Problem 1

Proof. Prove that $M = sup(E)$ iff M is an upper bound and there exists an element $x \in E$ such that $M - \epsilon < x$. Assume that there exists no $x \in E$ such that $x > M - \epsilon$ which would mean that for all $x \in E$, $x \leq M - \epsilon$. This would then make $M - \epsilon$ and upper bound. This creates a contradiction as if $M - \epsilon$ was an upper bound then $M < M - \epsilon$ would have to be true. However, we know that it is impossible since $\epsilon > 0$

□

Problem 2

Proof. Assume that U is open and that $h = 0$. Then for all $x \in U$ it will be that $(x, x) \subseteq U$. But if the range is (x, x) then $x \notin (x, x)$. In order for $x \in (a, b)$ it has to be so that $|b-a| \geq 2$. Which is why $h > 0$ so that when the intervals are $(x - h, x + h)$, There will at least be a large enough difference between the bounds so that $x \in (a, b)$ is true. □

Problem 3

Proof. Assume that u_1 and u_2 are open subsets. And that $U = u_1 \cap u_2$. Prove that U is also an open subset in $\mathbb{R}$. Saying that $U = u_1 \cap u_2$ means that for any $x \in U$, $x \in u_1$ and $x \in u_2$. Since both u_1, u_2 are open, it means that x is never at the boundaries.
Assume that U is not open. Then that statement $U = u_1 \cap u_2$ implies that x is on the bounds of either u_1 or u_2. But since we discussed earlier that x has to be on the interior of the bounds, then this is a contradiction. Hence proving that the intersection of open subsets is open in $\mathbb{R}$

□

Problem 4

Let us take some arbitrary open set with the intervals $(x - h, x + h)$ for all $h > 0$. To show a limit point, we need to find some point $y \in \mathbb{Q}$ such that $y \neq x$. We know that since $\mathbb{Q}$ is dense in $\mathbb{R}$ which means $(x-h, x+h)$ has an infinite number of rational numbers in between the bounds. Which means that the set of limit points of $\mathbb{Q}$ is $\mathbb{R}$

Problem 5

a) Show that A^0 is an open set.

Proof. We can say that x is an interior point such that $x \in (x - h, x + h) \subseteq A$. Let us take some interior point $p \in (x - h, x + h)$, this implies that p is also an interior point of A. We can then say that $(x - h, x + h) \subseteq A^0$ and that $p \in A^0$. This then implies that A^0 is also an open set.

Using the same argument as above. We can replace the open intervals with U such that it is an open subset of $\mathbb{R}$. That will then show that if $U \subseteq A$ then $U \subseteq A^0$

□