Problem 1

Proof. First we can recall the definitions of both $O(g(n))$ and $\Omega(g(n))$

$$O(g(n)) \in \{f(n)|\exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n)\}$$

$$\Omega(g(n)) \in \{f(n)|\exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n)\}$$

Now if we look at the intersection of both sets, i.e $O(g(n)) \cap \Omega(g(n))$, $O(g(n))$ is the set such that all $f(n) \leq cg(n)$ and the set $\Omega(g(n))$ is the set such that all $cg(n) \leq f(n)$. Thus the intersection of both sets will be where $c_1 g(n) \leq f(n) \leq c_2 g(n)$. We can then formally define the set as

$$\Theta(g(n)) \in O(g(n)) \cap \Omega(g(n)) = \{f(n)|\exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n)\}$$

Thus we have shown that both definitions are equivalent. $\square$

Problem 2

Since the log base is just a constant, I will just have them in base e to make my life easier.
a)

$$\begin{aligned}
\lim_{n\to\infty} \frac{2^n}{(nlg(n))^2} &= \lim_{n\to\infty} \frac{ln(2) \cdot 2^n}{2n \cdot lg(n)(lg(n)+1)} \\
&= \lim_{n\to\infty} \frac{2^n}{2lg(n) + 6lg(n) + 2} \\
&= \lim_{n\to\infty} \frac{n \cdot 2^n}{4lg(n) + 6} \\
&= \lim_{n\to\infty} n(n+1)2^n = \infty
\end{aligned}$$

Thus $f(n) = \omega(g(n))$

b)

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sqrt{n}}{lg(n)} &= \lim_{n\to\infty} \frac{\frac{1}{2}n^{-\frac{1}{2}}}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{1}{2}n^{\frac{1}{2}} = \infty
\end{aligned}$$

$f(n) = \omega(g(n))$

c)

$$\lim_{n\to\infty} \frac{lg(n)^{lg(n)}}{n^3} = \left(\lim_{n\to\infty} \frac{lg(n)}{n^3}\right)^{lg(n)}$$
$$= \lim_{n\to\infty} \left(\frac{\frac{1}{n}}{3n^2}\right)^{lg(n)} = 0$$

$f(n) = o(g(n))$

d) $f(n) = o(g(n))$ because 3^{2^n} grows at a lower rate than 2^{3^n}

e)

$$\lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{n!(n+1)}$$
$$= \lim_{n\to\infty} \frac{1}{n+1} = \infty$$

$f(n) = o(g(n))$

f)

$$test$$

Problem 3

Proof. **a)** If $f(n) \in O(g(n))$, this implies that there exists some constant c such that $c \in \mathbb{N}$ and $c > 1$, so that $f(n) \leq cg(n)$. From here we have it such that

$$\begin{aligned} f(n) \leq cg(n) \rightarrow lg(f(n)) &\leq lg(cg(n)) \\ = lg(f(n)) &\leq lg(c) + lg(g(n)) \\ \text{since } lg(g(n)) &\geq 1 \text{ we can do the following} \\ = lg(f(n)) &\leq lg(c) \cdot lg(g(n)) + lg(g(n)) \\ = lg(f(n)) &\leq (lg(c) + 1)lg(g(n)) \\ = lg(f(n)) &\leq O(lg(g(n))) \end{aligned}$$

Thus proven that $f(n) \in O(g(n))$ implies that $lg(f(n)) \in lg(O(g(n))$

□

Proof. **b)** $f(n) = O(g(n))$ does not imply that $2^{f(n)} \in O(2^{g(n)})$ This can be proven by counter example. Assume that $f(n) = 2n$ and let $g(n) = n$. We can see that $2n \in O(n)$ since

$$\lim_{n\to\infty} \frac{2n}{n} = 2$$

However, if we take the limit as such

$$\lim_{n\to\infty} \frac{2^{2n}}{2^n} = \lim_{n\to\infty} 2^n = \infty$$

Which means that $2^{f(n)} \notin O(2^{g(n)})$

$\square$

Problem 4

Proof. Let $Q(N)$ be some with degree k such that $k \in \mathbb{N}$ and $k > 0$. We can say that

$$Q(N) = c_1x^k + c_2x^{k-1} + ... + c_n$$

where $c_1, c_2, ..., c_n$ are coefficients. Similarly, we can say that $P(N)$ is another polynomial with degree n such that $n \in \mathbb{N}$ and $0 < n \leq k$, we can represent $P(N)$ as such

$$P(N) = d_1x^n + d_2x^{n-1} + ... + d_n$$

where $d_1, d_2, ..., d_n$ are a different set of coefficients. We can then take the limit of $\frac{P(n)}{Q(n)}$ as such

$$\lim_{n\to\infty} \frac{P(n)}{Q(n)} = \lim_{n\to\infty} \frac{c_1x^k + c_2x^{k-1} + ... + c_n}{d_1x^n + d_2x^{n-1} + ... + d_n}$$

If we take this limit, we know that both $P(n)$ and $Q(n)$ go to ∞. So we can use L'hoptials rule and derive both functions. Since, its a polynomial, the functions will keep deriving, until the coefficient of the highest degree term will be left (i.e c_1 or d_1).
In case that $n < k$, then the limit will be reduce to the following

$$\lim_{n\to\infty} \frac{0}{d_1} = 0$$

In the second case where $k = n$, then the limit will be reduced as such

$$\lim_{n\to\infty} \frac{c_1}{d_1}$$

If $\lim_{n\to\infty} \frac{P(n)}{Q(n)} = L$. Then we can say that $0 \leq L < \infty$. So $P(n) \in O(Q(n))$

$\square$

Problem 5

Case 1: a = 1 If $a = 1$ then the expression $f(n) = \Sigma_{i=1}^{n} a^i = n$. Since 1 to the power of anything will simply become 1, so the expression becomes a summation of 1, n n number of times. With this, we can say that $g(n) = n$, we then have

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n}{n} = \lim_{n\to\infty} 1 = 1$$

With this we can say that if a = 1, then $f(n) \in \Theta(n)$

Case 2: a > 1 The closed form for the summation above is essentially just the geometric series

$$\Sigma_{i=1}^{n} a^i = \frac{a^{n+1} - 1}{a - 1} = \frac{a \cdot a^n - 1}{a - 1} = \frac{a^{n+1}}{a - 1} - \frac{1}{a - 1}$$

We know that $\forall n, \Sigma_{i=1}^{n} a^i < \dfrac{a^{n+1}}{a - 1}$ and $\Sigma_{i=1}^{n} a^i > a^n$, so

$$\begin{aligned} c^n &< \frac{a^{n+1}}{a-1} - \frac{1}{a-1} \\ \frac{1}{a-1} &< \left(\frac{a}{a-1} - 1\right) a^n \\ 1 &< a^n \\ 0 &< n \end{aligned}$$

So $f(n) \in \Theta(a^n)$

Case 3: 0 < a < 1 In this case, we know that $0 < \Sigma_{i=1}^{n} a^i < 1$, so we can say that $f(n) \in \Theta(1)$