Problem 1

a)

Proof. base case: $n = 1$

$$9(10^{n-1} + 10^{n-2} + ... + 10 + 1) \leq 10^n$$

$$9(10^{1-1}) \leq 10^1$$

$$9 \leq 10$$

Inductive step: assume $9(10^{n-1} + 10^{n-2} + ... + 10 + 1) \leq 10^n$
Prove that $10^{n+1} \geq 9(10^n + 10^{n-1} + 10^{n-2} + ... + 10 + 1)$, using the assumption we can say that

$$\begin{aligned} 10^n \cdot 10 &\geq 9(10^{n-1} + 10^{n-2} + ... + 10 + 1) \cdot 10 \\ &= 9(10^n + 10^{n-1} + 10^{n-2} + ... + 100 + 10) \\ &\geq 9(10^n + 10^{n-1} + 10^{n-2} + ... + 10 + 1) \end{aligned}$$

This shows that $9(10^{n-1} + 10^{n-2} + ... + 10 + 1) \leq 10^n$ for all $n \in \mathbb{N}$

□

b) Prove that

$$\frac{9}{10^{m+1}} + \frac{9}{10^{m+2}} + ... + \frac{9}{10^n} \leq \frac{1}{10^m}$$

$$10^m \left(\frac{9}{10^{m+1}} + \frac{9}{10^{m+2}} + ... + \frac{9}{10^n} \right) \leq \frac{10^m}{10^m}$$

$$\frac{9}{10} + \frac{9}{100} + ... + \frac{9 \cdot 10^m}{10^n} \leq 1$$

$$10^n \left(\frac{9}{10} + \frac{9}{100} + ... + \frac{9 \cdot 10^m}{10^n} \right) \leq 10^n$$

$$9(10^{n-1} + 10^{n-2} + ... + 10^m) \leq 10^n$$

We can then say that

$$9(10^{n-1} + 10^{n-2} + ... + 10^m) \leq 9(10^{n-1} + 10^{n-2} + ... + 10 + 1)$$

Since the LHS is a sum from n to m whereas the RHS is a sum from n to 1. From this we can conclude that

Problem 2

a) Prove that $2^{n-1} + 2^{n-2} + ... + 1 \leq 2^n$

Proof. Base Case: $n = 1$

$$2^{1-1} \leq 2^1$$
$$1 \leq 2$$

Assume that $2^{n-1} + 2^{n-2} + ... + 1 \leq 2^n$
Prove that $2^n + 2^{n-1} + 2^{n-2} + ... + 1 \leq 2^{n+1}$
using the assumption we can say that

$$\begin{aligned}2^n \cdot 2 &\geq (2^{n-1} + 2^{n-2} + ... + 1) \cdot 2 \\ &= 2^n + 2^{n-1} + ... + 2 \\ &\geq 2^n + 2^{n-1} + ... + 1\end{aligned}$$

This shows that $2^{n-1} + 2^{n-2} + ... + 1 \leq 2^n$ is true for all $n \in \mathbb{N}$ □

b) Prove that $\frac{1}{k!} \leq \left(\frac{1}{2}\right)^{k-1}$

Proof. Base Case: $k = 2$

$$\frac{1}{2!} \leq \left(\frac{1}{2}\right)^{2-1}$$
$$\frac{1}{2} \leq \frac{1}{2}$$

Assume that $\frac{1}{k!} \leq \left(\frac{1}{2}\right)^{k-1}$ prove that $\frac{1}{(k+1)!} \leq \left(\frac{1}{2}\right)^{k}$ Using the assumption we can say that

$$\begin{aligned}\frac{1}{(k+1)!} &= \frac{1}{(k+1)k!} \\ &= \frac{1}{k+1} \cdot \frac{1}{k!} \\ &\leq \frac{1}{k+1} \cdot \left(\frac{1}{2}\right)^{k-1} = \frac{1}{2^{k-1}(k+1)} \\ &\leq \frac{1}{2^k}\end{aligned}$$

To show that the inequality is true we need to show that

$$2^{k-1}(k+1) \geq 2^k$$
$$2^{k-1}(k+1) \geq 2^{k-1} \cdot 2$$

we know that this statement is true because since $k \geq 2$ then the minimum value of $k+1$ is 3. So $2^{k-1}(k+1) \geq 2^k$ is true which then proves that $\frac{1}{k!} \leq \left(\frac{1}{2}\right)^{k-1}$ □

c)

Proof. Case 1: $n = 0$. We get $s_0 = 1$
Case 2: $n = 1$. We get $s_1 = 1 + 1 = 2$
Case 3: $n = 2$. we get $s_2 = 1 + 1 + \frac{1}{2} = 2.5$. We can then say that for any $n \in \mathbb{N}$ such that $n \geq 3$

$$s_n = 1 + 1 + \frac{1}{2} + ... + \frac{1}{(n-1)!} + \frac{1}{n!}$$

In order to show that $s_n \leq 3$ all we need to show is that (we take out the base case of n = 0)

$$1 + \frac{1}{2} + ... + \frac{1}{(n-1)!} + \frac{1}{n!} \leq 2$$

Using our result from a) we can do this

$$\begin{aligned}
1 + \frac{1}{2} + ... + \frac{1}{(n-1)!} + \frac{1}{n!} &\leq 1 + \frac{1}{2} + ... + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} \\
&= 2^{n-1} + \frac{2^{n-1}}{2} + ... + \frac{2^{n-1}}{2^{n-2}} + \frac{2^{n-1}}{2^{n-1}} \\
&= 2^{n-1} + 2^{n-2} + ... + 2 + 1 \leq 2^n \\
&= \frac{2^{n-1} + 2^{n-2} + ... + 2 + 1}{2^{n-1}} \leq \frac{2^n}{2^{n-1}} \\
&= 1 + \frac{1}{2} + ... + \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} \leq 2
\end{aligned}$$

From this we can conclude that for all $n \in \mathbb{N}$ that $s_n \leq 3$

□

Problem 3

prove that

$$x = 0.a_1a_2 \cdots a_na_1a_2 \cdots a_na_1a_2 \cdots$$

for all $n \in \mathbb{N}$. Let us multiply it by 10^n so we will end up with

$$x \cdot 10^n = a_1a_2 \cdots a_n.a_1a_2 \cdots a_na_1a_2 \cdots$$

$$x \cdot 10^n - x = a_1a_2 \cdots a_n$$

$$x(10^n - 1) = a_1a_2 \cdots a_n$$

$$x = \frac{a_1a_2 \cdots a_n}{10^n - 1}$$

This shows that x is a rational number.

Problem 4

Prove that $\{s_n\}$ converges such that $s_n = \sum_{k=0}^{n} \frac{a_k}{k!}$ $s_0 = a_0$
$s_1 = a_0 + a_1$
$s_2 = a_0 + a_1 + \frac{a_2}{2}$
$s_3 = a_0 + a_1 + \frac{a_2}{2} + \frac{a_3}{6}$
$s_n = a_0 + a_1 + \frac{a_2}{2} + \frac{a_3}{6} + \cdots + \frac{a_n}{n!}$
Each iteration of the sequence $\{s_n\}$ gets slightly larger, which would mean that the sequence is monotonically increasing. We also know that it is bounded as $\frac{a_k}{k!}$ approaches 0. since a_k is bounded and the denominator of the function is a factorial. as it approaches infinity it will converge towards 0. Which means that the summation is bounded. Which means that the sequence is convergent.

Problem 5