Problem 1

If we treat n as the index of the sequence. Then that means the statement

$$\lim_{n\to\infty} a_{2n} = L = \lim_{n\to\infty} a_{2n+1}$$

Shows that a_n converges. since a_{2n} Is all the even indexes of the sequence while a_{2n+1} takes care of all the odd indexes in the sequence. These can be treated as both subsequences of a_n. If all the subsequences of a_n converge, then it should be true that a_n also converges.

Problem 2

Assuming that $\{s_n\}$ is monotonically increasing and that $\{s_{n_k}\}$ and is converges. We want to show that $|s_n - s| \leq \epsilon$ for all $n \geq N$. Let us find some k such that $k \geq k - \epsilon$ and $k - \epsilon < s_{n_k} - s < 0$. So for any n we can then say that $s_{n_k} \leq s_n \leq s$. Then we can say

$$-\epsilon \leq s_{n_k} - s \leq s_n - s \leq 0$$

Which shows that s_n also converges

Problem 3

a) if $u_n = \dfrac{n^3}{e^n}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{u_{n+1}}{u_n} &= \lim_{n\to\infty} \frac{(n+1)^3}{e^{n+1}} \cdot \frac{e^n}{n^3} \\
&= \lim_{n\to\infty} \frac{1}{e} \cdot \frac{(n+1)^3}{n^3} \\
&= \frac{1}{e} < 1
\end{aligned}$$

So by the ratio test, it converges.

b) $b_n = n^{-1-\frac{1}{n}} = \dfrac{1}{n^{1+\frac{1}{n}}}$ and $a_n = \dfrac{1}{n^2}$

$a_n < b_n$ and since a_n diverges we know that b_n will also diverge.

c) I think you can show that this is divergent with the comparison test, but I am unsure what to use as a lower bound

d) $a_n = (n^{\frac{1}{n}} - 1)^n$

$$\begin{aligned}
\lim_{n\to\infty} |a_n|^{\frac{1}{n}} &= \lim_{n\to\infty} |(n^{\frac{1}{n}} - 1)^n|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} |n^{\frac{1}{n}} - 1| \\
&= 0
\end{aligned}$$

This shows that the sequence is absolutely convergent, which means that it is convergent.

Problem 4

a) If we know that $\sum_{n=1}^{\infty} a_n$ then we can say that

$$\sum_{n=1}^{\infty} a_n^2 = \left(\sum_{n=1}^{\infty} a_n\right)^2$$

Since a_n converges, we can do this, meaning that we can say a_n^2 also converges

b) Assuming a_n converges we can say

$$\frac{a_n}{1+a_n} = \frac{a_n}{1-a_n} = \frac{a_n}{1-\frac{1}{2}} = 2 \cdot a_n$$

Since a_n converges, using the comparison test we can say that $\dfrac{a_n}{1+a_n}$ also converges.

c) Using the same logic as the last problem we can say that

$$\frac{a_n^2}{1+a_n^2} = \frac{a_n^2}{1-a_n^2} = \frac{a_n^2}{1-\frac{1}{2}} = 2 \cdot a_n^2$$

in part a) we showed that a_n^2 is convergent. So through the comparison test we can once again show that $\dfrac{a_n^2}{1+a_n^2}$ is also convergent.

Problem 5

Not necessarily. Take the sequence $a_n = \dfrac{1}{n}$. We know that this sequence will converge towards 0 if we take the limit. However, it is a different story if it is in a series. We know that $\sum_{n=1}^{\infty} \dfrac{1}{n}$ is the harmonic series, which diverges.