)** the angles the vector makes with the positive X,Y and Z axes.

b) the angles the vector makes with the negative X ,Y and Z axes.

<u>Solution</u>

$\mathbf{A} = 1i + -4j + 3k = a_1 i + a_2 j + a_3 k \Rightarrow a_1 = 1$, $a_2 = -4$ and $a_3 = 3$

And $\|\mathbf{A}\| = \sqrt{1^2 + (-4)^2 + 3^2} = \sqrt{26}$

a) Positive X- axis **i** = (1,0,0) ||**i**|| = 1

$\mathbf{A}.i = \|\mathbf{A}\|\|\mathbf{i}\| \text{Cos}\alpha \Rightarrow a_1 = \|\mathbf{A}\| \cos\alpha \Rightarrow 1 = \sqrt{26} \cos\alpha$

$\text{Cos}\alpha = 1/\sqrt{26} \Rightarrow \alpha = \cos^{-1}(1/\sqrt{26}) \Rightarrow \alpha = 78^0$

Similarly

positive Y – axis **j** = (0,1,0) ||**j**|| = 1

$A.j = \|\mathbf{A}\|\|\mathbf{j}\| \text{Cos}\beta \Rightarrow a_2 = \|\mathbf{A}\| \cos\beta \Rightarrow -4 = \sqrt{26} \cos\beta$

$\cos\beta = -4/\sqrt{26} \Rightarrow \beta = \cos^{-1}(-4/\sqrt{26}) \Rightarrow \beta = 142^0$

Similarly

positive Z– axis **k** = (0,0,1) ||**k**|| = 1

$\mathbf{A}.k = \|\mathbf{A}\|\|kj\| \text{Cos}\gamma \Rightarrow a_2 = \|\mathbf{A}\| \cos\gamma \Rightarrow 3 = \sqrt{26} \cos\gamma$

$\cos\gamma = 3/\sqrt{26} \Rightarrow \gamma = \cos^{-1}(3/\sqrt{26}) \Rightarrow \gamma = 54^0$

b) **A** makes $180^0 - 78^0 = 102^0$, $180^0 - 142^0 = 38^0$ and $180^0 - 54^0 = 136^0$ with the negative X,Y Z axes .

Activity 1.3.5

In the above example **A** = 4i – 3k Find the angle the vector a makes with the negative X,Y Z axes .

Activity 1.3.6

Let A =2j -2k . Find the angle the vector a makes with the positive X,Y Z axes .

Activity 1.3.7

Let A =2j -2k . Find the angle the vector a makes with the positive X,Y Z axes .

Example 1.3.15: Show that $\text{Cos}^2\alpha + \text{Cos}^2\beta + \text{Cos}^2\gamma = 1$

Solution :

$\text{Cos}\alpha = a_1/\|A\|$, $\text{Cos}\beta = a_2/\|A\|$

and $\text{Cos}\gamma = a_3/\|A\|$

$\text{Cos}^2\alpha = \left(a_1/\|A\|\right)^2$, $\text{Cos}^2\beta = \left(a_2/\|A\|\right)^2$ and $\text{Cos}^2\gamma = \left(a_3/\|A\|\right)^2$

$$\text{Cos}^2\alpha + \text{Cos}^2\beta + \text{Cos}^2\gamma = \left(a_1/\|A\|\right)^2 + \left(a_2/\|A\|\right)^2 + \left(a_3/\|A\|\right)^2$$

$$= \frac{a_1^2 + a_2^2 + a_3^2}{\|A\|^2}$$

$$= 1$$

Therefore $\text{Cos}^2\alpha + \text{Cos}^2\beta + \text{Cos}^2\gamma$

The Projection of one vector onto another

Let **A** and **B** be two vectors and θ be the angle between them

P = Proj $_{\mathbf{B}}$ **A**

P = c**B**

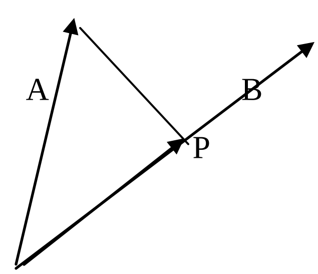
$(\mathbf{A} - \mathbf{P}) \perp \mathbf{P} \;\Rightarrow (\mathbf{A} - \mathbf{P}).\mathbf{P} = 0$

$\Rightarrow (\mathbf{A} - c\mathbf{B}).c\mathbf{B} = 0$

$\Rightarrow c(\mathbf{A.B} - c\mathbf{B.B}) = 0$

$\Rightarrow \mathbf{A.B} - c\mathbf{B.B} = 0$

$\Rightarrow c = \dfrac{A.B}{B.B}$



$P = c\mathbf{B} = \dfrac{A.B}{B.B}\mathbf{B}$

Example 1.3.16 : Let **A** = (1,2,3) and **B** = (-1,1,3)

Find

a) Proj $_{\mathbf{B}}$ **A**

b) Proj $_{\mathbf{A}}$ **B**

Solution

a) Proj$_{\mathbf{B}}$**A**

A.B = 1.(-1) + 2.1 + 3.3 = 10

B.B = (-1).(-1) + 1.1 + 3.3 = 11

Proj$_{\mathbf{B}}$**A** = $\frac{A.B}{B.B}$ **B** = $\frac{10}{11}(-1,1,3)$

b) Proj $_{\mathbf{A}}$ **B** = $\frac{B.A}{A.A}$ **A**

A.A = 1.1 + 2.2 + 3.3 = 14

Proj $_{\mathbf{A}}$ **B** = $\frac{B.A}{A.A}$ **A** = $\frac{10}{14}(1,2,3)$

Activity 1.3.8

Let **A** = (1,0,-1) and **B** = (2,5,3)

Find **a)** Proj $_{\mathbf{B}}$ **A**

b)) Proj $_{\mathbf{A}}$ **B**

Cos θ = $\| \frac{\Pr oj_B\ A}{\| A \|} \|$ **=** $| \frac{A.B}{B.B}$ **B** $| \frac{\| B \|}{\| A \|}$

$\Rightarrow$ Cos **θ** = $\frac{(A.B) \| B \|}{\| B \|^2 \| A \|}$

$\Rightarrow$ A.B = ||A||||B||cos θ

Moreover

|A.B| = ||A||||B|||cos θ| $\Rightarrow$|A.B| = ||A||||B|||cos θ|

$\Rightarrow$ |A.B| = ||A||||B|||cos θ| ≤ ||A||||B|| since |cos θ| ≤ 1

$\Rightarrow$ |A.B| ≤ ||A||||B||

Example1.3.17 : Let **A** = (2,-1,1) and **B** = (3,3,2).Find the cosine of the angle θ between **A** and **B.**

Solution : **A.B** = 2.3 + (-1)3 + 1.3 = 6

||**A**|| = $\sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

||**B**|| = $\sqrt{3^2 + 3^2 + 2^2} = \sqrt{9+9+4} = \sqrt{22}$

Cos θ = $\frac{A.B}{\| A \| \| B \|} = \frac{6}{\sqrt{6}\sqrt{22}}$ $\Rightarrow$ Cos θ = $\frac{\sqrt{6}}{\sqrt{22}}$

Activity 1.3.9: Let **A** = (1,0,2) and **B** = (-1,1,3).Find the cosine of the angle θ between **A** and **B**.

Theorem 1.3.6 : **(Triangle inequality)**

Let **A** and **B** be two vectors.Then || **A** + **B** || ≤ ||**A**|| + ||**B**||

Proof : || **A** + **B** ||^2= **A**^{2}+ 2**A.B** + **B** ^{2} = **A**^{2}+ **B**^{2} + 2**A.B**

≤ **A**^{2}+ **B**^{2} + 2|**A.B**| ≤ **A**^{2}+ **B**^{2} + ||**A**||||**B**||

= ||**A**||^2+ ||**B**||^2+ ||**A**||||**B**||

=(||**A**||+ ||**B**||) ^{2}

$\Rightarrow$|| **A + B** ||^2 ≤ (||**A**||+ ||**B**||) ^{2}

Taking square roots on both sides

$\| \mathbf{A} + \mathbf{B} \| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$

Resolution of a vector

If **a** and $\mathbf{a}^{\perp}$ (two non zero vectors) are perpendicular then any non zero vector **b** lying in the same plane as **a** and $\mathbf{a}^{\perp}$ can be expressed as a sum of the two vectors Proj $_{\mathbf{a}}\mathbf{b}$ and Proj $_{\mathbf{a}^{\perp}}\mathbf{b}$ which are parallel to **a** and $\mathbf{a}^{\perp}$ respectively.

Proj $_{\mathbf{a}}\mathbf{b} = \left(\dfrac{a.b}{\| a \|^2}\right)a$

Proj $_{\mathbf{a}^{\perp}}\mathbf{b} = (a^{\perp}.b/\|a^{\perp}\|^2)\, a^{\perp}$

$\mathbf{b} =$ Proj $_{\mathbf{a}}\mathbf{b}$ + Proj $_{\mathbf{a}^{\perp}}\mathbf{b}$

<u>Example1.3.18 :</u>Let $\mathbf{a} = 2i + 3j$, $\mathbf{a}^{\perp} = -3i + 2j$ and $\mathbf{b} = i + j$ Resolve **b** into vectors parallel to **a** and $\mathbf{a}^{\perp}$.

Solution : $\mathbf{a}.\ \mathbf{a}^{\perp} = 0$

$\mathbf{a}.\mathbf{b} = 2 + 3 = 5$ $\quad\quad$ $\mathbf{a}^{\perp}.\mathbf{b} = -3 + 2 = -1$, $\quad \|\mathbf{a}^{\perp}\|^2 = (-3)^2 + 2^2 = 13$

$\|\mathbf{a}\|^2 = a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2} = 4 + 9 + 0 = 13$

Proj $_{\mathbf{a}}\mathbf{b} = \dfrac{a.b}{\| a \|^2}\,\mathbf{a}$

$= \dfrac{5}{13}(2i + 3j)$

Proj $_{a\perp}\mathbf{b} = \left(\dfrac{a^{\perp}.b}{\| a^{\perp} \|^2}\right) a^{\perp}$

$= \dfrac{-1}{13}\ (-3i + 2j)$

Proj $_{\mathbf{a}}\mathbf{b}$ + Proj $_{a\perp}\mathbf{b} = \dfrac{5}{13}(2i + 3j) + \dfrac{-1}{13}(-3i + 2j) = i + j$

<u>Activity 1.3.10</u>: Let $\mathbf{a} = i + 2j$, $\mathbf{a}^{\perp} = -2i + j$ and $\mathbf{b} = 3i + 4j$ Resolve **b** into vectors parallel to **a** and $\mathbf{a}^{\perp}$.

1.4 The Cross product

<u>Definition 1.4.1 :</u> Let $\mathbf{A} = (a_1 , a_2 ,a_3)$.and $\mathbf{B} = (b_1 , b_2 ,b_3)$.be two vectors in three dimensional space . We define

$\mathbf{A} \times \mathbf{B} = i(a_2b_3 - b_3a_2) - j(a_1b_3 - a_3b_1) + k(a_1b_2 - a_2b_1)$

<u>Example 1.4.1</u> <u>:</u> Let $\mathbf{A} = (1,-1,3)$ and $\mathbf{B} = (2,1,1)$ Find

a) A x **B**
b) B x **A**

Solution :

A = (1,-1,3) = (a_1 , a_2 ,a_3) and a_1= 1 , a_2 = -1 , a_3 =3
B = (2,1,1) = (b_1 , b_2 ,b_3) and b_1= 2 , b_2 = 1 , b_3 =1

a) A x **B** = i(a_2b_3 - b_3a_2) – j(a_1b_3 - a_3b_1) + k(a_1b_2 - a_2b_1)
= i (-1.1 – 1.(-1)) – j(1.1 -3.2) + k(1.1 – (-1)2)
= i(-1 + 1) –j(1 – 6) + k(1 + 2)
= 0i + 5j + 3k

b) B x **A** = i(b_2a_3 - a_3b_2) – j(b_1a_3 - b_3a_1) + k(b_1a_2 - b_2a_1)
= i (1.3 – 3.1) – j(1.3 – 1.1) + k(2.(-1) – 1.1)
= i(3 - 3) –j (1 – 6) + k(-2 - 1)
= 0i - 5j - 3k
=-(0i + 5j + 3k) = - (**A** x **B**)

Using the definition of cross product one can show the following properties.

Theorem 1.4.1

1. **A** x **B** = - (**B** x **A**)
2. If **A** and **B** are parallel , then **A x B** = 0
3. **A**x(**B**+**C**) = **A** x **B** + **A** x **C** and (**B** + **C**) x **A** = **B** x **A** + **C** x **A**
4. For any $\alpha \in \Re$, (α**A)** x **B** = α(**A** x **B**) = **A** x (α**B**)
5. (**A** x **B**) x **C** = (**A.C**)**B** – (**B.C**)**A**
6. **A**x **B** is perpendicular to both **A** and **B**.
7. (**A** x **B**)^2= (**A.A**)(**B.B**) – (**A.B**)^2

Using (7) , $\|\mathbf{A} \times \mathbf{B}\|^2 = \|\mathbf{A}\|^2\|\mathbf{B}\|^2 - \|\mathbf{A}\|^2\|\mathbf{B}\|^2 \cos^2\theta$
$= \|\mathbf{A}\|^2\|\mathbf{B}\|^2 (1 - \cos^2\theta)$
$= \|\mathbf{A}\|^2\|\mathbf{B}\|^2 \sin^2\theta$
$\Rightarrow \|\mathbf{A} + \mathbf{B}\|^2 = \|\mathbf{A}\|^2\|\mathbf{B}\|^2 \sin^2\theta$

Taking square root on both sides
$\|\mathbf{A} + \mathbf{B}\| = \|\mathbf{A}\|\|\mathbf{B}\| |\sin\theta|$

Since $0 \le \theta \le 180$, $|\sin\theta| = \sin\theta$. Hence
$\|\mathbf{A} + \mathbf{B}\| = \|\mathbf{A}\|\|\mathbf{B}\| \sin\theta$

Remark : (**A** + **B**) x **C** ≠ **A** x (**B** x **C**)
Let **A** = i , **B** = i and **C** = j
(**A** x **B**) x **C** = 0i + 0j + ok and **A** x (**B** x **C**)= 0i – j + 0k

Example1.4.2 : It follows from (1) that
i × **i** = **j** × **j** = **k** × **k** = **0;** **i** × **j** = **k,** **j** ×**k** =**i,** **k** × **i** = **j**

Example 1.4.2: Let **u** and **v** be vectors in R^3, we have
(u ×**v).v = u.(v** ×**v) = u.0 =** 0**,**
(u ×**v).u = -(v** ×**u).u = -v. (u** ×**u) = -v.0 =** 0.

It follows that if $\mathbf{u} \times \mathbf{v} \neq 0$, then $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v** and to the plane determined by them.

Note: If two vectors are parallel then their cross product is zero because two parallel vectors form an angle of either 0 or 180. So sin(0) = sin(180) = 0.

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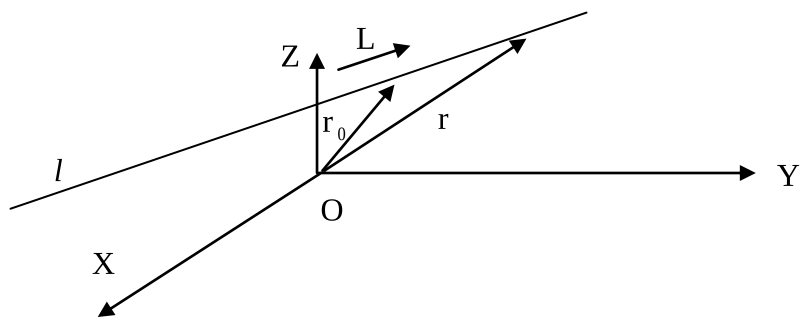
1.5 **Lines and planes**

Lines in spaces

Since we often think of vectors as directed line segments , lines and vectors are very much related. We will use vectors to describe lines.A line *l* and a vector L are parallel if L is parallel to the vector $\overrightarrow{P_0P}$ joining any two distinct points P_0 and P on *l* . Two points determine a unique line. From Euclidean Geometry that a line *l* in space is uniquely determined by a point Q on *l* and a vector L parallel to the line. From the above, a point P is on *l* if and only if $\overrightarrow{PQ}$ is parallel to L,

this means $\overrightarrow{PQ}$ = tL where $t \in \Re$



If $r_0 = x_0i + y_0j + z_0k$ and $r = xi+yi+zk$, then $\overrightarrow{PQ} = r - r_0$ that is $tL = r - r_0$

$$r = r_0 + tL$$

which is called the **vector equation** of *l*. Since r_0 can be any vector that joins the origin to a point on *l*, and Since L can be any vector parallel to *l*, there are many different vectors equations of a given line *l*.

Suppose we let L = ai + bj + ck

$r = r_0 + tL$ can be written as

$$xi+yj+zk = (x_0i+y_0j+z_0k) + t(ai+bj+ck)$$
$$= (x_0+at)i + (y_0+bt)j + (z_0+ct)k$$

Or equivalently

$$x= x_0+at,\ y= y_0+bt,\ z= z_0+ct$$

Which is called **parametric equations** of *l* and t is a parameter

Example1.5.1: Find a vector equation of the line that contains (1,1,3) and is parallel to i + j – 2k

Solution: -

L= i+j-2k=1i+1j-2k

r_0= li+lj+3k=i+j+3k

equation of the line *l* is r = r_0 + tL

= (i+j+3k)+t(i+j-2k)

= **(**1+t**)**i + (1+t)j + (3-2t)k

Activity1.5.1 : Find a vector equation of the line that contains (2,0,1) and is parallel to 2i – k

Example 1.5.2: - Find the parametric equations of the line that contains (1. 1, 3) and parallel to i+ j+k

Solution:

Take (x_0, y_0, z_0)= (1, 1,3)

And L = ai+bj+ck = i+ j+k

Hence the parametric equations of the line is $x = x_0+at = 1+1t$

$y= y_0+bt = 1+ t$

$z= z_0+ct = 3 + t$

Let $x = x_0+at$, $y= y_0+bt$ and $z = z_0+ct$ be a parametric equation of a line. Suppose all a, b, c are non - zero. If you solve each of these three equations for t, you will get

$t= \frac{x-x_0}{a}$, $t = \frac{y-y_0}{b}$ and $t = \frac{z-z_0}{c}$

$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ is called the **symmetric equations** of line *l*.

If a = 0 and b and c are non zero

$x= x_0$, $y= y_0+bt$, $z = z_0+ct$

$t = \frac{y-y_0}{b}$ $t= \frac{z-z_0}{c}$

$x = x_0$, $= \frac{y-y_0}{b} = \frac{z-z_0}{c}$ is the symmetric equations of line *l*.

Activity:1.5.2 - Find the symmetric equations of the line $x= x_0 + at$, $y = y_0+bt$, $z = z_0+ct$ where

a) b = 0
b) c = 0
c) b = c = 0

Example1.5.3 Find symmetric equations of the line *l* that contains the two points P= (1, 1,2) and Q=(2, 0,-1).

Solution :- First we find a vector L parallel to *l* .Since P and Q are two distinct points lying on *l*,

the vector $\overrightarrow{PQ}$ will serve as L .

L = $\overrightarrow{PQ} = (2-1)i + (0-1))j + (-1-2)k = i + -j - 3k$

Take a point on *l*. Let us take P = (1, 1,2)

$P = (1, 1,2) = (x_0\, y_0, z_0) \Rightarrow x_0 = 1\ y_0 = 1,\ z_0 = 2$

$L = i - j - 3k = ai + bj + ck \Rightarrow a = 1,\ b = -1,\ c = -3$

Symmetric equations of *l* are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Which becomes

$$\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z-2}{-3}$$

Activity1.5.3 : On **Example 1.5.3** above ,take a point on *l* as Q= (-2,1,-3).
Find its symmetric equation.

Many long and difficult computations can be simplified with the help of vector methods. Next we will compute the distance between a given line and a given point not on the line.

Theorem 1.5.1 : Let *l* be a line parallel to a vector L , and let Q be a point not on *l*. Then the distance D between Q and *l* is given by

$$D = \frac{\| L X \overrightarrow{PQ} \|}{\| L \|} \quad \text{where P is any point on } l$$

Proof: Let θ be the angle between L and $\overrightarrow{PQ}$, so that $0 \le \theta \le 180^0$ because angle between two vectors is between 0 and 180^0.

$$D = \| \overrightarrow{PQ} \| \sin\theta$$

$$\| L \text{ x } \overrightarrow{PQ} \| = \| L \| \| \overrightarrow{PQ} \| \, Sin\theta$$

$$= \|L\| \, D$$

It follows that $D = \dfrac{\| L X \overrightarrow{PQ} \|}{\| L \|}$

Example1.5.4: - Find the distance D from the point (3, -1, 1) to the line *l* with parametric equations $x = -1 + 2t, y = 2 + 3t, z = -t$

Solution

First we will show that $(3,-1,1) \notin l$.

$x = -1 + 2t$

$3 = -1 + 2t \Rightarrow t = 2$

If you put t = 2 in y = 2 + 3t ,we get y = 8 which is different from -1.

Hence$(3,-1,1) \notin l$

Put t = 0 . x = -1 + 2(0) = -1 , y = 2 + 3(0) = 2 and z = -0 = 0

(-1,2,0) is on *l* .

Take Q = (3,-1,1) , P =(-1,2,0)

$\overrightarrow{PQ}$ = (3- (-1) , -1 – 2, 1 – 0) = (4 , -3 ,1)

$\overrightarrow{PQ}$ = 4i + -3j + k

L = 2i + 3j +- k

$$L \times \overrightarrow{PQ} = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 4 & -3 & 1 \end{vmatrix} = i(3-3) - j(2-(-4)) + k(-6-12) = = -6i + -18k$$

$\|L\| = \| 2i + 3j +- k\| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{14}$

$\| L \times \overrightarrow{PQ} \| = \|-6i + -18k\| = \sqrt{(-6)^2 + (-18)^2} = \sqrt{360}$

$$D = \frac{\| L X \overrightarrow{PQ} \|}{\| L \|} = \frac{\sqrt{360}}{\sqrt{14}}$$

<u>Activity1.5.4 :</u> Find the distance D from the point (1, 1, 0) to the line *l* with parametric equations $x = -2 + 2t, y = 2 - t, z = 1 - t$

<u>Planes in space</u>

There is only one plane that contains a given point and perpendicular to a given line. Similarly, there is only one plane that contains a given point and perpendicular to a given non-zero vector. In other words, a plane is determined by a point and a non-zero vector. Let Q= (x_0, y_0, z_0) be a given point and N = ai + bj + ck be a non zero vector. Then a point P = (x,y,z) lies on the plane π that contains Q and is perpendicular to N if and only if the vector

$\overrightarrow{QP} = (x - x_0)i + (y - y_0)j + (z - z_0)k$ is perpendicular to N.

This means N. $\overrightarrow{QP} = 0$ which means $[ai + bj + ck][(x - x_0)i + (y - y_0)j + (z - z_0)k] = 0$

Which implies $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

The vector N is said to be **normal** to π

<u>Example1.5.5:</u> - Find the equation of the plane that contains the point (1,-1,3) and perpendicular to the vector 2i – j + 5k.

<u>Solution</u>: - The normal vector N= ai + bj + ck = 2i – j + 5k

a =2 , b = -1, and c = 5

$(x_0, y_0, z_0) = (1,-1,3) \Rightarrow x_0 = 1, y_0 = -1$ and $z_0 = 3$

We obtain the equation ; 2(x – 1) + -1(y – (-1)) + 5(z – 3) = 0

collecting terms 2x – y + 5z = 18

<u>Activity 1.5.5</u>: Find the equation of the plane that contains the point (2, 1,1) and perpendicular to the vector 4i +2j .

<u>Example1.5.6:</u> - Find an equation of the plane containing the points P=(1,-1,3) ,Q=(-1,4,1)and

R=(2,4,-2)

<u>Solution:</u> - The vectors $\overrightarrow{PQ}$ =-2i +5j +-2k and $\overrightarrow{PR}$ = i +5j + -5k are not parallel.

To find the normal vector take N= $\overrightarrow{PQ} \times \overrightarrow{PR}$

$$N = \begin{vmatrix} i & j & k \\ -2 & 5 & -2 \\ 1 & 5 & -5 \end{vmatrix} = i(-25 + 10) - j(10 + 2) + k(-10 - 5)$$

$$= -15i + -12j + -15k$$

N=-15i + -12j + -15k is normal to the plane and P =(1,-1,3) lies on the plane. The equation of the plane is

$$-15(x-1) + -12(y-(-1)) + -15(z - 3) = 0$$

or more simply

$$-15x + -12y + -15z = -48$$

Activity 1.5.6 : - Find an equation of the plane containing the points P =(2,1,1) , Q=(1,0,3)and R=(1,2,1)

Vector methods greatly simplified the calculation of distances between points and planes. First we find distance from a point to a plane. Using this concept we also find distance from a line to a plane and distance between two parallel planes.

Theorem 1.5.2 : - Let π be a plane with normal N, and let Q be any point not on π . Then the distance D between Q and π is given by

$$D = \frac{|N.\overrightarrow{QP}|}{\|N\|}$$

where P is any point on π .

Proof: Let θ be the angle between N and $\overrightarrow{QP}$, so that $0 \le \theta \le 180^0$.

Case (1) $0 \le \theta \le 90^0$

$$D = \|\overrightarrow{QP}\| \cos\theta$$

Case (2) $90^0 < \theta \le 180^0$

$$D = \|\overrightarrow{QP}\| \cos(180^0 - \theta) = \|\overrightarrow{QP}\| (-\cos\theta)$$

Since distance is non negative, we can write the above two cases as

$$D = \|\overrightarrow{QP}\| \, |\cos\theta|$$

From dot product,

$$N.\overrightarrow{QP} = \|N\|\|\overrightarrow{QP}\| (\cos\theta)$$

$$|N.\overrightarrow{QP}| = \|N\|\|\overrightarrow{QP}\| \, |\cos\theta|$$

$$= \|N\|D$$

Hence,

$$D = \frac{|N.\overrightarrow{QP}|}{\|N\|}$$

Example 1.5.7: Find the distance D between the point Q=(2,-1,3) and the plane

$$\pi : 2x - y + z = 3$$

Solution: - Q does not belong to π .

Take P =(1,1,2) P $\in$ π because 2(1) – 1 + 2 = 3

The normal vector is N= 2i – j + k

$\overrightarrow{QP}$ = (1-2)i+ (1-1)j +(2-(-3))k = -i + 5k

N. $\overrightarrow{QP}$ =(2i – j + k)(-i + 5k) = -1 + 5 = 4

|N. $\overrightarrow{QP}$ | = 4

||N|| = $\sqrt{2^2 + (-1)^2 + 1^2}$ = $\sqrt{6}$

Hence, D= $\frac{|N.\overrightarrow{QP}|}{\|N\|}$ = $\frac{4}{\sqrt{6}}$ = $\frac{4}{\sqrt{6}}$ x $\frac{\sqrt{6}}{\sqrt{6}}$ = $\frac{4\sqrt{6}}{6}$ = $\frac{2\sqrt{6}}{3}$

Example 1.5.8 Find the distance D between the point Q= (1,2,1) and the plane
π : x +2y +3z = 4

Solution: - Q does not belong to π .

Take P =(0,2,0) P $\in$ π because 0 + 2(2) + 3(0) = 4

The normal vector is N= 1i + 2j + 3k

$\overrightarrow{QP}$ = (0 -1)i+ (2-2)j +(0- 1)k = -1i + -1k

N. $\overrightarrow{QP}$ = (1i + 2j + 3k)(-1i + -1k) = -1 - 3 = -4

|N. $\overrightarrow{QP}$ | = 4

||N|| = $\sqrt{1^2 + 2^2 + 3^2}$ = $\sqrt{14}$

Hence, D= $\frac{|N.\overrightarrow{QP}|}{\|N\|}$ = $\frac{4}{\sqrt{14}}$ = $\frac{2\sqrt{14}}{7}$

Example1.5.9 : - Find the distance from the line *l* : x = 1 + 3t , y = -2 + t , z = 2t and the plane π : -2x + 4y + z = 3

Solution: - Since the line and the plane are parallel, take any point Q on *l*. Then the distance

from *l* to π is equal to the distance from Q to π .

Put t = 0 , then x = 1 , y = -2 , z = 0 .

Q = (1,-2,0)

P = (0,0,3) $\in$ π because (0,0,3) satisfies the equation -2x + 4y + z = 3

N = -2i+4j+k

D = $\frac{|N.\overrightarrow{QP}|}{\|N\|}$

$\overrightarrow{QP}$ = -i + 2j +3k

N. $\overrightarrow{QP}$ = (-2i+4j+k)(-i + 2j +3k) = 2 = 8 + 3 = 13

|N. $\overrightarrow{QP}$ | = 13

||N|| = $\sqrt{(-2)^2 + 4^2 + 1^2}$ = $\sqrt{21}$

$$D = \frac{|N.\overrightarrow{QP}|}{\|N\|} = \frac{13}{\sqrt{21}} = \frac{13\sqrt{21}}{21}$$

Activity 1.5.7:- Find the distance from the line l : x = 1 + t , y = 2 - t , z = 1 + 2t and the plane π : x - y + 2z = 1

Example1.5.10 : - Find the distance D between the parallel planes π_1= x – y + 2z and π_2 = -2x +2y -4z =6

Solution: - Take any point in one of the planes. The distance between the two planes is the same as the distance from one point in one plane to the other plane.

Take Q =(-1,0,1) $\in \pi_1$ (-1,0,1) $\notin \pi_2$

and π_2 = -2x +2y -4z =6

hence N = -2i +2j -4k

Take P=(0,3,0) $\in \pi_2$

$\overrightarrow{QP}$ = i +3j-k

N. $\overrightarrow{QP}$ = (-2i+2j -4k)(i – 3j – k) = -2 -6+ 4 = -4

|N. $\overrightarrow{QP}$ | =|-4| = 4

$$\|N\| = \sqrt{(-2)^2 + 2^2 + (-4)^2} = \sqrt{4+4+16} = \sqrt{24}$$

$$D = \frac{|N.\overrightarrow{QP}|}{\|N\|} = \frac{4}{\sqrt{24}} = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$$

$$D = \frac{|N.\overrightarrow{QP}|}{\|N\|} = \frac{5}{\sqrt{6}} = = \frac{5\sqrt{6}}{6} \text{ units}$$

1.6 Applications on area and volume

Area of a parallelogram

|| **A** x **B** || is the area of the parallelogram formed by the vectors A and B

Example 1.6.1 : Let **P** = (2,-1,3) , **Q** = (5,8,2) and **R** = (0,-1,3).Find the area of ΔPQR

Solution:

$\overrightarrow{PQ}$ = (3,9,-1) $\overrightarrow{PR}$ = (-2,0,0)

$$\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \begin{vmatrix} i & j & k \\ 3 & 9 & -1 \\ -2 & 0 & 0 \end{vmatrix} = i(9.0 - 0.(-1)) - j(3.0 - (-1)(-2)) + (3.0 - (-2)9)\,k$$

= 0i + 2j + 18k

Area of the triangle = $\sqrt{0^2 + 2^2 + 18^2} = \sqrt{328}$ square units

Example 1.6.2 : Find the area of a triangle with adjacent vectors **P**= (2, 3, -1) and **Q** = (1, 2, 2)

Solution:

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 1 & 2 & 2 \end{vmatrix} = (2(3)+2(1))\mathbf{i} - (2(2)+1(1))\mathbf{j} + (2(2)-1(3))\mathbf{k} = \mathbf{8i} - 5\mathbf{j} + \mathbf{k}$$

Area of a triangle = $\frac{1}{2} \| P \times Q \| = \frac{1}{2}\sqrt{64+25+1} = \frac{1}{2}\sqrt{90}$

Volume of a parallelpiped

The volume of the parallelpiped formed by the vectors **A**, **B** and **C** is |**A**.(**B** x **C**)|
|| **B** x **C** || = Area of the base of the paallelpiped
θ is the angle between **A** and **B** x **C**

Example 1.6.3: Find the volume of the parallelpiped formed by the vectors **A** = 3i+j-k ,
B = -i+2j + 4k and **C** = 2i-5j +3k

Solution

First let us find **B** x **C**

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} i & j & k \\ -1 & 2 & 4 \\ 2 & -5 & 3 \end{vmatrix} = i(2.3 - 4.(-5)) - j((-1).3 - 4.2) + ((-1)(-5) - 2.2)\,k$$

= 26j + 11j +1k

A.(B x **C)** = 3.26 + 1.11+(-1)1 **=** 88

Volume = |**A**.(**B** x **C**)|= |88| cubic units = 88 cubic units

VECTOR SPACES

Introduction:

One of the fundamental concepts of linear algebra is the concept of Vector space (or linear space. At the same time it is one of the more often used concepts algebraic structure in modern mathematics. For example, many function sets studied in mathematical analysis are with respect to their algebraic properties vector spaces. In analysis the notion ``linear space" is used instead of the notion ``vector space".

Moreover in many applications in mathematics, the sciences and engineering, the notion of a vector space arises. This idea is merely a carefully constructed generalization of R^n. The basic nmodels for vector spaces of n dimensional vectors. In studying the properties and structure of a vector space, you can study not only R^n, in particular, but many other important vector spaces. In this module we define the notion of a vector space in general and in later sections you study their structure.

If one considers geometrical vectors, and the operations one can perform upon these vectors such as addition of vectors, scalar multiplication, with some natural constraints such as closure of these operations, associativity of these and combinations of these operations, and so on, we arrive at a description of a mathematical structure which we call a vector space.

The "vectors" need not be geometric vectors in the normal sense, but can be any mathematical object that satisfies the following vector space axioms. Polynomials of degree $\le n$ with real-valued coefficients form a vector space, for example. It is this abstract quality that makes it useful in many areas of modern mathematics.

Objectives: At the end of this unit, you will be able to:

- Define what do we mean by a vector space and subspace
- Find the sum of two vector subspaces of a vector space.
- Find the generating set of a given vector space
- Differentiate linearly dependent and linearly independent vectors
- Find basis and dimension of a vector space
- Find the sum and direct sum of vector spaces

The axioms of vector spaces.

Mathematics is used to solve problems .One of the key ingredients in the problem solving process is the abstraction of the original problem in to mathematical smbols and ideas.The theory we are about to develop , that of vector spaces , is an abstraction .It is an abstraction from many mathematical models.

Vector spaces: Definition and Examples

Consider the following four mathematical objects
The XY plane
The set of all 2x2 matrices
The set of polynomials
The set of all solutions to a given system of homogeneous equations
These four have similarities that may not be immediately obvious(and all turn out to be vector spaces as we will soon see)

a. Each is a set of objects
b. Each has the property that any two objects in the set can be added and the result is an object in the same set
c. Each has the property that any object in the set can be multiplied by a number , and the result is an object in a set.

The three properties just mentioned form the nucleus of the definition of vector space.To complete the definition of vector space arount the above nucleus , we need to list the mathematical laws that must hold in every vector space. A precise definition of vector space will now be given.

<u>Definition of Vector Space</u>

We shall define vector spaces by using vector addition and scalar multiplication, that are subject to some conditions. We will try to define vector spaces based on the above operation. Based on that we check whether or not a given vector space is a vector space or not.

<u>Definition 4.1.1:</u> A **vector Space V** over a field **F** is a set of objects which satisfies the following properties

1.Given $u,v \in \mathbf{V}$, $u + v \in \mathbf{V}$
2.$\alpha \in F$ and $v \in \mathbf{V}$, $.\alpha v \in \mathbf{V}$
3. $\forall u, v \in \mathbf{V}$, $u + v = v + u$
4. $u,v,w \in \mathbf{V}$, $(u+ v) + w = u + (v+w)$
5. $0 + u = u + 0$ $\quad 0\in \mathbf{V}$, $\forall v \in \mathbf{V}$
6. $\forall u \in \mathbf{V}$ $\exists -u \in \mathbf{V}$, $u + -u = 0$
7.If $\alpha \in F$, $.\alpha(u+ v) = .\alpha u + .\alpha v$
8.If $\alpha , \beta \in F$, $(\alpha + \beta)u = \alpha u + \beta u \quad \forall u \in \mathbf{V}$
9. If $\alpha , \beta \in F$, $(\alpha \beta)u = \alpha(\beta u) \quad \forall u \in \mathbf{V}$
10. $\forall u \in \mathbf{V}$, $1.u = u.1 = u$

<u>Remark</u> : Elements of a vector space are called vectors

<u>Example 4.1.1</u>

Let $V = \Re^2$ over $\Re$.
$(a,b) +(c,d) = (a+c, b+d)$
$\alpha (a,b) = (a, \alpha b)$ .using the definition show that V is not a vector space over $\Re$.

<u>Solution</u>

1.Closed under addition
$(a,b) +(c,d) = (a+c, b+d) \in V$
2.Closed under scalar multiplication
$\alpha (a,b) = (a, \alpha b) \in V$
3.Commutativity
$(a,b) +(c,d) = (a+c, b+d) = (c + a,d + b)= (c,d)+(a,b)$
4.Associativity
$[(a,b)+(c,d)] + [(e,f)] =(a+c, b+d) + (e,f)$
$= ([a+c] + e ,[b+d] + f)$
$= (a+ c+ e ,b+d+f)$

$= (a + [c+e] , b +[d+f])$
$= (a, b) + [(c,d) + (e,f)]$

5.Existence of identity

$(a,b) +(0,0) = (a+0,b+0) = (a,b)$

$(0,0) + (a,b) = (0+a,0+b) = (a,b)$

$(0,0)$ is the identity element

6.Existence of inverse

$(a,b) + (-a,-b) =(a + -a ,b + -b) = (0,0)$

$(-a,-b) + (a,b) = (-a +a , -b + b) = (0,0)$

7.distributive law for addition in V

Let $v_1= (a,b)$ and $v_2= (c,d)$

$\alpha[v_1+v_2] = \alpha[(a,b) + (c,d)] =\alpha(a+c, b+d)$
$= (a+c,\alpha[b+d])$
$=(a+c , \alpha b + \alpha d)$
$= (a , \alpha b) + (c , \alpha d)$
$= \alpha(a,b) + \alpha(c,d)$
$= \alpha v_1+ \alpha v_2$

8.distributive law for addition in F

$(\alpha+ \beta)(a, b) = (a, (\alpha+ \beta)b) = (a, \alpha b + \beta b)$

$\alpha(a,b) + \beta(a , b) = (a, \alpha b) + (a , \beta b) = (2a , (\alpha+ \beta)b)$

Take $a= 1$, $b = 0$ and $\alpha = 1 = \beta$

$(\alpha+ \beta)(a, b) = (1 + 1)(1,0) = 2(1,0) = (1,0)$

$\alpha(a,b) + \beta(a , b) =1(1,0) +1(1,0) = (1,0) + (1,0) = (2,0)\neq(1,0)$

Hence $(\alpha+ \beta)(a, b))\neq \alpha(a,b) + \beta(a , b)$

9.Associative law for multiplication

$(\alpha\beta)(a,b) = (a, (\alpha\beta)b) = (a, \alpha(\beta b)) =\alpha(a, \beta b) = \alpha[\beta(a,b)]$

10. Multiplication by $1\in F$

$1v = 1(a,b) = (a,1b) = (a,b)$

Since property 9 is failed , V is not a vector space over $\Re$.

<u>Example 4.1.2</u>

1.**V** $=\Re^n = \{(x_1 , x_2 , x_3, \ldots, x_n) : x_i\in\Re \ i = 1,2,3 \ldots, n \}$
$F = \Re$ with the usual addition and scalar multiplication is a vector space.It is the most important example.

<u>Remark :</u> Vectors in $\Re^2$ can be visualized as points in the XY plane ; similarly , vectors in $\Re^3$ can be visualized as points in the XYZ space.We can also picture vectors in $\Re^2$ and $\Re^3$ by arrows.

2. . **V** $=Q^n = \{(x_1 , x_2 , x_3, \ldots, x_n) : x_i\in Q , i = 1,2,3 \ldots, n \}$
$F = \Re$ with the usual addition and scalar multiplication is not a vector space because take $\alpha= \sqrt{2}$ and $u = (1/3,1/3 ,1/3)$

$\alpha u = \sqrt{2}(1/3,1/3 ,1/3) = (\sqrt{2}/3 , \sqrt{2}/3 ,\sqrt{2}/3) \notin Q^3$

3. **V** $=\{(x_1, x_2, x_3, \ldots, x_{n-1}, 0) : x_i \in \Re, i = 1,2,3 \ldots, n-1\}$ $F = \Re$ with the usual addition and scalar multiplication is a vector space

4. **V** $=\{ a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x^1 + a_0 : a_i \in \Re\}$ $F = \Re$ with the usual addition and scalar multiplication is a vector space.

5. Let $V = \{ (x,y,z) : x,y,z \in \Re$ and $2x = y = 3z \}$ forms a vector space , where the operations + and . are as usual in $\Re^3$.
Note that $2x = y = 3z$ is the equation of the line passing through the origin and containing the point (3,6,2). Infact the set of all points on any line passing through the origin in $\Re^3$ forms a vector space.

6. V = The set of all solutions to the homogeneous system

$$\begin{cases} x + y + 3z = 0 \\ x - y - z = 0 \end{cases}$$

$= \{ (x,y,z) : x,y,z \in \Re$ such that $x + y + 3z = 0$ and $X - y - z = 0 \}$

The solutions here are all triples of the form (-a,-2a,a) and thus the set of vectors is $\{(-a,-2a,a) : a \in \Re\}$

Addition and scalar multiplication is defined as in $\Re^3$ is a vector space which is contained within a larger vector space $\Re^3$

<u>Activity 4.1.1:</u> In each of the following , Using the definition of a vector space find precisely which axioms in the definition of a vector space are violated. Take $V = \Re^2$ over the field $F = \Re$.

1.$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, 0)$, $\alpha.(x_1, x_2) = (\alpha x_1, 0)$
2..$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$, $\alpha.(x_1, x_2) = (\alpha x_1, 0)$
3..$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$, $\alpha.(x_1, x_2) = (\alpha x_1, 2\alpha x_2)$
4.$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$, $\alpha.(x_1, x_2) = (\alpha + x_1, \alpha + x_2)$

<u>Definition 4.1.2</u>: If u, v are vectors (elements of a vector space)
u – v = u + -v

<u>Theorem 4.1.2</u>: Let **V** be a vector space over a field F and $.\alpha \in F$ and $v \in$ **V**

1. $0v = 0$ and $\alpha.0 = 0$
2. $\alpha v = 0$ if and only if $\alpha = 0$ 0r $v = 0$
3. $(-1v) = -v$
4. $(-\alpha v) = -(\alpha v) = \alpha(-v)$
5. The identity element 0 is unique.
6. the inverse for every element v is unique.

<u>Proof</u> :

1. $0v + 0v = (0 + 0)v = 0v \Rightarrow -(0v) + 0v + 0v = -(0v) + 0v + 0$
$\Rightarrow 0v = 0$

$\alpha 0 + \alpha 0 = \alpha (0 + 0) = \alpha 0 \Rightarrow -(\alpha 0) + \alpha 0 + \alpha 0 = -(\alpha 0) + \alpha 0 + 0$

$\Rightarrow \alpha 0 = 0$

2. Suppose $\alpha v = 0$

($\Rightarrow$) We have to show $\alpha = 0$ or $v = 0$

Suppose $\alpha \neq 0$ $\alpha v = 0 \Rightarrow \alpha^{-1}(\alpha v) = \alpha^{-1}0$

$\Rightarrow (\alpha^{-1}\alpha)v) = 0$

$\Rightarrow 1v = 0$

$\Rightarrow v = 0$

($\Leftarrow$) Suppose $v \neq 0$. We have to show $\alpha = 0$

Suppose $\alpha \neq 0$ by the above $v = 0$ which is a contradiction . Hence $\alpha = 0$

Suppose $\alpha = 0$ or $v = 0$

By the above $ov = 0$ and $\alpha\, 0 = 0$

3. $0 = 0v = (1 + -1)v = 1v + -1v = V + (-1)v$ and $v + - v = 0$

$v + (-1)v = v + -v = 0 \Rightarrow v + -v + (-1)v = v + -v + -v$

$0 + (-1)v = 0 + -v \Rightarrow (-1)v = -v$

4. $0 = 0v = (\alpha + -\alpha)v = \alpha v + (-\alpha)v \Rightarrow \alpha(-v) = -(\alpha v)$

Similarly $0 = \alpha 0 = \alpha(v + -v) = \alpha v + \alpha(-v)$

$\Rightarrow \alpha(-v) =-(\alpha v) = (-\alpha)v$

Therefore , $\alpha(-v) = -(\alpha v) = (-\alpha)v$

5. Let $u \in$ **V** and $\forall\, v \in$ **V** $v + u = v$. We have to show $u = 0$

In particular choose $v = 0$

$0 + u = 0$ and $0 + u = u \Rightarrow u = 0$

6. Let $v \in$ **V** and suppose $u \in$ **V** such that $v + u = 0$ .We have to show $u = -v$

$v + -v = 0 \Rightarrow v + -v = 0 = v + u$

$\Rightarrow -v + (v + - v) = -v + (v + u)$

$\Rightarrow 0 + -v = 0 + u$

$\Rightarrow -v = u$

Activity 4.1.2

Consider the vector space $V = R^+$ over R with vector space operations defined as vector addition : $x + y = xy$ for all $x , y \in V$

Scalar multiplication : $\alpha x = x^\alpha$ for all $x \in V$ and $\alpha \in R$.

a) Find α such that $1 = \alpha 5$,
b) Find α and β such that $\alpha 3 + \beta 7 = 63$
c) The identity element.

Subspaces

One way of getting new vector spaces from a given vector space V is to look a subsets of V which forms vector spaces by themselves. A related notion to a vector space is that of a vector subspace. Suppose that V is a vector space and let $W \subseteq V$ be a subset. Not every subset W will itself be a vector space. Since V is a vector space, we know that we can add vectors in W and multiply them by scalars, but will only make W into a vector space in its own right if the results of these operations are back in W . Formally, we have

Definition 4.2.1: Let **V** be a vector Space over a field F and **W** $\subseteq$ **V** .

If W satisfies

S1. v, w $\in$ **W** , v + w $\in$ **W**

S2. $\alpha \in$ F and v $\in$ **W** , .αv $\in$ **W**

S3. The element 0 of v is also an element of **W**

Then we say W is **a subspace** of **V**

<u>Remark:</u> Every vector space has at least two subspaces, itself and the subspace {**0**} consisting only of the zero vector [Note that **0 +0 =0** and *c*.**0** = **0** in any vector space. The subspace {**0**} is called the **zero subspace.**

{0} and **V** are subspaces of **V** called the trivial subspaces of **V**.

<u>Example 4.2.1</u> : Show that **W** = {(x,y) : x + 2y = 0 } is a subspace of $\Re^2$ over $\Re$

<u>Solution :</u>

1. Let u, v $\in$ **W** , we have to show u + v $\in$ **W**

u = (x_1 , y_1) such that $x_1 + 2y_1 = 0$

v = (x_2 , y_2) such that $x_2 + 2y_2 = 0$

u + v = $(x_1 , y_1) + (x_2 , y_2) = (x_1 + x_2 , y_1 + y_2)$

$x_1 + 2y_1 + x_2 + 2y_2 = 0 \Rightarrow x_1 + x_2 + 2y_1 + 2y_2 = 0$

$\Rightarrow (x_1 + x_2) + 2(y_1 + y_2) = 0$

$\Rightarrow (x_1 + x_2 , y_1 + y_2) \in$ **W**

$\Rightarrow$ u + v $\in$ **W**

2. .Let $\alpha \in \Re$ and v $\in$ W WTS αv $\in$ **W**

v = (x,y) $\in$ **W** $\Rightarrow$ x + 2y = 0

α v = α(x,y) = (αx, αy)

x + 2y = 0 $\Rightarrow \alpha(x + 2y) = 0 \quad \forall \in \Re$

$\Rightarrow \alpha x + 2\alpha y = 0$

$\Rightarrow \alpha x + 2(\alpha y) = 0$

$\Rightarrow (\alpha x , \alpha y) = \alpha v \in$ **W**

3. 0 = (0,0)

0 + 2(0) = 0$\Rightarrow$ (0,0) $\in$ **W**

Hence **W** is a subspace of **V**

<u>Example 4.2.2</u> Let V = $\Re^3$ over $\Re$. .

W = {(a,b,c) : a $\geq$ 0}

Show that W is not a subspace of V.

<u>Solution :</u>

Let w = (1,-2,1) $\in$ W and α = -1

αw = -1(1,2,-1) = (-1,-2,1)$\notin$W

Hence W is not a subspace of V.

<u>Example 4.2.3</u>

U = The set of all solutions to the homogeneous system

$$\begin{cases} x + y + 3z = 0 \\ x - y - z = 0 \end{cases}$$

$= \{ (x,y,z) : x,y,z \in \Re$ such that $x + y + 3z = 0$ and $x - y - z = 0 \}$
$= \{ (-a,-2a,a): a \in \Re \}$
U is a subspace of V.

Example 4.2.4
U =The set $\{(a,0) : a \in \Re\}$ is a subspace of $\Re^2$ .Geometrically , this is the X-axis in the XY- plane.

Example 4.2.5
Let $W = \{(x_1, x_2, x_3, x_4, x_5): x_1, x_2, x_3, x_4, x_5, \in \Re$ and $x_3 + x_4 \geq 0 \}$
Is W a subspace of $\Re^5$?
Solution
Let $u, v \in W$. $u = (x_1, x_2, x_3, x_4, x_5): x_3 + x_4 \geq 0$
$v = (y_1, y_2, y_3, y_4, y_5) : y_3 + y_4 \geq 0$
$u+v=(x_1+y_1, x_2 + y_2, x_3+y_3, x_4+ y_4, x_5+y_5): x_3 + x_4 \geq 0$ and
$y_3 + y_4 \geq 0$
since $x_3 + x_4 \geq 0$ and $y_3 + y_4 \geq 0$, $x_3 + x_4 + y_3 + y_4 \geq 0$
$u + v \in W$
let $\alpha \in \Re$ and $u \in W$.
$\alpha u = (\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5)$ where $x_3 + x_4 \geq 0$
$\alpha x_3 + \alpha x_4 = \alpha (x_3 + x_4) < 0$ if $\alpha < 0$
Hence W is not a subspace of $\Re^5$.

Activity 4.2.1:
Let $W = \{(x_1,x_2,x_3): x_1,x_2,x_3 \in \Re$ and at least one $x_i = 0 \}$
Show that W a subspace of $\Re^3$?

Example 4.2.6:
Let $W = \{(x_1,x_2,x_3\ x_4, x_5): x_1, x_2,x_3,x_4,x_5, \in \Re$ and $x_2+ 2x_5= 0 \}$
Is W a subspace of $\Re^3$?

Solution :
Let $u, v \in W$.
a. $u = (x_1,x_2,x_3):x_1,x_2,x_3 \in \Re$ and $x_2+ 2x_5= 0$
$v = (y_1,y_2,y_3,y_4,y_5):y_1,y_2,y_3,y_4,y_5 \in \Re$ and $y_2+ 2y_5= 0$
$u + v = (x_1+y_1,x_2 + y_2, x_3+y_3,x_4+ y_4,x_5+y_5): x_2+ 2x_5= 0$ and
$y_2+ 2y_5= 0$
$x_2 + y_2 + 2(x_5+y_5)= x_2+ 2x_5 + y_2+ 2y_5= 0 + 0 = 0$
$u + v \in W$
b. Let $\alpha \in \Re$ and $u \in W$.
$\alpha u = \alpha(x_1,x_2,x_3,x_4,x_5):x_1,x_2,x_3,x_4,x_5 \in \Re$ and $x_2+ 2x_5= 0$
$= (\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5)\ x_1,x_2,x_3,x_4,x_5 \in \Re$ and $x_2+ 2x_5= 0$
$\alpha x_2+ 2\alpha x_5 = \alpha(x_2+ 2x_5) = \alpha 0 = 0$
$\Rightarrow \alpha u \in W$

c. $0 = (0,0,0,0,0) \in W$ because $0 + 2(0) = 0$
Therefore W a subspace of $\Re^5$.

Example 4.2.7 Let $V = \{ f : \Re \rightarrow \Re \}$

$W = \{ f : \Re \rightarrow \Re : f(x) = f(-x) \; \forall \in \Re \}$

Solution

a. Let $f_1, f_2 \in W$. We have to show $f_1 + f_2 \in W$

$f_1 \in W \Rightarrow f_1(x) = f_1(-x) \; \forall x \in \Re$

$f_2 \in W \Rightarrow f_2(x) = f_2(-x) \; \forall x \in \Re$

$(f_1 + f_2)(x) = f_1(x) + f_2(x) = f_1(-x) + f_2(-x) = (f_1 + f_2)(-x) \; \forall x \in \Re$

Hence $f_1 + f_2 \in W$

b. Let $f_1 \in W$ and $\alpha \in F$

$\alpha f_1(x) = \alpha f_1(-x) \Rightarrow \alpha f_1(x) = \alpha f_1(-x)$

c. The zero function is an element of W.

Hence W is a subspace of V.

Activity 4.2.2

Let $W = \{(x_1, x_2, x_3, x_4, x_5) : x_1, x_2, x_3, x_4, x_5, \in \Re$ and $x_1, =1 \}$

$= \{(1, x_2, x_3, x_4, x_5) : x_2, x_3, x_4, x_5, \in \Re \}$

Is W a subspace of $\Re^5$?

Example 4.2.8 :

U =The set $\{(a,b,b) : a,b \in \Re\}$ is a subspace of $\Re^3$.

Note that to belong to U , an ordered triple must have its second and third entries equal.

Example 4.2.9:

Show that $U = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a,b \in R \right\}$ is a subspace of M_{2x2} where M_{2x2} is the set of all 2x2 matrices.

Solution

1. Let $u, v \in U$. $u = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $v = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$ where $a,b,c,d \in \Re$

$$u + v = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix} \in U$$

2. Let $u \in U$ and $\alpha \in \Re$.Then $u = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $a,b \in \Re$

$$\alpha u = \alpha \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix} \in U$$

3. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in$ U

Example 4.2.10: Show that **W** = $\{(x,1,0) : x \in \Re\}$ is not a subspace of $\Re^3$ over $\Re$
Solution :
Since W is not closed under addition : (x,1,0) +(y,1,0) = (x+y,2,0) which does not have the required 1 as a second entry.This set also is not closed under scalar multiplication.

Theorem 4.2.1: The set of solutions of a homogeneous system of linear equations with n unknowns is a subspace of $\Re^n$.
Proof :

Let U be the set of solutions is not empty since at least it contains the zero vector (0,0,…..,0)
Suppose u= $(u_1, u_2, \ldots u_n)$ and v $=(v_1, v_2, \ldots, v_n)$ be two vectors in U. That is they are solutions to the system.We must show that their sum
$(u_1+v_1, u_2+v_2, \ldots, u_n+v_n)$ is also a solution.To this end
Suppose that
$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$
is any one of the equations .Substituting u +v in the x's in the left side , we obtain
$a_1(u_1+v_1) + a_2(u_2+v_2) + \ldots + a_n(u_n+v_n)$
$= (a_1u_1+a_2u_2+\ldots+a_nu_n) + (a_1v_1+a_2v_2+\ldots+a_nv_n)$
= 0 +0 since u= $(u_1, u_2, \ldots u_n)$ and v $=(v_1, v_2, \ldots, v_n)$ are solutions to the system.
= 0 , as desired
It remains to show that U is closed under scalar multiplication.
Let u $=(u_1, u_2, \ldots u_n)$ be a solution to the system and let α be a scalar. Substituting
$\alpha u = (\alpha u_1, \alpha u_2, \ldots, \alpha u_n)$ in to

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = a_1(\alpha u_1) + a_2(\alpha u_2) + \ldots + a_n(\alpha u_n)$$
$$= \alpha[a_1u_1 + a_2u_2 + \ldots + a_nu_n]$$
$$= \alpha.0 = 0$$

Therefore $\alpha u \in U$
Corollary 4.2.2: Lines and planes through the origin in $\Re^2$ and $\Re^3$ are subspaces.
Proof:
Lines and planes through the origin are sests of solutions of homogeneous system. .By the preceeding theorem they must be suspaces.
Activity 4.2.3
Determine whether the sets are subspaces of $\Re^3$.
1.$\{(2a,-5a,b): a,b \in \Re\}$
2.$\{a(2,0,1): a \in \Re\}$
3.$\{(2,0,1)+a(4,1,3): a \in \Re\}$

Activity 4.2.4
Determine whether the sets are subspaces of M_{2x2}
a. U = $\left\{ \begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix} : a \in R \right\}$

b. U= $\left\{ \begin{pmatrix} a & 0 \\ b & a+b \end{pmatrix} : a, b \in R \right\}$

Activity 4.2.5

Let V be a vector space and let u_1 and u_2 be two particular vectors in V.

Let U = $\{\alpha\, u_1 + \beta\, u_2 : \alpha , \beta \in \Re \}$

1.Prove that U is a subspace of V.

2. suppose that V =$\Re^3$. u_1 =(1,0,1) and u_2 =(1,1,0) .Describe U geometrically.

3.Suppose that V =$\Re^3$. u_1 =(1,0,1) and u_2 =(3,0,3) .Describe U geometrically.

Activity 4.2.6

Determne whether or not the point (7,1,3) is in the subspace {(3a-2b,b+a,b+a) , a,b$\in\Re$ } of $\Re^3$.

Remark :

1. In the definition of Subspace S1 says that W *is* closed under vector addition, whereas S2 says that W is closed under scalar multiplication.. For u and v in W , u + v belongs to V because V is a vector space. The question is whether u + v belongs to W, and S1 says that it does. Similarly, if w$\in$ W is a vector in W and $\alpha\in$ F is any scalar, then αw belongs to V because V is a vector space. The question is whether αw also belongs to W , and S2 says that it does.

2.You may ask whether we should not also require that the zero vector 0 also belongs to W . In fact this is guaranteed by A2, because for any w$\in$ W ,0w = 0 (why?) which belongs to W by A2 Similarly, if W $\subseteq$ V is a subspace, then if w$\in$ W, - w$\in$ W.

Theorem 4.2.3 : Let **W** $\neq\emptyset$ be s subset of a vector space **V**. the following Statements are equivalent.

1. **W** is a subspace of **V**
2. **W** is closed under Vector addition and scalar multiplication.
3. If u,v $\in$ **W** and α , $\beta \in$ F , then αu + βv $\in$ **W**

Proof : 1 $\Rightarrow$ 2 obviously true

2 $\Rightarrow$ 3 Suppose **W** is closed under Vector addition and scalar multiplication.

We have to show if u,v $\in$ **W** and α , $\beta \in$ F , then αu + βv $\in$ **W**

Suppose u,v $\in$ **W** and α , $\beta \in$ F , then αu $\in$ W and βv $\in$ **W**

αu $\in$ **W** , βv $\in$ **W** are two vectors hence αu + βv $\in$ **W** because **W** is closed under vector addition.

3 $\Rightarrow$1 For u,v $\in$ **W** and α , $\beta \in$ F , then αu + βv $\in$ **W**

We have to show **W** is a subspace of **V**,

Let u, v $\in$ **W** and , 1 $\in$ F

1u $\in$ **W** , 1v $\in$ **W** α = 1 , β = 1

αu + βv = 1u + 1v = u + v $\in$ **W**

Let u $\in$ **W** and $\alpha\in$ F WTS αu $\in$ **W**

αu = αu + 0v $\in$ **W** $\Rightarrow$ αu $\in$ **W**

0 = 0u + 0v = αu + βv $\in$ **W**

$\Rightarrow$ **W** is a subspace of **V**

Example 4.2.11

In each of the following , find out whether the subsets given form subspaces of a vector space V.

1. $V = \Re^3$, $S = \{(x,y,z) : 2x + y + z = 1\}$
2. $V = \Re^2$, $S = \{(x,y) : x \geq 0 \text{ and } y \geq 0\}$

Solution :

1. $0 = (0,0,0) \notin \mathbf{W}$ .Hence **S** is not a subspace of $\Re^3$.
2. $(1,1) \in S$ but $-1(1,1) = (-1,-1) \notin S$.

Activity 4.2.7

Show that $\mathbf{W} = \{(x,y) : x + 2y = 1 \}$ is not a subspace of $\Re^2$ over $\Re$.

Theorem 4.2.4: Let V be a vector space a field F.Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be subspaces of **V** . Then $\mathbf{W}_1 \cap \mathbf{W}_2$ is a subspace of **V**.

Proof :

1.Let $u, v \in \mathbf{W}_1 \cap \mathbf{W}_2 \Rightarrow u, v \in \mathbf{W}_1$ and $u, v \in \mathbf{W}_2$

Since $\mathbf{W}_1$ is a subspace of **V**, $u + v \in \mathbf{W}_1$ similarly , since $\mathbf{W}_2$ is a subspace of **V**, $u + v \in \mathbf{W}_2$

This implies $u + v \in \mathbf{W}_1 \cap \mathbf{W}_2$

2. let $v \in \mathbf{W}_1 \cap \mathbf{W}_2 \Rightarrow v \in \mathbf{W}_1$ and $v \in \mathbf{W}_2$

Since $\mathbf{W}_1$ is a subspace of **V** , $\alpha v \in \mathbf{W}_1$ similarly , since $\mathbf{W}_2$ is a subspace of **V** , $\alpha v \in \mathbf{W}_2$

$\Rightarrow \alpha v \in \mathbf{W}_1 \cap \mathbf{W}_2$

3. $0 \in \mathbf{W}_1$ and $0 \in \mathbf{W}_2 \Rightarrow 0 \in \mathbf{W}_1 \cap \mathbf{W}_2$

Hence $\mathbf{W}_1 \cap \mathbf{W}_2$ is sub space of **V**.

Example 4.2.12: Let $\mathbf{W}_1 = \{(x,y) : x + 2y = 0 \}$ and $\mathbf{W}_2 = \{(x,y) : 2x - y = 0 \}$.Show that $\mathbf{W}_1 \cup \mathbf{W}_2$ is not a subspace of $\mathbf{V} = \Re^2$

Solution:

To show that $\mathbf{W}_1 \cup \mathbf{W}_2$ is not a subspace of $\mathbf{V} = \Re^2$, we will show that one of the conditions of a subspace will be failed.

$u = (-2,1) \in \mathbf{W}_1$ and $v = (1,2) \in \mathbf{W}_2$ therefore $u, v \in \mathbf{W}_1 \cup \mathbf{W}_2$ But $u + v = (-2,1) + (1,2) = (-1,3)$ is not an element of $\mathbf{W}_1 \cup \mathbf{W}_2$

because $(-1,3)$ is neither an element of $\mathbf{W}_1$ nor $\mathbf{W}_2$

Example 4.2.13: Let $V = \Re^3$ over $\Re$

$U = \{(x,y,z) : x - 2z = 0\}$.Show that U is a subspace of V.

Solution

a. Let u_1 and $u_2 \in U \Rightarrow x_1 - 2z_1 = 0$ and $x_2 - 2z_2 = 0$

$\Rightarrow (x_1 + x_2) - 2(z_1 + z_2) = 0$

$\Rightarrow u_1 + u_2 \in U$

b. $\alpha u = \alpha(x,y,z) = (\alpha x, \alpha y, \alpha z) \in U$

$\alpha x - 2\alpha z = \alpha(x - 2z) = \alpha(0) = 0 \in U$

c. $0 = (0,0,0) \quad 0 - 2.0 = 0$

Hence U is a subspace of V

Activity 4.2.8: Let $V = \Re^3$ over $\Re$

$W = \{(x,y,z) : x = y ,\ z \in \Re \}$

Show that W is a subspace of V.

Example 4.2.14: Let $V = \Re^3$ over $\Re$

$U = \{(x,y,z) : x - 2z = 0\}$

$W = \{(x,y,z) : x = y , z \in \Re \}$

a. Find $U \cap W$

b.Show that $U \cup W$ is not a subspace of V.

Solution

a. Let $u \in U \cap W \Rightarrow u = (x,y,z)$ such that $x - 2z = 0$ and

$x = y$

$U \cap W = \{(x,y,z) : x = y = 2z \}$ which is a subspace of V.

b. $U \cup W$

Let $u = (2,4,1) \in U$ and $v = (3,3,9) \in W$

$u + v = (5,7,10) \notin U$ or $(5,7,10) \notin W$

$\Rightarrow u + v \notin U \cup W$

Hence $U \cup W$ is not a subspace of V.

Activity 4.2.9 :Let $V = \Re^2$ over $\Re$ and $U = [(1,1)]$ $W = [(1,2)]$

Show that $U \cup W$ is not a subspace of $\Re^2$.

Remark : Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be subspaces of $\mathbf{V}$. Then $\mathbf{W}_1 \cup \mathbf{W}_2$ is not a subspace of $\mathbf{V}$.

Definition 4.2.2: Let V be a vector space a field F. Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be subspaces of $\mathbf{V}$. Then $\mathbf{W}_1 + \mathbf{W}_2$ is defined as

$$\mathbf{W}_1 + \mathbf{W}_2 = \{ u_1 + u_2 \quad u_1 \in \mathbf{W}_1 \text{ and } u_2 \in \mathbf{W}_2 \}$$

Example 4.2.14:

Show that $\mathbf{W}_1 + \mathbf{W}_2$ is a subspace of $\mathbf{V}$.

Solution

a. Let $u,v \in \mathbf{W}_1 + \mathbf{W}_2 \Rightarrow u = u_1 + u_2 \quad u_1 \in \mathbf{W}_1$ and $u_2 \in \mathbf{W}_2$

$v = v_1 + v_2 \quad v_1 \in \mathbf{W}_1$ and $v_2 \in \mathbf{W}_2$

$u + v = (u_1 + u_2) + (v_1 + v_2) = (u_1 + v_1) + (u_2 + v_2) \in \mathbf{W}_1 + \mathbf{W}_2$

b. Let $u \in \mathbf{W}_1 + \mathbf{W}_2 \Rightarrow u = u_1 + u_2 \quad u_1 \in \mathbf{W}_1$ and $u_2 \in \mathbf{W}_2$

$\alpha u = \alpha(u_1 + u_2) = \alpha u_1 + \alpha u_2 \in \mathbf{W}_1 + \mathbf{W}_2$

c. $0 = 0 + 0 \in \mathbf{W}_1 + \mathbf{W}_2$

hence $\mathbf{W}_1 + \mathbf{W}_2$ is a sub space of $\mathbf{V}$.

Definition 4.2.3: Let $\mathbf{V}$ be a vector space over a field F and let $v_1 , v_2 \ldots v_n$ be elements of

V .Let $\alpha_1,\alpha_2 , \ldots ,\alpha_n \in F$.

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ is called a **linear combination** of $v_1, v_2 \ldots, v_n$

Example 4.2.15: The set of linear combinations of $v_1, v_2 \ldots v_n$ is a subspace of **V**.

Solution :

Let **W** = $\{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \quad \alpha_1,\alpha_2 , \ldots ,\alpha_n \in F$ and $v_1, v_2 \ldots v_n \in$ **V** $\}$.

Our aim is to show that W is a subspace of V.

1. Let u , v ∈ **W**

   $u = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

   $v = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n$

   $u + v = (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) + (\beta_1 v_1 + \beta_2 v_2 + \ldots +\beta_n v_n)$

   $= (\alpha_1+ \beta_1)v_1+ (\alpha_2+ \beta_2)v_2 + \ldots + (\alpha_n+ \beta_n)v_n \in$ **W**

2. Let v ∈ **W** and α ∈ F

   $\alpha v = \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n) =$

   $= \alpha(\alpha_1 v_1) + \alpha(\alpha_2 v_2) + \ldots + \alpha (\alpha_n v_n)$

   $=(\alpha\alpha_1)v_1 + (\alpha\alpha_2)v_2 + \ldots + (\alpha\alpha_n)v_n \in$ **W**

   3. $0 =0v_1+ 0v_2 + \ldots + 0v_n \in$ **W**

   Hence **W** is a sub space of **V**.

The subspace **W** is called the subspace generated by $v_1 ,v_2 , \ldots v_n$

Note: If **W** = **V**, i.e every element of **V** is a linear combination of $v_1 ,v_2 , \ldots v_n$ then we say that $v_1 ,v_2 , \ldots v_n$ generate **V** .

Let $S = \{ v_1 ,v_2 , \ldots v_n \}$ and $\alpha_1,\alpha_2 , \ldots ,\alpha_n \in F$

An element v∈ **V** of the form $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ is called a linear combination of elements of S.

Notation : L(S) denote the set of all finite linear combinations of elements of S.

L(S) is a subspace of **V**

Example 4.2.16:

V = $\Re^3$ over$\Re$

Let $v_1= (1,1,0) \quad v_2 = (0,1,0)$

$v_1 \ v_2 \in$ **V** and $\alpha_1,\alpha_2 \in F$

Let $W = \{ v : v = \alpha_1 v_1 + \alpha_2 v_2 \}$ is a sub space of **V**.

L(S) = L ({(1,1,0) , (0,1,0)})

Example 4.2.17 : Let **V** =$\Re^3$ over$\Re$

Let $v_1= (1,0,0) \quad v_2 = (0,1,0) \quad v_3 = (0,0,1)$

Every element of **V** is a linear combination of v_1 v_2 and v_3

Let (x,y,z) ∈ **V** $\quad (x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$= x\, v_1 + y\, v_2+ zv_3$

We say that (1,0,0) , (0,1,0) and (0,0,1) generates $\Re^3$

Example 4.2.18: Let u = $3x^2 + 8x – 5$, v = $2x^2 + 3x - 4$ and
w = $x^2 - 2x -3$ .Write u as a linear combination of v and w.

Solution

u =αv + βw = $\alpha(2x^2 + 3x - 4) + \beta(x^2 - 2x -3)$

$3x^2 + 8x – 5 = (2\alpha + \beta) x^2 + (3\alpha -2\beta) x + (- 4\alpha + -3 \beta)$

$2\alpha + \beta = 3$

$3\alpha -2\beta = 8$

$-4\alpha +-3\beta = -5$ Solving simultaneously $\alpha = 2$ and $\beta = -1$ satisfies the three equations

Therefore

u =2v + (-1) w

Activity 4.2.10 :Let u = (1,-3,2) , v = (2,-2,0) and w = (0,-1,1) Write (7,-1,0) as a linear combination of u, v and w.

Activity 4.2.11 : Find a subset $S \subset \mathbb{R}^2$ which is closed under scalar multiplication but not under vector addition. Do the same for $\Re^3$.

Activity 4.2.12: Consider the set $\Re^3$of ordered triples of real numbers and check whether or not the following subsets are subspaces

1. W = $\{(x,y,0) : x,y \in\Re \}$
2. W = $\{(x,y,1) : x,y \in\Re \}$

Example 4.2.19: Consider the set W consisting of all 2x3 matrices of the form

$$\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix},$$

where *a*, *b*, *c* and *d* are arbitrary real numbers. Then W is a subset of the vector space M_{23} (set of all 2x3 matrices). Show that W is a subspace of M_{23}.

Solution:

1) Let **u** = $\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & 0 & 0 \end{bmatrix}$ and **v** =$\begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & 0 & 0 \end{bmatrix}$ in W.

Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 & c_1 + c_2 \\ d_1 + d_2 & 0 & 0 \end{bmatrix} \quad \text{is in W}$$

2) If α is a scalar, then

$$\alpha\mathbf{u} = \begin{bmatrix} \alpha a_1 & \alpha b_1 & \alpha c_1 \\ \alpha d_1 & 0 & 0 \end{bmatrix} \text{ is in W}$$

3. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in W

Hence W is a subspace of M_{23}.

Example 4.2.20: Let W be the subset of R^3 consisting of all vectors of the form (1, *a, b*), where *a* and *b* are any real numbers.Show that W is not a subspace of R^3 .

Solution:

Let **u** = $(1, a_1, b_1)$ and **v** = $(1, a_2, b_2)$ be vectors in W. Then

u + v = $(1, a_1, b_1) + (1, a_2, b_2) = (2, a_1+a_2, b_1 + b_2)$ is not in W,

Since the first component of u + v is 2 which is different from 1. Hence W is not a subspace of R^3.

Example 4.2.21:

Let V be a the set of mxn matrices over a field F. For A an element of V let U be the set of all nx n matrices satisfying AX = 0 . Show that U is a subspace of V . (U is called the null space of A and is denoted here by N(A). It is sometimes called the solution space of A.)

Solution

(1) A0 = 0, so $0 \in N(A)$

(2) If $X, Y \in N(A)$, then AX = 0 and AY = 0, so A(X + Y) = AX + AY = 0 + 0 = 0 and so $X + Y \in N(A)$

(3) If $X \in N(A)$ and $\alpha \in F$, then $A(\alpha X) = \alpha (AX) = \alpha 0 = 0$, so $\alpha X \in N(A)$.

For example,

a. if A = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then N(A) = the set consisting of just the zero vector.

b. If A = $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, then N(A) is the set of all scalar multiples of $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Definition 4.2.4: Let **V** be a vector space over a field F and let $S \neq \emptyset$, $S \subseteq$ **V** . the subspace **L(S)** is called the **subspace generated by** S (or the subspace spanned by S)

Remark : 1.If $S = \emptyset$, define L(S) ={0 }

2. If L(S) = **V**, then S is called a set of generators of **V** or a spanning set of **V**

Example 4.2.22: Different sets may generate the same subspace

V $=\Re^3$ over$\Re$

S = { (1,0,0) , (0,1,0) , (0,0,1)} generate $\Re^3$.

S = { (1,0,0) , (0,1,0) , (0,0,1) , (1,-1,1)} generate $\Re^3$.

Notation : If S = { $v_1, v_2, \ldots v_n$ }.We shall denote L(S) by $[v_1, v_2, \ldots v_n]$

Example 4.2.23: Let U be the set of points on the plane with equation

$$X + 3Y + 2z = 0$$

Solution

$U = \{(a(-3,1,0) + b(-2,0,1) : a,b \in \Re\}$

U is a subspace of $\Re^3$ .every vector in U is written interms of two specific vectors (-3,1,0) and (-2,0,1) .We express this fact by saying that U is spanned by (-3,1,0) and (-2,0,1).That is

$U = [(-3,1,0) ,(-2,0,1)]$

Example 4.2.24: Let $V = \Re^3$. Let $S = \{ (1,1,0) , (0,-1,1) , (1,0,1)\}$.

Consider $v = (a,b,c) \in \Re^3$.

Determine the conditions on a,b,c so that $v \in [S] = [(1,1,0) , (0,-1,1) , (1,0,1)]$.

Solution

Let $v = (a,b,c) \in [S]$. Then $(a,b,c) = \alpha(1,1,0) + \beta(0,-1,1) + \gamma(1,0,1)$

Comparing coefficients

$$a = \alpha + \gamma$$
$$b = \alpha - \beta$$
$$c = \beta + \gamma$$

Solving the equations we get

$$a = b + c$$

So $(a,b,c) \in [S]$ if and only if $a = b + c$.

Consequently , $(2,-1,3) \in [S]$ while $(2,-1,2) \notin [S]$

Example 4.2.25: $\Re^2$ is spanned by (1,1) and (3,2).because $(a,b) \in \Re^2$ can be written as

$(a,b) = (-2a+3b)(1,1) + (a-b)(3,2)$

Activity 4.2.13

Show that $\Re^3$ is spanned by (1,0,2) ,(0,1,0) and (1,-1,1)

Theorem 4.2.5 : Let $v_1 ,v_2 , \ldots v_n$, w be vectors in the vector space **V** .Suppose

$w \in [v_1 ,v_2 , \ldots v_n]$ .Then

$[v_1 ,v_2 , \ldots v_n ,w] = [v_1 ,v_2 , \ldots v_n]$

Proof

i) $[v_1 ,v_2 , \ldots v_n] \subseteq [v_1 ,v_2 , \ldots v_n , w]$

Let $v \in [v_1 ,v_2 , \ldots v_n]$

$\Rightarrow v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

$= \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n + 0w$

$\Rightarrow v \in [v_1 ,v_2 , \ldots v_n ,w]$

ii) $[v_1 ,v_2 , \ldots v_n ,w] \subseteq [v_1 ,v_2 , \ldots v_n]$

let $v \in [v_1 ,v_2 , \ldots v_n ,w]$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n + \alpha w$

$= \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n + \alpha\{\beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n)$ since $w \in [v_1 ,v_2 , \ldots v_n]$

$= [\alpha_1 + \alpha\beta_1]v_1 + [\alpha_2 + \alpha\beta_2]v_2 + . . . + [\alpha_n + \alpha\beta_n]v_n$

$\Rightarrow v \in [v_1 ,v_2 , . . . v_n]$

From (i) and (ii)

$[v_1 ,v_2 , . . . v_n] = [v_1 ,v_2 , . . . v_n ,w]$

Corollary 4.2.6: Let $v_1 ,v_2 , . . . v_n ,w_1 ,w_2 , . . . w_n$ be vectors in the vector space **V** .Suppose $w_i \in [v_1 ,v_2 , . . . v_n]$ for i = 1,2,……, m .Then

$$[v_1 ,v_2 , . . . v_n] = [v_1 ,v_2 , . . . v_n ,w_1 ,w_2 , . . . w_m]$$

Proof : By using the above theorem and induction.

Example 4.2.26: Let V = $\Re^3$ over$\Re$

$U_1 = [(1,0,0) ,(0,1,0), (0,0,1)]$

$U_2 = [(1,0,0) ,(0,1,0), (0,0,1) , (1,-1,2)]$

$(1,-1,2) = 1 (1,0,0) + (-1) (0,1,0) + 2(0,0,1)$

Therefore , $U_1 = U_2$

Theorem 4.2.7: Suppose each of $x_1, x_2 ….., x_n$ is a linear combination of $y_1, y_2 …..,y_m$. Then any linear combination of $x_1, x_2 ….., x_n$ is a linear combination of $y_1, y_2 ….., y_m$

Proof:

Suppose $x_i= \alpha_{i1} y_1+ \alpha_{i2} y_2 + …+ \alpha_{im} y_m \qquad 1 \le i \le n$

Let z be a linear combination of the x_i's .That is

$Z = \beta_1x_1+ \beta_2x_2 + ….+\beta_nx_n$

$Z = \beta_1(\alpha_{11} y_1+\alpha_{12} y_2 + …+ \alpha_{1m} y_m)+\beta_2(\alpha_{21} y_1+ \alpha_{22} y_2 + + \alpha_{2m} y_m)$

$+…+\beta_n(\alpha_{n1} y_1+ \alpha_{n2} y_2 + + \alpha_{nm} y_m)$

$= (\beta_1\alpha_{11}+\beta_2\alpha_{21}+…+\beta_n\alpha_{n1}) y_1+(\beta_1\alpha_{12} +\beta_2 \alpha_{22})+… +$

$(\beta_1\alpha_{1m}+\beta_2\alpha_{2m} +…+\beta_n\alpha_{nm})y_m$

Which shows that z is a linear combination of $y_1, y_2 ….., y_m$

Ccorollary 4.2.8: Two Subspaces $[x_1, x_2 ….., x_n]$ and $[y_1, y_2 ….., y_m]$ are equal ifeach of $x_1, x_2 ….., x_n$ is a linear combination of $y_1, y_2 ….., y_m$ and each of $y_1, y_2 ….., y_m$. is a linear combination of $x_1, x_2 ….., x_n$.

Proof : It is left as an exercise for the student.

Example: If x and y are vectors in R^n, then [x ,y]= [x + y , x – y].

Solution.

Each of x + y and x – y is a linear combination of x and y .

Also x = ½ (x + y) + ½ (x – y) and

y = - ½ (x + y) + ½ (x – y)

so each of x and y is a linear combination of x + y and x - y .

Activity 4.2.14

1. Let U={(x,y,z): x ≥ 0 } .Is U is a subspace of R^3
2. Let V = $\mathbf{R}^2$ be the vector space over the field R with respect to the usual addition and scalar multiplication.
   a) Show that U= { (x, y) | x – 2y = 0 }, W = { (x, -x) | x $\in\mathbf{R}$ }are subspaces of V.
   b) Find an expression for U ∩ W and show that U ∩ W is a subspace of V.
   c) Is U ∪ W a subspace of V? Prove or give a counter example.

Activity 4.2.15

Which of the following subsets ofR^2 are subspaces?

(a)$\{(x,y) : x = 2y \}$

(b))$\{(x,y) : x = 2y$ and $2x = y \}$

(c))$\{(x,y) : x = 2y+ 1 \}$

(d))$\{(x,y) : xy =0 \}$

Activity 4.2.16

Write

1. (2,2,3) as a linear combination of $v_1 = (0,1,-1)$, $v_2 = (1,1,0)$, $v_3 = (1,0,3)$
2. (a,b,c) as a linear combination of $v_1 = (0,1,-1)$, $v_2 = (1,1,0)$, $v_3 = (1,0,3)$

4.3 Linear dependence and independence of vectors.

Suppose we are given a generating set A for a vector subspace U of vector space V . One question that naturally arises is : Is there any redundancy in A in the sense that some proper subset of A generates U? To answer this, we start with

Linear dependence and independence of vectors

Definition 4.3.1: Let **V** be a vector space over a field F and let $v_1, v_2, \ldots v_n$ be elements of **V** .We say that $v_1, v_2, \ldots v_n$ are **linearly dependent** over F if there exist $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero such that $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$. Otherwise we say the vectors are **linearly independent**

Remark : If $v_i = v_j$ for some i and j with $i \neq j$, then $v_1, v_2, \ldots v_n$ are linearly dependent since we can take $\alpha_i = 1$ and $\alpha_j = -1$ and every other α is zero.

Note : The empty set of vectors is linearly independent since the definition holds vacuously.

Example 4.3.1 . If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are **k** vectors in any vector space and $\mathbf{v}_i$ is the zero vector,

Then $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly dependent.

Solution: We can show by letting $c_i = 1$ and $c_j = 0$ for $j \neq i$ .Thus

$$0.\mathbf{v}_1 + 0.\mathbf{v}_2 + \cdots + \alpha_i \mathbf{v}_i + \cdots 0.\mathbf{v}_k = \mathbf{0} \text{ because } \mathbf{v}_i = \mathbf{0}.$$

Hence every set of vectors containing the zero vector is linearly dependent.

It is immediate from the definition that any set containing 0 is linearly dependent.

Example 4.3.2

1. A singleton set $\{x\}$ is linearly independent iff $x \neq 0$
2. $\{x,y\}$ is linearly independent iff none of x and y is a scalar multiple of the other.

Example 4.3.3: Let $\mathbf{V} = \Re^3$ over$\Re$

Show that the following sets are linearly independent

1. $v_1 = (1,1,0)$, $v_2 = (1,1,1)$

2. $v_1 = (1,1,0)$, $v_2 = (1,1,1)$, $v_3 = (0,1,-1)$
3. $v_1 = (0,1,-1)$, $v_2 = (1,1,0)$, $v_3 = (1,0,2)$

Solution

1. Suppose $\alpha_1 v_1 + \alpha_2 v_2 = \mathbf{0}$
   Then $\alpha_1(1,1,0) + \alpha_2 (1,1,1) =(0,0,0)$
   The above implies that
   $\alpha_1 + \alpha_2 = 0$ (i)
   $\alpha_1 + \alpha_2 = 0$ (ii)
   $\alpha_2 = 0$ (iii)
   Substituting (iii) in (i) we get $\alpha_1 = 0$
   Therefore $\{ v_1 , v_2\}$ is a linearly independent set.
2. Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \mathbf{0}$
   Then $\alpha_1(1,1,0) + \alpha_2 (1,1,1) + \alpha_3(0,1,-1) =(0,0,0)$
   The above implies that
   $\alpha_1 + \alpha_2=0$ (i)
   $\alpha_1 + \alpha_2 + \alpha_3= 0$ (ii)
   $\alpha_2 - \alpha_3= 0$ (iii)
   Adding (ii)and (iii) we get $\alpha_1 + 2\alpha_2 = 0$ equating with (i) ,
   $\alpha_1 = \alpha_2 = 0$ which implies $\alpha_3= 0$
   Therefore $\{ v_1 , v_2 ,v_3\}$ is a linearly independent set.
3. Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \mathbf{0}$
   Then $\alpha_1(0,1,-1) + \alpha_2 (1,1,0) + \alpha_3(1,0,2) =(0,0,0)$
   The above implies that
   $\alpha_2 + \alpha_3 = 0$ (i)
   $\alpha_1 + \alpha_2 = 0$ (ii)
   $-\alpha_1 +2\alpha_3= 0$ (iii)
   Solving simultaneously we get $\alpha_1 = \alpha_2 =\alpha_3= 0$ which Therefore $\{ v_1 , v_2 ,v_3\}$ is a linearly independent set.

Example 4.3.4: Show that the following sets are linearly dependent

1. $S= \{(-1,3) , (2,-6)\}$ is a linearly dependent set
2. $S^1= \{(-1,3) , (2,-6) , (1,4) \}$ is a linearly dependent set

Solution:

1. let $v_1=(-1,3)$ and $v_2 = (2,-6)$
   Suppose $\alpha_1 v_1 + \alpha_2 v_2 = \mathbf{0}$
   $-\alpha_1 + 2\alpha_2 = 0$ (i)
   $3\alpha_1 + -6\alpha_2 = 0$ (ii)

Solving simultaneously , we get 0 = 0.That is the system have infinitely many solutions. As an example , $\alpha_1 =2$ and $\alpha_2 =1$
Therefore $\{ v_1 , v_2\}$ is a linearly dependent set.

2. Let $v_1=(-1,3)$, $v_2 = (2,-6)$ and $v_3= (1,4)$
   Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \mathbf{0}$

$\alpha_1(-1,3) +\alpha_2(2,-6) + \alpha_3(1,4) = (0,\mathbf{0})$
Take $\alpha_1 =2$, $\alpha_2 =1$ and $\alpha_3= 0$
Therefore $\{ v_1 , v_2 ,v_3 \}$ is a linearly dependent set.

Activity 4.3.1:
Show S ={(-1,0,2), (-5,2,3),(3,-2,1)} is a linearly dependent set

Example 4.3.5:
Show S ={(1,1) , (-1,0) , (2,-1)}is a linearly dependent set
Solution
Suppose
$(0,0) = \alpha_1(1,1) + \alpha_2 (-1,0) + \alpha_3 (2,-1)$
$0 = \alpha_1 - \alpha_2 + 2\alpha_3$
$0 = \alpha_1 - \alpha_3 \Rightarrow \alpha_1 = \alpha_3$
$0 = \alpha_1 - \alpha_2 + 2\alpha_3 \Rightarrow 3\alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_2 = 3\alpha_1$
Therefore S ={(1,1) , (-1,0) , (2,-1)}is a linearly dependent set .
As a particular case
1(1,1) + 3(-1,0) + 1(2,-1) = (0,0)

Activity 4.3.2:
Show S ={(3,1,-4) , (2,2,-3) , (0,-4,1)}is a linearly dependent set

Example 4.3.6: Let V = $\Re^3$ over$\Re$
1.Show that the following sets are linearly dependent
{sinhx , coshx , e^x }
2.V = $P_2(x) = \{a_2x^2 + a_1x^1 + a_0 \ : \ a_i \in \Re \}$ over $\Re$
$\{x^2 + 4 , -2x^2 - 8x \ , x - 1 \}$ is a linearly dependent set
Solution:

1. Let v_1= sinhx , v_2 = coshx and $v_3 = e^x$
Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \mathbf{0}$.
$\alpha_1 \sinh x + \alpha_2 \cosh x + \alpha_3 e^x = \mathbf{0}$
$\alpha_1(\frac{e^x - e^{-x}}{2}) + \alpha_2 (\frac{e^x + e^{-x}}{2}) + \alpha_3 e^x = \mathbf{0}$
Collecting like terms
$e^x(\frac{\alpha_1 + \alpha_2}{2} + \alpha_3) + e^{-x}(\frac{\alpha_2 - \alpha_1}{2}) = 0$
$\frac{\alpha_1 + \alpha_2}{2} + \alpha_3 = 0$
$\frac{\alpha_2 - \alpha_1}{2} = 0$
Which implies that $\alpha_1 = \alpha_2$ and $\alpha_3 = -\alpha_1$
{sinhx , coshx , e^x } is a linearly dependent set.
2. $\{x^2 + 4 , -2x^2 - 8x \ , x - 1 \}$
Let $v_1 = x^2 + 4$ $v_2 = -2x^2 - 8x$ and $v_3 = x - 1$
Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \mathbf{0}$.
$\alpha_1(x^2 + 4) + \alpha_2 (-2x^2 - 8x) + \alpha_3 (x - 1) = \mathbf{0}$

$(\alpha_1 +\text{-}2\alpha_2)x^2 + (-8\alpha_2 + \alpha_3)x + (4\alpha_1 - \alpha_3) = \mathbf{0}$

$\alpha_1 +\text{-}2\alpha_2 = 0$ (i)

$-8\alpha_2 + \alpha_3 = 0$ (ii)

$4\alpha_1 - \alpha_3 = 0$ (iii)

Solving (ii) and (iii) simultaneously, we have $4\alpha_1 +\text{-}8\alpha_2 = 0$.Solving $4\alpha_1 +\text{-}8\alpha_2 = 0$ with $\alpha_1 +\text{-}2\alpha_2 = 0$ .We have

$\alpha_1 = 2\alpha_2$ .Finally solving with (iii) , we get $\alpha_3 = 8\alpha_2$

Hence we have $\alpha_1 = 2\alpha_2$ and $\alpha_3 = 8\alpha_2$

Which shows that $\{x^2+ 4 , -2x^2 - 8x , x - 1\}$ is a linearly dependent set

Example 4.3.7 :

Recall that

$S= \{(1,1, 0) , (1,1,1), (0,1,-1)\}$ is a linearly independent set

$S^1= \{(1,1,0) , (1,1, 1)\}$ is a linearly independent set

$S= \{(-1,3) , (2,-6)\}$ is a linearly dependent set

$S^1= \{(-1,3) , (2,-6) , (1,4)\}$ is a linearly dependent set

Remark :

Let S be a finite set of vectors in a vector space V .

a.If S is a linearly independent set and $S^{\prime} \subseteq S$, then S^1 is also a linearly independent set

b. If S is a linearly dependent set and $S \subseteq S^{\prime}$, then $S^{\prime}$ is also a linearly dependent set

c. If $0 \in S$, then S is a linearly dependent set .

d.If any element of S is a scalar multiple of the other then S is a linearly dependent set.

Example 4.3.8 . In the plane, any three or more vectors form a linearly dependent set, whereas any set consisting of one nonzero vector or any set consisting of two non-collinear vectors is linearly independent. The same holds in $\mathbb{R}^2$. In $\mathbb{R}^3$ four or more vectors are linearly dependent, whereas any two non-collinear vectors or any three non-coplanar vectors are linearly independent.

Theorem 4.3.1: Let V be a vector space over field F.Let $v_1 ,v_2 , . . . v_n$ be linearly independent elements of V .Let $\alpha_1 , \alpha_2 , . . . , \alpha_n , \beta_1 \ \beta_2 , . . . , \beta_n \in F$ such that $\alpha_1 v_1 + \alpha_2 v_2 + . . . + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + . . . + \beta_n v_n$ then $\alpha_i = \beta_I \quad i = 1,2, \ldots\ldots n$

Proof:

Suppose that

$\alpha_1 v_1 + \alpha_2 v_2 + . . . + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + . . . + \beta_n v_n$

$\Rightarrow (\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + . . . (\alpha_n - \beta_n)v_n = 0$

$\Rightarrow \alpha_1 - \beta_1 = 0 , \alpha_2 - \beta_2 = 0 , . . . ,\alpha_n - \beta_n = 0$

$\Rightarrow \alpha_i = \beta_I \quad i = 1,2, \ldots\ldots n$

4.4 Bases and dimensions of vector space

We have an intuitive idea of dimension.For instance , we think of lines as one dimensional, planes as two dimensional , and XYZ space as three dimensional.In order to make the idea of dimension precise , rather than intuitive , we need to introduced two concepts –spanning and independence . For example three vectors lying along the x,y and z axes respectively will span xyz space.If certain vectors both span a vector space V and are independent , then they provide a minimum collection of directions for movement in V , and their number in V's dimension.

A vector space may have many generating sets. For instance the sets A= {(1,0),(0,1)} and B= {(1,0),(0,1),(2,1)} are generating sets for $\Re^2$ Infact $\Re^2$ itself a generating set of $\Re^2$ .There is some repetition in B and $\Re^2$ in the sense that some proper subsets of these also generate $\Re^2$ where as A donot have this repetition .We study the latter type of generating sets in the sequel.

Bases

Definition 4.4.1: Let V be a vector space over a field F .The set $\{v_1, v_2, \ldots v_n\}$ of n vectors of V forms **a basis** for V if

B1. $v_1, v_2, \ldots v_n$ Span V and

B2. $\{v_1, v_2, \ldots v_n\}$ is linearly independent

Example 4.4.1: Let V = $\Re^3$ over $\Re$

Show S = {(1,0,0) , (0,2,0) ,(0,0,5) } forms a basis for V .

Solution

To show that S is a basis we have to show that S is a.Linearly independent set and Spans V

Suppose $\alpha_1(1,0,0) + \alpha_2(0,2,0) + \alpha_3,0,0,5) = 0$

$\Rightarrow \alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$

Hence S is a.Linearly independent set

Let $v \in V \Rightarrow v = (x,y,z)$

$$(x,y,z) = x(1,0,0) + y/2\,(0,2,0) + z/5\,(0,0,5)$$

Hence S = {(1,0,0) , (0,2,0) ,(0,0,5) } forms a basis for V

Example 4.4.2:.Let V = $P_2(x) = \{a_2x^2 + a_1x^1 + a_0 : a_i \in \Re \text{ for } i = 0,1,2\}$ over $\Re$.

Let S = $\{2, x\text{-}2, (x-2)^2\}$ Show that it is a basis

Solution

To show that S is a basis we have to show that S is a.Linearly independent set and Spans V.

$2\alpha_1 + \alpha_2(x-2) + \alpha_3(x-2)^2 = 0$

$\Rightarrow \alpha_3 x^2 + (\alpha_2 - 4\alpha_3)x + (4\alpha_3 - 2\alpha_2 + 2\alpha_1) = 0$

$\Rightarrow \alpha_3 = 0 \Rightarrow \alpha_2 = 0 \Rightarrow \alpha_1 = 0$

Span $P_2(x)$

$a_2x^2 + a_1x^1 + a_0 = \alpha_1(x-2)^2 + \alpha_2(x-2) + 2\alpha_3$

$\Rightarrow \alpha_1 = a_2$, $-4\alpha_1 + \alpha_2 = a_1$, $2\alpha_3 = a_0$

$\Rightarrow \alpha_1 = a_2$, $\alpha_2 = a_1 + 4a_2$, $\alpha_3 = a_0/2$

$a_2x^2 + a_1x^1 + a_0 = a_2(x-2)^2 + (a_1 + 4a_2)(x-2) + a_0/2$ (2)

Hence $\{2, x\text{-}2, (x-2)^2\}$forms a basis for V

Theorem 4.4.1 : Let V be a vector space
If $\{v_1, v_2, \ldots v_n\}$ generates V and $\{w_1, w_2, \ldots w_m\}$is linearly independent , then $m \le n$

Proof : We prove by contradiction.

Suppose not , $m > n$

$w_i \neq 0$, $i = 1,2, \ldots\ldots m$ otherwise linearly dependent

Since $\{v_1, v_2, \ldots v_n\}$ generates V , every element of V can be written as a linear combination of v_i's, in particular w_1 ,

$w_1 = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ not all α_i's are zero because

$w_1 \neq 0$

Suppose $\alpha_1 \neq 0$

$\Rightarrow \alpha_1^{-1} w_1 = v_1 + (\alpha_1^{-1}\alpha_2)v_2 + (\alpha_1^{-1}\alpha_3)v_3 + \ldots + (\alpha_1^{-1}\alpha_n)v_n$

$\Rightarrow v_1 = \alpha_1^{-1} w_1 + (-\alpha_1^{-1}\alpha_2)v_2 + (-\alpha_1^{-1}\alpha_3)v_3 + \ldots + (-\alpha_1^{-1}\alpha_n)v_n$

$\Rightarrow v_1 \in [w_1, v_2, \ldots v_n]$

$\Rightarrow [v_1, v_2, \ldots v_n] = [w_1, v_2, \ldots v_n]$

Similarly

$w_2 = \beta_1 w_1 + \beta_2 v_2 + \ldots + \beta_n v_n$ not all β_i's are zero because

$w_2 \neq 0$

If $\beta_2 = \beta_3 = \ldots = \beta_n = 0$, then $\beta_1 \neq 0$, $w_2 \neq 0$

$\Rightarrow w_2 + (-1)\beta_1 w_1 = 0$

$\Rightarrow (-1)\beta_1 w_1 + 1w_2 + 0\, w_3 \ldots + 0\, w_m = 0$

$\Rightarrow \{w_1, w_2, \ldots, w_m\}$ are linearly dependent which is a contradiction

$\Rightarrow \exists \beta_i \quad i \neq 0$ such that $\beta_i \neq 0$ .Let it be β_2 .

$w_2 = \beta_1 w_1 + \beta_2 v_2 + \ldots + \beta_n v_n$

$\Rightarrow \beta_2^{-1} w_2 = (\beta_2^{-1}\beta_1)w_1 + v_2 + (\beta_2^{-1}\beta_3)v_3 + \ldots\ldots + (\beta_2^{-1}\beta_n)v_n$

$\Rightarrow v_2 = (-\beta_2^{-1}\beta_1)w_1 + \beta_2^{-1} w_2 + (-\beta_2^{-1}\beta_3)v_3 + \ldots\ldots + (-\beta_2^{-1}\beta_n)v_n$

$\Rightarrow v_2 \in [w_1, w_2, v_3 \ldots, v_n]$

$\Rightarrow [v_1, v_2, \ldots v_n] = [w_1, v_2, \ldots v_n]$

$= [w_1, w_2, v_3 \ldots, v_n]$

Continuing in this manner ,

$[v_1, v_2, \ldots v_n] = [w_1, w_2, \ldots\ldots w_n] = V$

Since $m > n$ and $w_{n+1} \in V$ and $w_{n+1} \neq 0$

Since $\{v_1, v_2, \ldots v_n\}$ generates V and

$V = [v_1, v_2, \ldots v_n] = [w_1, w_2, \ldots\ldots w_n]$

$w_{n+1} = a_1 w_1 + a_2 w_2, \ldots\ldots + a_n w_n$ not all a_i's are zeroes

since $w_{n+1} \neq 0$

$\Rightarrow a_1 w_1 + a_2 w_2 \ldots + a_n w_n + (-1)\, w_{n+1} + 0\, w_{n+2} + 0 w_{n+3} + \ldots + 0\, w_m = 0$

$\Rightarrow \{w_1, w_2, w_3 \ldots, w_m\}$ is a linearly dependent set.. Which is a contradiction.

Corollary 4.4.2: Let V be a vectorspace and $\{v_1, v_2, \ldots v_n\}$ be a basis for V .then all other basis of V have n elements.

Proof:

Let $\{w_1, w_2, w_3 \ldots, w_m\}$ be another basis of V

i. Since $\{v_1, v_2, \ldots v_n\}$ basis of V and $\{w_1, w_2, w_3 \ldots, w_m\}$ is linearly independent, $m \leq n$.

ii. Since $\{w_1, w_2, w_3 \ldots, w_m\}$ is a basis of V and $\{v_1, v_2, \ldots v_n\}$ is linearly independent, $n \leq m$ From (i) and (ii), $m = n$

Example 4.4.3: Finding a basis contained in a finite generating set $\{v_1, v_2, \ldots v_n\}$

Let $U = \{(x,y,z,w) \in \Re^4 : 2x+z + w = 0\}$

Let A= {(1,0, 1,-3),(0,0,1,-1),**(**4,0, 1,-9),**(**0,1,0,0) }.Find a basis for **U.**

Solution

Take B = {(1,0, 1,-3)} .Since (1,0, 1,-3) $\neq$ (0,0,0,0),B is a linearly independent set.

(0,0,1,-1) $\notin$[(1,0,1,-3)](because one is not a scalar multiple of the other.)

C= {(1,0, 1,-3), (0,0,1,-1) } is a linearly independent set.

Is (4,0, 1,-9) $\in$[(1,0, 1,-3), (0,0,1,-1)]

Yes since (4,0, 1,-9) = 4(1,0, 1,-3) +-3(0,0,1,-1)

So ,[(4,0, 1,-9),(1,0, 1,-3), (0,0,1,-1)] = [(1,0, 1,-3), (0,0,1,-1)]

Is (0,1,0,0) $\in$[(1,0, 1,-3), (0,0,1,-1)]? No (verify)

therefore , {(1,0, 1,-3),(0,0,1,-1),(0,1,0,0) } is a linearly independent set.

Since the dimension of U is 3 and {(1,0, 1,-3),(0,0,1,-1),(0,1,0,0) } is a linearly independent set , {(1,0, 1,-3),(0,0,1,-1) , (0,1,0,0) } is a basis for U.

Activity 4.4.1 Find a basis for the solution set U to the system

$$x + y+5z+2w =0$$
$$x \;-\; 2z+4w = 0$$

Definition 4.4.2: Let V be a vector space .The number of elements in a basis of V is called the **dimension of V** , denoted by dimV.

Remark : If dim V is finite , then we say V is finite dimensional.. otherwise V is infinite dimensional.

We call a vector space finite dimensional if it has a finite basis . Clearly any vector space with finite generating set A is finite dimensional since A contains a minimal generating set.From now on we will consider only finite –dimensional vector spaces unless otherwise stated.

If V = {0} .it has no basis , then we say Dim V = 0

Example 4.4.4 . : Let V = $\Re^2$ over $\Re$

{(1,0),(0,1)} is a basis of V. Dim V = 2

Example 4.4.5 Let V = $P_2(x) = \{ a_3x^3 + a_2x^2 + a_1x^1 + a_0 : a_i \in \Re \}$ over $\Re$ $\{ x^3, x^2, x, 1 \}\}$ is a basis of V. Dim V = 3

Activity 4.4.2: Find the dimension of V where V = $\Re^3$ over$\Re$.Justify your reasons.

Example 4.4.6 Find the dimension of V where V = $P_2(x) = \{a_2x^2 + a_1x^1 + a_0 : a_i \in \Re$ for i =0,1,2 } over $\Re$.Justify your reasons.

Solution

Since $\{2, x-2, (x-2)^2\}$ forms a basis for V, Dimension of V is 3

Definition 4.4.3 : Let $\{v_1, v_2, \ldots v_n\}$ be a set of elements of a vector space V. Let $r \in N$ such that $r \leq n$.We say that $\{v_1, v_2, \ldots v_r\}$ is a maximal subset of linearly independent elements if $v_1, v_2, \ldots v_r$ are linearly independent and if in addition given any v_i $i > r$, the elements $v_1, v_2, \ldots v_r, v_i$ are linearly dependent.
Note: The above definition says that Let V be a vectorspace .A linearly independent subset $W = \{v_1, v_2, \ldots v_n\}$ of V is said to be a maximal set of linearly independent elements if any subset U of V such that $W \subset U$, then U is linearly dependent.

Example 4.4.7: Let $V = \Re^3$ over $\Re$
W = {(4,0,0) , (0,3,0) ,(0,0,-5) }.Show that W is a maximal set of linearly independent elements
Solution
Let $U \subseteq V$ such that $W \subset U \Rightarrow \exists u \in U$ such that $u \notin W$
Let U = {(4,0,0) , (0,3,0) ,(0,0,-5) , (a,b,c) }
We have to show that U is linearly dependent
(a,b,c) = a/4(4,0,0) + b/3(0,3,0) + c/-5 (0,0,-5)
1(a,b,c) + - a/4(4,0,0) +- b/3(0,3,0) + c/5 (0,0,-5) = (0,0,0)
$\Rightarrow$ U is linearly dependent
Hence W is a maximal set of linearly independent elements

Theorem 4.4.3: Let V be a vector space and $\{v_1, v_2, \ldots v_n\}$ be a maximal set of linearly independent elements in V .Then $\{v_1, v_2, \ldots v_n\}$a basis for V.
Proof : We have to show that$\{v_1, v_2, \ldots v_n\}$ generates V
Let $v \in V$, $\{v_1, v_2, \ldots v_n, v\}$is a linearly dependent set
$\exists$ $\alpha_1, \alpha_2, \ldots + \alpha_n, \alpha$ not all zero such that
$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n + \alpha v = 0$
$\alpha \neq 0$ if $\alpha = 0$, then $\{v_1, v_2, \ldots v_n\}$ is linearly dependent.
$\alpha \neq 0 \Rightarrow \alpha v = (-\alpha_1) v_1 + (-\alpha_2 v_2) + \ldots + (-\alpha_n) v_n$
$\Rightarrow v = (-\alpha_1/\alpha) v_1 + (-\alpha_2/\alpha) v_2 + .. + (-\alpha_n/\alpha) v_{1n}$
$\Rightarrow \{v_1, v_2, \ldots v_n\}$ generates V

Example 4.4.8: W = {(4,0,0) , (0,3,0) ,(0,0,-5) } is a basis of $\Re^3$ Since W is a maximal set of linearly independent elements

Theorem 4.4.4: Let V be a vector space and dim V = n .If $\{v_1, v_2, \ldots v_n\}$ is a linearly independent subset of V.then $\{v_1, v_2, \ldots v_n\}$ a basis for V.
Proof : Since dim v = n , any basis of V has n elements. let
$\{v_1, v_2, \ldots v_n\}$ be a linearly independent subset of V.
$v_1, v_2, \ldots v_n, v$ are linearly dependent .otherwise the number of elements in a basis (generators).hence $\{v_1, v_2, \ldots v_n\}$ is a maximal set of linearly independent vectors. By the above theorem $\{v_1, v_2, \ldots v_n\}$a basis for V.

Corollary 4.4.5: Let V be a vector space and W is a subspace of V .If dim V = dim W , then V = W

Proof : Let dim W = n .let $\{v_1, v_2, \ldots v_n\}$ be a basis of W . Then $[v_1, v_2, \ldots v_n] = W$

Since dim V = n and $\{v_1, v_2, \ldots v_n\}$ is a linearly independent subset of V , $\{v_1, v_2, \ldots v_n\}$ a basis for V by the above theorem $\Rightarrow [v_1, v_2, \ldots v_n] = V \Rightarrow V = W$

Theorem 4.4.6: Let V be a vector space and dim V = n and let W be a subspace of V.

(ii) $W \neq [0]$ has a basis

(iii) Dim W $\leq$ dim V = n

Proof : Since $W \neq [0]$, $\exists\ w_1 \neq 0$ such that $w_1 \in W$

a. If $W = [w_1]$, then $\{w_1\}$ is a basis of W

b. If $W \neq [w_1]$, then $\exists\ w_2 \in W$ such that $w_2 \notin [w_1]$

Since w_2 is not a scalar multiple of w_1 , $\{w_1, w_2\}$ is a linearly independent set.

If $[w_1, w_2]$, then $\{w_1, w_2\}$ is a basis of W.

Otherwise proceeding in this manner , we can find a linearly independent set $\{w_1, w_2, \ldots w_m\}$ such that

$W = [w_1, w_2, \ldots w_m]$ i.e $\{w_1, w_2, \ldots w_m\}$ is a basis of W.

At most m = n(the maximum is when m = n

Theorem 4.4.7(Extension of a linearly independent set in to a basis) Let V be a vector space and dim V = n . Let $v_1, v_2, \ldots v_r$ be linearly independent elements of V .Then there exists $v_{r+1}, v_{r+2}, \ldots v_n \in V$ such that $\{v_1, v_2, \ldots v_r\ v_{r+1}, \ldots v_n\}$ is a basis of V.

Proof : If r < n , then by definition of dimension $v_1, v_2, \ldots v_r$ cannot form a basis of v and hence can not generate V. Hence there exists $v_{r+1} \in V$ such that $v_{r+1} \notin [v_1, v_2, \ldots v_r]$.Then

$\{v_1, v_2, \ldots v_r, v_{r+1}\}$ is a linearly independent set.proceeding in this manner $v_{r+1}, v_{r+2}, \ldots, v_m$ such that

$v_1, v_2, \ldots v_r\ v_{r+1}, \ldots v_m$ are linearly independent. By the above theorem $m \leq n$ .If we take m to be maximal, then

$\{v_1, v_2, \ldots v_n\}$ is a linearly independent set.

$v_{m+1} \in V$ such that $v_{m+1} \notin [v_1, v_2, \ldots v_m]$ and hence

$v_1, v_2, \ldots v_{m+1}$ are linearly independent contradicting that m is maximal .Hence $\{v_1, v_2, \ldots v_n\}$ is a basis of V.

Example 4.4.9: Let V = $\Re^2$

Let W = {(1,3)} Extend W to a basis of V

Solution $(1,2) \notin [(1,3)]$

Hence U = {(1,2),(1,3)} a linearly independent set and maximal . Hence U is a basis.

Example 4.4.10 : Let $U = \{(x,y,z,w) \in \Re^4 : 2x+z+w=0\}$

Let A= {(1,0,-1,-1)} Extend A to a basis for **U.**

Solution

Check that B = {(0,1,0,0),(1,2,-1,-1),(1,0,0,-2)} is a basis for U

$(1,0,-1,-1) \in U$ since it satisfies $2x+z+w=0$

Hence $A \subseteq U$.A is a linearly independent set since $(1,0,-1,-1) \neq (0,0,0,0)$.
$(0,1,0,0) \notin [(1,0,-1,-1)]$
Set $A_1 = \{(1,0,-1,-1), (0,1,0,0)\}$
Next we check that whether or not
$(1,2,-1,-1) \in [(1,0,-1,-1), (0,1,0,0)]$
Since $(1,2,-1,-1) = (1,0,-1,-1) + 2(0,1,0,0)$,
$(1,2,-1,-1) \in [(1,0,-1,-1), (0,1,0,0)]$
Next we check that whether or not
$(1,0,0,-2) \in [(1,0,-1,-1), (0,1,0,0)]$.
Since $(1,0,0,-2) \notin [(1,0,-1,-1), (0,1,0,0)]$, $\{(1,0,-1,-1), (0,1,0,0),(1,0,0,-2)\}$ is a linearly independent set.Since the dimension of $[(1,0,-1,-1), (0,1,0,0),(1,0,0,-2)]$ is 3 which is equal to the dimension of U,we conclude that $\{(1,0,-1,-1), (0,1,0,0),(1,0,0,-2)\}$ is an extension of A to a basis for U.

Theorem 4.4.10: Given n vectors in $\Re^n$, let A be a matrix whose columns are the given vectors.
1. If rank of A = n , the vectors are a basis for $\Re^n$
2.If rank of A < n , the vectors are not a basis for the vectors are a basis for $\Re^n$

Corollary 4.4.11: if we have n independent vectors in $\Re^n$, then they form a basis for $\Re^n$
Proof: We form a matrix A whose columns are the n vectors.Since the vectors are independent , the rank of A is n.By the preceeding theorem, the vectors must be a basis for $\Re^n$

Example 4.4.11:
Show S = {(1,1,1) ,(0, 1,1) ,(0,1,-1) forms a basis for V = $\Re^3$.
Solution
Suppose
$(0,0,0) = \alpha_1(1,1,1) + \alpha_2 (0, 1,1) + \alpha_3 (0,1,-1)$
$0 = \alpha_1$
$0 = \alpha_2 + \alpha_3$
$0 = \alpha_2 - \alpha_3$
Solving simultaneously $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$
S = {(1,1,1) ,(0, 1,1) ,(0,1,-1)is a linearly independent set and maximal .Hence S is a basis.
Or to show S generates $\Re^3$.Let $(a,b,c) \in \Re^3$
$(a,b,c) = \alpha_1(1,0,0) + \alpha_2 (1,1,0) + \alpha_3 (1,1,1)$
$a = \alpha_1 + \alpha_2 + \alpha_3$
$b = \alpha_2 + \alpha_3$
$c = \alpha_3$
$\alpha_1 = a - b$, $\alpha_2 = b - c$ and $\alpha_3 = c$
$(a,b,c) = (a - b)(1,0,0) + (b - c)(1,1,0) + c (1,1,1)$
S generates $\Re^3$

Example 4.4.12:

Show $S = \{x - 1, x + 1, x^2\}$ is a basis for $V = P_2(x) = \{a_2x^2 + a_1x^1 + a_0 : a_i \in \Re$ for i $=0,1,2\}$ over $\Re$.

Solution

Suppose

$\alpha_1(x - 1) + \alpha_2(x + 1) + \alpha_3 x^2 = 0$

If $x = 1$, $2\alpha_2 + \alpha_3 = 0$

If $x = -1$, $-2\alpha_1 + \alpha_3$

If $x = 0$, $-\alpha_1 + \alpha_2$

Solving simultaneously $\alpha_1 = 0$, $\alpha_2 = 0$ and $\alpha_3 = 0$

$S = \{x - 1, x + 1, x^2\}$ is a linearly independent set and maximal . Hence S is a basis.

Example 4.4.13:

Let $V = \Re^3$ over $\Re$. $U = \{(a,b,c) : a = b \text{ and } 2b = c\}$

Find basis and dimension of U

Solution :

$U = \{(a,b,c) : a = b \text{ and } 2b = c\}$

$= \{(b,b,2b) : b \in \Re\}$

$= \{b(1,1,2) : b \in \Re\}$

$= [(1,1,2)]$

Dim U = 1 and $\{(1,1,2)\}$ is a basis of U.

Activity 4.4.3:

Let $V = \Re^3$ over $\Re$.

$U = \{(a,b,c) : a + b + c = 0\}$ Find basis and dimension of U

Example 4.4.14:

Let $V = \Re^3$ over $\Re$.

$U = \{(a,b,c) : a = b + 2c\}$ Find a basis and dimension of U.

Solution :

$U = \{(a,b,c) : a = b + 2c\}$

$= \{(b + 2c, b, c) : b,c \in \Re\}$

$= \{(b, b, 0) + (2c, 0, c) : b,c \in \Re\}$

$= \{b(1,1,0) + c(2,0,1) : b,c \in \Re\}$

$= [(1,1,0), (2,0,1)]$

Basis for U is $\{(1,1,0), (2,0,1)\}$ Dim U = 2

Activity 4.4.4

Show that (1,0,-1),(0,1,0) is a basis for $\{(a,b,-a) : a,b \in \Re\}$

Show that (-4,2,1,0),(1,3,0,1) is a basis for

$\{(-4a+b, 2a+3b, a, b) : a,b \in \Re\}$

Show that {(1,0, 1),(0,1,-2),(4,2,0)} is not a basis for

$\{(a+4c, b+2c, a-2b) : a,b,c \in \Re\}$

Show that $\left\{\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 5 \end{pmatrix}\right\}$ is a basis for M_{2x2}

Example 4.4.15: Let $V = M_{2x2}$. Let W_1 be the set of all symmetric matrices in V, and W_2 be the set of all skew- symmetric matrices in V.
Show that

a) W_1 and W_2 are subspaces of V.
b) $W_1 \cap W_2 = \{0\}$
a) $V = W_1 + W_2$

Solution

a)

i) $0^t = 0 = -0$. Hence $0 \in W_1$ and $0 \in W_2$

ii) Let $A \in W_1$ and $\alpha \in \Re$. $(\alpha A)^t = \alpha A^t = \alpha A$ since A is symmetric .

Therefore αA is symmetric

Let $A \in W_2$ and $\alpha \in \Re$. $(\alpha A)^t = \alpha A^t = \alpha(-A)$ since A is skew- symmetric .
$= -(\alpha A)$

Therefore αA is skew- symmetric

iii) Let $A,B \in W_1$. $A^t = A$ and $B^t = B$ since A and B are symmetric
$(A + B)^t = A^t + B^t = A + B$.

Therefore A + B is symmetric

Let $A,B \in W_2$. $A^t = -A$ and $B^t = -B$ since A and B are skew-symmetric
$(A + B)^t = A^t + B^t = -A + -B = -(A + B)$

Therefore A + B is Skew- symmetric

b.Now we will show $W_1 \cap W_2 = \{0\}$

Let $A \in W_1 \cap W_2$.$A \in W_1$ and $A \in W_2$. $A \in W_1$ implies that $A^t = A$.
Moreover $A \in W_2$ implies that $A^t = -A$
$A^t = A = -A$ implies $A = -A$.Thus $A = 0$
Therefore $W_1 \cap W_2 = \{0\}$

c.Let $B \in V$. We have

$B = \frac{1}{2}(B+B^t)+\frac{1}{2}(B-B^t)$. Let $B_1 = \frac{1}{2}(B+B^t)$ and $B_2 = \frac{1}{2}(B-B^t)$

$B_1^t = \frac{1}{2}(B+B^t)^t = \frac{1}{2}(B^t+(B^t)^t) = \frac{1}{2}(B^t+B) = B_1$ and

$B_2^t = \frac{1}{2}(B-B^t)^t = \frac{1}{2}(B^t-(B^t)^t) = \frac{1}{2}(B^t-B) = = -\frac{1}{2}(B-B^t) = -B_2$.

From the above we can conclude that B_1 is symmetric and B_2 is skew symmetric.
That is $B_1 \in W_1$ and $B_2 \in W_2$.

$B_1 + B_2 = \frac{1}{2}(B+B^t) + \frac{1}{2}(B-B^t) = B$

Therefore , $V = W_1 + W_2$

Example 4.4.16:

Let $V = \Re^3$ over $\Re$.
Let $U = \{(a,b,c) : c = 2b \}$ and $W = \{(a ,b ,c) : a + c = b \}$
Find U + W

Solution

$U + W = \{u + w : u \in U \text{ and } w \in W\}$

$= \{(a,b,c) + (a_1, b_1, c_1) : c = 2b \text{ and } a_1 + c_1 = b_1\}$

$= \{(a,b,2b) + (a_1, a_1 + c_1, c_1) : a, b, a_1, c_1 \in \Re\}$

$= \{(a,0,0) + (0,b,2b) + (a_1, a_1, 0) + (0, c_1, c_1) : a, b, a_1, c_1 \in \Re\}$

$= \{a(1,0,0) + b(0,1,2) + a_1(1,1,0) + c_1(0,1,1) : a, b, a_1, c_1 \in \Re\}$

$= [(1,0,0), (0,1,2), (1,1,0), (0,1,1)]$

Basis for U + W

Take (1,0,0)

$(0,1,2) \notin [(1,0,0)]$

Is $(1,1,0) \in [(1,0,0), (0,1,2)]$?

$(1,1,0) = \alpha(1,0,0) + \beta(0,1,2)$

There does not exist α and β . Hence $(1,1,0) \notin [(1,0,0), (0,1,,2)]$

So { (1,1,0) , (1,0,0), (0,1,,2)} is a linearly independent set.

Since U + W is a subspace of $V = \Re^3$

Dim $(U+W) \leq \dim V = 3$. Hence dim (U + W)= 3

Therefore { (1,1,0) , (1,0,0), (0,1,,2)} is a basis for U + W

Example 4.4.17:

Let $V = \Re^2$ over $\Re$.

Let U = [(1,3)] and W = [(2,1)].Find $U \cap W$

Solution :

$U \cap W = \{(a,b) : (a,b) \in U \text{ and } (a,b) \in W\}$

$= \{(a,b) : (a,b) = \alpha(1,3) \text{ and } (a,b) = \beta(2,1) : \alpha, \beta \in \Re\}$

$= \{(a,b) : (a,b) = \alpha(1,3) = \beta(2,1) : \alpha, \beta \in \Re\}$

$= \{(0,0)\}$

Activity 4.4.5:

Let $V = \Re^3$ over $\Re$.

U = {(a,b,c) : b =3c } and W={(a,b,c) : a + b =c } Find a basis and dimension of $U \cap W$.

Example 4.4.18:

Let $V = \Re^3$ over $\Re$.

Let W = [(1,0,2) , (2,-1,1) , (1,-1,-1)] Find a basis and dimension of W.

Solution :

Consider (1,0,2).

Is $(1,0,2) \in [(2,-1,1)]$? No Since (1,0,2) is not a scalar multiple of (2,-1,1)

Next consider (1,0,2) and (2,-1,1)

Is $(1,-1,-1) \in [(1,0,2), (2,-1,1)]$?

$(1,-1,-1) = \alpha(1,0,2) + \beta(2,-1,1)$

$\Rightarrow \alpha = -1$ and $\beta = 1$

Therefore $(1,-1,-1) \in [(1,0,2), (2,-1,1)]$

Therefore W = [(1,-1,-1),(1,0,2), (2,-1,1)] = [(1,0,2), (2,-1,1)]

Dim W = 2 and {(1,0,2), (2,-1,1) } is a basis for W.

Activity 4.4.6:

Determine the dimensions of the subspaces

A.The subspace of $\Re^4$consisting o of all 4 tuples that satisfy

$2x + 3y - z – 2w = 0$

b.The subspace of solutions of the system

$2x + y = 0$

$x + 3y = 0$

c. The subspace of solutions of the system

$x + 2y + 3z = 0$

$2x – y - z = 0$

$-x + 3y + 2z = 0$

d.All polynomials of the form $ax^4 + bx^2 + c$

Example 4.4.19: Let V be a vector space. We have two subsets of V, S_1 and S_2, having respectively 13 and 15 elements.

a.If S_1 and S_2 are both linearly independent, Find the dimension of V.

b.If S_1 generates V then S_2 is linearly independent.

Solution

a. The dimension of V is at least 15.

b. False

Definition 4.4.4: Let V be a vector space and $\{v_1, v_2, \cdots, v_n\}$ be a basis for V. Let $u \in$ V.If $u = a_1v_1 + a_2v_2 + \cdots + a_nv_n$, then $(a_1, a_2, \cdots, a_n)$ is called the **coordinate vector** of u with respect to the basis $\{v_1, v_2, \cdots, v_n\}$.

Example 4.4.20 Find the coordinates of (4,3) with respect to {(1,2), (2,-1)}.

Solution: {(1, 1), (2, -1)}is a basis of R^2 .

$$a(1, 2) + b(2,-1) = (4,3) \Rightarrow \begin{cases} a + 2b = 4 \\ a - b = 3 \end{cases}$$

which implies a = 2 and b = 1

Thus, (2,1) is the coordinate vector of (4,3) with respect to the basis {(1, 1), (2, -1)}.

Activity 4.4.7:

Find the coordinates of the following vectors with respect to the basis (1,0,1) ,(0,2,0),(1,0,-3) for R^3

a. (a,b,c)

b. (0,2,-4)

Activity 4.4.8

Find the coordinates of (1,3,0) with respect to the basis {(1,0,0) ,(0,3,0) ,(0,0,4)}

Solution

1(1,0,0) +1 (0,3,0) +0(0,0,4) = (1,1,0)

Example 4.4.21 The vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ form a basis for R^2, the vectors e_1, e_2 and e_3 form a basis for R^3 and, in general, the vectors $e_1, e_2, \cdots, e_n$ form a basis for R^n. Each of these sets of vectors is called the natural basis or standard basis for R^2, R^3, and R^n, respectively.

Activity 4.4.9 . Show that the set

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for the vector space V of all 2 x 2 matrices.

Exampele 4.4.22. Find the coordinates of $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ with respect to the basis

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

Solution

(2,0,1,0)

Activity 4.4.10

Determine whether or not the following sets are a basis for $\left\{ \begin{bmatrix} a & a+b \\ a-b & 0 \end{bmatrix}, a, b \in R \right\}$. of M_{2x2}

a. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

Chapter 2: Matrices, Determinants and Systems of Linear Equations

Introduction

Linear algebra begins with sets of linear equations in several variables. The number of variables may be large, or it may be unspecified. In either case we need an alphabet larger than the ones used in ordinary language. Thus we invent synthetic ones. is This unit provides an introduction to the basic concepts of matrices and system of linear equations. You will develop operations on matrices and will work with matrices according to the rules they obey; this will enable you to solve system of linear equations and to do other computational problems in a fast and efficient manner.

The concept of matrices has a wide range of application in our life. One of the main applications is to write systems of linear equation in a compact form, which enables you to solve them very easily and efficiently.

Objectives

After successful completion of this unit, you will be able to:

- Define a matrix
- Apply operations on matrices
- Describe properties of matrices
- Reduce matrices into reduced row echelon form
- Find the inverse of a matrix by using elementary row operations
- Solve system of linear equations by using Guassian elimination
- Find the inverse of a matrix.

2.1. Definition of matrices.

Definition 2.1.1: An *mxn* (m by n) matrix A is a rectangular array of *mn* real (or complex) numbers arranged in *m* horizontal rows and *n* vertical columns:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Remark : A matrix is a rectangular array of numbers, or symbols , like the above.The rows of a matrix are its horizontal lines: for example the second row of the matrix A has entries a_{21} ,a_{22} , ….,a_{2n} Note that the first subscript names the row from which the entry is taken .The columns of the matrix are its vertical lines; the first column of A has entries a_{11} ,a_{21} , ….,a_{m1} .The second subscript names the column from which the entry comes.
Notation: When the entries of a matrix are named by means of subscripts, the matrix can be abbreviated by telling the range of the subscripts.
The above matrix A will be written as A= $[a_{ij}]_{mxn}$ (i =1,2,…,m ; j = 1,2,…,m).

<u>**Example2.1.1**</u>.

$A = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix}$ is a 2x3 matrix with a_{21} = -1 , a_{23} =4

A = [1 0 -2 3] is a 1x4 matrix. with a_{13} = -2

A = [1] is a 1x1 matrix and a_{11} =1

<u>**Activity 2.1.1:**</u> Let A =[a_{ij}] be a matrix $\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$, and B =[b_{ij}] (i = 1,2 ;j = 1,2) with b_{ij} =a_{ji} . Write down B explicitely.

<u>**Definition 2.1.2**</u> :Let A be an *mxn* matrix. If $m = n$, we say that A is a **Square matrix** of order *n*.

<u>**Example 2.1.2**</u> : A= $\begin{pmatrix} 2 & -1 \\ 4 & 7 \end{pmatrix}$ is a 2x2 or square matrix of order 2

A= $\begin{bmatrix} 0 & 1 & 5 \\ 2 & 1 & -1 \\ 1 & 0 & 13 \end{bmatrix}$ is a 3x3 or square matrix of order 3.

<u>**Definition 2.1.3**</u> :The *mxn* zero matrix 0_{mxn} is the matrix with all entries equal to 0.

$$0_{mxn} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Definition 2.1.4:Two *mxn* matrices A= [a_{ij}] and B= [b_{ij}] are said to be **equal** if $a_{ij} = b_{ij}$ $1 \le i \le m$, $1 \le j \le n$, that is, if they have the same size and the corresponding components are equal.

Example 2.1.3: The matrices $\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 5 \\ 2 & 1 & -1 \\ 1 & 0 & 13 \end{bmatrix}$ are not equal since they do not have the same size.

There are three binary matrix operations These are matrix addition , multiplication of a matrix by a scalar and multiplication of two matrices.

2.2. Operations on matrices.

Matrix Addition

Definition 2.2.1: If A=[a_{ij}]and B=[b_{ij}] are *mxn* matrices that is A and B two matrices of the same size, then their sum written **A + B ,** is the matrix obtained by adding corresponding entries of A and B.

That is : If A + B = C, then C is the *mxn* matrix where C= [c_{ij}], such that $c_{ij} = a_{ij} + b_{ij}$ $(1 \le i \le m, 1 \le j \le n)$

Remark : If A and B do not have the same number of rows and columns , their sum is not defined.

Example2.2.1 : Let

$$A = \begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 4 & -1 \\ 7 & 0 & -2 \end{bmatrix}$$

Then

$$A+B = \begin{bmatrix} 0+6 & -1+4 & 4+-1 \\ -3+7 & 2+0 & 5+-2 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 3 \\ 4 & 2 & 3 \end{bmatrix}$$

The Second operation is multiplication of a matrix by a scalar(That is a number)

Scalar Multiplication:

Definition 2.2.2: If A= [a_{ij}] is an *mxn* matrix and α is a scalar, then the scalar multiple of A by α, **αA** is the *mxn* matrix B= [b_{ij}], where $b_{ij} = \alpha a_{ij}$ $(1 \le i \le m, 1 \le j \le n)$.

Remark : B is obtained from A by multiplying each element of A by α.

Example 2.2.2: Let A= $\begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix}$ and

(a) If α= 3 , then αA= 3A= $3\begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 12 \\ -9 & 6 & 15 \end{bmatrix}$.

(b) If α = -1, then αA = -1A = (-1)$\begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -4 \\ 3 & -2 & -5 \end{bmatrix}$.

Definition 2.2.3:

If A and B are *mxn* matrices, then we define

A - B = **A + (-1)B** and we call this the **difference** of A and B.

Example 2.2.3 : Let A= $\begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} 6 & 4 & -1 \\ 7 & 0 & -2 \end{bmatrix}$

Then find A - B

Solution

A - B = A+ (-1)B

(-1)B= $\begin{bmatrix} 6 & 4 & -1 \\ 7 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -6 & -4 & 1 \\ -7 & 0 & 2 \end{bmatrix}$

A- B = A+ (-1)B = $\begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} -6 & -4 & 1 \\ -7 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -6 & -5 & 5 \\ -10 & 2 & 7 \end{bmatrix}$

Activity 2.2.1

LetA= $\begin{bmatrix} 2 & -1 & -8 \\ -3 & 12 & 3 \end{bmatrix}$,B= $\begin{bmatrix} 1 & 0 & -3 \\ 5 & -1 & -2 \end{bmatrix}$ and C = $\begin{bmatrix} 0 & 1 & 3 \\ -1 & 3 & -2 \\ 0 & 1 & 6 \end{bmatrix}$ B. Then Find

a) A+B
b) A - B
c) A + C
d) A-2B

Next we state several elementary properties of matrix addition and scalar multiplication

Theorem 2.2.1: Properties of Matrix Addition and scalar Multiplication

Let A, B, C can be an *mxn* matrices and $\alpha, \beta \ \varepsilon \ \Re$. Then

M1 . A+B = B+A (Commutativity of Addition)

M2. A+(B+C) = (A+B)+C(Associativity of Addition)

M3. A+0 =0 +A = A
M4. A + (-A) = (-A)+A = 0
M5. α (A+B) = α A+ α B
M6. ($\alpha+\beta$)A = α A+ β A
M7. α (β A) = (α β)A
M8. 1A= A
M9. 0A = **0**

<u>Proof:</u>

Let A=[a_{ij}] ,B=[b_{ij}] and C = [c_{ij}] where $1 \le i \le m, 1 \le j \le n$

M1.A + B = [a_{ij} + b_{ij}] = [$b_{ij}+a_{ij}$] = B + A

M2. A+(B+C) =[a_{ij}] + [$b_{ij}+c_{ij}$]

=[a_{ij} +($b_{ij}+c_{ij}$)]

=[(a_{ij} + b_{ij})+ c_{ij}]= [a_{ij} + b_{ij}]+[c_{ij}] = (A + B) + C

M3. A + 0 = [a_{ij} + 0] = [a_{ij}] = A

M4. A + (-A) = [a_{ij} +(- a_{ij})] = [0] = 0

M5.α(A+B)= α[$a_{ij}+b_{ij}$] = [α a_{ij} +α b_{ij}] =[α a_{ij}] + [α b_{ij}] = αA + αB

M6. ($\alpha+\beta$)A = ($\alpha+\beta$) [a_{ij}]

= [($\alpha+\beta$) a_{ij}]

=[α a_{ij} + β a_{ij}]

=[α a_{ij}] + [β a_{ij}]

= α [a_{ij}] + β [a_{ij}]

= α A+ β A

M7, M8 and M9 are left as an exercise.

<u>Matrix Multiplication</u>

<u>Definition 2.2.4</u> : Let A = [a_{ij}] is an *mxn* matrix and B= [b_{jk}] is an n*xp* matrix, then the **product** of A and B, AB, is the *mx p* matrix C= [c_{ik}], defined by

$$c_{ik} = a_{i1}b_{1k}+a_{i2}b_{2k}+\ldots..+ a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk} \ (1 \le i \le m, 1 \le k \le p).$$

Note that the product of A and B is defined only when the number of columns of A is equal to the number of rows of B .

<u>Example 2.2.4</u> Let A = $\begin{pmatrix} 2 & -1 \\ 4 & 7 \end{pmatrix}$ and B = $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Find A.B

<u>Solution</u>

A is a 2x2 matrix and B is a 2x1 matrix hence multiplication of matrices is defined .

AB is a 2x1 matrix .Let C = (c_{ik}) where $(1 \le i \le 2, 1 \le k \le 1)$ be AB.

C = (c_{ik}) $(1 \le i \le 2, 1 \le k \le 1)$

i =1, k = 1 $c_{11} = a_{11}.b_{11} + a_{12}.b_{21}$ = 2.(-1) + (-1)1 = -3

i =2, k = 1 $c_{21} = a_{21}.b_{11} + a_{22}.b_{21}$ = 4.(-1) +7.1 = 3

Therefore **AB=** $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$

<u>**Example 2.2.5**</u> **Let A =** $\begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & 4 \\ 5 & 1 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & 2 \\ 2 & 1 & 4 \end{bmatrix}$

Find a. AB

b. BA

<u>**Solution**</u>

A is a 3x3 matrix and B is a 3x3 matrix hence multiplication of matrices is defined . AB is a 3x3 matrix .Let C = (c_{ik}) where $(1 \le i \le 3, 1 \le k \le 3)$ be AB.

C = (c_{ik}) $(1 \le i \le 3, 1 \le k \le 3)$

$c_{ik} = a_{i1}.b_{1k} + a_{i2}.b_{2k} + a_{i3}b_{3k}$

i =1, k = 1 , $c_{11} = a_{11}.b_{11} + a_{12}.b_{21} + a_{13}b_{31}$ = 0.1 +(-2)(-2) +(-1)2 =2

i =1 ,k = 2 , $c_{12} = a_{11}.b_{12} + a_{12}.b_{22} + a_{13}b_{32}$ = 0.0 +(-2)1 +(-1)1 =-3

i =1,,k = 3 , $c_{13} = a_{11}.b_{13} + a_{12}.b_{23} + a_{13}b_{33}$ = 0.3 +(-2)2 +(-1)4 =-8

i =2,,k = 1 , $c_{21} = a_{21}.b_{11} + a_{22}.b_{21} + a_{23}b_{31}$ = 1.1 +0(-2) +4.2 =9

i =2,,k = 2 , $c_{22} = a_{21}.b_{12} + a_{22}.b_{22} + a_{23}b_{32}$ = 1.0 +0.1 + 4.1 = 4

i =2,,k = 3 , $c_{23} = a_{21}.b_{13} + a_{22}.b_{23} + a_{23}b_{33}$ = 1.3 +0.2 +4.4 = 19

i =3,,k = 1 , $c_{31} = a_{31}.b_{11} + a_{32}.b_{21} + a_{33}b_{31}$ = 5.1 +1(-2) +1.2 =5

i =3,,k = 2 , $c_{32} = a_{31}.b_{12} + a_{32}.b_{22} + a_{33}b_{32}$ = 5.0 +1.1 + 1.1 = 2 i =3,,k = 3 , $c_{33} = a_{31}.b_{13} + a_{32}.b_{23} + a_{33}b_{33}$ = 5.3 +1.2 +1.4 = 21

Hence

A B = $\begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & 4 \\ 5 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & 2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 2 & -3 & -8 \\ 9 & 4 & 19 \\ 5 & 2 & 21 \end{bmatrix}$

Similarly

BA = $\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & 2 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & 4 \\ 5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 1 & 2 \\ 11 & 6 & 8 \\ 21 & 0 & 6 \end{bmatrix}$

From the above example we can conclude that AB $\neq$BA.

<u>Remark</u> : Matrix multiplication is not commutative.

<u>Activity2.2.2</u>

Let A = $\begin{bmatrix} 0 & -1 & 4 \\ -3 & 2 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 5 \\ 2 & -1 & 1 \end{bmatrix}$

Compute AB

<u>Theorem 2.2.2:</u> Properties of Matrix multiplication

If A, B, and C are compatible matrices under multiplication and addition, then

1. A(BC) = (AB)C
2. A(B+C) = AB + AC
3. (A+B)C = AC + BC

<u>Example 2.2.6</u>

Let A = $\begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix}$, B = $\begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & -3 \end{bmatrix}$ and C = $\begin{bmatrix} 4 & 3 & -2 \\ 0 & -3 & 11 \end{bmatrix}$

Compute

a. B + C
b. A(B + C)
c. AB
d. AC

<u>Solution</u>

a. B+C= $\begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & -3 \end{bmatrix} + \begin{bmatrix} 4 & 3 & -2 \\ 0 & -3 & 11 \end{bmatrix} = \begin{bmatrix} 1+4 & -1+3 & 5+2 \\ 3+0 & 1+-3 & -3+11 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 7 \\ 3 & -2 & 8 \end{bmatrix}$

b. A(B + C) = $\begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix} \begin{bmatrix} 5 & 2 & 7 \\ 3 & -2 & 8 \end{bmatrix} = \begin{bmatrix} 3.5+(-1)3 & 3.2+(-1)(-2) & 3.7+(-1)8 \\ 4.5+5.3 & 4.2+5.(-2) & 4.7+5.8 \end{bmatrix}$

$= \begin{bmatrix} 12 & 8 & 13 \\ 35 & -2 & 68 \end{bmatrix}$

c. AB = $\begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix} \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 3.1+(-1)3 & 3(-1)+(-1)1 & 3.5+(-1)(-3) \\ 4.1+5.3 & 4.(-1)+5.1 & 4.5+5.(-3) \end{bmatrix}$

$= \begin{bmatrix} 0 & -4 & 18 \\ 19 & 1 & 5 \end{bmatrix}$

e. AC = $\begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix} \begin{bmatrix} 4 & 3 & -2 \\ 0 & -3 & 11 \end{bmatrix} = \begin{bmatrix} 12 & 12 & -17 \\ 16 & -3 & 47 \end{bmatrix}$

From the above AB + AC = A(B + C)

Activity 2.2.3

Let A = $\begin{bmatrix} 1 & 9 & -2 \\ -1 & 0 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 2 & -1 & 0 \\ -1 & -2 & 3 \end{bmatrix}$ and C = $\begin{pmatrix} 0 & -1 \\ 2 & 3 \\ 5 & 1 \end{pmatrix}$

Compute

a. A + B
b. (A + B)C
c. AC
d. BC

Activity 2.2.4

Suppose A , B and C be matrices such that

AC = $\begin{bmatrix} 6 & -1 & 0 \\ -4 & 0 & 2 \end{bmatrix}$ and BC = $\begin{bmatrix} 5 & 1 & 2 \\ 3 & 2 & -3 \end{bmatrix}$. Find (A+B)C

Definition 2.2.5: Suppose that A is a square matrix and let n be a positive integer, then we

define $\mathbf{A^n = AA^{n-1}}$

That is $A^2 = AA$, $A^3 = AA^2$, $A^4 = AA^3$ and so on $A^n = AA^{n-1}$

Remark : $A^0 = I$

2.3. Types of matrices

The Transpose of a Matrix

There is another operation on matrices , this one is a unary operation.

Definition 2.3.1 :If A= $[a_{ij}]$ is an *mxn* matrix, then the *nxm* matrix $A^t = [b_{ij}]$,where $b_{ij} = a_{ji}$ $(1 \le i \le m, 1 \le j \le n)$is called the **transpose** of A.

Remark : If A is an m x n matrix , then A^t is an n x m matrix.

Example2.3.1: Let A= $\begin{bmatrix} 1 & 4 \\ 3 & -1 \\ -2 & 2 \end{bmatrix}$. Find A^t

Solution

Since A is an 3 x 2 matrix , A^t is a 2 x 3 matrix.

$A^t = [b_{ij}]$ where $b_{ij} = a_{ji}$ $(1 \le i \le 3, 1 \le j \le 2)$

$b_{11} = a_{11} = 1$, $b_{12} = a_{21} = 3$, $b_{13} = a_{31} = -2$,

$b_{21} = a_{12} = 4$, $b_{22} = a_{22} = -1$, $b_{23} = a_{32} = 2$

Therefore

$$A^t = [b_{ij}] = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ 4 & -1 & 2 \end{pmatrix}$$

Remark : The transpose of A is obtained by interchanging the rows by the columns of A

Activity 2.3.1:

Let A= $\begin{pmatrix} 2 & -1 & 0 & 6 \\ 4 & 3 & 5 & -2 \end{pmatrix}$. Find A^t

Theorem 2.3.1 Properties of Transpose

If α is a scalar and A and B are matrices, then

T1. $(A^t)^t = A$
T2. $(A+B)^t = A^t + B^t$
T3. $(AB)^t = B^t A^t$
T4. $(\alpha A)^t = \alpha A^t$

Remark : By using property T3 , one can show that $(ABC)^t = C^t B^t A^t$ and in general $(A_1 A_2 \ldots A_n)^T = A_n^t \ldots A_2^t A_1^t$

Example2.3.2: Let

$$\text{Let } A = \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & -3 \end{bmatrix}$$

Compute

1. A^t
2. B^t
3. $B^t A^t$
4. $(AB)^t$

Solution

a. $A^t = \begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix}$

b. $B^t = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 5 & -3 \end{bmatrix}$

c. $B^tA^t = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 5 & -3 \end{bmatrix} \begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix} = \begin{bmatrix} 0 & 19 \\ -4 & 1 \\ 18 & 5 \end{bmatrix}$

d. $AB = \begin{pmatrix} 3 & -1 \\ 4 & 5 \end{pmatrix} \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 18 \\ 19 & 1 & 5 \end{bmatrix}$

$$(AB)^t = B^tA^t = \begin{bmatrix} 0 & 19 \\ -4 & 1 \\ 18 & 5 \end{bmatrix}$$

Definition 2.3.2: A square matrix A =(a_{ij}) is called an **identity matrix** if a_{ij} =1 whenever i = j and a_{ij} = 0 whenever $i \neq j$ .It is then denoted by I_n or I when the order is clear from the context.

$$I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 2.3.3 :A square matrix in which all the diagonal elements are 0 is called a **diagonal matrix**..That is A =(a_{ij}) is a diagonal matrix iff a_{ij} = 0 whenever $i \neq j$.

Example 2.3.3: A = $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix

A = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a diagonal matrix

Definition 2.3.4 :A **diagonal** matrix in which all diagonal elements are equal is called a scalar matrix.

Example 2.3.4: A = $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a scalar matrix

A = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a scalar matrix

Remark : Scalar matrices can be written as αI for some scalar α and behaves like a scalar. That is $\alpha I + \beta I = (\alpha + \beta)I$ and
$\alpha I.A = \alpha A$

Definition 2.3.5: A square matrix A =(a_{ij}) is called an **upper triangular** matrix if all the elements below diagonal are 0 , that is a_{ij} = 0 whenever i > j

Definition 2.3.6: A square matrix A =(a_{ij}) is called a **lower triangular** matrix if all the elements above diagonal are 0 , that is a_{ij} = 0 whenever i < j

Definition 2.3.7: A square matrix A =(a_{ij}) is called **triangular** matrix if it is either upper triangular or lower triangular .

Example 2.3.5 : In the following two matrices

A= $\begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 7 \\ 0 & 0 & 8 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 5 & 0 \\ -2 & 9 & 4 \end{bmatrix}$

A is an upper triangular matrix while B is a lower triangular matrix

Definition 2.3.8:A matrix A= [a_{ij}] is called **symmetric** if $A^t = A$.
That is, A is symmetric if it is a square matrix for which $a_{ij} = a_{ji}$ for all *i*,*j*.

Example : $\begin{bmatrix} 2 & -1 & 4 \\ -1 & 3 & 7 \\ 4 & 7 & 8 \end{bmatrix}$ is a symmetric matrix.

Note Symmetry of a matrix refers to symmetry about the principal diagonal

Example 2.3.6:If A and B are symmetric matrices of the same order and α is a scalar then αA and A + B are symmetric matrices.

Solution:

a. $(\alpha A)^t = \alpha A^t = \alpha A$. Hence αA is a symmetric matrix.

b. $(A + B)^t = A^t + B^t = A + B$.Hence A + B is a symmetric matrix.

Remark : AB may not be symmetric.For example

Consider A = $\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & 4 \\ 4 & 1 \end{pmatrix}$ which are symmetric matrices.But AB = $\begin{pmatrix} 10 & 9 \\ 19 & 8 \end{pmatrix}$ is not symmetric.

Definition 2.3.9:A matrix A= [a_{ij}] is called **skew symmetric** if $A^t = -A$.

That is, A is skew symmetric if it is a square matrix for which $a_{ij} = -a_{ji}$ forall *i*, *j*.

Example2.3.7 :A= $\begin{bmatrix} 0 & -3 & 4 \\ 3 & 0 & 5 \\ -4 & -5 & 0 \end{bmatrix}$ is a skew symmetric matrix since $A^t = -A$.

Note: The main diagonal elements of a skew symmetric matrix must all be zero.

Example2.3.8: Let A be any square matrix. Then

a. $A + A^t$ is symmetric

b. $A - A^t$ is skew symmetric

c.A can be written as a sum of a symmetric and skew symmetric matrices.

Solution

a) $(A + A^t)^t = A^t + (A^t)^t = A^t + A$

b) $(A - A^t)^t = A^t - (A^t)^t = A^t - (A^t)^t = A^t - A = -(A - A^t)$.

Hence $A - A^t$ is a skew symmetric matrix.

c) Let B= $\frac{1}{2}(A + A^t)$, C = $\frac{1}{2}(A - A^t)$. Then A = B+C .Moreover since B is symmetric and C is skew symmetric, Any matrix can be written as a sum of symmetric and skew symmetric.

Activity 2.3.2

Give an example of a matrix which is

a) An upper triangular matrix
b) Whose transpose is an upper triangular matrix
c) A lower triangular matrix

d) Whose transpose is a lower triangular matrix
e) Symmetric
f) Skew symmetric

2.4 Elementary Row Operations and Inverse of a matrix.

In this section We shall study some operations called elementary operations which can be used to reduce any given matrix to one with a simple form thereby facilitating the solution of some problems to be solved for the original matrix.

Row echelon form

Definition2.4.1:An *mxn* matrix is said to be in **row echelon form** when it satisfies the following properties:

R1) All rows consisting entirely of zeros, if any, are at the bottom of the matrix.

R2) The leading non zero number in a row is 1.

R3) If two successive rows that do not consist entirely of zeros, then the leading entry of the lower row is farthest to the right than the leading entry of the upper row.

Example 2.4.1: The matrices

$$A=\begin{bmatrix}1&2&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},\ B=\begin{bmatrix}1&3&5&1&4\\0&0&1&0&1\\0&0&0&1&3\end{bmatrix},\ \text{and } C=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

are in row echelon form.

Example 2.4.2 The matrices

$$A=\begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&1&4\end{bmatrix},\ B=\begin{bmatrix}1&0&3&4\\0&1&-2&5\\0&0&2&2\\0&0&0&0\end{bmatrix}\text{and } C=\begin{bmatrix}1&0&5&4\\0&1&2&7\\0&1&3&6\\0&0&0&0\end{bmatrix}$$ are not in row

echelon form. Because in case of Matrix A Condition R1 failed , Matrix B Condition R2 failed and Matrix C Condition R2 failed

Activity 2.4.1

Give different examples of matrices which are in row echelon form.

Definition 2.4.2 An *mxn* matrix is said to be in reduced row echelon if its in the **row echelon form** and the elements above the first non zero entry any row are 0(the elements below are already 0).

Example 2.4.3 : Consider

$$A=\begin{bmatrix}1 & 2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\ B=\begin{bmatrix}1 & 3 & 5 & 1 & 4\\ 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 1 & 3\end{bmatrix},$$

A is in reduced row echelon form while B is not in reduced row echelon form.

Elementary operation

Definition2.4.3:An **elementary row operation** on a matrix is any one of the following operations:

a) Interchange two rows .
b) Multiply a row by a non zero constant
c) Replacing a row by the sum of that row and a scalar multiple of another row.

Notation :

$A_{i,j}$ Interchanging the i th with j th row of A.
$A_{i,} \to \alpha A_{i,}$ Multiply the i th row of A by α.
$A_{i,} \to A_i + \alpha A_j$ Add α times the j th row of A to the i th row of A.

Remark: An elementary column operation on a matrix is defined similarly.By an elementary operation we mean an elementary row operation or an elementary column operation.

Definition 2.4.4:An **elementary column operation** on a matrix is any one of the following operations:

a) Interchange two columns .
b) Multiply a column by a non zero constant
c) Replacing a column by the sum of that column and a scalar multiple of another column.

Definition 2.4.5: We say two matrices are **equivalent** if one is obtained from the other by a finite sequence of elementary operations.

Notation : If A and B are equivalent then we write A ~ B

Example 2.4.4: Let

$$A=\begin{bmatrix}1 & 3 & 5 & 1 & 4\\ -1 & 4 & 0 & 5 & 7\\ 3 & -2 & 5 & 6 & 3\end{bmatrix}\ A_{1,2}\begin{bmatrix}-1 & 4 & 0 & 5 & 7\\ 1 & 3 & 5 & 1 & 4\\ 3 & -2 & 5 & 6 & 3\end{bmatrix}\ A_{2,3}\begin{bmatrix}-1 & 4 & 0 & 5 & 7\\ 3 & -2 & 5 & 6 & 3\\ 1 & 3 & 5 & 1 & 4\end{bmatrix}$$

Hence , $\begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ -1 & 4 & 0 & 5 & 7 \\ 3 & -2 & 5 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 0 & 5 & 7 \\ 3 & -2 & 5 & 6 & 3 \\ 1 & 3 & 5 & 1 & 4 \end{bmatrix}$ Similarly

$A = \begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ -1 & 4 & 0 & 5 & 7 \\ 3 & -2 & 5 & 6 & 3 \end{bmatrix}$ $A_2 \rightarrow 2A_2$ $\begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ -2 & 8 & 0 & 10 & 14 \\ 3 & -2 & 5 & 6 & 3 \end{bmatrix}$ $A_3 \rightarrow A_3 + -1A_1$

$\begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ -2 & 8 & 0 & 10 & 14 \\ 2 & -5 & 0 & 5 & -1 \end{bmatrix}$ $A_{2,3}$ $\begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ 2 & -5 & 0 & 5 & -1 \\ -2 & 8 & 0 & 10 & 14 \end{bmatrix}$

Hence ,

$$\begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ -1 & 4 & 0 & 5 & 7 \\ 3 & -2 & 5 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 1 & 4 \\ 2 & -5 & 0 & 5 & -1 \\ -2 & 8 & 0 & 10 & 14 \end{bmatrix}$$

Note: A is column equivalent to B if and only if A^t is row equivalent to B^t.
Next we state a theorem without proof

<u>Theorem 2.4.1:</u> Any matrix can be reduced to a matrix in reduced echelon form by elementary operations.

<u>Example2.4.5</u>: Reduce $A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$

- **a.** In to row echelon form.
- **b.** In to reduced row echelon form.

<u>Solution</u>

a. $A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$ $A_{2,1}$ $\begin{bmatrix} -1 & 2 \\ 3 & 5 \end{bmatrix}$ $A_1 \rightarrow (-1)A_1$ $\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$

$A_2 \rightarrow A_2 + (-3)A_1$ $\begin{bmatrix} 1 & -2 \\ 0 & 11 \end{bmatrix}$ $A_2 \rightarrow (1/11)A_2$ $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

Hence $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ is the row echelon form of A.

b. $A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ $A_1 \rightarrow A_1 + (2)A_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the reduced row echelon form of A.

Example 2.4.6: Reduce $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

a. In to row echelon form.
b. In to reduced row echelon form.

Solution

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} A_2 \rightarrow A_2+(2)A_1 \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix} A_2 \rightarrow (-1)A_2 \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A_3 \rightarrow A_3+(-1)A_2 \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{bmatrix} A_3 \rightarrow (1/5)A_3 \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ is the row echelon form of A.

b.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} A_1 \rightarrow A_1+(1)A_2 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 \rightarrow A_2+(3)A_3 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_1 \rightarrow A_2+(1)A_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Activity 2.4.1

Find the reduced row echelon form of $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 3 & 3 & -1 \end{bmatrix}$

The Inverse of a Matrix

Definition 2.4.6:An *nxn* matrix ***A*** is called **nonsingular** (or **invertible**) if there exists an *nxn* matrix ***B*** such that

$$\mathbf{AB=BA=I_n.}$$

The matrix B is called an **inverse** of A denoted by A^{-1}. If there exists no such matrix B, then A is called **singular** (or **noninvertible**).

Example 2.4.7: Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \text{ and } B= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

Since AB = BA = I_2 ,we conclude that B is the *inverse* of A and hence that A is invertible..

Theorem 2.4.2 :If a matrix is invertible , then its inverse is unique.

Proof:Let B and C be inverses of A. Then

$$CA = AB= I_n.$$

Therefore,

$$C= CI_n = C(AB) = (CA)B = I_nB = B$$

Remark : Since the inverse of amatrix if it exists is unique,We write the inverse of A by A^{-1}. Therefore, $A^{-1} = A^{-1}A = I_n$.

Example 2.4.8: Let A = $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ .To find A^{-1}, let A^{-1} = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then we must have

$$A A^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so that

$$\begin{bmatrix} a+2c & b+2d \\ a+3c & b+3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the above two matrices to be equal the corresponding components should be equal. Hence

a+ 2c = 1   a + 3c = 0

b + 2d =0   b + 3d = 1

Collecting terms

a+ 2c = 1   b + 2d = 0

a + 3c = 0   b + 3d = 1

Solving simultaneously

a = 3, c = -1 , b = -2 , d= 1

Therefore

$$A^{-1}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Activity 2.4.2 :

By matrix multiplication show that $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$ is the inverse of $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

Theorem 2.4.3

1. If A is an invertible (nonsingular)matrix, then A^{-1} is invertible (nonsingular) and $(A^{-1})^{-1} = A$.
2. If A is an invertible (nonsingular)matrix, then $(A^t)^{-1} = (A^{-1})^t$.
3. If A and B are invertible (nonsingular) matrices, then AB

invertible (nonsingular) and $(AB)^{-1} = B^{-1} A^{-1}$.

Proof

1. Let A be an invertible matrix . there exists a matrix B such that $AB=BA=I_n$. $B^{-1} = A$ .Since $B = A^{-1}$, $B^{-1} = A$ implies that $(A^{-1})^{-1} = A$
2. $(A^t)(A^{-1})^t = (A^{-1}A)^t = I^t = I$ and $(A^{-1})^t(A^t) = (AA^{-1})^t = I^t = I$
Therefore , $(A^t)^{-1} = (A^{-1})^t$
3. $AB(B^{-1} A^{-1}) = A(BB^{-1})A^{-1} = AI A^{-1} = A A^{-1} = I$ .Similarly
$(B^{-1} A^{-1}) AB = B^{-1} (A^{-1} A)B = B^{-1} I B = B^{-1}B = I$
Therefore , $(AB)^{-1} = B^{-1} A^{-1}$

Remark: By repeated application the preceding theorem ,It can be shown that if
$A_1, \ldots A_n$ are invertible matrices of the same order then
$(A_1A_1 \ldots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \ldots\ldots A_2^{-1} A_1^{-1}$

Procedure for finding the inverse of a matrix.

The procedure for computing the inverse of matrix A is as follows.

Step1. Form the augmented matrix B = [$A \;\vdots\; I_n$] obtained by adjoining the identity matrix I_n to the given matrix A.
Step2. By using elementary row operations on B ,Transform the matrix A in to reduced row echelon form.
Step3.
a.If the reduced row echelon form. of A is the identity matrix , A is invertible and its inverse is the matrix to the right of the vertical bar..
b.If the reduced row echelon form. of A is not the identity matrix , A is not invertible.

Example 2.4.9 : Find the inverse of the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Solution :

Let A= $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

Step 1. B = $\begin{pmatrix} 1 & 2 & : & 1 & 0 \\ 1 & 3 & : & 0 & 1 \end{pmatrix}$

Step 2: B = $\begin{pmatrix} 1 & 2 & : & 1 & 0 \\ 1 & 3 & : & 0 & 1 \end{pmatrix}$ $B_2 \rightarrow B_2+(-1)B_1 \begin{pmatrix} 1 & 2 & : & 1 & 0 \\ 0 & 1 & : & -1 & 1 \end{pmatrix}$

$B_1 \rightarrow B_1+(-2)B_2 \begin{pmatrix} 1 & 0 & : & 3 & -2 \\ 0 & 1 & : & -1 & 1 \end{pmatrix}$

Therefore $\begin{pmatrix} 1 & 0 & : & 3 & -2 \\ 0 & 1 & : & -1 & 1 \end{pmatrix}$ is the reduced row echelon form of B.

Step 3 : Since the reduced row echelon form. of A is the identity matrix , A is invertible and its inverse is $\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

<u>Example 2.4.10</u> : Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 3 & 3 & -1 \end{bmatrix}$

<u>Solution :</u>

Let A= $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 3 & 3 & -1 \end{bmatrix}$

Step 1. B = $\begin{pmatrix} 1 & -1 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & -3 & : & 0 & 1 & 0 \\ 3 & 3 & -1 & : & 0 & 0 & 1 \end{pmatrix}$

Step2:B = $\begin{pmatrix} 1 & -1 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & -3 & : & 0 & 1 & 0 \\ 3 & 3 & -1 & : & 0 & 0 & 1 \end{pmatrix}$ $B_3 \rightarrow B_3+(-3)B_1$

$\begin{pmatrix} 1 & -1 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & -3 & : & 0 & 1 & 0 \\ 0 & 6 & -7 & : & -3 & 0 & 1 \end{pmatrix}$ $B_3 \rightarrow B_3+(-6)B_2$ $\begin{pmatrix} 1 & -1 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & -3 & : & 0 & 1 & 0 \\ 0 & 0 & 11 & : & -3 & -6 & 1 \end{pmatrix}$

$B_3 \rightarrow \frac{1}{11} B_3$ $\begin{pmatrix} 1 & -1 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & -3 & : & 0 & 1 & 0 \\ 0 & 0 & 1 & : & -3/11 & -6/11 & 1/11 \end{pmatrix}$

$B_2 \rightarrow B_2+(3)B_3$ $\begin{pmatrix} 1 & -1 & 2 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & -9/11 & -7/11 & 3/11 \\ 0 & 0 & 1 & : & -3/11 & --6/11 & 1/11 \end{pmatrix}$

$B_1 \rightarrow B_1+(-2)B_3$ $\begin{pmatrix} 1 & -1 & 0 & : & 17/11 & 12/11 & -2/11 \\ 0 & 1 & 0 & : & -9/11 & -7/11 & 3/11 \\ 0 & 0 & 1 & : & -3/11 & --6/11 & 1/11 \end{pmatrix}$

$$B_1 \rightarrow B_1+(1)B_2 \begin{pmatrix} 1 & 0 & 0 & : & 8/11 & 5/11 & 1/11 \\ 0 & 1 & 0 & : & -9/11 & -7/11 & 3/11 \\ 0 & 0 & 1 & : & -3/11 & -\,-6/11 & 1/11 \end{pmatrix}$$

Step 3 Since the reduced row echelon form. of A is the identity matrix , A is invertible

and its inverse is $\begin{bmatrix} 8/11 & -5/11 & 1/11 \\ -9/11 & -7/11 & 3/11 \\ -3/11 & -6/11 & 1/11 \end{bmatrix}$

Hence $A^{-1} = \begin{bmatrix} 8/11 & -5/11 & 1/11 \\ -9/11 & -7/11 & 3/11 \\ -3/11 & -6/11 & 1/11 \end{bmatrix} = 1/11 \begin{bmatrix} 8 & -5 & 1 \\ -9 & -7 & 3 \\ -3 & -6 & 1 \end{bmatrix}$

Activity 2.4.3

Find the inverse of

1. $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$
2. $A = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 0 & 4 \\ 1 & 1 & 1 & 4 \end{pmatrix}$

2.5 Rank of a matrix

Definition 2.5.1 :The **rank** of a matrix is the number of non zero rows the row echelon form

of the matrix.

Example 2.5.1: Let $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Find the rank of A.

Solution:

A is in row echelon form. A has two non zero rows. Hence the rank of A is 2.

Activity 2.5.1:

Let $A=\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Find the rank of A

<u>Activity 2.5.2</u>:

Let $A=\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Find the rank of A

<u>Remark :</u> If A is an mxn matrix , the the rank of A is less than or equal to the min(m,n)

<u>2.6 System of Linear Equations and methods of solving.</u>

<u>System of linear equations</u>

Consider the following m system of linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \qquad (*)$$

Now define the following matrices:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Hence linear system (*) above can be written in matrix form as

$A\mathbf{x} = \mathbf{b}$.

<u>Note :</u> If b is the zero matrix then the system is said to be homogeneous otherwise non homogeneous.

The matrix A is called the **coefficient matrix** of the equations in (*), and the matrix

$$B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots & b_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \vdots & b_m \end{bmatrix}$$

Obtained by adjoining **b** to A, is called the **augmented matrix** of (*). Like inveses of a matrix , the augmented matrix of (*) will be written as [$A \;\vdots\; \mathbf{b}$].

Example 2.6.1 :Consider the following system of three linear equations in two unknowns.

$$x - 2y + z = 12$$
$$-2x + \quad + z = 5$$

a. Write in the form of AX = b

b.Write the augmented matrix B

Solution

a.The system can be written as.

$$x - 2y + z = 12$$
$$-2x + 0y + z = 5.$$

$$A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad b = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$$

b. $B = \begin{pmatrix} 1 & -2 & 1 & : & 12 \\ -2 & 0 & 1 & : & 5 \end{pmatrix}$

Activity 2.6.1:Consider the following system of three linear equations in two unkowns.

$$3x_1 + 2x_2 + 7x_3 + 4x_4 = 0$$
$$x_1 - 3x_2 - 6x_3 + x_4 = 3$$
$$8x_1 \quad + x_3 - 3x_4 = 2$$

a) Write in the form of AX = b

b)Write the augmented matrix B

Solving Systems of linear equations

Consider the following m system of linear equations in n unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \qquad (*)$$

Our aim is now to solve the above system of linear equations

If $b_1 = b_2 = \ldots = b_n = 0$, the system is called homogeneous otherwise it is non **homogenous system.**

A solution of the above equation (*) is an n – tuple that satisfies the system. The solution set S is the family of all such n-tuples .Solving the system means describing the set S in some way that enables us to tell easily whether a given n – tuple belongs to S or not.

Remark : A homogenous system always has a solution where $x_1 = x_2 = \ldots = x_n = 0$, which is called a trivial solution.

Guassian elimination, named for the great Mathematician C.F.Guass , is the process of reducing a matrix to row echelon form.

Operations that leads to equivalent systems of equations

Each of the following operations performed on a system of linear equations produces an equivalent system .

i) Interchange two equations
ii) Multiply an equation by a non zero constant
iii) Add a multiple any equation to any other equation.

Given a system of m linear equations in n unknowns.Three questions arise

(i) Does the system have any solution
(ii) If so, how many?
(iii) What are the solutions?

The solution Set S of (*) is not changed to the standard form by these three operations

a. Two equations are interchanged
b. One equation is multiplied by anumber that is not 0.
c. One equation is changed by adding another one to it

The operations (a,b,c) have counterparts for matrices. Interchnging two linear equations is equivalent to interchnging two rows of the matrix of the system. Multiplying an equation by a number corresponds to multiplying the associated row of the matrix by the same number. Finally adding one equation to another is equivalent to adding one row to another row.

Guassian – Jordan method for solving Systems of linear equations

Consider the following m system of linear equations in n unknowns:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{aligned}$$

To solve the System

Step 1. Write in the form of A**x** = **b**.

Step2. Form the augmented matrix B = $[\,A \;\vdots\; \mathbf{b}\,]$.

Step3. Transform the augmented matrix to reduced row echelon form by using elementary row operations.

Step4. Compare the rank of A and the rank of B

a. If rank of A < rank of B , the system has no solution

b. If rank of A = rank of B , the system has a solution

(i) rank of A = rank of B = n , the system has exactly one solution.

(ii) rank of A = rank of B < n , the system has infinitely many solutions.

Step 5. Use back substitution

The above method is called Guass-Jordan elimination.

Example 2.6.2: Solve

$$x - 2y + z = 0$$
$$-2x + \quad + z = -1$$
$$2x - y + z = 2$$

Solution

Step 1: $A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 0 & 1 \\ 2 & -1 & 1 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $b = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

Step2 : B= $\begin{pmatrix} 1 & -2 & 1 & : & 0 \\ -2 & 0 & 1 & : & -1 \\ 2 & -1 & 1 & : & 2 \end{pmatrix}$

Step 3: B= $\begin{pmatrix} 1 & -2 & 1 & : & 0 \\ -2 & 0 & 1 & : & -1 \\ 2 & -1 & 1 & : & 2 \end{pmatrix}$ $B_2 \rightarrow B_2 + 2B_1$ $\begin{pmatrix} 1 & -2 & 1 & : & 0 \\ 0 & -4 & 3 & : & -1 \\ 2 & -1 & 1 & : & 2 \end{pmatrix}$

$B_3 \rightarrow B_3 + (-2)B_1$ $\begin{pmatrix} 1 & -2 & 1 & : & 0 \\ 0 & -4 & 3 & : & -1 \\ 0 & 3 & -1 & : & 2 \end{pmatrix}$ $B_2 \rightarrow -\frac{1}{4} B_2$ $\begin{pmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 1 & -\frac{3}{4} & : & \frac{1}{4} \\ 0 & 3 & -1 & : & 2 \end{pmatrix}$

$B_3 \rightarrow B_3 + (-3)B_2$ $\begin{pmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 1 & -\frac{3}{4} & : & \frac{1}{4} \\ 0 & 0 & \frac{5}{4} & : & \frac{5}{4} \end{pmatrix}$ $B_3 \rightarrow \frac{4}{5} B_3$ $\begin{pmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 1 & -\frac{3}{4} & : & \frac{1}{4} \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$

Now B is changed in to row echelon form.

Step 4. The rank of A = 3 (The left hand of the vertical bar) and the rank of B = 3 , the system has a solution. Since

rank of A = rank of B = 3 = n , the system has exactly one solution.

Step 5. By Using back substitution

$$\begin{pmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 1 & -\frac{3}{4} & : & \frac{1}{4} \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

$z = 1$, $y + -\frac{3}{4} z = \frac{1}{4} \Rightarrow y = 1$

$z = 1$, $y = 1$, $x + -2y + z = 0 \Rightarrow x = 1$

Hence $x = 1$, $y = 1$, $z = 1$

Remark: The linear systems with at least one solution are called consistent, otherwise the system is said to be inconsistent**.** A consistent system has either one solution or infinitely many solutions.

Theorem 2.6.1 : A system of linear equations with fewer equations than variables must have either an infinite number of solutions or no solution.(Such a system cannot have a unique solution)

Proof : $m < n$. Rank $A = r \le \min(m,n) = m < n$

$\Rightarrow$ rank $A < n$

$\Rightarrow$ It has no unique solution

Note: In a consistent system rank $A = r < n$, $n - r$ of the unknowns are assigned any values whatever, the other r unknowns are uniquely determined.

Example 2.6.3 : Solve

$$\begin{aligned} 2x + y + 2z &= 3 \\ 3x - y + 4z &= 7 \\ x + 3y &= -1 \end{aligned}$$

Solution

Step 1: $A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & -1 & 4 \\ 1 & 3 & 0 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $b = \begin{pmatrix} 3 \\ 7 \\ -1 \end{pmatrix}$

Step2 : B= $\begin{pmatrix} 2 & 1 & 2 & : & 3 \\ 3 & -1 & 4 & : & 7 \\ 1 & 3 & 0 & : & -1 \end{pmatrix}$

Step 3: B= $\begin{pmatrix} 2 & 1 & 2 & : & 3 \\ 3 & -1 & 4 & : & 7 \\ 1 & 3 & 0 & : & -1 \end{pmatrix}$ $B_{1,3}$ $\begin{pmatrix} 1 & 3 & 0 & : & -1 \\ 3 & -1 & 4 & : & 7 \\ 2 & 1 & 2 & : & 3 \end{pmatrix}$

$B_3 \rightarrow B_3 + (-2)B_1$ $\begin{pmatrix} 1 & 3 & 0 & : & -1 \\ 3 & -1 & 4 & : & 7 \\ 0 & -5 & 2 & : & 5 \end{pmatrix}$ $B_2 \rightarrow B_2 + (-3)B_1$ $\begin{pmatrix} 1 & 3 & 0 & : & -1 \\ 0 & -10 & 4 & : & 10 \\ 0 & -5 & 2 & : & 5 \end{pmatrix}$

$B_2 \rightarrow B_2 + (-2)B_3$ $\begin{pmatrix} 1 & 3 & 0 & : & -1 \\ 0 & 0 & 0 & : & 0 \\ 0 & -5 & 2 & : & 5 \end{pmatrix}$ $B_{3,2}$ $\begin{pmatrix} 1 & 3 & 0 & : & -1 \\ 0 & -5 & 2 & : & 5 \\ 0 & 0 & 0 & : & 0 \end{pmatrix}$

$B_2 \rightarrow -\frac{1}{5}B_2 \begin{pmatrix} 1 & 3 & 0 & : & -1 \\ 0 & 1 & -\frac{2}{5} & : & -2 \\ 0 & 0 & 0 & : & 0 \end{pmatrix}$

Now B is changed in to row echelon form.

Step 4. The rank of A = 2 (The left hand of the vertical bar) and the rank of B = 2 , the system has a solution. Since

rank of A = rank of B < 3 = n , the system has infinitely many solutions.

Step 5. By Using back substitution

r =rank of A = 2 , n = 3

n - r = 3 – 2 = 1 unknown will be assigned any value let us say t.

Let z = t

$y + -\frac{2}{5} z = -2 \Rightarrow y + -\frac{2}{5} t = -2 \Rightarrow y = -2 + \frac{2}{5} t$

$x + 3y = -1 \Rightarrow x = -1 + -3y \Rightarrow x = -1 + -3(-2 + \frac{2}{5} t) \Rightarrow x = -7 + -\frac{6}{5} t$

Hence the solution is $x = -7 + -\frac{6}{5} t$, $y = -2 + \frac{2}{5} t$, $z = t$ where $t \in \Re$

Example 2.6.4 : Solve

$x - y + z = 2$

$2x + y + z = 1$

$3x + 2z = 5$

Solution

Step 1: $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \end{pmatrix}$ $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $b = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$

Step2 : $B = \begin{pmatrix} 1 & -1 & 1 & : & 2 \\ 2 & 1 & 1 & : & 1 \\ 3 & 0 & 2 & : & 5 \end{pmatrix}$

Step 3: $B = \begin{pmatrix} 1 & -1 & 1 & : & 2 \\ 2 & 1 & 1 & : & 1 \\ 3 & 0 & 2 & : & 5 \end{pmatrix}$ $B_3 \rightarrow B_3+(-3)B_1 \begin{pmatrix} 1 & -1 & 1 & : & 2 \\ 2 & 1 & 1 & : & 1 \\ 0 & 3 & -1 & : & -1 \end{pmatrix}$

$B_2 \rightarrow B_2+(-2)B_1 \begin{pmatrix} 1 & -1 & 1 & : & 2 \\ 0 & 3 & -1 & : & -3 \\ 0 & 3 & -1 & : & -1 \end{pmatrix}$ $B_3 \rightarrow B_3+(-1)B_2 \begin{pmatrix} 1 & -1 & 1 & : & 2 \\ 0 & 3 & -1 & : & -3 \\ 0 & 0 & 0 & : & 2 \end{pmatrix}$

$B_2 \rightarrow \frac{1}{3}B_2 \begin{pmatrix} 1 & -1 & 1 & : & 2 \\ 0 & 1 & -\frac{1}{3} & : & -1 \\ 0 & 0 & 0 & : & 2 \end{pmatrix}$ $B_3 \rightarrow \frac{1}{2}B_3 \begin{pmatrix} 1 & -1 & 1 & : & 2 \\ 0 & 1 & -\frac{1}{3} & : & -1 \\ 0 & 0 & 0 & : & 1 \end{pmatrix}$

Now B is changed in to row echelon form.

Step 4. The rank of A = 2 (The left hand of the vertical bar) and the rank of B = 3, the system has a solution. Since
rank of A < rank of B , the system has no solution

<u>Activity 2.6.2</u>

1. Solve

$$2x - y + z = 3$$
$$x + y - 3z = 1$$
$$x + 2y = 2z = -11$$

2. . Solve

$$x - y + z = 6$$
$$2x + y + 3z = 1$$
$$x + y - 3z = 1$$

3. . Solve

$$x + y - z = 1$$
$$3x - y = 5$$
$$2x - 2y + z = 3$$

<u>Inverse of a matrix for solving Systems of linear equations</u>

Suppose A is an nxn matrix. Our aim is now to solve the equation **Ax =b** which is a system of *n* equations in *n* unknowns. Suppose that A is invertible . Then A^{-1} exists and we can multiply **Ax =b** by A^{-1} on both sides.We obtain

$$A^{-1}(\mathbf{Ax}) = A^{-1}\mathbf{b}$$

$$(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$$

$$I\mathbf{x} = A^{-1}\mathbf{b}$$

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Moreover, $\mathbf{x} = A^{-1}\mathbf{b}$ is clearly a solution to the given linear system. Thus if A is invertible , the solution is unique .

Using the above property ,

<u>Theorem 2.6.2</u>. If A is an nxn matrix, the homogeneous system

$$A\mathbf{x} = \mathbf{0}$$

has a nontrivial solution if and only if A is noninvertible.

Example 2.6.5: Solve

$$x + 2z = 1$$
$$x + 3z = -1$$

Solution

First write in the form of **AX = b**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{X} = \begin{pmatrix} x \\ z \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

By using elementary row operations , we can find

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$\begin{pmatrix} x \\ z \end{pmatrix} = A^{-1}\mathbf{b} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

Hence x = 5, z = -2 .

Example 2.6.6 : Solve

$$x - y + 2z = 1$$
$$y - 3z = 2$$
$$3x + 3y - z = 0$$

Solution

First write in the form of **AX = b**

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 3 & 3 & -1 \end{bmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

By using elementary row operations , we can find

$$A^{-1} = \frac{1}{11} \begin{bmatrix} 8 & -5 & 1 \\ -9 & -7 & 3 \\ -3 & -6 & 1 \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\mathbf{b} = \frac{1}{11} \begin{bmatrix} 8 & -5 & 1 \\ -9 & -7 & 3 \\ -3 & -6 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} -2 \\ -23 \\ -15 \end{pmatrix}$$

Activity 2.6.3 Solve the following systems of linear equations using the inverse of a matrix.

1. Solve

$$x - y = 2$$
$$2x + y = 1$$

2. $x + 2y - z = -2$

$$2x - y + z = 5$$
$$-x + y + z = 2$$

exercises

1.Let A = $\begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 0 & -1 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$.

Find a. AB.

b. A^t

c. B^t

d. $B^t\, A^t$

e. The rank of A

2. Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 0 \\ -2 & 7 & 4 \end{bmatrix}$$

is A a symmetric matrix?

1. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -3 & 5 \\ -2 & 6 & 1 \end{bmatrix}$$

Write A as a sum of symmetric and antisymmetric matrices..

4.Determine whether the matrices are in row or reduced row echelon forms.

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 5 & 6 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

5. Find the inverse of the following matrices.

a. A= $\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

b. A = $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

c. A = $\begin{bmatrix} 2 & 3 & 5 \\ 1 & -2 & 3 \\ 3 & -4 & 2 \end{bmatrix}$

6.Find the rank of the matrix

A= $\begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & -2 & 0 & 1 \\ 5 & -3 & 1 & 2 \end{pmatrix}$

7. Solve

$2x + y + 2z = 3$
$3x - y + 4z = 7$
$4x + 3y + 6z = 5$

8. Find for what values of α and β , the system

$2x + 4y + (\alpha + 3)z = 2$
$X + 3y + z = 2$
$(\alpha - 20x + 2y + 3z = \beta$

is consistent .

9. Solve

$2x + 6y + z = -1$
$3x + 9y + 2z = -1$
$-y + 3z = 4$

10.Let A be a 3x5 matrix and B be a 5x5 matrix .What is the size of AB?

11. Let A and B square matrices of the same order which of the following is true?

A. $(A + B)^{-1} = A^{-1} + B^{-1}$
B. $AB = 0 \Rightarrow |A| = 0$ or $|B| = 0$
C. $AB = 0 \Rightarrow |A| = 0$ and $|B| = 0$
D. $(AB)^{-1} = A^{-1} B^{-1}$

12.Solve

$x + 2y = 2$
$3x + 6y - z = 8$
$x + 2y + z = 0$
$2x + 5y - 2z = 9$

13.a. List all possible forms of 2x 2 reduced row echelon forms.

b. List all possible forms of 3 x 3 reduced row echelon forms

14.If $\begin{pmatrix} a+b & c+d \\ c-d & a-b \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 4 & 2 \end{pmatrix}$, find a,b,c,d

15. Verify that A = $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and B = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are two symmetric matrices such that AB is

Skew symmetric.

16.Let A = $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ where a $\neq$0, b $\neq$0 and c $\neq$0.Find the inverse of A.

17.If A^{-1} = $\begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}$, find A

18.Show that A= $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ is the inverse of B = $\begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$.

19.Let A = $\begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$. Find

a. $A.A^t$ b. $A^t.A$

20.Let A = $\begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$. Find a) A^2 b) A^3

21.Let A = $\begin{pmatrix} 1 & 3 \\ 4 & -3 \end{pmatrix}$. Find a non zero vector vector u = $\begin{pmatrix} x \\ y \end{pmatrix}$ such that Au =3u

22.Let A = $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.Find A^n

23.Find the inverse of A = $\begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$

DETERMINANTS

Determinant is a scalar associated with every square matrix .Their usefulness follows from two of their properties. First they can be used to compute areas and volumes, and secondly , that a zero determinant characterizes singular matrices. Computing areas and volumes brings determinants in to the formulas for changing variables inmultiple integrals .One of the most important uses of determinants within linear algebra is the study of eigenvalues.

Determinants also occur in cramer's rule for solving linear equations and can be used to give a formula for the inverse of an invertible matrix. In the calculus of several variables , the Jacobians used in transforming a multiple integral uses determinant. This use arises from the fact that determinant is the volume of the parallelpiped. Determinants are also useful in various other subjects like physics , Astronomy and statistics

Objectives:

After successful completion of this unit, you will be able to:

- Define determinant
- Find the determinant of a square matrix by using the definition and its properties.
- Find inverse of a square matrix using determinant
- Solve systems of linear equation
- Calculate area of a parallelogram
- Calculate volume of a parallelepiped

3.1. Definition of determinants

Every square matrix A= $\lfloor a_{ij} \rfloor_{n \times n}$; has a number associated to it is called its determinant. In this section we will define the determinant of a square matrix inductively and derive its properties .The determinant of a matrix A will be denoted by det A or |A|

Definition 3.1.1

i) Determinant of a 1x1 matrix

Let A = $[a_{11}]$ is a 1x1 matrix, then det A = |A| = a_{11}

<u>Example 3.1.1:</u> Find the determinant of A = (-2)

<u>Solution</u>

det A = |A| = a_{11} = -2

ii) **Determinant of a 2x2 matrix**

Let A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then det A = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ = $a_{11}\,a_{22} - a_{12}a_{21}$

<u>Example 3.1.2</u> : Find the determinant of A where A = $\begin{bmatrix} 2 & 4 \\ 0 & 6 \end{bmatrix}$

<u>Solution</u>

det A = |A| = det A = 2(6) – 4(0) = 12

Example 3.1.3

Find x if $\begin{vmatrix} x & 6 \\ 2 & x \end{vmatrix} = -4$

Solution:

$\begin{vmatrix} x & 6 \\ 2 & x \end{vmatrix} = 4 \quad \Rightarrow x^2$ - 12 = 4 $\quad \Rightarrow x^2$ = 16 $\quad \Rightarrow$ x = 4 or x = -4

Activity 3.1.1

1) If A= $\begin{bmatrix} 1 & -6 \\ 4 & -1 \end{bmatrix}$ then find det A.
2) Find x if $\begin{vmatrix} x & 2 \\ 2x & 3 \end{vmatrix} = 4$

iii) **The determinant of an nxn matrix**

The determinant of an nxn matrix can be computed in terms of (n-1) x (n-1) determinants .This expansion allows us to give a recursive definition of the determinant function.

Let A be an nxn matrix.and let A_{ij} denote the (n-1)x(n-1) matrix obtained by crossing out the ith row and the jth column of A.

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} (-1)^{i+j} aij \mid Aij \mid \quad (1 \le i \le n)$$

is called expansion along the ith row.

Example 3.1.4 : Find the determinant of the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 2 \\ 2 & 1 & 5 \end{bmatrix}$

Solution

$$\sum_{i=1}^{n} (-1)^{i+k} a_{ik} \mid A_{ik} \mid \quad (1 \le i \le n)$$

Let i = 1 , $\sum_{j=1}^{n} (-1)^{i+j} aij \mid Aij \mid$

$= (-1)^{1+1} a_{11}|A_{11}| + (-1)^{1+2} a_{12}|A_{12}| + (-1)^{1+3} a_{13}|A_{13}|$

$A_{11} = \begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}$, $|A_{11}| = 18$

$A_{12} = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}$, $|A_{12}| = 11$

$A_{13} = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$, $|A_{13}| = -5$

Det A $= (-1)^{1+1} a_{11}|A_{11}| + (-1)^{1+2} a_{12}|A_{12}| + (-1)^{1+3} a_{13}|A_{13}|$

$= a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}|$

$= 1.18 - 0.1 + (-1)(-5) = 23|$

Therefore det A = 23.

Hence we can write the above as

det A = $a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Activity 3.1.2

Find the determinant of $A = \begin{bmatrix} 2 & -1 & 4 \\ -1 & 4 & 3 \\ 7 & 1 & -1 \end{bmatrix}$

Example 3.1.5: -Calculate the determinant $\begin{vmatrix} -1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 0 \\ 3 & 1 & -1 & -3 \\ 0 & -1 & 1 & 3 \end{vmatrix}$

Solution

Det A = $\sum_{j=1}^{n} (-1)^{i+j} aij \mid Aij \mid$ ($1 \le i \le n$)

Let us expand along the second row

Det A = $\sum_{j=1}^{n} (-1)^{i+j} aij \mid Aij \mid = \sum_{j=1}^{n} (-1)^{2+j} a_{2j} \mid A_{2j} \mid$

$= (-1)^{2+1} a_{21}|A_{21}| + (-1)^{2+2} a_{22}|A_{22}| + (-1)^{2+3} a_{23}|A_{23}| + (-1)^{2+4} a_{24}|A_{24}|$

$= -1.0|A_{21}| + 1|A_{22}| - 1.(-2)|A_{23}| + 0|A_{24}|$

$= |A_{22}| + 2|A_{23}|$

Therefore det A = $|A_{22}| + 2|A_{23}|$

$$A_{22} = \begin{pmatrix} -1 & 0 & 1 \\ 3 & -1 & -3 \\ 0 & 1 & 3 \end{pmatrix} = 3$$

$$A_{23} = \begin{pmatrix} -1 & -2 & 1 \\ 3 & 1 & -3 \\ 0 & -1 & 3 \end{pmatrix} = 15$$

det A = $|A_{22}| + 2|A_{23}|$

$= 3 + 2(15) = 33$

As we can see from the above examples it is a little lengthy to calculate the determinant of a matrix. Next we start some properties which we can help us in calculating the determinant of a square matrix quickly**.**

3.2 Properties of determinants

Let A be an m x n matrix
Let A = $(A_1, A_2, A_3, \ldots A_i, A_{i+1}, \ldots, A_m)$
Or A = $({}^1A, {}^2A, \ldots, {}^iA, \ldots., {}^nA)$ Let A_i be the i^{th} row of the matrix A and ${}^jA$, be the j^{th} column of the matrix A

The following properties are true

Let A and B be square matrices of order n.

1. det I = 1

Example 3.2.1

Let I = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, det I = 1

2 . det AB = (det A) (det B)

Example 3.2.2

Let A= $\begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}$, $|A| = 18$ and B = $\begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}$, $|B| = 11$

$$AB = \begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 16 & 18 \\ 13 & 27 \end{pmatrix}$$

Det AB = 432 – 234 = 198 = |A||B|

Using 2 , one can show that det $A^k = (\det A)^k$ where k is a natural number

Activity 3.2.1

Give an example where det (A + B) ≠ det A + det B

3. If A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$

Example 3.2.3

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 5/18 & -2/18 \\ -1/18 & 4/18 \end{pmatrix}$$

$|A| = 18$, $|A^{-1}| = 1/18$

Example 3.2.4

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 5 \end{pmatrix}, \quad -3A = \begin{pmatrix} -12 & 6 \\ -3 & -15 \end{pmatrix}$$

$|A| = 22$, $|-3A| = 198 = (-3)^2(22)$

2. det A^t = det A

Example 3.2.5

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 5 \end{pmatrix} \quad A^t = \begin{pmatrix} 4 & 1 \\ -2 & 5 \end{pmatrix}$$

$|A| = 22$, $|A^t| = 22$

3. The determinant Vanishes if every element of some row (or column) is 0

   a..det $(A_1, A_2, \ldots, A_i, \ldots, A_n) = 0$ if $A_i = (0,0,0,\ldots,0)$

   b.det $({}^1A, {}^2A, \ldots, {}^iA, \ldots, {}^nA) = 0$ if ${}^iA = \begin{pmatrix} \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$

Example 3.2.6:

Let $A = \begin{pmatrix} 1 & 7 & 8 & 3 \\ 6 & -7 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 2 & 11 & 16 & 13 \end{pmatrix}$.|A| =0 .

Let A = $\begin{pmatrix} 1 & 0 & 8 & 3 \\ 6 & 0 & 2 & -1 \\ 9 & 0 & 5 & 6 \\ 2 & 0 & 16 & 13 \end{pmatrix}$. $|A| = 0$

4. The determinant vanishes if two rows (or columns) are equal or it one row (or column) is a scalar multiple of the other.
   a) i) det $(A_1, A_2, ..., A_i, ..., A_k, ..., An) = 0$ if $A_i = A_k$ $i \neq k$
      ii) det $({}^1A, {}^2A, ..., {}^iA, ..., {}^kA, ... {}^nA) = 0$ if ${}^iA = {}^kA$ $i \neq k$

Example 3.2.7:

Let A = $\begin{pmatrix} -1 & 2 & 3 & 5 \\ 6 & -7 & 2 & -1 \\ -1 & 2 & 3 & 5 \\ 2 & 11 & 16 & 13 \end{pmatrix}$. $|A| = 0$.

(the first row is equal the third row)

Let A = $\begin{pmatrix} 1 & 3 & 8 & 3 \\ 6 & -1 & 2 & -1 \\ 9 & 6 & 5 & 6 \\ 2 & 13 & 16 & 13 \end{pmatrix}$. $|A| = 0$

b) i) det $(A_1, A_2, ..., A_i, ..., A_k, ..., An) = 0$ if $A_i = \propto A_k$
   ii) det $({}^1A, {}^2A, ..., {}^iA, ..., {}^kA, ... {}^nA) = 0$ if ${}^iA = \propto({}^kA)$

Example 3.2.8

Let A = $\begin{pmatrix} 1 & -2 & -1 & 3 \\ 2 & -1 & -4 & -1 \\ 3 & -6 & -3 & 9 \\ -3 & 13 & 6 & 13 \end{pmatrix}$. $|A| = 0$ (the third row is equal to 3 times the first row)

Let A = $\begin{pmatrix} 1 & 3 & -2 & 3 \\ 2 & -1 & -4 & -1 \\ 0 & 6 & 0 & 6 \\ -3 & 13 & 6 & 13 \end{pmatrix}$. $|A| = 0$ (the third column is equal to -2 times the first column)

7. Interchanging two rows (or columns) multiplies the determinant by -1
   a. det $(A_1, ..., A_{i-1}, A_i, A_{i+1}, ..., A_{k,-1}, A_{k,}, A_{k+1}, ..., An)$
      $= -$ det $(A_1, ..., A_{i-1}, A_k, A_{i+1}, ..., A_{k-1}, A_i, A_{k+1}, ..., An)$
   b. det $({}^1A, ..., {}^{i-1}A, {}^iA, ..., {}^{k-1}A, {}^kA, {}^{k+1}A, ... {}^nA)$
      $= -$det $({}^1A, ..., {}^{i-1}A, {}^kA, {}^{i+1}A, ..., {}^{k-1}A, {}^iA, {}^{k+1}A, ... {}^nA)$

Example 3.2.9

Let A = $\begin{pmatrix} -1 & 0 & 1 \\ 3 & -1 & -3 \\ 0 & 1 & 3 \end{pmatrix}$. |A| = 3 .

Suppose B = $\begin{pmatrix} 0 & 1 & 3 \\ 3 & -1 & -3 \\ -1 & 0 & 1 \end{pmatrix}$. det B =- 3 (B is obtained by interchanging the first row and the third row of the matrix A

Example3.2.10

A = $\begin{pmatrix} -1 & -2 & 1 \\ 3 & 1 & -3 \\ 0 & -1 & 3 \end{pmatrix}$**.** Then A = 15

Let B = $\begin{pmatrix} 1 & -2 & 1 \\ -3 & 1 & 3 \\ 3 & -1 & 0 \end{pmatrix}$. Then |B| = -15 (B is obtained by interchanging the first column and the third column of the matrix A)

8. a. det $(A_1, A_2, ..., kA_i, ..., A_{n,}) = k \det (A_1, A_2, ..., A_i, ..., A_{n,})$
   b. det $({}^1A, {}^2A, ..., k\,{}^iA, ... , {}^nA) = k \det ({}^1A, {}^2A, ... , {}^iA, ... {}^nA)$

Example 3.2.11

Let A = $\begin{pmatrix} -1 & -2 & 1 \\ 3 & 1 & -3 \\ 0 & -1 & 3 \end{pmatrix}$. |A| = 15

a. If B = $\begin{pmatrix} -2 & -4 & 2 \\ 3 & 1 & -3 \\ 0 & -1 & 3 \end{pmatrix}$. then |B| = 2x15 = 30

(B is obtained by multiplying the first row the matrix A by 2.)

b.. If B = $\begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & -3 \\ 0 & 1 & 3 \end{pmatrix}$,then |B| = -1x15 = -15

(B is obtained by multiplying the second column of the matrix A by -1)

9. a. det $(A_1, A_2, .. A_{i-1}, A_i + A_i^1, A_{i+1}, ... , A_n)$

$= \det(A_1, A_2, \ldots A_{i-1}, A_i, A_{i+1}, \ldots, A_n) + \det(A_1, A_2, \ldots A_{i-1}, A_i^1, A_{i+1}, \ldots, A_n)$

b. $\det({}^1A, {}^2A, \ldots {}^{i-1}A., {}^iA + {}^iA^1, {}^{i+1}A \ldots, {}^nA)$

$= \det({}^1A, {}^2A, \ldots, {}^{i-1}A., {}^iA, {}^{i+1}A, \ldots, {}^nA) + \det({}^1A, {}^2A, \ldots {}^{i-1}A, {}^iA^1, {}^{i+1}A, \ldots {}^nA)$

Example 3.2.12

a. $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$

b. $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$

10. If B is obtained from A by adding the elements of i^{th} row (column) a scalar multiple of the corresponding elements of another row (or column), then det(A) = det(B) i.e

b) $\det(A_1, A_2, \ldots A_{i-1}, A_i, A_{i+1}, \ldots, A_n)$
$= \det(A_1, A_2, \ldots, A_{i-1}, A_i + \propto A_k, A_{i+1} \ldots, A_n)$

c) $\det({}^1A, {}^2A, \ldots {}^{i-1}A, {}^iA, {}^{i+1}A, \ldots {}^nA) =$
$\det({}^1A, {}^2A \ldots {}^{i-1}A, {}^iA + \propto {}^kA, {}^{i+1}A, \ldots, {}^nA)$

Example 3.2.13

Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Det A = det B where B= $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ (Because B is obtained from A by adding two times the first row of A to its second row.)

Det B = det C where C = $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{bmatrix}$ (Because C is obtained from B by adding one time the second row of B to its third row.)

From the above

Det A = det B = det C

11. If A is a triangular matrix, then det A = $a_{11}a_{22} \ldots a_{nn}$

Example 3.2.14 : Find the determinant of A where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution

$$\det \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 5 \end{bmatrix} = 5$$

Therefore det A = 5

Activity 3.2.2

Find the determinant of

a. $A = \begin{pmatrix} 1\,0\,0 \\ 0\,4\,0 \\ 0\,0\,6 \end{pmatrix}$ .b. $A = \begin{pmatrix} 1\,2\,3 \\ 0\,2\,5 \\ 0\,0\,6 \end{pmatrix}$ c. $A = \begin{pmatrix} 1\,2\,3 \\ 0\,4\,5 \\ 0\,0\,6 \end{pmatrix}$

Activity 3.2.3

Show that

$$\det \begin{pmatrix} 0 & 3 & 1 & 7 \\ 0 & 3 & 4 & 6 \\ 2 & 5 & 9 & -7 \\ 0 & 0 & 2 & 5 \end{pmatrix} = 102$$

3.3. Adjoint and inverse of a matrix.

Definition 3.3.1: - Let A = $\lfloor a_{ij} \rfloor$ be an nxn matrix. Let A_{ij} be the (n-1) x (n-1) matrix which is obtained by deleting the i^{th} row and the j^{th} column of A. The determinant det (A_{ij}) is called the **minor** of a_{ij}. The **cofactor** $A^{|}_{ij}$ of a_{ij} is defined as

$$\mathbf{A}^{|}_{\mathbf{ij}} = (-1)^{i+j} \det(A_{ij}) \qquad (1 \le i \le n, 1 \le j \le n)$$

Remark : A_{ij} is an (n-1) x (n-1) matrix

Example 3.3.1 : - Let $A = \begin{pmatrix} 1 & -2 & 1 \\ -3 & 1 & 3 \\ 3 & -1 & 0 \end{pmatrix}$

Then $A_{11} = \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix}$: $\mathbf{A}^{|}_{11} = (-1)^{1+1} \det A_{11} = \det A_{11} = 3$

$A_{21}=\begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$: $\mathbf{A}^{l}_{21}=(-1)^{2+1}\det A_{21}=(-1)\det A_{21}=1$

$A_{32}=\begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix}$: $\mathbf{A}^{l}_{32}=(-1)^{3+2}\det A_{32}=(-1)\det A_{32}=-6$

<u>**Definition 3.3.2**: -</u> Let A = $\lfloor a_{ij} \rfloor$ be an nxn matrix. The nxn matrix **adj A**, called the **adjoint** of A, is the transpose of the matrix whose (i, j)th element is the cofactor $\mathbf{A}^{l}_{\mathbf{ij}}$ of a_{ij}.

Thus

Adj A = $(\mathbf{A}^{l}_{ij})^{t}$. That is $adjA=\begin{bmatrix} A^{l}_{11} & A^{l}_{21} & \cdots & A^{l}_{n1} \\ A^{l}_{12} & A^{l}_{22} & \cdots & A^{l}_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ A^{l}_{1n} & A^{l}_{2n} & \cdots & A^{l}_{nn} \end{bmatrix}$

<u>**Example3.3.2**:</u> - Let A = $\begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix}$. Compute **adj A**

<u>**Solution**:</u>

The cofactor of A are A^{l}_{ij} of a_{ij} where $1 \le i,j \le 2$

$A_{11}=(3)$: $\mathbf{A}^{l}_{11}=(-1)^{1+1}\det A_{11}=\det A_{11}=3$

$A_{21}=(1)$: $\mathbf{A}^{l}_{21}=(-1)^{2+1}\det A_{21}=(-1)\det A_{21}=-1$

$A_{12}=(-3)$: $\mathbf{A}^{l}_{12}=(-1)^{1+2}\det A_{12}=(-1)\det A_{12}=3$

$A_{22}=(1)$: $\mathbf{A}^{l}_{22}=(-1)^{2+2}\det A_{22}=\det A_{2}=1$

adj A $=(A^{l}_{ij})^{t}=\begin{pmatrix} A^{l}_{11} & A^{l}_{21} \\ A^{l}_{12} & A^{l}_{22} \end{pmatrix}=\begin{pmatrix} 3 & 3 \\ -1 & 1 \end{pmatrix}$

<u>**Activity 3.3.1**</u>

Let A = $\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$. Compute adj A

<u>**Example 3.3.3**</u>: - Let $A=\begin{bmatrix} 2 & -1 & 4 \\ -1 & 4 & 3 \\ 7 & 1 & -1 \end{bmatrix}$. Compute adj A.

<u>**Solution**:</u> The <u>cofacto</u>r of A are A^{l}_{ij} of a_{ij} where $1 \le i,j \le 3$

$A_{11}=\begin{pmatrix} 4 & 3 \\ 1 & -1 \end{pmatrix}$: $\mathbf{A}^{l}_{11}=(-1)^{1+1}\det A_{11}=\det A_{11}=-7$

$A_{21}=\begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}$: $\mathbf{A}^{l}_{21}=(-1)^{2+1}\det A_{21}=(-1)\det A_{21}=3$

$A_{31}= \begin{pmatrix} -1 & 4 \\ 4 & 3 \end{pmatrix}$: $\mathbf{A}^{|}_{31}=(-1)^{3+1}\det A_{31}=\det A_{31}= -19$

$A_{12}= \begin{pmatrix} -1 & 3 \\ 7 & -1 \end{pmatrix}$: $\mathbf{A}^{|}_{12}=(-1)^{1+2}\det A_{12}=(-1)\det A_{12}= 20$

$A_{22}= \begin{pmatrix} 2 & 4 \\ 7 & -1 \end{pmatrix}$: $\mathbf{A}^{|}_{22}=(-1)^{2+2}\det A_{22}=\det A_{22}= -30$

$A_{32}= \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$: $\mathbf{A}^{|}_{32}=(-1)^{3+2}\det A_{32}= (-1)\det A_{32} = -10$

$A_{13}= \begin{pmatrix} -1 & 4 \\ 7 & 1 \end{pmatrix}$: $\mathbf{A}^{|}_{13}=(-1)^{1+3}\det A_{13}=\det A_{13}= -29$

$A_{23}= \begin{pmatrix} 2 & -1 \\ 7 & 1 \end{pmatrix}$: $\mathbf{A}^{|}_{23}=(-1)^{2+3}\det A_{23}= (-1)\det A_{23}= -9$

$A_{33}= \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$: $\mathbf{A}^{|}_{33}=(-1)^{3+3}\det A_{33}=\det A_{33} = 7$

$$\text{adj } A = \begin{pmatrix} A^{|}_{11} & A^{|}_{21} & A^{|}_{31} \\ A^{|}_{12} & A^{|}_{22} & A^{|}_{32} \\ A^{|}_{13} & A^{|}_{23} & A^{|}_{33} \end{pmatrix} = \begin{pmatrix} -7 & 3 & -19 \\ 20 & -30 & -10 \\ -29 & -9 & 7 \end{pmatrix}$$

<u>Theorem 3.3.1</u> : If A is an nxn matrix and det A $\neq$ 0, then $A^{-1} = \dfrac{adjA}{\det A}$

<u>Example 3.3.4</u> : Consider the Preceding examples

1. Let A $= \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$.

<u>Solution</u>

adj A $= \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}$ **,** det A = 9

$$A^{-1} = \frac{adjA}{\det A} = \frac{\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}}{9} = \begin{pmatrix} 5/9 & -1/9 \\ -1/9 & 2/9 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 5/9 & -1/9 \\ -1/9 & 2/9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

2.

$$A=\begin{bmatrix} 2 & -1 & 4 \\ -1 & 4 & 3 \\ 7 & 1 & -1 \end{bmatrix},$$ then det A= -136

$$A^{-1}=\frac{adj\,A}{\det A}=\frac{-1}{136}\begin{pmatrix} -7 & 3 & -19 \\ 20 & -30 & -10 \\ -29 & -9 & 7 \end{pmatrix}=\begin{pmatrix} 7/136 & -3/136 & 19/136 \\ -20/136 & -30/136 & -10/136 \\ 29/136 & 9/136 & -7/136 \end{pmatrix}$$

Activity 3.3.2

Using Adjoint find the inverse of

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$$

Theorem 3.3.2: A matrix A is invertible if and only if $\det(A) \neq 0$.

Proof: ($\Rightarrow$) Suppose A is invertible, then $AA^{-1} = I$

$\det AA^{-1} = \det I = 1$

$\det A \det A^{-1} = 1$ .Since the pproduct of two numbers is one , each of the numbers must be different from 0.Hence $\det A \neq 0$

$\Leftarrow$ Suppose $\det(A) \neq 0$, then $A^{-1}=\dfrac{adj\,A}{\det A}$, so A is invertible .

Corollary 3.3.3: If A is an invertible matrix , then the equation A **x** = b has exactly one solution

Proof: Suppose that A is invertible, then A^{-1} exists

$AX = b$

$A^{-1}(AX) = A^{-1}b$

$A^{-1}(AX) = A^{-1}b$

$X = A^{-1}b$

Hence the system has exactly one solution.

3.4. Cramer's rule for solving systems of linear equations.

The name cramer's rule is applied to a group of formulas giving solutions of systems of linear equations interms of determinants**.** To finish this section, we present a method of solving a system of n equations in n unknowns called Cramer's rule . The method is not used in practice. However it has a theoretical use as it reveals explicitly how the solution depends on the coeffcients of the augmented matrix.

CRAMER'S RULE

Let

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

be a linear system of *n* equations in *n* unknowns and let A = $\lfloor a_{ij} \rfloor$ be the coefficient matrix so that we can write the above equation as **AX = b,** where

$$A = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{1n} & a_{2n} & . & . & . & a_{nn} \end{pmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If det (A) $\neq$ 0, then the system has the unique solution.. The solution is

If $B_i = \begin{vmatrix} a_{11} & a_{12} & . & a_{1i-1} & b_1 & a_{1i+1} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2i-1} & b_2 & a_{2i+1} & . & a_{2n} \\ . & . & . & . & ' & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & a_{ni-1} & b_n & . & . & a_{nn} \end{vmatrix}$,

is the determinant of the matrix obtained from A by replacing the i[th] column of A by **b**.

$x_i = \dfrac{B_i}{\det A}$

To solve systems of linear equations by using determinants

Step 1. Write the system of linear equations in the form of AX = b
Step 2..Find the determinant of A.
Step 3 : Find each $x_i = \dfrac{B_i}{\det A}$ where B_i is the determinant of the matrix obtained from A by replacing the i[th] column of A by **b**.

Example 3.4.1: -Solve the following systems of linear equation by Using Cramer's rule

x – 3y = 2
2x + 7 y = 11

Solution

Step 1. A = $\begin{pmatrix} 1 & -3 \\ 2 & 7 \end{pmatrix}$, X = $\begin{pmatrix} x \\ y \end{pmatrix}$, b = $\begin{pmatrix} 2 \\ 11 \end{pmatrix}$

Step 2. det A = 13.

Step 3: **x =** $\dfrac{\begin{vmatrix} 2 & -3 \\ 11 & 7 \end{vmatrix}}{13} = \dfrac{47}{13}$ **,** **y =** $\dfrac{\begin{vmatrix} 1 & 2 \\ 2 & 11 \end{vmatrix}}{13} = \dfrac{7}{13}$

Therefore , x $= \dfrac{47}{13}$ **and y =** $\dfrac{7}{13}$

Activity 3.4.1 -Solve the following systems of linear equation by Using Cramer's rule

3x + y = 5

x - 4y = 3

Example 3.4.2: Solve the following systems of linear equation by Using Cramer's rule.

$$x_1 - x_2 + 2x_3 = 1$$
$$-2x_1 + x_2 - x_3 = 4$$
$$x_2 + 2x_3 = -3$$

Solution

Step 1. A = $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, X = $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, b = $\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$

Step 2. det A = 5.

Step 3. $x_1 = \dfrac{\begin{vmatrix} 1 & -1 & 2 \\ 4 & 1 & -1 \\ -3 & 1 & 2 \end{vmatrix}}{\det A} = \dfrac{22}{5}$, $x_2 = \dfrac{\begin{vmatrix} 1 & 1 & 2 \\ -2 & 4 & -1 \\ 0 & -3 & 2 \end{vmatrix}}{\det A} = \dfrac{21}{5}$

$$x_3 = \dfrac{\begin{vmatrix} 1 & -1 & 1 \\ -2 & 1 & 4 \\ 0 & 1 & -3 \end{vmatrix}}{\det A} = \dfrac{-3}{5}$$

Therefore , $x_1 = \dfrac{22}{5}$ $x_2 = \dfrac{21}{5}$ and $x_3 = \dfrac{-3}{5}$

Activity 3.4.2

Solve the following systems of linear equation by Using Cramer's rule

-2x + 3y- z = 1

x + 2y –z = 4

-2x –y + z =-3

Activity 3.4.3

Solve the following systems of linear equation by Using Cramer's rule

$2x + y + z = 6$
$3x + 2y - 2z = -2$
$x + y + 2z = 4$

3.5 Determinant : Cross product , Area and volume.

Cross product

Definition 3.5.1 Let **A** = (a_1 , a_2 ,a_3).and **B** = (b_1 , b_2 ,b_3).be two vectors in three dimensional space . We define

A x **B** = $i(a_2b_3 - b_3a_2) - j(a_1b_3 - a_3b_1) + k(a_1b_2 - a_2b_1)$

That is A x B = $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Example 3.5.2 : Let **A** = (1,0,2) and **B** = (1,2,4) Find

a)A x **B**
b)B x **A**

Solution :

A = (1,0,2) = (a_1 , a_2 ,a_3) and $a_1= 1$, $a_2 = 0$, $a_3 = 2$
B = (1,2,4) = (b_1 , b_2 ,b_3) and $b_1= 1$, $b_2 = 2$, $b_3 =4$

a) A x **B** = $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 1 & 2 & 4 \end{vmatrix} = -4i - 2j + 2k$

b) B x **A** = $\begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & b_3 \end{vmatrix} = (-1)\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (-1)(-4i - 2j + 2k)$

$= 4i + 2j - 2k$

From the above ,for any two vectors A and B, **A** x **B** = - (**B** x **A**)

Example 3.5.3: Suppose A and **B** are parallel ,Show that **A x B** = 0

Solution

If **A** and **B** are parallel , one is a scalar multiple of the other.

Let A =αB .Hence , $(a_1,a_2 ,a_3)=\alpha(b_1,b_2,b_3)=(\alpha b_1 ,\alpha b_2, \alpha b_3)$

A x **B** = $\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \alpha b_1 & \alpha b_2 & \alpha b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0$

Activity 3.5.1

Show that

1.$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$;

2. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

Area of a parallelogram

$\| \mathbf{A} \times \mathbf{B} \|$ is the area of the parallelogram formed by the vectors A and B

Example 3.5.4 : Let P = (2,-1,3) , Q = (5,8,2) and R = (0,-1,3).Find the area of parallelogram .

$\overrightarrow{PQ}$ = (3,9,-1) $\overrightarrow{PR}$ = (-2,0,0)

$$\| \overrightarrow{PQ} \times \overrightarrow{PR} \| = \begin{vmatrix} i & j & k \\ 3 & 9 & -1 \\ -2 & 0 & 0 \end{vmatrix} =$$

= i(9.0–0.(-1))–j(3.0 –(-1)(-2))+(3.0 –(-2)9)k

= 0i + 2j + 18k

Area of the parallelogram = $\sqrt{0^2 + 2^2 + 18^2} = \sqrt{328}$ square units

Example 3.5.5:

Find the area of a triangle with adjacent vectors P= (2, 3, -1) and Q = (1, 2, 2)

Solution:

$$P \times Q = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 1 & 2 & 2 \end{vmatrix} = (2(3)+2(1))\mathbf{i} - (2(2)+1(1))\mathbf{j} + (2(2)-1(3))\mathbf{k}$$

= 8i -5j +k

Area of a triangle = $\frac{1}{2} \| P \times Q \| = \frac{1}{2}\sqrt{64+25+1} = \frac{1}{2}\sqrt{90}$

Volume of a parallelpiped

The volume of the parallelpiped formed by the vectors **A**, **B** and **C** is |**A**.(**B** x **C**)|

$\| \mathbf{B} \times \mathbf{C} \|$ = Area of the base of the paallelpiped

θ is the angle beeween **A** and **B** x **C**

Example 3.5.6 : Find the volume of the parallelpiped formd by the vectors A = 3i+j-k ,

B = -i+2j + 4k and C = 2i-5j +3k

Solution

First let us find **B** x **C**

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} i & j & k \\ -1 & 2 & 4 \\ 2 & -5 & 3 \end{vmatrix} = \text{i}(2.3\text{–}4.(-5)) \text{–j}((-1).3\text{–}4.2)+((-1)(-5) – 2.2) \text{ k}$$

$= 26j + 11j +1k$

A.(B x **C)** = 3.26 + 1.11+(-1)1 **=** 88

Volume = |**A**.(**B** x **C**)|= |88| cubic units = 88 cubic units

Exercises

1. Let A = $\begin{pmatrix} -1 & 0 & 1 \\ 4 & 1 & 1 \\ -2 & -1 & -1 \end{pmatrix}$

   Find a. det A

2. . Let A = $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & -1 \end{pmatrix}$ and B = $\begin{pmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 1 & 4 & -1 \end{pmatrix}$

   Find a. det A^3

   b. detAB

   c. det$B^t A^t$

   d . det A^{-1}

   e. det$(A^t)^{-1}$

3. . Suppose $\begin{vmatrix} 1 & -1 & a & 2 \\ 3 & c & 3 & 6 \\ 4 & 1 & -1 & 3 \\ 2 & 2 & 5 & 8 \end{vmatrix} = 2.$

Find a. $\begin{vmatrix} 1 & -1 & a & 2 \\ 4 & 1 & -1 & 3 \\ 2 & 2 & 5 & 8 \\ 3 & c & 3 & 6 \end{vmatrix}$ b. $\begin{vmatrix} 1 & -1 & a & 2 \\ 7 & c+1 & 2 & 9 \\ 2 & 2 & 5 & 8 \\ 3 & c & 3 & 6 \end{vmatrix}$

4.. Let A = $\begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix}$. Find adj A

5. . Let A = $\begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}$. In the set of real numbers , find all values of x for which the matrix is invertible?

6. Let A be a 3x3 matrix and detA = -2 .Then
   (a) det 4A
   (b) $\det A^{-1}$
   (c) $\det A^{5}$
   (d) det(adjA)

7.Let A and B be square matrices of order n.. Show that det(AB)=det(BA).

8..Let A be an invertible matrix .Suppose det B = 4 .Find $\det(A^{-1}BA)$

9..Let A be a square matrix such that $A^t = A^{-1}$ .Then det A

10. Given that $\begin{pmatrix} 1 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 5\mathbf{I}_3$, what is the *determinant* of **A**?

11..Solve

$$\begin{aligned} x - y + 2z &= 1 \\ 2x + y + z &= -1 \\ x + y - z &= -2 \end{aligned}$$

by Cramer's rule.

12..If A = (1,1,3) , B = (1,-1,4) and C = (0,1,2) . Find the area of ΔABC.

13. Let **a** = 2i + j + k and **b =** j + 2k . The area of the parallelogram formed by the vectors **a** and **b**.

14.Let **a** = i + j + k and **b =**2i - j + 2k and **c** = i - j - k

Find

a. The volume of the parallelepiped formed by the vectors **a,b** and **c**

b.The total surface area of the parallelepiped formed by the vectors **a,b** and **c .**

15Show that for any invertible matrix nxn matrix A we have $\det(\text{adj}(A)) = (\det(A))^{n-1}$

16.Show that for any invertible matrix nxn matrix A thematrix adj(A) is also invertible and satisfies $(\text{adj}(A))^{-1} = \text{adj}(A^{-1})$.

17.Let $A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ a & 2 & 3 \end{pmatrix}$.Without expanding the determinant ,find the value of a for which det(A) = 0

18.Find the value of x for which $\begin{vmatrix} x-1 & 2 \\ 2 & x+1 \end{vmatrix}$ =0

19.Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & 3 & 2 \end{pmatrix}$

Find
a. adj(A)
b. A. adj(A)
c. A^{-1}

PART II: **CALCULUS**

Introduction:

This module encompasses four chapters. These are Limits, Continuity, Derivatives, Application of derivatives, Integrals and application of Integrals.

At the successful completion of these modules the students will be able to:

- Compute limits of various functions
- Check the continuity of functions
- Use the concept of limit to define derivative
- Compute derivatives of various functions
- Apply differential calculus to solve a real life problems.
- Use the concept of limits to define a definite integral
- Compute definite and indefinite integrals
- Apply Integral calculus to find area of regions bounded by two curves, compute volume of solids of revolution, compute length of plane curves
- Compute displacement and work done by force

Chapter 1: Revision on limits, continuity and differentiation

Objectives:

At the successful completion of these modules the students will be able to:

- Compute limits of various functions;
- Check the continuity of functions
- Use the concept of limit to define derivative
- Compute derivatives of various functions

1.1 Limit definition and examples

Definition:

Let f be a function defined on an open interval containing a (except possibly at a) and L be a real number. We say $\lim_{x\to a} f(x) = L$ if and only if for all values of x sufficiently close to a, but not necessarily equal to a, the corresponding values of $f(x)$ becomes arbitrarily close to the number L.

Remark: when we write $\lim_{x \to a} f(x) =$ L, we imply two statements.

- The limit exists
- The limit is L

One-sided limits

Definition of left hand limit:

Let f be a function defined on an open interval containing a (except possibly at) and L be a real number. We say $\lim_{x \to a^-} f(x) = L$ if and only if for all values of x sufficiently close to a from the left of $x = a$, but not necessarily equal to a, the corresponding values of $f(x)$ becomes arbitrarily close to the number L.

Definition of Right hand limit:

Let f be a function defined on an open interval containing a (except possibly at) and L be a real number. We say $\lim_{x \to a^+} f(x) = L$ if and only if for all values of x sufficiently close to a from the right of $x = a$, but not necessarily equal to a, the corresponding values of $f(x)$ becomes arbitrarily close to the number L.

Limit Theorems:

Suppose that c is a constant and $\lim_{x \to a} f(x) = L \quad and \quad \lim_{x \to a} g(x) = M$, where L and M are real numbers. Then

1. $\lim_{x \to a} \left[f(x) + g(x) \right] = L + M \qquad sum\ rule$
2. $\lim_{x \to a} \left[f(x) - g(x) \right] = L - M \qquad Difference\ rule$
3. $\lim_{x \to a} \left[cf(x) \right] = cL \qquad Cons\tan t\ multple\ rule$
4. $\lim_{x \to a} \left[f(x) g(x) \right] = LM \qquad product\ rule$
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \quad if\ \ M \neq 0 \qquad Qu\,otient\ rule$

Different types of Limits

a) Limit at infinity

Limit at infinity:

Definition 1: Let f be a function defined on some interval (a,∞). Then $\lim_{x\to\infty} f(x) = L$

Definition 2: Let f be a function defined on some interval $(-\infty,a)$. Then $\lim_{x\to-\infty} f(x) = L$

Example1: prove that $\lim_{x\to\infty}\frac{1}{x} = 0$

Example 2: prove that $\lim_{x\to-\infty}\frac{1}{x+4} = 0$

Definition: The line y = L is called a horizontal asymptote of the curve of y = f(x) if either

$\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

Definition: Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then $\lim_{x\to a} f(x) = \infty$

Definition: Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then $\lim_{x\to a} f(x) = -\infty$

Example1**:** Verify that:

$$\lim_{x\to 1}\frac{-1}{(x-1)^2}=-\infty$$

C) Infinite limits at infinity

Definition:

1. The statement $\lim_{x\to\infty} f(x)=\infty$ means that for any positive number M, there exists a positive number N such that: f(x) > M whenever x > N

2. The statement $\lim_{x\to\infty} f(x)=-\infty$ means that for any negative number M, there exists a positive number N such that: f(x) < M whenever x > N

3. The statement $\lim_{x\to-\infty} f(x)=-\infty$ means that for every negative number M, there exists a negative number N such that: f(x) < M whenever x < N.

Example 1**:** Prove that

a) $\lim_{x\to\infty}\sqrt{x+1}=\infty$ b) $\lim_{x\to-\infty}\left(4+x^3\right)=-\infty$

Example 2**: Cancellation technique**

Find the following limit

$$\lim_{x\to-2}\frac{x^2+x-2}{x+2}$$

Solution: Since $\lim_{x\to-2} x^2+x-2=0$ *and* $\lim_{x\to-2} x-3=0$

Direct substitution produces an indeterminate form 0/0.

In this case we rewrite the fraction by simplifying it

$$\frac{x^2+x-2}{x+2}=\frac{(x+2)(x-1)}{x+2}=x-1 \text{ for } x\neq 2$$

Therefore $\lim_{x\to-2}\frac{x^2+x-2}{x+2}=\lim_{x\to-2}(x-1)=-3$.

Example 3**: Rationalization Technique**

Find $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$

Solution: Since $\lim_{x\to 3}\sqrt{x+1}-2=0$ *and* $\lim_{x\to 3}(x-3)=0$

Direct substitution fails.
In this case we rewrite the fraction by rationalizing the numerator.

$$\frac{\sqrt{x+1}-2}{x-3}=\frac{\sqrt{x+1}-2}{x-3}\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\right)=\frac{1}{2+\sqrt{x+1}} \quad \text{for } x\neq 3$$

Therefore $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}=\lim_{x\to 3}\frac{1}{2+\sqrt{x+1}}=\frac{1}{2+2}=\frac{1}{4}$.

Example 4: **Cancellation technique applied to trigonometric functions.**

Evaluate: $\lim_{x\to 0}\frac{\tan x}{\sin x}$

Solution: $\frac{\tan x}{\sin x}=\frac{\frac{\sin x}{\cos x}}{\sin x}=\frac{\sin x}{(\sin x)(\cos x)}=\frac{1}{\cos x}$

Therefore $\lim_{x\to 0}\frac{\tan x}{\sin x}=\lim_{x\to 0}\frac{1}{\cos x}=\frac{1}{1}=1$

The squeezing Theorem: If $h(x)\leq f(x)\leq g(x)$ for all x in an open interval containing a, except possibly at a itself and if $\lim_{x\to a}h(x)=L=\lim_{x\to a}g(x)$, then $\lim_{x\to a}f(x)=L$

Theorem:

Theorems: Two special trigonometric limits

1. $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$
2. $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0$

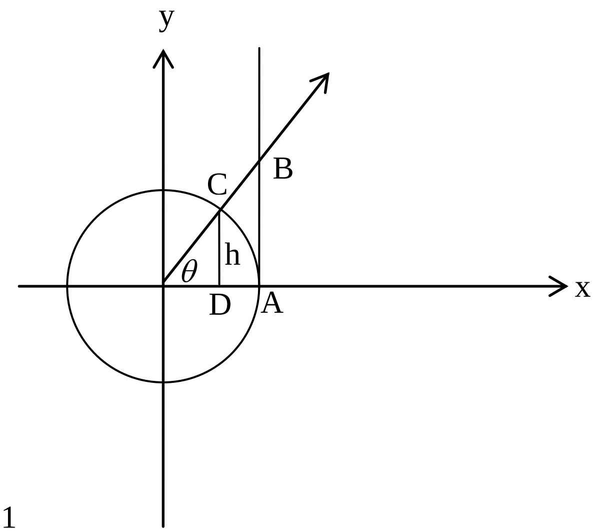


Fig.1

Proof: We prove the first limit and leave the proof of the second as an activity.

Suppose θ is an acute positive angle (measured in radians).
Figure 1 shows a circular section that is squeezed between ΔOAB *and* ΔOCD
We have:
$\sin\theta = h$ and $\tan\theta = AB$

Area of $\Delta OAB \geq area\ \ of\ \sec tor\ \ OAC \geq area\ of\ \ \Delta OCA$

$$\Rightarrow \frac{1}{2}OA.AB \geq \frac{1}{2}\theta \geq \frac{1}{2}OA.CD$$

$$\Rightarrow \frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

Multiplying each expression by $\dfrac{2}{\sin\theta}$ produces

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1$$

and taking reciprocals and reversing the inequalities yields

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$$

Since $\cos\theta = \cos(-\theta)$ *and* $\dfrac{\sin(-\theta)}{-\theta} = \dfrac{\sin\theta}{\theta}$, we conclude that this inequality is valid for all non zero θ in the open interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Since $\lim_{\theta\to 0}\cos\theta = 1$ *and* $\lim_{\theta\to 0} 1 = 1$

We can apply the Squeeze Theorem to conclude that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Continuity

In mathematics the term continuous has much the same meaning as it does in our everyday usage.

To say that a function is continuous at $x = a$ means that there is no interruption in the graph of f at $x = a$ .That is its graph is unbroken at a, there are no holes, jumps or gaps.

Definition of continuity

Continuity at a point: A function f is said to be continuous at a if the following three conditions are meet.

1. $f\ (a)$ is defined (a is in the domain of f)
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

Definition:
A function f is said to be continuous on an open interval (a,b) if it is continuous at each point in the interval.

Remark: A function f is said to be discontinuous at a if f is defined on an open interval containing a (except possibly at a) and f is not continuous at a.

Discontinuities fall in to two categories: removable and non removable (essential) discontinuity.

A discontinuity at $x = a$ is called **removable** if f can be made **continuous** by appropriately defining (re-defining) f at $x = a$, otherwise it is said to be **essential discontinuity.**

The discontinuity is removable if the limit of the function exists at that point.

Example1: Find the point of discontinuity of the following functions and determine whether the discontinuity is removable or essential.

a. $f(x)=\begin{cases}\dfrac{x^2-x-2}{x-2} & if \;\; x\neq 2\\ 1 & if \;\; x=2\end{cases}$ b. $g(x)=\begin{cases}1-x^2 & if \;\; x<0\\ -1 & if \;\; x=0\\ 2+x & if \;\; x>0\end{cases}$

Solution:

a. i. $f(2)=1$ is defined

ii. $\lim\limits_{x\to 2} f(x)=\lim\limits_{x\to 2}\dfrac{x^2-x-2}{x-2}=\lim\limits_{x\to 2}\dfrac{(x-2)(x+1)}{x-2}=\lim\limits_{x\to 2}(x+1)=3$

iii. But $\lim\limits_{x\to 2} f(x)=3\neq 1=f(2)$

Therefore f is discontinuous at $x = 2$.

f to be continuous on the entire real line we can redefine $f\ (2)$ equal to the $\lim\limits_{x\to 2} f(x)$.

Hence the discontinuity at $x = 2$ is removable.

b. i. g (0) = -1 is defined

ii. $\lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^-}(1-x^2)=1$ and $\lim\limits_{x\to 0^+} g(x)=\lim\limits_{x\to 0^+}(2+x)=2$

$\Rightarrow \lim\limits_{x\to 0} g(x)$ does not exist

Thus g is discontinuous at $x = 0$, the discontinuity at $x = 0$ is essential because it is not possible to redefine $g\ (0)$ to make it continuous at 0.

Definition: The greatest integer function $f(x)=[x]$ is the greatest integer less or equal to x.

Example 2**:** show that the greatest integer function $f(x)=[x]$ is discontinuous at all of the integers

$\lim\limits_{x\to n^-} f(x)=\lim\limits_{x\to n^-}[x]=n-1$ and $\lim\limits_{x\to n^+} f(x)=\lim\limits_{x\to n^+}[x]=n$

$\Rightarrow \lim\limits_{x\to n} f(x)=\lim\limits_{x\to n}[x]$ does not exist.

Therefore $f(x)=[x]$ is discontinuous and the discontinuity is essential

Since $\lim\limits_{x\to n}[x]$ does not exist.

One- Sided continuity

Definition: A function f is continuous from

i. the right at a if $\lim\limits_{x\to a^+} f(x) = f(a)$ and

ii. the left at a if $\lim\limits_{x\to a^-} f(x) = f(a)$

Example1: Let $f(x) = \begin{cases} 1+3x^2 & if \;\; x \geq -1 \\ 4-x^3 & if \;\; x < -1 \end{cases}$

Show that f is continuous from the right at -1 but discontinuous from the left at -1.

Solution: Let us compute the one sided limits:

$\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} \left(1+3x^2\right) = 1+3(-1)^2 = 4$

$\lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} \left(4-x^3\right) = 4-(-1)^3 = 5$

and f(-1) = 1+3(-1)^2 = 4

$\lim\limits_{x\to -1^+} f(x) = f(-1)$ and $\lim\limits_{x\to -1^-} f(x) = 5 \neq 4 = f(-1)$

Therefore, by definition f is continuous from the right at -1 but discontinuous from the left at -1.

Example2: Show that $f(x) = \sqrt{x-6}$ is continuous from the right at x = 6

Solution: $\lim\limits_{x\to 6^+} f(x) = \lim\limits_{x\to 6^+} \sqrt{x-6} = 0 = f(6)$

Therefore by definition f is continuous from the right at x = 6

Example 3: Show that $f(x) = [x]$ is continuous from the right but not continuous from the left at any integer n.

Solution: We have seen that $f(x) = [x]$ is discontinuous at all integers

Now let us compute the one - sided limits

$\lim\limits_{x\to n^-} f(x) = \lim\limits_{x\to n^-} [x] = n-1$ and $\lim\limits_{x\to n^+} f(x) = \lim\limits_{x\to n^+} [x] = n$

But $f(n) = [n] = n$

Hence we have $\lim\limits_{x\to n^+} f(x) = \lim\limits_{x\to n^+} [x] = n = f(n)$

Therefore by definition, f is continuous from the right but not continuous from the left at any integer n.

Continuity on a closed interval

Definition: A function f is continuous on the closed interval $[a,b]$ if it is

1. continuous on the open interval (a,b)
2. $\lim_{x\to a^+} f(x) = f(a)$ *and* $\lim_{x\to b^-} f(x) = f(b)$

Example: Discuss the continuity of $g(x) = \begin{cases} 5+x & if\ -1 \le x \le 2 \\ x^2-1 & if\ \ 2 < x \le 3 \end{cases}$

Solution: We know that the polynomial function given by 5 - x and x^2 -1 are continuous for all real x.

To conclude that g is continuous on the entire interval $[-1,3]$, we need only check the behavior of g when x = 2.

By taking the one sided limits when x = 2, we see that

$\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-}(5+x) = 7$ and

$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+}(x^2-1) = (2^2-1) = 3$

$\lim_{x\to 2^+} g(x) = 3 \ne 7 = \lim_{x\to 2^-} g(x)$

Therefore g is not continuous at x = 2.

Example2: Find the constant c that makes $f(x) = \begin{cases} x^2-c^2 & if\ \ x<4 \\ cx+20 & if\ x \ge 4 \end{cases}$ continuous on the entire real number.

Solution: F to be continuous on $(-\infty,\infty)$, it should be continuous at x = 4

That is:

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x)$$

$$\Rightarrow \lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(x^2-c^2) = 16-c^2 = \lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}(cx+20) = 4c+20$$

$$\Rightarrow 16-c^2 = 4c+20$$

$$\Rightarrow c^2+4c+4 = 0 \Rightarrow (c+2)^2 = 0 \Rightarrow c = -2$$

Therefore the value of c that makes f continuous on the entire real line is c= -2.

Theorem: If f and g are continuous at a and c is a constant then the following functions are also continuous at a.

1. $f + g$ 2. $f - g$ 3. cf 4. fg 5. $\frac{f}{g}$ if $g(a) \ne 0$

Each of the five parts of the theorem follows from the corresponding limit laws

Proof of 4:
Since f and g are continuous at a, we have $\lim_{x\to a} f(x) = f(a)$ *and* $\lim_{x\to a} g(x) = g(a)$
Then $\lim_{x\to a} fg(x) = \lim_{x\to a} f(x).\lim_{x\to a} g(x) = f(a).g(a) = fg(a)$
Therefore fg is continuous at a.

Remark:
The following functions are continuous at every number in their domain.

Polynomial functions	rational functions	root functions
Trigonometric functions	exponential functions	logarithmic functions

Theorem: If f is continuous at b and $\lim_{x\to a} g(x) = b$, *then* $\lim_{x\to a} f(g(x)) = f(b)$

In other words,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$$

1.2 The Intermediate Value Theorem (IVT)

The Intermediate Value Theorem (IVT)

If f is continuous on $[a,b]$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in (a,b) such that $f(c) = k$.

Remark:

1. A discontinuous function might not possess the IVT property
2. The Intermediate Value Theorem can be used to locate the zeros of a function that is continuous on a closed interval.

Corollary: If f is continuous on $[a,b]$ and f(a) and f(b) have opposite signs, then f has at least one zero on $[a,b]$

Example1: Find all values of c guaranteed by the IVT for the function

$$f(x)=\frac{x^2+x}{x-1} \quad on \quad \left[\frac{5}{2},4\right] \quad , where \quad f(c)=6$$

Solution: The domain of f is all real number except 1.

As f is a rational function it is continuous on $\left[\frac{5}{2},4\right]$.

We want to find the values of c such that $f(c)=6$

Then by the IVT, we have

$$f(c)=\frac{c^2+c}{c-1}=6$$
$$\Rightarrow c^2+c=6c-6 \Rightarrow c^2-5c+6=0$$
$$\Rightarrow (c-3)(c-2)=0 \Rightarrow c=3 \quad or \quad c=2$$

Therefore the values of c in $\left[\frac{5}{2},4\right]$ guaranteed by the IVT are 2and 3.

Example 2: Use the Intermediate value Theorem to show that $f(x)=x^3+2x-1$ has a root on $[0,1]$

Solution: Since f (x) is a polynomial function it is continuous on $[0,1]$.

And f (0) = -1< 0 and f(1) = 1^{3} +2(1) -1=3 -1 =2 > 0

Therefore by the corollary of IVT, f has at least one root on $[0,1]$.

EXERCISE ON LIMITS AND CONTINUITY

1. Show by means of examples that $\lim_{x\to a}\left[f(x)+g(x)\right]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

2. Evaluate the following limits if they exist.

a. $\lim_{x\to 1}\frac{1-x}{\sqrt{5-x^2}-2}$ b. $\lim_{h\to 0}\frac{(2+h)^2-8}{h}$

c. $\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}$ d. $\lim_{x\to -3}\frac{t^2-9}{2t^2+7t+3}$

3. a. If $1\le f(x)\le x^2+2x+2$, for all x find $\lim_{x\to -1} f(x)$

b. Show that $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

c. Find $\lim_{x\to 0} x^2\cos\frac{1}{x}$

4. a) Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left or neither?

a $f(x)=\begin{cases} x+1 & if\ x\le 1 \\ \frac{1}{x} & if\ 1<x<3 \\ \sqrt{x-3} & if\ x\ge 3 \end{cases}$

b. Find the values of k so that f is continuous on $(-\infty,\infty)$, where

$f(x)=\begin{cases} \frac{3-\sqrt{x}}{9-x} & if\ \ x\ne 9 \\ k & if\ x=9 \end{cases}$

c. Use continuity to evaluate $\lim_{x\to \pi}\sin(x+\sin x)$

5. Use the Intermediate Value Theorem to show that the following functions has at least one root in the indicated intervals

a. $f(x)=x^4+5x+3,\ \ [-1,0]$ b. $e^{-x}=x,\ \ [0,1]$ c. $x^4+x-3=0$, $(1,2)$

1.3 Differentiation

Here we briefly deals about geometric meaning of the derivatives, Rules of differentiation, derivative of algebraic and transcendental functions, the Chain Rule, implicit differentiation and higher order derivatives.

Objectives: By the completion of this chapter the students will be able to:

. Give the geometric meaning of derivative of a function

. Define the derivative of a function at a point and at an arbitrary point in its domain

. Find the slope of a tangent line to any curve at a given point

. Use techniques of differentiation to find the derivative of constant times a function, a sum, a product, of two or more differentiable functions and quotient of two differentiable functions.

. Apply the Chain Rule to find derivatives of composite functions

. Find derivatives of derivatives (higher order) derivatives.

Geometric meaning of derivative of a function

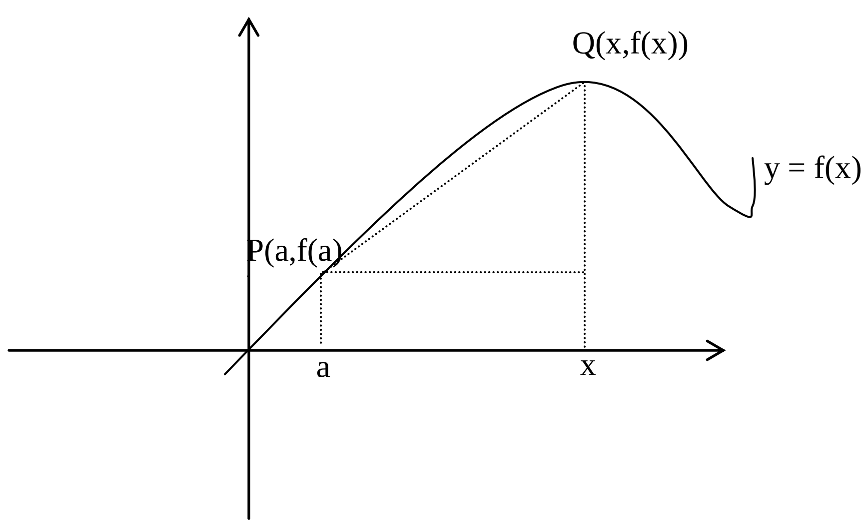
Tangent lines

Definition: The tangent line to the curve $y = f(x)$ at the point p (a, f (a)) is the line through p with slope

$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ provided that this limit exists.

And the equation of the tangent line to the given curve at p is given by

$y - f(a) = m(x - a)$.



Example1: Find the slope and the equation of the tangent line to the curve of $y = x^2 +1$ at the point (-1, 2).

Solution: Here we have given a = -1 and $f(x) = x^2 +1$. Then the slope to the curve at (-1,2) is.

$$m = \lim_{x \to -1} \frac{f(x) - f(1)}{x - (-1)} = \lim_{x \to -1} \frac{x^2 + 1 - 2}{x+1} = \lim_{x \to -1} \frac{x^2 - 1}{x+1} = \lim_{x \to -1} \frac{(x-1)(x+1)}{x+1} = \lim_{x \to -1} (x-1) = -2$$

Then using slope point form of equation of a line, we have y -2 =-2(x-(-1)) = -2x-2

There is another expression for the slope of a tangent line that is sometimes easier to use. Let h = x – a. Then x= a +h

So the slope of the secant line $\overleftrightarrow{PQ}$ of fig.3 is:

m = $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ ------- $(as \quad x \to a, \quad h \to 0)$

Definition of Derivative of a Function at a point

Therefore we have the following alternative definition of derivative.

> **Definition:** The derivative of a function f at a number a, denoted by $f'(a)$ is
>
> $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad (1) \quad \text{if this limit exists.}$$
>
> If we write $x = a + h$, then $h = x - a$ and $h \to 0 \;\; iff \;\; x \to a$

Example 1: Use the definition of derivative to find $f'(2)$, where $f(x) = x^2 - x$

Solution: By definition, we have

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x-2} = \lim_{x \to 2} \frac{x^2 - x - (2^2 - 2)}{x-2} = \lim_{x \to 2} \frac{x^2 - x - 2}{x-2}$$

$$= \lim_{x \to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x \to 2}(x+1) = 3$$

Therefore, $f'(2) = 3$

Example 2: Use the definition of derivative to find $f'(-1)$ where, $f(x) = x^3 + 1$

Solution: By definition, we have

$$f'(-1) = \lim_{x\to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x\to -1} \frac{x^3 + 1 - 0}{x + 1} = \lim_{x\to -1} \frac{(x+1)(x^2 - x + 1)}{x+1}$$

$$f'(-1) = \lim_{x\to -1} \frac{(x+1)(x^2 - x + 1)}{x+1} = \lim_{x\to -1}(x^2 - x + 1) = 3$$

The Derivative as a function

Definition: Let **y = f(x)** be a function. The **derivative of f** is the function whose value at x is the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

If this limit exists for each **x** in an open interval **I**, then we say that **f is differentiable on I**.

Note: Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we regard f' as a new function, called the derivative of f at x.

Notation:

In addition to f'(x), read "f prime of x", other notations are used to denote the derivative of y = f(x). The most common are:

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}(f(x)), \quad D_x(y)$$

Remark: The notation $\frac{dy}{dx}$ read as the derivative of y with respect to x.

Using limit notation:

$$\frac{dy}{dx} = \lim_{\Delta x\to} \frac{\Delta y}{\Delta x} = \lim_{\Delta x\to} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x)$$

Example 1: Let $f(x) = \frac{1}{\sqrt{x}}$. Then use the definition of derivative to find $f'(x)$

Solution: $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h}$

$$=\lim_{h\to 0}\frac{\dfrac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h}=\lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$$

$$=\lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}\cdot\left(\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right)=\lim_{h\to 0}\frac{x-x-h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$$

$$=\lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}=\frac{-1}{2x\sqrt{x}}=-\frac{1}{\sqrt{x^3}}$$

$$\therefore f'(x)=-\frac{1}{\sqrt{x^3}}$$

Example 2: Let $g(x)=x^3-x$. Then use the definition of derivative to find the formula for $g'(x)$

Solution: by definition, we have:

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{\left[(x+h)^3-(x+h)\right]-x^3}{h}$$

$$=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x-h-x^3}{h}$$

$$=\lim_{h\to 0}\frac{h\left(3x^2+3hx+h-1\right)}{h}=\lim_{h\to 0}\left(3x^2+3hx=h-1\right)=3x^2-1$$

$$\therefore g'(x)=3x^2-1$$

Definition: Let **y = f(x)** be a function and a be in the domain of **f**.

The **right-hand derivative of f at x = a** is the limit

$$f'_+(a)=\lim_{h\to a^+}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

and the **left-hand derivative of f at x = a** is the limit

$$f'_-(a)=\lim_{h\to a^-}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}$$

The function f is differentiable at a if $f'_-(a)=f'_+(a)$ and f differentiable on an interval if the derivative exists for each point in that interval.

Example 3: Show that $h(x)=|x|$ is not differentiable at 0.

Solution**:** If x > 0, then $|x|=x$

So $f_+{}'(0)=\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^+}\dfrac{x}{x}=\lim\limits_{x\to 0^+}1=1$

If x < 0, then $|x|=-x$

$f_-{}'(0)=\lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^-}\dfrac{-x}{x}\lim\limits_{x\to 0-}(-1)=-1$

$\Rightarrow$ the left and right hand side limits are not equal.

This implies, $f'(0)=\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}$ does not exist

Therefore, f is not differentiable at o.

Example 4: Assume that

$$f(x)=\begin{cases} 2+\sqrt{x} & if\;\; x\geq 1 \\ \dfrac{1}{2}x+\dfrac{5}{2} & if\;\; x<1 \end{cases}$$

Show that *f* is differentiable at *x* =1, i.e., use the limit definition of the derivative to compute $f'(1)$.

Solution*:* The derivative at *x*=1 is

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(1+\Delta x)-f(1)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{f(1+\Delta x)-\left(2+\sqrt{1}\right)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{f(1+\Delta x)-3}{\Delta x}$$

Note that Δx can be either positive or negative,

$1+\Delta x>1\;\; when\;\; \Delta x>0\;\;\; and\;\; 1+\Delta x<1\; when\;\;\; \Delta x<0$.

Thus

$$f\left(1+\Delta x\right)=\begin{cases}2+\sqrt{1+\Delta x} & if \quad \Delta x>0\\ \dfrac{1}{2}\left(1+\Delta x\right)+\dfrac{5}{2} & if \quad \Delta x<0\end{cases}$$

Further work requires the use of one-sided limits. First, the right-hand limit is

$$\lim_{\Delta x\to 0^+}\frac{f\left(1+\Delta x\right)-f\left(1\right)}{\Delta x}=\lim_{\Delta x\to 0^+}\frac{2+\sqrt{1+\Delta x}-3}{\Delta x}$$

$$=\lim_{\Delta x\to 0^+}\frac{\sqrt{1+\Delta x}-1}{\Delta x}$$

$$=\lim_{\Delta x\to 0^+}\frac{\sqrt{1+\Delta x}-1}{\Delta x}\ \frac{\sqrt{1+\Delta x}+1}{\sqrt{1+\Delta x}+1}$$

$$=\lim_{x\to\infty}\frac{\left(1+\Delta x\right)-1}{\Delta x\left(\sqrt{1+\Delta x}+1\right)}$$

$$=\lim_{\Delta x\to 0^+}\frac{\Delta x}{\Delta x\left(\sqrt{1+\Delta x}+1\right)}$$

$$=\lim_{\Delta x\to 0^+}\frac{1}{\sqrt{1+\Delta x}+1}$$

$$=\lim_{\Delta x\to 0^+}\frac{1}{\sqrt{1+0}+1}$$

$$=\frac{1}{2}.$$

The left-hand limit is

$$\lim_{\Delta x\to 0^-}\frac{f\left(1+\Delta x\right)-3}{\Delta x}=\lim_{\Delta x\to 0^-}\frac{\frac{1}{2}\left(1+\Delta x\right)+\frac{5}{2}-3}{\Delta x}$$

$$= \lim_{\Delta x \to 0^-} \frac{\frac{1}{2} + \frac{1}{2}\Delta x + \frac{5}{2} - 3}{\Delta x}$$

$$= \lim_{\Delta x \to 0^-} \frac{\frac{1}{2}\Delta x}{\Delta x}$$

$$= \frac{1}{2}$$

Thus, both one-sided limits exist and are equal, so that *f* is differentiable at *x* =1 with derivative equal to $\frac{1}{2}$

Rules of differentiation

> **Theorem 2:1** Derivative of a constant function.
>
> The derivative of a constant is zero, $\frac{d}{dx}(c) = 0$

Proof: Let f(x) = c.

Then $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$f'(x) = \lim_{h \to 0} \frac{c - c}{h}$$

$$f'(x) = \lim_{h \to 0} 0$$

$$f'(x) = 0$$

Example: Find the derivative of the following functions

a. $f(x) = \pi\sqrt{3}$ b. $g(x) = \frac{-e}{\sqrt[3]{7}}$

Solution:

a. $f(x)=\pi\sqrt{3}$ for all real number x and hence it is a constant function.

Therefore, $\frac{d}{dx}f(x)=\frac{d}{dx}\left(\pi\sqrt{3}\right)=0$

b. Again $g(x)=\frac{-e}{\sqrt[3]{7}}$ is a constant function.

Hence $\therefore \frac{d}{dx}g(x)=\frac{d}{dx}\left(\frac{-e}{\sqrt[3]{7}}\right)=0$

Theorem 2:2 The constant Multiple rule

If c is a constant and f is differentiable function, then

$$\frac{d}{dx}\left(cf(x)\right)=c\frac{d}{dx}f(x)=c\ f'(x)$$

Proof: Let $g(x)=cf(x)$.Then

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$g'(x)=\lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$

$$g'(x)=c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$g'(x)=c\ f'(x)$$

Therefore,

$$\frac{d}{dx}\left(cf(x)\right)=c\ f'(x)$$

Example: Find the derivative of the following functions.

a. $f(x)=\frac{2}{3}x^3$ b. $g(x)=2\sqrt{x}$

Solution**:**

By constant multiple rule, we have

a. $\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{2}{3}x^3\right)=\frac{2}{3}\left(3x^2\right)=2x^2$

b. $\frac{d}{dx}g(x)=\frac{d}{dx}\left(2\sqrt{x}\right)=2\left(\frac{d}{dx}x^{\frac{1}{2}}\right)=2\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$

Therefore, $\frac{d}{dx}\left(2\sqrt{x}\right)=\frac{1}{\sqrt{x}}$

Theorem 2:3 Sum and difference rules

The derivative of the sum (or difference) of two differentiable functions is the sum (or difference) of their derivatives.

i. $\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)=f'(x)+g'(x)$

ii. $\frac{d}{dx}\left[f(x)-g(x)\right]=\frac{d}{dx}f-\frac{d}{dx}g(x)=f'(x)-g'(x)$

Proof**:** The proof directly follows from the limit laws of sum and difference.

i. $$(f+g)'(x)=\lim_{h\to 0}\frac{\left[(f+g)(x+h)-(f+g(x))\right]}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)+g(x+h)-\left(f(x)+g(x)\right)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\quad by\ \lim it\ Sum\ rule$$

Therefore, $(f+g)'(x)=f'(x)+g'(x)$ Or

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)=f'(x)+g'(x)$$

Since f(x) - g(x) = f(x) + - g(x) the proof of ii is similar to that of i.

Remark: The sum and difference rules can be extended to cover the derivative of any finite number of functions

$Let\ \ p(x)=a_nx^n+a_{n-1}x^{n-1}+---+a_2x^2+a_1x+a_0$ be a polynomial. Then

$$p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + ---+ 2a_2 x + a_1 \text{ by using sum rule}$$

Example: Find the derivative of the following functions.

a. $f(x) = x^3 - 4x + 5$ b. $g(x) = -\frac{x^4}{2} + 3x^3 - 2x$

Solution: by using sum and constant rule, we get

a. $f'(x) = 3x^2 - 4$

b. $g'(x) = -\frac{4}{2}x^3 + 3(3x^2) - 2$

$$g'(x) = -2x^3 + 9x^2 - 2$$

Theorem 2:4 Product Rule:

The product of differentiable functions f and g is itself differentiable

Moreover the derivative of f.g is given by the first function times the derivative of the second plus the second function times the derivative of the first.

$$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$$

Proof: $(fg)'(x) = \lim_{h\to 0} \frac{fg(x+h) - fg(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h}\right]$$

$$= \lim_{h\to 0} f(x+h).\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x).\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$(fg)'(x) = f(x)g'(x) + g(x)f'(x)$$

Remark: The product rule can be extended to cover products involving more than two terms.

Example: Find the derivative of the following functions.

a. $f(x)=\left(\frac{1}{x^2}-\frac{3}{x^4}\right)\left(x+5x^3\right)$

b. $g(t)=\left(3t-2t^2\right)(5+4t)$

c. $h(x)=\left(x^2-x\right)\left(x^2+1\right)\left(x^2+x+1\right)$

Solution**:** a. First re-write f(x) as follows

$$f(x)=\left(x^{-2}-3x^{-4}\right)\left(x+5x^3\right)$$

$$f'(x)=\left(x^{-2}-3x^{-4}\right)\left(x+5x^3\right)'+\left(x^{-2}-3x^{-4}\right)\left(x+5x^3\right)$$

$$f'(x)=\left(x^{-2}-3x^{-4}\right)\left(1+5(3x^2)\right)+\left(-2x^{-3}-3(-4x^{-3})\right)\left(x+5x^3\right)$$

$$=\left(\frac{1}{x^2}-\frac{3}{x^4}\right)\left(1+15x^2\right)+\left(-\frac{2}{x^3}+\frac{12}{x^4}\right)\left(x+5x^3\right)$$

$$=\left(\frac{1}{x^2}+15-\frac{3}{x^4}-\frac{45}{x^2}\right)+\left(-\frac{2}{x^2}-10+\frac{12}{x^4}+\frac{60}{x^2}\right)$$

$$f'(x)=\frac{9}{x^4}+\frac{14}{x^2}+5$$

b. $g'(t)=\left(3t-2t^2\right)(5+4t)'+(5+4t)\left(3t-2t^2\right)'$

$$=\left(3t-2t^2\right)(4)+(5+4t)(3-4t)$$

$g'(t)=-24t^2+2t+15$ c.

$$h'(x)=\left(x^2-x\right)'\left(x^2+1\right)\left(x^2+x+1\right)+\left(x^2-x\right)\left(x^2+1\right)'\left(x^2+x+1\right)+\left(x^2-x\right)\left(x^2+1\right)\left(x^2+x+1\right)'$$

$$h'(x)=(2x-1)\left(x^2+1\right)\left(x^2+x+1\right)+\left(x^2-x\right)2x\left(x^2+x+1\right)+\left(x^2-x\right)\left(x^2+1\right)(2x+1)$$

Theorem 2:5: Quotient Rule

The quotient of two differentiable functions, f and g is itself differentiable at all values of x for which $g(x)\neq 0$.

Moreover, the derivative of $\frac{f}{g}$ is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator divided by the square of the denominator.

$$\left(\frac{f}{g}\right)'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

Proof: Suppose f and g are differentiable at all values of x for which $g(x)\neq 0$.
Then by definition we have:

$$\left(\frac{f}{g}\right)'(x)=\lim_{h\to 0}\frac{\left(\frac{f}{g}\right)(x+h)-\left(\frac{f}{g}\right)(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{hg(x)g(x+h)}$$

$$=\lim_{h\to 0}\frac{g(x)}{g(x)g(x+h)}\frac{f(x+h)-f(x)}{h}-\lim_{h\to 0}\frac{f(x)}{g(x)g(x+h)}\frac{g(x+h)-g(x)}{h}\qquad \text{Why?}$$

$$=\lim_{h\to 0}\frac{g(x)}{g(x)g(x+h)}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}-\lim_{h\to 0}\frac{f(x)}{g(x)g(x+h)}\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\qquad \text{Why?}$$

$$=\frac{g(x)}{(g(x))^2}f'(x)-\frac{f(x)}{(g(x))^2}g'(x)\qquad \text{Why?}$$

$$\left(\frac{f}{g}\right)'(x)=\frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

Example**:** Differentiate

a. $f(x)=\dfrac{2x^2-x}{1-3x^2}$ b. $g(t)=\dfrac{3-\frac{1}{t}}{t+5}$

Solution:

a. By using quotient Rule, we have

$$f'(x)=\frac{\left(1-3x^2\right)\left(2x^2-x\right)'-\left(2x^2-x\right)\left(1-3x^2\right)'}{\left(1-3x^2\right)^2}$$

$$f'(x)=\frac{\left(1-3x^2\right)(4x-1)-\left(2x^2-x\right)(-6x)}{\left(1-3x^2\right)^2}$$

$$f'(x)=\frac{-3x^2+4x+1}{\left(1-3x^2\right)^2}$$

b. First rewrite the expression as follows:

$$g(t)=\frac{3-\frac{1}{t}}{t+5}=\frac{3t-1}{t(t+5)}=\frac{3t-1}{t^2+5t}$$

Then by using quotient Rule, we get

$$g'(t)=\frac{\left(t^2+5t\right)(3t-1)(2t+5t)'}{\left(t^2+5t\right)^2}$$

$$=\frac{3\left(t^2+5t\right)-(3t-1)(2t+5)}{\left(t^2+5t\right)}$$

$$g'(t)=\frac{-3t^2+2t+5}{\left(t^2+5t\right)}$$

Derivatives of Trigonometric function

Recall that, we have seen

$\lim\limits_{x\to0}\frac{\sin x}{x}=1$ and $\lim\limits_{x\to0}\frac{\cos x-1}{x}=0$

In this section we will find the derivative of sine and cosine function.

Example 1: Let $f(x)=\sin x$.Then find $f'(x)$ by using the definition of derivative.

Solution: $f'(x)=\lim\limits_{h\to0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h - \sin h\, x + \sin h \cos x}{h}$$

$$= \lim_{h\to 0} \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \lim_{h\to 0} \frac{\sin h}{h} \text{ by product Rule}$$

$$= (\sin x)\ 0 + \cos x.1$$

$\lim_{h\to 0} \sin x = \sin x$ and $\lim_{h\to 0} \cos x = \cos x$ Since both are constants with respect to h

Therefore, $f'(x) = \cos x$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Similarly, it is easy to show that:

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Example 2**:** Differentiate each of the following function.

a. $f(x) = \tan x$ b. $g(x) = \csc x$

Solution: a) First rewrite $\tan x = \dfrac{\sin x}{\cos x}$ and then apply the quotient Rule to find $f'(x)$.

$$f'(x) = \frac{\cos x \frac{d}{dx}(\sin x) - (\cos x)}{\cos^2}$$

$$= \frac{\cos x . \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Hence, $f'(x) = \sec^2 x$

The Chain Rule:

If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product $F'(x) = f'(g(x)).g'(x)$.

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable function, then

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

To prove the Chain rules consider that, if $y = f(x)$ and x changes from a to $a + \Delta x$, we define the increment of y as:

$$\Delta y = f(a + \Delta x) - f(a).$$

Then by definition, we have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = f'(a)$$

If we define $\varepsilon = \begin{cases} \dfrac{\Delta y}{\Delta x} - f'(a) & if \ \Delta x \neq 0 \\ 0 & If \ \Delta x = 0 \end{cases}$

Then $\dfrac{\Delta y}{\Delta x} = \varepsilon + f'(a)$ for $\Delta x \neq 0$

$\Rightarrow \Delta y = f'(a)\Delta x + \varepsilon \ \Delta x$

Since $\lim\limits_{\Delta x \to 0} \varepsilon = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} - f'(a) = f'(a) - f'(a) = 0 = \varepsilon(0), \varepsilon$ becomes a continuous function of Δx.

Thus for a differentiable f we can write:

$\Delta y = f'(a)\Delta x + \varepsilon \ \Delta x$ ------- (1), where $\varepsilon \to 0$ as $\Delta x \to 0$ and ε is a continuous function of Δx.

This property of differentiable function enables us to prove the Chain Rule.

Proof of the Chain Rule:

Suppose $u = g(x)$ is differentiable at a and $y = f(u)$ be differentiable at $b = g(a)$.

If Δx is an increment in x and Δu and Δy are the corresponding increments in u and y, then we can use equation (1) to write

$\Delta u = g'(a)\Delta x + \varepsilon_1 \Delta x = \left(f'(b) + \varepsilon_1\right)\Delta u,$ where $\varepsilon_1 \to 0$ as $\Delta x \to 0$ ----- (2)

Similarly,

. $\Delta y f'(b)\Delta u + \varepsilon_2 \Delta u = \left(f'(b) + \varepsilon_2\right)\Delta u,$ where $\varepsilon_2 \to 0$ as $\Delta u \to 0$ ------ (3)

If we now substitute the expression for Δu from equation (2) in to equation (3),

We get:

$$\Delta y = f'(b)\Delta u + \varepsilon_2 \Delta u(g'(a) + \varepsilon_1)\Delta x$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = f'(b)\Delta u + \varepsilon_2 \Delta u(g'(a) + \varepsilon_1)$$

As $\Delta x \to 0$ equation 5 shows that $\Delta u \to 0$, so both $\varepsilon_2 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta x \to 0$

Therefore,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} f'(b)\Delta u + \varepsilon_2 \Delta u(g'(a) + \varepsilon_1)$$

$$= f'(b)g'(a) = f'(g(a)).g'(a)$$

Thus

$$\boxed{\left(f(g(x))\right)' = f'(g(x))g'(x)}$$

Example 1**:** Evaluate the derivative of the following functions.

a. $F(x) = \sin x^2$ b. $G(t) = \sqrt{t^2 + t}$ c. $H(\theta) = \sin\left(\tan\frac{1}{\theta}\right)$

Solution:

a. Let $g(x) = x^2, g'(x) = 2x$

and $f(t) = \sin t,\ f'(t) = \cos t$

Since $F(x) = \sin x^2 = f(g(x))$, then

$$F'(x) = f'(g(x))g'(x)$$

$$F'(x) \quad = \cos x^2(2x) = 2x\cos x^2$$

b. If we let $g(t) = (t^2 + t)$, then $g'(t) = 2t + 1$

and if we let $f(x) = \sqrt{x}$, then $f'(x) = \dfrac{1}{2\sqrt{x}}$

As $G'(t)=\dfrac{1}{2\sqrt{t^2+t}}(2t+1)=\dfrac{2t+1}{2\sqrt{t^2+t}}$

c. $H(\theta)$ can be expressed as a composition of three functions as shown below.

$$\text{If } h(\theta)=\frac{1}{\theta}, \qquad h'(\theta)=-\frac{1}{\theta^2}$$
$$g(x)=\tan x, \qquad g'(x)=\sec^2 x$$
$$f(t)=\sin t, \qquad f'(t)=\cos t$$

Then $H(\theta)=f(g(h(\theta)))$, then by the chain Rule,

$$H'(\theta)=f'(g(h(\theta)))g'(h(\theta))h'(\theta)$$

$$=\cos\left(\tan\left(\frac{1}{\theta}\right)\right)\sec^2\left(\frac{1}{\theta}\right)\left(\frac{1}{\theta^2}\right)$$

$$H'(\theta)=-\frac{1}{\theta^2}\cos\left(\tan\left(\frac{1}{\theta}\right)\right)\sec^2\left(\frac{1}{\theta}\right)$$

The power rule combined with the chain rule

If n is any real number and $u=g(x)$ is differentiable, then

$$\frac{d}{dx}\left(u^n\right)=nu^{n-1}\frac{du}{dx} \; or \; \frac{d}{dx}(g(x))^n=n(g(x))^{n-1}g(x)$$

Example 2: Differentiate each of the following functions.

a. $y=\left(\dfrac{6-5x}{x^2-1}\right)^2$ b. $y=e^{\sec 3\theta}$ c. $y=\sin(\cos(\sec x))$

Solution:

a. By using the power Rule together with the Chain Rule, we have

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{6-5x}{x^2-1}\right)^2=2\left(\frac{6-5x}{x^2=1}\right)\frac{d}{dx}\left(\frac{6-5x}{x^2-1}\right)$$

$$=\left(\frac{6-5x}{x^2-1}\right)\frac{-5(x^2-1)-2x(6-5x)}{(x^2-1)^2}$$

Thus,

$$\frac{dy}{dx}=(12-10x)\frac{5x^2-12x+5}{(x^2-1)}$$

b. Using the Chain rule, we get

$$\frac{dy}{d\theta}=\frac{d}{d\theta}\left(e^{\sec 3\theta}\right)$$

$$=e^{\sec 3\theta}\frac{d}{d\theta}(\sec 3\theta)=e^{\sec 3\theta}\sec 3\theta\tan 3\theta\frac{d}{d\theta}(3\theta)$$

$$\frac{dy}{d\theta}=3e^{\sec 3\theta}\sec 3\theta\tan 3\theta$$

c. By using the Chain Rule we have,

$$\frac{dy}{dx}=\frac{d}{dx}(\sin(\cos(\sec x)))$$

$$=\cos(\cos(\sec x))\frac{d}{dx}(\cos(\sec x))$$

$$=\cos(\cos(\sec x))(-\sin(\sec x))\frac{d}{dx}\sec x$$

$$\frac{dy}{dx}=-\cos(\cos(\sec x))\sin(\sec x)\sec x\tan x$$

The number e as a limit

The expression

$\lim_{h\to 0}\frac{e^h-1}{h}=1$ means that for very small values of h

$$e^h-1\approx h$$

$$e^h\approx 1+h$$

$$e\approx(1+h)^{\frac{1}{h}}$$

So,

Or, $$\boxed{e=\lim_{h\to 0}(1+h)^{\frac{1}{h}}=2.71828...}$$ if we say that $t=\frac{1}{h}\left(as\, h\to 0^{+}, t\to\infty,\right)$ and $h\to 0^{-}, t\to -\infty)$ .Hence we have

$$\boxed{e=\lim_{t\to \inf ty}\left(1+\frac{1}{t}\right)^{t}=2.71828...}$$

Derivative of a logarithmic function

In this section, all logarithmic functions have base a. For convenience, I didn't write this base number

Let $\mathrm{f}(x)=\log x$, then

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Leftrightarrow f'(x)=\lim_{h\to 0}\frac{\log(x+h)-\log(x)}{h}$$

$$\Leftrightarrow f'(x)=\lim_{h\to 0}\frac{\log(x+h)-\log(x)}{h}$$

$$\Leftrightarrow f'(x)=\lim_{h\to 0}\frac{\log(x+h/x)}{h}$$

$$\Leftrightarrow f'(x)=\lim_{h\to 0}\frac{1}{h}\log(x+h/x)$$

$$\Leftrightarrow f'(x)=\lim_{h\to 0}\log(x+h/x)^{\frac{1}{h}}$$

$$\Leftrightarrow f'(x)=\lim_{h\to 0}\log\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

$$\Leftrightarrow f'(x)=\lim_{h\to 0}\log\left(\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right)^{\frac{1}{x}}$$

$$\Leftrightarrow f'(x)=\lim_{h\to 0}\frac{1}{x}\log\left(\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right)$$

$$\Leftrightarrow f'(x) = \lim_{h\to 0} \frac{1}{x} \cdot \log\left(\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right)$$

$$\Leftrightarrow f'(x) = \frac{1}{x} \cdot \lim_{t\to 0} \log\left((1+t)^{\frac{1}{t}}\right) \qquad \text{(Letting } t = \frac{h}{x} \text{ and as } h \to 0, t \to 0\text{)}$$

$$\Leftrightarrow f'(x) = \frac{1}{x} \cdot \lim_{t\to 0} \log\left((1+t)^{\frac{1}{t}}\right)$$

$$\Leftrightarrow f'(x) = \frac{1}{x} \cdot \log = \frac{1}{x} \frac{1ne}{1na} = \frac{1}{x1na}$$

Thus,

Important cases

Let u be a differentiable function of x. Then

$$\boxed{\frac{d}{dx}\left(\log_a{}^x\right) = \frac{1}{x\ln a} \qquad \frac{d}{dx}(\log_a u) = \frac{1}{u\ln a}.\frac{du}{dx}}$$

Example: Find the Following derivatives:

a. $\frac{d}{dx}\log x$ b. $\frac{d}{dx}\left(\log_3 x^2 + 1\right)$

c. $\frac{d}{dx}\log_4\left(\frac{x+1}{x^2+1}\right)$ d. $\frac{d}{dx} ln\left(\frac{1-x}{1+x}\right)$

Solution:

a. $\frac{d}{dx}\log x = \frac{1}{x\ln 10}$

b. $\frac{d}{dx}\left(\log_3 x^2 + 1\right) = \frac{1}{(x^2+1)\ln 3}\frac{d}{dx}\left(x^2+1\right)$

$$= \frac{1}{(x^2+1)\ln 3}.2x$$

c and d are left as an exercise for the students

Derivatives of exponential functions

Let $f(x) = e^x$. Then Find $f'(x)$

Note that $\lim_{h \to 0}(1+h)^{\frac{1}{h}} = e$, implies that for very small values of h,

$$(1+h)^{\frac{1}{h}} \approx e$$

$$\Rightarrow 1 + h = e^h \Rightarrow h = e^h - 1 \qquad (1)$$

Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} e^x \cdot \lim_{h \to 0} \frac{e^h - 1}{h}$$ by product Rule.

$$f'(x) = \lim_{h \to 0} e^x . \lim_{h \to 0} \frac{h}{h} = e^x . \lim_{h \to 0} 1 = e^x$$ Since $1 + h = e^h \Rightarrow h = e^h - 1$ from 1 above

Therefore,

$$\frac{d}{dx}(e^x) = e^x$$

Now to find the derivative of $f(x) = a^x$ for $a > 0$, we can use the Chain rule.

We know that $a^x = e^{\ln a^x}$ by properties of logarithmic function.

Then we have,

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{\ln a^x}\right) = \frac{d}{dx}\left(e^{(\ln a)x}\right)$$

$$= e^{\ln a^x} \frac{d}{dx} \ln a(x) = a^x . \ln a$$

Therefore,

$$\frac{d}{dx}(a^x) = a^x . \ln a$$

In general, if f is differentiable at x and $a > 0$, then

$$\boxed{\frac{d}{dx}\left(a^{f(x)}\right)=a^{f(x)}\ln a.f'(x)}$$

Example: Evaluate

a. $\frac{d}{dx}\left(4^{\sin 3x}\right)$ b. $\frac{d}{d\theta}\left(\sqrt{e}\right)^{\sec-\theta}$

Solution:

a. $\frac{d}{dx}\left(4^{\sin 3x}\right)=4^{\sin 3x}\frac{d}{dx}(\sin 3x)$

$= 4^{\sin 3x}(3\cos 3x)$

b. $\frac{d}{d\theta}\left(\sqrt{e}\right)^{\sec-\theta}=\left(\sqrt{e}\right)^{\sec-\theta}\frac{d}{d\theta}(\sec-\theta)=-\left(\sqrt{e}\right)^{\sec-\theta}\sec-\theta.\tan-\theta$

Implicit differentiations

The function that we have meet so far can be described by expressing one variable explicitly in terms of another variable as in

$y=\sqrt{x^2+x},\quad y=x\sin x,\quad y=\frac{1+x}{1+x^2}.$

Such equations are said to be define explicitly.
However, suppose y is a differentiable function of x, and instead of having a formula for y in terms of x, we are given an equation such as

$x^3+y^3=2xy$ ------------- (1)

Because y does not appear alone on one side of the equation in (1), we say that the equation define y implicitly in terms of x.

If it is very difficult to express y as a function of x explicitly, we use a procedure called implicit differentiation.

Consider an equation involving x and y in which y is a differentiable function of x.

We can use the steps below to find $\frac{dy}{dx}$ by using implicit differentiation.

1. Differentiate both sides of the equation with respect to x.

2. Write the result so that all terms involving $\frac{dy}{dx}$ are on the left side of the equation and all other terms are on the right side of the equation.
3. Factor $\frac{dy}{dx}$ out of the terms on the left side of the equation.
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by the left-hand factor that does not contain $\frac{dy}{dx}$.

Example 1: Find $\frac{dy}{dx}$, where $x^2 + y^2 = 4$

Solution: Use implicit differentiation with respect to x.

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(4)$$

$$\Rightarrow \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = 0$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\frac{dy}{dx} = -2x,$$

$\frac{dy}{dx} = -\frac{x}{y}$, provided that $y \neq 0$

Example 2: Find y' if $x^3 + y^3 = 6xy$

Solution: using implicit differentiation, we have

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(6xy)$$

$$\Rightarrow \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) = 6y\frac{d}{dx}(x) + 6x\frac{dy}{dx}$$

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$\Rightarrow 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\Rightarrow \left(3y^2 - 6x\right)\frac{dy}{dx} = 6y - 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x}, \text{ provided that } y \neq \pm\sqrt{2x}$$

Example 3: Find $\frac{dy}{dx}$ if $\sin(x+y) = y^2 \cos x$

Solution: Use implicit differentiation with respect to x.

$$\frac{dy}{dx}(\sin(x+y)) = \frac{dy}{dx}\left(y^2 \cos x\right)$$

$$\Rightarrow \cos((x+y)\frac{d}{dx}(x+y) = y^2 \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}\left(y^2\right) \qquad \text{Why?}$$

$$\Rightarrow \cos(x+y)\left(\frac{d}{dx}(x) + \frac{d}{dx}(y)\right) = y^2(-\sin x) + \cos x(2y)\frac{dy}{dx} \qquad \text{Why?}$$

$$\Rightarrow \cos(x+y)\left(1+\frac{dy}{dx}\right) = -y^2 \sin x + 2y\cos x \frac{dy}{dx}$$

$$\Rightarrow \cos(x+y)\frac{dy}{dx} - 2y\cos x\frac{dy}{dx} = -\cos(x+y) - y^2 \sin x \qquad \text{Why?}$$

$$\Rightarrow (\cos(x+y) - 2y\cos x)\frac{dy}{dx} = -y^2 \sin x - \cos(x+y)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{-y^2 \sin x - \cos(x+y)}{\cos(x+y) - 2y\cos x} = \frac{y^2 \sin x + \cos(x+y)}{2y\cos x - \cos(x+y)}$$

Higher order-derivatives

If f is a differentiable function then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$

This new function f'' is called the second derivative of f.

Then we can define $f''(a)$ by the formula

$f''(a) = \lim\limits_{x\to a} \frac{f'(x) - f'(a)}{x-a}$, whenever this limit exists.

We call $f''(a)$ the second derivative of f at a. it is often read “f double prime of f at a”.

In general,

$$f''(x)=\lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h}, \text{ if this limit exists}$$

In this case, we say that f f' is differentiable at x.

Suppose $f^{(n-1)}(a)$ denote the $(n-1)$ st derivative at a for $n\geq 3$, then we can define $f^{(n)}(a)$ by the formula

$$f^{(n)}(a)=\lim_{h\to 0}\frac{f^{(n-1)}(x)-f^{(n-1)}(a)}{h}=\left(f^{(n-1)}\right)'(a)$$

Using Leibniz notation, we write the n^{th} derivative of $y=\text{f }(x)$ as

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)=\frac{d^n y}{dx^n}=\frac{d^n}{dx^n}\big(f(x)\big) \text{ for } n\geq 2.$$

The second derivative, the third derivative and so on are called higher derivatives to distinguish them from the first derivative.

Remark: f is n times differentiable if $f^{(n)}(x)$ exists for all x in the domain of f.

Example1: If $f(x)=2x^4-3x^2$, then find $f^{(5)}(x)$.

Solution: $f(x)=2x^4-3x^2$

$$f'(x)=8x^3-6x,\quad f''(x)=24x^2-6,\quad f^{(3)}(x)=48x,\quad f^{(4)}(x)=48,\quad f^{(5)}(x)=0$$

$$f^{(n)}(x)=0 \text{ for } n\geq 5$$

Example 2: Let $y=x\cos x.$ Then find $\dfrac{d^3y}{dx^3}$

Solution: $y=x\cos x$

$$\frac{dy}{dx}=\frac{dy}{dx}(x\cos x)=\cos x-x\sin x \qquad \text{Using the Product Rule}$$

$$\frac{d^2y}{dx^2}=\frac{dy}{dx}(\cos x-x\sin x)=-(\sin x+x\cos x)=-2\sin x-x\cos x$$

$$\frac{d^3y}{dx^3}=\frac{dy}{dx}(-2\sin x-x\cos x)=-2\cos x-(\cos x-x\sin x)=x\sin x-3\cos x$$

EXERCISE ON DIFFERENTIATION

1. Suppose $f(x) = 2x^2 + x$. Then

   a. Use the limit definition of derivative to find $f'(-1)$

   b. Use the result of a to find the equation of the tangent line to the curve $y = 2x^2 + x$ at $(-1,1)$

   c. find the equation of the normal line to the $y = 2x^2 + x$ at $(-1,1)$

   d. Find a point on the curve where the tangent line is horizontal.

2. Find the derivative of $g(x) = \sqrt{1+2x}$ by using the definition of derivative. State the domain of g and the domain of its derivative.

3. Find the equation of the tangent line to the graph of each equation at the prescribed point.

   a. $y = \sin^2\left(\dfrac{\pi x}{4}\right)$, x =1   b. $y = x\cos x$, $x = -\pi$

   c. $x^2 + y^2 = 13$, x = -2

4. For what values of x does the graph of $y = x + 2\sin x$ has horizontal tangent?

5. Evaluate the following limits, if it exists

   a. $\displaystyle\lim_{t\to 0}\frac{\tan 6t}{\sin 2t}$ b. $\displaystyle\lim_{h\to 0}\frac{\sqrt{16+h}-2}{h}$ c. $\displaystyle\lim_{x\to 1}\frac{x^{1000}-1}{x-1}$

6. Suppose that f (5) = 1, $f'(5) = 6,\ g(5) = -3$, *and* $g'(5) = 2$ .Then find

   a. $(fg)'(5)$ b. $\left(\dfrac{f}{g}\right)'(5)$ c. $\left(\dfrac{g}{f}\right)'(5)$

7. Determine whether or not $f'(0)$ exists for

$$f(x) = \begin{cases} x\sin\dfrac{1}{x} & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

8. If f is differentiable function, find an expression for the derivative of the following functions.

a. $y=\dfrac{f(x)}{x^2}$ b. $y=\dfrac{1+xf(x)}{\sqrt{x}}$

9. Find the derivative of the following functions

a. $g(x)=\sqrt[3]{x^2}+2\sqrt{x^3}$ b. $f(x)=\left(\dfrac{x-1}{x^3-\sqrt{x}}\right)$

c. $y=\dfrac{\sqrt{x}-1}{\sqrt{x}+1}$ d. $g(t)=\sqrt[3]{t}\,(t+1)$

e. $f(x)=x^2+\sec x+\sqrt{x}$ f. $f(x)=\sqrt{x}\cos x+x\cot x$

g. $g(x)=\dfrac{\sin x}{1-\cos x}$ h. $h(x)=\dfrac{x^2+\tan x}{3x+2\tan x}$

i. $g(t)=e^{\sqrt{x}}(x+\ln x)$ j. $y=\ln\left(\dfrac{x-1}{x+1}\right)$

10. Differentiate the function

a. $y=(2x-5)^4(8x^2-5)^{-3}$ b. $y=\cot^2(\sin\theta)$

c. $y=2\cos x^2+\sin^2 x$ d. $y=\left(\cos^{-1}(\sin\theta)\right)$

e. $y=\left(\dfrac{1}{t}\tan^{-1}\left(\dfrac{1}{t}\right)\right)$ f. $y=e^{4\tan\sqrt{x}}$

11. Assume that y is a differentiable function of x. Then find $\dfrac{dy}{dx}$ by implicit differentiation.

a. $x^2y+xy^2=3x$ b. $1+x=\sin(xy^2)$

c. $\sqrt{x+y}=1+x^2y^2$ d. $2(x^2+y^2)^2=25(x^2-y^2)$

e. $x=\ln\left(\dfrac{x}{y}\right)$ f. $\sin^{-1}(xy)=xy$

12. Use implicit differentiation to find an equation of the tangent line to the curve $x^2+y^2=(2x^2+2y^2-x)^2$ at the point $\left(0,\dfrac{1}{2}\right)$

13. Find $\dfrac{d^2y}{dx^2}$ of the following the functions.

a. $y=\tan^{-1}(x^2)$ b. $\sqrt{x}+\sqrt{y}=1$

d. $y=(\sinh^{-1}x)$ e. $y=x\ln\dfrac{1}{x}$

14. Find the derivative of the function

a. $y = x\tanh^{-1}x + \ln\sqrt{1-x^2}$ b. $y = x^2\sinh^{-1}(2x)$

15. Find $\frac{d^{103}}{dx^{103}}(\cos 2x)$ by finding the first few derivatives and observing the pattern that occurs.

16. Show that the families of curves $y = cx^2 \quad and \quad x^2 + 2y^2 = k$ are orthogonal trajectories of each other. (c, k are constants

1.4 L'Hôpital's Rule

While we study limits in the previous course of calculus we considered limits of quotients such as

$$\lim_{x\to -2}\frac{x^2-4}{x+2} \text{ and } \lim_{x\to 0}\frac{\sin x}{x}$$

and calculated the limits by using algebraic, geometric, and trigonometric methods even if the limits have the undefined form 0/0. In this section we develop another technique that employs the derivatives of the numerator and denominator of the quotient. This new technique is called L'Hôpital's rule. For the proof of this rule we need the following generalization of the Mean Value Theorem.

Theorem (Cauchy's formula)

If f and g are continuous on $[a,b]$ and differentiable on (a,b) and if $g'(x) \neq 0$ for every x in (a,b), then there is a number c in (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

The Indeterminate form $\frac{0}{0}$

Let L be a real number or ∞ or $-\infty$.

a) Suppose f and g be differentiable on (a,b) and $g'(x) \neq 0$ for $a < x < b$. If

$$\lim_{x\to a^+} f(x) = 0 = \lim_{x\to a^+} g(x) \text{ and } \lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = L = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$$

An analogous result holds if $lim_{x\to a^+}$ is replaced by $lim_{x\to b^-}$ or by $lim_{x\to c}$, where c is any number in (a, b). In the letter case f and g need not be differentiable at c.

b) Suppose f and g be differentiable on (a, b) and $g'(x) \neq 0$ for $x > a$. If

$$\lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x) \text{ and } \lim_{x\to\infty}\frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

An analogous result holds if $lim_{x\to\infty}$ is replaced by $lim_{x\to-\infty}$.

Example 1:

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} = \lim_{x\to 0}\frac{\cos x}{1} = 1.$$

$$\lim_{x\to 1}\frac{2\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{d}{dx}(2\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1}\frac{\frac{2}{x}}{1} = 2.$$

$$\lim_{x\to 0}\frac{e^x-1}{x^2} = \lim_{x\to 0}\frac{\frac{d}{dx}(e^x-1)}{\frac{d}{dx}(x^2)} = \lim_{x\to 0}\frac{e^x}{2x} = \infty.$$

Example 2: Evaluate $\lim_{x\to 0}\frac{1-3^x}{x}$.

Solution: Both the numerator and the denominator have the limit 0 as $\to 0$. Hence the quotient has the indeterminate form $0/0$ at $x = 0$. By L'Hôpital's rule

$$\lim_{x\to 0}\frac{1-3^x}{x} = \lim_{x\to 0}\frac{-3^x\,(\ln 3)}{1} = -\ln 3$$

Example 3: Evaluate $\lim_{x\to 0}\frac{\ln(1-x^2)}{\ln\cos 2x}$.

Solution: Observe that

$$\lim_{x\to 0}\ln(1-x^2) = 0 = \lim_{x\to 0}\ln\cos 2x$$

thus by L'Hôpital's rule we get

$$\lim_{x\to 0}\frac{\ln(1-x^2)}{\ln\cos 2x} = \lim_{x\to 0}\frac{\frac{-2x}{1-x^2}}{-2\tan 2x} = \lim_{x\to 0}\left(\frac{1}{1-x^2}\cdot\frac{-2x}{-2\tan 2x}\right)$$

$$= \lim_{x\to 0}\frac{x}{\tan 2x}, \text{ since } \lim_{x\to 0}\frac{1}{1-x^2} = 1$$

$$= \lim_{x\to 0}\left[\frac{x}{\sin 2x}\,.\,(\cos 2x)\right]$$

$$= \lim_{x\to 0}\frac{x}{\sin 2x}.\lim_{x\to 0}(\cos 2x) = \frac{1}{2}$$

Example 1: $\lim_{x\to 2}\frac{x^2+x}{x-1} = 6$

But what happens if both the numerator and the denominator tend to 0? It is not clear what the limit is. In fact, depending on what functions $f(x)$ and $g(x)$ are, the limit can be anything at all!

Example 2:

$$\lim_{x\to 0}\frac{x^2}{x} = \lim_{x\to 0} x = 0 . \qquad \lim_{x\to 0}\frac{-x}{x^3} = \lim_{x\to 0}\frac{-1}{x^2} = -\infty$$

$$\lim_{x\to 0}\frac{x}{x^2} = \lim_{x\to 0^+}\frac{1}{x} = \infty . \qquad \lim_{x\to 0}\frac{kx}{x} = \lim_{x\to 0} k = k$$

These limits are examples of **indeterminate forms** of type $\frac{0}{0}$.

Example 3: Evaluate the following limits using L'Hopital's Rule

a. $\lim_{x\to 1}\frac{\ln x}{x-1}$ b. $\lim_{x\to 0}\frac{\tan x - x}{x^3}$

Solution: a. since $\lim_{x\to 1}\ln x = 0 = \lim_{x\to 1}(x-1)$, the given expression is $\frac{0}{0}$ indeterminate form. Then by L'Hopital's Rule we have:

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1}\frac{\frac{1}{x}}{1} = \lim_{x\to 1}\frac{1}{x} = 1$$

b. $\lim_{x\to 0}(\tan x - x) = 0 \quad and \quad \lim_{x\to 0} x^3 = 0$

$\lim\limits_{x\to 0}\dfrac{\tan x - x}{x^3}$ is $\dfrac{0}{0}$ indeterminate form. Then by L'Hopital's Rule we have:

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\frac{d}{dx}(\tan x - x)}{\frac{d}{dx}(x^3)}$$

$$= \lim_{x\to 0}\frac{(\sec^2 x - 1)}{(3x^2)} \quad \text{This is again } \frac{0}{0} \text{ indeterminate form.}$$

We have to use L'Hopital's Rule to the expression $\lim\limits_{x\to 0}\dfrac{(\sec^2 x - 1)}{(3x^2)}$ and we get

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\frac{d}{dx}(\sec^2 x - 1)}{\frac{d}{dx}(3x^2)}$$

$$= \lim_{x\to 0}\frac{(2\sec^2 x\tan x - 1)}{(6x)}$$

$$= \frac{1}{3}\lim_{x\to 0}\left(\sec^2 x\frac{\tan x}{x}\right)$$

$$= \frac{1}{3}\lim_{x\to 0}\sec^2 x.\lim_{x\to 0}\frac{\tan x}{x}$$

$$= \frac{1}{3}.1.1 = \frac{1}{3}$$

Therefore,

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \frac{1}{3}$$

> Remark: Sometimes it may be necessary to apply L'Hopital's Rule more than once in the same problem.

Example 4: Evaluate $\lim\limits_{x\to\pi}\dfrac{\sin x}{1-\cos x}$

Solution: Don't mislead to use L'Hopital's Rule since $\lim\limits_{x\to\pi}\dfrac{\sin x}{1-\cos x}$ is not an indeterminate form.

We can evaluate the limit by quotient Rule and we get

$$\lim_{x\to\pi}\frac{\sin x}{1-\cos x}=\frac{\lim\limits_{x\to\pi}\sin x}{\lim\limits_{x\to\pi}(1-\cos x)}=\frac{0}{2}=0$$

But if you use L'Hopital's Rule by mistake you will obtain

$\lim\limits_{x\to\pi}\dfrac{\sin x}{1-\cos x}=\lim\limits_{x\to\pi}\dfrac{\cos x}{\sin x}=\lim\limits_{x\to\pi}\cot x$, in which the limit does not exist.

Indeterminate Product

If $\lim f(x)=0$ and $\lim g(x)=\infty$, then $\lim f(x).g(x)$ is called an indeterminate form of type $0.\infty$

We can deal this kind of limit by writing the product fg as a quotient:

$$fg=\frac{f}{1/g} \quad \text{or} \quad fg=\frac{g}{1/f}$$

These convert the given limit in to an indeterminate form of type $\dfrac{0}{0}$ *or* $\dfrac{\infty}{\infty}$ respectively

.Example 1: Evaluate the following limits using L'Hopital's Rule

a. $\lim\limits_{x\to0^+} x\ln x$ b. $\lim\limits_{x\to\infty} x^3e^{-x^2}$ c. $\lim\limits_{x\to1^+}\ln x\tan\left(\dfrac{\pi x}{2}\right)$

Solution: $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$

Therefore, $\lim_{x\to 0^+} x\ln x$ is $0.\infty$ Indeterminate form.

$$x\ln x = \frac{\ln x}{1/x}$$

Thus $\lim_{x\to 0^+} \frac{\ln x}{1/x}$ is $\frac{\infty}{\infty}$ indeterminate form.

Then by L'Hopital's Rule we have:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x}$$

$$= \lim_{x\to 0^+} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}1/x}$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$$

Therefore,

$$\lim_{x\to 0^+} x\ln x = 0$$

b. Since $\lim_{x\to\infty} x^3 = \infty$ and $\lim_{x\to\infty} e^{-x^2} = 0$, $\lim_{x\to\infty} x^3 e^{-x^2}$ is $0.\infty$ Indeterminate form

$$x^3 e^{-x^2} = \frac{x^3}{e^{x^2}}$$

Thus $\lim_{x\to\infty} \frac{x^3}{e^{x^2}}$ is $\frac{\infty}{\infty}$ indeterminate form.

Then by L'Hopital's Rule,

$$\lim_{x\to\infty}\frac{x^3}{e^{x^2}}=\lim_{x\to\infty}\frac{3x^2}{2xe^{x^2}}=\lim_{x\to\infty}\frac{3x}{2e^{x^2}},$$ this is again $\frac{\infty}{\infty}$ indeterminate form.

We have to use L'Hopital's Rule, once more

$$\lim_{x\to\infty}\frac{x^3}{e^{x^2}}=\lim_{x\to\infty}\frac{3x}{2e^{x^2}}=\lim_{x\to\infty}\frac{3}{4xe^{x^2}}$$

$$=0$$

Therefore,

$$\lim_{x\to\infty}\frac{x^3}{e^{x^2}}=0$$

c. $\lim_{x\to1^+}\ln x=0$ and $\lim_{x\to1^+}\tan\left(\frac{\pi x}{2}\right)=\infty$. Hence $\lim_{x\to1^+}\ln x\tan\left(\frac{\pi x}{2}\right)$ is

$0.\infty$ Indeterminate form.

$$\ln x.\tan\left(\frac{\pi x}{2}\right)=\frac{\ln x}{1/\tan\left(\frac{\pi x}{2}\right)}$$

Now, $\lim_{x\to1^+}\frac{\ln x}{1/\tan\left(\frac{\pi x}{2}\right)}$ is $\frac{0}{0}$ indeterminate form.

Thus by L'Hopital's, we have

$$\lim_{x\to1^+}\frac{\ln x}{1/\tan\left(\frac{\pi x}{2}\right)}=\lim_{x\to1^+}\frac{1/x}{-\frac{\pi}{2}\sec^2\left(\frac{\pi x}{2}\right)/\tan^2\left(\frac{\pi x}{2}\right)}$$

$$\lim_{x\to1^+}\frac{-2}{\pi x}\sin^2\left(\frac{\pi x}{2}\right)=\frac{-2}{\pi}$$

Therefore , $\lim_{x\to1^+} \ln x \tan\left(\frac{\pi x}{2}\right) = \frac{-2}{\pi}$

Indeterminate Differences:

If $\lim f(x)=\infty$ *and* $\lim g(x)=\infty$, then the limit $\lim\left(f(x)-g(x)\right)$ is called an indeterminate form of $\infty-\infty$

Remark: to find out the limits of such kind of expression, we have to convert in to a quotient by algebraic manipulation.

Exapme1: Compute the following limits if it exists

a. $\lim_{x\to\frac{\pi}{2}^-} (\sec x-\tan x)$

b. $\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)$

Solution:

a. Since $\lim_{x\to\frac{\pi}{2}^-}(\sec x)=\infty$ and $\lim_{x\to\frac{\pi}{2}^-}(\tan x)=\infty$, $\lim_{x\to\frac{\pi}{2}^-}(\sec x-\tan x)$ is $\infty-\infty$

$$(\sec x-\tan x)=\frac{1}{\cos x}-\frac{\sin x}{\cos x}=\frac{1-\sin x}{\cos x}$$

Thus $\lim_{x\to\frac{\pi}{2}^-}\frac{1-\sin x}{\cos x}$ is $\frac{0}{0}$ indeterminate form.

Then by L'Hopital's, we get:

$$\lim_{x\to\frac{\pi}{2}^-}(\sec x-\tan x)=\lim_{x\to\frac{\pi}{2}^-}\frac{1-\sin x}{\cos x}$$

$$=\lim_{x\to\frac{\pi}{2}^-}\frac{-\cos x}{-\sin x}$$

$$=\lim_{x\to\frac{\pi}{2}^-}\frac{\cos x}{\sin x}=\frac{0}{1}=0$$

Therefore, $\lim\limits_{x \to \frac{\pi^-}{2}} (\sec x - \tan x) = 0$

b. Since $\lim\limits_{x \to \infty}\left(\sqrt{x^2 + x}\right) = \infty$ and $\lim\limits_{x \to \infty} x = \infty$, $\lim\limits_{x \to \infty}\left(\sqrt{x^2 + x} - x\right)$ is $\infty - \infty$ indeterminate

form.

Rationalized the expression as follows:

$$\left(\sqrt{x^2 + x} - x\right) = \left(\sqrt{x^2 + x} - x\right).\left(\frac{\sqrt{x^2 + x} + x}{\sqrt{x^2 + x} + x}\right)$$

$$= \frac{x}{\sqrt{x^2 + x} + x}$$, this is $\frac{\infty}{\infty}$ indeterminate form.

Then, by L'Hopital's Rule we have:

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + x} + x} = \lim_{x\to\infty} \frac{1}{\frac{2x+1}{2\sqrt{x^2 + x}} + 1} = \lim_{x\to\infty} \frac{1}{\frac{x+1}{\sqrt{x^2 + x}}} = \frac{1}{2}$$

(Verify that $\lim\limits_{x\to\infty} \frac{2x+1}{2\sqrt{x^2 + x}} = 1$)

Therefore, $\lim\limits_{x \to \infty}\left(\sqrt{x^2 + x} - x\right) = \frac{1}{2}$

Indeterminate powers: Several indeterminate forms arise from the limit $\lim\left(f(x)\right)^{g(x)}$

1. $\lim f(x) = 0$ *and* $\lim g(x) = 0$ ----------- form 0^0

2. $\lim f(x) = \infty$ *and* $\lim g(x) = 0$ ----------- form ∞^0

3. $\lim f(x) = 1$ *and* $\lim g(x) = \infty$ ----------- form 1^∞

Each of these three cases can be treated by taking the natural logarithm

Let $y = \left[f(x)^{g(x)} \right]$.

Then $\ln y = g(x) \ln f(x)$

$y = e^{\ln y} = e^{g(x)\ln f(x)}$

Thus,

$\lim y = \lim \left[e^{g(x)\ln f(x)} \right]$

Example: Evaluate the following limits.

a. $\lim\limits_{x \to 0^+} x^{x^2}$ b. $\lim\limits_{x \to \infty} x^{\frac{1}{x}}$ c. $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x} \right)^x$

Solution: Rewrite each of the expression in exponential form

a. $x^{x^2} = e^{x^2 \ln x}$ .Then

$$\lim_{x \to 0^+} x^{x^2} = e^{\lim\limits_{x \to 0^+} x^2 \ln x}.$$

It is clear that $\lim\limits_{x \to 0^+} x^2 \ln x$ is $0.\infty$ indeterminate form.

$$x^2 \ln x = \frac{\ln x}{1/x^2}$$

$\lim\limits_{x \to 0^+} \frac{\ln x}{1/x^2}$ is $\frac{\infty}{\infty}$ indeterminate form

$$\lim_{x \to 0^+} \frac{\ln x}{1/x^2} = \lim_{x \to 0^+} \frac{1/x}{-2/x^3} = -\frac{1}{2} \lim_{x \to 0^+} x^2$$

$$= -\frac{1}{2}.0 = 0$$

Therefore,

$$\lim_{x \to 0^+} x^{x^2} = e^{\lim\limits_{x \to 0^+} x^2 \ln x} = e^0 = 1$$

b. $\lim\limits_{x \to \infty} x^{\frac{1}{x}}$ is ∞^0 indeterminate type

$$x^{\frac{1}{x}} = e^{\frac{1}{x}\ln x}$$

$$\lim_{x \to \infty} x^{\frac{1}{x}} = \lim_{x \to \infty} e^{\frac{1}{x}\ln x} = e^{\lim\limits_{x \to \infty} \frac{\ln x}{x}}$$

The limit $\lim\limits_{x \to \infty} \frac{\ln x}{x}$ is $\frac{\infty}{\infty}$ indeterminate form.

Then by L'Hopital's, we will get

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0.$$

Therefore,

$$\lim_{x \to \infty} x^{\frac{1}{x}} = e^{\lim\limits_{x \to \infty} \frac{\ln x}{x}} = e^0 = 1$$

c. $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ is 1^∞ indeterminate type.

$$\left(1 + \frac{1}{x}\right)^x = e^{x \ln\left(1 + \frac{1}{x}\right)}$$

$$\lim_{x \to \infty} e^{x \ln\left(1 + \frac{1}{x}\right)} = e^{\lim\limits_{x \to \infty}\left[x \ln\left(1 + \frac{1}{x}\right)\right]}$$

The limit $\lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right]$ is $\infty.0$ indeterminate form.

$$\left[x\ln\left(1+\frac{1}{x}\right)\right]=\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$$

$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$ is $\frac{0}{0}$ indeterminate form

Then by L'Hopital's Rule, we have:

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}=\lim_{x\to\infty}\frac{-x/x^2\,(x+1)}{-1/x^2}$$

$$=\lim_{x\to\infty}\frac{x}{x+1}=1$$

Therefore,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e^{\lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right]}=e^1=e$$

EXRECISES:

1. Evaluate the following limits Using L'Hopital's Rule

a. $\lim_{x\to 0}\frac{x-\sin x}{\tan^3 x}$ b. $\lim_{x\to 0}\frac{\sin^2 x}{\sin x^2}$ c. $\lim_{x\to\infty}\frac{\ln\left(x^2+1\right)}{x}$ d. $\lim_{x\to 1^+}\left(\frac{x}{x-1}\right)^x$

2. Find A so that $\lim_{x\to\infty}\left(\frac{x+A}{x-2A}\right)^x=5$

Chapter 2: Applications of the Derivative

Introduction:

Derivative has a lot of application in physical sciences, natural science and social science field of studies. We will see some of its application in this chapter.

The primary goal of this chapter is to examine the application of derivative in:

. Solving optimization problems
. Finding intervals of monotonocity
. Evaluating relative extreme values
. Evaluating absolute extreme values
. Curve sketching

Objectives: At the end of this chapter the students will be able to**:**

♠ State The Mean Value theorem
♠ State Maximum - Minimum value Theorem
♠ Find local maximum and local minimum values of a function
♠ Find absolute maximum and absolute minimum value of a continuous function
♠ Express related rate problems in mathematical formula
♠ Sketch graphs of functions

2.1 Maximum and Minimum Values of a continuous function

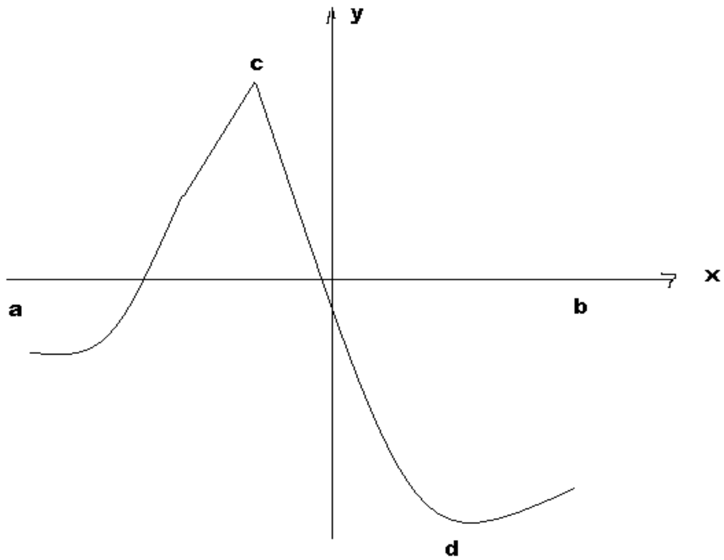
Consider the graph of some continuous function f on [a, b] as shown below.

a. Does the graph have the highest point on it? If so what is that point
   ____

b. Does the graph have the lowest point on it?
   ____

c. Does the graph contain both the lowest and the highest point on it?
   ____

d. Do you think that every continuous function on a closed interval has both the lowest and highest point on it? What do you call these points?
   ____

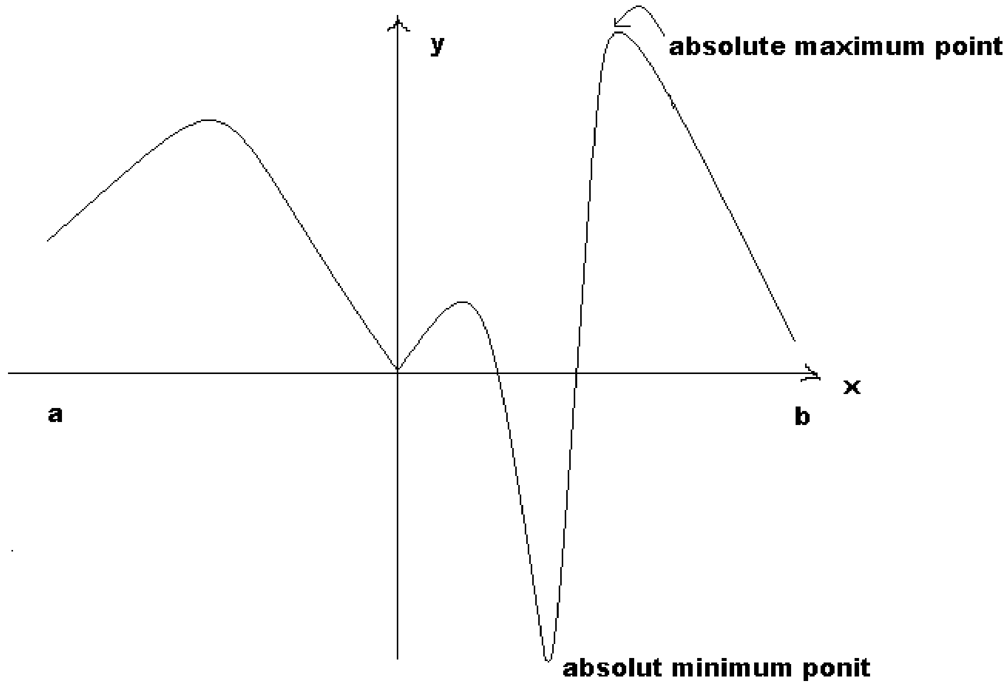


Definition1:

1. A function f has an absolute maximum (global maximum) value at c if $f(c) \geq f(x)$ for all x in D, where D is the domain of f.
   The number f (c) is called the maximum value of f on D.
2. Similarly , f has an absolute minimum (global minimum) value at c if $f(c) \leq f(x)$ for all x in D and the number f(c) is called the Minimum value of f on D.

Note: Either the absolute maximum or minimum value of f is called absolute extreme values.

The graph below shows the absolute maximum and absolute minimum points on the curve of a continuous function on [a, b]



Local Maximum and local minimum Values of a function

> **Definition 2:** A function f has a local maximum (or relative maximum) value at c if
> $f(c) \geq f(x)$ When x is near c (i.e. for all x in an open interval containing c)
> f has a local minimum (relative minimum) value at c if:
> $f(c) \leq f(x)$ for all x in an open interval containing c.

The Extreme Value theorem

If f is continuous on a closed interval [a, b], then f attains an absolute maximum f(c) and an absolute minimum value f(d) at some number c and d in [a, b].

Fermat's theorem:
If f has a local maximum or local minimum value at c and if $f'(c)$ exists, then $f'(c) = 0$.

Proof: Assume that f has a local maximum at c.
Then by definition 2 above, we have:
$f(c) \geq f(x)$ for all $x \in (c-h, c+h)$, where h is small positive number .
$\Rightarrow f(c) \geq f(c+h)$

$\Rightarrow f(c+h)-f(c)\le 0$ - - - Rewriting the preceding expression.

$\Rightarrow \dfrac{f(c+h)-f(c)}{h} \le 0$ if h > 0

Taking the limit on both sides, we have

$$f'(c) = \lim_{h\to 0}\frac{f(c=h)-f(c)}{h} \le \lim_{h\to 0} 0 = 0$$

This gives us,

$f'(c) \le 0$ - - - (7)

When h < 0, we get:

$\dfrac{f(c+h)-f(c)}{h} \ge 0$, then taking the limit on both sides gives

$$f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \le \lim_{h\to 0} 0 = 0$$

i.e. $f'(c) \le 0$ - - - (8)

From (7) and (8), we have:

$$0 \le f'(c) \le 0$$

Therefore, $f'(c)=0$

Critical Number:

Definition: A number c in the domain of a function f such that either $f'(c)=0$ or $f'(c)$ does not exist is said to be Critical Number of f.

Example 3: Find the critical number(s) of the function

a. $f(x)=x^{\frac{2}{3}}$ b. $f(x)=x^3+3x^2-24x$

Solution: First we have to determine the domain of the function

a. Domain of f is the set of real numbers.

$f'(x)=\dfrac{2}{3}x^{-\frac{1}{2}}=\dfrac{2}{3\sqrt{x}}$ - - - differentiating f

$f'(x)=0 \Rightarrow \dfrac{2}{3\sqrt{x}}=0$

$\Rightarrow 2=0$, which implies, there is no real number which makes $f'(x)$ zero.
$f'(x)$ does not exist at x = 0 which is in the domain of f.
Therefore, 0 is the only critical number of f.

b. Domain of f is the set of all real numbers.

$f'(x) = 3x^2 + 6x - 24$ exists for real number x.

$f'(x) = 0 \Rightarrow 3x^2 + 6x - 24 = 0$

$\Rightarrow x^2 + 2x - 8 = 0$

$\Rightarrow (x+4)(x-2) = 0$

$\Rightarrow x = -4 \text{ or } x = 2$

Hence the only critical numbers of f are -4 and 2.

Remark:
If f has a local maximum or minimum at c, then c is a critical number of f, but not every critical number gives rise to a maximum or a minimum value

2.2 Rolle's Theorem and Mean Value Theorem

To arrive at the Mean value Theorem, we first need the following result

Rolle's Theorem:
Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval [a, b]
2. f is differentiable on the open interval (a, b)
3. f(a) = f(b)

Then there is at least one number c in (a, b) such that $f'(c)=0$

In other words, there exists a point in the interval (a, b) which has a horizontal tangent.

Example1: Find a number c that satisfy the conclusion of Rolle's Theorem if the hypothesis are satisfied

a. $f(x)=x\sqrt{x+6}$ [-6, 0]

b. $g(x)=x^3-3x^2+2x$ [0, 2]

Solution:

a. Domain of f = $\{x : x + 6 \geq 0\} = \{x : x \geq -6\}$

i. If is continuous on [-6, 0] since every function is continuous on their domain

ii. $f'(x)=\dfrac{3x+12}{2\sqrt{x+6}}$ which exists for all $x \in (-6,0)$, i.e. f is differentiable f on

(-6, 0)

iii. f (-6) = 0 = f (0)

The three hypothesis of Rolle's Theorem are satisfied

Therefore, by Rolle's Theorem there exists a number c in (-6, 0) such that

$f'(c)=0$

Now, from (ii) above, we have

$f'(c)=\dfrac{3c+12}{2\sqrt{c+6}}=0$. This gives us:

$$\frac{3c+12}{2\sqrt{c+6}}=0$$

$3c+12=0$. So $c=-4$

Hence c = -4 $\in(-6,0)$ is the number which satisfies the conclusion of Rolle's Theorem.

b. since g is polynomial function

i. It is continuous [0,2]

ii. It is differentiable on (0, 2) and

iii. f (0) = 0^3-3(0^2) +2(0) = 0

f (2) = 2^3 – 3(2^2) + 2(2) = 8 -12 +4 = 0

That is f (0) = f (2)

The three hypothesis of Rolle's Theorem are satisfied.

Then Rolle's Theorem guaranteed that there is at least one number c in (0, 2) such that $f'(c)=0$

$f'(x)=3x^2-6x+2$. Then to find c solve $f'(c)=0$

$f'(c)=3c^2-6c+2=0$

$3c^2-6c+2=0$, using general quadratic formula to find c, we get

$$c=\frac{6+\sqrt{12}}{6}=1+\frac{\sqrt{3}}{3}\ or\quad c=\frac{6-\sqrt{12}}{6}=1-\frac{\sqrt{3}}{3}$$

Since both values of c are in (0, 2) the numbers which satisfy the conclusion of Rolle's Theorem are $c=1+\dfrac{\sqrt{3}}{3}$ *and* $c=1-\dfrac{\sqrt{3}}{3}$

Example 2: Let $f(x)=|x|$ for all c in [-1, 1]. Verify whether there is number c in [-1, 1] such that $f'(c)=0$. If there is no such number, does this contradict the conclusion of Rolle's Theorem? Explain.

Solution:

$$f(x)=|x|=\begin{cases} x & if\quad x>0 \\ -x & if\quad x<0 \end{cases}$$

Differentiating f gives us, $f'(x)=\begin{cases} 1 & if\quad x>0 \\ -1 & if\quad x<0 \end{cases}$

Then there is no number c in [-1, 1] such that $f'(c)=0$

This does not contradict Rolle's Theorem because f is not differentiable at 0.

That is $f_+'(0)=1$ and $f_-'(0)=-1$

This implies f is not differentiable at 0.

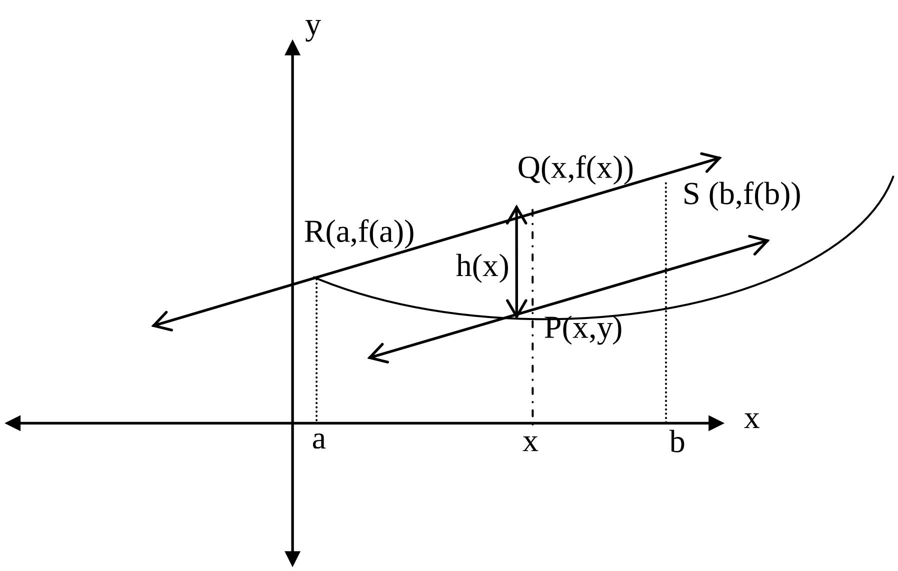
The Mean Value Theorem (MVT): Let f be a function that satisfies the following hypothesis:

1. f is continuous on the closed interval [a, b]
2. f is differentiable on the open interval (a, b)

Then there is at least one number c in (a, b) such that

$f'(c)=\dfrac{f(b)-f(a)}{b-a}$ Or f (b) – f (a) = $f'(c)(b-a)$

Geometrically, The Mean value Theorem states that there is a tangent line to the graph of f at some point between a and b which is parallel to the line joining the points (a, f (a)) and (b, f (b)).



MVT through this material stands for Mean Value Theorem

Proof: We apply Rolle's Theorem to the new function h defined by:

$$\boxed{h(x) = f(x) - y \qquad (5)}$$

But $\dfrac{y - f(a)}{x-a} = \dfrac{f(b) - f(a)}{b-a}$

$\Rightarrow y = f(a) + \dfrac{f(b) - f(a)}{b-a}(x-a)$. Substituting this value of y in equation (5) we get

$$h(x) = f(x) - [\, f(a) + \frac{f(b) - f(a)}{b-a}(x-a)]$$

Since f(x) is continuous on [a, b] and differentiable on (a, b), and f (a) is constant and $\dfrac{f(b) - f(a)}{b-a}(x-a)$ is a linear function and differentiable on $\Re$, then we have:

i. h(x) is continuous on [a, b] since it is a combination of continuous functions

ii. h(x) is differentiable on (a,b) as it is a combination of differentiable function.

iii. h (a) = 0 = h(b)

The three hypothesis of Rolle's Theorem are satisfied on [a,b].

Then by the conclusion of Rolle's Theorem, there exists at least one number c in (a, b) such that $h'(c) = 0$

But $h'(x) = f'(x) + \dfrac{f(b) - f(a)}{b-a}$

$\Rightarrow h'(c) = f'(c) + \dfrac{f(b) - f(a)}{b-a} = 0$

$\Rightarrow f'(c) + \dfrac{f(b) - f(a)}{b-a} = 0$

$\Rightarrow f'(c) = \dfrac{f(b) - f(a)}{b-a}$

Note:
If an object moves in a straight line with position at any time t is s = f (t) for t in [a, b], then the average velocity between t = a and t = b is:

$\frac{f(b)-f(a)}{b-a}$ and the velocity at t = c is $f'(c)$.

Then the mean Value Theorem tells us that at some time t = c between a and b the instantaneous velocity is equal to the average velocity.

Remark:

I. In general the Mean Value Theorem can be interpreted as saying that there is a number at which the instantaneous rate of change is equal to the average rate of change over an interval.

ii. MVT helps us to obtain information about a function from information about its derivatives.

Consequences of the Mean value theorem

The Mean Value Theorem can be used to establish some of the basic facts of differential calculus. One of these basics is the following theorem.

Theorem 3:1: If $f'(x)=0$ for all x in an interval (a, b), then f is constant on (a, b).

Proof: Use MVT to prove this theorem.
Let x_1, x_2 be any two arbitrary points in (a, b) with $x_1 < x_2$.

Since f is differentiable on (a, b) it is differentiable on (x_1, x_2) and also continuous on (x_1, x_2).Hence f satisfies The MVT on (x_1, x_2)
Then by The MVT, there is a number c in (x_1, x_2) such that:

$f'(c)=\frac{f(x_2)-f(x_1)}{x_2-x_1}=0$ --- (6) since $f'(x)=0$ for all x in an interval (a,b)

$\Rightarrow f(x_2)-f(x_1)=0$ --- multiply on both sides of (6) by $(x_2 - x_1)$

$\Rightarrow f(x_1)=f(x_2)$

Since x_1 and x_2 are arbitrary point in (a, b) f is constant on (a, b).

Corollary 3:1: If $f'(x)=g'(x)$ for all x in an interval (a, b) then f and g differ by a constant on (a, b), that is, f(x) = g(x) + c., where c is arbitrary constants

Proof: Let h(x) = f(x) – g(x) for all x in an interval (a,b).
Then $h'(x)=f'(x)-g'(x)=0$ since $f'(x)=g'(x)$ for all x in an interval (a, b)
But by theorem 1, above h is a constant on (a, b), that is:
h(x) = f(x) – g(x) = c, where c is constant

This implies, f(x) = g(x) + c

Example 1: Suppose that f (0) = -3 and $f'(x) \le 5$ for all values of x. Find the largest possible value of f (2).

Solution: We have given that f is differentiable and $f'(x) \le 5$ for all values of x.
Consider the interval [0, 2].
f is differentiable and continuous on [0, 2]. Then by the MVT, there exists a number c in (0, 2) such that:

$$f'(c) = \frac{f(2) - f(0)}{2-0} = \frac{f(2)+3}{2}$$, since f (0) = -3.

$f(2) = 2f'(c) - 3$, solving for f (2)

Since $f'(x) \le 5$ for all values of x and hence $f'(c) \le 5$, we get

$f(2) \le 7$

Therefore the largest possible value of f (2) is 7.

Example 2: Suppose that $f'(x) = g'(x)$ for all real number x and if $g(x) = x^2 + x - 1$ and f (1) = 3, then find f(x).

Solution: By corollary 3:1 f and g differ by a constant, that is
f(x) = g(x) + c where c is a constant to be determined
Then, we have: f(x) = $x^2 + x - 1$ + c - - - substituting the value of g(x).

$f(1) = 1^2 + 1 - 1 + c$ = 3 - - - since f (1) = 3
This gives 1+ c= 3, so c = 2.
Therefore, f(x) = $x^2 + x - 1$ + 2

Note: Care must be taken in applying Theorem 3:1.

Let $f(x) = \frac{x}{|x|} = \begin{cases} 1 & if \;\; x > 0 \\ -1 & if \;\; x < 0 \end{cases}$

The domain of f is $\mathbb{R}/\{0\}$ and $f'(x) = 0$ for all x in R/$\{0\}$.
But f is obviously not a constant function. This does not contradict Theorem 3:1 since R/$\{0\}$. is not an interval.
Notice that f is constant on $(-\infty, 0)$ and $(0, \infty)$.

Example 3: Prove the identity $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

Proof: Although calculus is not needed to prove this identity the proof using calculus is quite simple.

If $f(x) = \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, then

$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2}$ = 0 for all values of x. Therefore, f(x) = C is a constant.

To determine C, we put x =1 (because we can evaluate f (1) exactly.

$$C = f(1) = \tan^{-1}1 + \cot^{-1}1 = \frac{\pi}{2} + \frac{\pi}{2} = \frac{\pi}{2}.$$

Thus, $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

Example 4: For each of the following functions

i. verify whether the hypotheses of The Mean Value Theorem are satisfied
ii. If the hypotheses of The Mean value Theorem are satisfied, find a value that satisfies the conclusion on the given interval.

a. f(x) = x+$\frac{1}{x}$, $\left[\frac{1}{2},1\right]$ b. g(x) = $\frac{1}{2}x^2 - x^4$, $[-1,1]$

c. h(x) = $\frac{x}{x+2}$, $[1,4]$ d. $f(x) = \ln x$, $[1,e]$

Solution:

a. i. $f'(x) = 1 - \frac{1}{x^2}$ **,** which exists for all real number x except 0.

Hence, f is differentiable on $\left(\frac{1}{2},1\right)$ and continuous on $\left[\frac{1}{2},1\right]$

Therefore, f satisfies the hypothesis of The MVT.

ii. by the conclusion of the MVT , $\exists\, c \in \left(\frac{1}{2},1\right)$ such that

$$f'(c) = \frac{f(1) - f\left(\frac{1}{2}\right)}{1-\frac{1}{2}} = 2\left(f(1) - f\left(\frac{1}{2}\right)\right)$$

This implies,

$$1 - \frac{1}{c^2} = 2\left[2 - \frac{5}{2}\right] = -1$$

$$\frac{c^2-1}{c^2} = -1$$

Solving for c, we get $c = \pm\frac{1}{\sqrt{2}}$. Reject $c = -\frac{1}{\sqrt{2}} \notin \left[\frac{1}{2},1\right]$

Therefore, the number which satisfies the conclusion of The MVT on $\left[\frac{1}{2},1\right]$ is

$c=\frac{1}{\sqrt{2}}\cong 0.707\in\left[\frac{1}{2},1\right]$

b) i. Since f(x) is a polynomial it is differential on (-1,1) and continuous on [-1, 1]

$f'(x)=x-4x^3$

ii. Then by the conclusion of The MVT, $\exists\, c\in(-1,1)$ such that:

$$f'(c)=\frac{f(1)-f(-1)}{1-(-1)} \qquad \text{Conclusion of the MVT}$$

$$\Rightarrow c-4c^3=\frac{-\frac{1}{2}-\left(-\frac{1}{2}\right)}{2}=0$$

$$\Rightarrow c\left(1-4c^2\right)=0$$

Solving for c, we get c= o, or $c=\pm\frac{1}{2}$, all of them are in the interval (-1, 1).

Hence, the Values of c that satisfy the conclusion of the MVT in (-1, 1)

are $-\frac{1}{2},\ 0,\ \frac{1}{2}$.

c) i. $h'(x)=\frac{2}{(x+2)^2}$ which exists for all real number x except -2.

That is h is differentiable on (1, 4) and continuous on [1, 4].

Then by the conclusion of the MVT, $\exists c\in(1,4)$ such that:

$$h'(c)=\frac{h(4)-h(1)}{4-1} \qquad \text{- - - conclusion of MVT}$$

$$\Rightarrow\frac{2}{(c+2)^2}=\frac{\frac{2}{3}-\frac{1}{3}}{3}=\frac{1}{9} \quad \text{Why?}$$

$$\Rightarrow 18=c^2+2c+4$$

$\Rightarrow c^2+2c-14=0$. Solving this quadratic equation using quadratic formula we obtain,

$$c=\frac{-2\pm\sqrt{60}}{2}=-1\pm\sqrt{15}$$

Ignore $c=-1-\sqrt{15}$ since it is not in (1, 4)

Hence the number which satisfies the conclusion of The MVT in (1, 4) is $-1+\sqrt{15}$.

d. $f'(x)=\frac{1}{x}$ exists for all x in the interval [1, e], then by the conclusion of the MVT, $\exists\ c\in[1,e]$ such that:

$$f'(c)=\frac{f(e)-f(1)}{e-1}$$

$$\Rightarrow\frac{1}{c}=\frac{\ln e-\ln 1}{e-1}=\frac{1-0}{e-1}=\frac{1}{e-1}$$

$$\Rightarrow c=e-1$$

Hence the value of c that satisfies the conclusion of the MVT in [1, e] is e-1

Key Concepts

The Mean Value Theorem

1. If f is continuous on $[a,b]$ and differentiable on (a, b), then there exist a number c such that:

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

Rolle's Theorem

2. If $f(a)=f(b)$ in 1 above, we get

$$f'(c)=0$$

2.3 Monotonic functions

Consider the function $f(x)=3x^4-4x^3-12x^2+3$ on the interval [-2, 3]. We cannot find regions of which *f* is increasing or decreasing, relative maxima or minima, or the absolute maximum or minimum value of *f* on [-2, 3] by inspection. Graphing by hand is tedious and imprecise. Even the use of a graphing program will only give us an approximation for the locations and values of maxima and minima. We can use the first derivative of *f*, however, to find all these things quickly and easily.

Increasing and decreasing function

Definition: A function f is said to be:

i. Increasing on an interval I if $f(x_1)\le f(x_2)$ whenever $x_1<x_2$ for all x_1 , x_2 in I.

Increasing decreasing Test (ID Test)

Let *f* be continuous on an interval *I* and differentiable on the interior of *I*.

- If $f'(x) > 0$ for all $x \in I$, then *f* is *increasing* on *I*.
- If $f'(x) < 0$ for all $x \in I$, then *f* is *decreasing* on I.

ii. Decreasing on an interval I if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$ for all x_1 , x_2 in I.

Example: Find the interval where f is increasing and where it is decreasing

for the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$

Solution:

The function $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ has first derivative

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x+1)(x-2)$$

		-1		0		2	
12x	- - -		- - -	0	+ + +		+ + +
x+1	- - -	0	+ + +		+ + +		+ + +
x-2	- - -		- - -		- - -		+ + +
$f'(x)$	- - -		+ + +		- - -		+ + +

Thus, $f(x)$ is increasing on $(-1,0) \cup (2,\infty)$ and decreasing on $(-\infty,-1) \cup (0,2)$.

2.4 Search for local (relative) maxima and Minima

Relative extrema of *f* occur at **critical points** of *f*, values x_0 for which either

$f'(x_0) = 0$ or $f'(x_0)$ is undefined.

First Derivative Test:
Suppose that c is a critical number of a continuous function f.

1. If f' changes sign from positive to negative at c, then f has a local maximum value at c.
2. If f' changes sign from negative to positive at c, then f has a local minimum value at c.
3. If $f'(x)$ doesn't change sign at c, then f has neither local maximum nor local minimum value at c.

Example**:** Use the First derivative Test to find the local extreme of the following functions

a. $f(x) = 5 + 3x^2 + x^3$ *b.* $g(x) = \dfrac{x^2}{x^2+3}$ c. $h(x) = x\ln x$

Solution: a. $f'(x) = 6x + 3x^2$ ---- differentiating f to obtain f'

$f(x) = 0 \Rightarrow 3x\,(2+x) = 0$

$\Rightarrow x = 0 \text{ or } x = -2$

0 and -2 are the critical numbers of f.

Then Use sign Chart to observe the sign of f' on the

Intervals $(-\infty,-2)$, $(-2,0)$, $(0,\infty)$

		-2	0
	$x \in (-\infty,-2)$	$x \in (-2,0)$	$x \in (0,\infty)$
$3x$	—	— 0	+
2+x	— 0	+	+
$f'(x) = 3x(2+x)$	+ 0	— 0	+

Then, using the sign chart above and the First derivative Test,

$f(-2) = 5 + 3(-2)^2 + (-2)^3 = 9$ is local maximum value of f and

$f(0) = 5$ is local minimum value of f.

b. Use Quotient Rule to find $g'(x)$.

$$g'(x)=\frac{2x\left(x^2+3\right)-2x\left(x^2\right)}{\left(x^2+3\right)^2}=\frac{6x}{\left(x^2+3\right)^2}$$, this exists for all real values of x.

$g'(x)=0 \Rightarrow x=0$

Therefore, 0 is the only critical number of f.

Since $\left(x^2+3\right)^2$ is always positive, we have:

$g'(x)>0$ if x > 0 and $g'(x)<0$ if x < 0

Hence, g' changes sign from positive to negative at 0

As a result of this, g (0) = 0 is local maximum value of and g has no local minimum value.

c. Use product Rule to find $h'(x)$.

$$h'(x)=\ln x+1$$

$$h'(x) = 0 \Rightarrow \ln x+1 = 0$$

$$\Rightarrow \ln x = -1$$

$x = e^{-1} = \dfrac{1}{e}$, this is the only critical number of h.

We can see that, $h'(x)>0 \ \ for \ \ x>\dfrac{1}{e} \ \ and \ \ h'(x)< 0 \ \ for \ \ x<\dfrac{1}{e}$

Therefore by The first derivative Test, $h\left(\dfrac{1}{e}\right)=\dfrac{1}{e}\ln\left(\dfrac{1}{e}\right)=-\dfrac{1}{e}$ is local maximum value of f.

2.5 Absolute Maxima and Minima of a function

- If *f* has an extreme value on an *open* interval, then the extreme value occurs at a critical point of *f*.
- If *f* has an extreme value on a *closed* interval, then the extreme value occurs either at a critical point or at an endpoint.

According to the **Extreme Value Theorem**, if a function is continuous on a closed interval, then it achieves both an absolute maximum and an absolute minimum on the interval.

Example 1: Find the absolute maximum and minimum value of the function

$$f(x)=3x^4-4x^3-12x^2+3 \text{ on } [-2,3]$$

Solution: Since $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ is continuous on [-2, 3], *f* must have an absolute maximum and an absolute minimum on [-2, 3]. We simply need to check the value of *f* at the critical points *x* = -1, 0, 2 and at the endpoints *x* = -2 and *x* = 3:

$f(-2) = 35$; $f(-1) = -2$; $f(0) = 3$; $f(2) = -29$; $f(3) = 30$

Thus, on [-2, 3], *f*(*x*) achieves a maximum value of 35 at *x* = -2 and a minimum value of -29 at *x* = 2.

We have discovered a lot about the shape of $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$ without ever graphing it!

Example 2: Find the absolute maximum and minimum value of the following functions on the indicated interval.

a. $f(x) = x^3 - x^2 + 1$; $\left[-\frac{1}{2}, 4\right]$

b. $g(x) = (x-1)^3$; $[-1, 2]$

c. $h(x) = \dfrac{x^2 - 4}{x^2 + 4}$; $[-4, 4]$

Solution:

a. Since f is continuous on $\left[-\frac{1}{2}, 4\right]$, by the **Extreme Value Theorem** f has both the maximum and minimum value on $\left[-\frac{1}{2}, 4\right]$.

$f'(x) = 3x^2 - 2x$

Then set $f'(x) = 0$, to find the critical numbers

$f'(x) = 3x^2 - 2x = 0$

$x(3x - 2) = 0$

This implies,

x = 0 and x= $\frac{2}{3}$ are the critical numbers of f on $\left[-\frac{1}{2}, 4\right]$

Then compute f at x = 0, $\frac{2}{3}$, $\frac{-1}{2}$, 4

$$f(0)=1; \quad f\left(-\frac{1}{2}\right)=\frac{13}{8}; \ f\left(\frac{2}{3}\right)=-\frac{1}{27}; \quad f(4)=17$$

Therefore, $f(4)=17$ is the absolute maximum value and $f\left(\frac{2}{3}\right)=-\frac{1}{27}$

is the absolute minimum value of f on $\left[-\frac{1}{2},4\right]$.

b. $g'(x)=3(x-1)^2$

On equating $g'(x)=0$, we get x = 1

Thus x = 1 is the only critical number of g on $[-1,2]$.

$$g(-1)=(-1-1)^3=(-2)^3=8$$

$$g(1)=(1-1)^3=0$$

$$g(2)=(2-1)^3=(1)^3=1$$

Therefore, 8 is the absolute maximum value and 0 is the absolute value of g on $[-1,2]$.

c. $h'(x)=\dfrac{2x(x^2+4)-2x(x^2-4)}{(x^2+4)^2}=\dfrac{8x}{(x^2+4)^2}$

On setting $h'(x)=0$, we get x = 0

Thus x = 0 is the only critical number. Then evaluate h at x = 0, -4, 4.

$$h(0)=-1; \ h(-4)=\frac{12}{20}=\frac{3}{5}=h(4).$$

Therefore,

$\frac{3}{5}$ and -1 are respectively the absolute maximum and minimum value of f

on [-4, 4].

Key Concepts

Increasing or Decreasing?

- Let *f* be continuous on an interval *I* and differentiable on the interior of *I*.

  If $f'(x) > 0$ for all $x \in I$, then *f* is *increasing* on *I* and If $f'(x) < 0$ for

  all $x \in I$, then *f* is *decreasing* on I.

- **Relative Maxima and Minima**

  By the First Derivative Test, relative extrema occur where $f'(x)$ changes sign.

- **Absolute Maxima and Minima**

  If *f* has an extreme value on a closed interval, then the extreme value occurs either at a critical point or at an endpoint.

Concavity and the Second Derivative Test

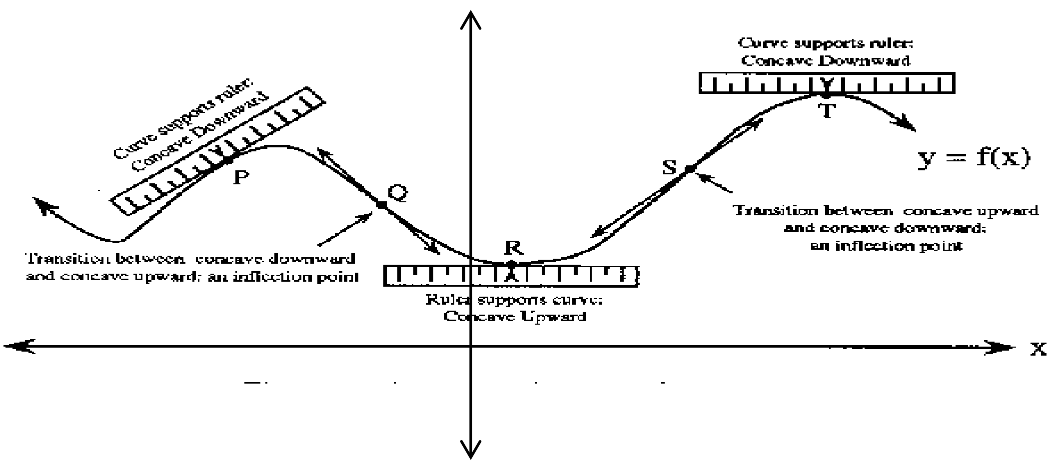
You are learning that calculus is a valuable tool. One of the most important applications of differential calculus is to find **extreme function values**. The calculus methods for finding the maximum and minimum values of a function are the basic tools of **optimization theory**, a very active branch of mathematical research applied to nearly all fields of practical endeavor. Although modern optimization theory is considerably more advanced, its methods and fundamental ideas clearly show their historical relationship to the calculus. In this section you will review how the second derivative of a function is related to the shape of its graph and how that information can be used to classify relative extreme values.

Concavity

The Second Derivative Test provides a means of classifying relative extreme values by using the sign of the second derivative at the critical number. To appreciate this test, it is first necessary to understand the concept of concavity.

The graph of a function f is **concave upward** at the point (c, $f(c)$) if $f'(c)$ exists and if for all x in some open interval containing c, the point (x, $f(x)$) on the graph of f lies above the corresponding point on the graph of the tangent line to f at c. This is expressed by the inequality $f(x) > [f(c) + f'(c)(x-c)]$ for all x in some open interval containing c. Imagine holding a ruler along the tangent line through the point (c, $f(c)$): if the ruler *supports the graph of* f near (c, $f(c)$), then the graph of the function is concave upward.

The graph of a function f is **concave downward** at the point (c, $f(c)$) if $f'(c)$ exists and if for all x in some open interval containing c, the point (x, $f(x)$) on the graph of f lies below the corresponding point on the graph of the tangent line to f at c. This is expressed by the inequality $f(x) < [f(c) + f'(c)(x-c)]$ for all x in some open interval containing c. In this situation *the graph of* f *supports the ruler*. This is pictured below.



Concavity Theorem: If the function f is twice differentiable at x = c, then the graph of f is concave up ward at $(c, f(c))$ if $f''(c) > 0$ and concave down ward if $f''(c) < 0$

Example

Suppose $f(x) = x^3 - 3x^2 + x - 2$. Let's determine where the graph of f is concave up and where it is concave down. Since f is twice-differentiable for all x, we use the result given above and first determine that $f''(x) = 6(x-1)$.

Thus, $f''(x) > 0$ if $x > 1$ and $f''(x) < 0$ if $x < 1$. By the Concavity Theorem, the graph of f is concave up for x > 1 and concave down for x < 1.

Inflection Points

Notice in the example above, that the concavity of the graph of *f* changes sign at x = 1. Points on the graph of *f* where the concavity changes from up-to-down or down-to-up are

called **inflection points** of the graph.

Inflection Point Theorem: If $f'(c)$ exist and $f''(c)$ change sign at x = c, then the point $(c, f(c))$ is an inflection point of the graph of f. If $f''(c)$ exists at the inflection point, then $f''(c)$ = 0

The following result connects the concept of inflection point to the derivatives properties of the function:

We return to our example, where $f(x) = x^3 - 3x^2 + x - 2$, the INFLECTION POINT THEOREM verifies that the graph of *f* has an inflection point at *x* = 1, since

$f''(1) = 0$.

The Second Derivative Test

The Second Derivative Test relates the concepts of critical points, extreme values, and concavity to give a very useful tool for determining whether a critical point on the graph of a function is a relative minimum or maximum.

The Second Derivative Test: Suppose c is a critical point at which $f'(c) = 0$ and that $f''(c)$ exists. Then f has a relative

i. maximum value at c if $f''(c) < 0$

ii. minimum value at c if $f''(c) > 0$

Example 1:

Let's find and classify the extreme points for the function f with values

$f(x) = x^3 - 3x^2 + x - 2$.

We find that $f'(x) = 3x^2 - 6x + 1$, and so there are two critical numbers where f '(c) = 0:

$$f''(x) = 6x - 6 = 6(x-1)$$

$$c_1 = 1 + \frac{\sqrt{6}}{3} \approx 1.82 \text{ and } c_2 = 1 - \frac{\sqrt{6}}{3} \approx 0.18$$

Notice that at c_1 we get that $f''(c_1) > 0$. Thus f has a relative minimum value at $x = 1 + \frac{\sqrt{6}}{3}$ and f'' (c_2) < 0, the Second Derivative Test informs us that f has a relative maximum at $x = 1 - \frac{\sqrt{6}}{3}$

Example 2: Find the local Extreme value of the following function using the Second Derivative Test

a. $f(x) = x^4 - 4x^3$ b. $g(x) = x + \sqrt{x-1}$ c. $h(x) = x^4 - 2x^2 + 3$

Solution:

a. $f'(x) = 4x^3 - 12x^2$

$= 4x^2(x-3)$

To find the critical numbers set $f'(x) = 0$

$4x^2(x-3) = 0$, this gives us, x = 0 or x = 3.

Thus x = 0, 3 are the critical numbers of f

$f''(x) = 12x^2 - 24x$

Then evaluating $f''(x)$ at the critical numbers, we get:

$f''(0) = 0$

$f''(3) = 12(3^2) - 24(3) = 36$

Then by the Second Derivative Test, f (3) = 36 is relative minimum Value of f and f has neither maximum nor minimum value at x = 0. We can use the First Derivative Test to see that $f'(x)$ doesn't change sign at x = 0.

b. $g'(x) = 1 + \frac{1}{2\sqrt{x-1}}$

Setting $g'(x) = 0$ gives us, x = $\frac{5}{4}$

As a result of this x = $\frac{5}{4}$ is the critical number of g.

$$g''(x) = \frac{-1}{4(x-1)^{\frac{3}{2}}}$$

$$g''\left(\frac{5}{4}\right) = \frac{-1}{4\left(\frac{5}{4}-1\right)^{\frac{3}{2}}} = \frac{-1}{4\left(\frac{1}{8}\right)} = -2$$

Then by the Second Derivative Test, g has a relative maximum value at x = $\frac{5}{4}$

That is:

$g\left(\frac{5}{4}\right) = \frac{5}{4} + \sqrt{\frac{5}{4}-1} = \frac{7}{4}$ is the relative maximum value of g.

c. $h'(x) = 4x^3 - 4x$

$= 4x(x^2 - 1)$

$= 4x(x-1)(x+1)$

$h'(x) = 0$ if x = 0 or x = 1 or x = -1

Thus the critical numbers of h are x = -1, 0, and 1.

$h''(x) = 12x^2 - 4$

Then, evaluating $h''(x)$ at these critical numbers gives us:

$h''(-1) = 8$

$h''(0) = -4$

$h''(1) = 8$

Therefore by the Second Derivative Test, h has a relative maximum value at x = 0 and a relative minimum value at x = 1and x =-1.

That is:

$h(-1) = 2 = h(1)$ is a relative minimum value and $h(0) = 3$ is a relative maximum value.

2.8. Rate of change

To understand this section, you should be familiar with the chain rule for derivatives or implicit differentiation.

If Q is a quantity that is varying with time, we know that the derivative measures how fast Q is increasing or decreasing .Specifically, if we let t stands for time, then we have the following

Rate of change of a quantity

Rate of change of Q = $\frac{dQ}{dt}$

ACTIVITY

The weight (in kg.) of rocket fuel in a rocket launcher is given by

W (t) = $\frac{1}{t} - \frac{4}{t^2}$, where t is time in seconds.

At time t = 10 seconds, the amount fuel in the launcher is

A. Decreasing at a rate of 0.002kg./s
B. Increasing at a rate of 0.002kg./s.
C. Increasing at a rate of 0.8kg. /s.
D. Not changing at all

In a related rates problem, we are given the rate of change of certain quantities, and are required to find the rate of change of related quantities.

The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate or use implicit differentiation on both sides with respect to time.

For this section, we have developed a simple, step- by –step approach to solve related rates problems which we shall illustrate with examples

Example 1: The area of a circular disc is growing at a rate of 12cm^2/s. How fast is the radius growing at the instant when it equals 10cm?

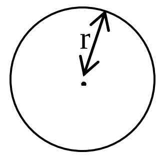
Step1: Identify the changing quantities, possibly with the aid of a sketch

Here the changing quantities are:

A. Derivative of the area B. radius of a disc
C. Time D. area of a disc

The changing quantities are the area of the disc and the radius of the disc

Here is a little sketch of the disc showing the changing quantities



r = radius of disc
A = area of disc

Note: At this stage, we do not substitute values for the changing quantities. That comes at the end.

Step2: Write down an equation that relates the changing quantities.
A formula that relates the area A and the radius r is

$$A = \pi r^2$$

Step3: Differentiate both sides of the equation with respect to t.
The derived equation is

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{2\pi r} \frac{dr}{dt}$$

Substituting the given values and solving for the unknown gives:

$$\frac{dr}{dt} = \frac{1}{2\pi (10cm)} 12cm^2 / s = \frac{3}{5\pi} cm / s$$

Example 2: Air is being pumped in a spherical balloon so that its volume increases at a rate of 100cm^3 /s. How fast is the radius of the balloon increasing when the diameter is 50cm?

Given: the rate of increase of the volume of air in the balloon, i.e,

$$\frac{dV}{dt} = 100cm^3 / s.$$

Required: The rate of increase of the radius when the diameter is 50 cm

(The radius r = 25cm), i.e, $\frac{dr}{dt} = ?$

Solution: Let V be the volume of the balloon and r be its radius.

$$V = \frac{4}{3} \pi r^3 \ .$$

Since the volume of the balloon depends up on the radius, we have

$$\frac{dV}{dr} = 4\pi r^2$$

Both the volume and the radius of the balloon are changing with time
Now, by Chain rule we get,

$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$ and solve for $\frac{dr}{dt}$. We obtain:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt}.$$

When r = 25cm and $\frac{dV}{dt} = 100 cm^3/s$, then

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2 cm^2} . 100 cm^3/s = \frac{1}{25\pi} cm/s$$

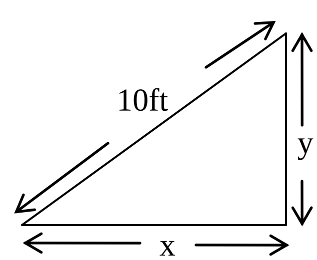
Therefore the radius of the balloon is increasing at the rate of $\frac{1}{25\pi} cm/s$ when the radius of the balloon is 25cm.

Strategy to solve related rate problems

1. Identify all given quantities and all quantities to be determined.
   If possible, make a sketch and label the quantities.
2. Write an equation that relates all variables whose rate of change is either given or to be determined.
3. Use the chain rule or implicit differentiation on both sides of the equation with respect to time.
4. Substitute in to the resulting equation all known values of the variables and their rates of change. Then solve for the required rate of change.

Example 3: A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?

Solution:
Let x be the distance from the bottom of the ladder to the wall at ant time t
Let y be the distance from the top of the ladder to the ground at any time t
Note: x and y are both function of time



We have given: $\frac{dx}{dt} = 1 ft/s$ and the length of the ladder is 10ft

Required: $\frac{dy}{dt}$, when x = 6ft

But from Pythagorean Theorem, we have

$x^2 + y^2 = (10)^2$ = 100. (1)

When x = 6, y = 8

Differentiate this equation implicitly on both sides, i.e,

$$\frac{d}{dt}\left(x^2 + y^2\right) = \frac{d}{dt}(100)$$

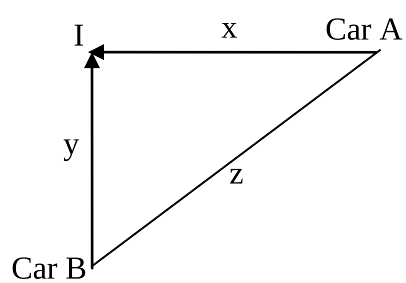
$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{6}{8}(1\,ft/s) = -\frac{3}{4}\,ft/s$$

Therefore the top of the ladder is sliding down at the rate of $\frac{3}{4}\,ft/s$ at the instant when the bottom of the ladder is 6ft from the wall.

Example 4:

Car A is traveling west at 50km/h, and car B is traveling north at 60km/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3km and car B is 0.4kmfrom the intersection?



Solution: At any time t, let x be the distance from car A to the intersection point I.
Let y be the distance from car B to point I and z is the distance between the two cars.

Given: $\frac{dx}{dt} = -50\,km/h$, $\frac{dy}{dt} = -60kk/h$

(Note: the derivatives are negative because x and y are decreasing)

Required: $\frac{dz}{dt}$, when x = 0.3km and y = o.4km

By Pythagoras Theorem in right triangle ABI, we have

$$\boxed{z^2 = x^2 + y^2 \quad (1)}$$

When x = 0.3 km and y = 0.4 km we get,

z = $\sqrt{(0.3)^2 + (0.4)^2}$ = $\sqrt{0.25} = 0.5$ km

Differentiate each side of (1) with respect to t.

$$\frac{d}{dt}\left(z^2\right)=\frac{d}{dt}\left(x^2+y^2\right)$$
$$\Rightarrow 2z\frac{dz}{dt}=2x\frac{dx}{dt}+2y\frac{dy}{dt}$$

$$\boxed{\Rightarrow \frac{dz}{dt}=\frac{1}{z}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right) \qquad (2)}$$

Then if we substitute x = 0.3, y = 0.4 and z = 0.5 in equation (2) we get

$$\frac{dz}{dt}=\frac{1}{0.5km}\left(0.3km\ \left(-50km/h\right)+0.4km\left(-60km/h\right)\right)=-78km/h$$

Therefore the cars are approaching at a rate of 78 km/h.

Example 5: All edges of a cube are expanding at a rate of 3cm/s. How fast is the volume changing when each edge is 1cm?

Solution: Let the edge of a cube be x cm at any time t.

Given: $\frac{dx}{dt}=3cm/s$

Required: $\frac{dV}{dt}$ at x = 1cm the volume of a cube of edge x cm is

$$\boxed{V(x) = x^3. \qquad (3)}$$

Since both the volume and the sides of the cube are changing with time, differentiate (3) with respect to t gives us

$$\boxed{\frac{dV}{dt}=\frac{d}{dt}\left(x^3\right)=3x^2\frac{dx}{dt} \qquad (4)}$$

Substituting x = 1cm and $\frac{dx}{dt}=3cm/s$, we obtain $\frac{dV}{dt}=3cm^3/s$

Therefore the volume is changing at a rate of $3cm^3/s$.

2.9 Curve sketching

The following checklist is intended as a guideline to sketch a curve y = f(x) by hand.
1. Determine the domain: The set of values of x for which f(x) is defined.
2. Determine the intercepts(x- and y-intercepts)

The y- intercept is f (0) and this tells us where the curve intersects the y- axis. To find the x- intercept, we set y = 0 and solve for x, and it is the point where the curve intersects the x-axis.

3. Determine the asymptotes:

I. Horizontal asymptotes:

If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line y = L, is a horizontal asymptote to the curve of y = f(x)

ii. Vertical Asymptotes:

The line y= a is a vertical asymptote if at least one of the following is true.

$$\lim_{x\to a^+} f(x) = \pm\infty \text{ or } \lim_{x\to a^-} f(x) = \pm\infty$$

4. Determine the interval for which f is increasing and f is decreasing by using ID Test (Increasing decreasing Test)

5. Determine relative extreme values either by using the First Derivative Test or the second Derivative Test
6. Determine the interval for which f is concave up and concave down and the inflection points.
7. Check for symmetric: by determining whether the function is even or odd.

Example1: Sketch the graph of $y = x^4 - 4x^3$

Solution: To sketch the graph we have to use the above guidelines.

Let f(x) = $x^4 - 4x^3$

1. Since f is polynomial function its domain is $(-\infty, \infty)$.

2. i. x- Intercept : solve for x by setting f(x) = 0

$$x^4 - 4x^3 = 0$$

$\Rightarrow x^3(x-4) = 0$, this gives us x = 0 or x = 4

Therefore the graph crosses the x-axis at the points (o, 0) and (4, 0)

ii. y – intercepts: is y = f(0) = 0

(0, 0) is the y intercept

3. Since $\lim_{x\to\infty}(x^4 - 4x^3) = \infty$ and $\lim_{x\to-\infty}(x^4 - 4x^3) = \infty$, the graph of y has no horizontal asymptote.

And also the limits $\lim_{x\to a}(x^4 - 4x^3)$, $\lim_{x\to a^-}(x^4 - 4x^3)$, $\lim_{x\to a^+}(x^4 - 4x^3)$ all exist, where a $\in (-\infty, \infty)$, the graph of y has no vertical asymptote

4. To determine the interval in which f is increasing and in which f is decreasing use ID Test

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

$f'(x) > 0$ if x > 3 and $f'(x) < 0$ if x < 3, since $4x^2 \geq 0$ for all real number x

Therefore, by the ID Test we have:

f is increasing on $[3, \infty)$ and decreasing on $(-\infty, 3]$

5. From example 2 a, above we have found that f has a relative minimum value at x = 3, i.e. $(3, -36)$ is relative minimum point.

6. Concavity and inflection point(s):

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

Use sign chart to see where $f''(x) > 0$ and $f''(x) < 0$.

0 2

	$x \in (-\infty, 0)$	$x \in (0, 2)$	$x \in (2, \infty)$
12x	− −	+ 0^+	+ 0 +
x - 2	− −	− 0 −	+ +
$f''(x) = 12x(x-2)$	+ +	− 0 −	+ 0 +

From the chart above, $f''(x) > 0$ for $x \in (-\infty, 0) \cup (2, \infty)$ and $f''(x) < 0$ for $x \in (0, 2)$

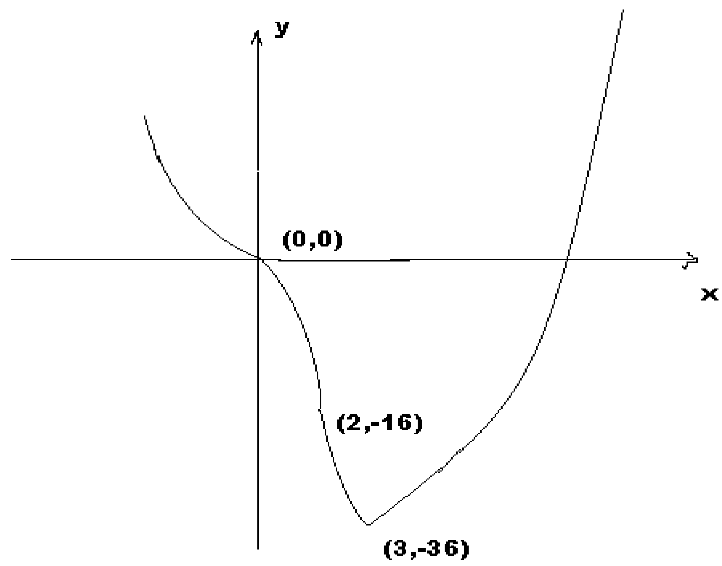
Therefore by the concavity test, the graph of f is concave up on and concave down on $(-\infty, 0) \cup (2, \infty)$ and $(0, 2)$ respectively.

The inflection points are (2,-16) and (0,0)

7. Observe that:

$$f(-x) = (-x)^4 - (-x)^3 = x^4 + x^3.$$

Thus f is neither odd nor even function.



The graph of $y = x^4 - 4x^3$

Example 2: Sketch the graph of the following functions

a. $y = \dfrac{2x^2}{x^2 - 1}$ b. $y = \dfrac{x}{(x-1)^2}$ c. $y = \dfrac{x^2}{\sqrt{x+1}}$

Solution:

a. Let $f(x) = \dfrac{2x^2}{x^2 - 1}$.

i. Domain of f $\{x : x \neq \pm 1\}$

ii. Intercepts

x – Intercept: (0, 0)
y – Intercept: (0, 0)

iii. Asymptotes

Horizontal asymptote:

Because $\lim\limits_{x\to\infty} \dfrac{2x^2}{x^2 - 1} = 2$, the line y = 2 is the horizontal asymptote to the curve.

Vertical asymptote:

Since $\lim_{x\to 1^+}\frac{2x^2}{x^2-1}=\infty$ and $\lim_{x\to -1^+}\frac{2x^2}{x^2-1}=\infty$, the lines x = -1 and x =1 are the vertical asymptotes to the given curve.

iv. Interval or union of intervals for which f is increasing and f is decreasing

$$f'(x)=\frac{-4x}{\left(x^2-1\right)^2}$$

Since for all $x\neq\pm 1$, $\left(x^2-1\right)^2>0$, $f'(x)>0$ if x < 0 and $f'(x)<0$ if x > 0.

Therefore, f is strictly increasing on $(-\infty,-1)\cup(-1,0)$ and strictly decreasing on $(0,1)\cup(1,\infty)$.

v. Local extreme values:

From (IV) above, we can see that $f'(x)$ changes sign from positive to negative at x = 0. Then by the First Derivative Test $f(0)=0$ is local maximum value of f and it does not have local minimum value.

vi. The interval for which the graph of f is concave up ward and for which it is concave down ward.

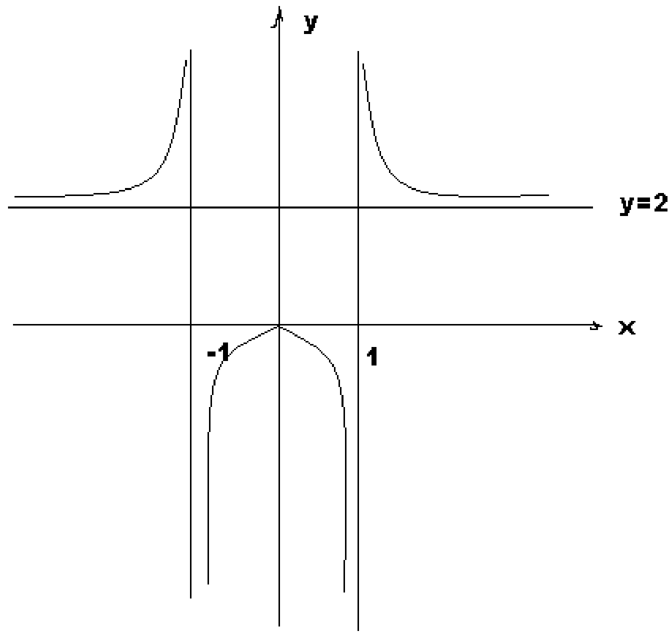
$$f''(x)=\frac{12x^2+4}{\left(x^2-1\right)^3}.$$

Since $12x^2+4>0$ for all x in the domain of f; $f''(x)>0$ if $\left(x^2-1\right)^3>0$ and $f''(x)<0$ if $\left(x^2-1\right)^3<0$.

Now use sign chart for $\left(x^2-1\right)^3$

		-1		1	
x -1	- - -		- - -	0	+ + +
x +1	- - -	0	+ + +		+ + +
$\left(x^2-1\right)^3$	+ + +	0	- - -	0	+ + +
$f''(x)=\frac{12x^2+4}{\left(x^2-1\right)^3}$	+ + +		- - -		+ + +

From the sign chart above, the graph of y = f(x) is concave up ward on $(-\infty,-1)\cup(1,\infty)$ and concave down ward on $(-1,1)$



The graph of $y=\dfrac{2x^2}{x^2-1}$

c. Let $y=g\left(x\right)=\dfrac{x^2}{\sqrt{x+1}}$

i. Domain of g = $\{x : x>-1\}$

ii. Intercepts:

x - Intercept: (0, 0)

y- Intercept: (0, 0)

iii. Asymptotes:

Vertical asymptote:

Since $\lim\limits_{x\to -1^+}\dfrac{x^2}{\sqrt{x+1}}=\infty$, the line x = -1 is the vertical asymptote to the graph of y = g(x).

Horizontal asymptote:

Since $\lim\limits_{x\to\infty}\dfrac{x^2}{\sqrt{x+1}}=\lim\limits_{x\to\infty}\sqrt{\dfrac{x^4}{x+1}}=\infty$, there is no horizontal asymptote to the given curve.

iv. Interval for which f is increasing and f is decreasing: Use (ID Test).

$$f'(x) = \frac{3x^2 + 4x}{2(x+1)^{\frac{3}{2}}} = \frac{x(3x+4)}{2(x+1)^{\frac{3}{2}}}$$

$f'(x) = 0$ only if $\dfrac{x(3x+4)}{2(x+1)^{\frac{3}{2}}}$ = 0. This gives us, x = 0 or x = $-\dfrac{4}{3}$. But $-\dfrac{4}{3}$ is not in the domain of g. Therefore x = 0 is the only critical number of g.Then using sign chart, we have

	-1		0	
x		- - -	0	+ + +
3x +4		+ + +		+ + +
$g'(x) = \dfrac{x(3x+4)}{2(x+1)^{\frac{3}{2}}}$		- - -		+ + +

From the sign chart above we can see that g is strictly increasing on $(0,\infty)$ and strictly decreasing on $(-1,0)$ by the ID Test.

v. Local extreme value:

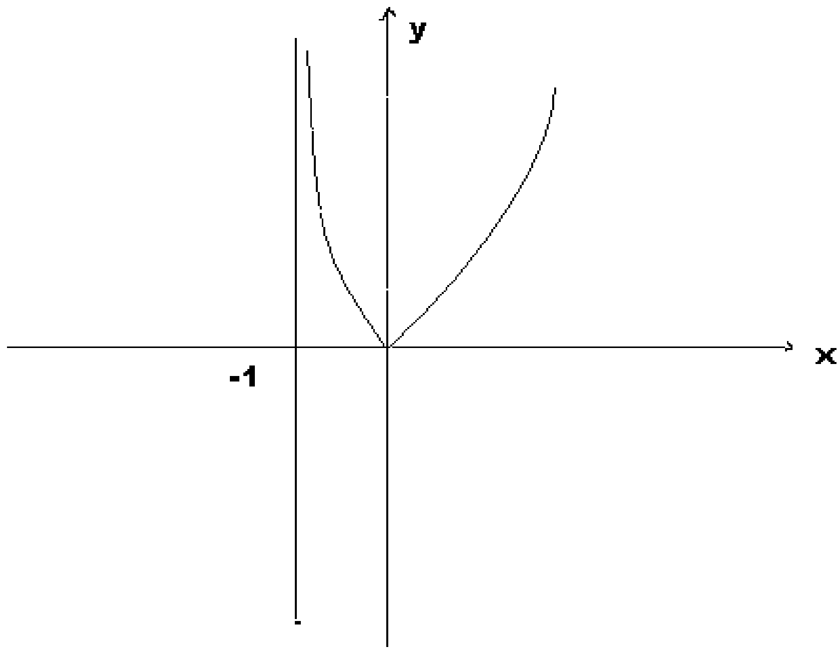
Since g' changes from negative to positive at x = 0, $g(0) = 0$ is a local minimum value of g by the First derivative test.

vi. The interval for which the graph of g is concave up ward and concave down ward. Using quotient Rule for differentiation and simplifying, we get

$$g''(x) = \frac{3x^2 + 8x + 8}{4(x+1)^{\frac{5}{2}}}$$

Both the numerator and the denominator are positive and hence $g''(x) > 0$ for x in the domain of g. Why? Try to explain.

Therefore, by the concavity Test, the graph of g is concave up ward on $\{x : x > -1\}$

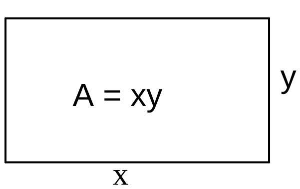


The graph of $y = \dfrac{x^2}{\sqrt{x+1}}$

2.7 Additional problems involving Absolute extrema

FINDING a maximum or a minimum has its application in pure mathematics, where we could find the largest rectangle that has a given perimeter. It also has its application to commercial problems, such as finding the least dimensions of a carton that is to contain a given volume.

Example 1. Find the dimensions of the rectangle that, for a given perimeter, will have the largest area.



Solution: Let the base of the rectangle be *x*, let its height be *y*, let *A* be its area,

and let *P* be the given perimeter. Then

$P = 2x + 2y$,

and

$A = xy$.

Since we are going to maximize *A*, we would like to have *A* as a function only of *x*. And we can do that because in the expression for *P* we can solve for *y*:

$$y = \frac{1}{2}(P - 2x) = \frac{1}{2}P - x.$$

Therefore,

$$A = x\left(\frac{1}{2}P - x\right) = \frac{1}{2}Px - x^2$$

On taking the derivative of *A* and setting it equal to 0,

$$\frac{dA}{dx} = \frac{1}{2}P - 2x = 0$$

$$x = \frac{1}{4}P$$

The base is one quarter of the perimeter. We can now find the value of y.

$$y = \frac{1}{2}(P - 2x) = \frac{1}{2}P - x$$

$$y = \frac{1}{2}P - \frac{1}{4}P = \frac{1}{4}P \quad \text{because } x = \frac{1}{4}P.$$

The height is also one quarter of the perimeter. That figure is a square! The rectangle that has the largest area for a given perimeter is a square.

Note: The value we found is a maximum, because the second derivative is negative.

All maximum-minimum problems follow this same procedure.

i. Write the function whose maximum or minimum value is to be determined. **(In the Example, we wrote A = xy.)**

ii. The resulting expression will typically contain more than one variable. Use the information given in the problem to express every variable in terms of a single variable. **(In the Example, we expressed y in terms of x.)**

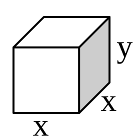
iii. Find the critical value of that single variable by taking the derivative and setting it equal to 0.
(In the Example, we took the derivative of A with respect to x.)

iv. If necessary, determine the values of the other variables.
(In the Example, we evaluated y by substituting the critical value of x).

In the following, notice how we follow these steps.

Example 2: A box having a square base and an open top is to contain 108 cubic feet. What should its dimensions be so that the material to make it will be a minimum? That is, what dimensions will cost the least?

$V = x^2y$



Solution: Let x be the side of the square base, and let y be its height. Then

Area of base = x^2

Area of the four bases = $4xy$

Let *M* be the total amount of material. Then

$M = x^2 + 4xy$

Now, how shall we express y in terms of x?

We have not yet used the fact that the volume must be 108 cubic feet. The volume is equal to

$x^2y = 108$

Therefore,

$y = \dfrac{108}{x^2}$ and therefore in the expression for M

$$4xy = 4x.\frac{108}{x^2} = \frac{432}{x}$$

$$M = x^2 + \frac{432}{x}$$

$$\frac{dM}{dx} = 2x - \frac{432}{x^2} = 0$$

This implies, on multiplying through by the denominator x^2

$$2x^3 - 432 = 0$$

$$x^3 = 216$$

$x = 6$ feet

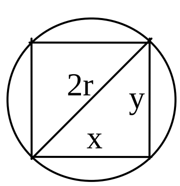
We can now evaluate y

$y = \dfrac{108}{x^2} = \dfrac{108}{6} = 3$ feet.

The dimensions that will cost the least are 6 feet and 3 feet.

Example 3: Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius r. Show, in fact, that area will be $2r^2$.

Solution: First, it should be clear that there is a rectangle with the maximum possible area.



Let A be the area of the rectangle with length x and height y.

A= xy

$$x^2 + y^2 = (2r)^2 = 4r^2$$

$$y = \sqrt{4r^2 - x^2}$$

$$A = xy = x\sqrt{4r^2 - x^2}$$

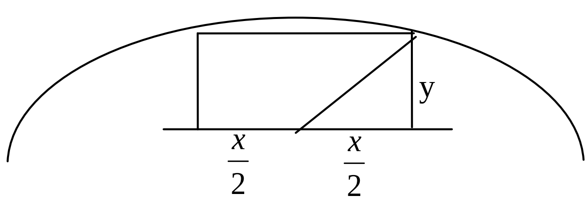
$$\frac{dA}{dx} = \sqrt{4r^2 - x^2} - \frac{2x^2}{2\sqrt{4r^2 - x^2}}$$

$$\frac{dA}{dx} = 0 \Rightarrow 8r^2 - 4x^2 = 0$$

Solving for x in terms of r we get $x = r\sqrt{2}$ and hence y = $r\sqrt{2}$

Therefore, $A = xy = r\sqrt{2}\left(r\sqrt{2}\right) = 2r^2$.

Example 4: Find the dimensions of the rectangle with the most area that can be inscribed in a semi-circle of radius *r*. Show, in fact, that the area of that rectangle is r^2.



Let *x* be the base of the rectangle, and let *y* be its height. Then, since *r* is the radius:

$$\frac{x^2}{2} + y^2 = r^2$$

$$\Rightarrow x^2 + 4y^2 = 4r^2$$

Therefore, $y = \frac{1}{2}\sqrt{4r^2 - x^2}$

Let *A* be the area we want to maximize. A = xy. That is,

A = $\frac{1}{2}x\sqrt{4r^2 - x^2}$

According to the product rule:

$$\text{A} = \frac{1}{2}x.\frac{-2x}{2\sqrt{4r^2 - x^2}} + \frac{1}{2}\sqrt{4r^2 - x^2}$$

On setting this equal to 0 and multiplying through by $2\sqrt{4r^2 - x^2}$

$$-x^2 + \left(4r^2 - x^2\right) = 0$$

This implies:

$$x^2 = 2r^2$$

$$x = \sqrt{2}r$$

This is the base of the largest rectangle. As for the height y

$$y = \frac{1}{2}\sqrt{4r^2 - 2r^2}$$

$$y = \frac{1}{2}\sqrt{2r^2} = \frac{\sqrt{2}}{2}r$$

The area of this largest rectangle, then, is

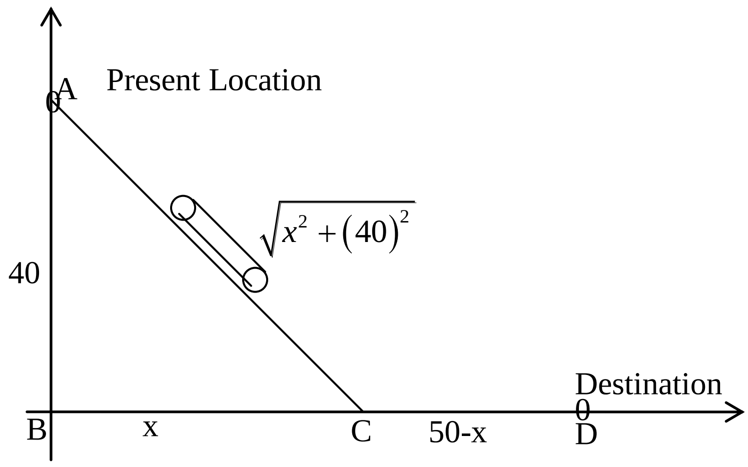
$$xy = \sqrt{2}\,r.\frac{\sqrt{2}}{2}r = r^2$$

Example 5: **Minimizing time of travel.**

A vehicle is on the desert at point A located 40km from a pint B, which lies on a long, straight road, as shown in the figure. The driver can travel at 45km/hr on the desert and 75km/hr on the road. The driver will win a prize if he or she arrives at the finish line at point D, 50km from B, in 84 minutes or less.

What route should the travel to minimize the time of travel?

Does the driver win the prize?



Solution: suppose the driver heads for a point C located x km down the road from B toward the destination as shown in the figure.

We want to minimize the time.

We know that:

$$\boxed{\text{Distance traveled = Velocity x time taken}}$$

$$\boxed{t = \frac{d}{v}}$$

$$T(x) = \frac{\sqrt{x^2+1{,}600}}{45} + \frac{50-x}{75}$$

But by Pythagoras's Theorem, distance from A to C is $\sqrt{x^2+(40)^2} = \sqrt{x^2+1{,}600}$

Time, $t = time\ from\ A to\ C\ +\ time\ from\ C\ to\ D$

$$= \frac{Distance\ from\ A\ to\ C}{Velocity\ from\ A\ to\ C} + \frac{distance\ from\ C to\ D}{Volocity\ from\ C\ to\ D}$$

and the distance from C to D is $50-x$, where x is the distance from B to C.

Therefore, $T(x) = \dfrac{\sqrt{x^2+1{,}600}}{45} + \dfrac{50-x}{75}$

The domain of T is $[0,50]$.

Let us find $T'(x)$

$$T'(x)=\frac{1}{45}\left(\frac{2x}{2\sqrt{x^2+1600}}\right)+\frac{1}{75}(-1)$$

$$T'(x)=\frac{1}{45}\left(\frac{x}{45\sqrt{x^2+1,600}}\right)-\frac{1}{75}$$

$T'(x)$ exist for all values of x, and

$$T'(x)=0\Rightarrow\frac{1}{45}\left(\frac{x}{45\sqrt{x^2+1,600}}\right)-\frac{1}{75}=0$$

$$\Rightarrow\frac{x}{\sqrt{x^2+1,600}}=\frac{45}{75}=\frac{3}{5}$$

$$\Rightarrow 5x=3\sqrt{x^2+1,600}$$

$$\Rightarrow 25x^2=9(x^2+1,600)$$

$$\Rightarrow 16x^2=9x1,600$$

$$\Rightarrow x^2=\frac{0x1,600}{16}=900$$

$$\Rightarrow x=\pm\sqrt{900}=\pm 30$$

Thus the only critical number is 30 (-30 is extraneous).

Evaluate T at 0, 30 and 50, we will get:

$$T(0)=\frac{\sqrt{0^2+1,600}}{45}+\frac{50-0}{75}\approx 1.5556hr\approx 93\min.$$

$$T(30)=\frac{\sqrt{(30)^2+1,600}}{45}+\frac{50-30}{75}\approx 1.377shrs.\approx 83\min.$$

$$T(50)=\frac{\sqrt{(50)^2+1,600}}{45}+\frac{50-50}{75}\approx 1.4229hr\approx 85\min.$$

The driver can minimize the total driving time by heading for a point that is 30 miles from the point B and then traveling on the road to point D.

The driver wins the prize because this minimal route requires only 83 minutes.

EXERCISE ON APPLICATIONS OF DERIVATIVE

1. Find an equation for the tangent and the normal line to the graph of

   $f(x) = \frac{1}{x}$ at the point where x = 2.

2. a. Find the derivative of $f(x) = x^2 - 3x$

   b. Show that the parabola whose equation is $y = x^2 - 3x$ has one horizontal tangent line. Find the equation of this line

   c. Find a point on the graph of f where the tangent line is parallel to the line

   $3x + y = 11$

3. Verify that for the given function f satisfies the hypothesis of the MVT, on the indicated interval [a, b] .Then find all numbers c between a and b for which

   $\frac{f(b) - f(a)}{b - a} = f'(c)$, if the hypotheses are satisfied.

   a. $f(x) = \sqrt{x}$, $[1, 4]$ b. $f(x) = 3x^2 + 2x + 5$, $[-1, 1]$

   c. $f(x) = \ln x$, $\left[\frac{1}{2}, 2\right]$ d. $f(x) = \sqrt[3]{x} - 1$, $[-8, 8]$

4. Let $f(x) = (x - 1)^{-2}$. Show that $f(0) = f(2)$ but there is no number c in (0, 2) such that $f'(c) = 0$ .Why does this not contradict Rolle's Theorem?

5. Suppose that $3 \le f'(x) \le 5$ for all values of x. Show that $18 \le f(8) - f(2) \le 30$.

6. Find the critical numbers for the given functions.

   a. $g(x) = \sqrt{x}(1 - x)$

   b. $f(x) = 4x^3 - 5x^2 - 8x + 20$

7. Find the absolute extrema of the function defined by the equations on the given interval

   a. $f(x) = x^4 - 2x^2 + 3$, [- 1, 2].

   b. $g(t) = (50 + t)^{2/3}$, [-50, 14]

8. A box with a square base is constructed so that the length of one side of the base plus the height is 10 in. What is the largest possible volume of such a box?

9. Find the critical number(s) for function $f(x) = 5 + 10x - x^2$ on [- 3, 3] and then tell whether each yields a minimum, maximum, or neither for the function

10. Find two numbers whose sum is 42 and whose product will be the largest.

11. $1200cm^2$ of material is available to make a box with a square base and an open top, find the largest possible volume

12. A painting is in an art gallery has height is and is hung so that its lower edge is a distance d for the eye of an observer (as shown in the figure). How far from the wall should the observer stand to get the best view? (Or where should the observer stand so as to maximize the angle θ subtended at his eye by the painting?

CHAPTER 3: Revision on Integration

In chapter 3, we have seen how to find the derivative of a function at a point and at an arbitrary point in its domain. If we think of discovering a function knowing its derivative, we are reversing the process of differentiation which is known as anti- differentiation or integration.

Integration can be used to solve physical problems, such as how long it takes for a sand bag to fall to the ground when dropped from a balloon.

Specific objectives: After the completion of this chapter, the students will be able to:

· Find the anti- derivative F of a continuous function f.

· Use the general power rule, Exponential rule, Logarithmic rule together with limit laws to calculate anti- derivative.

· Find lower sum and upper sum of a continuous function on [a, b] associated with a given partition.

· Evaluate definite integral using the different techniques of integration.

· Calculate the area bounded between two curves

· Use integration to find volume of solids of revolution

3.1 Antiderivatives

Up to this point in the module, we have been concerned primarily with this problem: given a function, find its derivative.

Many important application of calculus involve the inverse problem: given the derivative of a function, find the function.

A physicist who knows the velocity of a particle might wish to know its position at a given time. A biologist who knows the rate at a bacteria population is increasing might want to deduce what the size of the population will be at some future time.

I each case the problem is to find a function F whose derivative is a known function f. If such a function F exists, it is called an antiderivative of f.

Definition of antiderivative: A function F is an antiderivative of a continuous function f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem 4.1: If F is an antiderivative of f on an interval I, then F(x) + c is also an ant derivative of f on I, where c is arbitrary.

Proof: Suppose F is an antiderivative of f o I, i.e. $F'(x) = f(x)$

Then we want to show that F(x) + c is also an ant derivative of f on I.

$(F(x)+c)' = F'(x)+(c)' = F'(x) = f(x)$, by sum rule for differentiation.

Hence F(x) + c is also an antiderivative of f for all x in I.

For example, $F(x) = x^3$, $G(x) = x^3 + \sqrt{2}$, $H(x) = x^3 + \pi$ are antiderivative of $3x^2$ because the derivative of each is $3x^2$.

As it turns out, all anti derivatives of $3x^2$ are of the form $x^3 + c$.

Note: The process of antidifferentiation does not determine a single function, but rather a family of functions, each differing from the other by a constant.

Example 1: Find an antiderivative of each of the following functions.

a) $f(x) = 4$ b) $g(x) = -\sin x$ c) $h(x) = \frac{1}{x}$

d) $f(x) = x^n$ for $n \neq -1$

Solution: a) since the derivative of $F(x) = 4x + c$,where c is an arbitrary constant is 4. So the antiderivative of $f(x) = 4$ is $F(x) = 4x + c$.

b) Since $(\cos x + c)' = -\sin x$, the antiderivative of $f(x) = -\sin x$ is $F(x) = \cos x + c$

c) Recall that $\frac{d}{dx}(\ln x) = \frac{1}{x}$. So, on the interval $(0, \infty)$ the antiderivative of $\frac{1}{x}$

is lnx + c.

We can also see that:

$\frac{d}{dx}(|x| + c) = \frac{1}{x}$ for all $x \neq 0$.

Therefore the antiderivative of $\frac{1}{x}$ is $\ln|x| + c$ for all $x \neq 0$.

That is, $F(x) = \begin{cases} \ln x + c_1 & for \;\; x > 0 \\ \ln(-x) + c_2 & for \;\; x < 0 \end{cases}$ is the antiderivative of $f(x) = \frac{1}{x}$.

d) We use the power rule for differentiation to find an antiderivative of x^n .

If $n \neq -1$, then

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{(n+1)}{n+1}x^n = x^n \; .$$

Therefore, the antiderivative of $f(x) = x^n$ is $F(x) = \frac{x^{n+1}}{n+1}$ +c, for $n \neq -1$.

3.2 Indefinite Integrals

The antidifferentiation process is also called **Integration** and is denoted by the symbol $\int$ which is called an integral sign.

The symbol $\int f(x)dx$ is the indefinite integral of f(x), and it denotes the family of antiderivative of f(x). That is if $F'(x) = f(x)$ for all x, then we can write:

$\int f(x)dx = F(x) + c$ where, f(x) is the integrand and c is the constant of integration.

The differential dx in the indefinite integral identifies the variable of integration.

That is the symbol $\int f(x)dx$ denotes the “antiderivative “of f with respect to x.

Integral Notation of Antiderivative

Notation: $\int f(x)dx = F(x) + c$, where c is an arbitrary constant, means that F is an antiderivative of f.

Finding Antiderivatives:

The inverse relationship between the operations of integration and differentiation can be shown by symbolically as follows.

$$\boxed{\frac{d}{dx}\left(\int f(x)dx\right) = f(x)}$$

Differentiation is the inverse process of integration.

Proof: Let $\int f(x)dx = F(x)$ + c, where F(x) is an antiderivative of f, then

$$\frac{d}{dx}\left(\int f(x)dx\right) = \frac{d}{dx}(F(x)+c) = F'(x) = f(x)$$

$$\boxed{\int f'(x)dx = f(x)+c}$$

Integration is the inverse process of differentiation.

3.3 Some Integration Formulas

1. $\int kdx = kx + c$, where c and k are constants (Constant Rule)
2. $\int kf(x)dx = k\int f(x)dx$ (Constant Multiple Rule)
3. $\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$ (Sum Rule)
4. $\int (f(x)-g(x))dx = \int f(x)dx - \int g(x)dx$ (Difference Rule)
5. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \ \ if \ \ n \neq -1$ (Simple power Rule)

Table of Indefinite Integral

$$\int \cos x dx = \sin x + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \sec^2 x dx = \tan x + c$$

$$\int \csc^2 x dx = -\cot x + c$$

$$\int \sec x \tan x dx = \sec x + c$$

$$\int \csc x \cot x dx = -\csc x + c$$

$$\int \frac{1}{x} dx = \ln|x| + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

Remark: Every two antiderivative of $f(x)$ differ by a constant.

Example 1: Find the indefinite integral $\int 3xdx$

Solution: $\int 3xdx = 3\int xdx = 3\left(\frac{x^2}{2}\right) + c$

Note the general pattern of integration is similar to that of differentiation.

Original integral		Re write		Integrate		Simplify
$\int 3xdx$	$\Rightarrow$	$3\int x^1 dx$	$\Rightarrow$	$3\left(\frac{x^2}{2}\right) + c$	$\Rightarrow$	$\frac{3}{2}x^2 + c$

Example 2: **Rewriting before Integrating**

Find each of the following integral

a) $\int \frac{1}{x^3}dx$ b) $\int \sqrt[3]{x^2}dx$ c) $\int \frac{x+1}{\sqrt{x}}dx$

Solution:

a) Original integral | Rewrite | Integrate | Simplify

$\int \frac{1}{x^3}dx$ $\quad \int x^{-3}dx$ $\quad \frac{x^{-2}}{-2}+c$ $\quad \frac{-1}{2x^2}+c$

b) Original integral | Rewrite | Integrate | Simplify

$\int \sqrt[3]{x^2}dx$ $\quad \int x^{\frac{2}{3}}dx$ $\quad \frac{x^{\frac{2}{3}+1}}{\frac{3}{\frac{2}{3}+1}}+c$ $\quad \frac{3}{5}x^{\frac{3}{5}}+c$

c) Original integral | Rewrite as sum | rewriting using rational exponent

$\int \frac{x+1}{\sqrt{x}}dx$ $\quad \int\left(\frac{x}{\sqrt{x}}+\frac{1}{\sqrt{x}}\right)dx$ $\quad \int (x^{\frac{1}{2}}+x^{\frac{-1}{2}})dx$

Apply power rule | Simplifying

$\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}+\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+c$ $\quad \frac{2}{3}\sqrt{x^3}+2\sqrt{x}+c$

Example 3: Find the indefinite integral of

a) $\int\left(3x^4-5x^3-x+4\right)dx$ b) $\int\left(\sqrt{2}x^{\frac{5}{3}}-2\sec^2 x\right)dx$

Solution: a) Use the sum rule to integrate each part separately.

$$\int\left(3x^4-5x^3-x+4\right)dx=3\int x^4dx-5\int x^3dx-\int xdx+4\int 1dx$$

$$=3\left(\frac{x^5}{5}\right)-5\left(\frac{x^4}{4}\right)-\frac{x^2}{2}+4x+c$$

$$= \frac{3}{5}x^5 - \frac{5}{4}x^4 - \frac{1}{2}x^2 + 4x + c$$

b) Use the difference rule to integrate each part separately

$$\int\left(\sqrt{2}x^{\frac{5}{3}} - 2\sec^2 x\right)dx = \sqrt{2}\int x^{\frac{5}{3}}dx - 2\int \sec^2 x dx$$

$$= \sqrt{2}\,\frac{x^{\frac{8}{3}}}{\frac{8}{3}} - 2\tan x + c$$

$$= \frac{3\sqrt{2}}{8}x^{\frac{8}{3}} - 2\tan x + c$$

Example 4: Evaluate $\int \frac{\cos x}{\sin^2 x}dx$

Solution:

$$\int \frac{\cos x}{\sin^2 x}dx = \int \frac{1}{\sin x}\left(\frac{\cos x}{\sin x}\right)dx$$

$$= \int \csc x \cot x\, dx = -\csc x + c$$

Key Concepts

☺ If f(x) is continuous on [a, b] and $F'(x) = f(x)$ *for all* $x \in (a,b)$, then the set of all antiderivative of f is called an Indefinite Integral of f and denoted by

$$\int f(x)dx = F(x) + c$$

In fact, every anti derivative of f(x) can be written in the form $F(x) + c$,

for some c.

$$\frac{d}{dx}(F(x)) = f(x) \Leftrightarrow \int f(x)dx = F(x) + c$$

$$\frac{d}{dx}(g(x)) = g'(x) \Leftrightarrow \int g'(x)dx = g(x) + c$$

3.4 Techniques of Integration

3.4.1 Computing Integrals by substitution

Many integrals are most easily computed by means of a change of variables, commonly called a **u – substitution.**

Example 1: Compute $\int 2x(x^2-1)^4\,dx$

Solution: Let's us compute the integral by making the substitution

$u = x^2 - 1$, du = 2xdx. Then

$$\int 2x(x^2-1)^4\,dx = \int (x^2-1)2xdx$$

$$= \int u^4 du = \frac{u^5}{5} + c = \frac{1}{5}(x^2-1)^5 + c$$

We may check this by differentiation using the Chain rule.

$$\frac{d}{dx}\left(\frac{1}{5}(x^2-1)^5 + c\right) = \frac{5}{5}(x^2-1)^4 2x = 2x(x^2-1)^4$$

The substitution method amounts to applying the chain rule in reverse.

To compute $\int f(g(x)).g'(x)dx$, we let

$u = g(x)$, $du = g'(x)dx$. Then we get:

$\int f(g(x)).g'(x)dx = \int f(u)du = F(u) + c = F(g(x)) + c$, where F is an anti derivative of f and c is an arbitrary constant.

Example 2: Evaluate $\int \sin(2x)\cos(2x)dx$

Solution: Let $u = \sin(2x)$

$$du = 2\cos(2x)dx \quad \left(\cos(2x)dx = \frac{1}{2}du\right)$$

Then

$$\int \sin(2x)\cos(2x)dx = \int u.\frac{1}{2}du = \frac{1}{2}\int udu = \frac{1}{4}u^2 + c$$

$$= \frac{1}{4}\sin^2(2x) + c$$

Example 3: Compute $\int \frac{(x+2)}{\sqrt{x^2+4x}}dx$

Solution: Let us use substitution method.

Let $u = x^2 + 4x$. Then $du = (2x+4)dx = 2(x+2)dx$

$\frac{1}{2}du = (x+2)dx$

Then, $\int \frac{(x+2)}{\sqrt{x^2+4x}}dx = \int \frac{\frac{1}{2}du}{\sqrt{u}} = \frac{1}{2}\int u^{-\frac{1}{2}}du$

$$= \frac{1}{2}\left(2u^{\frac{1}{2}}\right) + c = \sqrt{x^2+4x} + c$$

Therefore, $\int \frac{(x+2)}{\sqrt{x^2+4x}}dx = \sqrt{x^2+4x} + c$

Remark: it is not always apparent until you try it whether or not a substitution will work.

Example 4: Evaluate $\int x\sqrt{x-3}dx$

Solution: To compute $\int x\sqrt{x-3}dx$, we will try:

$u = x - 3$, this implies x = u + 3

$du = dx$

So, $\int x\sqrt{x-3}dx = \int (u+3)\sqrt{u}\, du = \int (u+3)u^{\frac{1}{2}}du$

$$= \int \left(u^{\frac{3}{2}} + 3u^{\frac{1}{2}} \right) du$$

$$= \frac{2}{5}u^{\frac{5}{3}} + 2u^{\frac{3}{2}} + c$$

$$= \frac{2}{5}(x-3)^{\frac{5}{2}} + 2(x-3)^{\frac{3}{2}} + c$$

We can compute a definite integral using a substitution.

Example 2: Find $\int \sqrt{1+x^2}\, x^5 dx$

Solution: An appropriate substitution becomes more obvious if we factor

x^5 as $x^4.x$

Let u =1 + x^2

Then, du = 2xdx, so $\frac{1}{2}du = xdx$

x^2 = u -1, so x^4 = $(u-1)^2 = u^2 - 2u + 1$

Then,

$$\int \sqrt{1+x^2}\ \ x^5 dx = \int \sqrt{1+x^2}\ \ x^4.xdx$$

$$= \int \sqrt{u}\left(u^2 - 2u + 1\right).\frac{1}{2}du$$

$$= \frac{1}{2}\int \left[u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}} \right] du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}} - 2.\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} \right) + c$$

$$= \frac{1}{7}\left(1+x^2\right)^{\frac{7}{2}} - \frac{2}{5}\left(1+x^2\right)^{\frac{5}{2}} + \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} + c$$

Example 3: Evaluate $\int \tan xdx$

Solution: First we write tanx in terms of sinx and cosx

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$$

Then, substitute:

U = cosx

Du = -sinx dx, so –du = sinx dx

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int \frac{1}{u} du = -\ln|u| + c$$

$$= -\ln|\cos x| + c$$

Since $-\ln|\cos x| = \ln\left(|\cos x|^{1-}\right) = \ln\left|\frac{1}{\cos x}\right| = \ln|\sec x|$,

$We \quad have: \int \tan x dx = 1n|\sec x| + c$

Key Concept

☺ **. The substitution method: If** u = g(x) is a differentiable function and whose range is an interval I and f continuous on I, then,

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u)du = F(g(x)) + C, \ where \ \ F \ \ is\ an\ anti-derivative\ of \ \ f$$

3.4.2 Integration by Parts

We will use the Product Rule for derivatives to derive a powerful integration formula:

$(f(x)g(x)) = f(x)g'(x) + f'(x)g(x).$

Start with

$$f(x)g(x) = \int f(x)g'(x)dx + \int f'(x)g(x)dx$$

Integrate both sides

(We need not include a constant of integration on the left, since the integrals on the right will also have integration constants).

Solve for $\int f(x)g'(x)dx$, we get:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

This formula frequently allows us to compute a difficult integral by computing a much simpler integral. We often express the integration by parts formula as follows:

Let

$$\begin{array}{ll} u = f(x) & dv = g'(x)dx \\ du = f'(x)dx, & v = g(x) \end{array}$$

Then the formula becomes

$\boxed{\int u du = uv - \int v du.}$ Integration by parts formula

To integrate by parts, strategically choose u, dv and then apply the formula.

Example 1: Evaluate $\int xe^x dx$

Solution: Let

$$\begin{array}{ll} u = x & dv = e^x dx \\ du = dx & u = e^x \end{array}$$

Then by integration by parts,

$$\begin{aligned} \int xe^x &= xe^x - \int e^x dx \\ &= xe^x - e^x + C. \end{aligned}$$

Sometimes it is necessary to integrate twice by parts in order to compute an integral:

Example 3**:** Compute $\int e^x \cos x\, dx$

Let

$$\begin{array}{ll} u = e^x & dv = \cos x dx \\ du = e^x dx & v = \sin x \end{array}$$

Then $\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x dx$ (1)

It is not clear yet that we have accomplished anything, but now let's integrate the integral on the right-hand side by parts:

Now let

$$u = e^x \qquad dv = \sin x dx$$
$$du = e^x dx \qquad v = \cos x$$

$So, \int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$

Substituting this into (1) , we get:

$$\int e^x \cos x dx = e^x \sin x - \left[-e^2 \cos x + \int e^x cis\ x dx\right]$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x dx.$$

The integral $\int e^x \cos x dx$ appears on both sides of the equation, so we can solve for it:

$$2\int e^x \cos x dx = e^x \sin x + e^x \cos x.$$

Finally,

$$\int e^x \cos x dx = \frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + C.$$

Key concept

$\int u dv = uv - \int v du.$

Choose u, dv in such a way that:

1. u is easy to differentiate
2. dv is easy to integrate.
3. $\int v\, du$ is easier to compute than $\int u\, dv$

Sometimes it is necessary to integrate by parts more than once.

3.4.3 Integration by partial Fractions

Integrals which involves rational fraction for which substitution method is not convenient can be integrated by the method of integration called integration by **partial fraction.**

Example 1: Consider the integral

$$\int \frac{3x^3 - 2x^2 - 19x - 7}{x^2 - x - 6} dx.$$

The integrand is an improper rational function. By "long division" of polynomials, we can rewrite the integrand as the sum of a polynomial and a proper rational function "remainder"

$$\begin{array}{r|l} & 3x+1 \\ \hline x^2 - x - 6 & 3x^3 - 2x^2 - 19x - 7 \\ & 3x^3 - 3x^2 - 18x \\ & x^2 - x - 7 \\ & \underline{x^2 - x - 6} \\ & -1 \end{array}$$

$$\int \frac{3x^3 - 2x^2 - 19x - 7}{x^2 - x - 6} dx = \int (3x + 1 + \frac{-1}{x^2 - x - 6}) dx.$$

This looks much easier to work with! We can integrate 3x+1 immediately, but

What about $\frac{-1}{x^2 - x - 6}$?

Notice that:

$$\frac{-1}{x^2 - x - 6} = \frac{-1}{(x+2)(x-3)}$$

which suggests that we try to write $\frac{-1}{x^2 - x - 6}$ as the sum of two rational

functions of the from $\frac{A}{x+2}$ *and* $\frac{B}{x-3}$

$$\frac{-1}{x^2 - x - 6} = \frac{A}{x+2} + \frac{B}{x-3}$$

This is called the **partial Fraction Decomposition** for $\frac{-1}{x^2 - x - 6}$

Our goal now is to determine A and B. Multiplying both sides of the equation by (x + 2) (x - 3) to clear the fractions,

$$-1 = A(x-3) + B(x+2).$$

There are two methods for solving for A and B:

Method 1 **Method 2**

Collect like terms o the right:

-1 = (A+B)x + (-3A+2B)

Now equate coefficients of

Corresponding powers of X:

A+B =0, -3A+2B=-1

Solving this system,

A= 1/5, B = -1/5

The equation holds for all x.

Let x = -2:

-1=A (-2-3) +B (-2 + 2)

-1= -5A $\Rightarrow$ A =1/5

Now let x = 3:

-1 = A (3-3) + (3+2)

-1 = 5B $\Rightarrow$ B = -1/5

So

$$\frac{-1}{x^2-x-6} = \frac{1/5}{x-3} - \frac{1/5}{x-3}.$$

Returning to the original integral,

$$\int \frac{3x^3-2x^2-19x-7}{x^2-x-6}dx = \int\left(3x+1+\frac{[1/5]}{x+2}-\frac{[1/5]}{x-3}\right)dx$$

$$= \int(3x+1)dx + \frac{1}{5}\int\frac{1}{x+2}dx - \frac{1}{5}\int\frac{1}{x-3}dx$$

$$= \frac{3}{2}x^2 + x + \frac{1}{5}\ln|x+2| - \frac{1}{5}\ln|x-3| + C$$

$$= \frac{3}{2}x^2 + x + \frac{1}{5}\ln\left|\frac{x+2}{x-3}\right| + C.$$

In the next example, we have repeated factors in the denominator, as well as an irreducible quadratic factor.

Example2:Evaluate $\int \frac{x-1}{x^2(x^2+x+2)}dx.$

The integrand is a proper rational function, which we would like to decompose into proper rational functions of the form

$$\frac{A}{x}, \frac{B}{x^2}, and\ \frac{Cx+D}{x^2+x+1}$$

[Notice that we have two factors of x in the denominator of the integrand, leading to terms of the form $\frac{A}{x}\ and\ \frac{B}{x^2}$ in the decomposition. The factor $x^2 + x + 1$is irreducible and quadratic, so any proper rational function with $x^2 + x + 1$ as denominator has the form

$\frac{Cx+D}{x^2+x+1}$ where C or D may be zero.

Set $\frac{x-1}{x^2(x^2+x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+x+1}$

Multiplying throughout by $x^2(x^2+x+1)$,

$$x-1 = Ax(x^2+x+1) + B(x^2+x+1) + (Cx+D)x^2.$$

Since $x^2 + x + 1$ has no real roots, it is easiest to solve for A and B using Method 1: collecting like terms on the right,

$$x-1 = (A+C)x^3 + (A+B+D)x^2 + (A+B)x + B$$

Equating corresponding powers of x,

$$\begin{array}{ll} A+C=0 & A=2 \\ A+B+D=0 & B=-1 \\ A+B=1 & C=-2 \\ B=-1 & D=-1 \end{array}$$

That is, $\frac{x-1}{x^2(x^2+x+1)} = \frac{2}{x} - \frac{1}{x^2} - \frac{2x+1}{x^2+x+1}$

So

$$\begin{aligned} \int \frac{x-1}{x^2(x^2+x+1)}dx &= \int\left(\frac{2}{x} - \frac{1}{x^2} - \frac{2x+1}{x^2+x+1}\right)dx \\ &= 2\ln|x| + \frac{1}{x} - \ln\left|x^2+x+1\right| + C \\ &= \frac{1}{x} + \ln\left|\frac{x^2}{x^2+x+1}\right| + C. \end{aligned}$$

To see how the Method of partial fraction works in general, let's consider a rational function $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomial functions.

If f is proper $(Degree\ \ of\ \ P < Degree\ of\ \ Q)$, then we can express f as a sum of simpler fractions.

If f is improper $(Degree\ \ of\ P \geq Degree\ \ of\ \ Q)$, follow the following steps.

Step1: Divide P by Q using long division, until a remainder R(x) is obtained such that $Degree\,of\ R < Degree\,of\ Q\,and\ \ write$ f(x) as:

$$f(x)=\frac{P(x)}{Q(x)}=S(x)+\frac{R(x)}{Q(x)}$$ (1), where S and R are also polynomials

Step 2: Factor the denominator Q(x) as far as possible.

Step 3: Write the proper rational function $\frac{R(x)}{Q(x)}$ (of equation1) as a sum of

Partial fraction of the form

$$\frac{A}{(ax+b)^m}\,or\,\frac{Ax+B}{\left(ax^2+bx+c\right)^k}$$

Now we have to consider the following four possible cases:

Case 1: The denominator Q(x) is a product of distinct linear factors.

This means we can write:

$$Q(x)=(a_1x+b_1)(a_2x+b_2)---(a_mx+b_m)$$

And then

$$\frac{R(x)}{Q(x)}=\frac{A}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+---+\frac{A_m}{a_mx+b_m}, \qquad (2)$$

Where $A_1, A_2, ---, A_m$ are constants to be determined.

Example 1**:** Evaluate $\int\frac{x^2+2x-1}{2x^3+3x^2-2x}dx$

Solution: since the fraction is proper begin by factoring the denominator

$2x^3+3x^2-2x=x\left(2x^2+3x-2\right)=x(2x-1)(x+2)$ **(Distinct linear factors)**

Then the partial decomposition has the form:

$$\frac{x^2+2x-1}{2x^3+3x^2-2x}=\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}$$

To determine the values of A, B, C, we multiply both sides of this equation by x (2x-1)(x+2), obtaining:

$$x^2+2x-1=A(2x-2)(x+2)+Bx(x+2)+Cx(2x-1)$$

$$\Rightarrow x^2+2x-1=(2A+B+2C)x^2+(3A+2B-C)x-2A$$

Equating like powers of x, we get:

$$\begin{aligned} 2A+B+2C &= 1 \\ 3A+2B-C &= 2 \\ -2A \qquad &= 1 \end{aligned}$$

Solving, we get $\frac{1}{2}, B=\frac{1}{5}, C=-\frac{1}{10}$

So

$$\int \frac{x^2+2x-1}{2x^3+3x^2}dx = \int\left(\frac{1}{2}x+\frac{1}{5(2x-1)}-\frac{1}{10(x+2)}\right)dx$$

$$=\frac{1}{2}\ln|x|+\frac{1}{10}\ln|2x-1|-\frac{1}{10}\ln|x+2|+D$$

Case 2: Q(x**)** is a product of linear factors, (a_1x + b_1) is repeated K times in the factorization of Q(x). Then instead of the single term $\frac{A_1}{a_1x+b_1}$ in equation (2) we write:

$$\frac{A_1}{a_1x+b_1}+\frac{A_2}{(a_2x+b_2)^2}+----+\frac{A_k}{(a_kx+b_k)^k} \qquad (3)$$

Example 2: Evaluate $\int \frac{x^5+x-1}{x^4-x^3}dx$

Solution**:** The rational unction f(x) = $\frac{x^5+x-1}{x4-x^3}$ is improper**.**

By long division, we have:

$$\frac{x^5+x-1}{x^4-x^3}=(x+1)+\frac{x^3+x-1}{x^4-x^3}$$

Now applying, partial fraction decomposition procedures.

$$\frac{x3+x-1}{2x^4-x^3}=\frac{x^3+x-1}{x^3(x-1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x-1}$$

Multiplying both sides by the least common denominator $x^3(x-1)$produces

$$x^3+x-1=Ax^2(x-1)+Bx(x-1)+C(x-1)+cx^3$$
$$\Rightarrow x^3+x-1(Z+D)x^3+(-A+B)x^2+(-B+C)x-C$$

This gives the system of equations:

$$A+D=1$$
$$-A+B=0$$
$$-B+C=1$$
$$-C=1$$

Solving, we obtain,

A= 0, B = 0, C = 1, D = 1

$$\frac{x^3+x-1}{x^4-x^3}=\frac{1}{x^3}+\frac{1}{x-1}$$

Then,

$$\int\frac{x^5+x-1}{x^4-x^3}dx=\int\left(x+1+\frac{x^3+x-1}{x^4-x^3}\right)dx$$
$$=\int\left(x+1+\frac{1}{x^3}+\frac{1}{x-1}\right)dx$$
$$=\frac{x^2}{2}+x-\frac{1}{2x^2}+\ln|x-1|+C$$

Case 3: Q(x) contains irreducible factors, none of which is repeated.

If Q(x) has the factor ax^2 + bx + C, where $b^2-4ac<0$, in addition to the partial fraction in equation (2) and (3), the expression for $\frac{R(x)}{Q(x)}$ will have a term of the form

$\frac{Ax+B}{ax^2+bx+c}$, (4) where A and B are constants to be determined.

The integral in (4) can be integrated by completing the square and using the formula:

$$\boxed{\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C}$$

Example 3: Evaluate $\int\frac{2x^2-x+4}{x^3+4x}dx$

Solution: Since $x^3 + 4x = x(x^2 + 4)$ can not be factored further, we write

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Multiplying by $x(x^2 + 4)$, we have:

$$2x^2 - x + 4 = A(x^2 + 4) + Cx + 4A$$

Equating coefficients, we get:

$$\begin{aligned} &A + B = 2 \\ &C = -1 \qquad\qquad\qquad \Rightarrow A = 1, B = 1, C = -1 \\ &4A = 4 \end{aligned}$$

Then,

$$\int \frac{2x^2 - x + 4}{x^2 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x - 1}{x^2 + 4} \right) dx$$

$$= \int \frac{1}{x} dx + \int \frac{x}{x^2 + 4}\,dx - \int \frac{1}{x^2 + 4}\,dx$$

$$= \int \ln|x| + \frac{1}{2}\ln(x^2 + 4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + D$$

Case 4: Q(x) contains repeated irreducible quadratic factors:

This is left as an exercise for the students.

Key Concepts

Partial Fraction Decomposition of a Rational function

- If the rational function is improper, use “long division” of polynomials to write it as the sum of a polynomials and a proper rational function “remainder”
- Decompose the proper rational function as a sum of rational functions of the form

$$\frac{A}{(x - \alpha)^m} \qquad \textit{and}$$

$$\frac{Bx + C}{(x^2 + \beta x + \gamma)^k} \quad (x^2 + \beta x + \gamma \ \ \textit{irreducible})$$

- Where:

  Each factor $(x-\alpha)^m$ in the denominator of the proper rational function suggests terms

  $$\frac{A_1}{A\alpha}+\frac{A_2}{(x-\alpha)^2}+\ldots+\frac{A_m}{(x-\alpha)^m}$$

  Each factor $(x^2+\beta x+\gamma)^n$ Suggests terms

  $$\frac{B_1x+C_1}{(x^2+\beta x+\gamma)}+\frac{B_2x+C_2}{(x^2+\beta x+\gamma)^2}+\ldots+\frac{B_nx+C_n}{(x^2+\beta x+\gamma)}$$

Determine the (unique) values of all the constants involved. Use either Method 1 or Method 2, or a combination of both.

The partial fraction decomposition is often used to rewrite a complicated rational function integrand as a sum of terms, each of which is straightforward to integrate an integral after being rewritten in this form.

3.5 Definite Integral

Definition: A partition of [a,b] is a finite set ρ of points $x_0, x_{1,} x_2, x_3, ...x_n$ such that $a = x_0 < x_1 < x_2 < ... < x_n = b$

We denote it by writing $\rho = \{x_0, x_1, x_2, x_3, ..., x_n\}$.

By definition any partition of [a, b] must contain both a and b.

Except a and b, the number of point and their placement in [a, b] is arbitrary.

The n subintervals in to which partition $\rho = \{x_0, x_1, x_2, x_3, ..., x_n\}$ divide [a, b] are: $[x_0, x_1]$, $[x_{1,} x_2]$, $[x_{2,} x_3]$, ... , $[x_{n-1}, x_n]$.

The length $x_k - x_{k-1}$ of the k^{th} subinterval $[x_{k-1}, x_k]$ is denoted by Δx_k

$$\boxed{\Delta x_k = x_k - x_{k-1}}$$

Example: For the partition $\left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$, we have

$x_0 = 0,\quad x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2$.

$\Delta x_0 = \Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x_4 = \frac{1}{2}$

Now, let f(x) is continuous and none negative on $[a,b]$

Let $\rho = \{x_0, x_1, x_2, x_3, ..., x_n\}$ be a given partition of $[a,b]$.

Then the Maximum- Minimum Theorem implies for each k between 1 and n there exist smallest value m_k and largest value M_k of f on the k^{th} subinterval.

Based on this we have the following definition.

> The **lower sum** of f associated with ρ is given by:
>
> $$L_f(\rho) = m_1\Delta x_1 + m_2\Delta x_2 + ... + m_n\Delta x_n$$
>
> The **upper sum** of f associated with ρ is given by:
>
> $$U_f(\rho) = M_1\Delta x_1 + M_2\Delta x_2 + ... + M_n\Delta x_n$$

Example 1: Let $f(x) = x^2$ for $0 \le x \le 2$.

Then find $L_f(\rho)$ and $U_f(\rho)$ for the partition $\rho = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$.

Solution: The subintervals associated with ρ are:

$\left[0, \frac{1}{2}\right]$, $\left[\frac{1}{2}, 1\right]$, $\left[1, \frac{3}{2}\right]$, $\left[\frac{3}{2}, 2\right]$. Computing the maximum and the minimum values of f on each of this subinterval, we obtain:

$m_1 = f(0) = 0, m_2 = f\left(\frac{1}{2}\right) = \frac{1}{4}, m_3 = f(1) = 1, m_4 = f\left(\frac{3}{2}\right) = \frac{9}{4}$ are the minimum values of f.

$M_1 = f\left(\frac{1}{2}\right) = \frac{1}{4}, M_2 = f(1) = 1, M_3 = f\left(\frac{3}{2}\right) = \frac{9}{4}, M_4 = f(2) = 4$ are the maximum values of f.

$$\Delta x_0 = \Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x_4 = \frac{1}{2}$$

$$L_f(\rho) = m_1\Delta x_1 + m_2\Delta x_2 + m_3\Delta x_3 + m4\Delta x_4$$

$$= \frac{1}{2}(m_1 + m_2 + m_3 + m_4)$$

$$= \frac{1}{2}\left(0 + \frac{1}{4} + 1 + \frac{9}{4}\right) = \frac{7}{4}$$

$$U_f(\rho) = M_1\Delta x_1 + M_2\Delta x_2 + M_3\Delta x_3 + M_4\Delta x_4$$

$$= \frac{1}{2}(M_1 + M_2 + M_3 + M_4)$$

$$= \frac{1}{2}\left(\frac{1}{4} + 1 + \frac{9}{4} + 4\right) = \frac{15}{4}$$

Therefore, the lower and the upper sum of f associated with the partition of ρ are respectively $\frac{7}{4}$ and $\frac{15}{4}$

Example 2: compute the lower sum and the upper sum for $f(x) = \cos x$ on the partition

$$\rho = \left\{-\frac{\pi}{3}, -\frac{\pi}{6,}, 0, \frac{\pi}{6}, \frac{\pi}{3}\right\}$$

Solution: The subintervals associated with ρ are:

$$\left[-\frac{\pi}{3}, -\frac{\pi}{6}\right], \left[-\frac{\pi}{6}, 0\right], \left[0, \frac{\pi}{6}\right], \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$$

Computing the maximum and the minimum values of f on each of these subintervals, we found that:

$$m_1 = f\left(-\frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2},\ m_2 = f\left(-\frac{\pi}{6}\right) = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},$$

$$m_3 = f\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} = 0.86\ ,\ m_4 = f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$M_1 = f\left(-\frac{\pi}{6}\right) = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} = 0.86,\ M_2 = f(0) = \cos(0) = 1,$$

$$M_3 = f(0) = \cos(0) = 1\ ,\ M_4 = f\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = 0.86$$

$$\Delta x_0 = \Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta x_4 = \frac{\pi}{6}$$

Then,

$$L_f(\rho) = m_1\Delta x_1 + m_2\Delta x_2 + m_3\Delta x_3 + m4\Delta x_4$$

$$= \frac{\pi}{6}\left(\frac{1}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{\pi}{6}\left(1 + \sqrt{3}\right)$$

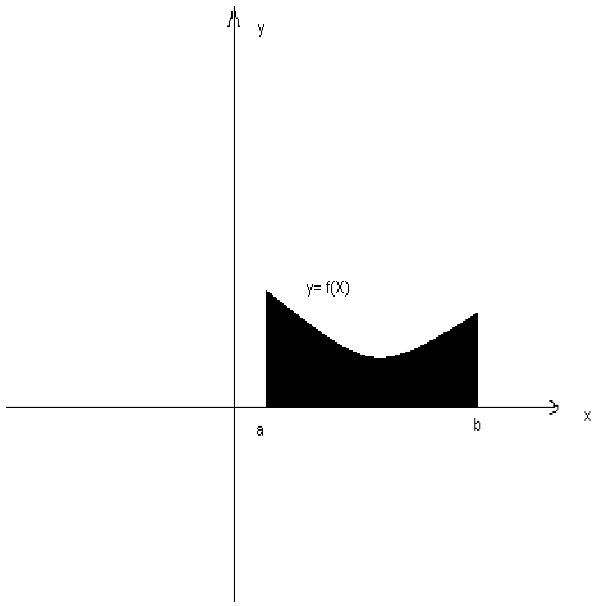
$$U_f(\rho) = M_1\Delta x_1 + M_2\Delta x_2 + M_3\Delta x_3 + M_4\Delta x_4$$

$$= \frac{\pi}{6}\left(\frac{\sqrt{3}}{2} + 1 + 1 + \frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}\left(2 + \sqrt{3}\right)$$

Therefore, the lower sum and the upper sum of f associated the given partition ρ are respectively $\frac{\pi}{6}\left(1+\sqrt{3}\right)$ and $\frac{\pi}{6}\left(2+\sqrt{3}\right)$.

3.5.1 Riemann Sum

Suppose that a function f is continuous and nonnegative on an interval $[a,b]$. Let's us compute the area of the region R bounded above by the curve $y = f(x)$, below by the x-axis on the sides by the lines x = a and x = b.



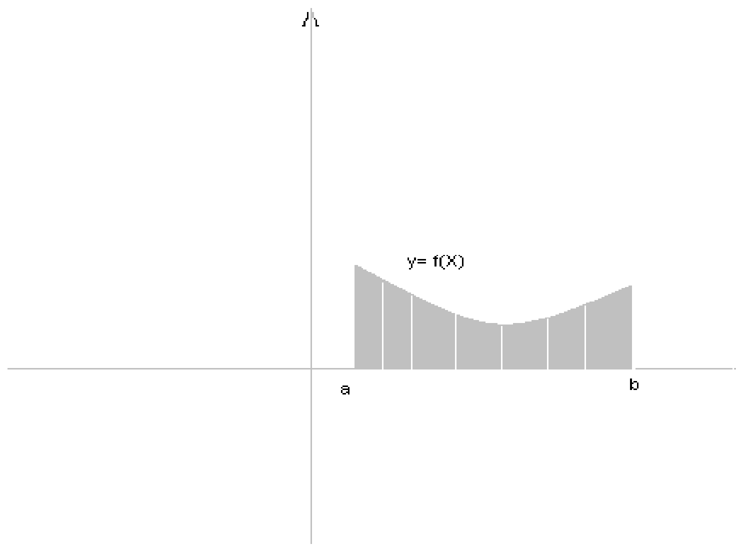
We will obtain this area as the limit of a sum of areas of rectangles as follows.

First divide the interval $[a,b]$ in to n subintervals

$[x_0, x_1]$, $[x_{1,} x_2]$, $[x_{2,} x_3]$, ... , $[x_{n-1}, x_n]$ where, $a = x_0 < x_1 < x_2 < ... < x_n = b$.

The intervals need not all be the same length, so call the lengths of the intervals Δx_1, Δx_2, Δx_3 ..., Δx_n respectively.

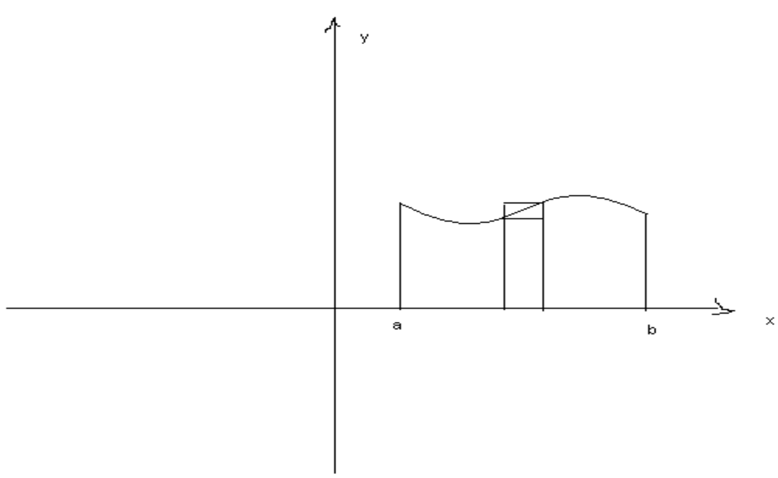
This partition divides the region in strips.



Next let us approximate each strip by a rectangle with height equal to the height of the curve $y = f(x)$ at some arbitrary point in the sub interval. That is for the first interval $[x_0, x_1]$ select some x_1^* contained in the sub interval and use $f(x_1^*)$ as the height of the first rectangle. The area of that rectangle is then becomes $f(x_1^*)\ \Delta x_1$.

Similarly for each subinterval $[x_{i-1}, x_i]$ we choose some x_i^* and calculate the area of the corresponding rectangle which is given by $f(x_i^*)\ \Delta x_i$.

The approximate area of the region R on [a, b] is the sum $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$ of these rectangles.



Depending on what points we select for x_i^*, our estimate may be too large or too small.

For example, if we choose each x_i^* to be the point in its subinterval giving the maximum height, we will overestimate the area of R (This is called an upper sum of f) .

If on the other hand, we choose each x_i^* to be the point in the subinterval giving the minimum height, we will under estimate the area of R. (This sum is call the lower sum of f).

When the point x_i^* is chosen randomly, the sum $\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$ is called Riemann sum and will given an approximation for the area of R that is in between the lower sum and the upper sum> The lower and the upper sums may be considered as a specific Riemann sums.

As we decrease the width of the rectangles we expect to be able to approximate the area of R better. In fact as $\Delta x_i \to 0$, we get the exact area of R which we denote by the definite integral $\int_a^b f(x)dx$.

That is,

$$\boxed{\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \left(\sum_{i=1}^{n} f(x_i)\Delta x_i \right)}$$

Remark:

This definition of definite integral still holds if f(x) assumes both positive and negative values on [a, b]. It even holds if f(x) has finitely many discontinuities but is bounded.

Key Concepts

Let f be continuous on [a, b] and let $x_0, x_1, x_{2,} \ldots, x_n$ be a partition of [a, b].

Foe each $[x_{i-1}, x_i]$, let x_i^* be in $[x_{i-1}, x_i]$, then the definite integral of f over [a, b] is defined by:

$$\boxed{\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \left(\sum_{i=1}^{n} f(x_i)\Delta x_i \right)}$$

If $f(x) \geq 0$ for all x in [a, b] , then the definite integral of f on [a, b] represents the area of the region under the curve $y = f(x)$ on the interval [a,b].

3.5.2 Properties of the definite Integrals

1. Identical limits of integration

$$\int_a^b f(x)dx = 0$$

2. Interchanging the limits of integration

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

1. The integral of a constant function f(x) = c is the constant times the length of

   the interval.

$$\int_a^b cdx = c(b-a)$$

4. The Integral of a sum is the sum of the integrals

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx \qquad \text{(Sum Rule)}$$

5. The integral of a constant time a function is the constant times the integral of a function.

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \qquad \text{(Constant multiple Rule)}$$

6. The integral of a difference if the difference of the integrals.

$$\int_a^b (f(x)-g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx \qquad \text{(Difference Rule)}$$

7. $\int_a^c f(x)dx \int_c^b f(x)dx = \int_a^b f(x)dx$, where c is any number between a nd b.

Proof: Only the proof of property 4 is given.

Following the same procedure try to prove the remaining properties.

By definition,

$$\int_a^b (f(x)+g(x))dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n}(f(x_i)+g(x_i))\Delta x_i\right)$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\Delta x_i + \sum_{i=1}^{n} g(x_i)\Delta x_i\right]$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i + \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\Delta x_i$$

$$= \int_a^b f(x)dx + \int_a^b g(x)dx$$

Comparison Property of the integral

8. If $f(x)\geq 0 \ \ for \ \ a\leq x\leq b$, then $\int_a^b f(x)dx \geq 0$

9. If $f(x)\geq g(x) \ \ for \ \ a\leq x\leq b$, $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

10. If $m\leq f(x)\leq M \ \ \ for \ \ a\leq x\leq b$, then $m\ (b-a)\leq \int_a^b f(x)dx \leq M\ (b-a)$

Proof of property 10:

Since $m\leq f(x)\leq M \ \ \ for \ \ a\leq x\leq b$, property 9 gives us:

$$\int_a^b m\,dx \leq \int_a^b f(x)dx \leq \int_a^b M\ dx$$

$\Rightarrow m(b-a)\leq \int_a^b f(x)dx \leq M\ (b-a)$ by property 3.

Example 1: If $\int_0^{10} f(x)dx = 17$ and $\int_0^8 f(x)dx = 12$, then find $\int_8^{10} f(x)dx$

Solution: By property 7, we have:

$$\int_0^8 f(x)dx + \int_8^{10} f(x)dx = \int_0^{10} f(x)dx$$

$$\Rightarrow \int_8^{10} f(x)dx = \int_0^{10} f(x)dx - \int_0^8 f(x)dx = 17 - 12 = 5$$

Example 2: Estimate the value of $\int_0^1 e^{-x^2}\,dx$.

Solution: Because $f(x) = e^{-x^2}$ is a decreasing function on [0, 1], its absolute maximum value is M = f (0) = 1 and its absolute minimum value is m = f (1) = e^{-1}.

Thus by property 10, we have

$$\frac{1}{e}(1-0) \le \int e^{-x^2}dx \le 1(1-0) = 1, \text{ that is } \frac{1}{e} \le \int e^{-x^2}dx \le 1.$$

3.5.3 Fundamental Theorem of Calculus

Fundamental Theorem of Calculus, Part I (FTC I):

Let f be continuous on [a, b], then the function g defined by $g(x) = \int_a^x f(t)dt$, $a \le x \le b$ is continuous on [a, b] and differentiable on (a, b) and $g'(x) = f(x)$

Proof: If x and x + h are in (a, b), then

$$g(x+h) - g(x) = \int_a^{x+h} f(t)dt - \int_a^x g(t)dt$$

$$= \left(\int_a^x f(t)dx + \int_x^{x+h} f(t)dt\right) - \int_a^x f(t)dt \text{ by property of definite integral}$$

$$g(x+h) - g(x) = \int_x^{x+h} f(t)dt$$

And so, for $h \ne 0$, we have

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt$$

For now let us assume h > 0. Since f is continuous on [x, x+h] by the Extreme Value Theorem, there are two numbers u and v in [x, x+h] such that:

$f(u) = m$ and $f(v) = M$ where m and M are the absolute maximum and the absolute minimum values of f on [x, x+h].

By property integrals, we have:

$$mh \le \int_x^{x+h} f(t)dt \le M\,h$$

$$\Rightarrow f(u)h \le \int_x^{x+h} f(t)dt \le f(v)h$$

Since h > 0, divide this inequality by h.

$$\Rightarrow f(u) \le \frac{1}{h}\int_x^{x+h} f(t)dt \le f(v)$$

$$\Rightarrow f(u) \le \frac{g(x+h)-g(x)}{h} \le f(v) \qquad \left(\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)dt\right)$$

Now as $h \to 0$, then $u \to x$ *and* $v \to x$ since u and v lie between x and x+h.

Therefore,

$\lim\limits_{h\to 0} f(u) = \lim\limits_{u\to x} f(u) = f(x)$ and $\lim\limits_{h\to 0} f(v) = \lim\limits_{v\to x} f(v) = f(x)$.

Then by squeezing Theorem, we have:

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f(x).$$

Example 1: Find $\dfrac{d}{dx}\displaystyle\int_0^x e^{t^2}\,dt$

Solution: Let $g(x) = \int_0^x e^{t^2}\,dt$. Since $f(t) = e^{t^2}$ is continuous on [0, x], then by the Fundamental Theorem of Calculus, we get:

$$g'(x) = \frac{d}{dx}\int_0^x e^{t^2}\,dt = f(x) = e^{x^2}$$

Example 2: Use the first part of the Fundamental Theorem of Calculus to find the derivative of the function:

a) $f(x)=\int_0^x \sin\left(\frac{\pi t^2}{2}\right)dt$ b) $g(y)=\int_2^y t^2 \sin t\, dt$

c) $f(x)=\int_x^2 \cos(t^2)dt$ d) $h(x)=\int_2^{\frac{1}{x}} \tan^{-1} t dt$

Solution: a) since $g(t)=\sin\left(\frac{\pi t^2}{2}\right)$ is continuous on $[0,x]$ for $x \geq 0$, then by the Fundamental Theorem of Calculus:

$$f'(x)=\sin\left(\frac{\pi x^2}{2}\right)$$

b) The function $f(t)=t^2 \sin t$ is continuous on $[2,t]$ for $t \geq 2$ .Then the FTC of part I, we have

$$g'(y)=y^2 \sin y$$

c) $f(x)=\int_x^2 \cos(t^2)dt = -\int_2^x \cos(t^2)dt$ by property of definite Integral.

Then by the FTC of part I, we obtain:

$$f'(x)=-\cos(x^2)$$

d) We use The FTC I together with the Chain rule.

Let $f(x)=\frac{1}{x}$ and $g(x)=\int_2^x \tan^{-1} t dt$

$f'(x)=\frac{-1}{x^2}$ and $gf(x)=\tan^{-1} x$. Then we can express h(x) as a composition of f and g, i.e.

$h(x)=g(f(x))=\int_2^{\frac{1}{x}} \tan^{-1} t dt$. Then by Chain rule

$$h'(x)=g'(f(x))f'(x)=g'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$= -\frac{1}{x^2}\tan^{-1}\left(\frac{1}{x}\right)$$

The Fundamental Theorem of Calculus, Part II (FTC II): If F is continuous on [a, b], then $\int_a^b f(x)dx = F(b) - F(a)$, where F is any anti derivative of f, that is a function such that $F' = f$

Proof: Let $g(x) = \int_a^x f(t)dt$. We know from part I that $g'(x) = f(x)$: that is g is an antiderivative of f. If F is any anti derivative of f on [a, b] then F and g differ by a constant. That is:

$F(x) = g(x) + c$, for a < x < b (1).

But both F and g are continuous on [a, b]. So by taking limits on both sides of (1), we have:

$$\lim_{x\to a^+} F(x) = \lim_{x\to a^+} (g(x)+c) \text{ and } \lim_{x\to a^-} F(x) = \lim_{x\to a^-} (g(x)+c)$$

$$\Rightarrow F(a) = g(a) + c \quad \text{and} \quad F(b) = g(b) + c \qquad (2)$$

If we put x = a in the formula for g(x), we get:

$$g(a) = \int_a^a f(t)dt = 0 \qquad (3)$$

Now using equation (2) and (3) we have:

$$F(b) - F(a) = g(b) + c - (g(a) + c) = g(b) - g(a) = g(b) = \int_a^b f(t)dt$$

Therefore,

$$F(b) - F(a) = \int_a^b f(t)dt$$

Example 1: Evaluate $\int_1^3 x^2 dx$

Solution: The function $f(x)=x^2$ is continuous on [1, 3] and we know that an antiderivative of x^2 is $F(x)=\frac{x^3}{3}$.

So, part II of FTC gives:

$$\int_1^3 x^2 dx = F(3)-F(1)=\frac{27}{3}-\frac{1}{3}=\frac{26}{3}$$

Notice that the FTC II says that we can use any anti derivative F of f. So we use the simplest one namely, $F(x)=\frac{x^3}{3}$ instead of $F(x)=\frac{x^3}{3}+4$ or $F(x)=\frac{x^3}{3}+c$

Notation: We often use the notation;

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a)$$

Example 2: Evaluate $\int_1^2 \frac{1}{x^2}dx$

Solution: $\int_1^2 \frac{1}{x^2}dx = \frac{x^{-1}}{-1}\Big|_1^2 = -\frac{1}{2}-(-1)=\frac{1}{2}$

Example 3: What is wrong with the following calculation?

$$\int_{-1}^2 \frac{1}{x^2}dx = \left(-\frac{1}{x}\right)\Big|_1^2 = -\frac{1}{2}-1=-\frac{3}{2}$$

Solution: This calculation must be wrong because the answer is negative but

$f(x)=\frac{1}{x^2}\geq 0$, which contradicts one of the properties of definite integral.

The FTC of Part II cannot be applied here since f is discontinuous at x= 0 in

[-1, 2]. Therefore $\int_{-1}^2 \frac{1}{x^2}dx$ does not exist

Example 4: Evaluate $\int_0^2 xe^{x^2}dx$

Solution:

Let $u = x^2$

$$du = 2xdx \Rightarrow xdx = \frac{1}{2}du$$

First we will compute the indefinite integral:

$$\int xe^{x^2}dx = \int e^{x^2}xdx = \int \frac{1}{2}e^u du = \frac{1}{2}e^u + c = \frac{1}{2}e^{x^2} + c$$

Now we have two approaches for the definite integral:

Approach 1

Substitution back to the original variable

$$\int xe^{x^2}dx = \frac{1}{2}e^{x^2}$$

So,

$$\int_0^2 xe^{x^2}dx = \left[\frac{1}{2}e^{x^2}\right]_0^2 = \frac{1}{2}\left(e^4 - 1\right)$$

Approach 2

Change the limit of Integration

Since $u = x^2$

u = 0 when x = 0

and u = 4 when x = 2

So, $\int_0^2 xe^{x^2}dx = \int_0^4 \frac{1}{2}e^u du = \left[\frac{1}{2}e^u\right]_0^4 = \frac{1}{2}\left(e^4 - 1\right)$

Thus we find that:

$$\int_0^2 xe^{x^2}dx = \frac{1}{2}\left(e^4 - 1\right)$$

Approach 2 works provided certain conditions on f and g meet:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

If

1. g' is continuous on [a, b]

2. f is continuous on the set of values taken by g on [a, b].

$Example\ 5: Evaluate \int_1^e \frac{\ln x}{x}\,dx$

$solution: Let\ u = \ln x$

$$du = \frac{1}{x}dx$$

$When\ x = 1, u = \ln 1 = 0; when\ x = e, u = \ln e = 1. Thus$

$$\int_1^e \frac{\ln x}{x}dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}$$

The next Theorems use the substitution rule for definite integrals to simplify the calculation of functions that possess symmetry properties.

Integrals of symmetric functions: suppose f is continuous on [a, b].

a. If f is even, then $f(-x) = f(x), and\ \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

b. If f is odd, then $f(-x) = -f(x), and\ \int_{-a}^{a} f(x)dx = 0$

Proof: We split the integral in to two:

$$\int_{-a}^{a} f(x)dxdx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$$
$$= -\int_0^a f(x)dx + \int_0^a f(x)dx \qquad (1)$$

In the first integral on the right side of (1), substitute u = -x (x = -u)

Then du = -dx and when x = 0,u = 0; when x = -a, u = a

Therefore,

$$-\int_0^{-a} f(x)dx = -\int_0^a f(-u)(-du) = \int_0^a f(-u)du$$

So equation 1 becomes:

$$\int_{-a}^{a} f(x)dx = \int_0^a f(-u)du + \int_0^a f(x)dx \qquad (2)$$

a. If f is even, then $f(-u) = f(u), so\ equation\ (2) gives:$

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

b. If f is odd, then $f(-u) = -f(u)$ *and so equation* (2) *gives*:

$$\int_{-a}^{a} f(x)dx = 0$$

Example 5: Evaluate

a. $\int_{-2}^{2}(x^6+1)dx$ b) $\int_{-1}^{1}\frac{\tan x}{1+x^2+x^4}dx$

Solution:

a. Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so,

$$\int_{-2}^{2}(x^6+1)dx = 2\int_{0}^{2}(x^6+1)dx$$

$$= 2\left[\frac{1}{7}x^7 + x\right]_0^2 = 2\left(\frac{128}{7}+2\right) = \frac{284}{7}$$

b. Since $f(x) = \dfrac{\tan x}{1+x^2+x^4}$ satisfies $f(-x) = -f(x)$, it is odd and so,

$$\int_{-1}^{1}\frac{\tan x}{1+x^2+x^4}dx = 0$$

☺ . **The substitution Rule for definite integrals:**

If g' is continuous on [a, b] and f is continuous on the set of values taken by g on [a, b], then

$$\int_{a}^{b} f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Integration by parts "works" on definite integrals as well:

$$\boxed{\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du}$$

Example 6: Evaluate $\int_0^1 \tan^{-1}x\,dx$

Let

$$u = \tan^{-1}(x) \qquad\qquad du = dx$$

$$du = \frac{1}{1+x^2}\,dx \qquad\qquad u = x$$

Then by integration by parts,

$$\int_0^1 \tan^{-1}(x) = x\tan^{-1}(x)\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}dx$$

$$= x\tan^{-1}(x)\Big|_0^1 - \frac{1}{2}\ln\left(1+x^2\right)\Big|_0^1$$

$$= \left(\frac{\pi}{4} - 0\right) - \left(\frac{1}{2}\ln(2) - 0\right)$$

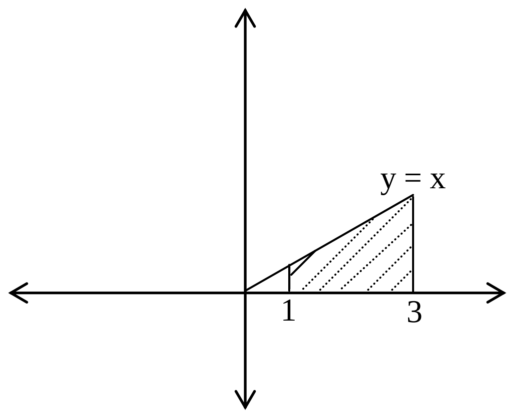
$$= \frac{\pi}{4} - \ln\left(\sqrt{2}\right)$$

Key Concepts

The Fundamental Theorem of Calculus: Suppose f is continuous on [a, b]

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$
2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any anti derivative of f , i.e. $F' = f$

Example 3: $\int_1^3 xdx = \left.\frac{x^2}{2}\right|_1^3 = \frac{1}{2}$



Area = Area (of larger Δ) – area (smaller triangle)
$= \frac{1}{2}(3^2) - \frac{1}{2}[1^2] = 4$

If we have chosen a different anti derivative $\frac{x^2}{2} + c$, the outcome would have been identical.

$$\left.\frac{x^2}{2} + c\right|_1^3 = \left(\frac{9}{2} + c\right) - \left(\frac{1}{2} + c\right) = 4$$

Chapter 4: Application of Integration

4.1: Computing Area between two curves

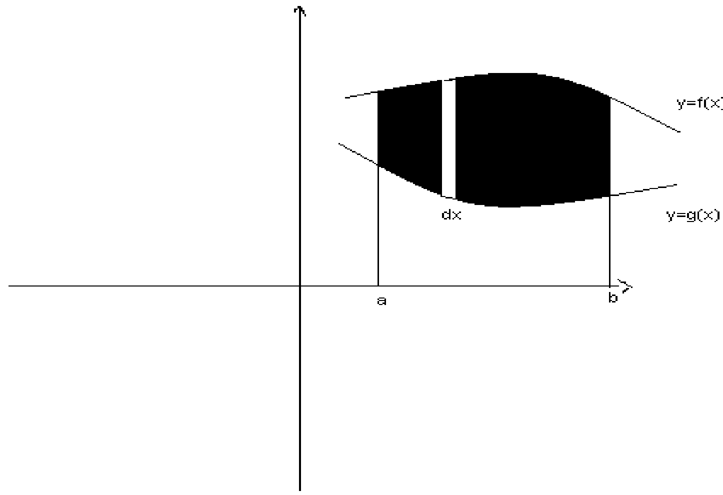
We have seen if y = f(x) is continuous and nonnegative on [a, b], then the area of the region under the curve y = f(x) on [a, b] is given by:

$$Area = \int_a^b f(x)dx\,.$$

Now we will discuss the area of the region bounded between two curves.

If $f(x) \geq g(x)$ for all x in [a, b], then the area A of the region between the graphs of f and g and x = a, x = b is given by:

$$A = \int_a^b (f(x) - g(x))dx$$



Area = (area under the curve f(x)) – (Area under the curve g(x))

We are trying to find the area between two curves y = f(x) and y = g(x) and the lines x = a and x = b.

We can see that if we subtract the area under the lower curve y = g(x) from the area under the upper curve y = f(x), then we will find the required area. This can be achieved in one step;

$$A = \int_a^b \left(f(x) - g(x) \right) dx$$

Alternative way to find the formula (First principles)

Another way of deriving this formula is as follows (the thinking here is important for understanding how we develop the later formulas in this section)

Each "typical" rectangle indicated as width dx and height f(x) – g(x), so its area is $\left(f(x) - g(x) \right) dx$.

If we add all these typical rectangles, starting from and finishing at b, the area is approximately:

$\sum_{x=a}^{b} \left(f(x) - g(x) \right) dx$. Now if we let $dx \to 0$, we can find the exact area by integration.

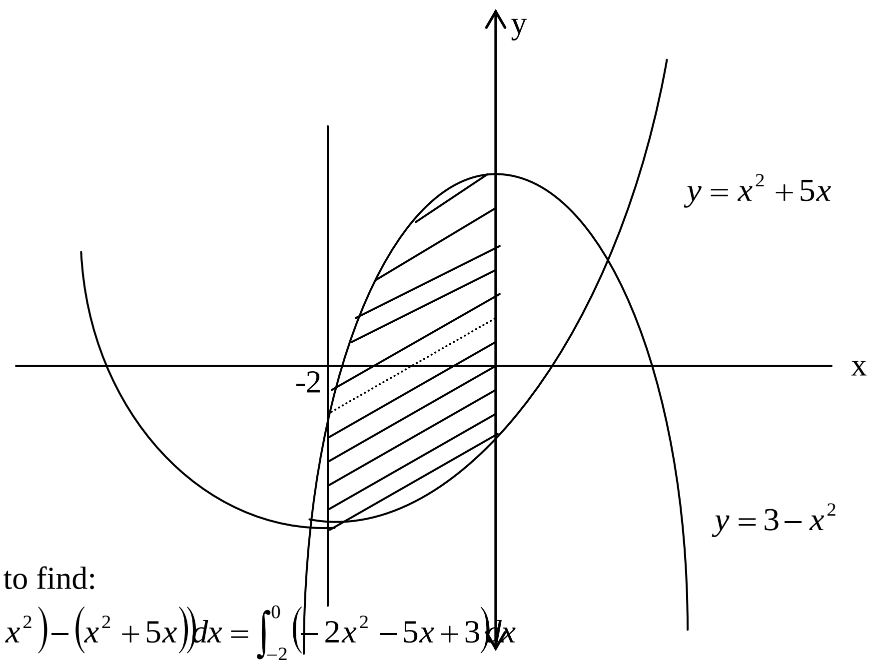
$$A = \int_a^b \left(f(x) - g(x) \right) dx$$

Likewise, we can sum vertically by re expressing both functions so that they are functions of y and we find:

$$A=\int_c^d \left(f(y)-g(y)\right)dy$$

Example1: Find the area between the curves $y=x^2+5x$ and $y=3-x^2$

between x = -2 and x = 0



So we need to find:

$$A=\int_{-2}^{0}\left(\left(3-x^2\right)-\left(x^2+5x\right)\right)dx=\int_{-2}^{0}\left(-2x^2-5x+3\right)dx$$

$$=\left[-\frac{2}{3}x^3-\frac{5}{2}x^2+3x\right]_{-2}^{0}=\frac{32}{3} \text{ square unit.}$$

Example 2: Find the area of the region bounded by the curves

$y=x^2$, $y=2-x$ and $y=1$

Solution: First sketch the region.

We need to take horizontal elements in this case.

So we need to solve $y=x^2$ for x

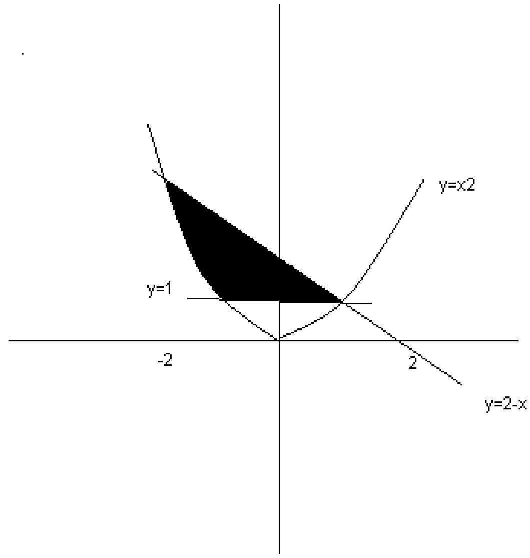
$x=\pm\sqrt{y}$

We need the left hand portion, so $x=-\sqrt{y}$

Notice that x = 2 – y to the right of $x=-\sqrt{y}$, so we choose:

$x_2=2-y$ and $x_1=-\sqrt{y}$.

The intersection of the graphs occurs at $(-2,4)$ and $(1,1)$



So we have c = 1 and d = 4

$$A = \int_1^4 \left((2-y) - \left(-\sqrt{y}\right)\right) dy$$

$$= \int_1^4 \left(2 - y + y^{\frac{1}{2}}\right) dy$$

$$= \left[2y - \frac{1}{3}y^2 + \frac{2}{3}y^{\frac{3}{2}}\right]_1^4 = 12 - \frac{7}{3} = \frac{29}{3}$$

A $= \dfrac{29}{3}$ sq. units

4.2 Computing Displacement

We have seen that given a position function $s(t)$ at any time t we can determine the velocity function and the acceleration function at any time t by finding the first and second derivative of the position function. Now we reverse the process, that is, knowing the velocity and the acceleration of a moving object any time t, we can recover the position function from the velocity as well as from the acceleration function.
That is

$v(t) = \dfrac{d\,s(t)}{dt}$ and $a(t) = \dfrac{dv(t)}{dt} = \dfrac{d\ \ s(t)}{dt^2}$

It follows (since integration is the opposite process to differentiation) that to obtain the displacement, s of an object at time t (given the expression for velocity, v) we would use:

$$\boxed{s(t) = \int v(t)dt}$$

Similarly the velocity of an object at time t , given the acceleration a(t) is given by:

$$\boxed{v(t) = \int a(t)dt}$$

Example 1: A proton moves in an electric field such that its acceleration (in cm/s^2) is:

$a(t) = -20(1+2t)^{-2}$, where t is in seconds.

Find the velocity as a function of time if v = 30cm s^{-1} when t = 0.

Solution:

$v = \int a(t)dt$

So, $v = \int \frac{-20}{(1+2t)^2} dt$

Put u = 1 + 2t, then $du = 2dt$

$v = \int \frac{-10}{u^2} du = \frac{10}{u} + c = \frac{10}{1+2t} + c$

When t = 0, v = 30, so, c = 20.

So the expression for velocity as a function of time is $v(t) = \frac{10}{1+2t} + 20$ cm/s.

Example 2: A flare is ejected vertically upwards from the ground at 15m/s.Find the height of the flare after 2.5 s.

Solution:

The object has acting on it the force due to gravity, so its acceleration is 9.8m s^{-2}.

$v(t) = \int a(t)dt = \int -9.8dt = -9.8t + c$

Now at t = 0, the velocity v = 15cm/s, so c = 15

So the expression for velocity becomes:

$v(t) = -9.8t + 15$

Now we need to find the displacement s.

$s(t) = \int v(t)dt = \int (-9.8t + 15)dt$

$= -4.9t^2 + 15t + k$.

But we have given that when t = 0 s = 0, this gives us k = 0

Therefore,

$s(t) = -4.9t^2 + 15t$.

Hence at time t = 2.5 s = 6.875 m.

Using Integration, we can obtain the well-known expression for displacement and velocity, given a constant acceleration a, initial displacement zero, and an initial velocity v_0:

$$v = \int a(t)dt$$

$$v = at + c$$

Since the velocity at t = 0 is v_0, we get c = v_0, so that

$$\boxed{v = v_0 + at}$$

Similarly,

$$s = \int v(t)dt = \int (v_0 + at)dt$$

$s = v_0 t + \frac{1}{2}at^2 + k$. Since the displacement at t = 0 is s = 0 we have k = 0 and so

$$\boxed{s = v_0 t + \frac{1}{2}at^2}$$

Key Concepts

If the object has position function $s = f(t)$, then

1. The velocity function is $v(t) = s'(t)$
2. The acceleration function $v(t) = s'(t)$

This means that, the position function is an antiderivative of the velocity function and the velocity function is an antiderivative of the acceleration function.

4.3 Computing work done by force

In this section we will be looking at the amount of work that is done by a force in moving an object. When a constant force, F, moving an object over a distance of d the work is,

$$W = Fd$$

However, most forces are not constant and will depend upon where exactly the force is acting. So, let's suppose that the force at any x is given by $F(x)$. Then the work done by the force in moving an object from $x = a$ to $x = b$ is given by

$$W = \int_a^b F(x)dx$$

Notice that if the force is constant we get the correct formula for a constant force.

$$W = \int_a^b F dx$$

$$= Fx|_a^b$$

$$= F(b - a)$$

where $b - a$ is simply the distance moved, or d.

Example 1: A spring has a natural length of 20 cm. A 40 N force is required to stretch (and hold the spring) to a length of 30 cm. How much work is done in stretching the spring from 35 cm to 38 cm?

Solution: This example will require Hooke's Law to determine the force. Hooke's Law tells us that the force required to stretch a spring a distance of x meters from its natural length is, $\mathrm{F(x) = kx}$ where $\mathrm{k > 0}$ is called the spring constant. It is important to remember that the x in this formula is the distance the spring is stretched from its natural length and not the actual length of the spring.

So, the first thing that we need to do is determine the spring constant for this spring. We can do that using the initial information. A force of 40 N is required to stretch the spring

$$30cm - 20cm = 10cm = 0.1cm$$

from its natural length. Using Hooke's Law we have,

$$40 = 0.10k \ \ => k = 400$$

So, according to Hooke's Law the force required to hold this spring x meters from its natural length is,

$$F(x) = 400x$$

We want to know the work required to stretch the spring from 35cm to 38cm. First, we need to convert these into distances from the natural length in meters. Doing that gives us $x's$ of 0.15m and 0.18m.

The work is then,

$$W = \int_{0.15}^{0.18} 400x dx$$

$$= 200x^2 |_{0.15}^{0.18}$$

$$= 1.98J$$

Example 2: A 20 ft cable weighs 80 lbs and hangs from the ceiling of a building without touching the floor. Determine the work that must be done to lift the bottom end of the chain all the way up until it touches the ceiling.
Answer: 400 ft/lb

4.4 Computing Volume of solids of revolution

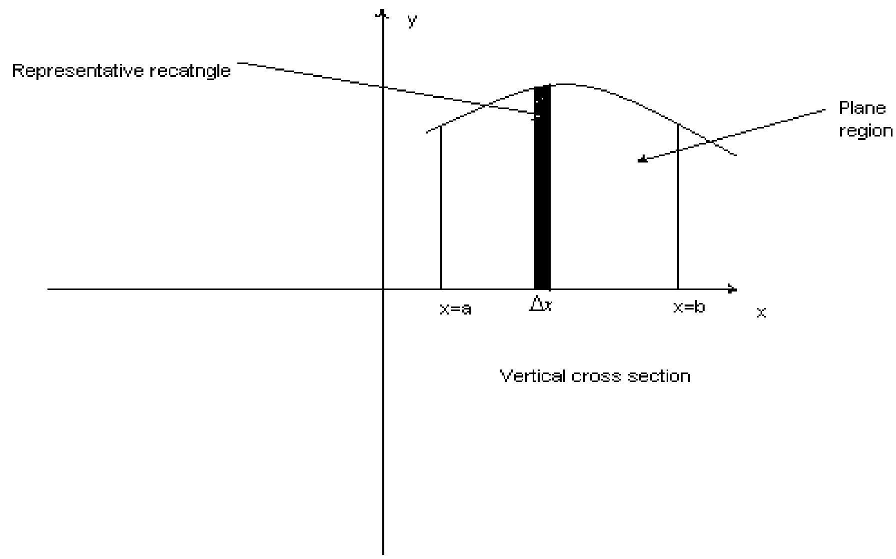
Another important application of the definite integral is its use in finding the volume of a three dimensional solid. In this section you will study a particular type of three dimensional solid one whose cross sections are similar. You will begin with solid of revolution.
A solid of revolution is formed by revolving a plane region about a line. The line is called **the axis of revolution.**

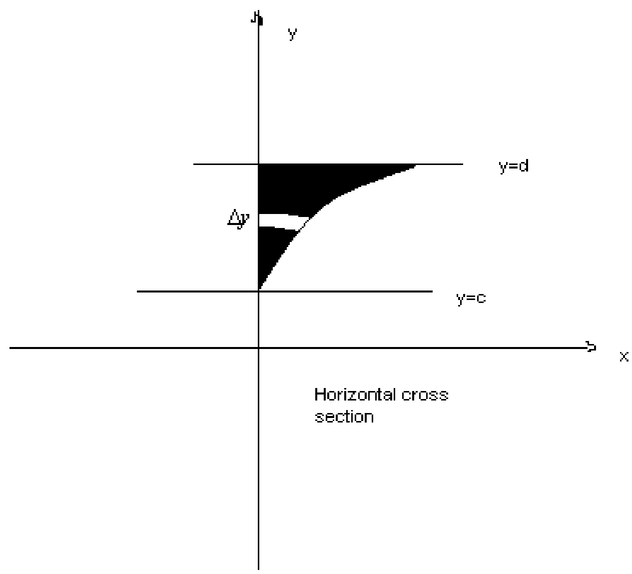
Volume by the disk method:

The disk method:

The volume of the solid formed by revolving the region bounded by the graph of f and the x- axis between x = a and x = b is given by:

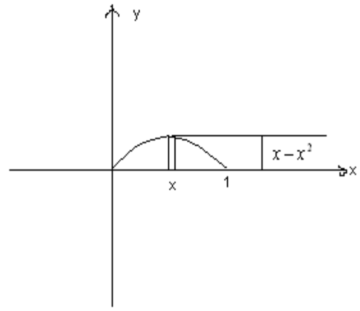
$$volum = \int_a^b \pi\left(f(x)\right)^2 dx$$





Example 1: Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = x - x^2$ and the x- axis about the x – axis .

Solution: First Sketch the graph.



Sketch a representative rectangle whose height is f(x) and whose width is Δx . From this rectangle the radius of the solid is $r = f(x) = x - x^2$.
Using the disk method, we have:

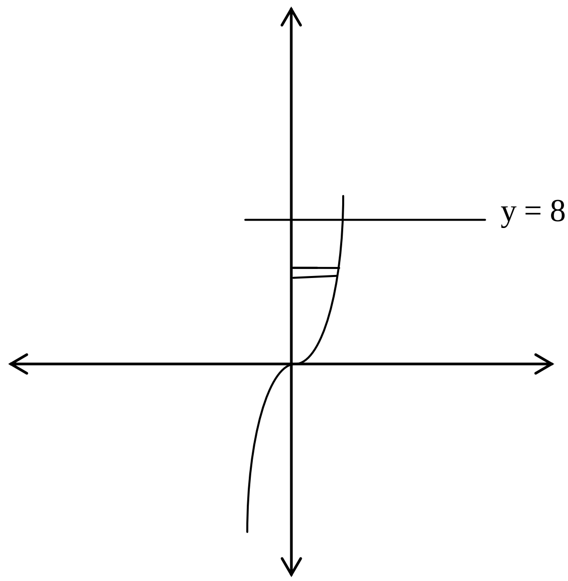
$$V = \pi\int_0^1 f(x)^2\,dx = \pi\int_0^1 (x - x^2)^2\,dx$$

$$= \pi\int_0^1 (x^4 - 2x^3 + x^2)dx$$

$$= \pi\left[\frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 = \frac{\pi}{30} \approx 0.105$$

Example 3: Find the volume of the solid formed by revolving the region bounded by $x^4 - 2x^3 + x^2$ $y = x^3$, y = 8, y = 0 and x = 0 about the y- axis.

Solution: First sketch the region.
The region has horizontal cross section.
The radius of the disk at any point between y = 0 and y = 8 is
$x = \sqrt[3]{y} = f(y)$



Then using the disk method, we have:

$$V = \pi\int_0^8 \left(f(y)\right)^2 dy = \int_0^8 \left(\sqrt[3]{y}\right) dy$$

$$= \pi\int_0^8 y^{\frac{2}{3}} dy$$

$$= \pi\left[\frac{3}{5} y^{\frac{3}{5}}\right]_0^8 = \frac{96}{5}\pi$$

Therefore, V $= \dfrac{96}{5}\pi$

The Washer Method

We can extend the disk method to find the volume of a solid of revolution with a hole. Consider a region that is bounded by the graphs of f and g as shown in the figure below. If the region is revolved about the x – axis, then the volume of the resulting solid can be found by applying the disk method to f and g and subtracting the results.
That is,

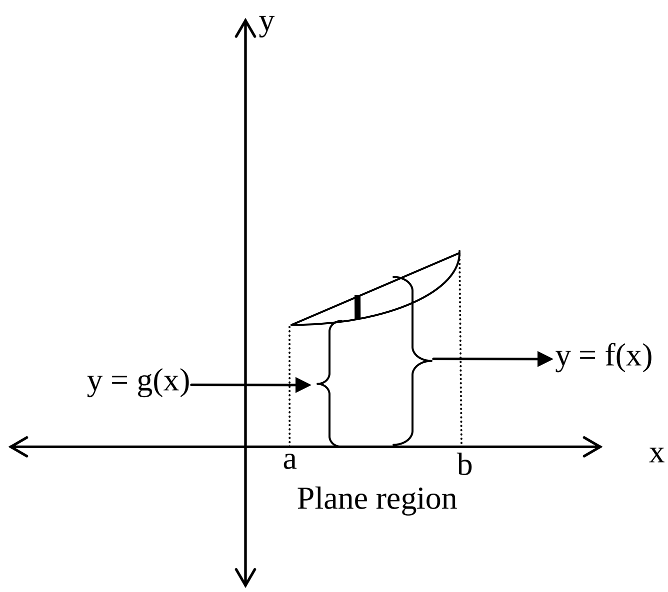
Volume $= \pi\int_a^b f(x)^2 dx - \pi\int_a^b g(x)^2 dx = \pi\int_a^b \left[f(x)^2 - g(x)^2\right] dx$

The Washer Method:
Let f and g be continuous and non-negative on the closed interval, then the volume of the solid formed by revolving the region bounded by the graphs of f and g between x = a and x = b is:

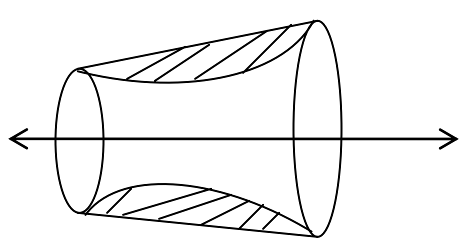
$$V = \pi\int_a^b \left[f(x)^2 - g(x)^2\right] dx.$$

f(x) is the outer radius and g(x) is the inner radius

Geometrical description

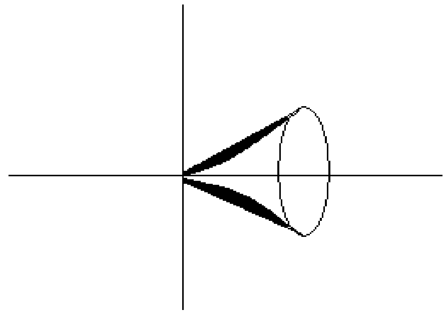


Plane region



Solid of revolution
with hole

Example 1: Find the volume of the solid formed by revolving the region bounded by the curves $y = x$ *and* $y = x^2$ about the x – axis.



Solution: The two curves intersect at x = 0 and x = 1.
The outer radius is x and the inner radius is x^2.
Then by the washer method, the volume of the solid is:

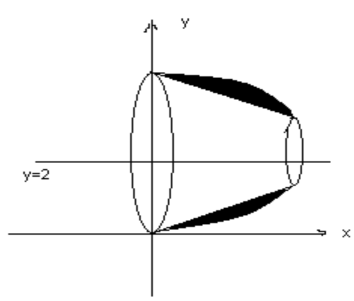
$$V = \pi \int_0^1 \left(x^2 - \left(x^2 \right)^2 \right) dx = \pi \int_0^1 \left(x^2 - x^4 \right) dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{2}{3}\pi$$

Therefore,

$$V = \frac{2}{3}\pi .$$

Example 2: Find the volume of the solid formed by revolving the region enclosed by the curves $y = x$ *and* $y = x^2$ about the line y = 2.



Solution: The outer radius of the disk is $2-x^2$ and the inner radius of the disk is 2 – x. Then by the washer method, the volume of the resulting solid is:

$$V=\pi\int_0^1\left[\left(2-x^2\right)^2-(2-x)^2\right]dx$$
$$=\pi\int_0^1\left(x^4-5x^2+4x\right)dx$$
$$=\pi\left[\frac{x^5}{5}-\frac{5}{3}x^3+2x^2\right]_0^1=\frac{8}{15}\pi$$

Therefore, $V=\dfrac{8}{15}\pi$

4.5. Computing Length of plane curves

Definition: Let a function : $[a,b]\to\mathbb{R}$, and a smooth curve C be given by $y=f(x)$, $x\in[a,b]$. Then the arc length of C is equal to

$$\ell(C)=\int_a^b\sqrt{1+(f'(x))^2}dx$$

Definition: Let a function : $[c,d]\to\mathbb{R}$, and a smooth curve C be given by $x=g(y)$, $y\in[c,d]$. Then the arc length of C is equal to

$$\ell(C)=\int_c^d\sqrt{1+(g'(y))^2}dy$$

Examples: (1) Let $y=x$, $x\in[2,4]$. Find its arc length.
Solution:

$$\ell(C)=\int_2^4\sqrt{1+(x')^2}dx=\int_2^4\sqrt{1+(1)^2}dx=\int_2^4\sqrt{2}dx=2\sqrt{2}\ \text{units}$$

(2) Let $y=\frac{2x^6+1}{8x^2}$, $x\in[1,2]$. Find its arc length.
Solution:

$$\ell(C)=\int_1^2\sqrt{1+\left(x^3-\frac{1}{4x^3}\right)^2}\,dx=\frac{123}{32}\text{ units}$$

EXRECISE ON INTEGRATION AND ITS APPLICATION

1. Find the following indefinite Integrals.

a) $\int\left(2x+3x^{1.7}\right)dx$ b) $\int\frac{1}{x\sqrt{x}}dx$ c) $\int\left(\sqrt[4]{x^3}+\sqrt[3]{x^4}\right)dx$

d) $\int\frac{x^2-1}{\sqrt{x^3}}dx$ e) $\int\left(2t^2-1\right)^2dt$

2. Find a function f that satisfies the given conditions.

a) $f'(x) = 3\sqrt{x} + 3$, $f(1) = 4$ b) $f'(x) = \sqrt{x}(6+5x)$, $f(1) = 10$

c) $f''(t) = \dfrac{3}{\sqrt{t}}$, $f(4) = 20$, $f'(0) = 4$

3. Evaluate the following integrals.

a) $\displaystyle\int \frac{x}{(x^2+1)^2}\,dx$ b) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ c) $\displaystyle\int \frac{1}{x\ln x}\,dx$

d) $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$ e) $\displaystyle\int \sqrt{x}\sin\left(1+x^{\frac{3}{2}}\right)dx$ f) $\displaystyle\int_0^1 x^2(1+2x^3)^5\,dx$

4. Find the general solution of $F'(x) = 2x - 2$, and find the particular solution that satisfies the initial condition F (1) = 2.

5. A particle moves in straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity $v(0) = 9\,cm/s$ and its initial displacement $s(0) = 9\,cm$. Find its position function.

6. A ball is thrown up ward with a speed of $48\,ft/s$ from the edge of a cliff 432 ft above the ground. Then

a) Find its height above the ground t seconds later.
b) When does it reach its maximum height?
c) When does it hit the ground?

7. Evaluate the integral

a) $\displaystyle\int x^2(\sin\pi x)\,dx$ b) $\displaystyle\int_1^2 \frac{\ln x}{x^2}\,dx$ c) $\displaystyle\int \cos x\ln(\sin x)\,dx$

c) $\displaystyle\int_0^1 \frac{t}{e^{2t}}\,dt$ d) $\displaystyle\int \sqrt{1-4x^2}$ f) $\displaystyle\int_{\sqrt{2}}^2 \frac{1}{x^3\sqrt{x^2-1}}\,dx$

g) $\displaystyle\int \frac{\sqrt{t^2-1}}{t^3}\,dt$ h) $\displaystyle\int \frac{dx}{(x^2+2x+2)^2}\,dx$ i) $\displaystyle\int \sqrt{x}e^{x\sqrt{x}}\,dx$

j) $\displaystyle\int \frac{dt}{\sqrt{t(\sqrt{t}+1)}}$ k) $\displaystyle\int_{-1}^0 \frac{dx}{\sqrt[3]{1-2x}}$

8) Show that the improper integral $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converges for all real number p > 1 and diverges for all real number $p \geq 1$.

9. Find the area of the region bounded by the curves $y = e^x$, y = x and by the lines x = 0 and x = 1

10. Find the area of the region bounded by the following curves.

a) $x = 4y - y^2$ and x = 2y – 3 b) $y = e^x - 1$, $y = x^2 - x$, x = 1

11. Find the area of the region bounded by the curves $y = 1 - x^2$ and $y = 3x^2$ between x = 0 and x = 1

12. Find the volume of the solid formed by revolving the region bounded by the graph of the equations about the x – axis.

   a) $y = \sqrt{\sin x}$, x = 0 and $x = \pi$ b) $y = \sqrt{25 - x^2}$ and $g(x) = \frac{3}{2}x$

13. Find the volume of the solid formed by revolving the region bounded by the graphs of the equations about the y – axis.

   a) $y = \sqrt{4 - x}$, y = 0 , x = 0 b) $x = \frac{1}{2}y$, x = 0 , y = 0

   c) $y = x^2 + 1$, y = 0 , x = 0 and x = 1

Answer Key

1. a) $x^2 + \frac{30}{7}x^{0.7} + c$ b) $\frac{-1}{2\sqrt{x}} + c$ c) $\frac{8}{7}x^{\frac{7}{4}} + c$

   d) $\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{\sqrt{x}} + c$ e) $\frac{4}{3}t^3 - 2t^2 + t + c$

2. a) $f(x) = \frac{1}{2}x^{\frac{3}{2}} + 3x + \frac{1}{2}$ b) $f(x) = 4x^{\frac{3}{2}} + \frac{1}{2}x^{\frac{5}{2}} + \frac{11}{2}$

   d) $f(t) = -4\sqrt{t^3} + 4t + 36$

3. a) $-\frac{1}{2(x^2+1)} + c$ b) $-2\cos\sqrt{x} + c$ c) $\ln|\ln x| + c$

   d) 2 e) $-\frac{2}{3}\sin\left(1 + x^{\frac{3}{2}}\right)$ f) $\frac{182}{9}$

4. $F(x) = x^2 - 2x + 3$

5. $s(t) = t^3 + 2t^2 - 6t + 9$

6. a) $v(t) = -32t + 48$

b) The ball reached its maximum height at t = 1.5 seconds.

c) The ball hits the ground after 6.95 seconds.

7. a) $\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi^2}x\sin\pi x + \frac{2}{\pi^3}\cos\pi x + c$ b) $\frac{1}{2} - \frac{1}{2}\ln 2$

c) $\sin x(\ln(\sin x - 1)) + c$ d) $\frac{1}{4} - \frac{3}{4}e^{-2}$

e) $\frac{1}{4}\sin^{-2}(2x) + \frac{1}{2}x\sqrt{1-4x^2}$ + c f) $\frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}$

g) $\frac{1}{6}\sec^{-1}\left(\frac{t}{3}\right) - \frac{\sqrt{t^2-9}}{2t^2} + c$ h) $\frac{1}{2}\tan^{-1}(x+1) + \frac{x+1}{x^2+2x+2} + c$

i) $\frac{2}{3}e^{x^{\frac{3}{2}}} + c$ j) $2\ln(1+\sqrt{t}) + c$ k) $-\frac{3}{4} - \sqrt[3]{9}$

8. The improper diverges for all real numbers p > 1 and diverges for all other values of p.

9. Area $= e - \frac{3}{2}$ sq. units

10. a) Area $= \frac{125}{6}$ sq. units b) Area $= \frac{32}{3}$ sq. units

11. Area $= \frac{10}{3}$ sq. units.

12.

a) Volume $= \frac{256}{3}\pi$ cubic units

b) Volume $= 2\pi$ cubic units

13.

a) Volume $= \frac{256}{15}\pi$ cubic units

b) Volume $= \frac{2}{3}\pi$ cubic units

c) Volume $= \frac{3}{2}\pi$ cubic units

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