List of definitions (For problems 2 and 4)

Definition 1. *An integer n is even if $n = 2a$ for some $a \in \mathbb{Z}$*

Definition 2. *An integer n is off if $n = 2a + 1$ for some $a \in \mathbb{Z}$*

Definition 3. *Two integers have the same parity if they are both even or they are both odd. Otherwise, they have opposite parity*

Definition 4. *Suppose a and b are integers. We say that a divides b, written as $a|b$ if $b = ac$ for some $c \in \mathbb{Z}$. We can also say that a is a divisor of b, or b is a multiple of a*

Problem 2

Proof. By the definition of an odd number, let $x = 2a + 1$ for some $a \in \mathbb{Z}$

$$\begin{aligned}
x^3 &= (2a+1)^3 \\
&= (2a+1)(2a+1)(2a+1) \\
&= (2a+1)(4a^2+4a+1) \\
&= 8a^3 + 8a^2 + 2a + 4a^2 + 4a + 1 \\
&= 8a^3 + 12a^2 + 6a + 1 \\
&= 2(4a^3 + 6a^2 + 3a) + 1 \\
&\text{Let } n = 4a^3 + 6a^2 + 3a \\
&= 2n + 1
\end{aligned}$$

(note: the substitution with n is no necessary but makes the answer more clear)
From the definition of an odd number, it is proven that if x is odd, then x^3 is also odd.

Now to prove the other direction, we need to assume that x^3 is an odd number to show that x is also an odd number. If we take the definition of an odd number we can do the following.

$$\begin{aligned}
x^3 &= 2a + 1 \\
x^3 - 1 &= 2a \\
&\text{We then factor (x-1) from the left side} \\
(x-1)(x^2+x+1) &= 2a
\end{aligned}$$

Here, we can use problem 16 from chapter 4, we know that x^2 and x have the same parity. The sum of two integers that are the same parity is even, so the $+1$ in the second half of the product will be odd. However, we know that the equation should be even, since the left side is $2a$. Now we know that $(x-1)$ must be even for the product to be even, so we have it such that $(x-1) = 2a$ and that $x = 2a+1$. So we can say that if x^3 is odd, then x is also odd. $\square$

Problem 4

Proof. By the definition of an add number, let $x = 2a + 1$ for some $a \in \mathbb{Z}$ and let $y = 2b + 1$ for some $b \in \mathbb{Z}$.

$$\begin{aligned} xy &= (2a+1)(2b+1) \\ &= 4ab + 2a + 2b + 1 \\ &= 2(2ab + a + b) + 1 && \text{Let } b = 2ab + a + b \\ &= 2n + 1 \end{aligned}$$

(note: the substitution with n is no necessary but makes the answer more clear)
By the definition of an odd number, we can say that if x and y are odd numbers, then xy is also an odd number.

... □

Problem 6

Suppose $a, b, c \in \mathbb{Z}$. If $a|b$ and $a|c$ then $a|(b + c)$. For this proof we will use the following definiton

Definition. *Suppose a and b are integers. We say that a divides b, written as $a|b$ if $b = ac$ for some $c \in \mathbb{Z}$. We can also say that a is a divisor of b, or b is a multiple of a*

Proof. From the above statements, we know that since $a|b$ and $a|c$, that $b = am$ and $c = an$ for $m, n \in \mathbb{Z}$. So we have

$$\begin{aligned} b + c &= am + an \\ &= a(m + n) \\ \textbf{let } & j = m + n \\ &= aj \end{aligned}$$

So by definition we can say that $a|(b + c)$

□

Problem 8

Suppose a is an integer, If $5|2a$, then $5|a$. We will use the following definition

Definition. *Suppose a and b are integers. We say that a divides b, written as $a|b$ if $b = ac$ for some $c \in \mathbb{Z}$. We can also say that a is a divisor of b, or b is a multiple of a*

Proof. By definition we can say that since $5|2a$ that $2a = 5n$ for $n \in \mathbb{Z}$

$$\begin{aligned}
2a &= 5n \\
&\text{Let } n = 2m \text{ since we know that 2a is even} \\
2a &= 10m \\
a &= 5m
\end{aligned}$$

By definition, we can see that if $5|2a$, then $5|a$

□

Problem 14

If $n \in \mathbb{Z}$ then $5n^2 + 3n + 7$ is odd. We will use the following definitions.

Definition 5. *An integer n is even if $n = 2a$ for some $a \in \mathbb{Z}$*

Definition 6. *An integer n is off if $n = 2a + 1$ for some $a \in \mathbb{Z}$*

Proof. **Case 1:** First let us assume that n is even, so by definition, we can write it as $n = 2a$ for some $a \in \mathbb{Z}$. we then have

$$\begin{aligned}
5n^2 + 3n + 7 &= 5(2a)^2 + 3(2a) + 7 \\
&= 20a^2 + 6a + 7 \\
&= 20a^2 + 6a + 6 + 1 \\
&= 2(10a^2 + 3a + 3) + 1 \\
\textbf{Let } & m = 10a^2 + 3a + 3 \\
&= 2m + 1
\end{aligned}$$

By definition, if n is an even number, then $5n^2 + 3n + 7$ is also an odd number.
Case 2: Second let us assume that n is an odd number, so by definition, we can write it as $n = 2a + 1$ for some $a \in \mathbb{Z}$. We then have

$$\begin{aligned}
5n^2 + 3n + 7 &= 5(2a + 1)^2 + 3(2a + 1) + 7 \\
&= 5(4a^2 + 4a + 1) + 6a + 3 + 7 \\
&= 20a^2 + 20a + 5 + 6a + 10 \\
&= (20a^2 + 26a + 14) + 1 \\
&= 2(10a^2 + 13a + 7) + 1 \\
\textbf{Let } & m = 10a^2 + 13a + 7 \\
&= 2m + 1
\end{aligned}$$

By definition, if n is an odd number, then $5n^2 + 3n + 7$ is also an odd number.

Since we have shown for both cases that $5n^2+3n+7$ is an odd number. We can say that for all $n \in Z$ that $5n^2 + 3n + 7$ is odd.

$\square$

Problem 18

Suppose x and y are positive real number. If $x < y$ then $x^2 < y^2$.

Proof. Since we know that $x < y$. We can say two things by manipulating the inequality

$$x \cdot x < y \cdot x$$

$$x \cdot y < y \cdot y$$

From these two inequalities we can create one larger one as such

$$xx < xy < yy$$

From the inequality above we can then say that

$$xx < yy$$

$$x^2 < y^2$$

$\square$

Thus we have shown that if $x < y$ then $x^2 < y^2$

Problem 21

If p is prime and k is an integer for which $0 < k < p$, then p divides *pchoosek*

Proof. From the formula $\binom{p}{k} = \dfrac{p!}{(p-k)!k!}$, we get $p! = \binom{p}{k}(p-k)!k!$. Now, since the prime number p is a factor of $p!$ on the left, it must also be a factor of $\binom{p}{k}(p-k)!k!$ on the right. Thus the prime number p appears in the prime factorization of $\binom{p}{k}(p-k)!k!$. As $k!$ is a product of numbers smaller than p, its prime factorization contains no p's. Similarly the prime factorization of $(p-k)!$ contains no p's. But we noted that the prime factorization of $\binom{p}{k}(p-k)!k!$ must contain a p, so the prime factorization of $\binom{p}{k}$ contains a p. Thus $\binom{p}{k}$ is a multiple of p, so p divides $\binom{p}{k}$ $\square$