Problem #1

(a) $Pr(50 \le x \le 70) = \int_{50}^{70} \frac{1}{78-42} \, dx = \frac{5}{9}$

(b) Pr (A = Two of the first seven customer is between 50 and 70) $= C_6^2 \cdot (\frac{5}{9})^2 (1-\frac{5}{9})^{6-2}$

≈ 0.181

Pr (A and the 7th customer is between 50 and 70)

$= Pr(A) \cdot Pr(50 \le x \le 70)$

≈ 0.10

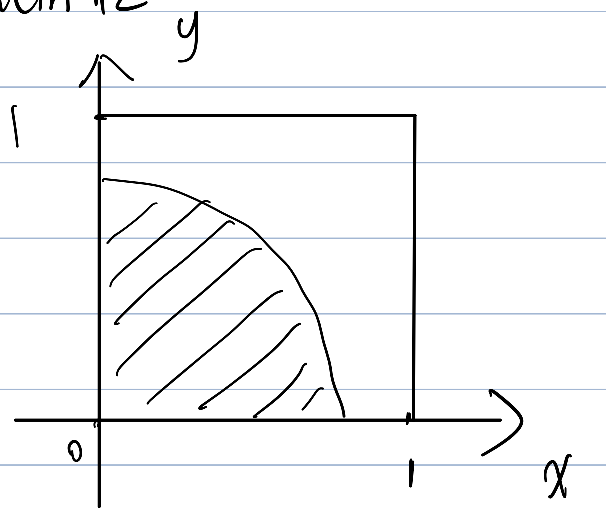
(c) Pr (Three of the 7 people in bank is between 50 and 70)

$= C_7^3 (\frac{5}{9})^3 (1-\frac{5}{9})^{7-3}$

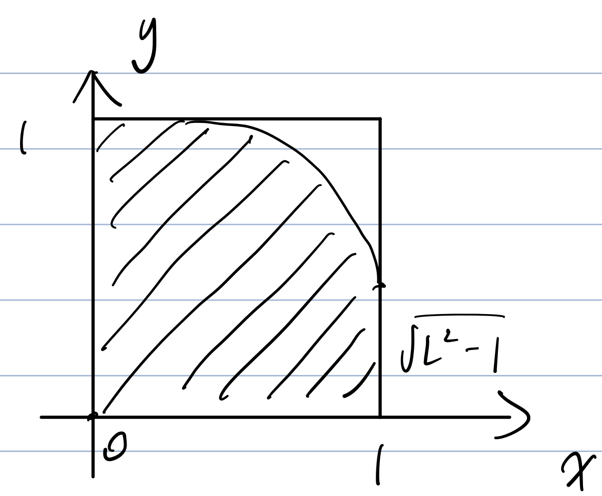
≈ 0.234

Problem #2

(a)



$0 \le L \le 1$



$1 \le L \le \sqrt{2}$

when $0 \le L \le 1$, $P = \frac{\pi}{4} L^2$

when $1 \le L \le \sqrt{2}$, we get

$$P = 2 \cdot 1 \cdot \sqrt{L^2 - 1} + \left(\frac{\pi}{2} - 2 \tan^{-1} \sqrt{L^2 - 1} \right) \cdot \frac{\pi L^2}{2\pi}$$

$$= \frac{\pi}{4} L^2 + \sqrt{L^2-1} - L^2 \tan^{-1}(\sqrt{L^2-1})$$

The Result of Monte-Carlo Simulation is shown in the code.

(b) Distance $= \int_0^1 \int_0^1 \sqrt{x^2+y^2} \, dx dy$

$$= \int_0^1 \frac{1}{2} x \sqrt{x^2+y^2} + \frac{1}{2} y^2 \ln(x + \sqrt{x^2+y^2}) \Big|_0^1 dy$$

$$= \frac{1}{4}\sqrt{2} + \frac{1}{4} \ln(1+\sqrt{2}) + \frac{1}{2} \int_0^1 y^2 \ln(1+\sqrt{1+y^2}) dy$$

$$= \frac{1}{4}(\sqrt{2} + \ln(1+\sqrt{2})) + \frac{1}{18} + \frac{1}{2} \int_0^1 y^2 \ln(1+\sqrt{1+y^2}) dy$$

$$\approx 0.765$$

The simulation Result is shown in the code

Problem #3

(a) $\int_{-\infty}^{+\infty} f(x) dx = \int_{-a}^{0} \frac{b(x+a)}{a} dx + \int_0^{+\infty} b e^{-\lambda x} dx$

$$= bx + \frac{b}{2a} x^2 \Big|_{-a}^{0} - \frac{b}{\lambda} e^{-\lambda x} \Big|_0^{+\infty}$$

$= \quad \frac{ab}{\quad} \quad \frac{b}{\quad} = 1$

$$ab - \frac{ab}{2} + \frac{b}{\lambda} = 1$$

$$\Rightarrow \quad \frac{ab}{2} + \frac{b}{\lambda} = 1$$

(b) For $x \le -a$, we get $F(x) = 0$

and for $-a \le x \le 0$, we get

$$F(x) = \int_{-a}^{x} b(x+a)/a \, dx$$

$$= bx + \frac{b}{2a} x^2 \Big|_{-a}^{x}$$

$$= bx + \frac{b}{2a} x^2 + ab - \frac{b}{2a} \cdot a^2$$

$$= bx + \frac{b}{2a} x^2 + \frac{ab}{2}$$

for $x \ge 0$, we get

$$F(x) = \int_{-a}^{0} b(x+a)/a \, dx + \int_{0}^{x} b e^{-\lambda x} dx$$

$$= \frac{ab}{2} + \left(-\frac{b}{\lambda} e^{-\lambda x}\right)\Big|_{0}^{x}$$

$$= \frac{ab}{2} + \frac{b}{\lambda} - \frac{b}{\lambda} e^{-\lambda x}, \quad \text{since } \frac{ab}{2} + \frac{b}{\lambda} = 1$$

$$\Rightarrow F(x) = \begin{cases} 0 & , x < -a \\ bx + \frac{b}{2a} x^2 + \frac{ab}{2} & , -a \le x \le 0 \\ 1 - \frac{b}{\lambda} e^{-\lambda x} & , 0 < x \end{cases}$$

Problem #4

For a single game, the probability that the game ends in this round is

$$P_e = P_1 P_2 (1-P_3) + P_1 P_3 (1-P_2) + P_2 P_3 (1-P_1)$$
$$+ P_1 (1-P_2)(1-P_3) + P_2 (1-P_1)(1-P_3) + P_3 (1-P_1)(1-P_2)$$

the probality that the game ends at the xth round is

$$P = (1-P_e)^{x-1} P_e$$

$$E(X) = \sum_{x=1}^{\infty} x \cdot (1-P_e)^{x-1} P_e = \frac{1}{P_e}$$

For $P_1 = \frac{1}{4}$, $P_2 = \frac{1}{2}$ and $P_3 = \frac{3}{4}$, we have

$$P = \frac{1}{4} \cdot \frac{1}{2} \cdot (1-\frac{3}{4}) + \frac{1}{4} \cdot \frac{3}{4} \cdot (1-\frac{1}{2}) + \frac{1}{2} \cdot \frac{3}{4} \cdot (1-\frac{1}{4})$$
$$+ \frac{1}{4} \cdot (1-\frac{1}{2}) \cdot (1-\frac{3}{4}) + \frac{1}{2} \cdot \frac{3}{4} \cdot (1-\frac{1}{4}) + \frac{3}{4} \cdot (1-\frac{1}{4}) \cdot (1-\frac{1}{2})$$

$$= \frac{13}{16}, \quad \Rightarrow E(X) = \frac{16}{13}$$

Problem #5

If this equation yields real results for s,

the $B^2 - 4AC \geq 0$

And since A, B, C are exponentially distributed

$$f_A(a) = f_B(b) = f_C(c) = \lambda e^{-\lambda x}, \quad x > 0$$

$$P(B^2 - 4AC \geq 0) = \int^{\infty} \int^{\infty} \int^{\infty} \lambda e^{-\lambda a} \lambda e^{-\lambda b} \lambda e^{-\lambda c}$$

$$\int_0 \int_0 \int_{2\sqrt{ac}} \cdots \, db\,da\,dc$$

$$= -\int_0^{\infty}\int_0^{\infty} \lambda^2 e^{-\lambda a} e^{-\lambda c} \left(-e^{-\lambda b}\right)\Big|_{2\sqrt{ac}}^{\infty} da\,dc$$

$$= -\int_0^{\infty}\int_0^{\infty} \lambda^2 e^{-\lambda a} e^{-\lambda c} e^{-\lambda 2\sqrt{ac}}\, da\,dc$$

$$= \lambda^2 \int_0^{\infty}\int_0^{\infty} e^{-(\lambda a + \lambda c + \lambda 2\sqrt{ac})}\, da\,dc$$

$$= \lambda^2 \int_0^{\infty}\int_0^{\infty} e^{-(\sqrt{\lambda a} + \sqrt{\lambda c})^2}\, da\,dc$$

$$= \frac{\lambda^2}{3\sqrt{\lambda}^3\sqrt{\lambda}} = \frac{1}{3}$$

And the simulation result is 0.33

Problem #6

(a) $$Pr(A \text{ wins the game}) = \sum_{i=0}^{\infty} (1-P_A)^i (1-P_B)^i P_A$$

$$= \lim_{i\to\infty} \frac{P_A \times \left(1-(1-P_A)^i(1-P_B)^i\right)}{1-(1-P_A)(1-P_B)}$$

$$= \frac{P_A}{1-(1-P_A)(1-P_B)}$$

$$Pr(B \text{ wins the game}) = \sum_{i=1}^{\infty} (1-P_A)^i (1-P_B)^i \frac{P_B}{1-P_B}$$

$$= \lim_{i \to \infty} \frac{(1-p_A)p_B \times (1-(1-p_A)^i(1-p_B)^i)}{1-(1-p_A)(1-p_B)}$$

$$= \frac{(1-p_A)p_B}{1-(1-p_A)(1-p_B)}$$

we let $\dfrac{(1-p_A)p_B}{1-(1-p_A)(1-p_B)} = \dfrac{p_A}{1-(1-p_A)(1-p_B)}$

$$(1-p_A)p_B = p_A$$

$$p_B = \frac{p_A}{1-p_A}$$

(b)

$$E = \sum_{i=0}^{\infty} (2i+1)(1-p_A)^i(1-p_B)^i p_A$$

$$+ \sum_{i=1}^{\infty} (2i+2)(1-p_A)^i(1-p_B{}^i) \frac{p_B}{1-p_B}$$

$$= (2p_A + 2(1-p_A)p_B) \sum_{i=0}^{\infty} i(1-p_A)^i(1-p_B)^i$$

$$+ (p_A + 2(1-p_A)p_B) \sum_{i=0}^{\infty} ((1-p_A)^i(1-p_B)^i$$

$$= \frac{(2p_A + 2((1-p_A)p_B)(1-p_A)(1-p_B)}{(1-(1-p_A)(1-p_B))^2} + \frac{p_A + 2(1-p_A)p_B}{1-(1-p_A)(1-p_B)}$$

$$= \frac{2-p_A}{p_A + p_B - p_A p_B}$$

(c) see the code

Simulation problem #1

According to the simulation result, the max profit is about \$310, when λ is in $[0.65, 0.7]$