Problem 1

We need to show that $[\{a_n\}] = [\{1\}]$

$$\lim_{n\to\infty}(1-a_n) = 0$$

$$1 - a_n = \frac{1}{10^n} = \left(\frac{1}{10}\right)^n$$

$$\lim_{n\to\infty}\left(\frac{1}{10}\right)^n$$

$$\lim_{n\to\infty}\frac{1}{10^n} = 0$$

This shows that the sequence will converge to (1, 1, 1, 1, ...)

Problem 2

We know that every Cauchy sequence of real numbers is bounded and convergent. We know that a sequence is bounded if there exists a number $M \in \mathbb{Q}$, $M > 0$ such that $a_n \leq M$ for all $n \in \mathbb{N}$. If a sequence of rational numbers does not converge towards another rational numbers. Then it is bound to converge towards a real number. Since $\mathbb{Q} \subset \mathbb{R}$, and any sequence of real numbers is bounded. Then any sequence of rational numbers should also be bounded.

Problem 3

let a_n be a convergent sequence of rational numbers. and let $\lim_{n\to\infty} a_n = a$. Assume that $a < 0$ so $a + \epsilon < 0$. Since we assumed that $\lim_{n\to\infty} a_n = a$, then for some $j \in \mathbb{Q}$ we can then say

$$a - \epsilon < a_n < a + \epsilon \ \forall n \geq j$$

If we let $k = max\{j, m\}$ we can then say that $a_n > 0$. However this is a contradiction since earlier we stated that $a_n < a + \epsilon < 0$. This means that $\lim_{n\to\infty} a_n > 0$ and $a > 0$.

From here let us say that $a_n = x_n - y_n > 0$ and that $\lim_{n\to\infty} a_n = x - y$. So from what we showed earlier we can say that $x - y > 0$ so $x > y$

Problem 4

Assuming that $\mathbb{Q}$ is dense in $\mathbb{R}$. If we take two real numbers $x, y \in \mathbb{R}$ such that $x \neq y$. Let us say that the first rational number attained is q_1 where $q_1 \in (x, y)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we can then continue this by saying the second number is $q_2 \in (x, q_1)$. From this we can create the sequence $q_n \in (x, q_{n-1})$ or $q_{n+1} = (x, q_n)$. This can also go the other way if we change q_2 to $q_2 \in (q_1, y)$. So we would then get $q_{n+1} \in (q_n, y)$.