Problem 1

Using the limit definitions of little o. We know that if $h(n) = o(f(n))$ then $\lim_{n\to\infty} \dfrac{h(n)}{f(n)} = 0$. We can then show that $f(n) + h(n) = \Theta(f(n))$ as such:

$$\begin{aligned}\lim_{n\to\infty} \frac{f(n)+h(n)}{f(n)} &= \lim_{n\to\infty} \frac{f(n)}{f(n)} + \lim_{n\to\infty} \frac{h(n)}{f(n)} \\ &= \lim_{n\to\infty} 1 + \lim_{n\to\infty} 0 \\ &= 1 + 0 = 1\end{aligned}$$

We have then shown that

$$0 < \lim_{n\to\infty} \frac{f(n)+h(n)}{f(n)} < \infty$$

So we know that $f(n) + h(n) = \Theta(f(n))$

Problem 2

a)

Loop Invariant: At the start of each iteration of the outer loop for some value j. The first j elements of the array are sorted, and all the original values are still in the array.

Initialization: During the very first iteration of the outer loop(i.e j = 1), The very first elements of the array A[1] will be sorted, all the original values are still in the array.

Maintenance: When the j^{th} iteration starts, the previous $j-1$ iterations have the elements from A[1] to A[j-1] already sorted since during this iteration we are placing the current largest value into the A[j-1] position in the array. All original values are still in the array

Termination: When $j = n$ the loop will terminate. Assuming the "maintenance" step is true, then all the values from A[1] to A[n-1] are sorted an the last element of the array A[n] should be the smallest value element. We can then say that all elements in the array are properly sorted, and all the original values are still in the array.

b)

Since the outer loop has a number of $n-1$ iterations, and for each inner loop, there is an average of $\dfrac{n}{2}$ iterations since the inner loop decreases as j increases in value.

In this case we have $f(n) = (n-1)(\dfrac{n}{2})$. We can then say that $g(n) = \Theta(n^2)$

$$\begin{aligned}\lim_{n\to\infty} \frac{(n^2/2 - n/2)}{n^2} &= \lim_{n\to\infty} \frac{n/2 - 1/2}{n} \\ &= \frac{1}{2}\end{aligned}$$

So by definition we can say that the runtime of the reverse sort is $\Theta(n^2)$

Problem 3

a) For $n \geq 1$ show that $\sum_{k=1}^{n}(k)(k+1) = \dfrac{(n)(n+1)(n+2)}{3}$. We can use weak induction to prove this statement.

Proof. **Base Case: n = 1** we have

$$\sum_{k=1}^{1}(n)(n+1) = \frac{(1)(2)(3)}{3}$$
$$(1)(2) = (1)(2)$$
$$2 = 2$$

Inductive Hypothesis: assume that $\sum_{k=1}^{n}(k)(k+1) = \dfrac{(n)(n+1)(n+2)}{3}$ is true

Inductive step: prove for all $n+1$, so show that $\sum_{k=1}^{n+1}(k)(k+1) = \dfrac{(n+1)(n+2)(n+3)}{3}$

$$\begin{aligned}
\sum_{k=1}^{n+1}(k)(k+1) &= (n+1)(n+2) + \sum_{k=1}^{n}(k)(k+1) \\
&= (n+1)(n+2) + \frac{(n)(n+1)(n+2)}{3} \\
&= \frac{3(n+1)(n+2)}{2} + \frac{(n)(n+1)(n+2)}{3} \\
&= \frac{3(n+1)(n+2) + (n)(n+1)(n+2)}{3} \\
&= \frac{(n+1)(n+2)(n+3)}{3}
\end{aligned}$$

Thus we have proven that $\sum_{k=1}^{n}(k)(k+1) = \dfrac{(n)(n+1)(n+2)}{3}$ for all $n+1$ by induction

□

b) Prove that for some set $S = \{1, 2, 3, ..., n\}$ such that $n \geq 1$ and $|S| = n$, that the number of subsets with an **odd** cardinality is 2^{n-1}. We can prove this statement with a strong induction proof.

Proof. **Base case:** When $n = 1$, we have the set $S = \{1\}$. So our subsets with odd cardinality will be $\{1\}$, since the only other subset is the empty set, and it has a cardinality of 0. We also have $2^{1-1} = 2^0 = 1$, so the base case is satisfied.

Inductive Hypothesis: Assume that when $|S| = n$ that the number of odd cardinality subsets is 2^{n-1}. Prove for all $n+1$ that if $|S| = n+1$ that the number of odd cardinality subsets are 2^n

Inductive Step: Let $|S| = n+1$ and let $x \in S$. We also can construct another set S' such that $S' = S - \{x\}$ and $|S'| = k$. We can then call the set $E \subset S'$ as all the subsets with even cardinality, and the set $O \subset S'$ as all the subsets with odd cardinality. By definition the size of both sets should be 2^{n-1}. We can then take the set E, and for each subset, we can take $E \cup \{x\}$. We then have all the odd subsets O that do not contain the element $\{x\}$ and all odd cardinality subsets that do contain $\{x\}$ as $E \cup \{x\}$. If we add their cardinalities together we have $2^{n-1} + 2^{n-1} = 2^n$. Thus we have shown that the numbre of odd cardinality subsets of a set S, such that $|S| = n$ is 2^{n-1}

$\square$

Problem 4

a)

Level	no. Problems	Size of each problem	Amount of "work" at this level
0	1	n	5n
1	9	n/3	15n
...	...	...	...
t	9^t	$n/3^t$	$3^t \cdot 5n$
...	...	...	...
leaf	n^2	c	$\Theta(n^2)$

b)

$$T(n) = c \qquad \textbf{iff } n = 0$$
$$T(n) = 9T(\frac{n}{3}) + 5n \qquad \textbf{iff } n \geq 1$$

Let us guess that $T(n) = O(n^2)$. Then we can say that $T(n) \leq kn^2$ such that $\forall n > n_0$ and $k > 0$.

Base cases

$$T(1) = 9c + 5 \leq k + 5$$
$$T(2) = 9c + 10 \leq 4k + 10$$

Inductive step: since $\frac{n}{3} < n$we have it such that

$$\begin{aligned} T(n) = 9T(\frac{n}{3}) + 5n &\leq 9kn^2 + 5n \\ &\leq 9k\left(\frac{n}{3}\right)^2 + 5(\frac{n}{3}) \\ &\leq 9k \cdot \frac{n^2}{9} + \frac{5n}{3} \\ &\leq kn^2 + n \\ &\leq n^2 \end{aligned}$$

So we have shown that $T(n) = O(n^2)$

Problem 5

a) $\Theta(n^3)$
b) cant be used
c) $\Theta(\sqrt{n}\log(n))$
d) cant be used
e) $\Theta(n)$
f) cant be used