Problem 1

$X = \{f : \mathbb{R} \to \mathbb{R}\}$

i) $(X, +)$ Is a group.
Closure: take two functions $f, g \in X$. Since f and g are real functions, which means that $f + g \in X$
Associativity: pointwise addition is associative since $x, y, z \in X$ we have $(x + y) + z = x + (y + z)$
Identity: pointwise addition has the identity element of the constant function $f(x) = 0$
Inverses: For every $f \in X$ there is an inverse element of $-f$

ii) $(X, \cdot)$ is a group if you exclude the constant function $f(x) = 0$ since that does not have an inverse.
Closure: take two functions $f, g \in X$. Since f and g are real functions, which means that $f \cdot g \in X$
Associativity: pointwise multiplication is associative since $x, y, z \in X$ we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
Identity: pointwise multiplication has the identity element of the constant function $f(x) = 1$
Inverses: For every $f \in X$ there is an inverse element of $\frac{1}{f}$

iii) $(X, \circ)$ is not a group. This is because the operation is not associative. if you have 2 functions $f, g \in X$ then $f \circ g \neq g \circ f$ Which means that it can not be a group.

Problem 2

Proof. Remember the set S from the lecture:

$$S = \{a \in \mathbb{N} | a = qb + r \text{ for some } q, r \in \mathbb{Z}, 0 \leq r < b\}$$

Since we have proved that the division algorithm holds for all $a > 0$, all we need to do is show that it holds for $-a$. So, what we want to find is

$$-a = q'b + r'$$

base case: a = 0
this is true for all b if $q = 0, r = 0$, we get:

$$0 = 0 \cdot b + 0$$

assume that $a = qb + r$ this means that, and take $q' = -q$ and $r' = -r$

$$-a = -qb - r$$
$$-a = q'b + r'$$

This shows that $-a \in S$ □

Problem 3

a) Invalid
b) Invalid
c) Invalid
d) Valid
e) Invalid
f) Valid
g) Valid
h) Valid

Problem 4

According to the lectures, I will use Axiom **(8)** which states that For all $x, y, z \in \mathbb{Z}$, if $x < y$ and $z > 0$, then $xz < yz$.

Case 1: $a > 0$ and $b > 0$
if we take $a > 0$ and multiply b then we get $ab > 0$, which means that $ab \neq 0$

Case 2: ($a > 0$ and $b < 0$) or ($a < 0$ and $b > 0$)
if we multiply $a > 0$ and b together, we will end up with $ab < 0$ which means that $ab \neq 0$

Case 3: $a < 0$ and $b < 0$
If we multiply $a < 0$ and b then we will get $ab > 0$ which means that $ab \neq 0$.

this means it has to be the case that either a or b is 0

Problem 5

Assume that $a = -1$ and $b = 1$ this means that $ab = -1$ which means that $ab \neq 1$. This is also true if the values of a and b are swapped.
This shows that it has to be the case that $a = b = 1$ or $a = b = -1$. and we know that if that is the case $ab = 1$ through axiom **(8)**

Problem 6

$S = \{n | n \neq 0, n \in \mathbb{Z}\}$ This set satisfies Closure and Inverses but does not cover identity since we have removed 0 from the set of integers.
$S = \{n | n \geq 0, n \in \mathbb{Z}\}$ This set satisfies Closure and the Identity element, but by removing the negative integers we have removed all the inverses for all elements.

Problem 7

i) In order for H to be a subgroup of G it needs to have elements within it to satisfy the 3 axioms. For example, H can not have an identity element if it has no elements, thus it needs to be non-empty.

ii) Assuming that the set H satisfies the Inverses axiom, then that means each element will have its inverse counterpart. Which means that according to axiom of closure that if $a, b \in G$ then $a \cdot b \in G$ which means that $a \cdot b^{-1} \in G$ must also be true.

Problem 8

We can show this by proof by contradiction. Assume some $x \in \bigcap_{i=0}^{\infty} m_i\mathbb{Z}$
Suppose that $x \neq 0$. This means that $x = m_i \cdot y$ for some $y \in \mathbb{Z}$. This means that $m_i|x$ for all i. This is contradiction as we know that all non-zero integers have a finite number of possible divisors. However, this is stating that there is an infinite number of divisors. Which means that it must be true that:

$$\bigcap_{i=0}^{\infty} m_i\mathbb{Z} = \{0\}$$

Problem 9

Assume that $A \nsubseteq B$ or $B \nsubseteq A$ and that $A \cup B$ is a subgroup of $\mathbb{Z}$
This means that there is some element $a \in A$ and $a \notin B$, likewise $b \in B$ and $b \notin A$.
However, since $A \cup B$ is a group then we can say that $ab \in A \cup B$. which means that either $ab \in A$ or $ab \in B$.
If $ab \in A$ then that means is must be true that $b \in A$ which is a contradiction.
This proves that $A \subseteq B$ or $B \subseteq A$ in order for $A \cup B$ to be a subgroup of $\mathbb{Z}$

Problem 10

a) $16\mathbb{Z} \cap 12\mathbb{Z} = 48\mathbb{Z}$
b) $5Z + 7Z = \mathbb{Z}$
c) $3\mathbb{Z} + (-3)\mathbb{Z} = 3\mathbb{Z}$
d) $12\mathbb{Z} \cap (3\mathbb{Z} + 9\mathbb{Z}) = 12\mathbb{Z}$
e) $5\mathbb{Z} + (10\mathbb{Z} \cap 55\mathbb{Z}) = 5\mathbb{Z}$

Problem 11

In order to prove that if H and K are subgroups of G, then $H \cap K$ is also a subgroup of G, we can refer to the three axioms.
i) Assume some identity element e. if $e \in H$ and $e \in K$ then we should also have $e \in H \cap K$
ii) Let $x \in H \cap K$. This would mean that $x^{-1} \in H$ and $x^{-1} \in K$. Since both H and K are subgroups then we can say that $g^{-1} \in H \cap K$.
iii) let $x, y \in H \cap K$. This means that $x \cdot y \in H$ and $x \cdot y \in K$. Which then means that $x \cdot y \in H \cap K$
This shows that if H and K are subgroups of G, then $H \cap K$ is also a subgroup of G.

Problem 12

Let $n, m \in \mathbb{Z}$. There will be some integers x and y such that $nx + my = 1$.
$d = \gcd(n, m)$. This means that $d|a$ and $d|b$. We can then say that $d|(nx + my)$
We can then turn this to $d|1$ since $nx + my = 1$. This means that the gcd is 1.
In order to show that n and m^2 are relatively prime we can change the equation as such:

$$nx + m^2y = 1$$

Since $n, m \in \mathbb{Z}$ we can alter the equation.

$$nx + m \cdot (m \cdot y) = 1$$

set $m \cdot y = z$

$$nx + mz = 1$$

With this we can still do the same proof and it would show that if n and m are relatively prime then n and m^2 are also relatively prime.